Hence, to obtain the required formula, it is sufficient to show

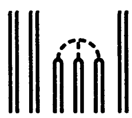
$$\iota_d \check{Z}(U_+)^{3d} = (-1)^d + (\text{terms of degree} > 0), \tag{7.9}$$

$$\iota_d \check{Z}(\mathcal{L}_D) = D + (\text{terms of degree} > d). \tag{7.10}$$

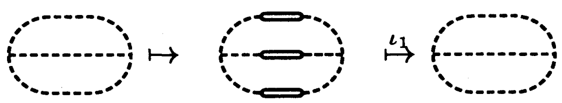
For the proof of (7.9), see [27]. Further we obtain (7.10) by Lemma 7.12 below.

□

Lemma 7.12.

$\check{Z}(\mathcal{L}_D) =$  $+$ (terms of $\sharp$ { trivalent vertices } ≥ 2)

Proof. We obtain the formula by long calculation along the definition of $\hat{Z}$. For example, for the dashed θ curve D, we show rough pictures of the calculation below. Recall that $\mathcal{L}_D$ is a linear sum of links with 3 components in this case (with $3d$ components in general).



D $\qquad$ $\hat{Z}(\mathcal{L}_D) \sim \check{Z}(\mathcal{L}_D)$

For the detailed proof, see [22]. □

8 Quantum invariants and the universal perturbative invariant

8.1 Quantum $SO(3)$ invariant constructed from quantum invariants of framed links

Let V_m be the m dimensional irreducible representation of sl_2 and M the 3-manifold obtained from S^3 by Dehn surgery along a framed link L.

Theorem 8.1 ([12]). Let r be an odd integer ≥ 3, and put $q = \exp(2\pi\sqrt{-1}/r)$. Then

$$\frac{\sum [m] Q^{sl_2;V_m}(L)}{\left(\sum [m] Q^{sl_2;V_m}(U_+)\right)^{\sigma_+} \left(\sum [m] Q^{sl_2;V_m}(U_-)\right)^{\sigma_-}} \in \mathbb{C}$$

is invariant under Kirby moves I and II. Hence it becomes a topological invariant of M; we denote it by $\tau_r^{SO(3)}(M)$. Here the summations in the above formula run over all odd integers m with $1 \le m \le r-2$, $Q^{sl_2;V_m}$ is the quantum (sl_2, V_m) invariant, $U_\pm$ are the unknots with framing ± 1 and $[m]$ is the quantum dimension of the representation V_m, that is, $[m] = (q^{m/2} - q^{-m/2})/(q^{1/2} - q^{-1/2})$. Further $\sigma_\pm$ are the numbers of positive and negative eigenvalues of the linking matrix of L.

Further we have the following theorem.

Theorem 8.2. Let r be an odd prime and M a rational homology 3-sphere.

(1) ([29])

$$\tau_r^{SO(3)}(M) \in \mathbb{Z}[q].$$

(2) ([31]) There exists the unique power series $\tau^{SO(3)}(M) \in \mathbb{Q}[[h]]$ such that

$$\begin{aligned}&(\text{coefficient of } s^d \text{ in } \tau^{SO(3)}(M)|_{h=\log(1+s)}) \\ &\equiv \left(\frac{|H_1(M;\mathbb{Z})|}{r}\right)(\text{coefficient of } s^d \text{ in } \tau_r^{SO(3)}(M)|_{q=s+1}),\end{aligned}$$

modulo r for any odd prime integer r and any d satisfying $0 \le d \le (r-3)/2$. Here $\left(\frac{\cdot}{r}\right)$ denotes the Legendre symbol.

As for (1) of the theorem, it is non-trivial whether $\tau_r^{SO(3)}(M)$ belongs to $\mathbb{Z}[q]$ after dividing it by normalization factors, though it is easy to show $\sum [m] Q^{sl_2;V_m}(L)$ belongs to $\mathbb{Z}[q]$.

As for (2) of the theorem, we consider the correspondences $q = e^h$ and $q - 1 = s$. In the left hand side of the formula, we expand it as a power series of an indeterminate s. On the other hand, in the right hand side, since q is an rth root of unity, low coefficients in the expansion in $s = q - 1$ in $\mathbb{Z}[q]$ are well defined modulo r.

As for $PSU(N)$, $\tau_r^{PSU(N)}(M)$ is defined by Kohno and Takata [18]. Further Takata and Yokota [37] showed $\tau_r^{PSU(N)}(M) \in \mathbb{Z}[q]$; it is an extension of Theorem 8.2 (1). We expect an extension of (2) as

Conjecture 8.3. For $\tau_r^{PSU(N)}(M)$, there exists the unique power series

$$\tau^{PSU(N)}(M) \in \mathbb{Q}[[h]]$$

which satisfies the same properties as in Theorem 8.2 (2).

We call $\tau^{PSU(N)}(M)$ the *perturbative invariant* of M. We expect that it should recover from the universal perturbative invariant $\hat{\Omega}(M)$ as

Conjecture 8.4 ([27]). For any rational homology 3-sphere M, the following equality holds:

$$\tau^{PSU(N)}(M) = \frac{1}{|H_1(M;\mathbb{Z})|^{N(N-1)/2}} \hat{W}_{sl_N}(\hat{\Omega}(M)).$$

At this point in time, we have

Theorem 8.5 ([32]). The above conjecture is true for $N = 2$.

We have the following corollary, which was directly proved in [21].

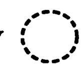
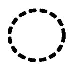
Corollary 8.6. Put $\tau^{SO(3)}(M) = \sum \lambda_n h^n$. Then each λ_n is a finite type invariant of degree $3n$. Further its weight system is equal to W_{sl_2}.

Proof. The proof is obtained in the same way as in the proof of Theorem 5.2. □

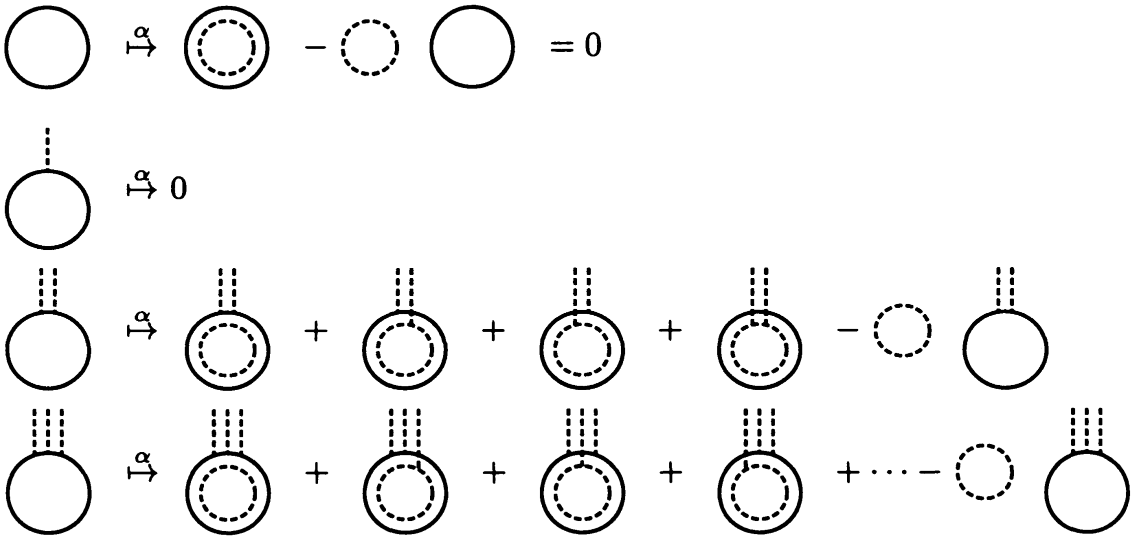
By the above corollary, we see that there exist many finite type invariants, though we had known only a few examples of finite type invariants including the Casson invariant, before getting the corollary.

8.2 Expression of j_n by a map α

In this section, we consider a map α which expands j_n in some sense. Define the map $\alpha : \mathcal{A}(S^1) \to \mathcal{A}(S^1)$ by

$\alpha =$ (replace one $\bigcirc$ by ) $\circ \Delta -$  id, (8.1)

where the second part means the disjoint union of a dashed loop. This is a well-defined map of $\mathcal{A}(S^1)$ to itself. For example, we show some simple cases below.



Note that, as in the above examples, the first term cancels with the last term. Further a dashed loop with a trivalent vertex vanishes by the AS relation. Hence we have a remarkable property of α that it decreases the number of vertices on a solid circle at least by two. Therefore we have

Lemma 8.7. If the number of vertices on a solid circle of a chord diagram $D \in \mathcal{A}(S^1)$ is less than $2m$, then $\alpha^m(D) = 0$.

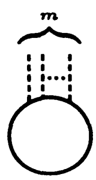
If we want to calculate $\Omega(M)$ along its definition, we would compute the tree T_m. However it will be a hard calculation. To avoid it, we prepare the following proposition.

Proposition 8.8. There exists a power series

$$p(\alpha) = \sum_{i=1}^{\infty} c_i \alpha^i \in \mathbb{Q}[[\alpha]]$$

such that each c_i belongs to $\mathbb{Z}[1/2, 1/3, \cdots, 1/(2i+1)]$ and

$$W_{sl_2}(j_1(D_m)) = W_{sl_2}((\varepsilon \circ p(\alpha))(D_m))$$

for any $m = 0, 1, \cdots$, where we put $D_m =$  .

Remark 8.9. The coefficients of α^i are concretely determined as

$$c_1 = \frac{1}{4}, \quad c_2 = -\frac{1}{16 \cdot 3}, \quad c_3 = -\frac{1}{8 \cdot 9 \cdot 5}, \quad c_4 = -\frac{1}{64 \cdot 5 \cdot 7}.$$

In general, we expect a better evaluation of denominators of c_i as

$$c_i \in \mathbb{Z}[1/2, 1/3, \cdots, 1/(2i-1)]$$

than the condition in the proposition. As for precise values of c_i, Thang Le suggested that c_i might be expressed by using the Bernoulli numbers.

Proof of Proposition 8.8. For any power series $p(\alpha) = \sum_{i=1}^{\infty} c_i \alpha^i$, by Lemma 8.7, we have

$$p(\alpha)(D_m) = p_k(\alpha)(D_m),$$

for each $m \leq 2k$, where we put $p_k(\alpha) = \sum_{i=1}^{k} c_i \alpha^i$. Hence it is sufficient to show the existence of an infinite series of scalars $c_1, c_2, c_3, \cdots$ satisfying

$$W_{sl_2}(j_1(D_m)) = W_{sl_2}\big((\varepsilon \circ p_k(\alpha))(D_m)\big) \tag{8.2}$$

for each k and for each $m \leq 2k$. We show (8.2) by induction on k as follows.

Suppose that (8.2) holds for $k-1$, *i.e.*, there exists a finite series $c_1, \cdots, c_{k-1}$ satisfying

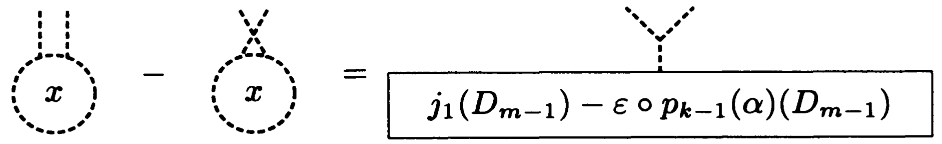
$$W_{sl_2}(j_1(D_m)) = W_{sl_2}\big((\varepsilon \circ p_{k-1}(\alpha))(D_m)\big), \tag{8.3}$$

for each $m \leq 2k-2$. Then, by Lemma 8.7, the required formula (8.2) holds for $m \leq 2k-2$, even if we put c_k to be any value.

Further we show that (8.2) holds for $m = 2k-1$ as follows; note that the right hand side does not still depend on a choice of c_k, as in the above case, by Lemma 8.7. Put

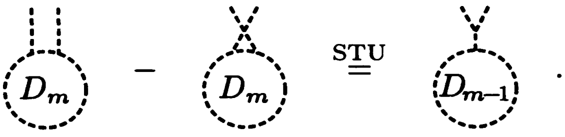
$$x = j_1(D_m) - (\varepsilon \circ p_{k-1}(\alpha))(D_m). \tag{8.4}$$

Then $W_{sl_2}(x)$ belongs to $(sl_2)^{\otimes m}$; recall that we define $W_{sl_2}(x)$ to be the image of $1 \in \mathbb{C}$ in $(sl_2)^{\otimes m}$ by the linear map defined in Section 3. By interchanging two of m dashed ends, we have



(8.5)

where the equality follows from the STU relation:



(8.6)

Since the right hand side in (8.5) vanishes, $W_{sl_2}(x)$ is invariant under the change of two adjacent dashed ends. Hence it is invariant under any change of dashed ends. Therefore $W_{sl_2}(x)$ belongs to the invariant space $((sl_2)^{\otimes m})^{\mathfrak{S}_m, sl_2}$ with respect to the action of the symmetric group $\mathfrak{S}_m$ and the action of the tensor product of m copies of the adjoint action of sl_2; we have the invariance with respect to the latter action, since $W_{sl_2}(x)$ is the image of $1 \in \mathbb{C}$ by an intertwiner, where $\mathbb{C}$ is the trivial representation of sl_2. The invariant space is one dimensional if m is even, and is the null vector space if m is odd, by invariant theory; for example, see [9]. Since m is odd in this case, $W_{sl_2}(x)$ vanishes, because it belongs to the null vector space. Hence, by (8.4), we obtain (8.2) for $m = 2k - 1$, since we have $(p_{k-1}(\alpha))(D_m) = (p_k(\alpha))(D_m)$ by Lemma 8.7.

We show that (8.2) holds for $m = 2k$ for a suitably chosen c_k as follows. We put x as in (8.4) again, and repeat the same argument as above. In this case $W_{sl_2}(x)$ belongs to one dimensional vector space, since m is even. Further $W_{sl_2}((\varepsilon \circ \alpha^k)(D_m))$ is non-zero in the space; in fact, we see below that it does not vanish, when dashed ends are closed. Hence we put

$$W_{sl_2}(x) = c_k W_{sl_2}\big((\varepsilon \circ \alpha^k)(D_m)\big)$$

for some scalar c_k. Therefore we have

$$\begin{aligned} W_{sl_2}\big(j_1(D_m)\big) &= W_{sl_2}\big((\varepsilon \circ p_{k-1}(\alpha))(D_m) + x\big) \\ &= W_{sl_2}\big((\varepsilon \circ (p_{k-1}(\alpha) + c_k \alpha^k))(D_m)\big). \end{aligned}$$

Putting $p_k(\alpha) = p_{k-1}(\alpha) + c_k \alpha^k$, we obtain (8.2) for $m = 2k$.

We evaluate the factors of the denominator of c_k by induction on k, as follows. Let Θ^k be the chord diagram consisting of a solid circle with k isolated dashed chords. Since Θ^k has $2k$ dashed chords on S^1, we have

$$W_{sl_2}(j_1(\Theta^k)) = W_{sl_2}(\varepsilon \circ p_k(\alpha)(\Theta^k)).$$

Hence we have

$$c_k W_{sl_2}((\varepsilon \circ \alpha^k)(\Theta^k)) = W_{sl_2}(j_1(\Theta^k)) - \sum_{i=1}^{k-1} c_i W_{sl_2}((\varepsilon \circ \alpha^i)(\Theta^k)). \tag{8.7}$$

By definition of j_1, $j_1(\Theta^k)$ is equal to the chord diagram obtained from the tree T_{2k} by closing $2k$ ends with k isolated chords. Hence the first term in the right hand side of (8.7) belongs to $\mathbb{Z}[1/2, 1/3, \cdots, 1/(2k-1)]$. Further the second terms belong to $\mathbb{Z}[1/2, 1/3, \cdots, 1/(2k-3)]$ by the hypothesis of induction. On the other hand, we calculate the left hand side as follows. $(\varepsilon \circ \alpha^k)(\Theta^k)$ consists of terms such that each of k α's decreases exactly two chord of Θ^k; note that the other terms vanish. Each α makes a dashed loop with two dashed segments. W_{sl_2} takes it to 4. Hence we have

$$W_{sl_2}((\varepsilon \circ \alpha)(\Theta^k)) = 4^k W_{sl_2}(j_k(\Theta^k)) = 4^k(2k+1)!!.$$

By (8.7), we obtain $c_k \in \mathbb{Z}[1/2, 1/3, \cdots, 1/(2k+1)]$. □

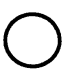
8.3 Expression of α by representations

In this section we consider the representation a corresponding to the map α. Define a to be $V_3 - 3 \cdot V_1 \in R(sl_2)$, where $R(sl_2)$ denotes the representation ring of sl_2 with integral coefficients. We have a relation between α and a as

Lemma 8.10. For any $D \in \mathcal{A}(S^1)$, we have

$$W_{sl_2}(\varepsilon \circ \alpha^n(D)) = W_{sl_2; a^n}(D).$$

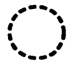
Proof. There are the following correspondences:

replace one  by  $\longleftrightarrow$ substitute the adjoint representation,

$$\text{taking a 2-parallel} \longleftrightarrow \text{taking a tensor of representation,}$$

$$\text{taking } \varepsilon \longleftrightarrow \text{substitute the trivial representation.}$$

More precisely, we have

$$W_{sl_2}\big((\text{replace one } \bigcirc \text{ by }$$  $$)(D)\big) = W_{sl_2;V_3}(D), \tag{8.8}$$

$$W_{sl_2;R_1,R_2}\big(\Delta(D)\big) = W_{sl_2;R_1\otimes R_2}(D), \tag{8.9}$$

$$W_{sl_2}\big(\varepsilon(D)\big) = W_{sl_2;V_1}(D). \tag{8.10}$$

By (8.8) and (8.9), we have

$$W_{sl_2;R}(\alpha(D)) = W_{sl_2;R\otimes a}(D) \tag{8.11}$$

by definition of α and a. Applying (8.11) repeatedly to the initial condition (8.10), we obtain the required formula. $\square$

Lemma 8.11. Suppose r is an odd prime number. Then we have

$$-2a^{(r-3)/2} \underset{(r)}{=} \sum mV_m, \tag{8.12}$$

$$a^{(r-1)/2} \underset{(r)}{=} 0, \tag{8.13}$$

where the sum in the first formula runs over all odd m in $1 \leq m \leq r-2$. Here $\underset{(r)}{=}$ denotes the equivalence relation in $R(sl_2)$ generated by the following two relations;

$$\big(\text{elements divisible by } r \text{ in } R(sl_2)\big) \sim 0,$$

$$V_r \sim V_{2r} \sim V_{3r} \sim \cdots \sim 0.$$

By this lemma, we cut off higher terms of a modulo r in a polynomial in a; recall that we cut off higher terms of α in a power series of α by Lemma 8.7.

We give examples of Lemma 8.11 below. We have

$$\begin{aligned} V_m \cdot a &= V_m \otimes V_3 - 3V_m \\ &= V_{m-2} + V_m + V_{m+2} - 3V_m \end{aligned}$$

$$= V_{m-2} - 2V_m + V_{m+2},$$

where the second equality is obtained by the decomposition formula of representation for sl_2. We pictorially denote the above equality by

$$\underset{1}{\overset{V_m}{\bullet}} \cdot a = \underset{1}{\overset{V_{m-2}}{\bullet}}\ \underset{-2}{\overset{V_m}{\bullet}}\ \underset{1}{\overset{V_{m+2}}{\bullet}}.$$

Here each number under a dot denotes the coefficient of the representation corresponding to the dot. Note that this formula also holds for negative m by regarding V_{-m} as $-V_m$. We begin with

$$a^0 = \underset{1}{\overset{V_1}{\bullet}},$$

$$a^1 = \underset{1}{\overset{V_1}{\bullet}} \cdot a = \underset{1}{\overset{V_{-1}}{\bullet}}\ \underset{-2}{\overset{V_1}{\bullet}}\ \underset{1}{\overset{V_3}{\bullet}} = \underset{-3}{\overset{V_1}{\bullet}}\ \underset{1}{\overset{V_3}{\bullet}}.$$

We show (8.12) in Lemma 8.11 for $r = 7$ as

$$a^2 = \underset{1}{\overset{V_{-3}}{\bullet}}\ \underset{-4}{\overset{V_{-1}}{\bullet}}\ \underset{6}{\overset{V_1}{\bullet}}\ \underset{-4}{\overset{V_3}{\bullet}}\ \underset{1}{\overset{V_5}{\bullet}}$$

$$= \underset{10}{\overset{V_1}{\bullet}}\ \underset{-5}{\overset{V_3}{\bullet}}\ \underset{1}{\overset{V_5}{\bullet}}$$

$$\overset{\times(-2)}{\longmapsto} \underset{-20}{\overset{V_1}{\bullet}}\ \underset{10}{\overset{V_3}{\bullet}}\ \underset{-2}{\overset{V_5}{\bullet}} \underset{(7)}{=} \underset{1}{\overset{V_1}{\bullet}}\ \underset{3}{\overset{V_3}{\bullet}}\ \underset{5}{\overset{V_5}{\bullet}}.$$

Further, as for (8.13), we have

$$a^3 = \underset{1}{\overset{V_{-5}}{\bullet}}\ \underset{-6}{\overset{V_{-3}}{\bullet}}\ \underset{15}{\overset{V_{-1}}{\bullet}}\ \underset{-20}{\overset{V_1}{\bullet}}\ \underset{15}{\overset{V_3}{\bullet}}\ \underset{-6}{\overset{V_5}{\bullet}}\ \underset{1}{\overset{V_7}{\bullet}}$$

$$= \underset{-35}{\overset{V_1}{\bullet}}\ \underset{21}{\overset{V_3}{\bullet}}\ \underset{-7}{\overset{V_5}{\bullet}}\ \underset{1}{\overset{V_7}{\bullet}} \underset{(7)}{=} 0.$$

The proof of Lemma 8.11 for general r is left to the reader; it is shown by a similar calculation as above.

8.4 Proof of Theorem 8.5

In this section, we prove Theorem 8.5, which states the universality of $\hat{\Omega}$ for the perturbative $SO(3)$ invariant. For simplicity, we show the theorem assuming that L is a knot. We begin with the following lemma.

Lemma 8.12. For the power series $p(\alpha)$ given in Proposition 8.8, we have

$$j_n(D) = \frac{1}{n!}(\varepsilon \circ p(\alpha)^n)(D).$$

Proof. By definition of Δ, we have

$$(\Delta^{(k_1)} \otimes \Delta^{(k_2)}) \circ \Delta = \Delta^{(k_1+k_2+1)}. \tag{8.14}$$

Since $\alpha^i(D)$ is a linear sum of chord diagrams $\Delta^{(k)}(D)$ possibly replaced some solid circles with dashed ones, we have the following formula by (8.14),

$$\big((\varepsilon \circ p_1(\alpha)) \otimes (\varepsilon \circ p_2(\alpha))\big) \circ \Delta = \varepsilon \circ \big(p_1(\alpha)p_2(\alpha)\big) \tag{8.15}$$

for any two power series $p_1(\alpha)$ and $p_2(\alpha)$, where $p_1(\alpha)p_2(\alpha)$ implies the usual product as power series.

Further, since $j_n = (1/n!)j_1 \circ \Delta^{(n-1)}$ holds by definition of j_n, we have

$$j_n(D) = \frac{1}{n!}\big((\varepsilon \circ p(\alpha))^{\otimes n} \circ \Delta^{(n-1)}\big)(D)$$

by replacing j_1 with the power series $p(\alpha)$ by Proposition 8.8, noting that we need n copies of $p(\alpha)$ since the solid circle in D becomes n solid circles by $\Delta^{(n-1)}$. By applying the formula (8.15) $n-1$ times, we obtain the required formula. □

Sketch of the proof of Theorem 8.5. Let L be a framed link. Applying the above lemma to the computation of $j_n\big(\check{Z}(L)\big)$, we have

$$h^n \cdot \hat{W}_{sl_2}(j_n\check{Z}(L)) = \frac{1}{n!}\hat{W}_{sl_2}\big((\varepsilon \circ p(\alpha)^n)(\check{Z}(L))\big), \tag{8.16}$$

where the first h^n is derived from the fact that the map j_n decreases the degree of chord diagrams by n. We consider terms of at most finite degree in the following of this proof. Then we can reduce the power series $p(\alpha)^n = c_1^n\alpha^n + nc_1^{n-1}c_2\alpha^{n+1} + \cdots$ to a finite sum by Lemma 8.7. Moreover using Lemma 8.10, we replace the right hand side of (8.16) with

$$\frac{c_1^n}{n!}\hat{W}_{sl_2;\alpha^n}\big(\check{Z}(L)\big) + \frac{nc_1^{n-1}c_2}{n!}\hat{W}_{sl_2;\alpha^{n+1}}\big(\check{Z}(L)\big) + \cdots + O(h^{n+(r-1)/2}). \tag{8.17}$$

Let r be an odd prime ≥ 5. Putting $n = (r-3)/2$, we have

$$a^n \underset{(r)}{=} \sum mV_m$$
$$a^{n+1} \underset{(r)}{=} a^{n+2} \underset{(r)}{=} \cdots \underset{(r)}{=} 0,$$

by Lemma 8.11. Hence (8.17) is congruent to

$$\frac{c_1^n}{n!} \sum m W_{sl_2;V_m}(\check{Z}(L)) + O(h^{n+(r-1)/2}) \tag{8.18}$$

modulo r. By the formula $W_{sl_2;V_m}(\nu) = [m]/m$ and Theorem 3.3, (8.18) is equal to

$$\begin{aligned} &\frac{c_1^n}{n!} \sum [m] W_{sl_2;V_m}(\hat{Z}(L)) + O(h^{n+(r-1)/2}) \\ &= \frac{c_1^n}{n!} \sum [m] Q^{sl_2;V_m}(L) + O(h^{n+(r-1)/2}). \end{aligned}$$

Further we replace j_n with ι_n using

$$\hat{W}_{sl_2}(\iota_n \check{Z}(L)) \underset{(r)}{=} \hat{W}_{sl_2}(j_n \check{Z}(L)) + O(h^{(r-1)/2})$$

which is obtained by the congruence $W_{sl_2}(\bigcirc) = 3$ and $\iota_n(\bigcirc) = -2n$ modulo r. Hence we have the formula[7]

$$h^n \cdot \hat{W}_{sl_2}(\iota_n \check{Z}(L)) \underset{(r)}{=} \frac{c_1^n}{n!} \sum [m] Q^{sl_2;V_m}(L) + O(h^{n+(r-1)/2}). \tag{8.19}$$

Let M be the 3-manifold obtained by Dehn surgery along L. Suppose that M is a rational homology 3-sphere. Further, as in [31], we can assume that L is algebraically split. Using the formula $\Omega_n(M)^{(d)} = |H_1|^{n-d}\Omega_d(M)^{(d)}$ for any $n \geq d$, we have the following formula by definition of $\hat{\Omega}$,

$$|H_1|^n \hat{\Omega}(M) = \Omega_n(M) = \frac{\iota_n \check{Z}(L)}{\left(\iota_n \check{Z}(U_+)\right)^{\sigma_+} \left(\iota_n \check{Z}(U_-)\right)^{\sigma_-}}$$

where we put $H_1 = H_1(M;\mathbb{Z})$. Further we have

$$|H_1|^n \hat{W}_{sl_2}(\hat{\Omega}(M))$$

[7]To obtain the formula, if we expanded $\check{Z}(L)$ directly, r might appear in the denominator, though we calculated the formulas modulo r. As in the text, we technically avoid the difficulty as follows. We replace $\check{Z}(L)$ with quantum invariants, before taking modulo r. Since the quantum invariants have integral coefficients, we calculate the formulas taking modulo r.

$$
\begin{aligned}
&= \hat{W}_{sl_2}\left(\frac{\iota_n \check{Z}(L)}{\left(\iota_n \check{Z}(U_+)\right)^{\sigma_+}\left(\iota_n \check{Z}(U_-)\right)^{\sigma_-}}\right) \\
&\underset{(r)}{=} \frac{\sum [m] Q^{sl_2;V_m}(L)}{\left(\sum [m] Q^{sl_2;V_m}(V_+)\right)^{\sigma_+}\left(\sum [m] Q^{sl_2;V_m}(V_-)\right)^{\sigma_-}} + O(h^{(r-1)/2}) \\
&= \tau_r^{SO(3)}(M) + O(h^{(r-1)/2}) \\
&\underset{(r)}{=} \left(\frac{|H_1|}{r}\right)\tau^{SO(3)}(M) + O(h^{(r-1)/2}),
\end{aligned}
$$

where we obtain the second equality by (8.19), obtain the third equality by the definition of $\tau_r^{SO(3)}(M)$ and obtain the fourth equality by Theorem 8.2 (2).[8] Here $(\frac{\cdot}{r})$ denotes the Legendre symbol. Further the formula $\left(\frac{f}{r}\right) \equiv f^{(r-1)/2}$, where f is not divisible by r, is known in number theory. Hence we have

$$|H_1|^n \hat{W}_{sl_2}(\hat{\Omega}(M)) \underset{(r)}{=} |H_1|^{n+1}\tau^{SO(3)}(M) + O(h^{(r-1)/2}).$$

Since this formula holds for infinitely many r, we obtain the required formula.

□

Summary for results in Sections 6 to 9. As mentioned in Section 0, we expect the notion of finite type and the existence of the universal quantum invariant for the quantum invariants $\tau_r^G(M) \in \mathbb{C}$. However, unlike the case of knots, values of the quantum invariants of 3-manifolds do not belong to a graded set. That is a reason of the technical difficulty to define finite type invariants and the universal quantum invariant for the quantum invariants themselves. Instead of them, we consider the perturbative invariants,[9] whose values belong to the graded set $\mathbb{Q}[[h]]$. We define finite type invariants and the universal

[8] To be precise, we should have expanded the formulas in power series of s, putting $h = \log(s+1)$.

[9] In this lecture note we use the terminology "perturbative invariant" in the sense of Section 8.1, which is obtained from quantum invariants by number theoretical limit. On the other hand the terminology has originally been used for invariants obtained from path integral formula of quantum invariants by perturbative expansion around flat connections. Rozansky [35, 36] gave rigorous definition of perturbative invariants along this approach. His and our definitions can be shown to be equal together under some assumption of integrality of coefficients of quantum invariants; see [36] for numerical examples.

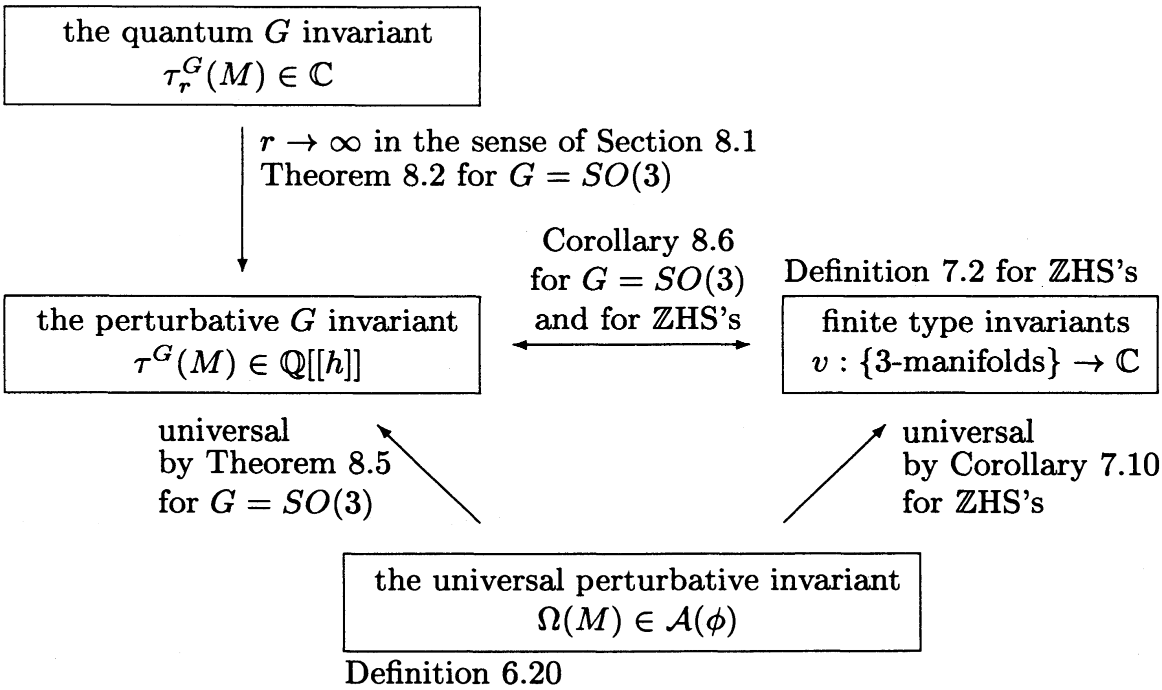


Figure 8.1: Invariants of 3-manifolds and the relations between them

invariant for the perturbative invariants; see Figure 8.1 for relations between these invariants. In the figure we also show the present attainments.

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