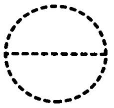
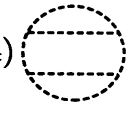
Example 6.21 ([27]). For the 3-manifold $M_{n,k}$ obtained from S^3 by integral surgery along $(2, n)$ torus knot with k framing, the universal perturbative invariant $\Omega(M_{n,k})$ is given by

$$\Omega(M_{n,k}) = \exp\Big(\frac{1}{48}(3n^2 - k^2 + 3k - 5)$$ 

$$+\frac{1}{2^7 \cdot 3^2}(12n^4 - 12kn^3 + 3k^2n^2 - 15n^2 + 12kn - 4k^2 + 4)$$ 

$$+(\text{terms of degree} \geq 3)\Big).$$

In general $\Omega(M)$ can be expressed as the exponential of a linear sum of connected chord diagrams; see [27].

7 Finite type invariants and the universal perturbative invariant

7.1 Finite type invariants of integral homology 3-spheres

Let M be an oriented integral homology 3-sphere, that is, $H_*(M; \mathbb{Z}) = H_*(S^3; \mathbb{Z})$. A framed link $\mathcal{L} = (L, f)$ is an unoriented link $L = \cup_{i=1}^n L_i$ in M with framing $f = (f_1, f_2, \cdots, f_n)$, with $f_i \in \mathbb{Z}$. We call $\mathcal{L}$ *algebraically split* if the linking number of L_i and L_j is zero for each pair (i, j). We call $\mathcal{L}$ *unit-framed* if all framings of $\mathcal{L}$ are ± 1. By $M_{\mathcal{L}}$ we denote the closed oriented 3-manifold obtained from M by Dehn surgery along $\mathcal{L}$ with respect to the framing.

Remark 7.1. Let M be an integral homology 3-sphere and $\mathcal{L}$ a framed link in M. Then the following two conditions are equivalent:

(1) $\mathcal{L}$ is algebraically split and unit-framed.

(2) $M_{\mathcal{L}'}$ is an integral homology 3-sphere for any sublink $\mathcal{L}'$ in $\mathcal{L}$.

Proof. If $\mathcal{L}$ is algebraically split and unit-framed, then we can easily verify the condition (2). Conversely, suppose that the condition (2) holds. Then we have

$f_i = \pm 1$ since $M_{\mathcal{L}_i}$ is an integral homology 3-sphere, where $\mathcal{L}_i$ is a sublink (L_i, f_i) of $\mathcal{L}$. Further the linking number of L_i and L_j is zero since $M_{\mathcal{L}_i \cup \mathcal{L}_j}$ is an integral homology 3-sphere. Therefore $\mathcal{L}$ is algebraically split and unit-framed. $\square$

In the following of this section, we assume that a framed link $\mathcal{L}$ is algebraically split and unit-framed. Let $\mathcal{M}$ be the vector space over $\mathbb{C}$ freely spanned by homeomorphism classes of oriented integral homology 3-spheres. We put

$$(M, \mathcal{L}) := \sum_{\mathcal{L}' \subset \mathcal{L}} (-1)^{\#\mathcal{L}'} M_{\mathcal{L}'} \in \mathcal{M},$$

where the sum runs over all sublinks $\mathcal{L}'$ in $\mathcal{L}$ including the empty link. Further let $\mathcal{M}_d$ be the vector subspace of $\mathcal{M}$ spanned by $(M, \mathcal{L})$ such that M is an integral homology 3-sphere, and $\mathcal{L}$ is an algebraically split and unit-framed link with d components in $\mathcal{M}$.

Recall that a Vassiliev invariant of degree d is a linear map $\mathcal{K} \to \mathbb{C}$ which vanishes in $\mathcal{K}_{d+1}$, where $\mathcal{K}_{d+1}$ is the vector subspace spanned by linear sums of 2^{d+1} knots obtained by crossing changes at $d+1$ crossings. Here, instead of "crossing", we consider "Dehn surgery" to obtain the vector subspace $\mathcal{M}_{d+1}$ of $\mathcal{M}$. In analogue of the definition of Vassiliev invariants, we have

Definition 7.2. A map $v : \mathcal{M} \to \mathbb{C}$ is called a finite type invariant of degree d if $v|_{\mathcal{M}_{d+1}} = 0$.

Though the property of finite type for integral homology 3-spheres might be defined in different ways, we should expect that the definition is related to chord diagrams in the same way as Vassiliev invariants. In fact, we have the following theorem for our finite type invariant defined above.

Theorem 7.3 ([8]). We have $\mathcal{M}_{3d+1}/\mathcal{M}_{3d+2} = 0$ and $\mathcal{M}_{3d+2}/\mathcal{M}_{3d+3} = 0$. Further there exists a surjection

$$\mathcal{A}(\phi)^{(d)} \to \mathcal{M}_{3d}/\mathcal{M}_{3d+1},$$

where $\mathcal{A}(\phi)^{(d)}$ is a subspace of $\mathcal{A}(\phi)$ spanned by chord diagrams of degree d.

We show a sketch of the proof of the theorem below. We begin with the following lemma.

Lemma 7.4. Let $\mathcal{K} \cup \mathcal{L}$ be an algebraically split and unit-framed link in an integral homology 3-sphere M. Suppose $\#\mathcal{K} = 1$ and $\#\mathcal{L} = d$. Then we have $(M, \mathcal{L}) = (M_{\mathcal{K}}, \mathcal{L})$ in $\mathcal{M}_d/\mathcal{M}_{d+1}$.

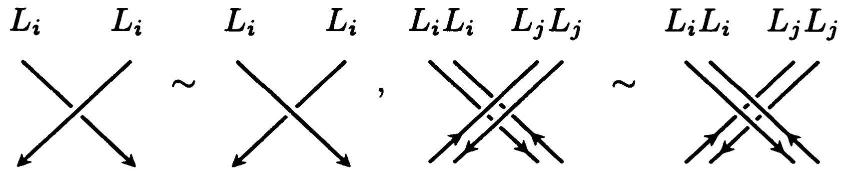
Proof. Since $(M, \mathcal{L}) - (M_{\mathcal{K}}, \mathcal{L}) = (M, \mathcal{K} \cup \mathcal{L})$ belongs to $\mathcal{M}_{d+1}$, we obtain this lemma. $\square$

Lemma 7.5. For any $(M, \mathcal{L})$ in $\mathcal{M}_d$ there exists some framed link $\mathcal{L}'$ in the 3-sphere S^3 such that $(M, \mathcal{L}) = (S^3, \mathcal{L}')$ in $\mathcal{M}_d/\mathcal{M}_{d+1}$.

Proof. This lemma is shown using the above lemma for a sequence of surgeries along knots, getting S^3 from M. $\square$

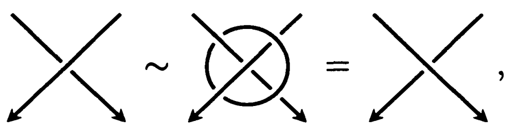
Therefore $\mathcal{M}_d/\mathcal{M}_{d+1}$ is spanned by $(S^3, \mathcal{L})$ such that $\#\mathcal{L} = d$. In the following of this proof, it is sufficient to consider the equivalence relation among framed links in S^3 given by $\mathcal{M}_{d+1}$. Here we denote the equality $(S^3, \mathcal{L}) = (S^3, \mathcal{L}')$ in $\mathcal{M}_d/\mathcal{M}_{d+1}$ by $\mathcal{L} \sim \mathcal{L}'$.

Lemma 7.6.



(7.1)

Proof. Use the following relation,



(7.2)

where the middle picture implies the 3-manifold after Dehn surgery along the trivial knot winding around the crossing. $\square$

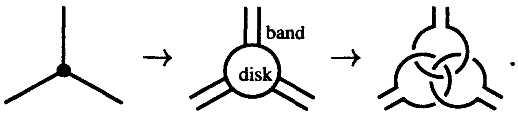
Recall that, in the case of knots, we collapsed the chords and obtain singular knots with d-crossings in Proposition 4.3, to get a linear map $\varphi : \mathcal{A}(S^1) \to \mathcal{K}_d/\mathcal{K}_{d+1}$. Instead of φ, we consider the following map in this case.

Proposition 7.7 ([30]). The map

$$\psi : \mathbb{Q}\left\{\begin{array}{r}\text{the uni-trivalent graphs such that}\\ \#\text{edge} = d \text{ and each trivalent}\\ \text{vertex has a cyclic order}\end{array}\right\} \to \mathcal{M}_d/\mathcal{M}_{d+1}$$

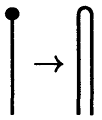
is well defined, and surjective.

Proof. For a uni-trivalent graph D, we make a link L as follows. For a trivalent vertex we associate the Borromean ring in the following way. We use a given cyclic order and have a ribbon graph by replacing a vertex with a disk and replacing an edge with a band. Further we replace the disk by the Borromean ring and replace the band by its boundaries as



(7.3)

For a univalent vertex we consider the following correspondence.



(7.4)

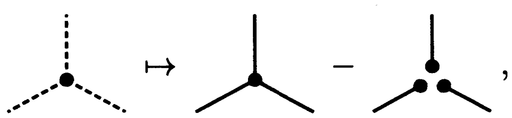
Then, for a uni-trivalent graph D with d edges, we obtain a link L with d components.

We define $\psi(D)$ to be the class in $\mathcal{M}_d/\mathcal{M}_{d+1}$ represented by $(S^3, \mathcal{L}) \in \mathcal{M}$, where the framed link $\mathcal{L}$ is the link L given above with $+1$ framings. By Lemma 7.6, $\psi(D)$ does not depend on the embedding of the ribbon graph. For the proof of surjectivity, see [30]. $\square$

Proof of Theorem 7.3. We define a map d of

$$\mathbb{Q}\left\{\begin{array}{r}\text{the uni-trivalent graphs such that}\\ \#\text{edge} = d \text{ and each trivalent}\\ \text{vertex has a cyclic order}\end{array}\right\}$$

to itself by



where the above formula implies that we replace each dashed trivalent vertex by the linear sum in the right hand side; if there are k trivalent vertices, we obtain a linear sum of 2^k uni-trivalent graphs by the map. Then $\psi \circ d$ is a surjection. Further it takes graphs with univalent vertices to zero, and it also takes the AS and IHX relations for trivalent vertices to zero; see [8] for the detailed proof. Hence $\psi \circ d$ induces the required map $\mathcal{A}(\phi)^{(d)} \to \mathcal{M}_{3d}/\mathcal{M}_{3d+1}$. □

In the same way as the case of Vassiliev invariants, we have the following identification,

$$\{\text{the finite type invariants of degree } d\} = (\mathcal{M}/\mathcal{M}_{d+1})^*.$$

Hence, by taking the dual of formulas in Theorem 7.3, we obtain

Corollary 7.8. We have

$$\begin{aligned} & \{\text{the finite type invariants of degree} \quad 3d\} \\ = & \{\text{the finite type invariants of degree} \quad 3d+1\} \\ = & \{\text{the finite type invariants of degree} \quad 3d+2\}. \end{aligned}$$

Furthermore there exists an injection

$$(\mathcal{A}(\phi)^{(d)})^* \hookleftarrow \frac{\{\text{the finite type invariants of degree } 3d\}}{\{\text{the finite type invariants of degree } 3d+1\}}.$$

For a finite type invariant v of degree $3d$, we call its image in $(\mathcal{A}(\phi)^{(d)})^*$ the *weight system* of v.

7.2 Universality of the universal perturbative invariant among finite type invariants

In this section we show the following theorem. The procedure here is analogous to that in Section 4.

Theorem 7.9 ([22]). The surjective map

$$\mathcal{A}(\emptyset)^{(d)} \to \mathcal{M}_{3d}/\mathcal{M}_{3d+1}$$

given in Theorem 7.3 is an isomorphism.

As a corollary, we have

Corollary 7.10. For any non-negative integer d and any finite type invariant v of degree d, there exists a map $W : \mathcal{A}(\emptyset) \to \mathbb{C}$ such that $v(M) = W(\Omega(M))$ for any integral homology 3-sphere M.

The proof of the corollary is obtained in the same way as the case of Vassiliev invariants. The corollary implies that any finite type invariant factors through Ω. Therefore we call Ω the ***universal finite type invariant.***

Proof of Theorem 7.9. We denote by M_D the image of a chord diagram D by the above surjection given in Theorem 7.3. As in Section 4, we can reduce the proof to the following formulas,

$$\Omega(M_D) = D + (\text{terms of degree} > d), \tag{7.5}$$

$$\Omega(\mathcal{M}_{3d+3}) \subset \mathcal{A}(\phi)^{(\geq d+1)}, \tag{7.6}$$

for any $D \in \mathcal{A}(\phi)^{(d)}$ and $M_D \in \mathcal{M}_{3d}$; as for (7.6) note $\mathcal{M}_{3d+1} = \mathcal{M}_{3d+3}$ by Theorem 7.3. By Lemma 7.11 below, we obtain (7.5) noting $\Omega(M)^{(\leq d)} = \Omega_d(M)$ for integral homology spheres M, see [27]. We omit the proof of (7.6); see [22] for its proof.[6] $\square$

Lemma 7.11. If $D \in \mathcal{A}(\emptyset)^{(d)}$, then we have

$$\Omega_d(M_D) = D \in \mathcal{A}(\phi)/D_{>d}$$

Outline of the proof. As in the proof of Theorem 7.3, we break trivalent vertices of a trivalent graph D to obtain a linear sum of uni-trivalent graphs. Further, as in the proof of Proposition 7.7, we replace it with the linear sum of links L_D; we show the procedure pictorially as

(7.7)

[6] It is not a trivial corollary of Lemma 7.11; note that $\mathcal{M}_{3d+3}$ is spanned, not by $\{M_D\}$ for $D \in \mathcal{A}(\phi)^{(d+1)}$, but by $\{M_D\} \cup \mathcal{M}_{3d+6}$.

We put $L_D = \sum_L \epsilon_L L$, where $\epsilon_L = \pm 1$. Let $\mathcal{L}_D$ be L_D with all framings $+1$. Thus, by construction of M_D, we have

$$M_D = (S^3, \mathcal{L}_D) = \sum_{\mathcal{L}}^{2^{2d}} \epsilon_{\mathcal{L}} \sum_{\mathcal{L}' \subset \mathcal{L}}^{2^{3d}} (-1)^{\sharp \mathcal{L}'} S^3_{\mathcal{L}'} \tag{7.8}$$

where $2d$ is the number of vertices, $3d$ is the number of edges of D, and $\mathcal{L}'$ is a sublink of $\mathcal{L}$.

Here the range of the first summation $\sum^{2^{2d}}$ in (7.8) implies the set of choices; whether we break each of $2d$ trivalent vertices of D or not. Further, as for the second summation $\sum^{2^{3d}}$, the choices are whether we choose each of $3d$ edges of D, or remove it. By regarding in such a way, the two summations become independent, and we replace the order of them as

$$M_D = \sum^{2^{3d}} \sum^{2^{3d}} \epsilon_{\mathcal{L}} (-1)^{\sharp \mathcal{L}'} S^3_{\mathcal{L}'}.$$

If $\mathcal{L}'$ is a proper sublink of $\mathcal{L}$ (namely, there is an edge which is not chosen), then the second summation

$$\sum^{2^{3d}} \epsilon_{\mathcal{L}} (-1)^{\sharp \mathcal{L}'} S^3_{\mathcal{L}'}$$

vanishes, since the Borromean ring becomes unlink if one of the components of the Borromean ring is removed; we consider the Borromean ring corresponding to a vertex at an end of the removed edge. We see that the right two pictures in (7.7) cancel together, if we remove one of the middle three components. Therefore the sum reduces to the sum for $\mathcal{L}' = \mathcal{L}$ as

$$M_D = \sum_{\mathcal{L}} \epsilon_{\mathcal{L}} (-1)^{\sharp \mathcal{L}} S^3_{\mathcal{L}}.$$

By the linearity of Ω_d we have

$$\begin{aligned} \Omega_d(M_D) &= \sum_{\mathcal{L}} \epsilon_{\mathcal{L}} (-1)^d \frac{\iota_d \check{Z}(\mathcal{L})}{\iota_d \check{Z}(U_+)^{3d}} \\ &= (-1)^d \frac{\iota_d \check{Z}(\mathcal{L}_D)}{\iota_d \check{Z}(U_+)^{3d}}. \end{aligned}$$

Hence, to obtain the required formula, it is sufficient to show

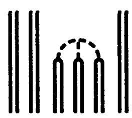
$$\iota_d \check{Z}(U_+)^{3d} = (-1)^d + (\text{terms of degree} > 0), \tag{7.9}$$

$$\iota_d \check{Z}(\mathcal{L}_D) = D + (\text{terms of degree} > d). \tag{7.10}$$

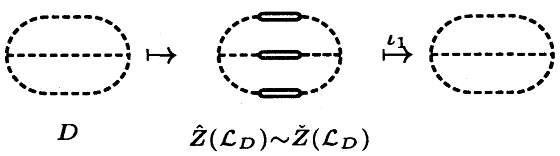
For the proof of (7.9), see [27]. Further we obtain (7.10) by Lemma 7.12 below.

$\square$

Lemma 7.12.

$\check{Z}(\mathcal{L}_D) =$  $+ (\text{terms of } \sharp \{ \text{ trivalent vertices } \} \geq 2)$

Proof. We obtain the formula by long calculation along the definition of $\hat{Z}$. For example, for the dashed θ curve D, we show rough pictures of the calculation below. Recall that $\mathcal{L}_D$ is a linear sum of links with 3 components in this case (with $3d$ components in general).



For the detailed proof, see [22]. $\square$

8 Quantum invariants and the universal perturbative invariant

8.1 Quantum $SO(3)$ invariant constructed from quantum invariants of framed links

Let V_m be the m dimensional irreducible representation of sl_2 and M the 3-manifold obtained from S^3 by Dehn surgery along a framed link L.

Theorem 8.1 ([12]). Let r be an odd integer ≥ 3, and put $q = \exp(2\pi\sqrt{-1}/r)$. Then

$$\frac{\sum [m] Q^{sl_2; V_m}(L)}{\left(\sum [m] Q^{sl_2; V_m}(U_+)\right)^{\sigma_+} \left(\sum [m] Q^{sl_2; V_m}(U_-)\right)^{\sigma_-}} \in \mathbb{C}$$