Part II
Axioms for a vertex algebra

In Section 4, we will begin with describing the set of axioms for a vertex algebra following [B2] and some of the straightforward consequences. A vertex algebra of local fields is introduced as a corollary to the result in Section 3. The notion of a module for a vertex algebra and the notion of an intertwiner are also described briefly. In section 5, we will explain the state-field correspondence, which is the characteristic feature of vertex algebras that separates those acting on themselves from those acting on modules. This property together with the mutual locality can be used to characterize the vertex algebra structures on a given vector space by using the vertex algebra of local fields as a tool. In Section 6, we will explain the role of the translation operator (or the derivation), which is one of the main ingredients in the formulation by Goddard.

4 Axioms and their consequences

In this section, we first review the definition of a vertex algebra following Borcherds [B2] and describe some consequences of the axioms. We next describe the vertex algebra of local fields as an application of the results of the preceding section. We also explain the notion of a module for a vertex algebra.

4.1 Borcherds' axioms for a vertex algebra

According to Borcherds [B2], a vertex algebra is defined as follows[10]:

Definition 4.1.1. A *vertex algebra* is a vector space V equipped with countably many bilinear binary operations

$$\begin{array}{ccc} V \times V & \longrightarrow & V \\ (a, b) & \longmapsto & a_{(n)}b, \quad (n \in \mathbb{Z}), \end{array}$$

and a vector $\mathbf{1} \in V$ subject to the following conditions:

(B0) For each pair of vectors $a, b \in V$, there exists a nonnegative integer n_0 such that

$$a_{(n)}b = 0 \quad \text{for all} \quad n \geq n_0.$$

[10]Originally in [B1, Section 4], the properties (B0), (4.2.3),(4.2.2),(4.2.6) and (4.3.3) given below are taken as the axioms for a vertex algebra.

(B1) **(Borcherds identity)**[11] For all vectors $a, b, c \in V$ and all integers $p, q, r \in \mathbb{Z}$,

$$\sum_{i=0}^{\infty} \binom{p}{i} (a_{(r+i)}b)_{(p+q-i)}c$$
$$= \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} \left(a_{(p+r-i)}(b_{(q+i)}c) - (-1)^r b_{(q+r-i)}(a_{(p+i)}c) \right) .$$

(B2) For any $a \in V$,

$$a_{(n)}\mathbf{1} = \begin{cases} 0, & (n \geq 0), \\ a, & (n = -1). \end{cases}$$

The vector $\mathbf{1}$ is called the vacuum vector of V. Note that, because of (B0), each side of the Borcherds identity in (B1) is a finite sum. We also note that (B2) does not contain a condition on $a_{(n)}\mathbf{1}$ for $n \leq -2$. However, if $a_{(-2)}\mathbf{1}$ are specified for all $a \in V$, in other words, if we know the endomorphism

$$\begin{aligned} T: \ V &\longrightarrow \ V \\ a &\longmapsto a_{(-2)}\mathbf{1}, \end{aligned}$$

then $a_{(n)}\mathbf{1}$ are uniquely determined by T as we will see in (4.2.2) below.

It is sometimes convenient to introduce the generating series

$$Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$$

and consider the map

$$\begin{aligned} Y: \ V &\longrightarrow (\operatorname{End} V)[[z, z^{-1}]] \\ a &\longmapsto Y(a, z) \end{aligned} .$$

Then the vertex algebra structure is specified by the triple $(V, \mathbf{1}, Y)$. Note that the axiom (B0) says that the series $Y(a, z)$ is a field on V for each $a \in V$.

Let V and W be vertex algebras. A linear map $f : V \longrightarrow W$ is said to be a homomorphism of vertex algebras if $f(\mathbf{1}) = \mathbf{1}$ and $f(a_{(n)}b) = f(a)_{(n)}f(b)$ hold for all $a, b \in V$ and $n \in \mathbb{Z}$. The latter condition is also written as $f(Y(a, z)b) =$

[11] This identity is nothing but the Cauchy-Jacobi identity of Frenkel-Lepowsky-Meurman [FLM, (8.8.29) and (8.8.41)], while the special case (4.3.3) is due to Borcherds [B1]; we here follow the terminology in [K].

$Y(f(a), z)f(b)$. A homomorphism of vertex algebras is said to be an isomorphism of vertex algebras if it is a linear isomorphism.

For a vertex algebra V, we set

$$V_{(n)}V = \mathrm{Span}\,\{\, a_{(n)}b \mid a, b \in V \,\}.$$

Note that $V = V_{(-1)}V$ by (B2).

4.2 Consequences of axioms

For the convenience of the reader, we derive various properties of vertex algebras from the axioms. First, putting $a = b = c = \mathbf{1}$ and $p = q = r = -1$ in (B1), we immediately see

$$\mathbf{1}_{(-2)}\mathbf{1} = 0, \quad \text{i.e.,} \quad T\mathbf{1} = 0. \tag{4.2.1}$$

Then the substitution $b = c = \mathbf{1}$ and $p = 0, q = -2, r = n$ shows by (B2)

$$(a_{(n)}\mathbf{1})_{(-2)}\mathbf{1} = -na_{(n-1)}\mathbf{1}.$$

Therefore, for $n \leq -1$, we inductively deduce

$$a_{(n)}\mathbf{1} = T^{(-n-1)}a$$

where $T^{(k)} = T^k/k!$. Thus (B2) is completed as

$$a_{(n)}\mathbf{1} = \begin{cases} 0, & (n \geq 0), \\ T^{(-n-1)}a, & (n \leq -1). \end{cases} \tag{4.2.2}$$

Now, let $b = c = \mathbf{1}, p = -1, q = n, r = 0$ in (B1). Then, by (B2) and (4.2.1), we have

$$\mathbf{1}_{(n)}a = \begin{cases} 0, & (n \neq -1), \\ a, & (n = -1). \end{cases} \tag{4.2.3}$$

Next, let $b = \mathbf{1}, p = n, q = 0, r = -2$ in (B1) and replace c by b. Then by (4.2.3),

$$(Ta)_{(n)}b = -na_{(n-1)}b. \tag{4.2.4}$$

Further, $c = \mathbf{1}, p = 0, q = -2, r = n$ yields by (B2)

$$a_{(n)}(Tb) = T(a_{(n)}b) + na_{(n-1)}b. \tag{4.2.5}$$

Namely, $[T, a_{(n)}] = -na_{(n-1)}$, called the *translation covariance*. Then comparing (4.2.4) and (4.2.5) we see that T is a derivation for all the products:

$$T(a_{(n)}b) = (Ta)_{(n)}b + a_{(n)}(Tb).$$

Finally, $c = \mathbf{1}, p = -1, q = 0, r = n$ gives rise to

$$b_{(n)}a = \sum_{i=0}^{\infty}(-1)^{n+i+1}T^{(i)}(a_{(n+i)}b) \tag{4.2.6}$$

which is called the *skew symmetry* .

Note that the relation (4.2.4) shows that $V_{(n-1)}V \subset V_{(n)}V$ if $n \neq 0$, thus

$$\cdots \subset V_{(-3)}V \subset V_{(-2)}V \subset V_{(-1)}V = V, \quad V_{(0)}V \subset V_{(1)}V \subset V_{(2)}V \subset \cdots .$$

In terms of the generating series $Y(a,z)$, the properties (4.2.2)–(4.2.6) are rewritten as follows:

$$\begin{aligned} &Y(a,z)\mathbf{1} = e^{Tz}a, \quad Y(\mathbf{1},z) = \mathrm{id}_V, \quad Y(Ta,z) = \partial_z Y(a,z), \\ &[T, Y(a,z)] = \partial_z Y(a,z), \quad Y(b,z)a = e^{Tz}Y(a,-z)b. \end{aligned}$$

Remark 4.2.1. Let V be a vertex algebra satisfying the following condition: For any $a \in V$, there exists a nonnegative integer n_0 such that $a_{(n)}V = 0$ for all $n \geq n_0$. Then, by (4.2.5) we actually have $a_{(n)}V = 0$ for all $n \geq 0$.

Note 4.2.2. The structure of a vertex algebra as in the remark is described as follows ([B1]): It has a structure of commutative associative algebra with respect to the multiplication defined by $ab = a_{(-1)}b$, and T is a derivation (cf. Subsection 8.4). Conversely, any commutative associative algebra with a derivation T has a vertex algebra structure with respect to

$$a_{(n)}b = \begin{cases} 0 & (n \geq 0), \\ (T^{(-n-1)}a)b & (n \leq -1). \end{cases}$$

Any finite-dimensional vertex algebra is described in this way ([B2, p.416]), since it obviously satisfies the condition in the remark.

4.3 Structure of Borcherds identity

Let $B(p,q,r)$ denote one of the three terms of the Borcherds identity:

$$B(p,q,r)=\sum_{i=0}^{\infty}\binom{p}{i}(a_{(r+i)}b)_{(p+q-i)}c,\quad \sum_{i=0}^{\infty}(-1)^i\binom{r}{i}a_{(p+r-i)}(b_{(q+i)}c),$$
$$\text{or}\quad \sum_{i=0}^{\infty}(-1)^{r+i}\binom{r}{i}b_{(q+r-i)}(a_{(p+i)}c).$$

Then, straightforward calculation shows

$$B(p+1,q,r)=B(p,q+1,r)+B(p,q,r+1). \tag{4.3.1}$$

Therefore, the Borcherds identity for two of the indices $(p+1,q,r)$, $(p,q+1,r)$ and $(p,q,r+1)$ imply the Borcherds identity for the other index. This proves

Proposition 4.3.1. *The Borcherds identity for all p and q with fixed r and for all q and r with fixed p imply the Borcherds identity for all p,q and r.*

Now let us consider special cases of the Borcherds identity. Setting $r=0$ and $p=0$ respectively, we have

$$[a_{(p)},b_{(q)}]=\sum_{i=0}^{\infty}\binom{p}{i}(a_{(i)}b)_{(p+q-i)} \tag{4.3.2}$$

and

$$(a_{(r)}b)_{(q)}=\sum_{i=0}^{\infty}(-1)^i\binom{r}{i}(a_{(r-i)}b_{(q+i)}-(-1)^r b_{(q+r-i)}a_{(i)}) \tag{4.3.3}$$

called the *commutator formula* and the *associativity formula*. Here we have omitted the overall c. We next take an r_0 such that $a_{(r)}b=0$ for all $r\ge r_0$. Then for such r, we have,

$$\sum_{i=0}^{\infty}(-1)^i\binom{r}{i}\left(a_{(p+r-i)}b_{(q+i)}-(-1)^r b_{(q+r-i)}a_{(p+i)}\right)=0,\quad (r\ge r_0) \tag{4.3.4}$$

which is called the *commutativity*. This exactly means that the fields $Y(a,z)$ and $Y(b,z)$ are mutually local for any $a,b\in V$. Finally, take a p_0 such that $a_{(p)}c=0$ for all $p\ge p_0$. Then, for such p, we have

$$\sum_{i=0}^{\infty}\binom{p}{i}(a_{(r+i)}b)_{(p+r-i)}c=\sum_{i=0}^{\infty}(-1)^i\binom{r}{i}a_{(p+r-i)}(b_{(q+i)}c),\quad (p\ge p_0) \tag{4.3.5}$$

called the *duality* or the *associativity*. This time we can not omit c because the choice of p_0 depends on it.

The relations (4.3.2)–(4.3.5) are rewritten in terms of generating series as:

$$[a_{(p)}, Y(b,z)] = \sum_{i=0}^{\infty} \binom{p}{i} Y(a_{(i)}b, z) z^{p-i},$$

$$Y(a_{(r)}b, z) = Y(a,z)_{(r)} Y(b,z),$$

$$Y(a,y)Y(b,z)(y-z)^r = Y(b,z)Y(a,y)(y-z)^r, \quad (r \geq r_0),$$

$$Y(Y(a,y)b,z)(y+z)^p c = Y(a, y+z)Y(b,z)(y+z)^p c, \quad (p \geq p_0).$$

Remark 4.3.2. (4.3.3) and (4.3.4) in particular shows that the fields $Y(a,z)$ and $Y(b,z)$ on V are mutually local with OPE

$$Y(a,y)Y(b,z) \sim \sum_{i=0}^{\infty} \frac{Y(a_{(i)}b,z)}{(y-z)^{i+1}}.$$

As an immediate consequence of Proposition 4.3.1, we have

Proposition 4.3.3. *The axiom (B1) is equivalent to either (4.3.2) or (4.3.4) and either (4.3.3) or (4.3.5).*

We here note the remarkable correspondence between the coefficients and indices in the Borcherds identity and the coefficients and exponents of the expansions:

$$(x-y)^r(y-z)^q(x-z)^p = \sum_{i=0}^{\infty} \binom{p}{i} (x-y)^{r+i}(y-z)^{p+q-i},$$

$$\sum_{i=0}^{\infty}(-1)^i \binom{r}{i} (x-z)^{p+r-i}(y-z)^{q+i}, \text{ and } \sum_{i=0}^{\infty}(-1)^{r+i}\binom{r}{i}(y-z)^{q+r-i}(x-z)^{p+i}.$$

With this resemblance in mind, we see the relation (4.3.1) corresponds to

$$\begin{aligned}(x-y)^r&(y-z)^q(x-z)^{p+1} \\ &= (x-y)^r(y-z)^{q+1}(x-z)^p + (x-y)^{r+1}(y-z)^q(x-z)^p.\end{aligned}$$

Note 4.3.4. In [K], it is considered a vector space equipped with bilinear binary operations $(a,b) \longmapsto a_{(n)}b$ for nonnegative integers n and a linear map $T: V \longrightarrow V$ satisfying (B0), (4.2.4),(4.2.6), and (4.3.2), and is called a *conformal algebra* [12]. The same notation is also considered independently in [P] and is called a *vertex Lie algebra*.

[12]The axioms for a conformal algebra in the sense of [K] is nothing to do with a conformal structure.

4.4 Some categorical constructions

Let us describe the notion of subalgebras and quotient algebras of a vertex algebra V.

A *vertex subalgebra* of V is a subspace of V containing the vacuum vector $\mathbf{1}$ of V closed under the binary operations of V. In other words, a subspace U of V is a vertex subalgebra if it is equipped with a structure of a vertex algebra such that the inclusion $U \longrightarrow V$ is a homomorphism of vertex algebras. The image of any homomorphism of vertex algebras is a vertex subalgebra.

For example, let G be a group that acts on V as automorphism of a vertex algebra. Then the fixed-point space $V^G = \{a \in V \mid g(a) = a, \text{ for all } g \in G\}$ is a vertex subalgebra.

An ideal of V is a subspace $J \subset V$ such that $J_{(n)}V \subset J$ for all $n \in \mathbb{Z}$. The kernel of any homomorphism of vertex algebras is an ideal. If J is an ideal of V, then $TJ = J_{(-2)}\mathbf{1} \subset J$. Hence, by the skew symmetry,

$$V_{(n)}J \subset \sum_{i=0}^{\infty} T^i(J_{(n+i)}V) \subset \sum_{i=0}^{\infty} T^i J \subset J$$

for any $n \in \mathbb{Z}$.

Therefore, for an ideal $J \subset V$, the vertex algebra structure on V induces the one on the quotient space V/J. The vertex algebra V/J is called a quotient vertex algebra of V. In other words, a quotient space W of V is a quotient vertex algebra if it is equipped with a structure of a vertex algebra such that the projection $V \longrightarrow W$ is a homomorphism of vertex algebras.

We will describe the notion of the direct product and the tensor product of vertex algebras. Let $(V, \mathbf{1}_V, Y_V)$ and $(V', \mathbf{1}_{V'}, Y_{V'})$ be vertex algebras.

The direct product $V \times V'$ of vector spaces has a structure of a vertex algebra by setting

$$\mathbf{1}_{V\times V'} = (\mathbf{1}_V, \mathbf{1}_{V'}), \quad (a, a')_{(n)}(b, b') = (a_{(n)}b, a'_{(n)}b').$$

The vertex algebra $V \times V'$ is called the direct product of vertex algebras V and V'. Then the projections

$$V \longleftarrow V \times V' \longrightarrow V'$$

are homomorphisms of vertex algebras, which give a product in the category of vertex algebras.

On the other hand, consider the tensor product $V \otimes V'$ of vector spaces. We set

$$\mathbf{1}_{V\otimes V'} = \mathbf{1}_V \otimes \mathbf{1}_{V'}, \quad (a\otimes a')_{(n)}(b\otimes b') = \sum_{i\in\mathbb{Z}} (a_{(i)}b) \otimes (a'_{(n-i-1)}b').$$

Note that the latter is written in terms of the generating series as

$$Y_{V\otimes V'}(a\otimes a', z)(b\otimes b') = Y_V(a,z)b \otimes Y_{V'}(a',z)b'$$

where $Y_{V\otimes V'}(a\otimes a', z) = \sum_{n\in\mathbb{Z}} (a\otimes a')_{(n)} z^{-n-1}$. Then by Theorem 2.3.1, $Y_{V\otimes V'}(a\otimes a', z)$ and $Y_{V\otimes V'}(b\otimes b', z)$ are mutually local field on $V\otimes V'$ with the associativity

$$\begin{aligned}
&Y_{V\otimes V'}(a\otimes a', z)_{(n)} Y_{V\otimes V'}(b\otimes b', z) \\
&= \sum_{i\in\mathbb{Z}} (Y_V(a,z)_{(i)} Y_V(b,z)) \otimes (Y_{V'}(a',z)_{(n-i-1)} Y_{V'}(b',z)) \\
&= \sum_{i\in\mathbb{Z}} Y_V(a_{(i)}b, z) \otimes Y_{V'}(a'_{(n-i-1)}b', z) \\
&= Y_{V\otimes V'}((a\otimes a')_{(n)}(b\otimes b'), z),
\end{aligned}$$

which imply the Borcherds identity by Proposition 4.3.3. Hence $(V\otimes V', \mathbf{1}_{V\otimes V'}, Y_{V\otimes V'})$ is a vertex algebra, called the *tensor product of vertex algebras* V and V'. Then the subspaces $\{a\otimes \mathbf{1}_{V'} \mid a \in V\}$ and $\{\mathbf{1}_V \otimes a' \mid a' \in V'\}$ are vertex subalgebras identified with V and V' respectively. The inclusions

$$V \longrightarrow V\otimes V' \longleftarrow V'$$

are homomorphisms of vertex algebras characterized by the following universal property:

Proposition 4.4.1. *Let V and V' be vertex algebras and let $i : V \longrightarrow V\otimes V'$, $i' : V' \longrightarrow V\otimes V'$ be the embeddings. Then, for any vertex algebra W and any homomorphisms $f : V \longrightarrow W$ and $f' : V' \longrightarrow W$ satisfying $[f(V)_{(m)}, f'(V')_{(n)}] = 0$ for all $m, n \in \mathbb{Z}$, there exists a unique homomorphism $g : V\otimes V' \longrightarrow W$ such that $g\circ i = f$ and $g \circ i' = f'$.*

Proof. The uniqueness follows from

$$\begin{aligned}
g(u\otimes u') &= g((u\otimes \mathbf{1})_{(-1)}(\mathbf{1}\otimes u')) \\
&= g(u\otimes \mathbf{1})_{(-1)} g(\mathbf{1}\otimes u') \\
&= (g\circ i)(u)_{(-1)} (g\circ i')(u') \\
&= f(u)_{(-1)} f'(u').
\end{aligned}$$

For the existence, let g be the linear map defined by $g(u \otimes u') = f(u)_{(-1)} f'(u')$. Then obviously it satisfies $g \circ i = f$ and $g \circ i' = f'$. It is a homomorphism of vertex algebras; by $f(u)_{(n)} f'(u') = 0$, $(n \geq 0)$, $f(u)_{(-1)} f'(u') = f'(u')_{(-1)} f(u)$ and the Borcherds identity for $p = 0, q = n, r = -1$, we have

$$\begin{aligned}
g((u \otimes u')_{(n)}(v \otimes v')) &= \sum_{i \in \mathbb{Z}} g((u_{(i)}v) \otimes (u'_{(n-i-1)}v')) \\
&= \sum_{i \in \mathbb{Z}} f(u_{(i)}v)_{(-1)} f'(u'_{(n-i-1)}v') \\
&= \sum_{i \in \mathbb{Z}} f'(u')_{(n-i-1)} (f'(v')_{(-1)} (f(u)_{(i)} f(v)) \\
&= \sum_{i \in \mathbb{Z}} f'(u')_{(n-i-1)} (f(u)_{(i)} (f'(v')_{(-1)} f(v))) \\
&= \sum_{i \geq 0} f'(u')_{(n-i-1)} (f(u)_{(i)} (f(v)_{(-1)} f'(v'))) \\
&\quad + \sum_{i \geq 0} f(u)_{(-1-i)} (f'(u')_{(n+i)} (f(v)_{(-1)} f'(v'))) \\
&= (f(u)_{(-1)} f'(u'))_{(n)} (f(v)_{(-1)} f'(v')) \\
&= g(u \otimes u')_{(n)} g(v \otimes v').
\end{aligned}$$

□

4.5 Vertex algebras of local fields

We now return to the situation of Section 3. Let M be a vector space and consider the fields on M.

A set of fields on M is said to be *pairwise mutually local* if any pair of fields in the set, not necessarily distinct, are mutually local.

Now, let $A(z), B(z)$ and $C(z)$ be mutually local fields on M, while $I(z)$ the identity field. Then we have already shown that there exists a nonnegative integer n_0 such that $A(z)_{(n)} B(z) = 0$ for all $n \geq n_0$ (2.1.2), that the Borcherds identity holds (Corollary 3.2.2):

$$\begin{aligned}
&\sum_{i=0}^{\infty} \binom{p}{i} (A(z)_{(r+i)} B(z))_{(p+q-i)} C(z) \\
&= \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} \left(A(z)_{(p+r-i)} (B(z)_{(q+i)} C(z)) - (-1)^r B(z)_{(q+r-i)} (A(z)_{(p+i)} C(z)) \right),
\end{aligned}$$

and that we have (1.4.5)

$$A(z)_{(n)}I(z) = \begin{cases} 0 & (n \geq 0), \\ A(z) & (n = -1). \end{cases}$$

They are precisely Borcherds' axioms for a vertex algebra. Therefore, we have established the following theorem[13].

Theorem 4.5.1. *Let M be a vector space, and let $\mathcal{O}$ be a vector space consisting of fields on M which are pairwise mutually local. If $\mathcal{O}$ is closed under the residue products and contains the identity field $I(z)$, then the residue products equip $V = \mathcal{O}$ with a structure of vertex algebra with the vacuum vector $\mathbf{1} = I(z)$.*

By a *vertex algebra of local fields* on M, we mean a vector space of pairwise mutually local fields on M containing the identity field on which a vertex algebra structure is provided by the residue products.

Note 4.5.2. In [LZ1], Lian and Zuckerman introduced the notion of a *quantum operator algebra*. In terms of our notations and terminologies, a quantum operator algebra is a pair $(M, \mathcal{O})$ of a vector space M and a space of fields $\mathcal{O}$ containing the identity fields such that $\mathcal{O}$ is closed under the residue products. A quantum operator algebra is said to be *commutative* if any pair of fields in $\mathcal{O}$ are mutually local in our sense. Therefore, Theorem 4.5.1 shows that a commutative quantum operator algebra gives rise to a vertex algebra of local fields.

Definition 4.5.3. The space $\langle \mathcal{S} \rangle$ *generated* by a set of fields $\mathcal{S}$ is the linear span of the fields constructed by successive application of the residue products to the fields in $\mathcal{S}$ as well as the identity field $I(z)$.

The space $\langle \mathcal{S} \rangle$ is the smallest space closed under the residue products that contains $\mathcal{S} \cup \{I(z)\}$.

Consider the space $\langle \mathcal{S} \rangle$ generated by $\mathcal{S}$. By successive use of Lemma 2.1.4, we have

Proposition 4.5.4. *If a set $\mathcal{S}$ of fields is pairwise mutually local, then so is the space $\langle \mathcal{S} \rangle$.*

In particular, let $\mathcal{S}$ be a set of pairwise mutually local fields and let $\mathcal{O}$ be the space $\langle \mathcal{S} \rangle$ generated by $\mathcal{S}$. Then $\mathcal{O}$ is closed under the residue products and

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[13] The special case for a maximal pairwise local space of fields is given by Li [Li2, Theorem 3.2.10]. However, his proof indeed applies to the statement in the theorem.

contains the identity field; it is a vertex algebra. It follows from the associativity formula (4.3.3) and the formulas (1.4.2), (1.4.3) and (1.4.4) that the space $\langle \mathcal{S} \rangle$ is the linear span of the fields of the form

$$A^1(z)_{(n_1)}A^2(z)_{(n_2)}\cdots A^\ell(z)_{(n_\ell)}I(z)$$

where $n_1,\ldots,n_{\ell-1} \in \mathbb{Z}, n_\ell \in \mathbb{Z}_{<0}, A^1(z),\ldots,A^\ell \in \mathcal{S}$. Here we understand the nested products as $A(z)_{(m)}B(z)_{(n)}C(z) = A(z)_{(m)}(B(z)_{(n)}C(z))$.

Remark 4.5.5. Let V_0 be a vector space and suppose given a series of binary operations satisfying (B0) and (B1). Then the image $\mathcal{V}_{Y_0}$ of the corresponding map $Y_0 : V_0 \longrightarrow (\operatorname{End} V_0)[[z,z^{-1}]]$ is pairwise mutually local by (4.3.4). Therefore the space $V = \langle \mathcal{V}_{Y_0} \rangle$ generated by $\mathcal{V}_{Y_0}$ is a vertex algebra with respect to the residue products by the theorem. Thus we have constructed a canonical map from V_0 to a vertex algebra V, which preserves the binary operations by (4.3.3). In particular, if Y_0 is injective, then V_0 is canonically embedded in a vertex algebra.

Note 4.5.6. Li's original proof of the above theorem is as follows. For each $A(z) \in \mathcal{O}$, consider

$$Y(A(z),\zeta) = \sum_{n\in\mathbb{Z}} A(z)_{(n)}\zeta^{-n-1} \in (\operatorname{End}\mathcal{O})[[\zeta,\zeta^{-1}]].$$

Then he shows that the map $Y : \mathcal{O} \longrightarrow (\operatorname{End}\mathcal{O})[[\zeta,\zeta^{-1}]]$ satisfies Goddard's axioms (G0)–(G3) (see Subsection 6.2) where $V = \mathcal{O}, \mathbf{1} = I(z)$ and $T = \partial_z$. (In particular, the locality (G1) follows from one of his result which we have explained in Proposition 3.4.3.) Then by the equivalence of Goddard's axioms and Borcherds' axioms, which he shows by direct calculation (cf. Note 6.2.2), the result follows.

4.6 Modules for a vertex algebra

Let V be a vertex algebra. Following Borcherds [B1], we define the notion of a module for V, or a V-module, as follows:

Definition 4.6.1. A V-module is a vector space M equipped with countably many bilinear maps

$$\begin{array}{ccc} V\times M & \longrightarrow & M \\ (a,v) & \longmapsto & a_{[n]}v,\ (n\in\mathbb{Z}) \end{array}$$

satisfying the following conditions:

(M0) For each pair $(a,v) \in V\times M$, there exists a nonnegative integer n_0 such that $a_{[n]}v = 0$ for all $n \geq n_0$.

(M1) For all $a, b \in V$, $v \in M$ and $p, q, r \in \mathbb{Z}$,

$$\sum_{i=0}^{\infty} \binom{p}{i} (a_{(r+i)}b)_{[p+q-i]}v$$
$$= \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} (a_{[p+r-i]}b_{[q+i]}v - (-1)^r b_{[q+r-i]}a_{[p+i]}v).$$

(M2) For any $v \in M$,

$$\mathbf{1}_{[n]}v = \begin{cases} 0 & (n \neq -1), \\ v & (n = -1). \end{cases}$$

In particular, V itself is a V-module by setting $a_{[n]}b = a_{(n)}b$, called the *adjoint module*.

We note that (M1) for $b = \mathbf{1}, p = 0, q = n, r = -2$ together with (M2) implies

(4.6.1) $$(Ta)_{[n]}v = -na_{[n-1]}v.$$

All the results in Subsection 4.3 hold for the Borcherds identity (M1) for a V-module. For example,

(4.6.2) $$[a_{[p]}, b_{[q]}] = \sum_{i=0}^{\infty} \binom{p}{i} (a_{(i)}b)_{[p+q-i]}$$

in the analogue of (4.3.2),

(4.6.3) $$(a_{(r)}b)_{[q]} = \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} (a_{[r-i]}b_{[q+i]} - (-1)^r b_{[q+r-i]}a_{[i]})$$

is the analogue of (4.3.3), and

(4.6.4) $$\sum_{i=0}^{\infty} (-1)^i \binom{r}{i} (a_{[p+r-i]}b_{[q+i]} - (-1)^r b_{[q+r-i]}a_{[p+i]}) = 0, \quad (r \geq r_0)$$

is the analogue of (4.3.4). Here r_0 is an integer such that $a_{(r)}b = 0$ for all $r \geq r_0$. Then (4.6.3) and (4.6.4) for all $p, q, r \in \mathbb{Z}$ imply the Borcherds identity (M1).

Now, let M be a V-module and set $Y_M(a, z) = \sum_{n \in \mathbb{Z}} a_{[n]}z^{-n-1}$ for each $a \in V$. Then (4.6.3) and (4.6.4) are respectively written as

(4.6.5) $$Y_M(a_{(r)}b, z) = Y_M(a, z)_{(r)}Y_M(b, z),$$

(4.6.6) $$(y - z)^r [Y_M(a, y), Y_M(b, z)] = 0, \ (r \geq r_0).$$

Then (M0) and (4.6.6) show that $\mathcal{O}_M = \{Y_M(a,z) \mid a \in V\}$ is a space of pairwise mutually local fields, and (4.6.5) implies that $\mathcal{O}_M$ is closed under the residue products. Note that (M2) says that $Y_M(\mathbf{1}, z)$ is the identity field. Hence, by Theorem 4.5.1, $\mathcal{O}_M$ is a vertex algebra of local fields on M. Then (4.6.5) also says that the map $Y_M : V \longrightarrow \mathcal{O}_M$ is a surjective homomorphism of vertex algebras.

Conversely, let M be a vector space and let $\mathcal{O}$ be a vertex algebra of local fields on M. We set $A(z)_{[n]}v = A_n v$, $(n \in \mathbb{Z})$ for $A(z) \in \mathcal{O}$ and $v \in M$. Then, since $\mathcal{O}$ is pairwise mutually local,

$$\sum_{i=0}^{\infty}(-1)^i\binom{r}{i}\left(A(z)_{[p+r-i]}B(z)_{[q+i]} - (-1)^r B(z)_{[q+r-i]}A(z)_{[p+i]}\right) v$$

$$= \sum_{i=0}^{\infty}(-1)^i\binom{r}{i}\left(A_{p+r-i}B_{q+i} - (-1)^r B_{q+r-i}A_{p+i}\right) v = 0.$$

On the other hand, by the definition (1.4.1) of the residue products,

$$\begin{aligned}
&(A(z)_{(r)}B(z))_{[q]}v \\
&= \sum_{i=0}^{\infty}(-1)^i\binom{r}{i}(A_{r-i}B_{q+i} - (-1)^r B_{q+r-i}A_i)v \\
&= \sum_{i=0}(-1)^i\binom{r}{i}(A(z)_{[r-i]}B(z)_{[q+i]} - (-1)^r B(z)_{[q+r-i]}A(z)_{[i]})v.
\end{aligned}$$

Therefore, since (4.6.3) and (4.6.4) imply $(M1)$, and $(M0)$ and $(M2)$ are obvious, M is a $\mathcal{O}$-module. Therefore, any homomorphism $f : V \longrightarrow \mathcal{O}$ gives rise to a V-module structure on M. (By replacing $\mathcal{O}$ by the image of f, we may suppose that f is surjective). Thus we have established the following also stated in [Li2]:

Proposition 4.6.2. *Let V be a vertex algebra and let M be a vector space. Then, giving a V-module structure on M is equivalent to giving a homomorphism of vertex algebras from V to a vertex algebra of local fields on M.*

Note 4.6.3. In view of Note 4.5.2, this proposition says that a commutative quantum operator algebra is nothing but a pair of a vertex algebra and a module for it.

We close this section with a brief account of intertwiners (cf. [TK],[FHL]).

Definition 4.6.4. Let L, M, N be modules for a vertex algebra V. An *intertwiner* of type $\binom{N}{L\ M}$ is a set of countably many bilinear maps

$$\begin{array}{ccc}
L \times M & \longrightarrow & N \\
(u, v) & \longmapsto & u_{\{q\}}v \quad, q \in \Lambda + \mathbb{Z}
\end{array}$$

where Λ is a finite subset of $\mathbf{k}$, satisfying the following conditions:
(I0) For each pair $(u,v) \in L \times M$ and $\lambda \in \Lambda$, there exists an n_0 such that $u_{\{q\}}v = 0$ if $q \in \lambda + n_0 + \mathbb{N}$.
(I 1) For all $a \in V, u \in L, v \in M, p, r \in \mathbb{Z}, q \in \Lambda + \mathbb{Z}$

$$\sum_{i=0}^{\infty} \binom{p}{i} (a_{[r+i]}u)_{\{p+q-i\}}v$$
$$= \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} (a_{[p+r-i]}u_{\{q+i\}}v - (-1)^r u_{\{q+r-i\}}a_{[p+i]}v).$$

(I 2) For all $(u,v) \in L \times M$ and $q \in \Lambda + \mathbb{Z}$,

$$(Tu)_{\{q\}}v = -qu_{\{q-1\}}v.$$

In particular, for a V-module M, the maps

$$\begin{array}{ccc} V \times M & \longrightarrow & M \\ (a,v) & \longmapsto & a_{[n]}v, \quad n \in \mathbb{Z} \end{array}$$

give an intertwiner of type $\binom{M}{V\ M}$.

All the results in Subsection 4.3 also hold for the Borcherds identity (I 1) for an intertwiner, such as

$$(a_{[r]}u)_{\{q\}} = \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} (a_{[r-i]}u_{\{q+i\}} - (-1)^r u_{\{q+r-i\}}a_{[i]}),$$
$$\sum_{i=0}^{\infty} (-1)^i \binom{r}{i} (a_{[p+r-i]}u_{\{q+i\}} - (-1)^r u_{\{q+r-i\}}a_{[p+i]}) = 0, \quad (r \geq r_0).$$

In terms of the generating series $I(u,z) = \sum_q u_{\{q\}}z^{-q-1}$, they are respectively written as

$$I(a_{[r]}u, z) = \operatorname*{Res}_{y=0} Y_N(a,y)I(u,z)(y-z)^r - \operatorname*{Res}_{y=0} I(u,z)Y_M(a,y)(y-z)^r \quad \text{and}$$
$$(y-z)^r\,(Y_N(a,y)I(u,z) - I(u,z)Y_M(a,y)) = 0, \ (r \geq r_0),$$

which might be written as

$$I(a_{[r]}u, z) = Y(a,z)_{(r)}I(u,z) \quad \text{and}$$
$$(y-z)^r[Y(a,y), I(u,z)] = 0$$

symbolically.

5 State-field correspondence

In this section, we will describe the state-field correspondence of a vertex algebra and the characterization of vertex algebras due to Lian-Zuckerman. Throughout this section, we suppose given a vector space V and a nonzero vector $|I\rangle \in V$.

5.1 Creative fields

We begin with the notion of creative fields. Let M be a vector space and suppose given a nonzero vector $|I\rangle \in M$.

Definition 5.1.1. A field $A(z)$ on M is *creative* with respect to $|I\rangle$ if $A(z)|I\rangle \in M[[z]]$.

In other words, $A(z)$ is creative if and only if $A_n|I\rangle = 0$ for all $n \geq 0$.

We define $\tilde{\mathcal{O}} = \tilde{\mathcal{O}}(M, |I\rangle)$ to be the set of all creative fields on M with respect to $|I\rangle$. For a creative field $A(z) \in \tilde{\mathcal{O}}$, we set

$$|A\rangle = \lim_{z\to 0}(A(z)|I\rangle) = A_{-1}|I\rangle$$

and call it the *state* corresponding to the field $A(z)$.

Consider the map

$$\begin{array}{rccc} s: & \tilde{\mathcal{O}} & \longrightarrow & M \\ & A(z) & \longmapsto & |A\rangle \end{array}$$

which assigns the state to a creative field.

Lemma 5.1.2. *The space $\tilde{\mathcal{O}}$ is closed under the residue products and we have*

$$s(A(z)_{(m)}B(z)) = A_m|B\rangle$$

for all $A(z), B(z) \in \tilde{\mathcal{O}}$.

Proof. For $A(z), B(z) \in \tilde{\mathcal{O}}$, we have

$$\begin{aligned} &\Big(A(z)_{(m)}B(z)\Big)|I\rangle \\ &\quad = \sum_{n\in\mathbb{Z}}\left(\sum_{i=0}^{\infty}(-1)^i\binom{m}{i}\left(A_{m-i}B_{n+i} - (-1)^m B_{m+n-i}A_i\right)\right)|I\rangle z^{-n-1} \\ &\quad = \sum_{n\leq -1}\left(\sum_{i=0}^{-n-1}(-1)^i\binom{m}{i}A_{m-i}B_{n+i}|I\rangle\right)z^{-n-1}. \end{aligned}$$

Hence $A(z)_{(m)}B(z)$ belongs to $\tilde{\mathcal{O}}$ and we have

$$s(A(z)_{(m)}B(z)) = \lim_{z\to 0}(A(z)_{(m)}B(z)|I\,\rangle) = A_m B_{-1}|I\,\rangle = A_m|B\rangle.$$

□

The following lemma, which will be used in the next section, is a part of the statement known as Goddard's uniqueness theorem:

Lemma 5.1.3. *Let $\mathcal{O}$ be a subspace of $\tilde{\mathcal{O}}$ which is pairwise mutually local. Then, if the map $s|_{\mathcal{O}} : \mathcal{O} \to M$ is surjective, then the map*

$$\begin{array}{rccc} s|_{\mathcal{O}}: & \mathcal{O} & \to & M[[z]] \\ & A(z) & \mapsto & A(z)|I\,\rangle \end{array}$$

is injective.

Proof. Suppose $A(z)|I\,\rangle = 0$ and take any $u \in M$. Since $s|_{\mathcal{O}}$ is surjective, there is a field $U(z) \in \mathcal{O}$ such that $|U\rangle = s(U(z)) = u$. Then, by the locality,

$$z^n A(z)u = \lim_{y\to 0}(z-y)^n A(z)U(y)|I\,\rangle = \lim_{y\to 0}(z-y)^n U(y)A(z)|I\,\rangle = 0$$

for sufficiently large n. Since u is arbitrary, we have $A(z) = 0$. □

5.2 State-field correspondence

Let V be a vertex algebra with the vacuum vector $|I\,\rangle$. Consider the generating series $Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)}z^{-n-1}$ and put

$$\mathcal{V}_Y = \{Y(a,z)\,|\,a \in V\}.$$

Then the axiom (B2) says that $\mathcal{V}_Y$ is a subspace of $\tilde{\mathcal{O}} = \tilde{\mathcal{O}}(V, |I\,\rangle)$ and the map

$$\begin{array}{rccc} Y: & V & \longrightarrow & \mathcal{V}_Y \\ & a & \longmapsto & Y(a,z) \end{array}$$

is the inverse of the state map

$$s|_{\mathcal{V}_Y} : \mathcal{V}_Y \longrightarrow V.$$

In particular, they are isomorphisms of vector spaces.

Remark 5.2.1. Since the map Y is recovered from the space $\mathcal{V}_Y$ as the inverse of $s|_{\mathcal{V}_Y}$, a vertex algebra structure $(V, |I\,\rangle, Y)$ is uniquely determined by the subspace $\mathcal{V}_Y \subset \tilde{\mathcal{O}}$

Now, we have $Y(a_{(n)}b, z) = Y(a, z)_{(n)}Y(b, z)$ by (4.3.3) and $Y(|I\rangle, z) = I(z)$ by (4.2.3). Therefore, we obtain the following theorem:

Theorem 5.2.2 (State-field correspondence). *Let $(V, |I\rangle, Y)$ be a vertex algebra. Then the residue products equip $\mathcal{V}_Y$ with a structure of a vertex algebra with the vacuum vector being the identity field $I(z)$ such that the map*

$$s|_{\mathcal{V}_Y} : \mathcal{V}_Y \longrightarrow V$$

is an isomorphism of vertex algebras.

Note that the formula (4.2.4) means that the differentiation ∂_z corresponds to the translation operator T under the state-field correspondence.

Now, let M be a vector space and suppose given a nonzero vector $|I\rangle \in M$. Let $\mathcal{O}$ be a vertex algebra of local fields on M which is creative with respect to $|I\rangle$. Suppose that the state map

$$s|_{\mathcal{O}} : \mathcal{O} \longrightarrow M$$

is injective. Set $V = s(\mathcal{O}) \subset M$. Then, since the map is an isomorphism onto V, we can introduce a structure of a vertex algebra on V through this isomorphism:

$$|A\rangle_{(n)}|B\rangle = s(A(z)_{(n)}B(z)).$$

Then, by Lemma 5.1.2, we have

$$|A\rangle_{(n)}|B\rangle = A_n|B\rangle, \quad \text{i.e.,} \quad Y(|A\rangle, z) = A(z).$$

Hence

Theorem 5.2.3. *Let $\mathcal{O}$ be a vertex algebra of local fields on M which is creative with respect to $|I\rangle \in M$. If the state map $s|_{\mathcal{O}}$ is injective, then the image $V = s(\mathcal{O})$ has a unique structure of a vertex algebra with the vacuum vector $|I\rangle$ such that*

$$Y(|A\rangle, z) = A(z)$$

for any $|A\rangle \in V$, which is isomorphic to $\mathcal{O}$ as vertex algebras, and this endows M with a structure of a V-module.

5.3 Characterization of the image (I)

Let us characterize vertex algebra structures on V with the vacuum vector $|I\rangle$ by means of the subspaces $\mathcal{V}_Y = \{Y(a, z)|a \in V\}$ of $\tilde{\mathcal{O}} = \tilde{\mathcal{O}}(V, |I\rangle)$.

We consider the following set of conditions on a subspace $\mathcal{V} \subset \tilde{\mathcal{O}}$:

(L1) $\mathcal{V}$ is pairwise mutually local.

(L2) $\mathcal{V}$ is closed under the residue products.

(L3) The map $s|_{\mathcal{V}} : \mathcal{V} \to V$ is an isomorphism of vector spaces.

Proposition 5.3.1. *If a subspace $\mathcal{V} \subset \tilde{\mathcal{O}}$ satisfies (L1)–(L3), then the residue products equip $\mathcal{V}$ with a structure of a vertex algebra with the vacuum vector being the identity field $I(z)$.*

Proof. Let $\mathcal{V} \subset \tilde{\mathcal{O}}$ satisfy (L1)–(L3). Then, since $\mathcal{V}$ is pairwise mutually local by (L1), the residue products satisfy (B0) and (B1) by (2.1.2) and Corollary 3.4.2. Now let $I(z)$ be the unique field in $\mathcal{V}$ such that $s(I(z)) = |I\rangle$. Then, for any $A(z) \in \mathcal{V}$, we have

$$s(A(z)_{(n)}I(z)) = A_n|I\rangle = \begin{cases} 0 & (n \geq 0), \\ |A\rangle & (n = -1) \end{cases}$$

by Lemma 5.1.2 and the creativity of $A(z) \in \mathcal{V}$. Since $A(z)_{(n)}I(z) \in \mathcal{V}$ by (L2), it follows from (L3) that

$$A(z)_{(n)}I(z) = \begin{cases} 0 & (n \geq 0), \\ A(z) & (n = -1), \end{cases}$$

which is nothing else but (B2). Hence the residue products equip $\mathcal{V}$ with the structure of a vertex algebra with the vacuum vector being $I(z)$. In particular, by (4.2.3),

$$I_n|A\rangle = s(I(z)_{(n)}A(z)) = \begin{cases} 0 & (n \neq -1), \\ |A\rangle & (n = -1). \end{cases}$$

Since $|A\rangle$ is arbitrary in V, we see that $I(z)$ is the identity field. □

Therefore, by Theorem 5.2.3, we obtain the following theorem stated by Lian-Zuckerman[14] [LZ2, Theorem 5.6]:

Theorem 5.3.2. *Let V be a vector space, and let $|I\rangle \in V$ be a nonzero vector. If a subspace $\mathcal{V} \subset \tilde{\mathcal{O}} = \tilde{\mathcal{O}}(V, |I\rangle)$ satisfies (L1)–(L3), then there exists a unique vertex algebra structure on V with the vacuum vector $|I\rangle$ such that $\mathcal{V} = \mathcal{V}_Y$, where the map Y is given by the inverse of $s|_{\mathcal{V}}$.*

[14]To be precise, Lian-Zuckerman assumed in addition that $\mathcal{V}$ contains the identity field. However, it follows from the other assumptions as we saw in Proposition 5.3.1

Conversely, if $(V, |I\rangle, Y)$ is a vertex algebra, then the space $\mathcal{V} = \mathcal{V}_Y \subset \tilde{\mathcal{O}}$ satisfies (L1)–(L3), Therefore, the conditions (L1)–(L3) on a subspace $\mathcal{V} \subset \tilde{\mathcal{O}}$ characterize vertex algebra structures on V with the vacuum vector $|I\rangle$. More precisely,

Corollary 5.3.3. *The correspondence $Y \longmapsto \mathcal{V}_Y = \{\, Y(a,z) \,|\, a \in V \,\}$ from the set of vertex algebra structures on V with the vacuum vector $|I\rangle$ to the set of the subspaces $\mathcal{V} \subset \tilde{\mathcal{O}}$ satisfying (L1)–(L3) is bijective, and the inverse correspondence is given by $\mathcal{V} \longmapsto Y_{\mathcal{V}} = (\, s|_{\mathcal{V}})^{-1}$.*

In other words, giving a vertex algebra structure $(V, |I\rangle, Y)$ is equivalent to giving a subspace $\mathcal{V} \subset \tilde{\mathcal{O}}$ satisfying (L1)–(L3).

6 Goddard's axioms and the existence theorem

In this section, we will describe the characterization of vertex algebras by the axioms essentially given by Goddard [G], where the translation operator T plays an essential role. We also describe the existence theorem due to Frenkel-Kac-Radul-Wang as a consequence of still another characterization.

Throughout this section, we suppose given a vector space V and a nonzero vector $|I\rangle \in V$.

6.1 Translation covariance

Let us first consider the role played by the translation operator. Suppose given an endomorphism $T : V \longrightarrow V$ such that $T|I\rangle = 0$.

Definition 6.1.1. A field $A(z)$ is *translation covariant* with respect to T, if

$$[T, A(z)] = \partial A(z) \tag{6.1.1}$$

is satisfied.

Remark 6.1.2. The property (6.1.1) is equivalent to

$$e^{yT} A(z) e^{-yT} = A(y+z)|_{|y|<|z|} \,,$$

where the right-hand side means

$$A(y+z)|_{|y|<|z|} = \sum_{n\in\mathbb{Z}} A_n \; (y+z)^n|_{|y|<|z|} = \sum_{n\in\mathbb{Z}} \sum_{i=0}^{\infty} \binom{n}{i} A_n y^i z^{n-i}.$$

We say that a set of fields $\mathcal{S}$ is translation covariant if all the fields in $\mathcal{S}$ are translation covariant with respect to the same T.

Lemma 6.1.3. *If two fields $A(z)$ and $B(z)$ are translation covariant, then so are the residue products $A(z)_{(n)}B(z)$.*

Proof. It is obvious by $[T, A(z)_{(n)}B(z)] = [T, A(z)]_{(n)}B(z) + A(z)_{(n)}[T, B(z)]$ and $\partial(A(z)_{(n)}B(z)) = \partial A(z)_{(n)}B(z) + A(z)_{(n)}\partial B(z)$. $\square$

Now consider the map

$$\begin{array}{rccc} \imath: & V[[z]] & \longrightarrow & V \\ & u(z) & \longmapsto & u(0) \end{array}$$

which assigns the initial value $u(0) = u_{-1}$ to a series $u(z) = \sum_{n\leq -1} u_n z^{-n-1}$. If a field $A(z)$ is creative and translation covariant, then the series $A(z)|I\,\rangle \in V[[z]]$ satisfies the differential equation

$$\partial_z(A(z)|I\,\rangle) = T(A(z)|I\,\rangle) \tag{6.1.2}$$

whose solution is uniquely determined by the initial value $\imath(A(z)|I\,\rangle) = |A\rangle$ (cf. [K, Remark 4.4a]). More precisely,

Lemma 6.1.4. *If a space of fields $\mathcal{V}$ is creative and translation covariant, then the map*

$$\begin{array}{rccc} \imath|_{\mathcal{V}|I\,\rangle}: & \mathcal{V}|I\,\rangle & \longrightarrow & V \\ & A(z)|I\,\rangle & \longmapsto & |A\rangle \end{array}$$

is injective.

Proof. By the assumptions, we have

$$\sum_{i=0}^{\infty} iA_{-i-1}|I\,\rangle z^{i-1} = \partial A(z)|I\,\rangle = [T, A(z)]|I\,\rangle = TA(z)|I\,\rangle = \sum_{i=0}^{\infty} TA_{-i-1}|I\,\rangle z^{-i}.$$

Equating the coefficients, $(i+1)A_{-i-2}|I\,\rangle = TA_{-i-1}|I\,\rangle, (i \geq 0)$. Therefore, if $|A\rangle = 0$, then we inductively deduce $A_{-i-1}|I\,\rangle = 0, (i = 0, 1, \dots)$, namely $A(z)|I\,\rangle = 0$. $\square$

Remark 6.1.5. If $A(z)$ is creative and translation covariant, then, by solving the differential equation (6.1.2), we have $A(z)|I\,\rangle = e^{zT}|A\rangle$.

6.2 Goddard's axioms

Suppose given a linear map

$$\begin{array}{rrcc} Y: & V & \longrightarrow & (\operatorname{End} V)[[z, z^{-1}]] \\ & a & \longmapsto & Y(a,z) \end{array},$$

and consider the following set of axioms[15] on $(V, |I\rangle, Y)$:

(G0) For any $a, b \in V$, $Y(a,z)b \in V((z))$.

(G1) For any $a, b \in V$, $Y(a,z)$ and $Y(b,z)$ are mutually local.

(G2) For any $a \in V$, $Y(a,z)|I\rangle$ is a formal power series with the constant term a.

(G3) There exists an endomorphism $T: V \longrightarrow V$ such that $T|I\rangle = 0$ and

$$[T, Y(a,z)] = \partial Y(a,z)$$

for all $a \in V$.

Let $(V, |I\rangle, Y)$ satisfy the axioms (G0)–(G3). For each $a \in V$, we define the endomorphism $a_{(n)} \in \operatorname{End} V$ by the expansion $Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$. Now, let $a, b \in V$, and consider the series $Y(a,z)_{(n)}Y(b,z)$ and $Y(a_{(n)}b, z)$. Then they are creative and translation covariant. Since

$$\lim_{z\to 0} Y(a,z)_{(n)}Y(b,z)|I\rangle = a_{(n)}b = \lim_{z\to 0} Y(a_{(n)}b, z)|I\rangle,$$

we have $Y(a,z)_{(n)}Y(b,z)|I\rangle = Y(a_{(n)}b,z)|I\rangle$ by lemma 6.1.4. Then by Lemma 5.1.3 applied to $\mathcal{V} = \{\, Y(a,z) \mid a \in V \,\}$, we obtain the associativity formula

$$Y(a,z)_{(n)}Y(b,z) = Y(a_{(n)}b, z),$$

which, together with the locality, implies the Borcherds identity (B1)

$$\sum_{i=0}^{\infty} \binom{p}{i} \left(a_{(r+i)}b\right)_{(p+q-i)} c$$
$$= \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} \left(a_{(p+r-i)}(b_{(q+i)}c) - (-1)^r b_{(q+r-i)}(a_{(p+i)}c)\right).$$

[15]These are essentially the axioms considered by Goddard [G]. More precisely, he assumed $Y(a,z)|I\rangle = e^{zT}a$ instead of $[T, Y(a,z)] = \partial Y(a,z)$ in (G3). However, they are equivalent under the axioms (G0)–(G2) by Lemma 5.1.3 and Remark 6.1.5. (cf. [FKRW, Section 3], [K, Subsection 1.3].)

by Proposition 4.3.3.

The other implications are easy, and we have (cf. [Li, Proposition 2.2.4], [K, Proposition 4.8]):

Theorem 6.2.1. *Let V be a vector space, and let $|I\rangle \in V$ be a nonzero vector. Then, a linear map*

$$Y : V \longrightarrow (\operatorname{End} V)[[z, z^{-1}]]$$

gives rise to a vertex algebra structure on V with the vacuum vector $|I\rangle$ if and only if $(V, |I\rangle, Y)$ satisfies Goddard's axioms (G0)–(G3).

Note 6.2.2. Li's proof of this result is as follows. Let $(V, |I\rangle, T)$ satisfy (G0)–(G3). First note that $e^{zT}Y(a,y) = Y(a, y+z)e^{zT}$ by the translation covariance (G3). Then, for any $a, b \in V$, we have

$$\begin{aligned}(y-z)^m Y(b,z)Y(a,y)|I\rangle &= (y-z)^m Y(a,y)Y(b,z)|I\rangle \\ &= (y-z)^m Y(a,y)e^{zT}b = (y-z)^m e^{zT}Y(a, y-z)b\end{aligned}$$

for $m \gg 0$. Letting $y = 0$ and dividing the both sides by $(-z)^m$, we have $Y(b,z)a = e^{zT}Y(a,-z)b$. Then for any $a, b \in V$, we have

$$\begin{aligned}&(x+z)^m Y(a, x+z)Y(b,z)c \\ &= (x+z)^m Y(a, x+z)e^{zT}Y(c,-z)b = (x+z)^m e^{zT}Y(a,x)Y(c,-z)b \\ &= (x+z)^m e^{zT}Y(c,-z)Y(a,x)b = (x+z)^m Y(Y(a,x)b, z)c\end{aligned}$$

for $m \gg 0$, which is the duality (cf. [G, Theorem 3]). Finally, he shows by calculation involving the delta function that the locality and the duality imply the Borcherds identity. This last step is simplified by our consideration in Subsection 4.3 (cf. Proposition 4.3.3).

6.3 Characterization of the image (II)

In Subsection 5.3, we gave a set of conditions on a subspace $\mathcal{V}$ of $\tilde{\mathcal{O}} = \tilde{\mathcal{O}}(V, |I\rangle)$ to be the image $\mathcal{V}_Y$ of a vertex algebra $(V, |I\rangle, Y)$. Let us rewrite them using the translation covariance.

To this end, let us consider the map $s|_{\mathcal{V}} : \mathcal{V} \longrightarrow V$. It decomposes as

$$\begin{array}{ccccc}\mathcal{V} & \longrightarrow & \mathcal{V}|I\rangle & \longrightarrow & V \\ A(z) & \longmapsto & A(z)|I\rangle & \longmapsto & |A\rangle.\end{array}$$

Then by combining Lemma 5.1.3 and Lemma 6.1.4, we have

Lemma 6.3.1. *Let $\mathcal{V}$ be a subspace of $\tilde{\mathcal{O}}$. If the map $s|_{\mathcal{V}} : \mathcal{V} \longrightarrow V$ is surjective, and if $\mathcal{V}$ is pairwise mutually local and translation covariant, then the map $s|_{\mathcal{V}}$ is bijective.*

The following proposition illustrates the role of T.

Proposition 6.3.2. *If $\mathcal{V} \subset \tilde{\mathcal{O}}$ is pairwise mutually local and the map $s|_{\mathcal{V}} : \mathcal{V} \longrightarrow V$ is surjective, then the following conditions are equivalent:*

(a) *$\mathcal{V}$ is closed under the residue products and the map $s|_{\mathcal{V}} : \mathcal{V} \longrightarrow V$ is bijective.*

(b) *$\mathcal{V}$ is translation covariant.*

Proof. Suppose that the condition (a) holds. Then $\mathcal{V}$ satisfies (L1)–(L3), and it coincides with the image $\mathcal{V}_Y$ of a vertex algebra $(V, |I\,\rangle, Y)$ by Theorem 5.3.2. Hence $\mathcal{V}$ is translation covariant.

Conversely, suppose that (b) holds. Then the map $s|_{\mathcal{V}} : \mathcal{V} \longrightarrow V$ is bijective by Lemma 6.3.1. Now, let $\tilde{\mathcal{V}} = \langle \mathcal{V} \rangle$ be the space of fields generated by $\mathcal{V}$. Then, $\tilde{\mathcal{V}}$ is pairwise mutually local and translation covariant, so we have $s|_{\tilde{\mathcal{V}}} : \tilde{\mathcal{V}} \longrightarrow V$ is also bijective. Therefore $\mathcal{V} = \tilde{\mathcal{V}}$ and $\mathcal{V}$ is closed under the residue products. $\square$

Thus we are led to the following set of conditions on a subspace $\mathcal{V} \subset \tilde{\mathcal{O}}$ of creative fields:

(T1) $\mathcal{V}$ is pairwise mutually local.

(T2) The map $s|_{\mathcal{V}} : \mathcal{V} \longrightarrow V$ is surjective.

(T3) $\mathcal{V}$ is translation covariant.

Then the above proposition says that the set of conditions (T1)–(T3) is equivalent to the set of conditions (L1)–(L3). Therefore, by Theorem 5.3.2,

Theorem 6.3.3. *Let V be a vector space, and let $|I\,\rangle \in V$ be a nonzero vector. If a subspace $\mathcal{V} \subset \tilde{\mathcal{O}}$ satisfies (T1)–(T3), then there exists a unique vertex algebra structure on V with the vacuum vector $|I\,\rangle$ such that $\mathcal{V} = \mathcal{V}_Y$, where the map Y is given by the inverse of $s|_{\mathcal{V}} : \mathcal{V} \longrightarrow V$.*

6.4 Existence theorem

An immediate consequence of the last theorem is the following existence theorem ([FKRW, Proposition 3.1], [K, Theorem 4.5]):

Corollary 6.4.1 (Existence theorem). *Let V be a vector space, and let $|I\rangle \in V$ be a nonzero vector. If a subset $\mathcal{S}$ of $\tilde{\mathcal{O}} = \tilde{\mathcal{O}}(V, |I\rangle)$ satisfies*

(S1) *$\mathcal{S}$ is pairwise mutually local.*

(S2) *The set $\{ A^1_{-j_1-1} \cdots A^k_{-j_k-1}|I\rangle \mid k, j_1, \ldots, j_k \in \mathbb{N}, A^1(z), \ldots, A^k(z) \in \mathcal{S} \}$ spans V.*

(S3) *$\mathcal{S}$ is translation covariant for some endomorphism $T : V \longrightarrow V$.*

Then there exists a unique vertex algebra structure on V with the vacuum vector $|I\rangle$ such that $Y(|A^\lambda\rangle, z) = A^\lambda(z)$ for any $A^\lambda(z) \in \mathcal{S}$.

Just apply the theorem to $\mathcal{V} = \langle \mathcal{S} \rangle$ to get the corollary. We note that the map Y is given by

$$Y(A^{\lambda_1}_{-j_1-1} \cdots A^{\lambda_k}_{-j_k-1}|I\rangle, z) = {}^{\circ}_{\circ}\, \partial^{(j_1)} A^{\lambda_1}(z) \cdots \partial^{(j_k)} A^{\lambda_k}(z) \,{}^{\circ}_{\circ},$$

which span the space $\mathcal{V}$.