Chapter 1

Geometric Structures

The aim of this memoir is to give an introduction to the statement and main ideas in the proof of the "Orbifold Theorem" announced by Thurston in late 1981 ([83], [81]). The Orbifold Theorem shows the existence of geometric structures on many 3-dimensional orbifolds, and on 3-manifolds with a kind of topological symmetry.

The main result implies a special case of the following Geometrization Conjecture proposed by Thurston in 1976 as a framework for the classification of 3-manifolds. For simplicity, we state the conjecture only for compact, orientable 3-manifolds.

Conjecture 1.1 (Geometrization Conjecture). ([81]) *The interior of every compact 3-manifold has a canonical decomposition into pieces having a geometric structure.*

The kinds of *decomposition* needed are:

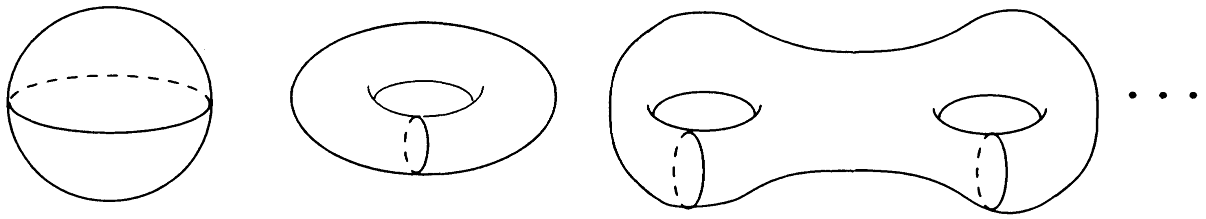
1. prime (or connected sum) decomposition, which involves cutting along separating 2-spheres and capping off the pieces by gluing on balls.
2. torus decomposition, which involves cutting along certain incompressible non-boundary parallel tori.

The meaning of *canonical* is that the pieces obtained are unique up to ordering and homeomorphism. The spheres used in the decomposition are not unique up to isotopy, but the tori are unique up to isotopy.

A *geometric structure* on a manifold is a complete Riemannian metric which is locally homogeneous (i.e. any two points have isometric neighbourhoods). A *geometric decomposition* is a decomposition of this type into pieces whose interior have a geometric structure. There are essentially eight kinds of geometry needed; of these hyperbolic geometry is the most common and the most interesting.

1.1 Geometry of surfaces

We would like to generalize the well-known topological classification of closed 2-manifolds (surfaces). The *orientable* surfaces are just:



2-sphere S^2 | 2-torus T | T#T(connected sum)
$\chi=2$ | $\chi=0$ | $\chi=-2$
genus g=0 | g=1 | g=2

The *non-orientable* closed surfaces (those containing Möbius strips) are: the real projective plane $P = \mathbb{R}P^2$, the Klein bottle $K = P\#P$, $P\#P\#P, \cdots$. (See [58] or [4] for details.)

These surfaces are easy to distinguish by their orientability and Euler characteristic given by

$$\chi = \#(vertices) - \#(edges) + \#(faces),$$

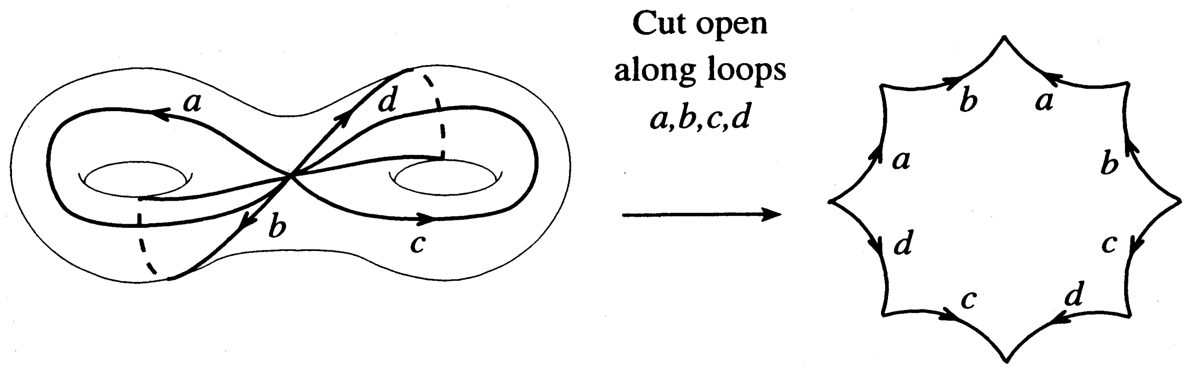
for any decomposition of the surface into polygons.

It has been known since the nineteenth century that there is a very close relationship between geometry and topology in dimension two. Each surface can be given a spherical, Euclidean or hyperbolic structure, that is, a Riemannian metric of constant curvature $K = +1, 0$, or -1. Further, the topology of the surface is determined by the geometry via the Gauss-Bonnet formula:

$$2\pi\chi(M) = \int_M K\,dA.$$

Exercise 1.2. Prove this formula for constant curvature surfaces, using the fact that the angle sum of a (geodesic) triangle in a space of constant curvature K is $\pi + KA$, where A is the area of the triangle.

Example 1.3. A 2-sphere clearly has a spherical metric — just take the round sphere S^2 in Euclidean 3-space $\mathbb{E}^3$. A torus can be given a Euclidean metric: take a square (or any parallelogram) in the Euclidean plane $\mathbb{E}^2$ and glue together opposite edges. (Note that the corners fit together to form a small Euclidean disk since the angles of the parallelogram add up to 2π.) Similarly, a closed surface of genus $g \geq 2$ can be given a hyperbolic metric by taking a regular $4g$-gon in the hyperbolic plane $\mathbb{H}^2$ with angles $2\pi/4g$ and gluing together edges in pairs in the usual combinatorial pattern.



Exercise 1.4. Show that there exists a regular $4g$-gon in the hyperbolic plane $\mathbb{H}^2$ with angles $2\pi/4g$ for each $g \geq 2$.

1.2 Geometry of 3-manifolds

It now seems natural to ask whether there is a similar division of 3-manifolds into different geometric types, but this question was not considered until the work of Thurston starting in about 1976.

Question: What kinds of geometries are needed to deal with 3-manifolds?

We would like to find geometric structures (or metrics) on 3-manifolds which are *locally homogeneous*. Roughly, this means the space should look locally the same near every point; more precisely: any two points have isometric neighbourhoods. Our spaces should also be *complete* as metric spaces, i.e. every Cauchy sequence converges. Intuitively, this means you can't fall off the edge of the space after going a finite distance!

First, we obviously have 3-dimensional spaces of constant curvature: Euclidean geometry $\mathbb{E}^3$, spherical geometry S^3 and hyperbolic geometry $\mathbb{H}^3$. These geometries look the same near every point and in every direction. Many examples of 3-manifolds with these geometries are discussed in Thurston's book [82].

Not all closed 3-manifolds can be modelled on the constant curvature geometries. For example, the universal cover of a 3-manifold with one of these geometries is topologically either $\mathbb{R}^3$ or S^3. So $S^2 \times S^1$, with universal cover $S^2 \times \mathbb{R}$, cannot have such a geometry. Nevertheless, it does have a very nice homogeneous metric: take the natural product metric on $S^2 \times S^1$.

Formally, we say that a manifold M has a *geometric structure* if it admits a complete, locally homogeneous Riemannian metric. This gives a way of measuring the length of smooth curves by integrating an element of arc length ds, and we can talk about geodesics, angles, volume etc. Then the universal cover X of M has a complete *homogeneous* metric, i.e. the isometry group G acts transitively on X. Further, the stabilizer $G_x = \{g \in G : gx = x\}$ of each point $x \in X$ is *compact*, since it is a closed subgroup of $O(n)$. Then the manifold M is isometric to a quotient space X/Γ, where Γ is a discrete subgroup of G. (This can be proved by an "analytic continuation" argument using the "developing map" discussed below, see also [80]).

1.3 Thurston's eight geometries

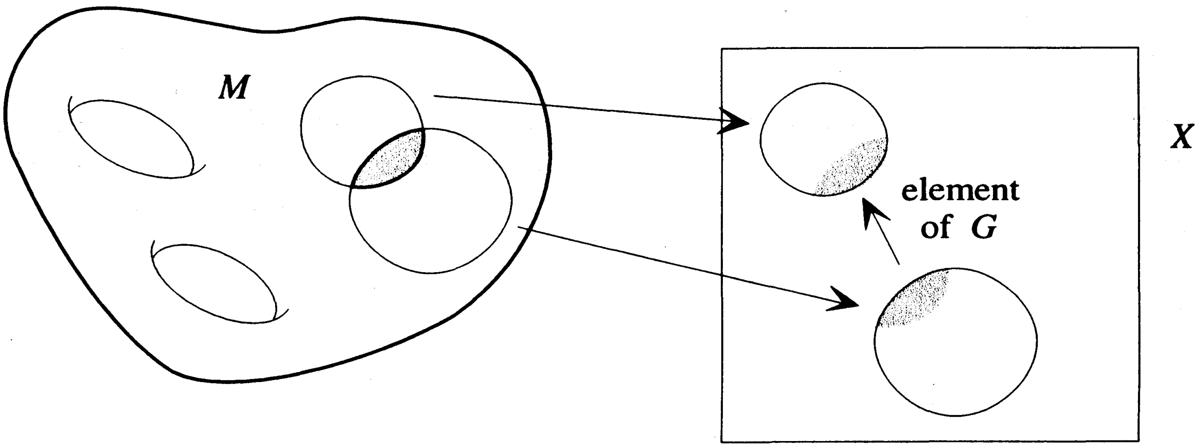
Following the viewpoint of Klein's Erlangen program (from 1872), we can also regard *geometry* as the study of the properties of a space X which are invariant under a group of transformations G. The geometry (G, X) is *homogeneous* if G acts transitively on X. The geometry is *analytic* if each transformation in G is uniquely determined by its restriction to any nonempty open subset of X. For example, groups of isometries of manifolds are analytic.

The geometries needed for studying 3-manifolds are pairs (G, X) where X is a simply connected space, and G is a group acting transitively on X with compact point stabilizers. To avoid redundancy, we require that G is a *maximal* such group. Finally, we restrict to geometries which can model compact 3-manifolds: G contains a discrete subgroup Γ such that X/Γ is compact.

Thurston showed that there are *exactly eight* such geometries on 3-manifolds. The most familiar 3-dimensional geometries are the constant curvature geometries: Euclidean geometry $\mathbb{E}^3$ (of constant curvature $K = 0$), spherical geometry S^3 (of constant curvature $K = +1$), and hyperbolic geometry $\mathbb{H}^3$ (of constant curvature $K = -1$). The other geometries are the product geometries $S^2 \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^1$; and three "twisted products" called *Nil*, *PSL* and *Solv* geometries. (See [73], [81] for detailed discussions of these geometries.)

1.4 Developing map and holonomy

A (G, X) *geometric structure* on a manifold M is given by a covering of M by open sets U_i and diffeomorphisms $\phi_i : U_i \to X$ to open subsets of X, giving coordinate charts on M, such that all the transition maps are restrictions of elements in G. (If G acts by isometries on X, this means that M is locally isometric to X.)

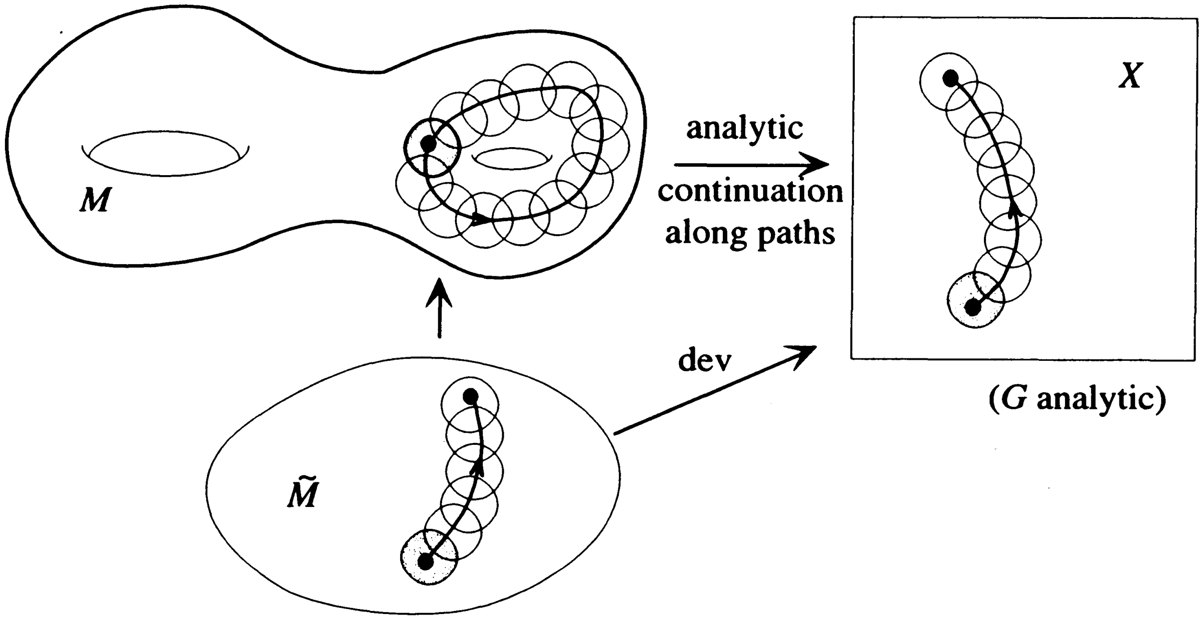


Given an analytic (G, X) structure on M, analytic continuation of coordinate charts gives a "global coordinate chart", called a *developing map*

$$\text{dev} : \tilde{M} \to X$$

defined on the universal cover $\tilde{M}$ of M.

This is constructed as follows: Begin with an embedding $\phi_1 : U_1 \subset M \to X$ giving a coordinate chart on M. If $\phi_2 : U_2 \to X$ is another coordinate chart with $U_1 \cap U_2$ connected and non-empty, there is a unique $g \in G$ such that $g \circ \phi_2 = \phi_1$ on $U_1 \cap U_2$. So ϕ_1 extends to a map $\phi : U_1 \cup U_2 \to X$, with $\phi = \phi_1$ on U_1 and $\phi = g \circ \phi_2$ on U_2. In this way, we can extend ϕ_1 by analytic continuation along paths in M.



Since the result of the analytic continuation only depends on the homotopy class of the path involved, we obtain a well defined map dev : $\tilde{M} \to X$. Then dev is a local diffeomorphism satisfying the equivariance condition

$$\mathrm{dev} \circ \gamma = h(\gamma) \circ \mathrm{dev}$$

for each deck transformation γ in $\pi_1(M)$, where

$$h : \pi_1(M) \to G$$

is a homomorphism called the *holonomy representation* for the geometric structure. (See Thurston [82, Chapter 3] for more details.)

Note that dev and h are not uniquely defined; changing the original coordinate chart ϕ_1 by an element $g \in G$ gives a new developing map $g \circ \mathrm{dev}$ with corresponding holonomy representation $g \circ h \circ g^{-1}$. It can be shown that the pair (dev, h) determines the (G, X)-structure on M.

As a simple example, let M be the Euclidean surface obtained as follows. Let D the subset of the Euclidean plane bounded by two distinct rays starting at a point x and making an angle θ. Identify the two sides of D by an isometry and delete the point x. The resulting surface is not complete. The image of the holonomy is discrete if and only if θ is a rational multiple of π. The developing map has image the complement of x and is not injective.

The developing map is an important tool for analyzing (G, X) structures. For instance, it can be used to prove the following important completeness criterion (see [82]).

Theorem 1.5. *Let M be a manifold with a geometric structure modelled on a geometry (G, X) where G is a group of isometries of X. Then M is complete as a metric space if and only if the developing map* $\mathrm{dev} : \tilde{M} \to X$ *is a covering map.*
If X is simply connected, then such a complete manifold M is isometric to X/Γ where the holonomy group $\Gamma = h(\pi_1 M) \cong \pi_1(M)$ is a discrete subgroup of $\mathrm{Isom}(X)$ *which acts freely and properly discontinuously on X.*

The proof of the Orbifold Theorem involves deformations through *incomplete* structures. This means that discrete group techniques cannot be used; however, the developing map and holonomy again play a key role.

1.5 Evidence for the Geometrization Conjecture

We begin by restating Thurston's Geometrization Conjecture. For simplicity, we will assume that all manifolds are *compact* and *orientable.*

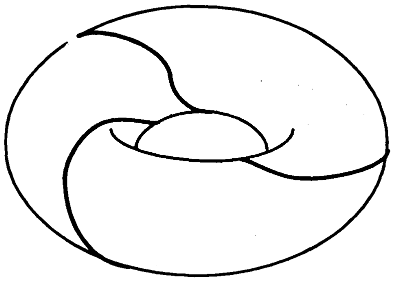
Conjecture 1.6 (Geometrization Conjecture). *Let M be a compact, orientable, prime 3-manifold. Then there is a finite collection of disjoint, embedded incompressible tori in M (given by the Johannson, Jaco-Shalen torus decomposition), so that each component of the complement admits a geometric structure modelled on one of the eight geometries discussed in section 1.3.*

There is a great deal of evidence for this conjecture. Here are some of the main results.

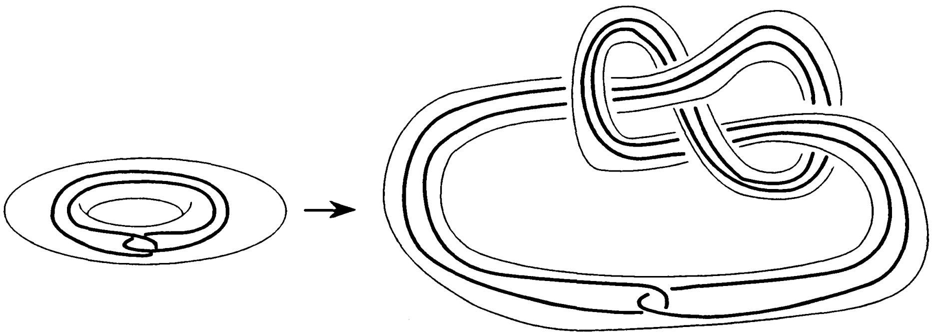
Theorem 1.7 (Thurston). *The Geometrization Conjecture is true if M is a Haken manifold.*

Recall that M is *Haken* if it is irreducible and contains an incompressible surface. For example, this theorem applies if M is irreducible and ∂M contains a surface $\not\cong S^2$. This result is proved by a difficult argument using hierarchies; Thurston developed many wonderful new geometric ideas and techniques to carry this out. The article of Morgan in [64] provides a good overview of the proof.

An important application of this result is to knot complements. Let K be a knot (i.e. an embedded circle) in $S^3 = \mathbb{R}^3 \cup \infty$. Then K is called a *torus knot* if it can be placed on the surface of a standard torus. It is easy to see that $S^3 - K$ is then a Seifert fibre space with ≤ 2 exceptional fibres (see section 2.8).



A knot K' is called a *satellite knot* if it is obtained by taking a non-trivial embedding of a circle in a small solid torus neighbourhood of a knot K. (*Non-trivial* means that the circle is not contained in a 3-ball, and is not isotopic to K). Then $S^3 - K'$ contains an incompressible torus T which is the boundary of the solid torus around K. (This follows since the exterior of every non-trivial knot has incompressible boundary, by the loop theorem.)



Corollary 1.8. *Let K be a knot in S^3. Then $S^3 - K$ has a geometric structure if and only if K is not a satellite knot. Further, $S^3 - K$ has a hyperbolic structure if and only if K is not a satellite knot or a torus knot.*

Thus, "most" knot complements are hyperbolic. Similarly, "most" link complements are hyperbolic.

The unresolved cases of the Geometrization Conjecture are for closed, orientable, irreducible 3-manifolds M which are non-Haken. (It is known that this is a large collection of 3-manifolds!) Such manifolds fall into 3 categories:

1. Manifolds with $\pi_1(M)$ finite.
2. Manifolds with $\pi_1(M)$ infinite, and containing a $\mathbb{Z} \times \mathbb{Z}$ subgroup.
3. Manifolds with $\pi_1(M)$ infinite, and containing no $\mathbb{Z} \times \mathbb{Z}$ subgroup.

For manifolds of type (1), the Geometrization Conjecture reduces to the following:

Conjecture 1.9 (Orthogonalization Conjecture). *If $\pi_1(M)$ is finite, then M is spherical; thus M is homeomorphic to S^3/Γ where Γ is a finite subgroup of $O(4)$ which acts on S^3 without fixed points.*

(This includes the Poincaré Conjecture as the special case where $\pi_1(M)$ is trivial.)

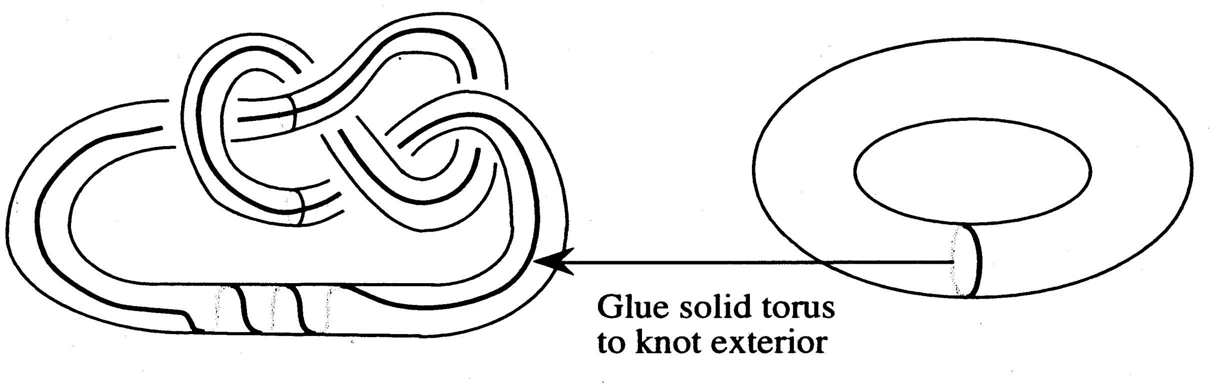
For manifolds of type (2), the recent work of [30] and [15] shows that the manifolds are either *Seifert fibre spaces* (see 2.8) or Haken; hence the Geometrization Conjecture holds.

For manifolds of type (3), the Geometrization Conjecture reduces to the following:

Conjecture 1.10 (Hyperbolization Conjecture). *If M is irreducible with $\pi_1(M)$ infinite and containing no $\mathbb{Z}\times\mathbb{Z}$ subgroup, then M is hyperbolic.*

Further important evidence for the hyperbolization conjecture is provided by Thurston's "Hyperbolic Dehn surgery theorem", to be discussed in more detail in 5.6 below.

Definition 1.11. First we recall the construction of 3-manifolds by *Dehn filling.* Let M be a compact 3-manifold with boundary consisting of one or more tori $T_1, \ldots, T_k$. We can then form *closed* 3-manifolds by attaching solid tori to $T_1, \ldots, T_k$ using arbitrary diffeomorphisms between their boundary tori. These 3-manifolds are said to be obtained from M by *Dehn filling.* The resulting manifold depends only on the isotopy classes of the *surgery curves* $\gamma_i \subset T_i$ which bound discs in the added solid tori; we denote it $M(\gamma_1, \ldots, \gamma_k)$. If L is any link in S^3, we can apply this construction to the link exterior M, obtained by removing an open tubular neighbourhood of L from S^3. Then we say that the resulting 3-manifold is obtained from S^3 by *Dehn surgery* along L.



Theorem 1.12. (Lickorish [56], Wallace [87]). *Every closed, orientable 3-manifold can be obtained by Dehn surgery along some link in S^3.*

Before stating Thurston's result, we note that an orientable hyperbolic 3-manifold which has *finite volume* but is non-compact is homeomorphic to the interior of a compact 3-manifold $\overline{M}$ which is compact with boundary $\partial\overline{M}$ consisting of tori. We then call M a *cusped* hyperbolic manifold, and we write $M(\gamma_1, \ldots, \gamma_k)$ for the manifolds obtained by Dehn filling on $\overline{M}$.

Theorem 1.13 (Hyperbolic Dehn Surgery Theorem). *If M is a cusped hyperbolic 3-manifold, then "almost all" manifolds obtained from M by Dehn filling are hyperbolic. (More precisely, only a finite number of surgeries must be excluded for each cusp.)*

In particular, if M has one cusp, then the Dehn filled manifolds $M(\gamma)$ are closed hyperbolic manifolds for all but a finite number of isotopy classes of surgery curves γ.

Since every closed 3-manifold can be obtained by Dehn filling from a hyperbolic link complement ([56], [87], [65]), this shows that in some sense "most" closed 3-manifolds are hyperbolic! (However, it is not currently known how to make this into a precise statement.)

In fact the number of non-hyperbolic surgeries is usually very small. The worst known case is the figure knot complement which has 10 non-hyperbolic surgeries (see section 5.7).

The Geometrization Conjecture, and the special cases proved so far, has had a profound effect on 3-manifold topology, including major roles in the solution of several old conjectures (e.g. The Smith Conjecture, see [64].)

The existence of a geometric structure on a given manifold provides a great deal of information about that manifold. For example, its fundamental group is residually finite, so has a solvable word problem. For hyperbolic 3-manifolds, the Mostow rigidity theorem shows that the hyperbolic structure is unique; thus geometric invariants of hyperbolic 3-manifolds are actually topological invariants. This provides very powerful tools for understanding 3-dimensional topology.

1.6 Geometric structures on 3-manifolds with symmetry

In late 1981, Thurston announced that the Geometrization Conjecture holds for 3-manifolds with a kind of topological symmetry (see also Thurston's Theorem A stated on page 153 of the appendix.)

Theorem 1.14 (Symmetry Theorem). (Thurston [83]) *Let M be an orientable, irreducible, closed 3-manifold. Suppose M admits an action by a finite group G of orientation preserving diffeomorphisms such that some non-trivial element has a fixed point set of dimension one. Then M admits a geometric decomposition preserved by the group action.*

Later, we'll state a more general version in terms of *orbifolds* (see section 2.13 and Thurston's Theorem B stated on page 153).

This theorem has many applications to the study of group actions on 3-manifolds and to the existence of geometric structures on 3-manifolds. Here is an important special case.

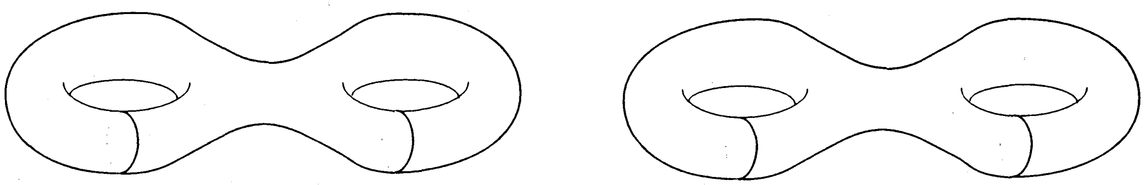
Theorem 1.15. *Assume M and G are as in the symmetry theorem above. If M contains no incompressible tori, then M has a geometric structure such that this action of the group G is by isometries. In particular, the fixed point set of each group element is totally geodesic.*

Taking $M = S^3$ and the group to be cyclic gives the **Smith Conjecture:** If ϕ is a periodic, non-free, orientation preserving diffeomorphism of S^3 then ϕ is conjugate to a rotation. In particular, the fixed point set of ϕ is an unknotted circle.

1.7 Some 3-manifolds with symmetry

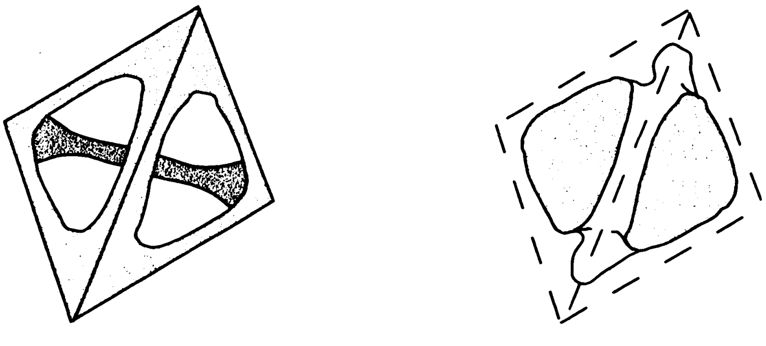
Next we give some applications to 3-manifolds, constructing various classes of manifolds with symmetry to which the main theorem can be applied. Rolfsen's book [69] is an excellent general reference for these constructions.

Every orientable 3-manifold has a *Heegaard decomposition:* a representation as the union of two handlebodies glued together along their boundaries.



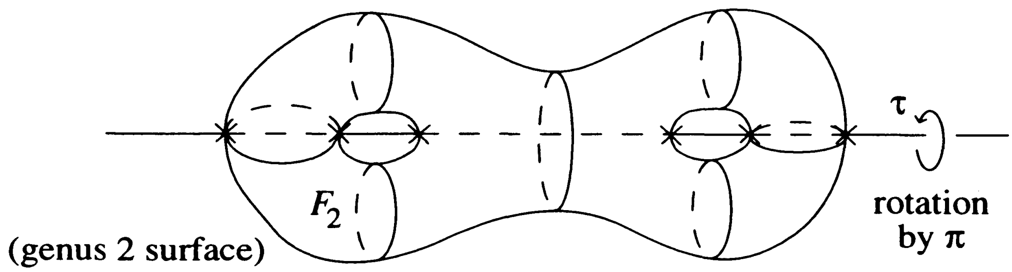
Glue boundaries together by a homeomorphism

Proof. Triangulate the manifold and take a regular neighbourhood of the 1-skeleton as one handlebody, and its complement (a neighbourhood of the dual 1-skeleton) as the other handlebody. $\square$



Example 1.16. The only manifold with a genus 0 Heegaard decomposition is S^3, while the only manifolds with genus 1 Heegaard decompositions are the lens spaces and $S^2 \times S^1$. The manifolds S^3 and $S^2 \times S^1$ are clearly geometric. Every lens space $L(p,q)$ is the quotient of $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ by the cyclic group $\mathbb{Z}_p$ of isometries generated by $(z_1, z_2) \mapsto (tz_1, t^q z_2)$, where $t = \exp(2\pi i/p)$. So $L(p,q)$ has a spherical structure.

Example 1.17. Heegaard decompositions of genus 2 have a certain kind of 2-fold symmetry which can be seen as follows. A surface F_2 of genus two has a special "hyper-elliptic" involution τ which takes every simple closed curve to an isotopic curve (possibly reversing orientation).



Using Lickorish's result that every diffeomorphism is isotopic to a product of Dehn twists around simple closed curves [56], we obtain:

Proposition 1.18. *Every diffeomorphism $\phi : F_2 \to F_2$ is isotopic to ϕ' such that $\phi'\tau = \tau\phi'$.*

Extending τ over each handlebody and adjusting the gluing map by this proposition, we obtain an involution on any manifold M with a genus 2 Heegaard decomposition. (The fixed point set is 1-dimensional, and covers a 3-bridge knot in the quotient $M/\tau = S^3$.)

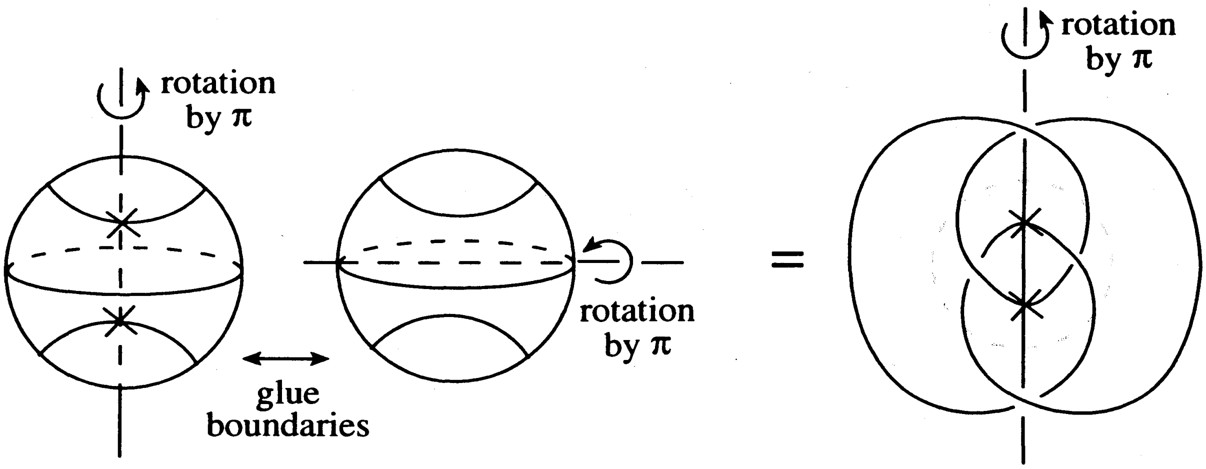
Corollary 1.19. *Every 3-manifold of Heegaard genus two admits a geometric decomposition.*

Proof. If the genus 2 manifold is irreducible then theorem 1.14 applies directly. Otherwise, a result of Haken [34] shows that the manifold is a connected sum of two genus one manifolds, which are geometric by example 1.16. □

Next we consider the construction of 3-manifolds by *Dehn surgery* as described in 1.11. If a link in S^3 has a suitable kind of symmetry, the manifolds obtained by Dehn surgery on the link often exhibit a similar kind of symmetry.

Example 1.20. The *two bridge* (or *rational*) knots and links in S^3 are the links which can be cut by a 2-sphere into two pairs of unknotted, unlinked arcs. Alternatively, these are the links which have a projection with exactly 2 local maxima and 2 local minima. These have been extensively studied and classified by Schubert [72].

Each 2-bridge link has a 2-fold symmetry: it can be arranged in $\mathbb{R}^3$ so that it is invariant under a 180 degree rotation. Further, this involution always extends over the solid torus added in Dehn surgery.



Corollary 1.21. *Every Dehn surgery on a 2-bridge knot in S^3 gives a manifold admitting a geometric decomposition.*

Proof. If the Dehn surgered manifold M is irreducible this follows from the symmetry theorem. Otherwise, note that the Heegaard genus of M is at most 2 (**Exercise**). So theorem 1.19 applies. □

A similar argument proves

Corollary 1.22. *Every Dehn filling on a bundle over the circle with once-punctured torus as fibre admits a geometric decomposition.*

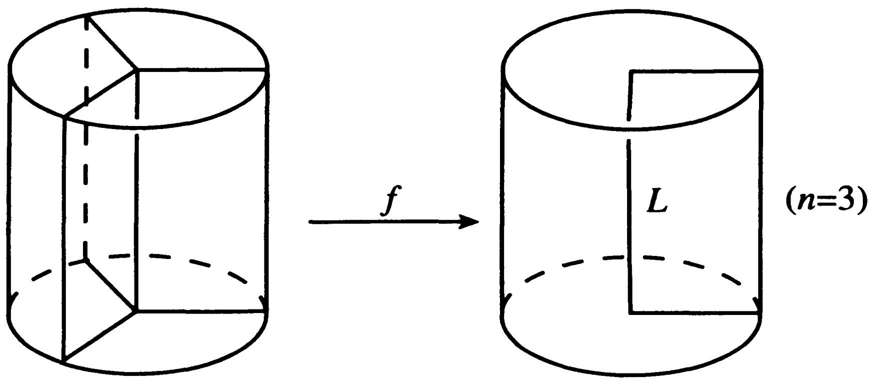
Proof. Such a manifold again has a suitable 2-fold symmetry. (**Exercise.**)

1.8 3-manifolds as branched coverings

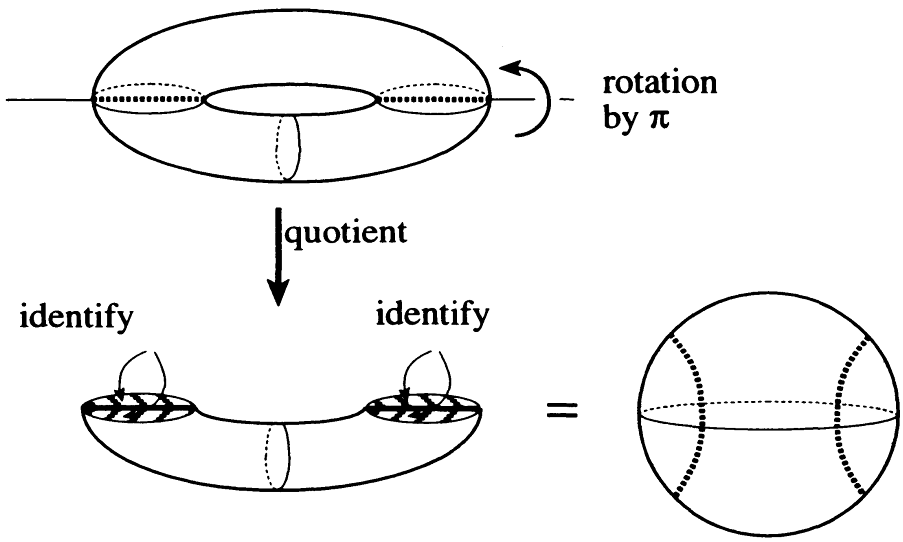
Finally we consider the construction of 3-manifolds by *branched coverings.*

Let M and N be 3-manifolds. A map $f : M \to N$ is a *branched covering, branched over* $L \subset N$, if

1. the restriction $f : M - f^{-1}(L) \to N - L$ is a covering, and
2. any point $x \in f^{-1}(L)$ has a neighbourhood homeomorphic to $D \times I$, where D is the unit disc in $\mathbb{C}$, on which f has the form $f : D \times I \to D \times I, (z,t) \mapsto (z^n, t)$, for some integer $n \geq 2$.



Example 1.23. A solid torus is the 2-fold branched covering of a 3-ball branched over two unknotted, unlinked arcs as shown below.



Theorem 1.24. (Alexander [2]). *Every closed orientable* 3*-manifold can be obtained as a branched covering of* S^3 *branched along a link.*

In fact, one can always take the branching set in S^3 to be the figure eight knot by results of Hilden et. al. [42].

A branched covering f is *regular* if the covering transformations act transitively on the fibres $f^{-1}(x)$; then N is the quotient of M by the group of covering transformations. (The branched coverings in the above theorems are typically irregular.)

Theorem 1.25. *Any regular branched covering space, M, of S^3 branched over a knot or link L has a geometric decomposition.*

Proof. If M is irreducible then this follows from the symmetry theorem. Otherwise it follows from the Orbifold Theorem (see 2.58 below). Let Q be the quotient orbifold with underlying space S^3 and singular locus L; each component of L is labelled with the branch index on that component. It is an easy exercise to check the hypotheses of the Orbifold Theorem in this setting. Thus Q has a geometric decomposition. Lifting this gives a geometric decomposition of M. □

For example, any manifold of Heegaard genus two is a (regular) 2-fold covering of S^3 branched over a 3-bridge knot or link (see Example 1.17).

A knot K in S^3 is *prime* if there is no 2-sphere which meets K transversally in exactly two points and separates K into two non-trivial knotted arcs. A *Conway sphere* is an incompressible 4-punctured sphere in the exterior of the knot whose boundary consists of four meridians of the knot. Now let M be the n-fold cyclic branched cover of S^3, branched along the knot K. It follows from the Orbifold Theorem that M is prime if and only if K is prime. Also if M contains an essential torus then either K is a torus knot or satellite knot, or else there is a Conway sphere and covering degree is 2.

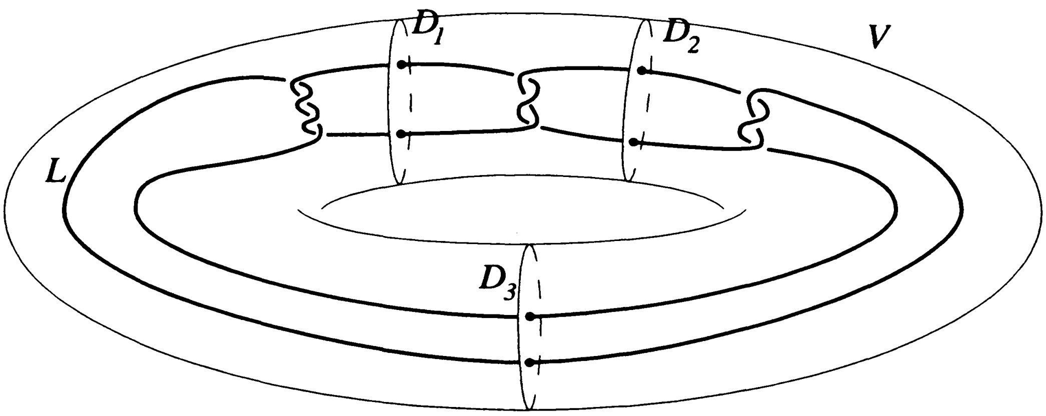
Corollary 1.26. *Let K be any knot in S^3 which is not a torus knot or a satellite knot. Then the n-fold cyclic branched cover of S^3 over K is hyperbolic for all $n \geq 3$, except for the 3-fold cover of the figure eight knot (which has a Euclidean structure).*

Proof. The preceding remarks show that the branched cover is irreducible and atoroidal hence has a geometric structure by the symmetry theorem. Dunbar has classified those orbifolds with non-hyperbolic geometric structures and underlying space the three sphere. This provides the one exception noted in the theorem. (See Dunbar [26], Bonahon-Siebenmann [9],[10].) □

Remark 1.27. The condition on K holds if and only if $S^3 - K$ has a hyperbolic structure.

In contrast, the 2-fold branched coverings of many knots do not admit hyperbolic structures. For example, each 2-bridge knot has a 2-fold branched cover which is a lens space, so has a spherical structure. (The branched covering is obtained by gluing together two copies of the example in 1.23.)

Definition 1.28. A *Montesinos knot or link* is a link L contained in an unknotted solid torus V in S^3 such that there are disjoint meridional disks $D_1, \cdots, D_n$ in V which separate V into components whose closures are balls and with the property that L intersects each such ball in a pair of unknotted arcs. In addition it is required that the boundary of V be incompressible in $V - L$; in other words, every meridional disc of V intersects L. The 2-fold cover of S^3 branched over a Montesinos knot or link is a Seifert fibre space.



For knots with non-trivial Conway decomposition the 2-fold cover contains an essential torus.

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