Introduction

The theory of manifolds of dimension three is very different from that of other dimensions. On the one hand we do not even have a conjectural list of all 3-manifolds. On the other hand, if Thurston's Geometrization Conjecture is true, then we have a very good structure theory.

The topology of compact surfaces is well understood. There is a well known topological classification theorem, based on a short list of easily computable topological invariants: orientability, number of boundary components and Euler characteristic. For closed surfaces (compact with no boundary) the fundamental group is a complete invariant. The geometry of surfaces is also well understood. Every closed surface admits a metric of constant curvature. Those with curvature $+1$ are called spherical, or elliptic, and comprise the sphere and projective plane. Those with curvature 0 are Euclidean and comprise the torus and Klein bottle. The remainder all admit a metric of curvature -1 and are called hyperbolic. The Gauss-Bonnet theorem relates the topology and geometry

$$\int_F K \, dA = 2\pi\chi(F)$$

where K is the curvature of a metric on the closed surface F of Euler characteristic $\chi(F)$. In particular this implies that the sign of a constant curvature metric is determined by the sign of the Euler characteristic. However in the Euclidean and hyperbolic cases, there are many constant curvature metrics on a given surface. These metrics are parametrized by a point in a Teichmüller space.

The topology of 3-dimensional manifolds is far more complex. At the time of writing there is no complete list of closed 3-manifolds and no *proven* complete set of topological invariants. However if Thurston's Geometrization Conjecture were true, then we would know a complete set of topological invariants. In particular for irreducible atoroidal 3-manifolds, with the exception of lens spaces, the fundamental group is a complete invariant. However this group, on its own, does not provide a practical method of identifying a 3-manifold. On the other hand, once the geometric structure has been found then there are geometrical invariants which can be practically calculated and completely determine the manifold.

A *geometric structure* on a manifold is a complete, locally homogeneous Riemannian metric: every two points have isometric neighbourhoods. The universal cover of such a manifold is a *homogeneous space* and is thus the quotient of a Lie group by a compact subgroup. In dimension two it is a classical result that every surface admits a geometric structure. There are eight geometries needed for compact 3-manifolds. The connected sum of two geometric three manifolds is usually not geometric. However the Geometrization Conjecture states that every closed 3-manifold can be decomposed (in a way to be described) into geometric pieces.

The first step in the decomposition of orientable 3-manifolds is into irreducible pieces by cutting along *essential* 2-spheres and capping off the resulting boundaries by attaching 3-balls. This theory was worked out by Kneser and refined by Milnor. For 3-dimensional manifolds the irreducible pieces obtained are unique. The corresponding statement in higher dimensions is false. Some important classes of 3-manifolds which were studied early on include the following:

- The quotient of the 3-sphere by a finite group of isometries acting freely (a spherical space form). These include the lens spaces (quotients of the round 3-sphere by a cyclic group of isometries) which provide the only known examples of distinct irreducible, atoroidal 3-manifolds with the same fundamental group. The famous Poincaré homology 3-sphere is the quotient of the 3-sphere by the binary icosahedral group (the double cover in $SU(2)$ of the icosahedral subgroup of $SO(3)$.)
- The Seifert fibre spaces. These are compact 3-manifolds which can be foliated by circles and were classified by Seifert. A special case is a *circle bundle* over a closed surface F. If F and the total space M are both orientable this bundle is determined by its Euler class $e \in \mathbb{Z}$. In general, the quotient space obtained by collapsing each circle to a point is a two dimensional *orbifold.* All Seifert fibre spaces have a geometric structure.
- The 10 Euclidean 3-manifolds fit into the general theory of flat manifolds developed by Bieberbach. Bieberbach showed that a compact Euclidean manifold of dimension n is finitely covered by an n-torus. Bieberbach's results also apply to Euclidean orbifolds, producing the 219 types of 3-

dimensional crystallographic groups known to chemists.

- The *mapping cylinder* construction produces an n-manifold M from any automorphism θ of an $(n-1)$-manifold F as the quotient $M = F \times [0,1]/(x,1) \equiv (\theta(x),0)$. In the case that F is a 2-torus, the automorphism is determined up to isotopy by an element of the group $GL(2,\mathbb{Z})$. These give 3-manifolds with the *Solv*, *Nil* and Euclidean geometries. When the genus of F is more than 1 there is a (possibly trivial) torus decomposition into geometric pieces.
- A *Haken* manifold, M, is a compact, irreducible 3-manifold which contains a closed embedded surface with infinite fundamental group that injects under the map induced by inclusion into the fundamental group of M. Haken manifolds include many important classes of 3-manifolds, and a great deal is now known about these manifolds through the work of Haken, Waldhausen, Thurston and many others. In particular they have geometric decompositions. However, Hatcher [38] showed that all but finitely many Dehn surgeries on a knot give a non-Haken manifold. More recently, Cooper and Long [20] showed that all but finitely many such fillings give a 3-manifold containing an essential *immersed* surface.

The next step in the classification program is to decompose along *essential* embedded tori. The JSJ decomposition (of Jaco-Shalen and Johannson) gives a canonical splitting of a compact 3-manifold by cutting out a maximal Seifert fibred piece.

Thurston [80] introduced the idea of "hyperbolic Dehn surgery" which is a method of *continuously* changing one 3-manifold into another with a different topology. The intermediate spaces are **cone-manifolds** with a hyperbolic metric everywhere except along a knot or link called the *singular locus*. The set of manifolds form a discrete subset, contained in the larger subset of *orbifolds*. This method of continuously changing topology and geometry only works in dimension three. The computer program SnapPea developed by Jeff Weeks [88] allows one to put this philosophy into practice. Many insights and theorems have developed from this point of view.

Roughly speaking an **orbifold** is the quotient of a manifold by a finite group of diffeomorphisms. Actually an orbifold has the *local structure* of such a space. It is the natural object to consider when one is studying *discrete symmetry groups*. Compact two dimensional orbifolds are classified in a similar way to surfaces, using an orbifold version of Euler characteristic. This classification encompasses the classification of the regular solids (finite subgroups of the orthogonal group $O(3)$), the classification of the 17 wallpaper groups, and of periodic tessellations of the hyperbolic plane. There are, however, four families of *bad* or *non-geometric* two-dimensional orbifolds that do not arise globally as the quotient of a manifold by a finite group. However they do arise quite naturally as the base-orbifolds of certain Seifert fibrations. In fact the base orbifold of a Seifert fibration is *bad* if and only if the fibration is not isotopic to one with the fibres geodesic in a geometric structure on the Seifert fibre space.

The *Orbifold Theorem* characterizes when a 3-dimensional orbifold with 1-dimensional singular locus has a *geometric structure*, in other words, when it is the quotient of a homogeneous space by a discrete group of isometries. This theorem has many consequences, for example an irreducible, atoroidal, closed orientable 3-manifold which admits a symmetry with 1-dimensional fixed set is geometric. It follows that all 3-manifolds of Heegaard genus two have a geometric decomposition.