Introduction

The aim of this paper is to study the identities satisfied by two-dimensional chiral quantum fields and to apply them to the basis of the theory of vertex algebras.

The notion of a vertex algebra was introduced by R.E. Borcherds in [B1] as a purely algebraic structure: A vertex algebra is a vector space equipped with a series of binary operations

$$V \times V \longrightarrow V, \quad (a, b) \longmapsto a_{(n)}b,$$

where n runs over the set of integers, subject to certain axioms (see Section 4 for the precise definition). Vertex algebras are essentially infinite-dimensional objects, since a finite-dimensional vertex algebra is merely a finite-dimensional commutative associative algebra equipped with a derivation. Conversely, any commutative associative algebra with a derivation can be regarded as a vertex algebra. This and some other evidences imply that the notion of a vertex algebra is a generalization of that of a commutative associative algebra, though it also resembles a Lie algebra in some points.

Now, in a famous book of Frenkel, Lepowsky and Meurman [FLM], some of the properties of a vertex algebra are expressed in a unified manner: In our notation,

$$(1) \quad \sum_{i=0}^{\infty} \binom{p}{i} (a_{(r+i)}b)_{(p+q-i)}c$$
$$= \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} \Big(a_{(p+r-i)}(b_{(q+i)}c) - (-1)^r b_{(q+r-i)}(a_{(p+i)}c) \Big).$$

This identity is usually written in terms of generating series

$$Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}, \quad a \in V$$

of the binary operations, involving the delta function, and is called the Cauchy-Jacobi identity. We will call it the Borcherds identity following Kac [K] since important cases are already given by Borcherds.

The axioms are related to some aspects of two-dimensional quantum field theory initiated by Belavin, Polyakov and Zamolodchikov [BPZ] in theoretical physics. In conformal field theory, one assumes that a theory is decomposed into the holomorphic part and the anti-holomorphic part, called the chiral and the anti-chiral part respectively, and the algebra formed by the Fourier modes of certain quantum fields (operators) involved in the chiral part is called the chiral algebra of the theory. The most characteristic property of the chiral algebra is the one called the crossing-symmetry or the duality, which is expressed in terms of the four-point functions of fields. The same property is sometimes expressed in terms of the operator products, and is called the associativity of the operator algebra. This property is almost identical to the Borcherds identity of a vertex algebra, if we understand the vector space underlying a vertex algebra to be the space of chiral fields.

In the case when the fields are realized as generating series of linear operators, the property called the mutual locality and some other properties imply the associativity, as discussed by Goddard [G] in a context close to vertex (operator) algebras. This point of view is later adopted by Li [Li2] and Frenkel, Kac, Radul and Wang [FKRW] as the basis of the theory of vertex algebras.

A closely related but different point of view, due to Lian and Zuckerman, is to consider a vector space of fields (quantum operators) closed under the multiplication

$$A(z)_{(n)}B(z) = \operatorname*{Res}_{y=0}\,[A(y),B(z)](y-z)^n, \tag{2}$$

which we call the residue product, and to take it as an abstract definition of the notion of a quantum operator algebra. If the fields are mutually local in the sense above, then the algebra is called a commutative quantum operator algebra.

This latter point of view is the place we are to begin with in this paper. Namely, we study the properties satisfied by chiral quantum fields acting on a vector space with respect to the multiplication (2), and find that the usual Jacobi identity implies the equality

$$\begin{aligned}
&\sum_{i=0}^{\infty}\binom{p}{i}[[A(x),B(y)],C(z)](x-y)^{r+i}(y-z)^{p+q-i}\\
&=\sum_{i=0}^{\infty}(-1)^i\binom{r}{i}[A(x),[B(y),C(z)]](x-y)^{p+r-i}(y-z)^{q+i}\\
&\qquad-\sum_{i=0}^{\infty}(-1)^{r+i}\binom{r}{i}[B(y),[A(x),C(z)]](y-z)^{q+r-i}(x-z)^{p+i},
\end{aligned}\tag{3}$$

if $A(z), B(z)$ and $C(z)$ are mutually local fields. By taking the residue of the both sides, we obtain the Borcherds identity. Therefore, a commutative quantum operator algebra gives rise to a vertex algebra by letting V be the space of fields of the quantum operator algebra and forgetting the vector space on which the fields act. Though this fact was already mentioned implicitly in [Li2] in a weaker form as a consequence of some results on vertex algebras, we see here that the fact is independent of any results on vertex algebras, and it is strong enough to play conversely the role of the basis of the theory of vertex algebras.

Thus, we have intended to reconstruct the whole of fundamental part of the theory of vertex algebras using our identity (3).

The paper is divided into three parts: the first part studies the properties of generating series of linear endomorphisms acting on a vector space, the second part consists of a detailed account of fundamental aspects of the theory of vertex algebras, and the last part is a review of some topics and examples. Detail of the contents will be given at the beginning of each part.

This paper is an expanded version of our research paper entitled "On axioms for a vertex algebra and the locality of quantum fields" distributed in July 1997, which is basically the parts I and II of the present version. Our equality (3) seems to be new, and some of our proofs differ from those found in the literature. Since the authors believe that the paper would be meaningless without applications, they have included some known results as applications of our consideration. However, as a result, the paper came to contain main known statements, and the authors were suggested to rewrite the paper into an expository paper so that a beginner can benefit from reading it. According to the suggestion, we have rearranged the paper and added part III to cover further topics and examples. We have also added some notes and examples in the first and the second parts.

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