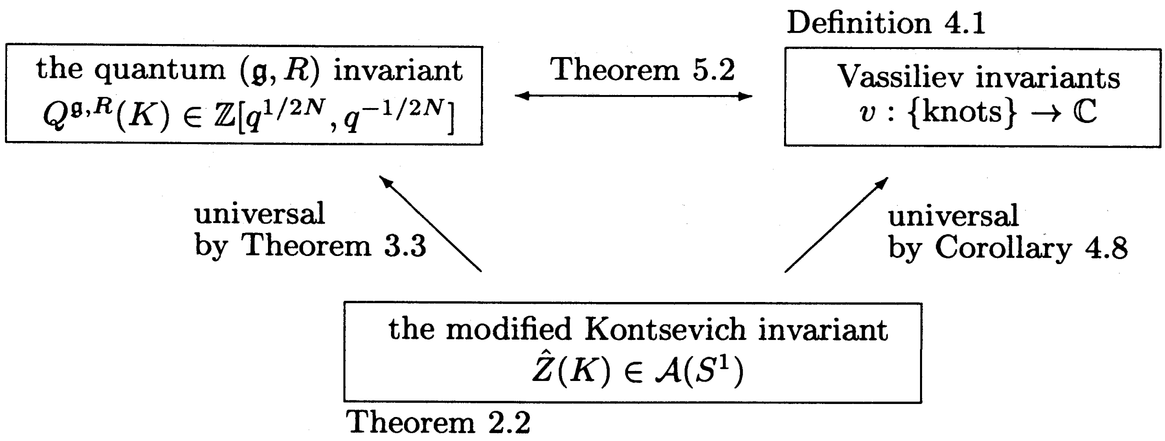


Figure 5.1: Three kinds of invariants of knots and the relations between them

with the weight system $\hat{W}_{\mathfrak{g},R}$ derived from the substitution of $\mathfrak{g}$ and R into chord diagrams. The other is the universality among Vassiliev invariants; each Vassiliev invariant v is expressed as

$$v = W \circ \hat{Z}$$

with some weight system W.

As a corollary of the two universalities, we obtain a relation between quantum invariants and Vassiliev invariants; the coefficients of the quantum $(\mathfrak{g}, R)$ invariant are Vassiliev invariants and their weight systems are equal to $W_{\mathfrak{g},R}$.

6 The universal perturbative invariant of 3-manifolds

So far we have dealt with invariants of knots and links. From now on we will consider invariants of 3-manifolds. The purpose of this section is to construct an invariant of 3-manifolds which has the universal property that the perturbative quantum invariants of 3-manifolds recover from it. So we call it the ***universal perturbative invariant*** of 3-manifolds.

6.1 Properties of $\hat{Z}(L)$

We will construct invariants from $\hat{Z}(L)$ in Section 6.4. To show the invariance under Kirby moves, we need the following properties of $\hat{Z}(L)$.

Proposition 6.1 ([24, Theorem 5.1]). Let L be a framed link, C a component of L and C' the component of $\sqcup^l S^1$ corresponding to C. Then we obtain the following formulas, where Δ, S and ε are defined in Section 2.
(1) Let L' be the link obtained from L by taking 2-parallel along C. Then we have

$$\hat{Z}(L') = \Delta_{(C')}(\hat{Z}(L)).$$

(2) Let L'' be the link obtained from L by reversing the orientation of C. Then we have

$$\hat{Z}(L'') = S_{(C')}(\hat{Z}(L)).$$

(3) Let L''' be the link obtained from L by removing C. Then we have

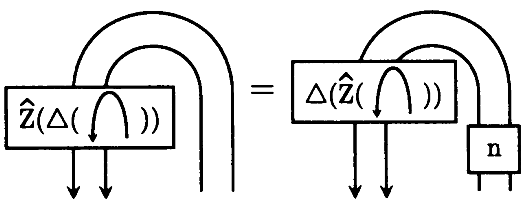
$$\hat{Z}(L''') = \varepsilon_{(C')}(\hat{Z}(L)).$$

Proof. We prove these relations by fixing a link diagram of L and decomposing the link into elementary quasi-tangle diagrams. In such a way, we can easily prove (2) and (3) by the definitions of $\hat{Z}, S_{(C')}, \varepsilon_{(C')}$ and properties of Φ. So we prove only (1) for elementary quasi-tangle diagrams in the following of this proof.

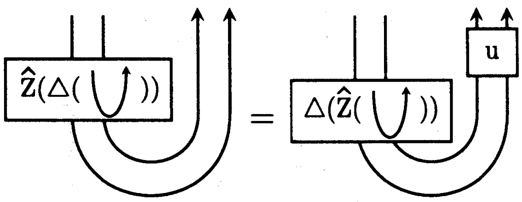
For the diagrams $|\backslash|$ and $|/|$, the proof is trivial by the definition of $\hat{Z}$.

For the diagrams  and , the proof is a consequence of the hexagon relation.

It remains to show the proof for the diagrams $\cup$ or $\cap$. We define n to be the ratio of $\Delta(\hat{Z}(\ \cap\ ))$ to $\hat{Z}(\Delta(\ \cap\ ))$ and u that of $\Delta(\hat{Z}(\ \cup\ ))$ to $\hat{Z}(\Delta(\ \cup\ ))$ as follows.

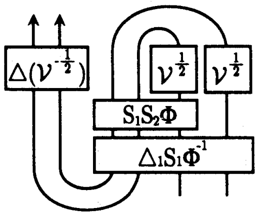


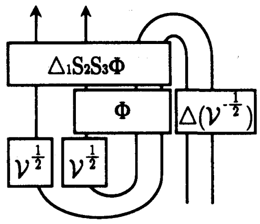
(6.1)



(6.2)

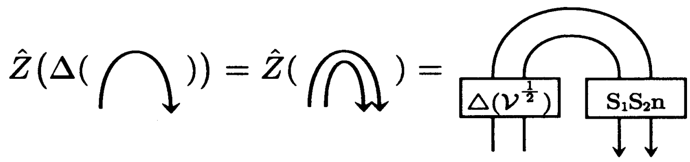
Alternatively we also give a constructive definition of u and n as follows.

$n =$ 

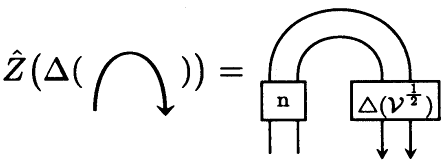
$u =$ 

As in (6.1) and (6.2), n and u are error terms for commutation of $\hat{Z}$ and Δ at maximal and minimal critical points respectively. Note that, if n and u were trivial chord diagrams, then the proof would be completed. However n and u are not necessarily trivial.

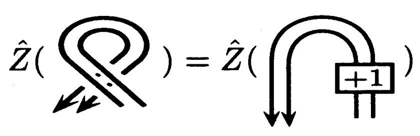
As for the other orientation at critical points, we express error terms for $\Delta(\hat{Z}($ ∩ $))$ and $\hat{Z}(\Delta($ ∩ $))$, and for $\Delta(\hat{Z}($ ∪ $))$ and $\hat{Z}(\Delta($ ∪ $))$, by using n and u as follows. By Proposition 6.1 (2), we have



Further it is equal to the right picture in (6.3) below; this is shown by comparing both sides of (6.4).

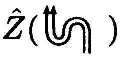


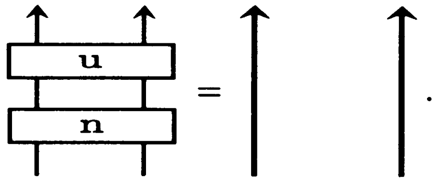
(6.3)



(6.4)

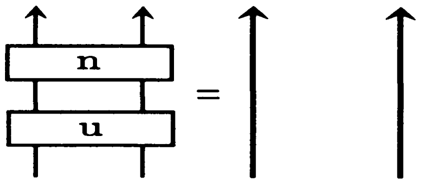
A similar formula holds for ∪ . Therefore, for any orientation, the error terms at maximal and minimal points are equal to n and u respectively.

By comparing both sides of $\hat{Z}($$) = \hat{Z}(\uparrow\uparrow)$, we obtain



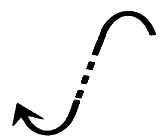
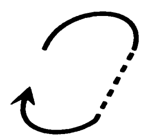
$$\tag{6.5}$$

Since the coefficient of degree 0 part of u is not zero, the inverse of u exists. By using the existence of u^{-1}, we obtain the following relation from (6.5).

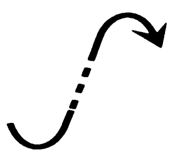
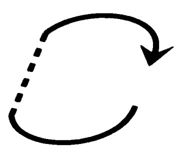


$$\tag{6.6}$$

By (6.5), error terms for the following pairs of critical points cancel together.

 and 

Further that holds for the following pairs by (6.6).

 and 

Hence, for a link diagram, all error terms cancel with each other. This completes the proof of (1). □

Let L be a link with l components. We define $\check{Z}(L)$ to be $\hat{Z}(L)\sharp\nu^{\otimes l}$, where this formula implies that $\check{Z}(L)$ is obtained from $\hat{Z}(L)$ by taking connected sum of ν to each component of L.

Proposition 6.2 ([24, Theorem 7.3], [23]). Let L be an oriented framed link, and L' a framed link obtained from L by KII move, that is, the handle slide move in [13]. Then $\check{Z}(L')$ can be obtained from $\check{Z}(L)$ by replacing the left picture in Figure 6.1 with the right picture.

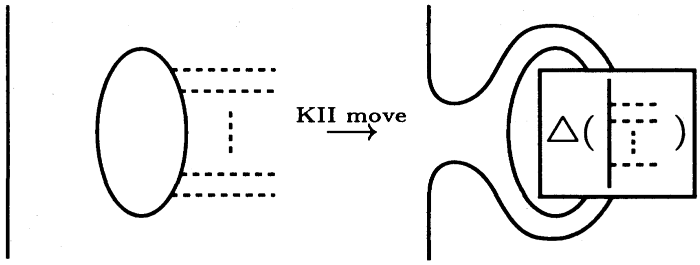


Figure 6.1: KII move on chord diagrams

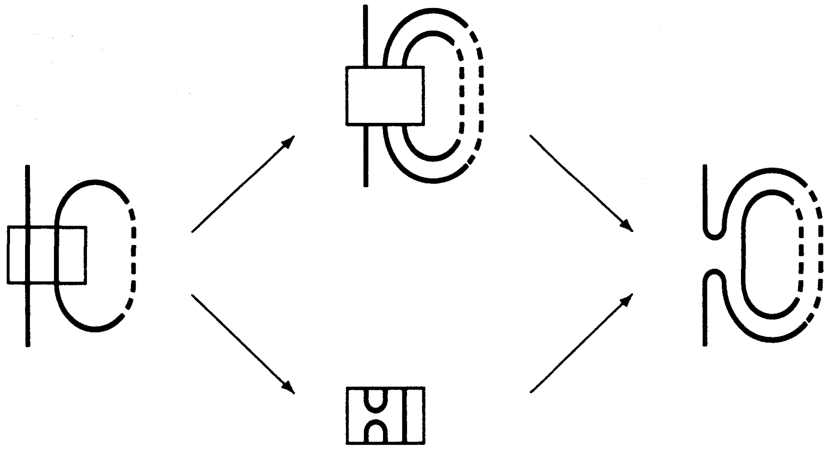
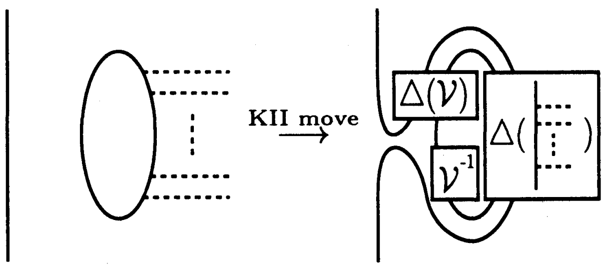


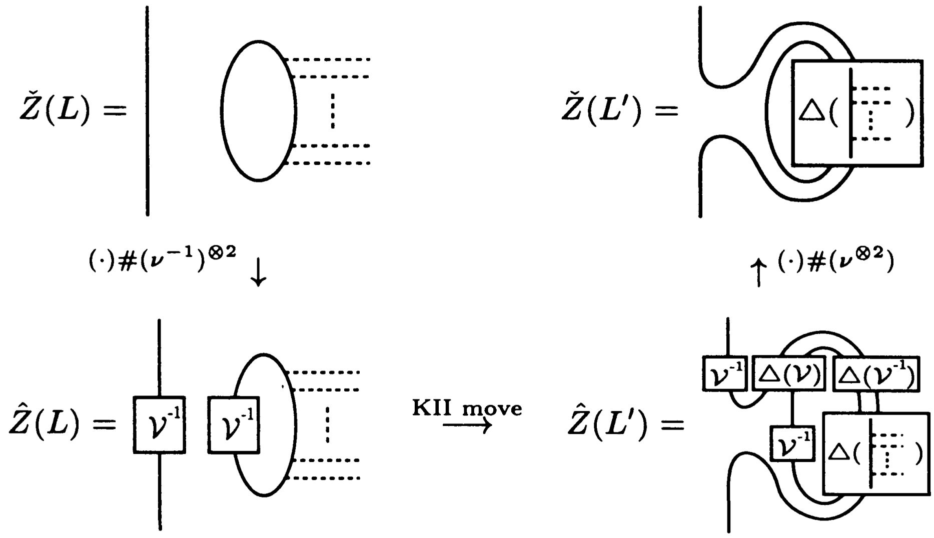
Figure 6.2: KII move on links

Proof. We decompose KII move of a link into the following successive operations; we choose two adjacent vertical segments in L (after deforming it by isotopy if necessary), replace them by the quasi-tangle obtained by connecting the quasi-tangles in the statement of Lemma 6.3 below vertically, and take two parallel of the remaining part of the right one of the above segments, see Figure 6.2.

By Proposition 6.1 (1) (extended to quasi-tangles) and Lemma 6.3 below, we have the change of $\hat{Z}(L)$ as follows.

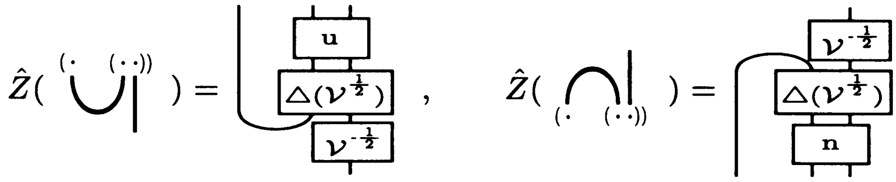


Hence we have the change of $\check{Z}(L)$ under KII move as follows.

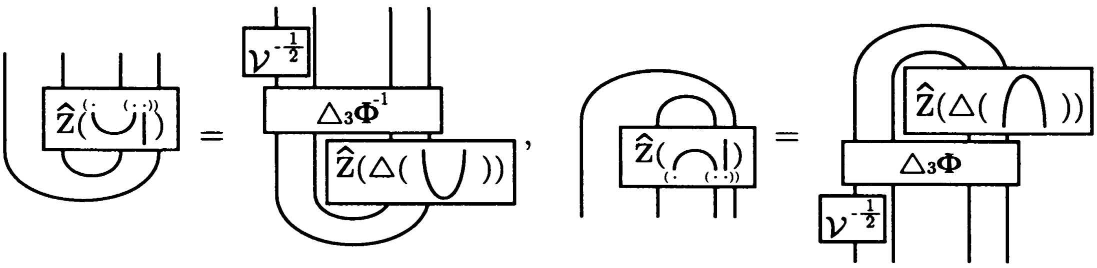


Then we have the desired result. □

Lemma 6.3.

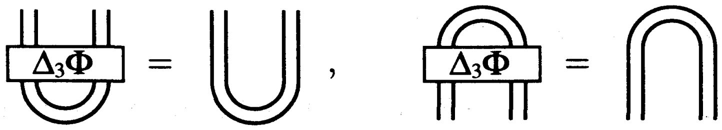


Proof. We express each of the quasi-tangles in the lemma as a part of a decomposition of each of  and  into elementary quasi-tangle diagrams. By such expressions, we have the following formulas of chord diagrams.

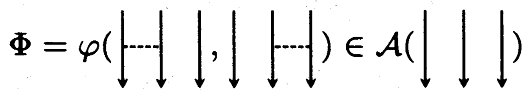


As for the first formula, $\Delta_3\Phi^{-1}$ and $\hat{Z}(\Delta(\cup))$ commute together, since $\Delta(D)$ commutes with any element of $\mathcal{A}(\uparrow\uparrow)$ [26, Theorem 1] for any chord diagram D on $\uparrow$ possibly with dashed external vertices. We push down $\Delta_3\Phi^{-1}$. Further, by Lemma 6.4 below, it vanished in the formula. In the same way, $\Delta_3\Phi$ vanishes in the second formula. Hence we have the required formulas by applying (6.2) and (6.1) to the above formulas. □

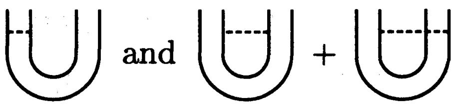
Lemma 6.4.



Proof. As in [25], Φ has the following expression



with a certain power series $\varphi(A, B)$ of non-commutative indeterminates A and B such that $\varphi(A, B)$ is congruent to 1 modulo the relation $AB = BA$, see Section 2.2 for definition of Φ. Since

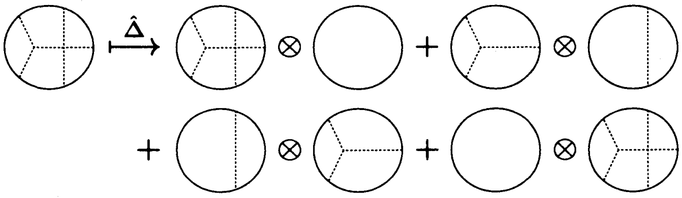


commute with each other, we obtain the required formula. □

We define the map

$$\hat{\Delta} : \mathcal{A}(\sqcup^l S^1) \to \mathcal{A}(\sqcup^l S^1) \otimes \mathcal{A}(\sqcup^l S^1)$$

as follows; see also [3] for its definition. For a chord diagram D, we define $\hat{\Delta}(D)$ to be the sum of $D' \otimes D''$ where D' runs over all subdiagram of D, and D'' is its complement. Here we mean by subdiagram of D a chord diagram obtained from D by removing some of dashed connected components. We show an example for $\hat{\Delta}$ below.



Since $\hat{\Delta}$ plays a similar role as the comultiplication in a Hopf algebra, we call $\alpha \in \mathcal{A}(\sqcup^l S^1)$ *group-like* if $\hat{\Delta}(\alpha) = \alpha \otimes \alpha$, and call α *primitive* if $\hat{\Delta}(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha$, as terminology of Hopf algebra.

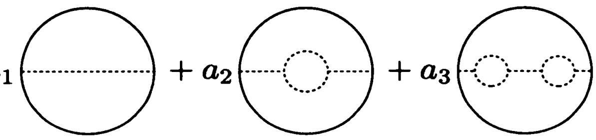
Proposition 6.5 ([27, Theorem 3.7]). For an oriented framed link L with l components, $\hat{Z}(L)$ is group-like in $\sqcup^l \mathcal{A}(S^1)$.

As a corollary, we have

Corollary 6.6. $\check{Z}(L)$ is also group-like.

Proof. By applying Proposition 6.5 to the trivial knot, we see that ν is group-like; noting that $\hat{Z}(\text{the trivial knot}) = \nu$. Since $\hat{\Delta}$ is multiplicative with respect to the connected sum, we have the desired result by the definition of $\check{Z}(L)$. □

Remark 6.7. For a knot K, Proposition 6.5 gives a strong restriction to the value of $\hat{Z}(K)$ as follows. In this case, $\hat{Z}(K)$ belongs to $\mathcal{A}(S^1)$. Here $\mathcal{A}(S^1)$ becomes a Hopf algebra such that the multiplication is connected sum, and the comultiplication is $\hat{\Delta}$, see [3]. It is known, see for example [1], that a non-zero group-like element of a graded Hopf algebra can be expressed as the exponential image of a primitive element. By Proposition 6.5, $\hat{Z}(K)$ is equal to the exponential image of a primitive element. By definition, a primitive element in $\mathcal{A}(S^1)$ is a linear sum of chord diagrams with one connected dashed component. Hence $\hat{Z}(K)$ is expressed as

$\hat{Z}(K) = \exp\Big(a_1$  $+ \cdots\Big),$

where $a_1, a_2, a_3, \cdots$ are scalar invariants of K. We expect that these scalar invariants are fundamental invariants of K. For example, a_1 is equal to half the framing of K and a_2 is equal to the second coefficient of the Conway polynomial.

Proof of Proposition 6.5. Let l be the number of components of L, and let X_1 and X_2 be two copies of $\sqcup^l S^1$. We define the map $p : \mathcal{A}(X_1 \sqcup X_2) \to \mathcal{A}(X_1) \otimes \mathcal{A}(X_2)$ by

$$p(D) = \begin{cases} 0, & \text{if } D \text{ has a chord which connects } X_1 \text{ and } X_2, \\ D_1 \otimes D_2, & \text{if } D \text{ is the disjoint union of } D_1 \in \mathcal{A}(X_1) \text{ and } D_2 \in \mathcal{A}(X_2). \end{cases}$$

By definition of $\hat{\Delta}$, we have $\hat{\Delta} = p \circ \Delta_{(C_1)} \circ \cdots \circ \Delta_{(C_l)}$, where C_i denotes the ith component of $\sqcup^l S^1$. Therefore, by Proposition 6.1 (1), we have

$$\hat{\Delta}(\hat{Z}(L)) = p \circ \Delta_{(C_1)} \circ \cdots \circ \Delta_{(C_l)}(\hat{Z}(L)) = p \circ \hat{Z}(L^{(2)}), \tag{6.7}$$

where $L^{(2)}$ denotes the 2-parallel of L along all components of L.

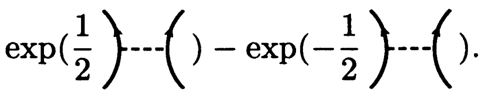
Let $L \sqcup L$ be the split union of two copies of L. Then, by definition of p, we have

$$p(\hat{Z}(L \sqcup L)) = \hat{Z}(L) \otimes \hat{Z}(L). \tag{6.8}$$

By (6.7) and (6.8), it is sufficient to show

$$p(\hat{Z}(L^{(2)}) - \hat{Z}(L \sqcup L)) = 0. \tag{6.9}$$

We calculate $\hat{Z}(L^{(2)}) - \hat{Z}(L \sqcup L)$ as follows. We obtain $L^{(2)}$ from $L \sqcup L$ by crossing changes between the first L and the second L. By each of the crossing changes, the value of $\hat{Z}$ changes by the following difference,



It consists of chord diagrams with chords connecting X_1 and X_2. Hence it vanishes by p. Therefore we obtain (6.9). □

6.2 Replacing solid circles with dashed graphs

In this section we construct a series of the maps $\iota_n : \mathcal{A}(\sqcup^l S^1) \to \mathcal{A}(\phi)$ for $n = 1, 2, \cdots$, which replace S^1 with dashed graphs.

We consider chord diagrams with support m ordered points labeled by 0, 1, 2, $\cdots$, $m-1$. That is, the chord diagrams are oriented uni-trivalent dashed graphs whose m univalent vertices are on the m fixed points respectively. We denote by $\mathcal{A}(m)$ the vector space over $\mathbb{C}$ spanned by such chord diagrams subject to the AS and IHX relations. In particular, we put $\mathcal{A}(0) = \mathcal{A}(\phi)$.

We define $T_m \in \mathcal{A}(m)$ as follows. We put $T_0 = T_1 = 0$, and put T_2 be a dashed chord connecting the two ends 0 and 1. The definition for $m \geq 3$ is the following. For an element τ in the symmetric group $\mathfrak{S}_{m-2}$ acting on the set $\{1, 2, \cdots, m-2\}$, we define $T_\tau \in \mathcal{A}(m)$ as follows; see also Figure 6.3 for a pictorial definition. *i.e.*, we put a dashed line between 0 and $m-1$, and make $m-2$ trivalent vertices on the line, labeled by $1', 2', \cdots, (m-2)'$ in order from

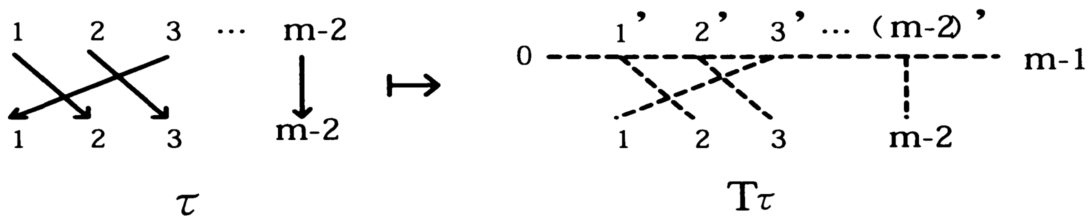


Figure 6.3: The definition of T_τ

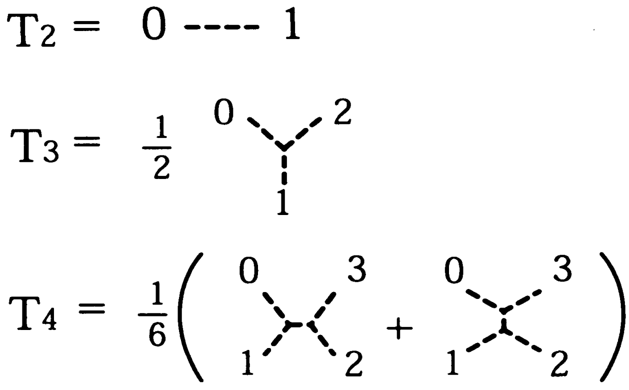


Figure 6.4: Simple cases of T_m

the end 0 to the end $m-1$. For each i, we connect i' and $\tau(i)$ by a chord. We put T_τ to be the uni-trivalent graph obtained in this way.

Further we define $T_m \in \mathcal{A}(m)$ by

$$T_m = \sum_{\tau \in \mathfrak{S}_{m-2}} \frac{(-1)^{r(\tau)}}{(m-1)\binom{m-2}{r(\tau)}} T_\tau,$$

where we denote by $r(\tau)$ the number of k satisfying $\tau(k) > \tau(k+1)$, and $\binom{m-2}{r(\tau)}$ the binomial coefficient. We show some simple cases in Figure 6.4; note that we use the IHX relation to obtain T_4 in the figure.

We have following two propositions, proved in [27]. We omit the proofs of them.

Proposition 6.8 ([27]). The chord diagram T_m is symmetric by the action of the dihedral group given by

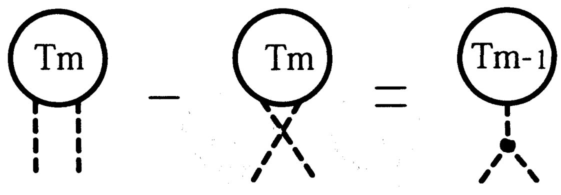


Figure 6.5: T_m satisfies a relation link the STU relation

(1) T_m is invariant by the rotations.

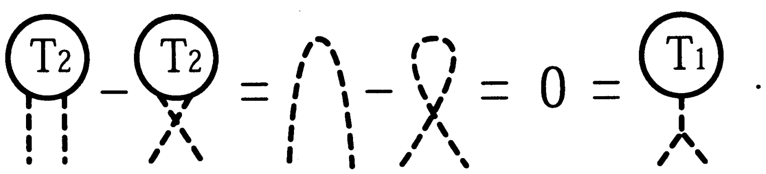
(2) Each inversion takes T_m to $(-1)^m T_m$.

As for (2), the factor $(-1)^m$ is due to reversing vertex-orientation of T_m; recall the AS relation, though T_m itself is symmetric also by the inversions as a linear sum of graphs.

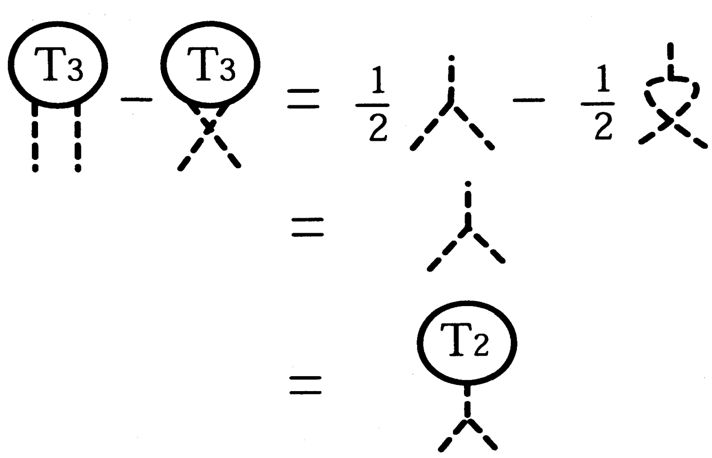
Proposition 6.9 ([27]). The difference between T_m and the element of $\mathcal{A}(m)$ obtained by changing any adjacent two univalent vertices is equal to T_{m-1} with one extra trivalent vertex as shown in Figure 6.5.

Instead of proving Proposition 6.9, we check the formula shown in Figure 6.5 for $m = 2, 3, 4$, by concrete calculations. For the complete proof of the proposition, see [27].

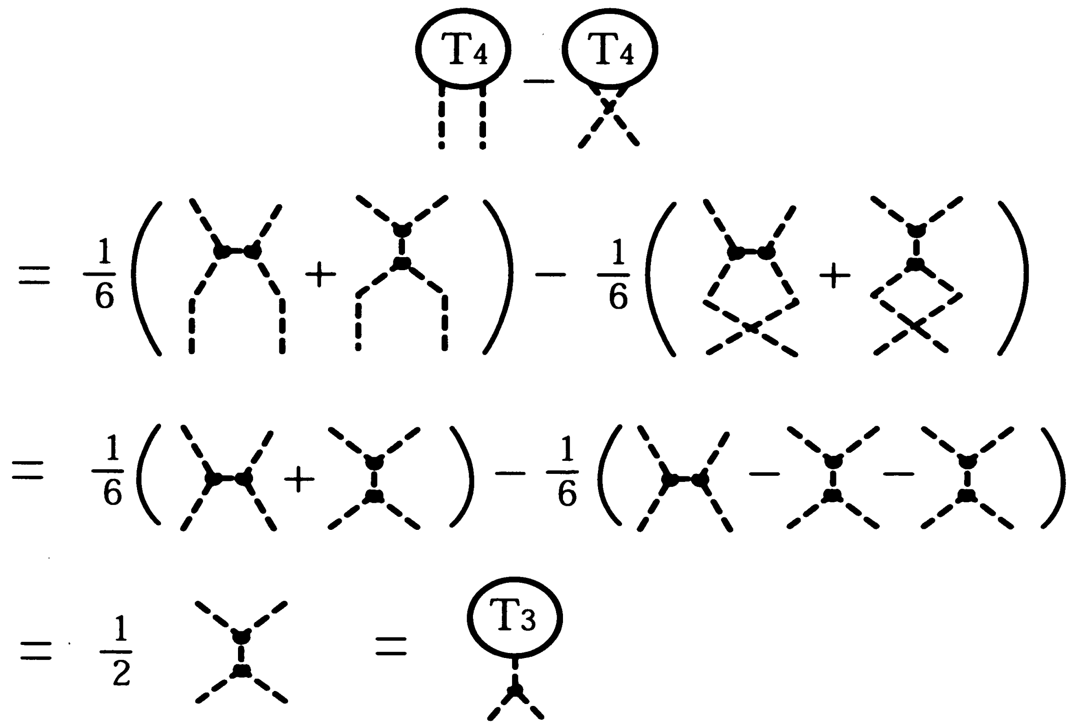
For $m = 2$, we have



For $m = 3$, we have



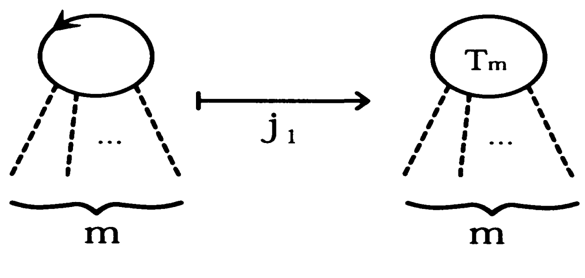
For $m = 4$, we have



Let $\mathring{\mathcal{A}}(X)$ be the space of chord diagrams with support X including trivial dashed loops, while we do not allow dashed loops in $\mathcal{A}(X)$. We define

$$j_1 : \mathcal{A}(\sqcup^l S^1) \longrightarrow \mathring{\mathcal{A}}(\phi)$$

to be the linear map replacing a solid circle with m univalent vertices by T_m as



This is well defined by Propositions 6.8 and 6.9.

Remark 6.10. If the solid circle has the opposite orientation, we replace it by reversed T_m; it is equal to $(-1)^m T_m$ by Proposition 6.9.

Further we define the map $j_n : \mathcal{A}(\sqcup^l S^1) \to \mathring{\mathcal{A}}(\phi)$ as follows. We consider the map $\mathcal{A}(\sqcup^l S^1) \longrightarrow \mathcal{A}(\sqcup^{nl} S^1)$ obtained by acting Δ to each S^1 of $\mathcal{A}(\sqcup^l S^1)$, $n-1$ times. We define j_n to be $1/n!$ times the composition of the above map and j_1.

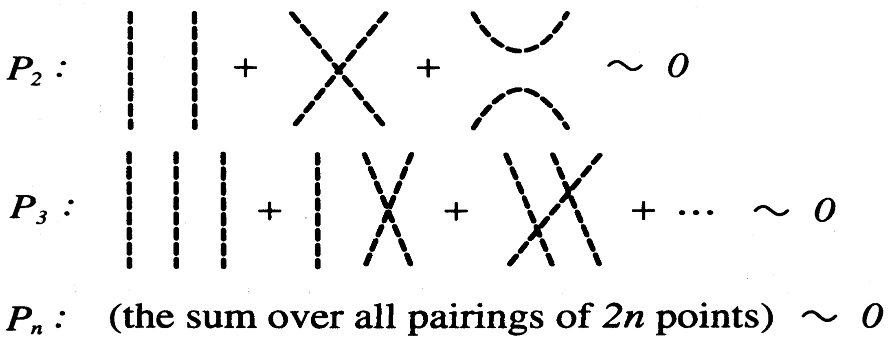


Figure 6.6: Definition of the relations P_n

Remark 6.11. For a chord diagram D on $\sqcup^l S^1$ which has less than $2n$ univalent vertices on a component S^1, we have $j_n(D) = 0$ by $T_0 = T_1 = 0$ and the definition of Δ.

6.3 Relation P_n

In this section we introduce a series of relations P_n in $\mathring{\mathcal{A}}(X)$. These relations will lead to the invariance of $\check{Z}(L)$ under KII move.

Let X be a compact oriented 1-manifold. For any integer $n > 1$, the relations P_n in $\mathring{\mathcal{A}}(X)$ is defined as follows. Let P_2 be the equivalence relation shown in the first formula in Figure 6.6. The left hand side of the first formula in the figure is the sum over all pairings of 4 points. Similarly we define the equivalence relation P_n so that the sum over all pairings of $2n$ points is equivalent to zero.

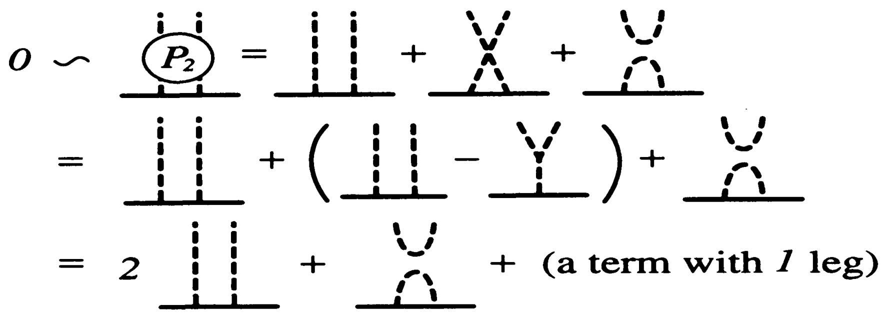
Before proving the invariance of $\check{Z}(L)$ in Proposition 6.13 below, we prepare the following lemma.

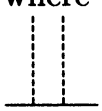
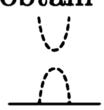
Lemma 6.12 ([27, Lemma 3.1]). Let C be a component of X and n a positive integer. Then, with the equivalence relation P_{n+1}, any chord diagram is equivalent to a linear sum of chord diagrams each of which has at most $2n$ univalent vertices on C.

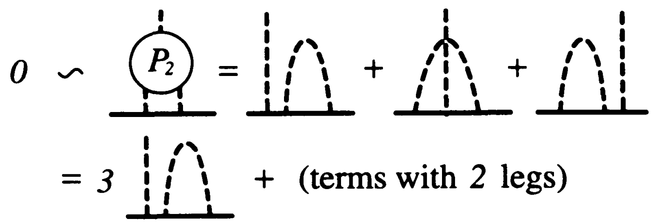
Proof. Let D be a chord diagram on X which has l univalent vertices on C. We prove this lemma by induction on l. It is sufficient to show that, if $l > 2n$, then

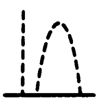
D is equivalent, modulo P_{n+1}, to a linear sum of chord diagrams, each of which has at most $l-1$ univalent vertices on C. In the following of this proof, we say that a circle S^1 in a chord diagram has l *legs* if the number of the univalent vertices on C.

The case $n = 1$. Suppose that D has more than 2 legs. We show that D is equivalent, modulo P_2, to a linear sum of chord diagrams with at most 2 legs. We have

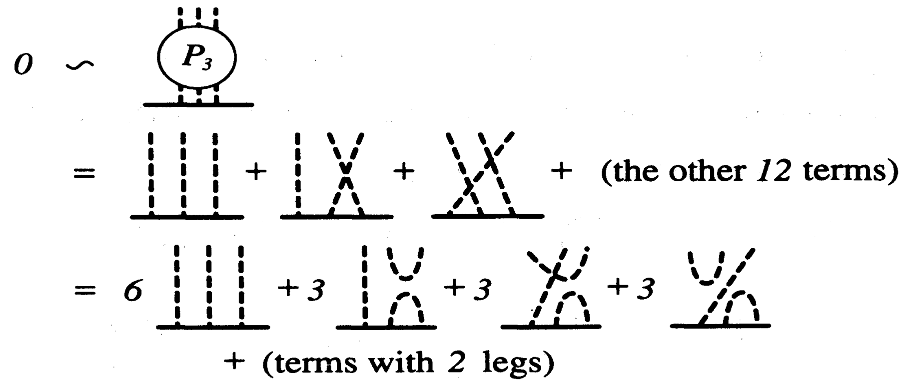


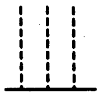
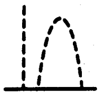
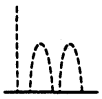
where we obtain the second equality by the STU relation. Hence we reduce  to  modulo a term with 1 leg. Further we have

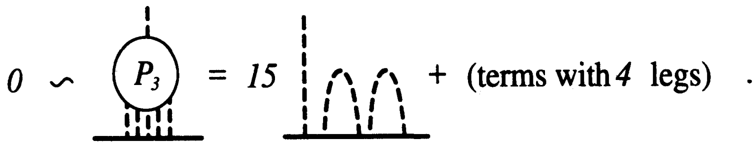


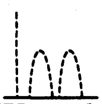
by using the STU relation. Hence we can reduce  to terms with less legs. Therefore we can reduce D to terms with less legs, completing this case.

The case $n = 2$. Suppose that D has more than 4 legs. It is sufficient to show that D is equivalent, modulo P_3, to a linear sum of chord diagrams with at most 4 legs. We have



We reduce  to  modulo terms with 2 legs. So we reduce D to terms including . Further we have



Hence we reduce  to terms with less legs, completing this case.

The case $n \geq 3$. We reduce any chord diagram with more than $2n$ legs to terms with less legs modulo P_{n+1}, in the same way as above. □

By the above lemma, we have the following proposition.

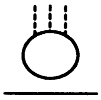
Proposition 6.13 ([27, Proposition 3.1]). Let L be any oriented framed link with l components and n any positive integer. Then the equivalence class $[\check{Z}(L)]$ including $\check{Z}(L)$ in $\overset{\circ}{\mathcal{A}}(\sqcup^l S^1)/L_{<2n}, P_{n+1}$ is invariant under Kirby move II, where we denote by $L_{<2n}$ the equivalence relation such that any chord diagram including a solid circle with less than $2n$ univalent vertices is equivalent to zero.

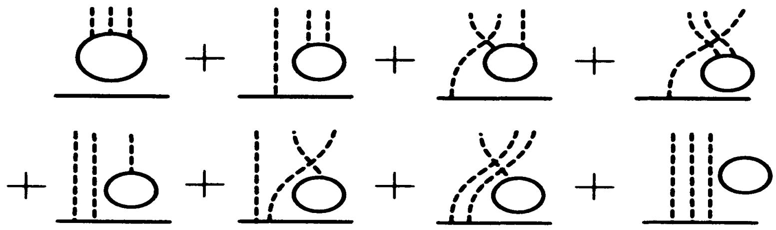
Proof. We show the proof for the case $n = 1$; for general case, see [27].

Let D be the chord diagram in the left picture in Figure 6.1, C the closed solid component of D in the figure. KII move takes D to the right picture in the figure; we denote it by D' in this proof. Let m be the number of the univalent vertices of D on C. We show that D and D' are equivalent modulo P_2 and $L_{<2}$ for each m.

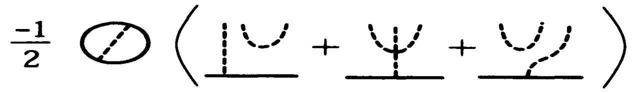
If $m = 0$ or 1, then both of D and D' are equivalent to 0 by $L_{<2}$.

If $m = 2$, then D' is a sum of D and chord diagrams with 0 or 1 univalent vertex on C; the latter vanishes modulo $L_{<2}$.

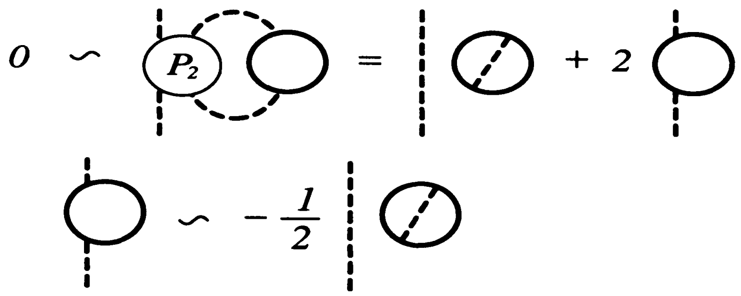
If $m = 3$, then, for $D =$  , the linear sum D' is equal to



where the first term is equal to D and the four terms in the second line vanish by $L_{<2}$. The remaining three terms are equivalent to

 (6.10)

where, to obtain the formula, we use the following relation derived from P_2.



Further (6.10) vanishes by P_2, completing this case.

If $m > 3$, we reduce the proof to the case $m \leq 3$ by decreasing univalent vertices on C by Lemma 6.12. □

Remark 6.14. The formula (6.10) is a reason why the relation P_2 was introduced in [23]. The relation P_{n+1} is also derived in such a way, starting from the working hypothesis $L_{<2n}$.

Let X be a compact oriented 1-manifold and D a chord diagram on X. Recall that the degree of D is half the number of univalent and trivalent vertices in

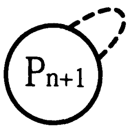
D. Let $D_{>n}$ denote the relation in $\mathring{\mathcal{A}}(X)$ such that terms of degree $> n$ are equivalent to 0. Then the following lemma holds.

Lemma 6.15. The identity map on $\mathcal{A}(\phi)$ induces the following isomorphism between the quotient spaces

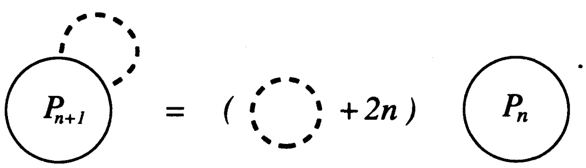
$$\mathcal{A}(\phi)/D_{>n} \longrightarrow \mathring{\mathcal{A}}(\phi)/D_{>n}, P_{n+1}, O_n,$$

where O_n denotes the equivalence relation such that a trivial dashed loop is equivalent to $-2n$.

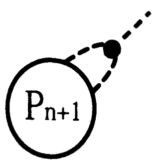
Proof. We also denote by P_{n+1} the formula defining the relation P_{n+1}. Since a dashed loop can be removed, we have a natural isomorphism between $\mathring{\mathcal{A}}(\phi)/D_{>n}, O_n$ and $\mathcal{A}(\phi)/D_{>n}$. Hence it is sufficient to show that $\mathring{\mathcal{A}}(\phi)/D_{>n}, O_n$ is isomorphic to $\mathring{\mathcal{A}}(\phi)/D_{>n}, P_{n+1}, O_n$. We show that an element of $\mathring{\mathcal{A}}(\phi)$ including P_{n+1} vanishes modulo $D_{>n}$ and O_n as follows, by dividing the shape outside of P_{n+1} into the following three cases.

Case 1. If there exists a chord without trivalent vertices connecting two ends of P_{n+1} as 

, then we have

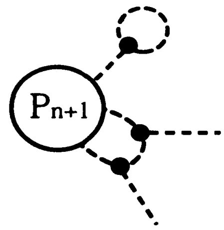


This vanishes by O_n.

Case 2. If there exists a chord with one trivalent vertex connecting two ends of P_{n+1} as 

, then it vanishes by the AS relation and the interior symmetry of P_{n+1}.

Case 3. Otherwise, we show the degree of a chord diagram including P_{n+1} is greater than n by counting argument as follows. For each end point of P_{n+1}, we associate at least one trivalent vertex as in the following picture.



Hence the number of trivalent vertices is at least $2n+2$. Therefore the chord diagram has degree $> n$, and it vanishes by $D_{>n}$. $\square$

Since $x \overset{P_{n+1}}{\sim} y$ implies $j_n(x) \overset{P_{n+1}}{\sim} j_n(y)$, the map $j_n : \mathcal{A}(\sqcup^l S^1) \to \mathring{\mathcal{A}}(\phi)$ induces the following map

$$\mathring{\mathcal{A}}(\sqcup^l S^1)/L_{<2n}, P_{n+1}, O_n \to \mathring{\mathcal{A}}(\phi)/P_{n+1}, O_n, \tag{6.11}$$

noting that $L_{<2n}$ vanishes by j_n, see Remark 6.11. Further we consider

$$\mathring{\mathcal{A}}(\phi)/P_{n+1}, O_n \longrightarrow \mathring{\mathcal{A}}(\phi)/D_{>n}, P_{n+1}, O_n \overset{=}{\longleftarrow} \mathcal{A}(\phi)/D_{>n}, \tag{6.12}$$

where the first map is the projection, and the second map is an isomorphism given in Lemma 6.15. We define the map

$$\iota_n : \mathcal{A}(\sqcup^l S^1) \longrightarrow \mathcal{A}(\phi)/D_{>n}$$

by putting $\iota_n(z)$ to be the image of $[z] \in \mathring{\mathcal{A}}(\sqcup^l S^1)/L_{<2n}, P_{n+1}, O_n$ by the composition of the maps (6.11) and (6.12). Since $[\check{Z}(L)]$ is invariant under Kirby move II, so is $\iota_n(\check{Z}(L))$.

Remark 6.16. By the same argument as in the proof of Lemma 6.12, any solid circle can be changed into Θ^n modulo P_{n+1} and $L_{<2n}$, where Θ^n denotes the left picture in Figure 6.7; we often depict it as in the right picture, noting that they are equivalent by the STU relation. Hence any element $z \in \mathcal{A}(\sqcup^l S^1)/P_{n+1}, L_{<2n}, O_n$ can be presented by a disjoint union of Θ^n and a dashed trivalent graph. The image $\iota_n(z)$ is equal to the dashed trivalent graph up to a scalar multiple; the proof is left to the reader.

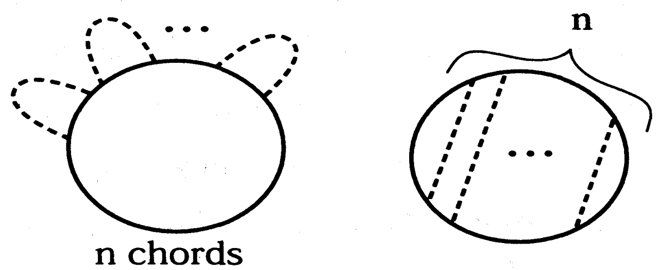


Figure 6.7: Two pictures of Θ^n

6.4 Invariants of 3-manifolds

In this section we construct a series of topological invariants $\Omega_n(M) \in \mathcal{A}(\phi)/D_{>n}$ of a 3-manifold M for each positive integer n and unify the series $\Omega_n(M)$ into an invariant $\Omega(M) \in \mathcal{A}(\phi)$; we call it the ***universal perturbative invariant*** of M.

We have an algebra structure in $\mathcal{A}(\phi)$ such that the product is induced from the disjoint union of chord diagrams.

Theorem 6.17 ([27, Theorem 3.7]). Let L be a framed link, and M the 3-manifold obtained from S^3 by Dehn surgery along L. Then

$$\Omega_n(M) = (\iota_n(\check{Z}(U_+))^{-\sigma_+}(\iota_n(\check{Z}(U_-))^{-\sigma_-}(\iota_n(\check{Z}(L))) \in \mathcal{A}(\phi)/D_{>n} \qquad (6.13)$$

is a topological invariant of M for any positive integer n, where $U_\pm$ denotes the trivial knot with ± 1 framing and $\sigma_\pm$ the number of positive and negative eigenvalues of the linking matrix of L.

Proof. It is sufficient to show the invertibility of $\iota_n(\check{Z}(U_\pm))$ in $\mathcal{A}(\phi)/D_{>n}$ and the invariance of the right hand side of (6.13) under the changes of the orientation of L and under Kirby moves I and II.

Invariance under changes of the orientation of L. Suppose L' is a link obtained from L by reversing the orientation of a component C of L. Then, by Proposition 6.1 (2), $\hat{Z}(L')$ differs from $\hat{Z}(L)$ by $S_{(C)}$, and so does for $\check{Z}(L)$ and $\check{Z}(L')$, since $S(\nu) = \nu$. Let D be a chord diagram in $\check{Z}(L)$ with m univalent vertices on C. Then the factor $(-1)^m$ appears by acting $S_{(C)}$. On the other hand, $(-1)^m$ also appears when we replace the circle with T_m, see Remark 6.10. Then both $(-1)^m$ cancel each other, and we obtain the invariance.

Invariance under KI move. We obtain the invariance, since the change of $\iota_n(\check{Z}(L))$ under the move cancels with the change of $\sigma_\pm$.

Invariance under KII move. We obtain the invariance by Proposition 6.13.

The invertibility of $\iota_n(\check{Z}(U_\pm))$. The map $\hat{\Delta}$ induces the following map

$$\begin{aligned}&\mathring{\mathcal{A}}(\sqcup^l S^1)/P_{n+1}, O_n, L_{<2n} \\ &\to \mathring{\mathcal{A}}(\sqcup^l S^1)/P_{n_1+1}, O_{n_1}, L_{<2n_1} \otimes \mathring{\mathcal{A}}(\sqcup^l S^1)/P_{n_2+1}, O_{n_2}, L_{<2n_2}\end{aligned}$$

between the quotient spaces; we denote it by $\hat{\Delta}_{n_1,n_2}$, where we put $n = n_1 + n_2$. We have

$$\hat{\Delta} \circ \iota_n = (\iota_{n_1} \otimes \iota_{n_2})\Delta_{n_1,n_2}, \tag{6.14}$$

by definition of ι_n. By using (6.14) $n-1$ times repeatedly, we have

$$\hat{\Delta}^{(n-1)} \circ \iota_n = \iota_1^{\otimes n} \circ \hat{\Delta}^{(n-1)}, \tag{6.15}$$

where we define $\hat{\Delta}^{(k)} : \mathcal{A}(X) \to \mathcal{A}(X)^{\otimes(k+1)}$ by the recursive formula $\hat{\Delta}^{(k)} = (\hat{\Delta}^{(k-1)} \otimes id) \circ \hat{\Delta}$ and $\hat{\Delta}^{(1)} = \hat{\Delta}$. More precisely, $\hat{\Delta}^{(n-1)}$ in the right hand side is the map induced between the quotient spaces. Since $\check{Z}(U_\pm)$ is group-like by Corollary 6.6, we have

$$\hat{\Delta}^{(n-1)}(\check{Z}(U_\pm)) = \check{Z}(U_\pm)^{\otimes n}.$$

Hence, by (6.15), we have

$$\hat{\Delta}^{(n)} \circ \iota_n(\check{Z}(U_\pm)) = \left(\iota_1 \check{Z}(U_\pm)\right)^{\otimes n}.$$

Since the constant term of $\iota_1 \check{Z}(U_\pm)$ is non-zero (see [23]), that of $\iota_n(\check{Z}(U_\pm))$ is also non-zero. Hence it is invertible. □

$\Omega_n(M)$ has the following properties.

Proposition 6.18. For any positive integers n_1 and n_2, we have

$$\hat{\Delta}_{n_1,n_2}(\Omega_{n_1+n_2}(M)) = \Omega_{n_1}(M) \otimes \Omega_{n_2}(M).$$

Proof. This is shown by (6.14) and the property that $\check{Z}(L)$ is group-like. □

Proposition 6.19. Let $\Omega_n(M)^{(d)}$ be the degree d part of $\Omega_n(M)$. Then

$$\Omega_n(M)^{(d)} = k^{n-d}\Omega_d(M)^{(d)},$$

where $k = \Omega_1(M)^{(0)} \in \mathbb{C}$.

Proof. Let $\hat{\varepsilon} : \mathcal{A}(\phi) \to \mathbb{C}$ be the map which picks up the degree 0 part of an element of $\mathcal{A}(\phi)$. Since $(\hat{\varepsilon} \otimes id) \circ \hat{\Delta}_{1,n-1}$ is equal to the projection

$$\mathcal{A}(\phi)/D_{>n} \longrightarrow \mathcal{A}(\phi)/D_{>n-1}$$

we have

$$(\hat{\varepsilon} \otimes id) \circ \hat{\Delta}_{1,n-1}(\Omega_n(M)) = \Omega_n(M)^{(n-1)} \in \mathcal{A}(\phi)/D_{>n-1}.$$

On the other hand, by Proposition 6.18, we have

$$(\hat{\varepsilon} \otimes id) \circ \hat{\Delta}_{1,n-1}(\Omega_n(M)) = \hat{\varepsilon}(\Omega_1(M)) \otimes \Omega_{n-1}(M) \in \mathbb{C} \otimes \mathcal{A}(\phi)/D_{>n-1}.$$

Comparing the above two formulas, we obtain

$$k\Omega_{n-1}(M) = \Omega_n(M)^{(\leq n-1)},$$

since $k = \hat{\varepsilon}(\Omega_1(M))$ by definition of $\hat{\varepsilon}$. By using the above formula repeatedly, we obtain the required formula. □

By Proposition 6.19, independent invariants obtained from the series $\Omega_n(M)$ are $\Omega_n(M)^{(n)}$, noting that $\Omega_n(M)^{(<n)}$ is obtained from $\Omega_{n-1}(M)$.

Definition 6.20. We define an invariant $\Omega(M)$ by

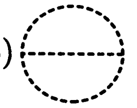
$$\Omega(M) = 1 + \sum_{d=1}^{\infty} \Omega_d(M)^{(d)} \in \mathcal{A}(\phi).$$

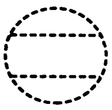
We call $\Omega(M)$ the *universal perturbative invariant* (or the *LMO invariant*) of M. For a rational homology 3-sphere M, we also define $\hat{\Omega}(M)$ by

$$\hat{\Omega}(M) = 1 + \sum_{d=1}^{\infty} \frac{\Omega_d(M)^{(d)}}{|H_1(M;\mathbb{Z})|^d} \in \mathcal{A}(\phi),$$

where $|\cdot|$ denotes the order of the set.

Example 6.21 ([27]). For the 3-manifold $M_{n,k}$ obtained from S^3 by integral surgery along $(2, n)$ torus knot with k framing, the universal perturbative invariant $\Omega(M_{n,k})$ is given by

$$\Omega(M_{n,k}) = \exp\Big(\frac{1}{48}(3n^2 - k^2 + 3k - 5)$$ 

$$+\frac{1}{2^7 \cdot 3^2}(12n^4 - 12kn^3 + 3k^2n^2 - 15n^2 + 12kn - 4k^2 + 4)$$ 

$$+(\text{terms of degree} \geq 3)\Big).$$

In general $\Omega(M)$ can be expressed as the exponential of a linear sum of connected chord diagrams; see [27].

7 Finite type invariants and the universal perturbative invariant

7.1 Finite type invariants of integral homology 3-spheres

Let M be an oriented integral homology 3-sphere, that is, $H_*(M;\mathbb{Z}) = H_*(S^3;\mathbb{Z})$. A framed link $\mathcal{L} = (L, f)$ is an unoriented link $L = \cup_{i=1}^{n} L_i$ in M with framing $f = (f_1, f_2, \cdots, f_n)$, with $f_i \in \mathbb{Z}$. We call $\mathcal{L}$ *algebraically split* if the linking number of L_i and L_j is zero for each pair (i, j). We call $\mathcal{L}$ *unit-framed* if all framings of $\mathcal{L}$ are ± 1. By $M_{\mathcal{L}}$ we denote the closed oriented 3-manifold obtained from M by Dehn surgery along $\mathcal{L}$ with respect to the framing.

Remark 7.1. Let M be an integral homology 3-sphere and $\mathcal{L}$ a framed link in M. Then the following two conditions are equivalent:

(1) $\mathcal{L}$ is algebraically split and unit-framed.

(2) $M_{\mathcal{L}'}$ is an integral homology 3-sphere for any sublink $\mathcal{L}'$ in $\mathcal{L}$.

Proof. If $\mathcal{L}$ is algebraically split and unit-framed, then we can easily verify the condition (2). Conversely, suppose that the condition (2) holds. Then we have