0 Introduction

Discovery of the Jones polynomial[1] [10] was the beginning of the series of discoveries of new many invariants of knots (and links), known as quantum invariants at the present. After the Jones polynomial, Turaev [39] defined link invariants derived from solutions of the Yang-Baxter equation obtained from representations of quantum groups, and Kirillov-Reshetikhin [14] constructed link invariants directly by using representations of the quantum group $U_q(sl_2)$. Further it became known that, for each Lie algebra $\mathfrak{g}$ and each representation R of it, there exists an invariant of links derived from the quantum group $U_q(\mathfrak{g})$ of $\mathfrak{g}$ and the representation of the quantum group obtained by deforming R. We call the invariant the *quantum* $(\mathfrak{g}, R)$ *invariant* of links.

In 1989, Witten proposed his famous formula in [42], written by using a path integral based on Chern-Simons gauge theory. The integral is over the infinite dimensional set of G connections on a 3-manifold, where G is a fixed Lie group. In particular, when the 3-manifold is S^3 including a link, his formula expresses a quantum invariant of the link. Though nobody gives a geometric regularization of the formula yet, the formula gave us a new viewpoint to understand quantum invariants. Further the formula also gave a predict of existence of an invariant of 3-manifolds for each Lie group G. We call the invariant the *quantum* G *invariant* of 3-manifolds.

In a combinatorial viewpoint, also as predicted by Witten's formula, the quantum G invariant of a 3-manifold should be obtained as a linear sum of quantum invariants of a link in S^3 such that the 3-manifold is obtained by Dehn surgery along the link. Along the idea, Reshetikhin and Turaev [34] gave a rigorous construction of the quantum $SU(2)$ invariant of 3-manifolds. It was a new invariant of 3-manifolds obtained by a new construction, different from "classical invariants" such as homology groups or the fundamental group. Though the construction is not a natural geometric realization of Witten's idea,

[1] It is interpreted as the quantum (sl_2, V) invariant of knots later, where V is the vector representation of the Lie algebra sl_2.

it let us confident of the existence of rigorous constructions of the quantum G invariant for other Lie groups G. In fact, after the quantum $SU(2)$ invariant, many researchers gave rigorous constructions of the quantum G invariant of 3-manifolds by various combinatorial approaches [6, 12, 17, 18, 19, 28, 38, 41, 43].

1985	Jones	Jones polynomial
1988	Turaev	link invariants obtained from solutions of the Yang-Baxter equation
1989	Kirillov-Reshetikhin	link invariants derived from representations of $U_q(sl_2)$
⋮	many researchers	rigorous construction of quantum invariants of links
1989	Witten	proposal of the path integral formula for quantum invariants
1991	Reshetikhin-Turaev	the quantum $SU(2)$ invariant of 3-manifolds
⋮	many researchers	rigorous construction of quantum invariants of 3-manifolds

Table 1: Brief history of quantum invariants

Since there are many Lie groups and Lie algebras, we had obtained many quantum invariants of knots and 3-manifolds. The invariants often behave independently, *i.e.*, they might not have a linear order with respect to strength of the invariants. So our next aim is to assemble and control them. We have the following two approaches to do it.

(1) Characterize them with a common property.

(2) Unify them into an invariant.

By the first approach, we obtain the notion of finite type. Typical examples of finite type invariants are the coefficients of the quantum invariants. Finite type invariants of knots are known as Vassiliev invariants, see for example [3].

Finite type invariants of integral homology 3-spheres are defined in [30].

By the second approach, we obtain the universal quantum invariant whose value belongs to the space of chord diagrams. The quantum invariants are obtained from the universal invariant by substituting Lie algebras and their representations into chord diagrams. The universal quantum invariant of knots is known as Kontsevich invariant [20]; we study the modified one [26] in this lecture. Though the universal quantum invariant of 3-manifolds is not found yet, we have the universal invariant [27] among the perturbative quantum invariants of 3-manifolds.

Our aim in this lecture is to give rigorous definitions and basic properties of the 6 kinds of invariants; quantum invariants, finite type invariants and the universal quantum invariant of knots and 3-manifolds respectively. We also give formulations of the 6 relations between them. We show a rough picture including the topics in this lecture in Figure 0.1.

Our standpoint. In this lecture we often consider knots and 3-manifolds in a combinatorial viewpoint, *i.e.*, we often regard the set of knots as the set of equivalence classes of knot diagrams modulo Reidemeister moves RI, RII and RIII; for example see [5] for the definition of the moves, and regard the set of 3-manifolds as the set of equivalence classes of framed links in S^3 modulo Kirby moves KI and KII [13]. As formulative expressions, we have

$$\{\text{knots}\}/\text{isotopy} = \{\text{knot diagrams}\}/\text{isotopy of } \mathbb{R}^2, \text{RI, RII, RIII},$$

$$\{\text{3-manifolds}\}/\text{homeomorphic} = \{\text{framed links}\}/\text{isotopy, KI, KII}.$$

The above identifications are quite available when we consider invariants of knots and 3-manifolds, since, for our purpose, we often deal with the sets of knots and 3-manifolds, not each knot or each 3-manifold. In other words the above formulas give virtual definitions of knots and 3-manifolds in this lecture.

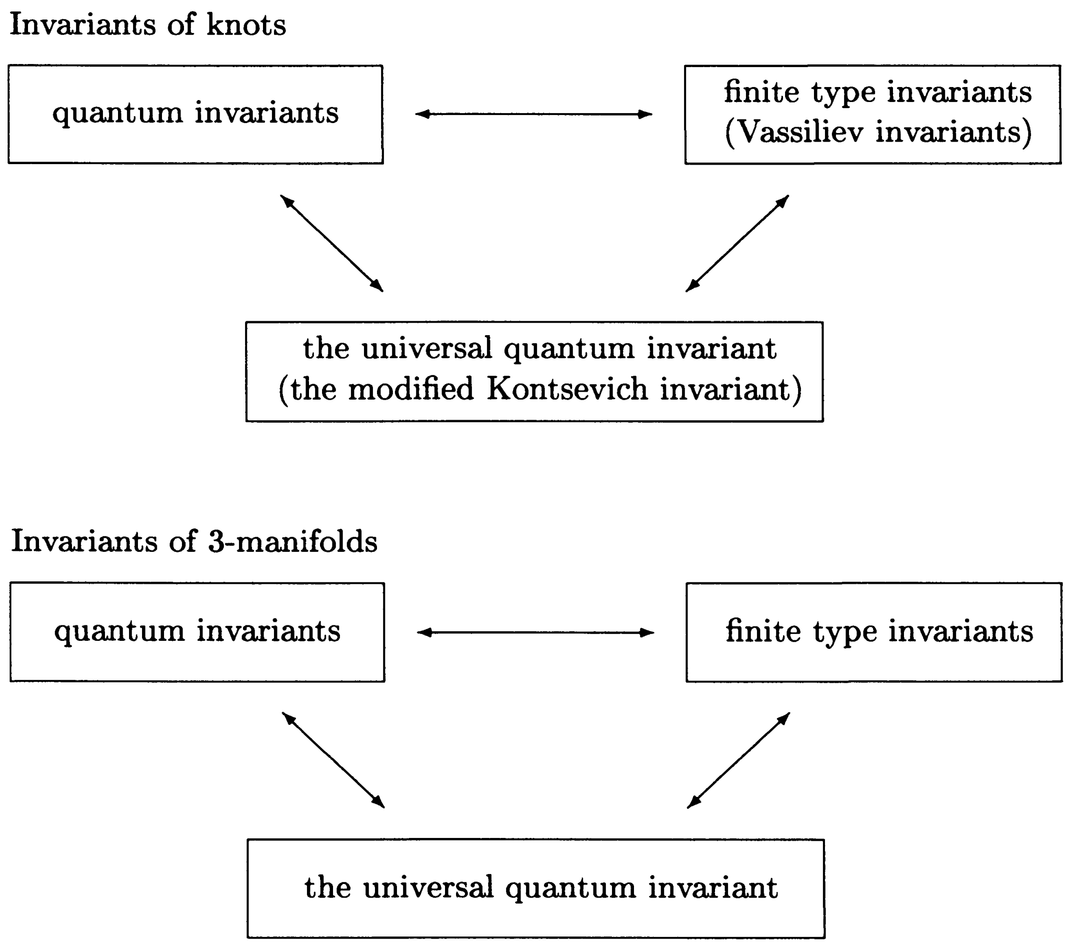

Figure 0.1: The 6 kinds of invariants. We expect the 6 relations between them