Chapter 3

Cone-Manifolds

To find a geometric structure on a topological 3-orbifold Q, we will typically start with a complete hyperbolic structure on $Q - \Sigma$ (a Kleinian group) and try to *deform* this to a hyperbolic structure on the 3-orbifold Q (another Kleinian group). The intermediate stages will be hyperbolic metrics with cone-type singularities — **3-dimensional hyperbolic cone-manifolds.**

3.1 Definitions

An *n-dimensional cone-manifold* is a manifold, M, which can be triangulated so that the link of each simplex is piecewise linear homeomorphic to a standard sphere and M is equipped with a complete path metric such that the restriction of the metric to each simplex is isometric to a geodesic simplex of constant curvature K. The cone-manifold is *hyperbolic*, *Euclidean* or *spherical* if K is -1, 0, or $+1$.

Remark: We could allow more general topology, for example M a rational homology n-manifold. Most arguments still apply in that setting.

The *singular locus* Σ of a cone-manifold M consists of the points with no neighbourhood isometric to a ball in a Riemannian manifold. It follows that

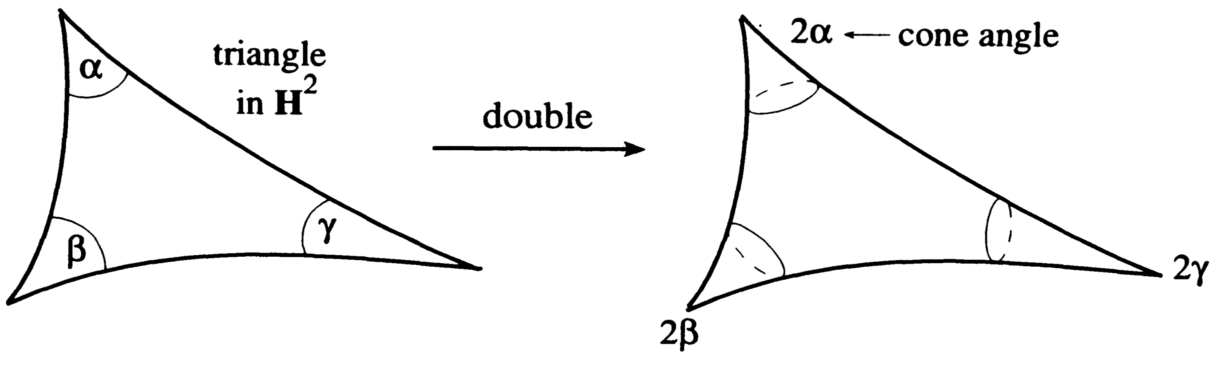
- Σ is a union of totally geodesic closed simplices of dimension $n-2$.
- At each point of Σ in an open $(n-2)$-simplex, there is a *cone angle* which is the sum of dihedral angles of n-simplices containing the point.
- $M - \Sigma$ has a smooth Riemannian metric of constant curvature K, but this metric is *incomplete* if $\Sigma \neq \emptyset$.

We will see that many of the techniques and results from Riemannian geometry also apply to cone-manifolds. Useful references on the geometry of cone-manifolds and more general spaces of piecewise constant curvature include [12] and [47].

Example 3.1.
Let Δ be the lune (or bigon) in S^2 contained between two geodesics making an angle α. The double of Δ gives a cone-manifold structure on S^2 with two *cone points* each with cone angle 2α. This cone-manifold is called a *football.*

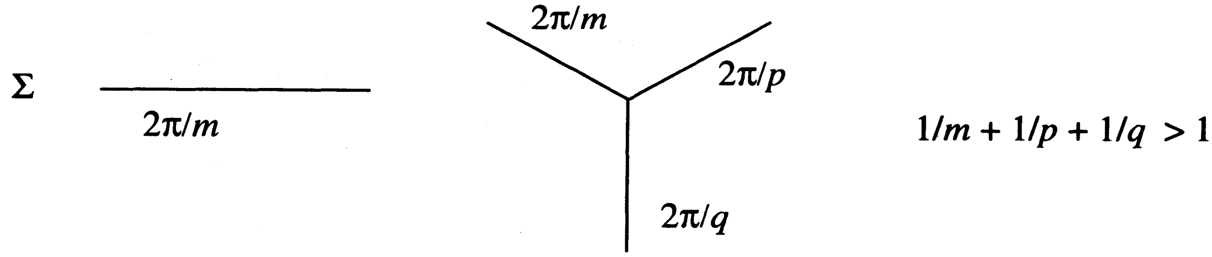
Example 3.2.
Let Δ be a triangle with angles $0 < \alpha, \beta, \gamma < \pi$ in
- $\mathbb{H}^2$ if $\alpha + \beta + \gamma < \pi$,
- $\mathbb{E}^2$ if $\alpha + \beta + \gamma = \pi$,
- S^2 if $\alpha + \beta + \gamma > \pi$.

Doubling the triangle Δ gives a cone-manifold structure on S^2 with three *cone points* with cone angles $2\alpha, 2\beta, 2\gamma$. These cone-manifolds are called *turnovers* and are hyperbolic, Euclidean or spherical respectively. These cone-manifolds are denoted $S^2(2\alpha, 2\beta, 2\gamma)$.



Remark: In the spherical case, not all angles satisfying $\alpha + \beta + \gamma > \pi$ are possible: $\pi - \alpha, \pi - \beta, \pi - \gamma$ represent the edge lengths of the "dual" or "polar" spherical triangle, so must satisfy three triangle inequalities (see, for example, [5]).

Example 3.3.
A closed, orientable, constant curvature orbifold of dimension 2 or 3 is a cone-manifold with cone angles of the form $2\pi/m$, where $m \geq 2$ is an integer. The singular locus locally looks like:

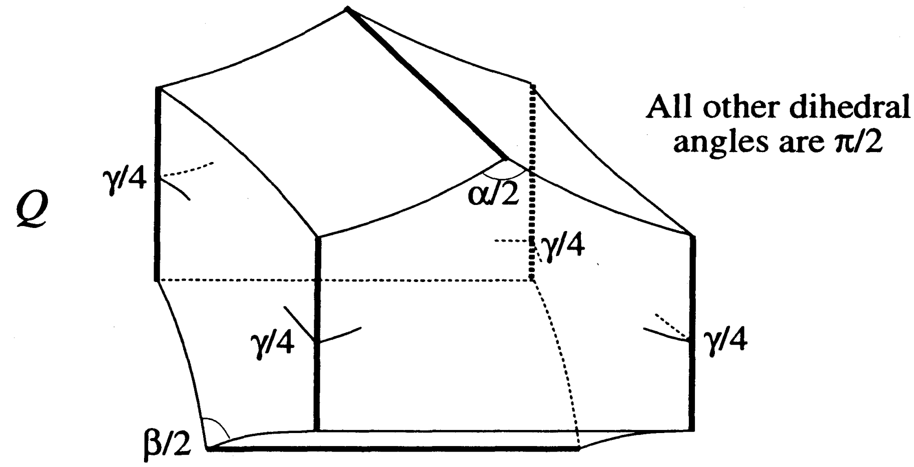


In 3.6 below, we will see that if M is an orientable 3-dimensional cone-manifold with cone angles at most π then the singular locus is again a trivalent graph, and the link of every point in the singular locus is a football or turnover.

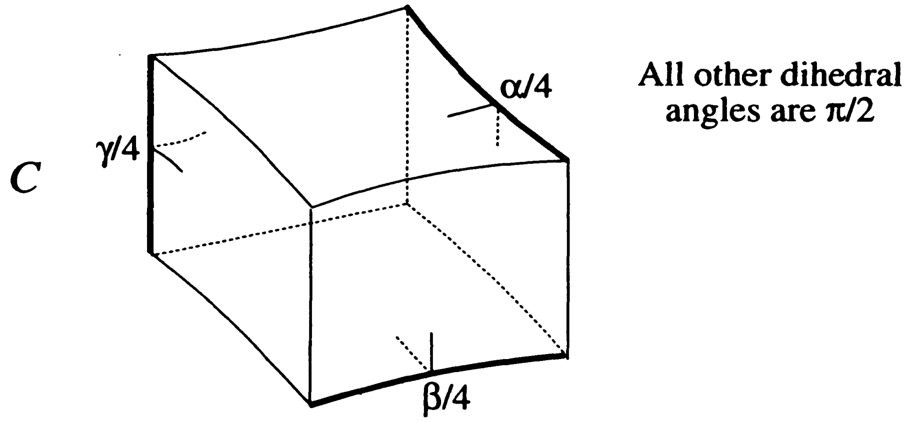
Example 3.4.
Let $T^3 = S^1 \times S^1 \times S^1$ be a 3-torus and let Σ consist of three simple closed curves in orthogonal directions, say $S^1 \times p \times q, q \times S^1 \times p, p \times q \times S^1$ where p, q are distinct points in S^1. Then $T^3 - \Sigma$ is homeomorphic to the complement of the Borromean rings, as can be seen in 2.32.

We will construct hyperbolic cone-manifold structures on T^3 with arbitrary cone angles $0 \leq \alpha, \beta, \gamma < 2\pi$ along the three components of Σ. (Cone angle "zero" corresponds to a cusp.) Further, these hyperbolic structures on T^3 degenerate as any angle approaches 2π.

We begin with a polyhedron Q in $\mathbb{H}^3$ as shown below.

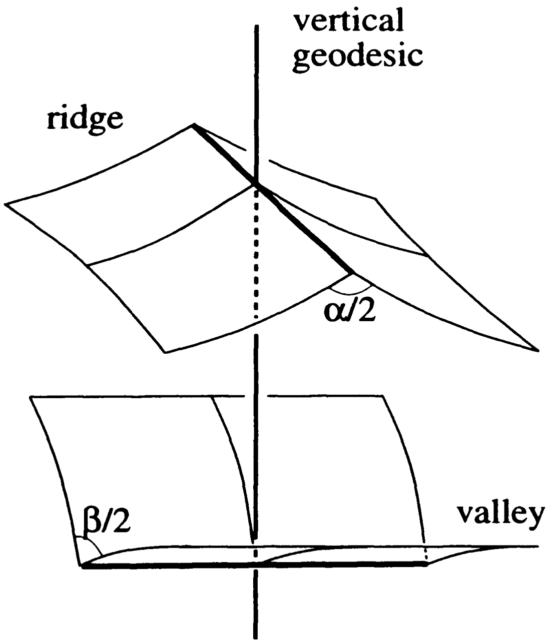


Such a polyhedron can be constructed whenever α, β, γ are less than 2π. For example, Q can be obtained from four copies of a hyperbolic cube C with dihedral angles as shown below; such a cube exists by a theorem of Andreev [3] which characterizes the convex polyhedra in $\mathbb{H}^3$ with all dihedral angles $\leq \pi/2$.



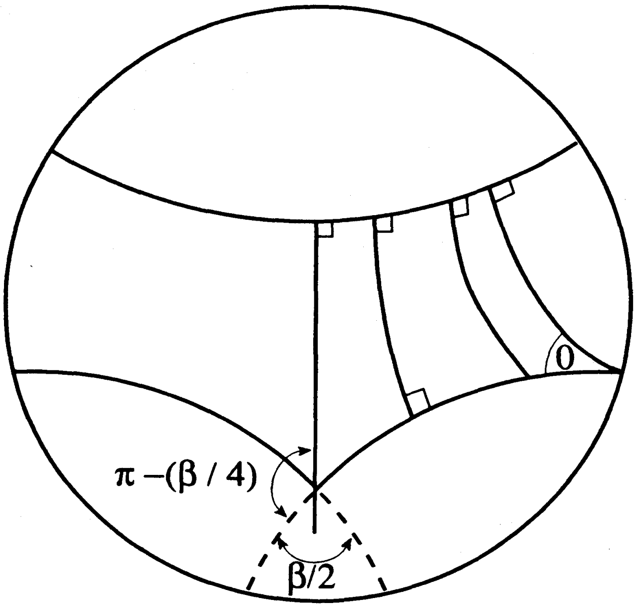
Gluing together the opposite vertical faces of Q gives a hyperbolic structure on $T^2 \times I$ bent along a curve with angle $\alpha/2$ on top and a curve with angle $\beta/2$ on the bottom, and with a vertical axis with cone angle γ. Finally, doubling this gives a hyperbolic cone-manifold structure on T^3 with cone angles α, β, γ along the components of Σ.

Here is a direct construction of Q, which also gives some idea of the variation in the shape of Q as the angles α, β, γ are varied. Begin with a "vertical" geodesic in $\mathbb{H}^3$ and two pairs of planes: one pair meeting in a ridge line with angle $\alpha/2$, the other meeting along a valley line with angle $\beta/2$. Further, the geodesics defining the ridge and the valley should be orthogonal to the vertical geodesic and to each other (after vertical translation).



There is a one parameter family of planes orthogonal to the ridge meeting one side of the valley; with angles of intersection varying monotonically from

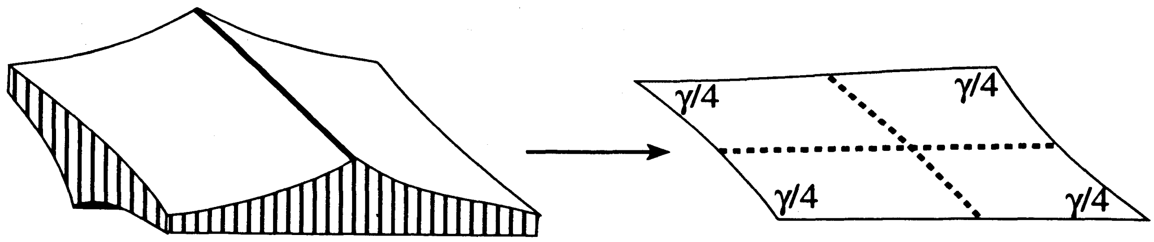
0 to $\pi - \beta/4$. Hence, there is a unique such plane for which the angle is $\pi/2$.



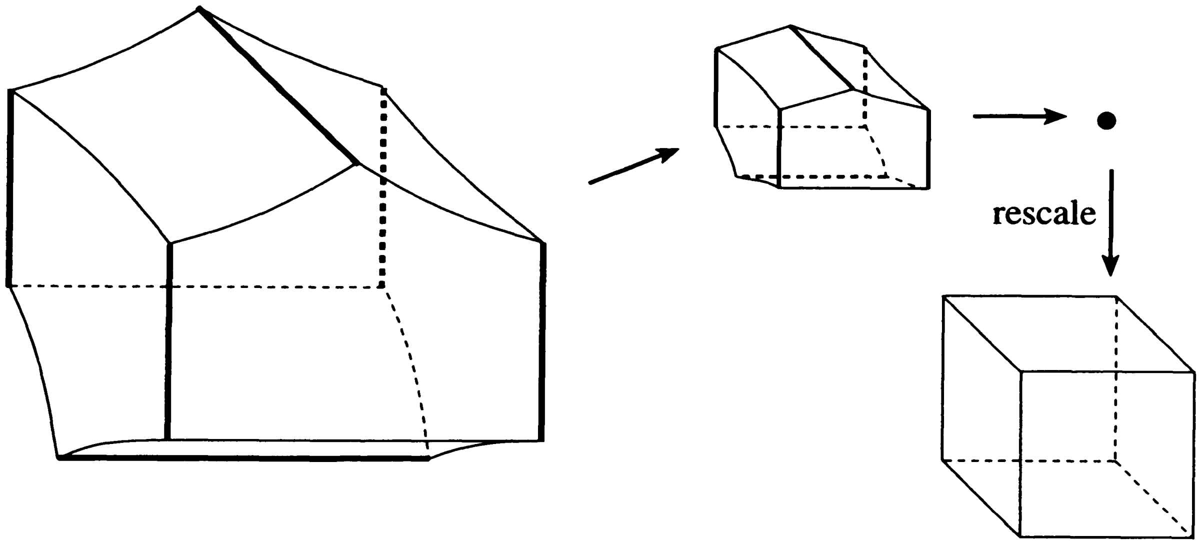
Similarly, we find three other vertical planes; by symmetry these meet at the same angle γ. Finally, it is easy to see that the angle γ varies from $\pi/2$ to 0 as the vertical distance from the valley to ridge varies from 0 to ∞.

From this description of Q we can see how the hyperbolic cone-manifold structures on T^3 degenerate as any cone angle approaches 2π.

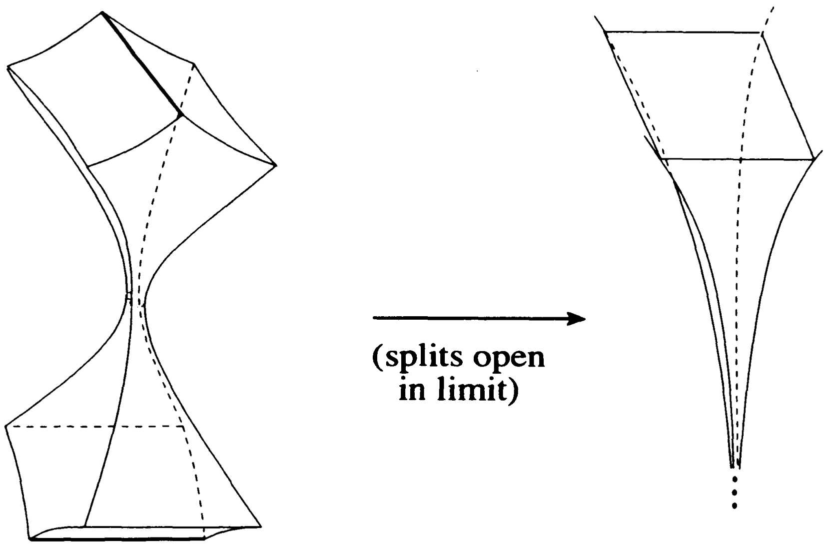
(a) As $\alpha, \beta \to 2\pi$, with $\gamma < 2\pi$ fixed, the polyhedra flatten out to give a 2-dimensional hyperbolic limit: a hyperbolic torus with one cone point of angle γ.



(b) As $\alpha, \beta, \gamma \to 2\pi$, the geometric limit (see chapter 6) can be either a point, circle or a line, depending on the exact mode of convergence. In each case there is a limiting Euclidean structure after rescaling.



(c) If $\gamma \longrightarrow 2\pi$ with $\alpha, \beta < 2\pi$ fixed, then the hyperbolic structures have diameter going to infinity, while two tori (with cone angle γ) become smaller and smaller, looking more and more like Euclidean tori as $\gamma \longrightarrow 2\pi$. The limiting polyhedra give a hyperbolic structure on the manifold obtained from T^3-(two horizontal curves) by splitting open along two incompressible tori.



(This process is a 3-dimensional analogue of the process of creating a cusp in a hyperbolic surface by pinching a curve to a point, in going to the boundary of Teichmüller space. Compare example 6.9.)

Exercise 3.5.
(a) Show that there are hyperbolic cone-manifold structures on S^3 with the Borromean rings as the singular locus, with arbitrary cone angles satisfying

$0 \leq \alpha, \beta, \gamma < \pi$. [Hint: reassemble 8 copies of a cube C of the type described above.]
(b) Describe how these hyperbolic structures degenerate as any angle approaches π.
(c) Show that $T^3 - \Sigma$ is homeomorphic to the complement of the Borromean rings in S^3.

Many other examples of 3-dimensional hyperbolic cone-manifolds arise from Thurston's theory of hyperbolic Dehn surgery. A detailed discussion of this will be given in section 5.6 below.

3.2 Local structure

Each point in an n-dimensional cone-manifold has a neighbourhood called a *standard cone neighbourhood* which is an *open cone* $\mathrm{Cone}_K(S; R)$ with constant curvature K and radius R, based on a *spherical cone-manifold* S of dimension $(n-1)$.

Topologically, $\mathrm{Cone}_K(S; R)$ is $S \times [0, R)$ with $S \times \{0\}$ collapsed to a point. The metric is

$$ds^2 = dr^2 + s_K^2(r)\, d\theta^2$$

where $d\theta^2$ denotes the metric on S, $r \in [0, R)$ and

$$s_K(r) = \begin{cases} \frac{1}{\sqrt{K}} \sin(\sqrt{K} r) & \text{if } K > 0, \\ r & \text{if } K = 0, \\ \frac{1}{\sqrt{|K|}} \sinh(\sqrt{|K|} r) & \text{if } K < 0. \end{cases}$$

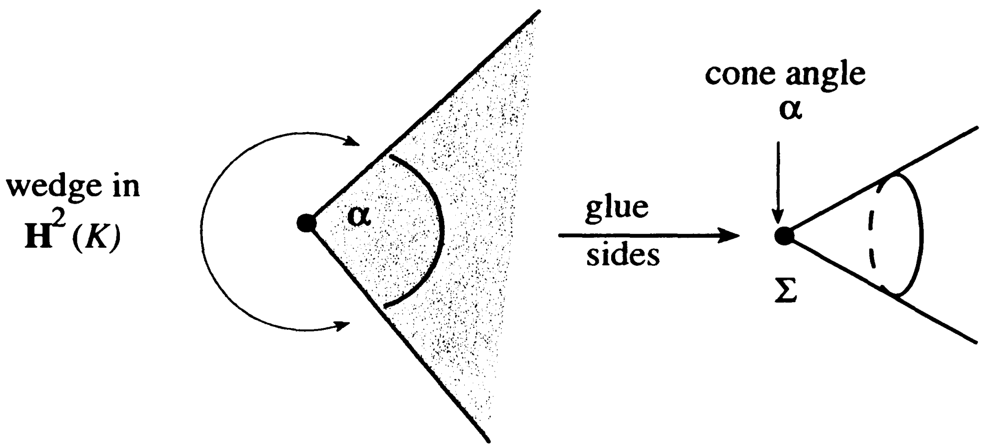
If $K \leq 0$ then $\mathrm{Cone}_K(S; R)$ is defined for $0 < R \leq \infty$. If $K > 0$ then $\mathrm{Cone}_K(S; R)$ is defined for $0 < R \leq \frac{\pi}{\sqrt{K}}$ and we define the *suspension* $\mathrm{Susp}_K(S)$ of S to be the completion of $\mathrm{Cone}_K(S; R)$ where $R = \frac{\pi}{\sqrt{K}}$. This suspension is obtained by gluing together two closed cones of radius $\frac{\pi}{2\sqrt{K}}$; the centres of these cones are called the *suspension points.* These cones and suspensions are analogues of standard balls and spheres for constant curvature cone-manifolds.

3.3 Standard cone neighbourhoods

Given a point p in a n-dimensional cone-manifold, let $S_r(p)$ denote the set of points at distance r from p. Then for r sufficiently small, $S_r(p)$ with the induced path metric is an $(n-1)$-dimensional cone-manifold of constant positive curvature.

Let $\mathbb{H}^n(K)$ denote the complete, simply connected n-manifold of constant curvature K. Then each *non-singular point* of an n-dimensional cone-manifold has a neighbourhood which is a metric ball in $\mathbb{H}^n(K)$.

If $n = 2$, each *singular point* has a cone neighbourhood which is a metric ball as shown below:



In polar coordinates the metric has the form:

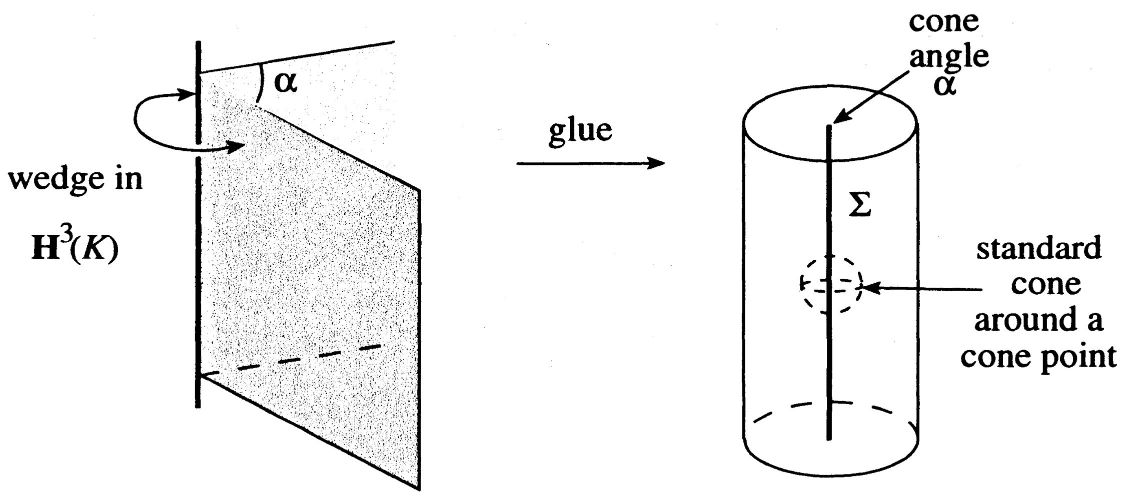
$$ds^2 = dr^2 + s_K^2(r)d\theta^2,$$

where $0 \leq \theta \leq \alpha$, $r \geq 0$.

If $n = 3$, then each point has a neighbourhood which is a cone on a spherical 2-orbifold. We will see in 4.2 that every orientable spherical cone 2-manifold with cone angles at most π is either a sphere, a football (as in example 3.1) or a turnover (as in example 3.2).

Proposition 3.6. *Every point in the singular locus of an orientable 3-dimensional cone-manifold with cone angles at most π has neighbourhood which is the cone on a football or turnover. Thus the singular locus is a trivalent graph.*

A cone neighbourhood of *singular point* which is not a vertex looks like:

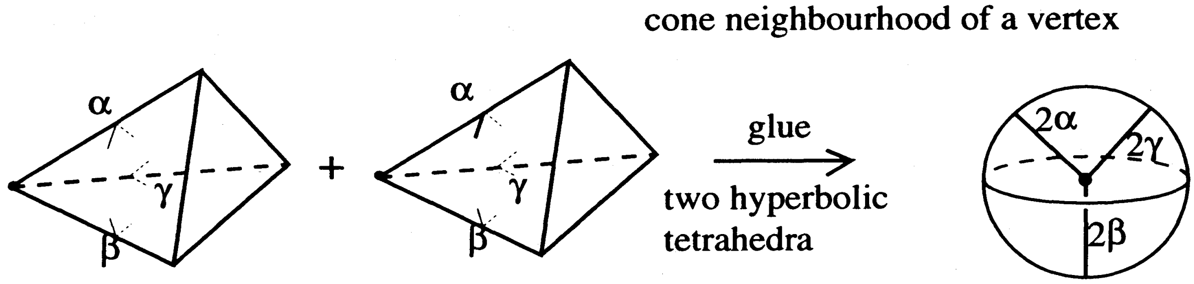


In cylindrical coordinates the metric has the form:

$$ds^2 = dr^2 + s_K^2(r)d\theta^2 + c_K^2(r)dz^2,$$

where $0 \leq \theta \leq \alpha$, $r \geq 0$, $z \in \mathbb{R}$, and $c_K(r) = s'_K(r)$. (For example, $s_K(r) = \sinh(r), c_K(r) = \cosh(r)$ in the hyperbolic case where $K = -1$.)

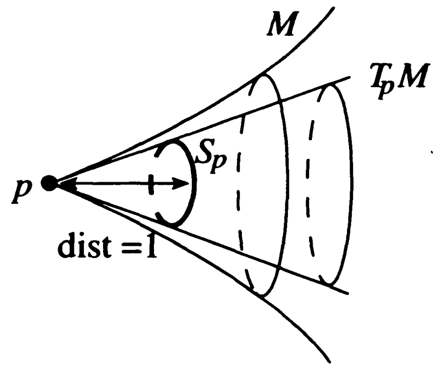
A vertex has a neighbourhood which is obtained by taking two isometric tetrahedra in $\mathbb{H}^3(K)$ and identifying along the three faces containing the vertex. A standard cone neighbourhood is a metric ball centred on the vertex inside this neighbourhood.



3.4 Geodesics

A cone-manifold has a *path metric* in the sense of Gromov and Alexandrov: the *distance* between two points is the infimum of lengths of paths joining the points. A *geodesic* in a cone-manifold is a curve which is locally length minimizing. At each point p of a cone-manifold M there is a *tangent cone* T_pM isometric to a Euclidean cone: T_pM is the union of the Euclidean tangent cones to all the n-dimensional simplices containing p. The subset

S_p of unit tangent vectors at p is a spherical cone-manifold — the visual sphere at p.



Next we outline some important results on geodesics in cone-manifolds; see [47] and [12] for more details.

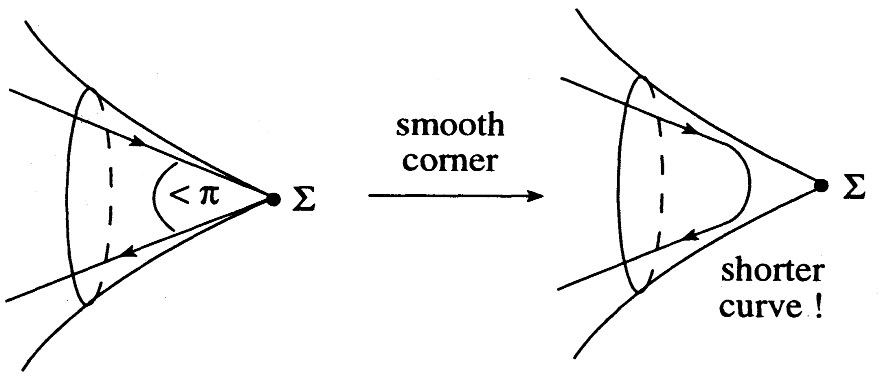
Lemma 3.7 (Hopf-Rinow). *Let M be a complete, connected cone-manifold. (i) Then any two points in M can be joined by a geodesic of length equal to the distance between the points.*
(ii) Given any vector v in T_pM, there is a constant speed geodesic g_v in M with initial tangent vector v, and g_v is uniquely defined in some neighbourhood of p.

Proof. Part (i): Take a sequence of constant speed curves $\gamma_i : [0, 1] \to M$ joining p to q, with $\text{length}(\gamma_i) \to \text{dist}(p, q)$ as $i \to \infty$. This is bounded and equicontinuous, so has a convergent subsequence by Ascoli-Arzela. The limit is a geodesic. Part (ii) is easy. □

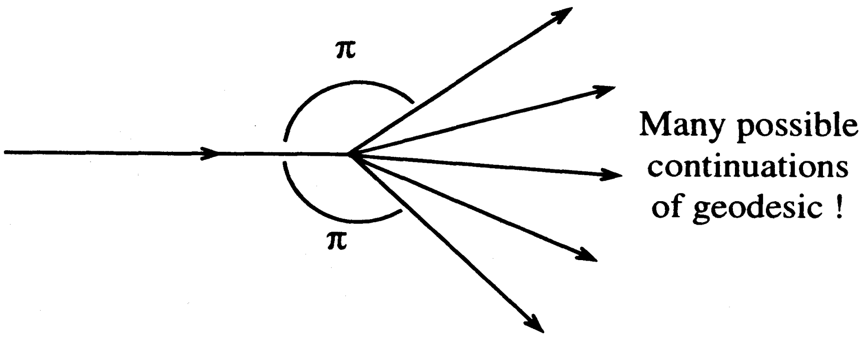
Exercise 3.8. Check the details in the above argument.

Lemma 3.9 (Lemma A). *In a cone-manifold with all cone angles $< 2\pi$, if the interior of a geodesic contains a point in the singular locus then the entire geodesic is contained in a single stratum of the singular locus.*

First we will sketch the main idea of the proof in the 2-dimensional case. If a curve meets Σ at a point p and is not entirely contained in Σ, then (the smallest) angle between the incoming and outgoing arcs at p is $< \pi$. Hence, the curve can be shortened by smoothing the corner at p.



Note: This result fails if cone angles larger than 2π are allowed. In this case length minimizing geodesics may pass through cone points. In fact, there is a pencil of extensions of any geodesic at any cone point with angle $> 2\pi$, consisting of all outgoing geodesic arcs making an angle $\geq \pi$ with the incoming arc.



Lemma 3.9 implies that if a vector v is not tangent to Σ then the geodesic g_v can be extended until it meets the singular locus. If v is tangent to Σ then g_v is contained in Σ and can be extended until it meets a different strata of the singular locus.

Lemma 3.10 (Lemma B). *A connected spherical cone-manifold of dimension at least* 2 *and curvature* 1 *with all cone angles* $< 2\pi$ *has diameter* $\leq \pi$, *with equality if and only if the cone-manifold is a suspension. Further, the only two points at distance* π *apart are the suspension points.*

Proof. We prove Lemmas 3.10 and 3.9 simultaneously by induction on dimension, following an argument of Thurston (compare [44]). Let (A_n), (B_n) be the statements of Lemmas A and B for n-dimensional cone-manifolds. The statement (A_1) is trivially true. Lemma B does not apply in dimension 1.

Assume (A_{n-1}) and (B_{n-1}) are true for $n \geq 2$. Let g be a length minimizing geodesic in an n-dimensional cone-manifold. Suppose p is a singular point in the interior of g, and let v_- and v_+ be the unit tangent vectors to g at p, directed away from p. Then v_- and v_+ give two points in the unit tangent cone S_p, which is a spherical cone-manifold of dimension $(n-1)$. The angle between these tangent vectors is the distance between v_- and v_+ measured in S_p so is $\leq \pi$, by the hypothesis on cone angles if $n = 2$ and by the induction hypothesis (B_{n-1}) if $n > 2$. If this angle is $< \pi$ then the length of g could be reduced by smoothing the corner at p, so g would not be locally length minimizing. If the angle is equal to π then $n > 2$ and it follows from (B_{n-1}) that g is tangent to Σ at p, hence contained in Σ. This proves (A_n).

To prove (B_n), we consider a length minimizing geodesic g in an n-dimensional spherical cone-manifold S of curvature 1. By (A_n), either the interior of g is disjoint from Σ or the entire geodesic g is contained in a single stratum of Σ. Elementary spherical geometry therefore shows that g has length $\leq \pi$; hence S has diameter $\leq \pi$. Further, if g has length π then its interior has a neighbourhood which is a suspension. Now suppose that S contains two points p, q at distance π apart. Let U be the set of unit tangent vectors $v \in S_p$ such that the geodesic g_v with initial tangent vector v is length minimizing on $[0, \pi]$ and joins p to q. Then U is an open subset of S_p by our previous remark. Suppose that v is a vector in the closure of U. We claim that the geodesic g_v is defined on $[0, \pi]$. If not, then g_v ends at some singular point $p_0 \in \Sigma$. Let g' be a shortest geodesic from p_0 to q. Then $g_v \cup g'$ has length $\leq \pi$ since v lies in $\overline{U}$. But this curve has a corner at p_0, so can be shortened to give a path from p to q of length less than π, contradicting the choice of p and q. Since the exponential map is continuous, it now follows that $v \in U$. Hence, $U = S_p$ and M is the suspension of S_p, with p and q as suspension points. □

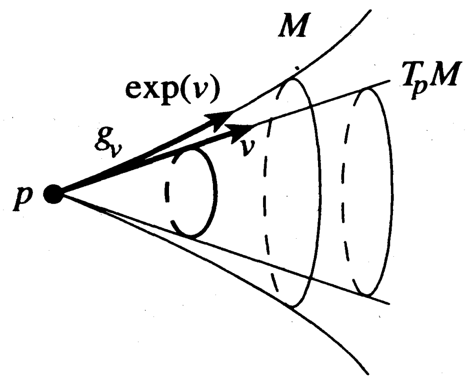
3.5 Exponential map

Recall that g_v denotes the constant speed geodesic starting at p with initial tangent vector v. Let $\mathcal{D}_p \subset T_pM$ be the subset consisting of all v in T_pM such that g_v *is defined and globally length minimizing up to time* 1. We also say that $g_v([0,1])$ is a *segment.*

Then there is a well-defined, continuous *exponential map*

$$exp : \mathcal{D}_p \to M,$$

defined by $exp(v) = g_v(1)$.



From the previous lemmas we obtain:

Lemma 3.11. *If M is a complete, connected cone-manifold with all cone angles $< 2\pi$ then the exponential map $exp : \mathcal{D}_p \to M$ is* onto.

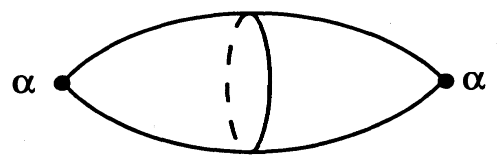
To study $\partial\mathcal{D}_p$ and the *cut locus* $exp(\partial\mathcal{D}_p)$, we need to understand: *How can a geodesic from p stop minimizing length?*

If $f : M \to N$ is a continuous map between cone-manifolds, then it is possible to consider directional derivatives at each point of M. If all such directional derivatives exist, then one obtains a map $df : T_pM \to T_{f(p)}N$. We say that q is a *conjugate point* of p in M if $dexp(v) = 0$ for some non-zero $v \in TM_p$ such that $exp(v) = q$.

Lemma 3.12 (Key Lemma). *A point $q \in M$ lies in the cut locus $exp(\partial\mathcal{D}_p)$ if and only if at least one of the following cases occurs:*
(1) There are two minimizing geodesics from p to q in M.
(2) q is a conjugate point of p.
(3) q is in the singular locus Σ and there is a length minimizing geodesic from p to q which doesn't extend past the point q.

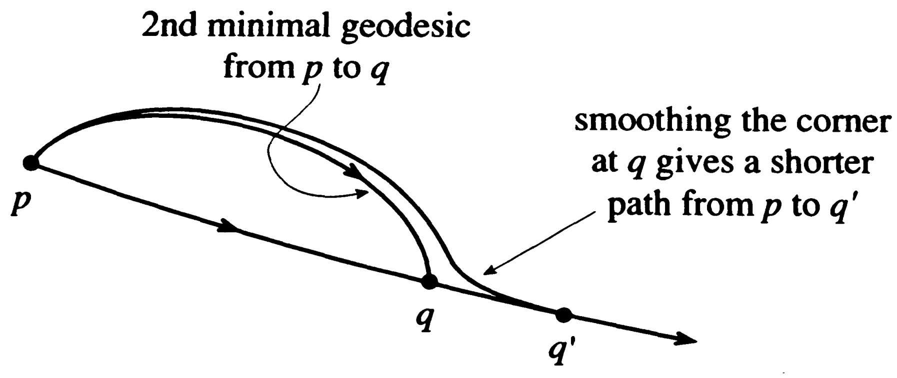
Proof. Use the usual arguments from Riemannian geometry plus Lemma 3.9.

Note: (2) is very special in our context: it only occurs when the curvature $K > 0$, and the diameter of the cone-manifold is $\pi/\sqrt{K}$.



Further, if (2) occurs then (1) also occurs for our constant curvature spaces.

Picture for case (1):

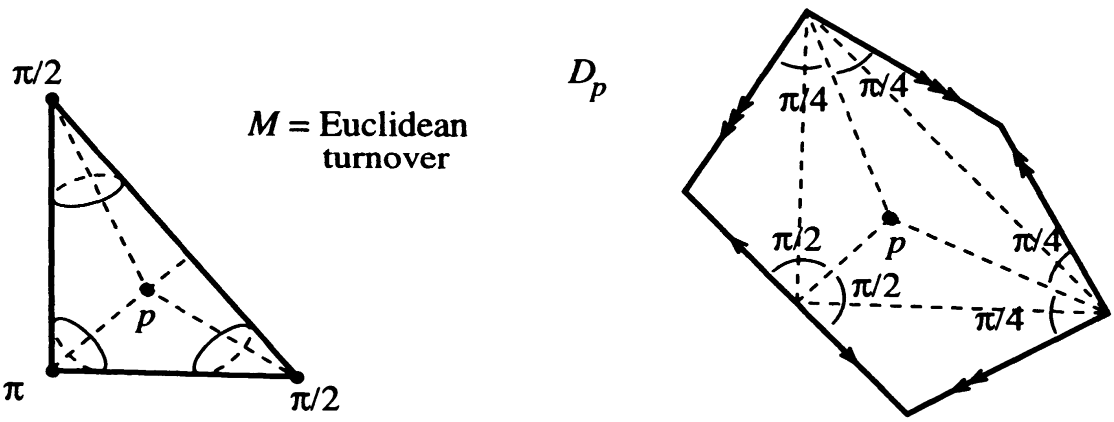


3.6 Dirichlet domains

Define the *(open) Dirichlet domain* $\mathring{D}_p$ at a point p as the subset of $q \in M$ such that there is a unique minimal geodesic from p to q, and if $q \in \Sigma$ then the entire geodesic is in Σ.

By the previous lemma, this is just the image of the interior of $\mathcal{D}_p$ under the exponential map: $\mathring{D}_p = exp(\mathring{\mathcal{D}}_p)$. Hence $\mathring{D}_p$ is a union of geodesic segments starting at p, so is *star-shaped* with respect to p.

Example 3.13. A Dirichlet domain for the Euclidean orbifold $S^2(2,4,4)$ (the double of a triangle with angles $\pi/2, \pi/4, \pi/4$.)



Let M be a cone-manifold of curvature K, let p be a point in M, and S_p the unit tangent cone (or visual sphere) at p. Define the *global cone* C_p at p to be the infinite cone $\mathrm{Cone}_K(S_p; \infty)$ if $K \leq 0$, or the suspension $\mathrm{Susp}_K(S_p)$ if $K > 0$.

Now assume that p *has no conjugate points.* Then the exponential map $exp : \mathcal{D}_p \to M$ has no critical points, and we can pull back the metric on M to obtain a constant curvature metric on $\mathcal{D}_p$.

Since $\mathcal{D}_p$ is star-shaped, it also embeds by a local isometry into the global cone C_p. The image D_p of this embedding will be called the *Dirichlet domain* for M based at the point p. Then D_p is isometric to $\mathcal{D}_p$, and there is a natural map

$$q : D_p \to M$$

which is *onto* by Lemma 3.11 if all cone angles are at most 2π.

If p is a non-singular point, then the global cone C_p is just the simply connected n-manifold $\mathbb{H}^n(K)$ of constant curvature K, and D_p is a subset of $\mathbb{H}^n(K)$. If M is a complete non-singular manifold, then D can be obtained by taking the usual Dirichlet region based at any preimage of p in the universal cover $\tilde{M} \cong H^n(K)$. Note that this is always a convex polyhedron in $H^n(K)$.

Special Case: If p has a conjugate point, then Lemma 3.10 shows that M is a suspension and p is a suspension point. In this case, we will regard $C_p = \mathrm{Susp}_K(S_p) = M$ as the Dirichlet domain for M at p, and consider this to be a convex polyhedral subset of C_p.

Next we examine the structure of the Dirichlet domain in more detail. The following result, in particular the convexity conclusion, will play a key role at several places in the proof of the orbifold theorem.

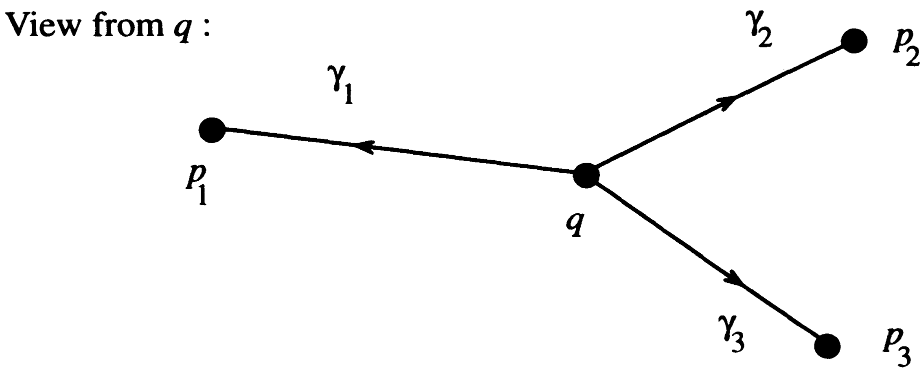
Proposition 3.14. *Let D be the Dirichlet domain based at a point p of a cone-manifold M of constant curvature K.*
(1) If M has cone angles $< 2\pi$, then D is a star-shaped (geodesic) polyhedron *in the global cone C_p. (i.e. D is a union of totally geodesic simplices with one vertex at p.) Further, M is obtained from D by identifying pairs of faces by isometries.*
(2) If M has cone angles $\leq \pi$, then the polyhedron D is convex *(i.e. any two points of D can be joined by a minimal geodesic lying in D.)*

Before giving a detailed proof, we will sketch the main ideas in the 2-dimensional case.

Sketch of proof (assuming $K \leq 0$): We look at the local picture near a point q in the cut locus $exp(\partial\mathcal{D}) \subset M$.

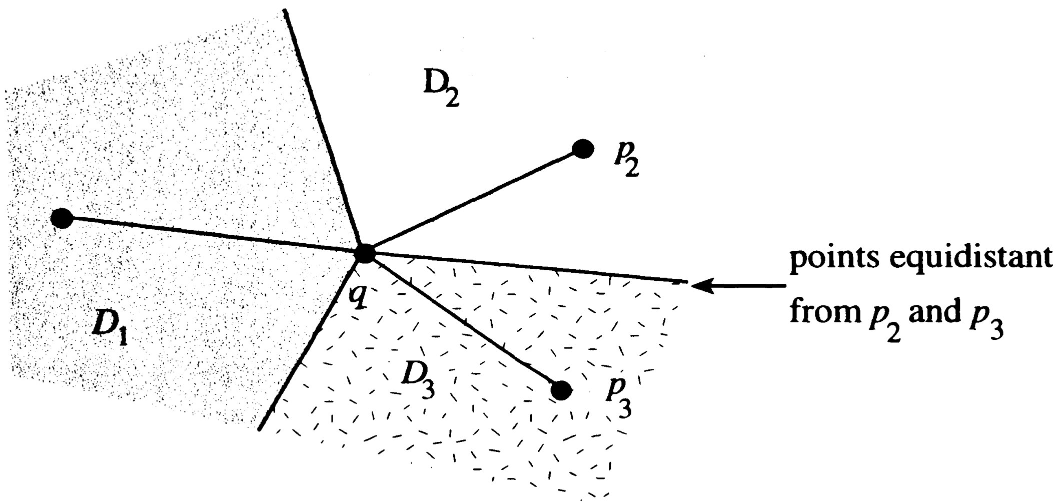
Case 1: q is non-singular. Then there are at least two shortest geodesic in M from p to q, say $\gamma_1, \ldots, \gamma_k$.

These come out from q in k different directions, and end at points $p_1, \ldots, p_k$ (corresponding to different "lifts" of p.)



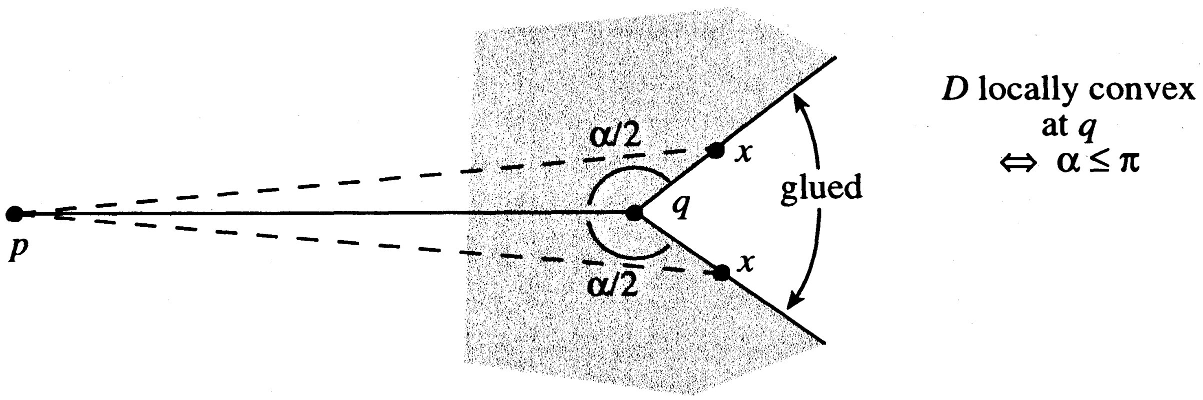
Then the shortest geodesics from p to points x near q are perturbations of $\gamma_1, \ldots, \gamma_k$ joining $p_1, \ldots, p_k$ to x. So the part of $\overset{\circ}{D}$ near q consists of the *Voronoi regions*

$$D_i = \{x \colon d(x, p_i) \leq d(x, p_j) \text{ for all } j \neq i\}.$$



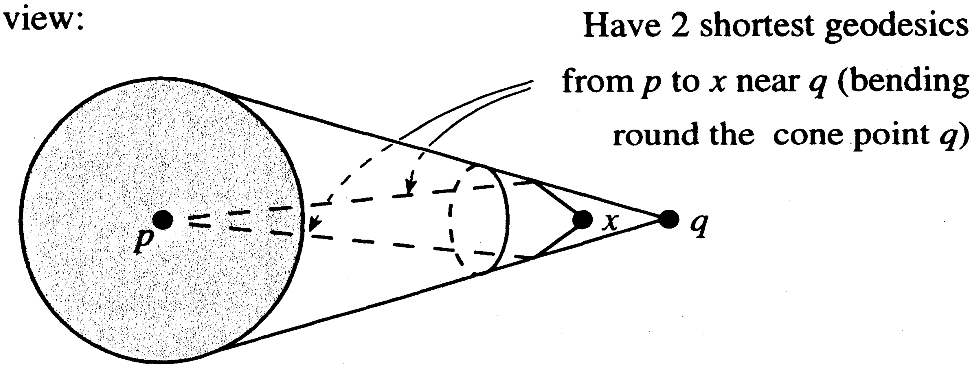
These are bounded by planes equidistant from two points, so the boundary of D is polyhedral and locally convex near q.

Case 2: q is singular and there is one shortest geodesic from p to q. Then a wedge of angle 2π − cone angle is excluded from the Dirichlet domain near Σ.



Again the boundary of D is polyhedral near q. It is locally convex if and only if the cone angle is $\leq \pi$.

Another view:



Case 3: q is singular and there are at least two shortest geodesics from p to q.
Exercise. Prove case 3 by combining the arguments from cases 1 and 2.

Proof of 3.14. We will prove a slightly more general result by induction on $\dim M$. Given a finite set of points $P = \{p_i\}$ in a cone-manifold M then the (open) *Voronoi region at p_i* is the set $V(p_i)$ consisting of points $q \in M$ satisfying:
(1) $d(q, p_i) \leq d(q, p_j)$, for all $j \neq i$,
(2) there is a unique shortest geodesic γ from q to P, and
(3) if $q \in \Sigma$, then the global cone C_q is a suspension, and the unit tangent vector to γ at q gives a suspension point in C_q.
We also define the *cut locus of P*, $\mathrm{Cut}(P)$, to be the complement of $\cup_i V(p_i)$ in M.

If P contains a single point p, then the cut locus is $\mathrm{Cut}(p) = exp(\partial\mathcal{D}_p)$, and $V(p)$ is just the complement of $\mathrm{Cut}(p)$ in M, i.e. an "open Dirichlet domain" at p.

We inductively prove the following statement.

Convex(n): Let M be a cone-manifold of dimension n with cone angles $\leq 2\pi$. Let $P = \{p_1, \ldots, p_k\}$ be a set of points in M. Also assume that if $k = 1$ then the point in P has no conjugate point in M.

(a) Then each Voronoi region $V(p_i)$ is a (geodesic) polyhedron contained in the global cone at p_i.

(b) Further, if M has cone angles $\leq \pi$, then each Voronoi region is convex.

Proof. If $\dim M = 1$, then each $V(p_i)$ is an interval so the result is clear.

To show that Convex($n-1$) $\Rightarrow$ Convex(n), we study the local structure of the cut locus $\mathrm{Cut}(P)$ near a point q_0. The condition on P means that that q_0 is not conjugate to any point p_i. Hence, there exists $\delta > 0$ such that only finitely many geodesics of length $< d(p, q_0) + \delta$ join q_0 to P. In particular, there are finitely many shortest geodesics from q to P for all q near q_0 and these are obtained by *small perturbations* of the shortest geodesics from q_0 to P. (Compare [8], Lemma 1.3.)

Let γ be a shortest geodesic joining q_0 to a point $p \in P$, with length d_0. Then if γ' is a geodesic sufficiently close to γ joining a point q to p, then the law of cosines in $H^n(K)$ expresses length(γ') as an explicit function depending only on d_0, the distance from q_0 to q, and the angle $\theta = \angle pq_0q$ between γ and q_0q. Further, length(γ') is an increasing function of θ for $0 \leq \theta \leq \pi$, if we fix d_0 and $d(q_0, q)$.

Let $v_1, v_2, \ldots, v_k \in S_{q_0}$ be initial unit tangent vectors of the shortest geodesics from q_0 to P. Then the observation above gives us an explicit local description of the sets $V(p_i)$ near q_0 as cones on the Voronoi regions of the collection of points $\{v_1, v_2, \ldots, v_k\}$ on S_{q_0}. (Note that this is a spherical cone-manifold of dimension $n-1$.) To finish the argument, we look separately at the geometry of these cones for the cases where $n = 2$ and $n \geq 3$.

If $n = 2$, S_{q_0} is circle of length equal to the cone angle α of M at q_0. If we have at least two shortest geodesics from q_0 to P, then we have $k \geq 2$ points v_i on a circle of length $\alpha \leq 2\pi$. Hence each Voronoi region $V(v_i)$ has length $\leq \pi$, and the cone on this is polyhedral with angle $\leq \pi$ at the cone point.

If q_0 is a *singular* point of M, it is possible that there is only *one* shortest geodesic from q_0 from P. In this case, the Voronoi region of the corresponding point on S_{q_0} is an interval of length α, and the cone on this is polyhedral with angle α at the cone point. This will be locally convex if and only if $\alpha \leq \pi$.

So for $n = 2$ we see that the Voronoi regions in M are locally polyhedral, and have corner angles $\leq \pi$, *provided* that all cone angles are $\leq 2\pi$.

If $n \geq 3$, we use the inductive hypothesis directly. By Lemma 3.12 and the following remarks, the points $v_1, \ldots, v_k$ satisfy the assumption for Convex$(n-1)$ applied to the $(n-1)$-dimensional spherical cone-manifold S_{q_0}. By induction each of these Voronoi regions $V(v_i)$ is polyhedral and is convex if the cone angles are $\leq \pi$. But the cone on such a region is always polyhedral and locally convex if $n \geq 3$. This shows that the $V(p_i)$ satisfies the conclusions of Convex(n) locally near each point q_0 in $Cut(P)$ and completes the proof by induction. □

3.7 Area and volume of cone-manifolds

For 2-dimensional cone-manifolds we have the following following version of the Gauss-Bonnet theorem relating topology and geometry:

Theorem 3.15. *Let M be a closed, 2-dimensional cone-manifold of constant curvature K with n cone points with cone angles $\theta_1, \ldots, \theta_n$. Then*

$$\int_M K dA + \sum_i (2\pi - \theta_i) = 2\pi\chi(M). \tag{1}$$

Exercise 3.16. Prove this Gauss-Bonnet formula.

In particular, this gives an explicit formula relating the area A to the cone angles for a 2-dimensional cone-manifold of constant curvature K:

$$KA = 2\pi\chi(M) - \sum_i (2\pi - \theta_i).$$

For higher dimensional cone-manifolds, there is a remarkable formula for the *variation* of volume when constant curvature cone-manifolds are deformed. The basic ingredient needed is the following theorem, originally proved in the spherical case by Schläfli [70] in 1858. For modern proofs see, for example, [86], [63], or [21].

Theorem 3.17 (Schläfli Formula for Polyhedra). *Let P_t be a smooth one-parameter family of polyhedra in the simply-connected n-dimensional space $\mathbb{H}^n(K)$ of constant curvature K. (Thus the faces of X_t are totally geodesic planes in $\mathbb{H}^n(K)$ which vary smoothly with t.) Then the derivative of the volume V_t of P_t satisfies the equation:*

$$(n-1)K\,\frac{dV_t}{dt} = \sum_F V_{n-2}(F)\frac{d\theta_F}{dt} \tag{2}$$

where the sum is over all codimension-2 faces of X_t, and V_{n-2} denotes the $(n-2)$-dimensional volume, and θ_F denotes the dihedral angle at F.

In the 2-dimensional case, the theorem is equivalent to the Gauss-Bonnet theorem for constant curvature polygons. The formula becomes

$$K\,\frac{dA_t}{dt} = \sum \frac{d\theta_v}{dt}$$

where θ_v is the angle at vertex v and A denotes area. Integrating this gives

$$KA = \sum \theta_v + \text{constant.}$$

Exercise 3.18. Determine the constant of integration by considering the case where the polygon flattens out to a straight line segment.

Remark: As Milnor notes in [63], the theorem also applies to the case of hyperbolic polyhedra with some ideal vertices. When $n \neq 3$, no change in the statement of the theorem is needed. In the 3-dimensional case, some edge lengths $V_{n-2}(F)$ become infinite. However, the theorem remains valid if we remove small horoball neighbourhoods of the ideal vertices before measuring edge lengths. (The right hand side of (2) is easily seen to be independent of the choice of horoballs, using the fact that the sum of dihedral angles at an ideal vertex is constant.)

Exercise 3.19. Use the Schläfli formula for ideal simplices to prove that the regular ideal simplex is the unique simplex of maximal volume in $\mathbb{H}^3$.

Following Hodgson [43] we can apply the Schläfli formula to study the volume of constant curvature cone-manifolds. The following theorem shows that the variation in volume for a family of cone-manifold structures is completely determined by the changes in geometry *along the singular locus.*

Theorem 3.20 (Schläfli Formula for Cone-Manifolds).
Let C_t be a smooth family of cone-manifold structures of constant curvature K. Assume that the underlying space and singular locus are of fixed topological type. Then the derivative of volume V_t of C_t satisfies

$$(n-1)\,K\,\frac{dV_t}{dt} = \sum_i V_{n-2}(\Sigma_i)\,\frac{d\theta_i}{dt} \tag{3}$$

where the sum is over all strata Σ_i of the singular locus Σ, and θ_i is the cone angle along Σ_i.

Proof. Divide C_t into geometric simplices, varying smoothly with t, such that the singular locus remains a subcomplex. Applying the Schläfli formula (2) to each simplex and adding shows that the variation of volume is given by

$$(n-1)\,K\,\frac{dV_t}{dt} = \sum_F V_{n-2}(F)\,\frac{d\theta_F}{dt}$$

summed over all codimension 2 faces of the triangulation of C_t, where θ_F denotes the cone angle along F. However, at any non-singular face F the cone angle is 2π for all t so $\dfrac{d\theta_F}{dt} = 0$. So the right hand side reduces to a sum over faces F in the singular locus Σ. □

As a corollary we see that the volumes of cone-manifolds satisfy the following remarkable monotonicity property:

Corollary 3.21. *Let C_t be a smooth family of cone-manifolds of constant curvature K. If $K > 0$, then the volume increases strictly monotonically as any cone angle is increased. If $K < 0$, then the volume decreases strictly monotonically as any cone angle is increased.*