1 Preliminaries

1.1 Chord diagrams

In this section we introduce definitions and some properties of chord diagrams.

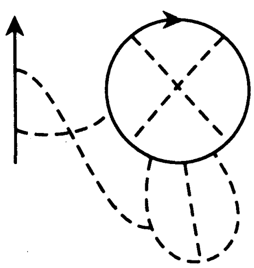
A *uni-trivalent* graph is a graph, every vertex of which is either univalent or trivalent, where a vertex of a graph is *univalent* (resp. *trivalent*) if there is one edge (resp. there are three edges) of the graph adjacent to the vertex. For a compact oriented 1-dimensional manifold X (possibly with boundary), a *chord diagram* on X is the manifold X together with a uni-trivalent graph whose univalent vertices are on X and whose trivalent vertices are vertex-oriented. Here a trivalent vertex is *vertex-oriented* if a cyclic order of three edges around the vertex is fixed. The *degree* of a chord diagram is half the number of univalent and trivalent vertices of the chord diagram. In figures, we draw X by solid lines and the graph by dashed lines, and each vertex-orientation is fixed in counterclockwise orientation in the plane. For definition of chord diagrams, see also [3, 27].

Definition 1.1. Let X be a compact oriented 1-manifold. We define the vector space $\mathcal{A}(X)$ by

$$\mathcal{A}(X) = \mathbb{C}\{\text{chord diagrams on } X\}/\text{AS}, \text{IHX}, \text{STU} \tag{1.1}$$

where the AS, IHX and STU relations are shown in Figure 1.1.

Let X be the disjoint union of an interval and a circle. We show an example of a chord diagram below.



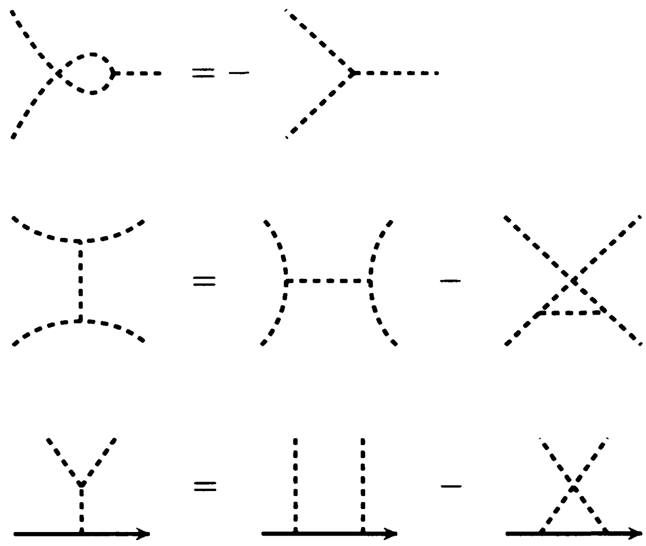
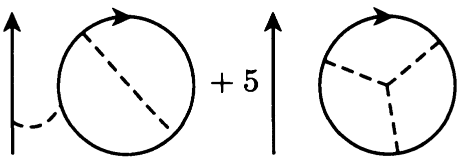


Figure 1.1: Definition of the AS, IHX and STU relations

Further we show an example of an element in $\mathcal{A}(X)$ below.



Proposition 1.2. $\mathcal{A}(X)$ satisfies the following properties.

(1) The vertically connecting operation gives an algebra structure in $\mathcal{A}(\downarrow)$. Further the algebra is commutative.

(2) We have an isomorphism $\mathcal{A}(\downarrow) \to \mathcal{A}(S^1)$ as a linear map, given by closing the interval, *i.e.*, by attaching the two end points of the interval to obtain a circle.

(3) Fixing a component C of X, we have an action of $\mathcal{A}(S^1)$ on $\mathcal{A}(X)$, given by taking connected sum of S^1 to C.

For proof of this proposition, see [3]. Here we illustrate (1) of the proposition by the following example. As shown in Figure 1.2 a dashed line with a univalent vertex on a solid interval can go through a chord diagram on the solid interval. Since a chord diagram near a solid interval consists of a disjoint union of dashed lines, two chord diagrams on a solid interval commutes together by the relation in Figure 1.2 as shown in Figure 1.3.

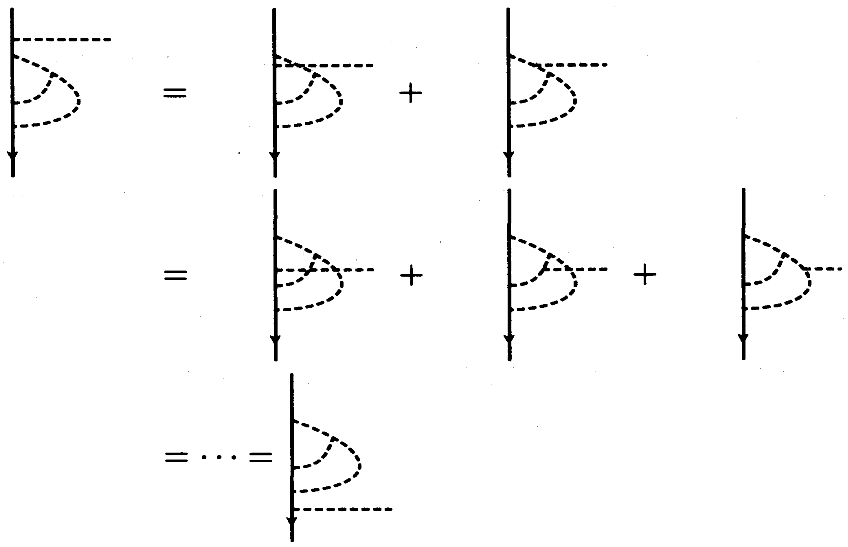


Figure 1.2: A dashed line goes through a chord diagram on a solid interval. The first and second equalities are derived from the STU and IHX relations respectively.

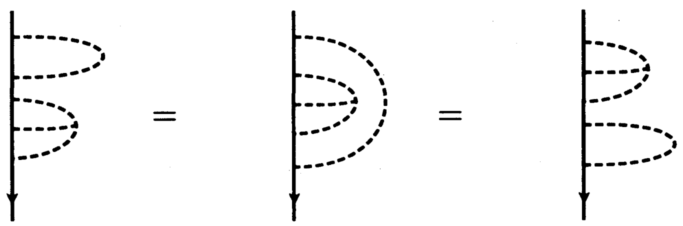


Figure 1.3: Two chord diagrams on a solid interval commute together. The equalities are derived from the relation shown in Figure 1.2

Remark 1.3. Generally, the vertically connecting operation also gives an algebra structure in $\mathcal{A}\Big(\underbrace{\downarrow\downarrow\cdots\downarrow}_{n}\Big)$. However it is non-commutative for $n > 1$, unlike the case $n = 1$.

1.2 Quasi-tangles

We give a combinatorial construction of the modified Kontsevich invariant in Section 2. The invariant will be defined for a quasi-tangle which is a tangle with the following extra structure defined as follows.

Definition 1.4. A *tangle* is a compact 1-manifold properly embedded in $\mathbb{R} \times \mathbb{R} \times [0,1]$ such that the boundary of the embedded 1-manifold is a discrete subset of $\{0\} \times \mathbb{R} \times \{0,1\}$. A ***quasi-tangle*** is a tangle with an order of connecting its end points in each of two lines $\{0\} \times \mathbb{R} \times \{0,1\}$ as shown in the top and bottom series of end points in Figure 1.4.

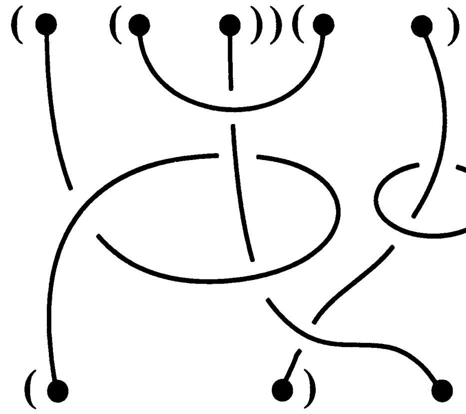


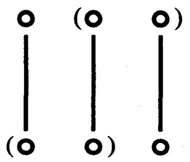
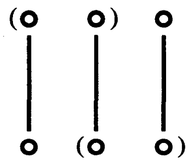
Figure 1.4: An example of a quasi-tangle

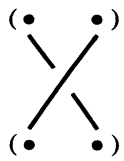
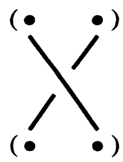
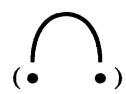
We show an example of a quasi-tangle in Figure 1.4.

Any quasi-tangle diagram can be decomposed into elementary quasi-tangle diagrams given as below, after deforming the diagram by isotopy of the plane if necessary. Here we mean by a *diagram* the projective image of a tangle in the plane.

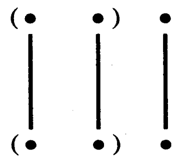
Definition 1.5. An *elementary quasi-tangle diagram* is one of the quasi-tangle diagrams given in the following (1), (2) and (3).

(1) The trivial quasi-tangle diagrams. A *trivial quasi-tangle diagram* is a trivial tangle diagram as a tangle diagram and has the same orders of connection of upper and lower end points, where a trivial tangle diagram is a tangle diagram given as $\{\text{points}\} \times [0,1] \subset \mathbb{R} \times [0,1]$, *i.e.*, a tangle diagram consisting only of vertical straight lines without crossings.

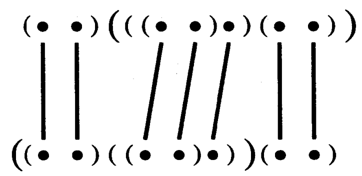
(2) The diagrams  or , where  denotes a trivial quasi-tangle. They are trivial tangles such that the orders of connections are changed between both sides.

(3) , ,  and .

We show an example of trivial quasi-tangles below.



Further we show an example of (2) below.



The following lemma can be obtained easily.

Lemma 1.6. Any quasi-tangle diagram, in particular link diagram, is isotopic (as diagrams in the plane) to a union of elementary quasi-tangle diagrams obtained by attaching them vertically along their end points and by taking disjoint union of them horizontally.

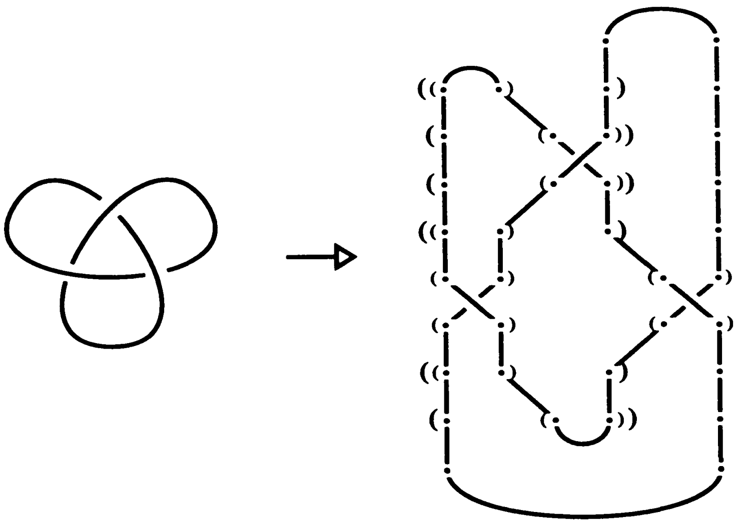
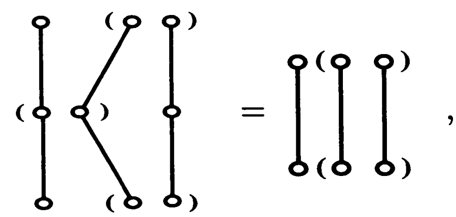


Figure 1.5: A decomposition of a knot diagram into elementary quasi-tangle diagrams

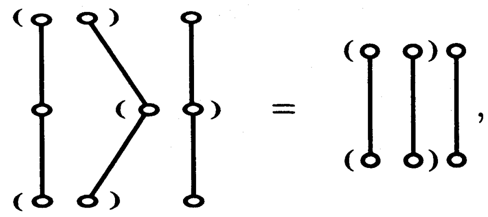
We call such expression of a quasi-tangle diagram by a union of elementary diagrams a *decomposition* of a quasi-tangle diagrams into elementary diagrams. We show an example of a decomposition of a knot diagram in Figure 1.5.

Moves for quasi-tangle diagrams with decompositions. There are many ways to decompose a link diagram into quasi-tangle diagrams in general. We introduce the following moves for letting the ways be related to each other. There are the following three types for the moves.

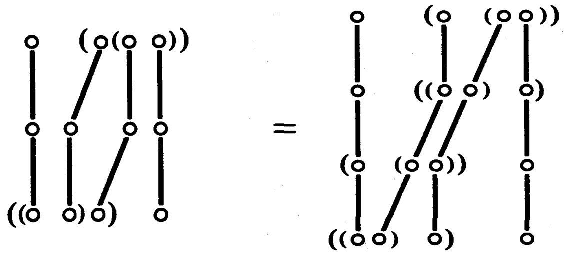
Moves of type 1 are the following (1.2) to (1.6),



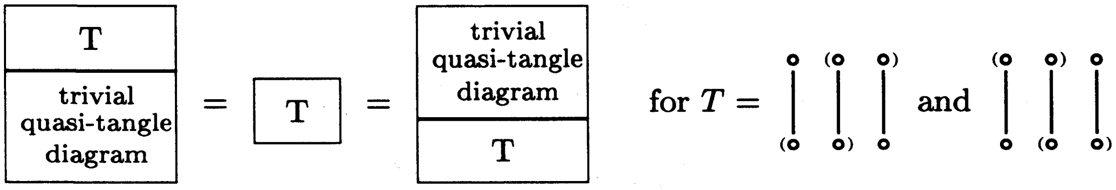
(1.2)



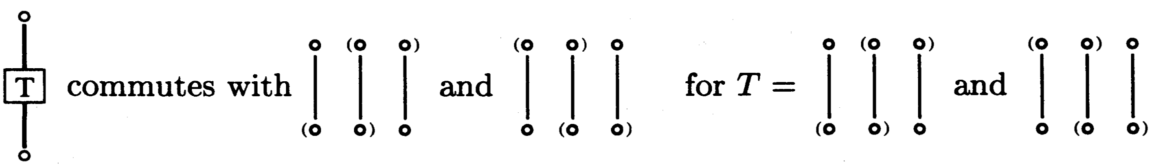
(1.3)



(1.4)

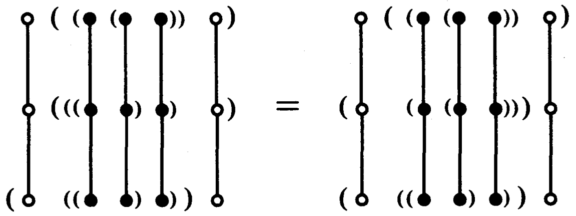


(1.5)



(1.6)

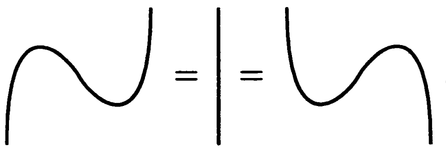
where we call the move (1.4) the ***pentagon relation***. We show an example of the move (1.6) below.



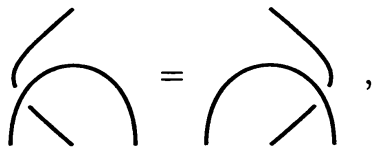
Moves of type 2 are the following (1.7) to (1.11),

(1.5) for the other elementary quasi-tangle diagrams T, (1.7)

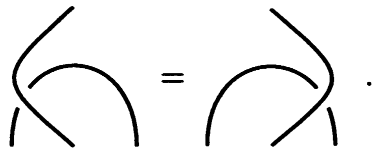
(1.6) for the other elementary quasi-tangle diagrams T, (1.8)



(1.9)

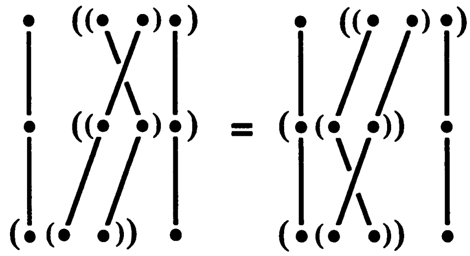


(1.10)



(1.11)

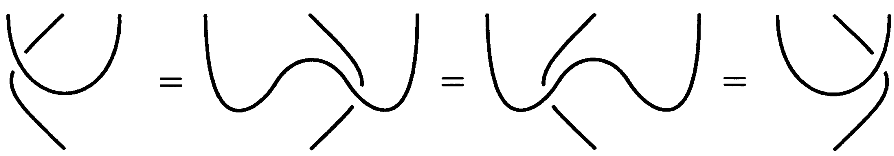
We show an example of the move (1.8) below.



Remark 1.7. As for the latter three moves (1.9) to (1.11), we fix quasi-tangle decompositions in both sides in any way. Note that all fixing are equivalent to each other modulo moves of type 1.

Moves of type 3 are Reidemeister moves RI, RII and RIII; for example see [5] for the definition of the moves.

Remark 1.8. The moves (1.10) and (1.11) are moves such that a vertical path moves over a maximal critical point. We realize the moves such that a vertical path moves over a minimal critical point by moves of types 1 and 2 as follows.



Theorem 1.9. A link diagram determines a link. This correspondence induces

the following bijection,

$$\{\text{link diagrams with decompositions}\}/\text{moves of types 1, 2 and 3}$$
$$= \{\text{links}\}/\text{isotopy}.$$

Outline of the proof. We show outline of the proof in the following three steps.
Step1. We show

$$\begin{aligned}&\{\text{link diagrams with decompositions}\}/\text{moves of type1}\\&= \{\text{link diagrams}\}/\text{restricted isotopy of } \mathbb{R}^2,\end{aligned} \tag{1.12}$$

where the isotopy in the right hand side is restricted such that the height function of $\sqcup^l S^1$,

$$\sqcup^l S^1 \to \mathbb{R}^2 \to \mathbb{R},$$

is preserved. Here the first map is an immersion of $\sqcup^l S^1$ expressing a link diagram, and the second map is the projection to the height coordinate. The formula (1.12) is reduced to the following formula,

$$\{\text{the trivial quasi-tangles with decompositions}\}/\text{moves of type1}$$
$$= \{\text{the trivial quasi-tangles}\},$$

which is shown by elementary calculations.
Step2. The following formula holds

$$\begin{aligned}&\{\text{link diagrams with decompositions}\}/\text{moves of types 1 and 2}\\&= \{\text{link diagrams}\}/\text{isotopy of } \mathbb{R}^2.\end{aligned} \tag{1.13}$$

This equality is shown by (1.12) in Step 1 and results in [40].
Step 3. We obtain the required formula by (1.13) in Step 2 and Reidemeister's theorem, see [5]. □

2 The modified Kontsevich invariant

2.1 Definition of the modified Kontsevich invariant

Let L be an oriented framed link with l components in S^3. We fix a link diagram of L such that the framing of L is expressed by the blackboard framing of the link