$v : \mathcal{K}_d \to \mathbb{C}$. Further this map induces a linear map $\mathcal{K}_d/\mathcal{K}_{d+1} \to \mathbb{C}$, since v is of degree d *i.e.*, $v|_{\mathcal{K}_{d+1}} = 0$. By composing φ, we obtain the weight system W_k of v. Further we have the inverse $[\hat{Z}]$ of φ by Theorem 4.5. These maps are written in a diagram as:

$$
\begin{tikzcd}
\mathcal{K} \supset & \mathcal{K}_d \arrow[rr, "v"] \arrow[d, "\text{projection}"'] & & \mathbb{C} \\
& \mathcal{K}_d/\mathcal{K}_{d+1} \arrow[urr] \arrow[rr, shift right, "{[\hat{Z}]}"'] & & \mathcal{A}(S^1)^{(d)} \arrow[u, "W_d"'] \arrow[ll, shift right, "\varphi"']
\end{tikzcd}
$$

where we obtain the commutativity of this diagram by Theorem 4.5 and the definition of the weight system. Thus we have $(v - W_d \circ [\hat{Z}])|_{\mathcal{K}_d} = 0$ in the diagram. Hence the map

$$v - W_d \circ \hat{[Z]} : \{\text{knots}\} \to \mathbb{C}$$

is a Vassiliev invariant of degree $d-1$. We put W_{d-1} to be the weight system of $v - W_d \circ [\hat{Z}]$.

For $k = d-2$, we put W_{d-2} to be the weight system of $v - (W_d + W_{d-1}) \circ [\hat{Z}]$; it is a Vassiliev invariant of degree $d-2$ by the same argument as above.

For $k = d-3, d-4, \cdots$, we can go on similarly for the rest. □

5 Vassiliev invariants and quantum invariants

We have seen the relations between quantum invariants and the modified Kontsevich invariant in Section 3, and between Vassiliev invariants and the modified Kontsevich invariant in Section 4. In this section, we see a relation between quantum invariants and Vassiliev invariants.

Theorem 5.1 **([4]).** For a framed knot K, the coefficient of h^d in $Q^{\mathfrak{g},R}(K)|_{q=e^h}$ is a Vassiliev invariant of degree d as an invariant of K.

Proof. In a construction of $Q^{\mathfrak{g},R}(K)|_{q=e^h}$, we associate positive and negative crossings with R-matrices $\mathcal{R}_+$ and $\mathcal{R}_-$, respectively. These two R-matrices coincide at $h=0$. Thereby $\mathcal{R}_+ - \mathcal{R}_-$ is a matrix whose entries are divisible by h in $\mathbb{C}[[h]]$, and this $\mathcal{R}_+ - \mathcal{R}_-$ corresponds to a singular point of a singular knot in the definition of Vassiliev invariants given in Section 4. If K is a singular knot with $d+1$ singular points, $Q^{\mathfrak{g},R}(K)$ is divisible by h^{d+1}. Hence the coefficient of h^d is 0 for such singular knots. $\square$

Theorem 5.2 ([33, Theorem 5.1]). The weight system of the Vassiliev invariant

$$K \mapsto \text{the coefficient of } h^d \text{ in } Q^{\mathfrak{g},R}(K)|_{q=e^h}$$

is equal to the weight system $W_{\mathfrak{g},R}$ derived from the substitution of $\mathfrak{g}$ and R into chord diagrams.

Proof. We give another proof than that in [33].

By results in Section 3, we have the following commutative diagram.

$$\begin{tikzcd} & \mathcal{K} \arrow[dl, "\hat{Z}"'] \arrow[dr, "Q^{\mathfrak{g},R}(\cdot)|_{q=e^h}"] \arrow[d, phantom, "\circlearrowleft"] & \\ \mathcal{A}(S^1) \arrow[rr, "\hat{W}_{\mathfrak{g},R}"'] & & \mathbb{C}[[h]] \end{tikzcd} \tag{5.1}$$

By restricting it to $\mathcal{K}_d$ and $\mathcal{K}_{d+1}$, we obtain the following commutative diagrams (5.2) and (5.3), respectively.

$$\begin{tikzcd} & \mathcal{K}_d \arrow[dl, "\hat{Z}"'] \arrow[dr, "Q^{\mathfrak{g},R}(\cdot)|_{q=e^h}"] \arrow[d, phantom, "\circlearrowleft"] & \\ \mathcal{A}(S^1)^{(\geq d)} \arrow[rr, "\hat{W}_{\mathfrak{g},R}"'] & & h^d \cdot \mathbb{C}[[h]], \end{tikzcd} \tag{5.2}$$

$$\begin{tikzcd} & \mathcal{K}_{d+1} \arrow[dl, "\hat{Z}"'] \arrow[dr, "Q^{\mathfrak{g},R}(\cdot)|_{q=e^h}"] \arrow[d, phantom, "\circlearrowleft"] & \\ \mathcal{A}(S^1)^{(\geq d+1)} \arrow[rr, "\hat{W}_{\mathfrak{g},R}"'] & & h^{d+1} \cdot \mathbb{C}[[h]]. \end{tikzcd} \tag{5.3}$$

By dividing the diagram (5.2) by (5.3), we obtain the following commutative diagram:

$$\begin{tikzcd}
& \mathcal{K}_d/\mathcal{K}_{d+1} \arrow[dl, shift left, "{[\hat{Z}]}"] \arrow[dr, "v"] \arrow[d, phantom, "\circlearrowleft"] & \\
\mathcal{A}(S^1)^{(d)} \arrow[ur, shift left, "\varphi"] \arrow[ur, phantom, bend left, "\circlearrowleft"] \arrow[rr, "W_{\mathfrak{g},R}"'] & & \mathbb{C}
\end{tikzcd}, \tag{5.4}$$

where v is the map obtained by dividing the right maps in (5.2) and (5.3) as

$$\mathcal{K}_d/\mathcal{K}_{d+1} \longrightarrow \frac{h^d \cdot \mathbb{C}[[h]]}{h^{d+1} \cdot \mathbb{C}[[h]]} \cong \mathbb{C}, \tag{5.5}$$

and φ is the map given in Proposition 4.3. Note that, for a singular knot $K \in \mathcal{K}_d$, the image $v(K)$ is equal to the coefficient of h^d in $Q^{\mathfrak{g},R}(K)|_{q=e^h}$.

Taking the dual of the above diagram (5.4), we have the following commutative diagram:

$$\begin{tikzcd}
& \dfrac{\{\text{the Vassiliev invariants of degree } d\}}{\{\text{the Vassiliev invariants of degree } d-1\}} \arrow[dl, "\varphi^*"'] \arrow[d, phantom, "\circlearrowleft"] & \\
(\mathcal{A}(S^1)^{(d)})^* & & \mathbb{C} \arrow[ul, "v^*"'] \arrow[ll, "{W_{\mathfrak{g},R}}^*"]
\end{tikzcd}$$

We compare two images of $1 \in \mathbb{C}$ in $(\mathcal{A}(S^1)^{(d)})^*$. On one hand, $v^*(1)$ is the Vassiliev invariant induced by the linear map (5.5). Hence the image of $v^*(1)$ in $(\mathcal{A}(S^1)^{(d)})^*$ is equal to the former weight system in the statement of the theorem. On the other hand, the image of $1 \in \mathbb{C}$ by ${W_{\mathfrak{g},R}}^*$ is the map $W_{\mathfrak{g},R}$ itself. The required equality is the equality of these two images of 1; it is derived from the commutativity of the above diagram. □

Summary for results in Sections 2 to 5. As mentioned in Section 0, we gave three kinds of invariants of knots; quantum invariants, finite type invariants (Vassiliev invariants) and the universal quantum invariant (the modified Kontsevich invariant), and showed the relations between them in Sections 2 to 5, see Figure 5.

The modified Kontsevich invariant has two universalities. One is the universality among quantum invariants; for each Lie algebra $\mathfrak{g}$ and each representation R of it, the quantum $(\mathfrak{g}, R)$ invariant $Q^{\mathfrak{g},R}$ is expressed as

$$Q^{\mathfrak{g},R} = \hat{W}_{\mathfrak{g},R} \circ \hat{Z}$$

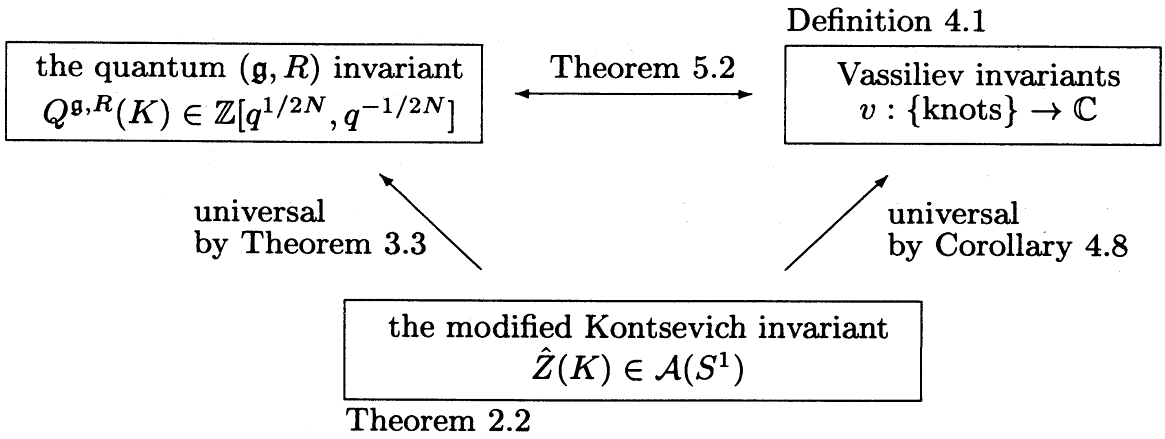

Figure 5.1: Three kinds of invariants of knots and the relations between them

with the weight system $\hat{W}_{\mathfrak{g},R}$ derived from the substitution of $\mathfrak{g}$ and R into chord diagrams. The other is the universality among Vassiliev invariants; each Vassiliev invariant v is expressed as

$$v = W \circ \hat{Z}$$

with some weight system W.

As a corollary of the two universalities, we obtain a relation between quantum invariants and Vassiliev invariants; the coefficients of the quantum $(\mathfrak{g}, R)$ invariant are Vassiliev invariants and their weight systems are equal to $W_{\mathfrak{g},R}$.

6 The universal perturbative invariant of 3-manifolds

So far we have dealt with invariants of knots and links. From now on we will consider invariants of 3-manifolds. The purpose of this section is to construct an invariant of 3-manifolds which has the universal property that the perturbative quantum invariants of 3-manifolds recover from it. So we call it the *universal perturbative invariant* of 3-manifolds.

6.1 Properties of $\hat{Z}(L)$

We will construct invariants from $\hat{Z}(L)$ in Section 6.4. To show the invariance under Kirby moves, we need the following properties of $\hat{Z}(L)$.