the following bijection,

$$\{\text{link diagrams with decompositions}\}/\text{moves of types 1, 2 and 3}$$
$$= \{\text{links}\}/\text{isotopy}.$$

Outline of the proof. We show outline of the proof in the following three steps.
Step1. We show

$$\begin{aligned}&\{\text{link diagrams with decompositions}\}/\text{moves of type1}\\ &= \{\text{link diagrams}\}/\text{restricted isotopy of } \mathbb{R}^2,\end{aligned} \tag{1.12}$$

where the isotopy in the right hand side is restricted such that the height function of $\sqcup^l S^1$,

$$\sqcup^l S^1 \to \mathbb{R}^2 \to \mathbb{R},$$

is preserved. Here the first map is an immersion of $\sqcup^l S^1$ expressing a link diagram, and the second map is the projection to the height coordinate. The formula (1.12) is reduced to the following formula,

$$\{\text{the trivial quasi-tangles with decompositions}\}/\text{moves of type1}$$
$$= \{\text{the trivial quasi-tangles}\},$$

which is shown by elementary calculations.
Step2. The following formula holds

$$\begin{aligned}&\{\text{link diagrams with decompositions}\}/\text{moves of types 1 and 2}\\ &= \{\text{link diagrams}\}/\text{isotopy of } \mathbb{R}^2.\end{aligned} \tag{1.13}$$

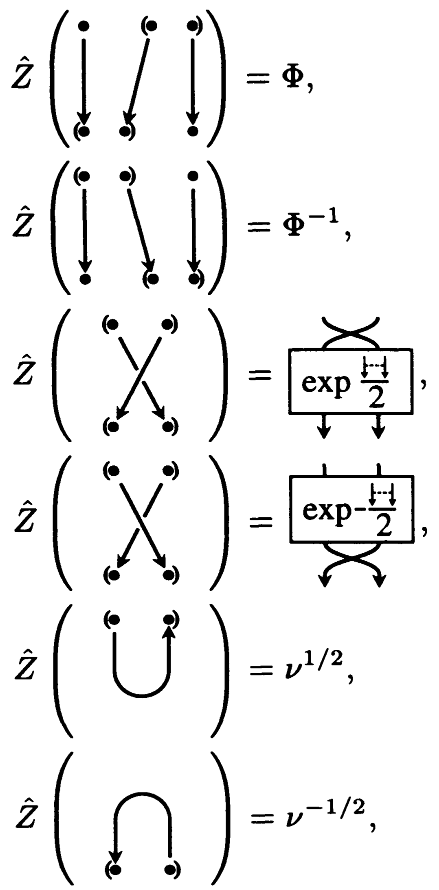
This equality is shown by (1.12) in Step 1 and results in [40].
Step 3. We obtain the required formula by (1.13) in Step 2 and Reidemeister's theorem, see [5]. □

2 The modified Kontsevich invariant

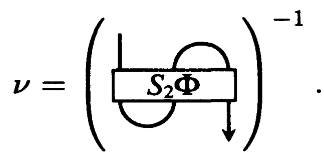
2.1 Definition of the modified Kontsevich invariant

Let L be an oriented framed link with l components in S^3. We fix a link diagram of L such that the framing of L is expressed by the blackboard framing of the link diagram. Further we fix a decomposition of the diagram into elementary quasi-tangle diagrams. We define the modified Kontsevich invariant $\hat{Z}(L) \in A(\sqcup^l S^1)$ by gluing $\hat{Z}(T)$ for elementary quasi-tangle diagram T, where we give $\hat{Z}(T)$ for each elementary quasi-tangle T as follows.

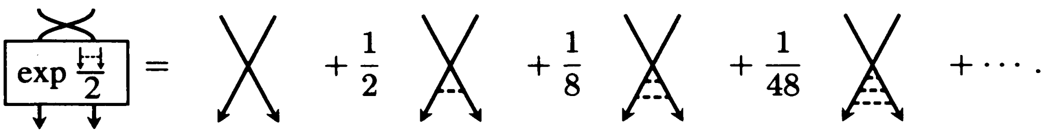
Firstly we give $\hat{Z}(T)$ for the following six elementary quasi-tangle diagrams,

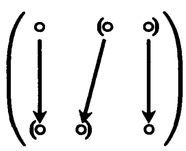
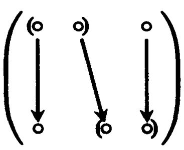
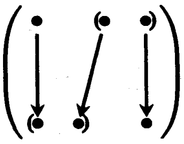
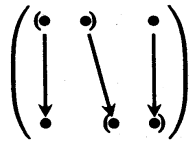


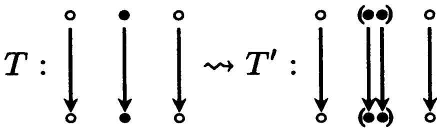
where we define Φ later in Section 2.2, and ν is defined by



Here S_2 denotes the antipode S, defined below, acting on the second string. Further the right hand side of the third formula means



Secondly we give $\hat{Z}$  and $\hat{Z}$  by applying the comultiplication Δ, defined below, to $\hat{Z}$  and $\hat{Z}$ ; recall that each  denotes a trivial elementary quasi-tangle. Let C be a component of a quasi-tangle diagram T, let C' be the corresponding component of X, let T' be the diagram obtained from T by taking 2-parallel along C as shown below.

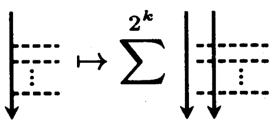


We set $\hat{Z}(T')$ by

$$\hat{Z}(T') = \Delta_{(C')}(\hat{Z}(T)),$$

where $\Delta_{(C')}$ is given by

$$\Delta_{(C')} : A(X) \to A(X')$$



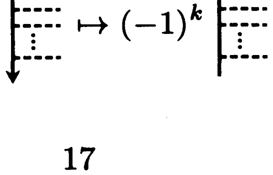
where X' is the 1-manifold obtained from X by replacing C' with two copies of C'. Here the sum in the right hand side runs over all possible 2^k configurations of connecting each dashed chord to either of the two solid vertical lines, where k is the number of univalent vertices on the vertical line in the left picture. For definition of $\Delta_{(C')}$ see also [3, 27].

Thirdly we give $\hat{Z}(T)$ for T with other orientations, by applying the antipode S, defined below, to $\hat{Z}(T)$ given above. Let T'' be the diagram with the opposite orientation on C. We set $\hat{Z}(T'')$ by

$$\hat{Z}(T'') = S_{(C')}(\hat{Z}(T)) \tag{2.1}$$

where $S_{(C')}$ is given by

$$S_{(C')} : A(X) \to A(X'')$$



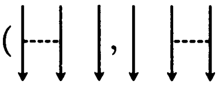
(2.2)

where X'' is X with opposite orientation on C', and k is the number of univalent vertices on the vertical line in the left picture. For definition of $S_{(C')}$ see also [3, 27]. When X is a set of vertical lines, for the ith component C_i of X, we often denote $S_{(C_i)}$ simply by S_i.

Lastly we define $\hat{Z}(L)$ for an oriented framed link L with a fixed diagram and a fixed decomposition into elementary quasi-tangle diagrams $\{T_i\}$, by gluing $\hat{Z}(T_i)$ defined as above.

2.2 Associator Φ

In the above definition of $\hat{Z}(L)$, we used an undefined element $\Phi \in \mathcal{A}(\downarrow\downarrow\downarrow)$. We define Φ to be a solution of the four properties (2.3) to (2.6) below, though the solution is not unique. We call it an ***associator***, since Φ corresponds to a quasi-tangle changing connection order of its end points. It is known [26] that $\hat{Z}(L)$ is uniquely determined for a link L, not depending on the choice of the solutions of the four properties, though $\hat{Z}(T)$ depends on the choice for a quasi-tangle T. When we consider $\hat{Z}(T)$ later in this lecture, we put Φ to be, what is called, Knizhnik-Zamolodchikov associator, whose concrete expression is given in [25] by

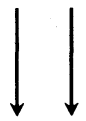
$\Phi = \varphi($  $)$

with a certain power series $\varphi(A,B)$ of non-commutative variables A and B defined as[2]

$$\varphi(A,B) = 1 + \frac{1}{24}[A,B] - \frac{\zeta(3)}{(2\pi\sqrt{-1})^3}([A,[A,B]] + [B,[A,B]]) + \cdots$$

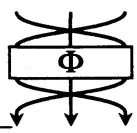
where $\zeta(3)$ is a special value of the zeta function $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$ and we put $[X,Y] = XY - YX$. Instead of the definition, we use the following 4 properties of Φ; we will prove the isotopy invariance of $\hat{Z}\,(L)$ based on the properties.

The first property is

$$\varepsilon_1\Phi = \varepsilon_2\Phi = \varepsilon_3\Phi =$$  . (2.3)

Here, for a component C of X, the map $\varepsilon_{(C)} : A(X) \to A(X - C)$ is defined as follows. For a chord diagram D, we put $\varepsilon_{(C)}(D) = 0$ if there exist chords on C, $\varepsilon_{(C)}(D) = D - C$ otherwise. When X is a set of vertical lines, we often denote by ε_i the map $\varepsilon_{(C_i)}$ for the ith component C_i of X.

The second property is

 $= \Phi^{-1}$. (2.4)

---

[2]We define the power series $\varphi(A,B)$, following [25], as follows. We define the multiple zeta function by

$$\zeta(a_1, a_2, \cdots, a_k) = \sum_{n_1 < n_2 < \cdots < n_k \in \mathbf{N}} n_1^{-a_1} n_2^{-a_2} \cdots n_k^{-a_k}.$$

For $\mathbf{a} = (a_1, \cdots, a_l)$ and $\mathbf{b} = (b_1, \cdots, b_l)$ we put

$$\eta(\mathbf{a},\mathbf{b}) = \zeta(\underbrace{1,1,\cdots,1}_{a_1-1}, b_1+1, \underbrace{1,1,\cdots,1}_{a_1-1}, b_2+1, \cdots, \underbrace{1,1,\cdots,1}_{a_l-1}, b_l+1),$$

$$|\mathbf{a}| = a_1 + a_2 + \cdots + a_l,$$

$$\binom{\mathbf{a}}{\mathbf{b}} = \binom{a_1}{b_1}\binom{a_2}{b_2}\cdots\binom{a_l}{b_l},$$

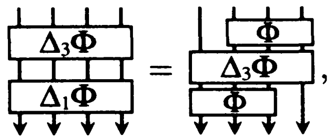
$$(A,B)^{(\mathbf{a},\mathbf{b})} = A^{a_1}B^{b_1}\cdots A^{a_l}B^{b_l}.$$

Further we define

$$\varphi(A,B) = 1 + \sum_{l=1}^{\infty} \sum_{\mathbf{a},\mathbf{b},\mathbf{p},\mathbf{q}} (-1)^{|\mathbf{b}|+|\mathbf{p}|} \eta(\mathbf{a}+\mathbf{p}, \mathbf{b}+\mathbf{q}) \binom{\mathbf{a}+\mathbf{p}}{\mathbf{p}} \binom{\mathbf{b}+\mathbf{q}}{\mathbf{q}} B^{|\mathbf{q}|}(A,B)^{(\mathbf{a},\mathbf{b})} A^{|\mathbf{p}|}$$

where the second sum is taken over $\mathbf{a},\mathbf{b},\mathbf{p},\mathbf{q}$ such that the sum of their length is l and entries of them are non-negative integers.

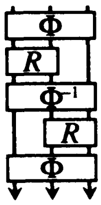
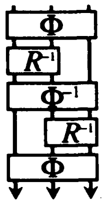
The third property is

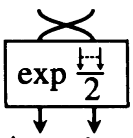


(2.5)

which is called the ***pentagon relation***.

The fourth property is

$\Delta_1 R =$  and $\Delta_1 R^{-1} =$  (2.6)

where we put $R =$ . These formulas are called the *hexagon relations*, and Δ_1 is the map Δ acting on the first string; the string from the upper right to the lower left.

The property (2.3) implies that the degree 0 part of Φ does not vanish. This guarantees the existence of Φ^{-1}, ν^{-1} and $\nu^{1/2}$; they are computed like computation of formal power series.

Remark 2.1. At the present, the properties (2.5) and (2.6) can be proved only by using Kontsevich integral. We expect that the existence of Φ satisfying the above four relations should be proved in a combinatorial way.

2.3 Isotopy invariance of $\hat{Z}(L)$

We show the following theorem in this section.

Theorem 2.2. $\hat{Z}(L)$ is an isotopy invariant of a framed oriented link L.

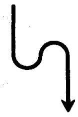
Outline of the proof. By Theorem 1.9 it is sufficient to show that $\hat{Z}(L)$ is invariant under the moves in the theorem, using the above properties of Φ. We show outlines of proofs of the invariance under the moves. The detailed proofs are left to the reader.

Invariance under (1.2) and (1.3). The invariance is derived from $\Phi \cdot \Phi^{-1} = \Phi^{-1} \cdot \Phi = 1$.

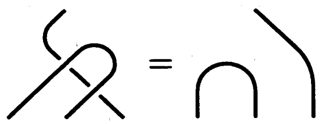
Invariance under (1.4). We obtain the invariance by the pentagon relation (2.5) of Φ.

Invariance under (1.5) and (1.7). The invariance is trivially obtained by definition of $\hat{Z}$.

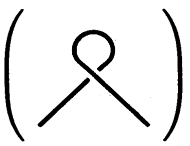
Invariance under (1.6) and (1.8). We obtain the invariance by definition of Δ.

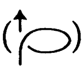
Invariance under (1.9). We obtain the invariance under the move  $=$  by definition of ν. Further, applying (2.4) to the above invariance, we obtain the invariance under the move  $=$ .

Invariance under (1.10) and (1.11). By definition of Δ and S, we have the invariance of $\hat{Z}$ under the following change.



Applying the first formula in (2.6) to the above invariance, we obtain the invariance under the move (1.10). The invariance under (1.11) is also obtained from the second formula in (2.6) similarly.

Invariance under RI. The invariance is obtained as follows. By calculating $\hat{Z}$ , we obtain

$\hat{Z}($  $) = \hat{Z}(\uparrow)\sharp e^{\Theta/2},$ (2.7)

see also [27], where "$\sharp$" denotes the connected sum of chord diagrams, and Θ denotes the chord diagram consisting of a solid circle together with a dashed chord, see Figure 2.1. Since $e^{\Theta/2}$ is central in $\mathcal{A}(\uparrow)$, we obtain the invariance[3] under RI.

[3] To be precise, we obtain the exact change of $\hat{Z}$ under the move RI. Since the change is central, we obtain the isotopy invariance of $\hat{Z}(L)$ for framed links L.

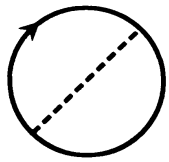


Figure 2.1: The chord diagram Θ

Invariance under RII. The invariance is derived from $R \cdot R^{-1} = R^{-1} \cdot R = 1$.
Invariance under RIII. We obtain the invariance by the hexagon relations (2.6).

□

3 The modified Kontsevich invariant and quantum invariants

In this section we show that quantum invariants recover from the modified Kontsevich invariant; we expect the recovery because of the following historical development from quantum invariants to the modified Kontsevich invariant.

T. Kohno [16] gave an expression of quantum invariants using an iterated integral solution of the Knizhnik-Zamolodchikov equation [15]. Based on Kohno's work, Drinfeld [7] led the universal version of the Knizhnik-Zamolodchikov equation; the solution of it consists of chord diagrams, not depending on a Lie algebra and its representation, and the ordinary solution recovers from the "universal" solution by substituting a Lie algebra and its representation to chord diagrams. After that, Kontsevich gave a definition of an invariant (the Kontsevich invariant) of knots using the universal solution written by the iterated integral. Further J. Murakami and Le [25] gave a combinatorial construction of the invariant (the modified Kontsevich invariant), modifying the invariant for framed links; we denote it by $\hat{Z}(L)$ for an oriented framed link L. Therefore quantum invariants should recover from the modified Kontsevich invariant by substituting a Lie algebra and its representation into chord diagrams.