Proof-Theoretic Methods in Nonclassical Logic — an Introduction

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1 Introduction

This is an introduction to proof theory of nonclassical logic, which is directed at people who have just started the study of nonclassical logics, using proof-theoretic methods. In our paper, we will discuss only its proof theory based on sequent calculi. So, we will discuss mainly cut elimination and its consequences. As this is not an introduction to sequent systems themselves, we will assume a certain familiarity with standard sequent systems **LK** for the classical logic and **LJ** for the intuitionistic logic. When necessary, readers may consult e.g. Chapter 1 of *Proof Theory* [43] by Takeuti, Chapters 3 and 4 of *Basic Proof Theory* [45] by Troelstra and Schwichtenberg, and Chapter 1 of the present Memoir by M. Takahashi [41] to supplement our paper. Also, our intention is not to give an introduction of nonclassical logic, but to show how the standard proof-theoretic methods will work well in the study of nonclassical logic, and why certain modifications will be necessary in some cases. We will take only some modal logics and substructural logics as examples, and will give remarks on further applications. Notes at the end of each section include some indications for further reading.

An alternative approach to proof theory of nonclassical logic is given by using natural deduction systems. As it is well-known, natural deduction systems are closely related to sequent calculi. For instance, the normal form theorem in natural deduction systems corresponds to the cut elimination theorem in sequent calculi. So, it will be interesting to look for results and techniques on natural deduction systems which are counterparts of those on sequent calculi, given in the present paper.

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1.1 What is proof theory?

The main concern of proof theory is to study and analyze structures of proofs. A typical question in it is "what kind of proofs will a given formula A have, if it is provable?", or "is there any *standard* proof of A?". In proof theory, we want to derive some logical properties from the analysis of structures of proofs, by anticipating that these properties must be reflected in the structures of proofs. In most cases, the analysis will be based on combinatorial and constructive arguments. In this way, we can get sometimes much more information on the logical properties than with semantical methods, which will use set-theoretic notions like *models*, *interpretations* and *validity*.

On the other hand, the amount of information which we will get from these syntactic analyses depends highly on the way of formalizing a given logic. That is, in some formulations structures of proofs will reflect logical properties more sensitively than in others. For instance, Hilbert-style formal systems were popular in the first half of this century, which consist usually of many axiom schemes with a few rule of inference, including *modus ponens*. This formulation is in a sense quite flexible and is convenient for representing various logics in a uniform way. On the other hand, the arbitrariness of the choice of axioms of a given logic in Hilbert-style formulation tells us the lack of sensitivity to logical properties, although to some extent the proof theory based on Hilbert-style formal systems had been developed.

Then, G. Gentzen introduced both natural deduction systems and sequent calculi for the classical logic and the intuitionistic one in the middle of 1930s. The main advantage of these two kinds of systems comes from the fact that there is a *standard proof* of a given formula as long as it is provable. This fact is certified by *normal form theorem* in natural deduction systems and by *cut elimination theorem* in sequent calculi. Moreover, it turned out that "standard" proofs reflect some of logical properties quite well. For instance, one can derive the decidability, and moreover the decision procedure, for a propositional logic by using the existence of "standard" proofs. In this way, we can get much more information on logical properties from proofs in these formulations. This fact has led to the success of proof theory based on natural deduction systems and sequent calculi, in particular to consistency proofs of Peano arithmetic and its extensions.

1.2 Proof-theoretic methods in nonclassical logic

Our goal of the present paper is to show the usefulness (and the limitations) of proof-theoretic methods by taking appropriate examples from sequent calculi for nonclassical logic, in particular for modal and substructural logics. As will be explained later, proof-theoretic methods will give us deeper results than semantical ones do, in particular in their computational aspects. For instance, the cut elimination theorem for a propositional logic implies often its decidability, and the proof will give us an effective decision procedure (see Sections 4 and 5). Also, Maehara's method of proving Craig's interpolation theorem will give us an interpolant effectively (see Section 6).

The study of computational aspects of logics is particularly important in their applications. Modal logics cover epistemic logics, deontic logics, dynamic logics, temporal logics and so on, which will give theoretical foundations of artificial intelligence and software verification. Also, substructural logics include linear logic and Lambek calculus, which will give us a theoretical basis of concurrent processes (see [31]) and categorial grammar, respectively. This is the reason why we will focus on modal and substructural logics in the present paper.

We note here that most of automatic theorem provers at present are designed and implemented on the basis of Hilbert-style formal systems because of the *modularity* of these systems, while many of interactive theorem provers are based on Gentzen-style systems.

But, when we gain, we may also lose. As mentioned above, proof-theoretic methods work well mostly in the case where the existence of "standard" proofs is guaranteed. But as a matter of fact, this happens to hold only for a limited number of nonclassical logics. On the other hand, one may have much more interest in the study of a class of nonclassical logics, than in that of particular logics. In such a case, proof-theoretic methods will be useless. Semantical and algebraic methods will work well instead, although results obtained by them will usually lack constructive and computational features. For example, Maksimova proved in [27] that the interpolation theorem holds only for seven intermediate propositional logics, i.e. logics between the intuitionistic logic and the classical. We can use proof-theoretic methods to show that the interpolation theorem holds for each of these seven logics. But, when we show that the interpolation theorem doesn't hold for any other logics, there will be no chance of using these proof-theoretic methods. In this sense, we can say that proof-theoretic methods and semantical ones are complementary to each other. In fact, sometimes it happens that a general result shown by a semantical method was originally inspired by a result on particular logics proved by a proof-theoretic method.

The present paper consists of eight sections. In Section 2, we will introduce sequent calculi for logics discussed in this paper. Among others, sequent calculi for standard substructural logics will be given. As this is an introductory paper, extensions of sequent calculi like labelled deduction systems, display logics and systems for hypersequents will not be treated here. Cut elimination theorem will be discussed in Section 3. After a brief outline of the proof for **LJ**, we will indicate how we should modify the proof when we show the cut elimination theorem for substructural logics. An important consequence, and probably the most important consequence, of cut elimination theorem is the subformula property. In the end of Section 3, we will introduce Takano's idea of proving the subformula property of sequent calculi for some modal logics, for which the cut elimination theorem doesn't hold.

In Sections 4 and 5, the decision problem will be discussed. It will be shown that the contraction rule will play an essential role in decidability. In many cases, Craig's interpolation theorem can be derived from the cut elimination theorem by using Maehara's method. This is outlined in Section 6. Another application of the cut elimination theorem in the author's recent joint paper [30] with Naruse and Bayu Surarso gives a syntactic proof of Maksimova's principle of variable separation. The topic is outlined in Section 7. A short, concluding remark is given in the last section.

1.3 Notes

We will give here some basic literature on proof theory. Hilbert-Ackermann [20] developed the first, comprehensive study of proof theory based on a Hilbert-style formulation. In his paper [17] in 1935 Gentzen introduced both natural deduction systems and sequent calculi for the classical logic and the intuitionistic. The paper became the main source of studies on proof theory developed later. As for further information on natural deduction, see e.g. [37, 40, 45, 10]. Another way of formalizing logics, which is closely related to one using sequent calculi, is the tableau method. For the details, see e.g. [16]. As for recent developments of proof theory in general, consult [6].

2 Sequent Calculi for Nonclassical Logic

We will introduce several basic systems of sequent calculi discussed in our paper. First, we will introduce sequent calculi **LK** and **LJ** for the classical predicate logic and the intuitionistic one, respectively. Then, we will introduce some of sequent calculi for modal logics, by adding rules or initial sequents for the modal operator $\Box$, and also sequent calculi for substructural logics, by deleting some of *structural rules* either from **LK** or from **LJ**.

2.1 Sequent calculi LK and LJ

We assume here that the language $\mathcal{L}$ of **LK** and **LJ** consists of logical connectives $\wedge, \vee, \supset$ and $\neg$ and quantifiers $\forall$ and $\exists$. The first-order formulas of $\mathcal{L}$ are defined in the usual way. A *sequent* of **LK** is an expression of the form $A_1, \ldots, A_m \to B_1, \ldots, B_n$, with $m, n \geq 0$. As usual, Greek capital letters Σ, Λ, Γ etc. denote (finite, possibly empty) sequence of formulas. Suppose that a sequent $\Sigma \to \Pi$ is given. Then, any formula in Σ (Π) is called an *antecedent* (a *succedent*) of the sequent $\Sigma \to \Pi$.

In the following, we will introduce first a formulation of the sequent calculus **LK**, which is slightly different but not in an essential way from the usual one. Initial sequents of **LK** are sequents of the form $A \to A$. Rules of inference of **LK** consist of the following;

Structural rules:

Weakening rule:

$$\frac{\Gamma, \Sigma \to \Delta}{\Gamma, A, \Sigma \to \Delta}\ (w \to) \qquad \frac{\Gamma \to \Lambda, \Theta}{\Gamma \to \Lambda, A, \Theta}\ (\to w)$$

Contraction rule:

$$\frac{\Gamma, A, A, \Sigma \to \Delta}{\Gamma, A, \Sigma \to \Delta}\ (c \to) \qquad \frac{\Gamma \to \Lambda, A, A, \Theta}{\Gamma \to \Lambda, A, \Theta}\ (\to c)$$

Exchange rule:

$$\frac{\Gamma, A, B, \Sigma \to \Delta}{\Gamma, B, A, \Sigma \to \Delta}\ (e \to) \qquad \frac{\Gamma \to \Lambda, A, B, \Theta}{\Gamma \to \Lambda, B, A, \Theta}\ (\to e).$$

Cut rule:

$$\frac{\Gamma \to A, \Theta \quad \Sigma, A, \Pi \to \Delta}{\Sigma, \Gamma, \Pi \to \Delta, \Theta}$$

Rules for logical connectives:

$$\frac{\Gamma \to A, \Theta \quad \Pi, B, \Sigma \to \Delta}{\Pi, A \supset B, \Gamma, \Sigma \to \Delta, \Theta}\ (\supset\to) \qquad \frac{\Gamma, A \to B, \Theta}{\Gamma \to A \supset B, \Theta}\ (\to\supset)$$

$$\frac{\Gamma, A, \Sigma \to \Delta}{\Gamma, A \wedge B, \Sigma \to \Delta}\ (\wedge 1 \to) \qquad \frac{\Gamma, B, \Sigma \to \Delta}{\Gamma, A \wedge B, \Sigma \to \Delta}\ (\wedge 2 \to)$$

$$\frac{\Gamma \to \Lambda, A, \Theta \quad \Gamma \to \Lambda, B, \Theta}{\Gamma \to \Lambda, A \wedge B, \Theta}\ (\to \wedge)$$

$$\frac{\Gamma, A, \Sigma \to \Delta \quad \Gamma, B, \Sigma \to \Delta}{\Gamma, A \vee B, \Sigma \to \Delta}\ (\vee \to)$$

$$\frac{\Gamma \to \Lambda, A, \Theta}{\Gamma \to \Lambda, A \vee B, \Theta}\ (\to \vee 1) \qquad \frac{\Gamma \to \Lambda, B, \Theta}{\Gamma \to \Lambda, A \vee B, \Theta}\ (\to \vee 2)$$

$$\frac{\Gamma \to A, \Theta}{\neg A, \Gamma \to \Theta}\ (\neg \to) \qquad \frac{\Gamma, A \to \Theta}{\Gamma \to \neg A, \Theta}\ (\to \neg)$$

Rules for quantifiers:

$$\frac{\Gamma, A[t/x], \Sigma \to \Delta}{\Gamma, \forall x A, \Sigma \to \Delta}\ (\forall \to) \qquad \frac{\Gamma \to \Lambda, A[z/x], \Theta}{\Gamma \to \Lambda, \forall x A, \Theta}\ (\to \forall)$$

$$\frac{\Gamma, A[z/x], \Sigma \to \Delta}{\Gamma, \exists x A, \Sigma \to \Delta}\ (\exists \to) \qquad \frac{\Gamma \to \Lambda, A[t/x], \Theta}{\Gamma \to \Lambda, \exists x A, \Theta}\ (\to \exists).$$

Here, $A[z/x]$ ($A[t/x]$) is the formula obtained from A by replacing all free occurrence of x in A by an individual variable z (a term t, respectively), but avoiding the *clash* of variables. Also, in rules for quantifiers, t is an arbitrary term and z is an arbitrary individual variable not occurring in the lower sequent. (Usually, the cut rule is considered to be one of the structural rules. For convenience sake, we will not include it among the structural rules in our paper.)

Sequents of the sequent calculus **LJ** for the intuitionistic logic are expressions of the form $A_1, \ldots, A_m \rightarrow B$ where $m \geq 0$ and B may be empty. Initial sequents and rules of inference of **LJ** are obtained from those of **LK** in the above, first by deleting both $(\rightarrow c)$ and $(\rightarrow e)$ and then by assuming moreover that both Λ and Θ are empty and that Δ consists of at most one formula. *Proofs* and the *provability* of a given sequent in **LK** or **LJ** are defined in a usual way. As usual, we say that a formula A is provable in **LK** (**LJ**), if the sequent $\rightarrow A$ is provable in it. For formulas A and B, we say that A is *logically equivalent* to B in **LK** (**LJ**) when both $A \supset B$ and $B \supset A$ are provable in **LK** (**LJ**, respectively).

In the following, sometimes we will concentrate only on propositional logics. In such a case, **LK** and **LJ** mean the sequent calculi in the above, which deal only with sequents consisting of propositional formulas and which have only rules for logical connectives. To emphasize this, we call them, propositional **LK** and **LJ**.

2.2 Adding modal operators

We will discuss two types of nonclassical logic in this paper. Logics of the first type can be obtained from classical logic by adding new logical operators or connectives. Particularly interesting nonclassical logics of this type are *modal logics*. Modal logics usually have unary logical operators $\Box$ and $\Diamond$, called *modal operators*. There are many variations of modal logics. Some will contain many modal operators and some will be based on logics other than classical logic.

We will discuss here sequent calculi for some standard modal *propositional* logics as a typical example. In both of them, we assume that $\Diamond A$ is an abbreviation of $\neg\Box\neg A$. Thus, we suppose that our language $\mathcal{L}_\Box$ for modal logics consists of (the propositional part of) $\mathcal{L}$ with a unary operator $\Box$, which is usually read as "it is necessary" and called the *necessity* operator. Now, consider the following axiom schemes:

K: $\Box(A \supset B) \supset (\Box A \supset \Box B)$,
T: $\Box A \supset A$,
4: $\Box A \supset \Box\Box A$,
5: $\Diamond A \supset \Box\Diamond A$.

A Hilbert-style system for the modal logic **K** is obtained from any Hilbert-style system for the classical propositional logic (with *modus ponens* as its single rule of inference) by adding the axiom scheme K and the following *rule of necessitation*;

from A *infer* $\Box A$.

Hilbert-style systems for modal logics **KT**, **KT4** and **KT5** are obtained from **K** by adding *T*, *T* and *4*, and *T* and *5*, respectively. Traditionally, **KT4** and **KT5** are called **S4** and **S5**, respectively.

Now, we will introduce sequent calculi for these modal logics. A sequent calculus **GK** for **K** is obtained from **LK** by adding the following rule;

$$\frac{\Gamma \to A}{\Box\Gamma \to \Box A}\ (\Box).$$

Here, $\Box\Gamma$ denotes the sequence of formulas $\Box A_1, \ldots, \Box A_m$ when Γ is $A_1, \ldots, A_m$. Moreover, $\Box\Gamma$ is the empty sequence when Γ is empty. Consider moreover the following three rules;

$$\frac{A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}\ (\Box \to)$$

$$\frac{\Box\Gamma \to A}{\Box\Gamma \to \Box A}\ (\to \Box 1). \qquad \frac{\Box\Gamma \to \Box\Delta, A}{\Box\Gamma \to \Box\Delta, \Box A}\ (\to \Box 2).$$

Note here that the rule $(\Box)$ can be derived when we have both $(\Box \to)$ and $(\to \Box 1)$ and that $(\to \Box 1)$ is obtained from $(\to \Box 2)$ by assuming that Δ is empty. The sequent calclus **GKT** is obtained from **GK** by adding the rule $(\Box \to)$. The sequent calculi **GS4** and **GS5** are obtained from **LK** by adding the rules $(\Box \to)$ and $(\to \Box 1)$, and the rules $(\Box \to)$ and $(\to \Box 2)$, respectively. We can show the following.

Theorem 1 *For any formula A, A is provable in* **K** *if and only if* $\to A$ *is provable in* **GK**. *This holds also between* **KT** *and* **GKT**, *between* **S4** *and* **GS4**, *and between* **S5** *and* **GS5**.

2.3 Deleting structural rules

Nonclassical logics of the second type can be obtained either from classical logic or from intuitionistic logic, by deleting or restricting structural rules. They are called *substructural logics*. *Lambek calculus* for categorial grammar, *linear logic* introduced by Girard in 1987, *relevant logics*, *BCK logics* (i.e. logics without contraction rules) and *Łukasiewicz's many-valued logics* are examples of substructural logics. Lambek calculus has no structural rules and linear logic has only exchange rules. Relevant logics have usually the contraction rules, but lack weakening rules. Łukasiewicz's many-valued logics form a subclass of logics without contraction rules.

Our basic calculus **FL** (called the *full Lambek calculus*) is, roughly speaking, the system obtained from the the sequent calculus **LJ** by deleting all structural rules. We will also extend our language for the following reasons.

In the formulation of the classical and the intuitionistic logic, we will sometimes introduce propositional constants like $\top$ and $\bot$ to express the constantly *true* proposition and *false* proposition, respectively. When we have these constants, we will take the following sequents as the additional initial sequents:

1. $\Gamma \to \top$
2. $\Gamma, \bot, \Sigma \to \Delta$

where Γ, Σ and Δ may be empty. (We can dispense with $\top$, as it is logically equivalent to $\bot \supset \bot$.) It is easy to see that for any formula A which is provable in **LK** or **LJ**, A is logically equivalent to $\top$ in it. Note here that we need the weakening rule to show that $\top \supset A$ is provable, though $A \supset \top$ follows always from the above 1. We can also show by the help of the weakening rule that $\neg A$ is logically equivalent to $A \supset \bot$ in either of **LK** and **LJ**.

These facts suggest to us that we may need special considerations for logics without the weakening rule. To make the situation clearer, let us introduce two more propositional constants t and f, and assume the following initial sequents and rules of inference for them:

3. $\to \mathsf{t}$
4. $\mathsf{f} \to$

$$\frac{\Gamma, \Sigma \to \Delta}{\Gamma, \mathsf{t}, \Sigma \to \Delta}\ (\mathsf{t}w) \qquad\qquad \frac{\Gamma \to \Lambda, \Theta}{\Gamma \to \Lambda, \mathsf{f}, \Theta}\ (\mathsf{f}w)$$

These initial sequents and rules mean that t (f) is the *weakest* (*strongest*) proposition among provable formulas (contradictory formulas, respectively) and that $\neg A$ is logically equivalent to $A \supset \mathsf{f}$. It can be easily seen that t and $\top$ (and also f and $\bot$) are logically equivalent to each other, when we have the weakening rule.

In **LK**, we can show that a sequent $A_1, \ldots, A_m \to B_1, \ldots, B_n$ is provable if and only if the sequent $A_1 \wedge \ldots \wedge A_m \to B_1 \vee \ldots \vee B_n$ is provable. Thus, we can say that commas in the antecedent of a given sequent mean conjunctions, and commas in the succedent mean disjunctions. But, to show the equivalence of the above two sequents, we need both weakening and the contraction rules. Thus, in substructural logics we cannot always suppose that commas express conjunctions or disjunctions.

To supplement this defect, we may introduce two, additional logical connectives $*$ and $+$. The connective $*$ is usually called a *multiplicative conjunction* (in linear logic) or a *fusion* (in relevant logics), for which we assume the following rules of inference:

$$\frac{\Gamma \to A, \Lambda \quad \Sigma \to B, \Theta}{\Gamma, \Sigma \to A * B, \Lambda, \Theta}\ (\to *) \qquad \frac{\Gamma, A, B, \Sigma \to \Delta}{\Gamma, A * B, \Sigma \to \Delta}\ (* \to)$$

Also, the connective $+$ is called the *multiplicative disjunction*, which is the dual of $*$, for which we assume the following rules of inference:

$$\frac{\Gamma \to \Pi, A, B, \Sigma}{\Gamma \to \Pi, A + B, \Sigma}\ (\to +) \qquad \frac{\Gamma, A \to \Lambda \quad \Sigma, B \to \Theta}{\Gamma, \Sigma, A + B \to \Lambda, \Theta}\ (+ \to)$$

Then, it is not hard to see that a sequent $A_1, \ldots, A_m \to B_1, \ldots, B_n$ is provable if and only if $A_1 * \ldots * A_m \to B_1 + \ldots + B_n$ is provable. Note that we cannot introduce + in sequent calculi in which succedents of sequents consist of at most one formula.

Now, we will give a precise definition of our propositional calculus **FL**. The system **FL** is obtained from the propositional part of **LJ** by first deleting all structural rules and then by adding initial sequents and rules for propositional constants $\top, \bot$, t and f, and also those for the logical connective $*$.

In **FL**, we can show that a sequent $A * B \to C$ is provable if and only if the sequent $A \to B \supset C$ is provable. In sequent calculi like **FL**, where we don't assume the exchange rule, it will be natural to introduce another implication, say $\supset^\star$, for which it holds that $A * B \to C$ is provable if and only if the sequent $B \to A \supset^\star C$ is provable. For this purpose, it suffices to assume the following rules of inference:

$$\frac{A, \Gamma \to B}{\Gamma \to A \supset^\star B}\ (\to \supset^\star) \qquad \frac{\Gamma \to A \quad \Pi, B, \Sigma \to \Delta}{\Pi, \Gamma, A \supset^\star B, \Sigma \to \Delta}\ (\supset^\star \to)$$

We will consider extensions of **FL** which are obtained by adding some of the structural rules. Let e, c, and w denote the exchange, the contraction and the weakening rules, respectively. By taking any combination of suffixes e, c, and w, we denote the calculus obtained from **FL** by adding structural rules corresponding to these suffixes. (We allow to add neither $(\to c)$ nor $(\to e)$. Also, we may add $(\to w)$ only if Λ, Θ is empty.) For instance, $\mathbf{FL_{ew}}$ denotes the system **FL** with both the exchange and the weakening rules. The system $\mathbf{FL_e}$ is a sequent calculus for intuitionistic linear logic. Also, $\mathbf{FL_{ecw}}$ is essentially equivalent to **LJ**. Note that the exchange rule is admissible in $\mathbf{FL_{cw}}$ by the help of either rules for conjunction or rules for multiplicative conjunction. On the other hand, we can show that the cut elimination theorem doesn't hold for it, similarly to the case for $\mathbf{FL_c}$ (see Lemma 7). So, we will leave the system $\mathbf{FL_{cw}}$ out of consideration in the rest of the paper.

Each of the calculi given above determines a substructural logic which is weaker than intuitionistic logic, since it is essentially a system obtained from **LJ** by deleting some structural rules. In the same way as this, we can define a propositional calculus $\mathbf{CFL_e}$ as the system obtained from the propositional part of **LK** by first deleting both the weakening and the contraction rules (but not the exchange rules) and then by adding initial sequents and rules for propositional constants $\top, \bot$, t and f, and also those for the logical connective $*$ and $+$.

The system $\mathbf{CFL_e}$ is a sequent calculus for (classical) linear logic (without Girard's *exponentials* introduced in [18]), and is essentially equivalent to the system called **MALL**. By adding the weakening rule and the contraction rule, respectively, to $\mathbf{CFL_e}$, we can get systems, $\mathbf{CFL_{ew}}$ and $\mathbf{CFL_{ec}}$. (In subsystems of **LK**, differences of the order of formulas in the antecedents and the succedents in some rules of inference like ($\rightarrow *$) may give us essentially different systems, when we don't assume the exchange rule. For this reason, we consider here only extensions of $\mathbf{CFL_e}$.) We call the extensions of either **FL** or $\mathbf{CFL_e}$, mentioned in the above, each of which is obtained by adding some of structural rules, *basic substructural logics.*

When we lack either the weakening rule or the contraction rule, we cannot show the following *distributive law* in general;

$$A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C).$$

Relevant logics form a subclass of substructural logics which usually have the contraction rule, but don't have the weakening rule. Moreover, we assume the distributive law in many of them. This fact causes difficulties in obtaining cut-free systems for them and also certain complications in applying proof-theoretic methods to them. (See 5.4. Also, see [34].)

Sometimes in the following, to avoid unnecessary complications, we will identify a given sequent calculus for a logic **L** with the logic **L** itself when no confusion will occur.

2.4 Notes

For more information on the calculi **LK** and **LJ** and also on basic proof theory for them, consult [43]. A small, nonessential difference in the formulation of these systems is the order of occurrences of formulas in antecedents or in succedents in rules. This is because we want to use these rules also for sequent calculi without the exchange rules.

To get general information on modal logic, see e.g. [5, 22, 8]. Sequent calculi for **S4** and **S5** are introduced and discussed by Curry, Maehara, Ohnishi-Matsumoto and Kanger etc. from the middle of 1950s. Tableau calculi for modal logics are closely related to the sequent calculi for them. A concise survey of tableau calculi for modal logic is given in Goré [19]. For information on historical matters on calculi for modal logics, consult e.g. Zeman [47].

Until now, there is no textbook on substructural logics. The book "Substructural Logics" [11], which consists of papers on various fields within the scope of substructural logics, will show that the study of substructural logics is a *crossover* of nonclassical logics. Girard's paper [18] on linear logic contains a lot of ideas and his view on the subject. Troelstra's book [44], on the other hand, will show clearly where the study of linear logic stands in the study of logic developed so far. See also the paper by Okada [31] in the present Memoir. As for relevant logics, consult either two volumes of Anderson and Belnap's book [1, 2] or a concise survey by Dunn [12]. In [35, 32], one can see some proof-theoretic approaches to substructural logics, in particular, logics without the contraction rule. For general information on Lambek calculus, see [29].

3 Cut Elimination Theorem

Gentzen proved the following cut elimination theorem for both **LK** and **LJ**, which is a fundamental result in developing proof theory based on sequent calculi.

Theorem 2 *For any sequent S, if S is provable in* **LK**, *it is provable in* **LK** *without using the cut rule. This holds also for* **LJ**.

A proof with no applications of the cut rule is called a *cut-free proof.* When the cut elimination theorem holds for a sequent system **L**, we say that **L** is a *cut-free system.* In this section, we will first explain basic ideas of the proof of the cut elimination theorem by taking **LJ** as an example. Then, we will show that a slight modification of the proof gives us the cut elimination theorem for most of basic substructural logics.

When the cut elimination theorem holds, we can restrict our attention only to cut-free proofs and analyses of cut-free proofs sometimes bring us important logical properties hidden in proofs, as will be explained later. This is the reason why the cut elimination theorem is considered to be fundamental.

3.1 Elimination of the cut rule in LJ

In this subsection, we will explain the idea of elimination of applications of the cut rule and give the outline of the proof. Here, we will consider the cut elimination theorem for **LJ**. The cut elimination for **LK** can be shown in the same way.

Suppose that a proof P of a sequent $\mathcal{S}$ is given. (In this case, $\mathcal{S}$ is called the *endsequent* of the proof P.) We will try to modify the proof P by removing applications of the cut rule in it, but without changing its endsequent $\mathcal{S}$. We note first that the cut rule is not a derivable rule in the system obtained from **LJ** by deleting the cut rule. That is, it is impossible to replace the cut rule in a uniform way by repeated applications of other rules, as each application of the cut rule will play a different role in a given proof. Therefore, we have to eliminate the cut rule depending on how it is applied.

Thus, it will be necessary to eliminate applications of the cut rule inductively. Now, take one of uppermost applications of the cut rule in P, i.e. an application of the cut rule such that the subproof of P over each of its upper sequents ($\Gamma \to A$ and $A, \Pi \to D$ in the following case) contains no cut, which we suppose, is of the following form;

$$\frac{\Gamma \to A \quad A, \Pi \to D}{\Gamma, \Pi \to D} \ (cut)$$

Recall here that the formula A in the above cut is called the *cut formula* of this application of the cut rule. Let k_1 (and k_2) be the total number of sequents appearing in the proof of the left upper sequent $\Gamma \to A$ (and the right upper sequent $A, \Pi \to D$, respectively). Then, we call the sum $k_1 + k_2$, the *size* of the proof above this cut rule.

Our strategy of eliminating this cut is either *pushing the cut up* or *replacing the cut formula by a simpler one.* We will explain this by giving some examples.

1. Pushing the cut up.

Consider the following cut, for instance.

$$\cfrac{\cfrac{B, \Gamma \to A}{B \wedge C, \Gamma \to A} \quad A, \Pi \to D}{B \wedge C, \Gamma, \Pi \to D} \; (cut)$$

This cut can be replaced by the following, which has the same endsequent, but the size of the proof above the cut rule becomes smaller by one;

$$\cfrac{\cfrac{B, \Gamma \to A \quad A, \Pi \to D}{B, \Gamma, \Pi \to D} \; (cut)}{B \wedge C, \Gamma, \Pi \to D}$$

2. Replacing the cut formula by a simpler one.

Let us consider the following cut.

$$\cfrac{\cfrac{\Gamma \to B \quad \Gamma \to C}{\Gamma \to B \wedge C} \quad \cfrac{B, \Pi \to D}{B \wedge C, \Pi \to D}}{\Gamma, \Pi \to D} \; (cut)$$

This cut can be replaced by the following, which has the same endsequent, but clearly the cut formula below is simpler than that in the above;

$$\cfrac{\Gamma \to B \quad B, \Pi \to D}{\Gamma, \Pi \to D} \; (cut)$$

Now, by applying these two kinds of replacement (which we call *reductions*) repeatedly, we will eventually come to such an application of the cut rule that either at least one of the upper sequents is an initial sequent, or the cut formula is introduced by the weakening rule just above the cut rule. In the former case, the cut rule will be either of the following forms;

$$\cfrac{\Gamma \to A \quad A \to A}{\Gamma \to A} \; (cut)$$

or of the following;

$$\cfrac{A \to A \quad A, \Pi \to D}{A, \Pi \to D} \; (cut)$$

In either case, the lower sequent is the same as one of upper sequents, and hence is provable without using this application of the cut rule. Similarly, we can show that the lower sequent is provable without using the cut rule, when the cut formula is introduced by the weakening rule.

So far, so good. But, there is only one case where the above reduction does not work. This is caused by the contraction rule. In fact, consider the following cut;

$$\dfrac{\Gamma \to A \quad \dfrac{A, A, \Pi \to D}{A, \Pi \to D}\,(c\to)}{\Gamma, \Pi \to D}\,(cut)$$

One may reduce this to;

$$\dfrac{\dfrac{\Gamma \to A \quad \dfrac{\Gamma \to A \quad A, A, \Pi \to D}{A, \Gamma, \Pi \to D}\,(cut)}{\Gamma, \Gamma, \Pi \to D}\,(cut)}{\Gamma, \Pi \to D}\,some(e\to)(c\to)$$

Here, while the size of the proof above the upper application of the cut rule becomes smaller, the one above the lower application will not. Thus, some modifications of the present idea will be necessary.

3.2 Mix rule

To overcome the above difficulty, Gentzen took the following modified form of the cut rule, called the *mix rule*, instead of the cut rule.

Mix rule:

$$\dfrac{\Gamma \to A, \Theta \quad \Pi \to \Delta}{\Gamma, \Pi_A \to \Delta, \Theta_A}\,(mix),$$

where Π has at least one occurrence of A, and both Π_A and Θ_A are sequences of formulas obtained from Π and Θ, respectively, by deleting all occurrences of A. (For **LJ**, we assume that Θ is empty and Δ consists of at most one formula.) We call the formula A, the *mix formula* of the above application of the mix rule.

Let $\mathbf{LK}^\dagger$ and $\mathbf{LJ}^\dagger$ be sequent calculi obtained from **LK** and **LJ** by replacing the cut rule by the mix rule. We can show the following.

Lemma 3 *For any sequent S, S is provable in* $\mathbf{LK}^\dagger$ ($\mathbf{LJ}^\dagger$) *if and only if it is provable in* **LK** (**LJ**, *respectively*).

Proof. We will show this for $\mathbf{LJ}^\dagger$ and **LJ**. To show this, it is enough to prove that the cut rule is derivable in $\mathbf{LJ}^\dagger$ and conversely that the mix rule is derivable in **LJ**. First, consider the following cut;

$$\dfrac{\Gamma \to A \quad A, \Pi \to D}{\Gamma, \Pi \to D}$$

Then, the following figure shows that this cut is derivable in $\mathbf{LJ}^\dagger$:

$$\dfrac{\dfrac{\Gamma \to A \quad A, \Pi \to D}{\Gamma, \Pi_A \to D}\,(mix)}{\Gamma, \Pi \to D}\,some(w \to)$$

Conversely, take the following mix;

$$\dfrac{\Gamma \to A \quad \Pi \to \Delta}{\Gamma, \Pi_A \to D}$$

Then, this mix is derivable in **LJ**, as the following figure shows.

$$\dfrac{\Gamma \to A \quad \dfrac{\Pi \to D}{A, \Pi_A \to D}\,some(c \to)(e \to)}{\Gamma, \Pi_A \to D}\,(cut)$$

Now, we will return back once again to the beginning. So, we suppose that a proof P of a sequent $\mathcal{S}$ in **LJ** is given. By using the way in the proof of the above Lemma, we can get a proof Q of a sequent $\mathcal{S}$ in $\mathbf{LJ}^{\dagger}$, which may contain some applications of the mix rule, but no applications of the cut rule. This time, we will try to eliminate applications of the mix rule inductively, by taking one of uppermost applications of the mix rule in Q. Each reduction introduced in the previous subsection works completely well after this modification. In particular,

$$\dfrac{\Gamma \to A \quad \dfrac{A, A, \Pi \to D}{A, \Pi \to D}\,(c \to)}{\Gamma, \Pi_A \to D}\,(mix)$$

will be reduced to;

$$\dfrac{\Gamma \to A \quad A, A, \Pi \to D}{\Gamma, \Pi_A \to D}\,(mix)$$

Clearly, the size of the latter proof becomes smaller than that of the former. But, the replacement of the cut rule by the mix rule forces us to introduce a new notion of the "simplicity" of a proof. To see this, let us consider the following application of the mix rule, where we assume that $A \wedge B$ doesn't occur in Γ.

$$\dfrac{\Gamma \to A \wedge B \quad \dfrac{\dfrac{A, B \to D}{A, A \wedge B \to D}}{A \wedge B, A \wedge B \to D}}{\Gamma \to D}\,(mix)$$

The above application of the mix rule can be replaced by the following;

$$\dfrac{\Gamma \to A \wedge B \qquad \dfrac{\dfrac{\Gamma \to A \wedge B \quad \dfrac{A, B \to D}{A, A \wedge B \to D}}{A, \Gamma \to D}\,(mix)}{A \wedge B, \Gamma \to D}}{\dfrac{\Gamma, \Gamma \to D}{\Gamma \to D}\, some(c \to)(e \to)}\,(mix)$$

It is easy to see that the size of the proof above the lower application of the mix rule becomes greater than before. Hence, we need to introduce the notion of the *rank*, instead of the *size*, to measure the simplicity of the proof. Roughly speaking, the rank of a given application of the mix rule with the mix formula C is defined as follows. We consider such *branches* in a given proof that start from a sequent $\mathcal{S}$ which contains an occurrence of C in the succedent and ends at the left upper sequent of this mix. (The occurrence of C in $\mathcal{S}$ must be an *ancestor* of the mix formula C.) The *left rank* r_1 of this mix rule is the maximal length of these branches. Similarly, the *right rank* r_2 is the maximal length of branches which end at the right upper sequent of this mix. Then, we call the sum $r_1 + r_2$, the *rank* of this mix rule. (As for the precise definition of the rank, see [43].)

In the above case, the right rank r_2 of the mix rule is 2 in the upper proof. On the other hand, both of the right ranks of the two applications of the mix rule are 1 in the lower proof, while all of the left ranks are the same.

What remains to show is whether the procedure described in the above will eventually terminate. For a given formula A, define the *grade* of A by the number of occurrences of logical connectives in it. Take one of uppermost applications of the mix rule in a given Q. Define the *grade* of this application of the mix by the grade of its mix formula.

Then, we can show that after any of these reductions, either the grade becomes smaller, or the rank becomes smaller while the grade remains the same. Thus, by using the *double induction* on the grade and the rank, we can derive that each uppermost application of the mix is eliminable. This completes the proof of the *mix elimination theorem* of $\mathbf{LJ}^\dagger$. So, we have a proof R of $\mathcal{S}$ in $\mathbf{LJ}^\dagger$ without any application of the mix rule. It is obvious that R can be regarded as a proof of $\mathcal{S}$ in **LJ** without any application of the cut rule. Thus, we have the cut elimination theorem for **LJ**.

3.3 Cut elimination theorem for nonclassical logics

By modifying the proof of the cut elimination theorem for **LK** and **LJ** in the previous subsection, we can get the cut elimination theorem for some important nonclassical logics. For sequent calculi for modal logics introduced in Section 2, we have the following.

Theorem 4 *The cut elimination theorem holds for* **GK**, **GKT** *and* **GS4**.

On the other hand, the cut elimination theorem doesn't hold for **GS5**, as the following Lemma shows.

Lemma 5 *The sequent* $p \to \Box\neg\Box\neg p$ *is provable in* **GS5**, *but is not provable in* **GS5** *without applying the cut rule.*

Proof. The following is a proof of $p \to \Box\neg\Box\neg p$ in **GS5**.

$$\dfrac{\dfrac{\dfrac{\Box\neg p \to \Box\neg p}{\to \neg\Box\neg p, \Box\neg p}}{\to \Box\neg\Box\neg p, \Box\neg p} \quad \dfrac{\dfrac{p \to p}{\neg p, p \to}}{\Box\neg p, p \to}}{p \to \Box\neg\Box\neg p}\ (cut)$$

Now, suppose that $p \to \Box\neg\Box\neg p$ has a cut-free proof. Then, consider the last inference in the proof. By checking the list of rules of inference of **GS5** (except the cut rule, of course), we can see that only the weakening rule and the contraction rule are possible. By repeating this, we can see that this cut-free proof contains only sequents of the form, $p \to$, $\to \Box\neg\Box\neg p$ and $p, \ldots, p \to \Box\neg\Box\neg p, \ldots, \Box\neg\Box\neg p$. Thus, it can never contain any initial sequent. This is a contradiction.

For basic substructural logics, we have the following.

Theorem 6 *Cut elimination holds for* **FL**, $\mathbf{FL_e}$, $\mathbf{FL_w}$, $\mathbf{FL_{ew}}$, $\mathbf{FL_{ec}}$, *and* $\mathbf{FL_{ecw}}$. *It holds also for* $\mathbf{CFL_e}$, $\mathbf{CFL_{ew}}$, $\mathbf{CFL_{ec}}$ *and* $\mathbf{CFL_{ecw}}$.

We will give here some remarks on Theorem 6. As the discussions in 3.1 show, it suffices for us to show the cut elimination theorem (not the mix elimination theorem) directly for basic substructural logics without the contraction rule, i.e. **FL**, $\mathbf{FL_e}$, $\mathbf{FL_w}$, $\mathbf{FL_{ew}}$, $\mathbf{CFL_e}$ and $\mathbf{CFL_{ew}}$, by using double induction on the grade and the *length*. For, the source of obstacles in proving the cut elimination theorem directly is the contraction rule.

Next, consider substructural logics $\mathbf{FL_{ec}}$ and $\mathbf{CFL_{ec}}$, each of which has both the exchange and the contraction rules, but not the weakening rule. As we have shown in 3.2 (see the proof of Lemma 3), we need both the weakening and the contraction rules, to replace the cut by the mix rule. But, this difficulty can be overcome by introducing a generalized form of the mix rule. For example, the generalized form of the mix rule for $\mathbf{FL_{ec}}$ is given as follows;

$$\frac{\Gamma \to A \quad \Pi \to C}{\Gamma, \tilde{\Pi}_A \to C}\ (g - mix)$$

where Π has at least one occurrence of A, and $\tilde{\Pi}_A$ is a sequence of formulas obtained from Π by deleting *at least one* occurrence of A.

This replacement of the cut rule by the generalized mix rule works well only when we have the exchange rule. In fact, we have the following.

Lemma 7 *Cut elimination doesn't hold for* $\mathbf{FL_c}$. *In fact, it doesn't hold even for the implicational fragment of* $\mathbf{FL_c}$.

3.4 Some consequences of the cut elimination theorem

There are many important consequences of the cut elimination theorem. As *decidability* and the *interpolation theorem* will be discussed in detail in the later sections, we will here discuss some other consequences.

1) Subformula property

The first important consequence of the cut elimination theorem is the *subformula property.* Recall here that any formula of the form $A[t/x]$ is regarded as a subformula of both of $\forall xA$ and $\exists xA$ for any term t.

Theorem 8 *In a cut-free proof of* **LK**, *any formula appearing in it is a subformula of some formula in the endsequent. Hence, if a sequent* S *is provable in* **LK** *then there is a proof of* S *such that any formula appearing in it is a subformula of some formula in* S.

Proof. To show this, it is enough to check that in every rule of inference of **LK** except the cut, every formula appearing in the upper sequent(s) is a subformula of some formulas in the lower sequent. (This doesn't hold always for the cut rule, as the cut formula may not be a subformula of some formulas in the lower sequent.)

This argument holds also for every cut-free system discussed in 3.3. The importance of the subformula property will be made clear in the rest of this paper. We say that a proof in a sequent calculus **L** has the *subformula property* if it contains only formulas which are subformulas of some formulas in its endsequent. Also, we say that the *subformula property* holds for **L** when any provable sequent $\mathcal{S}$ in **L** has a proof of $\mathcal{S}$ with the subformula property.

2) Conservativity

The next result is on *conservative extensions.* Suppose that $\{o_1, \ldots, o_n\}$ is a set of logical connectives and quantifiers of our language. Let $J = \{o_1, \ldots, o_n\}$. By the *J-fragment* of **LK**, we mean the sequent calculus whose initial sequents and rules of inference are the same as those of **LK** except that we will take only rules of inference for logical connectives in J. A formula A is said to be a *J-formula* when it contains only logical connectives and quantifiers from J. We can show the following.

Theorem 9 *For any nonempty set* J *of logical connectives and quantifiers,* **LK** *is a conservative extension of the* J*-fragment of* **LK**. *More precisely, for any sequent* S *consisting only of* J*-formulas,* S *is provable in* **LK** *if and only if it is provable in the* J*-fragment of* **LK**. *In particular, the predicate calculus* **LK** *is a conservative extension of the propositional calculus* **LK**.

Proof. The if-part of the Theorem is trivial. So suppose that a sequent $\mathcal{S}$ which consists only of J-formulas is provable in **LK**. Consider a cut-free proof P of $\mathcal{S}$ in **LK**. By the subformula property of **LK**, P contains only J-formulas. Then we can see that there is no chance of applying a rule of inference for a logical connective or a quantifier other than those from J. Thus, P can be regarded as a proof in the J-fragment of **LK**.

By this theorem, it becomes unnecessary to say in which fragment of **LK** a given sequent is provable, as long as it is provable in **LK**. It is easy to see that this theorem holds for any other system as long as the subformula property holds for it.

3) Disjunction property

A logic **L** has the *disjunction property* when for any formulas A and B, if $A \vee B$ is provable in **L** then either A or B is provable in it. Classical logic doesn't have the disjunction property, as $p \vee \neg p$ is provable but neither of p and $\neg p$ are provable. On the other hand, the following holds.

Theorem 10 *Intuitionistic logic has the disjunction property.*

Proof. Suppose that the sequent $\to A \vee B$ is provable in **LJ**. It suffices to show that either $\to A$ or $\to B$ is provable in it. Consider any cut-free proof P of $\to A \vee B$. Then the last inference in P will be either $(\to w)$ or $(\to \vee)$. If it is $(\to w)$ then the upper sequent must be $\to$. But this is impossible. Hence, it must be $(\to \vee)$. Then the upper sequent is either $\to A$ or $\to B$. This completes the proof.

By using the similar argument, we have the following.

Theorem 11 *Each of* **FL**, $\mathbf{FL_e}$, $\mathbf{FL_w}$, $\mathbf{FL_{ew}}$ *and* $\mathbf{FL_{ec}}$ *has the disjunction property.*

As mentioned before, classical logic doesn't have the disjunction property. Then, where does the above argument break for classical logic? The answer may be obtained easily by checking the proof of $\to p \vee \neg p$ in **LK**. In this case, the last inference is $(\to c)$. This suggests that we can use the same argument as in the above when we have no contraction rules. Thus, we have the following result, in addition.

Theorem 12 *Both* $\mathbf{CFL_e}$ *and* $\mathbf{CFL_{ew}}$ *have the disjunction property.*

An immediate consequence of the above theorem is that the sequent $\to p \vee \neg p$ is not provable in $\mathbf{CFL_{ew}}$. Note that $\mathbf{CFL_{ew}}$ is obtained from $\mathbf{FL_{ew}}$ by adding $\neg\neg A \to A$ as additional initial sequents. On the other hand, the system obtained from $\mathbf{FL_{ew}}$ by adding $\to A \vee \neg A$ as initial sequents becomes classical logic.

In the same way as in **LK**, we can show that $\mathbf{CFL_{ec}}$ doesn't have the disjunction property. On the other hand, we note that the presence of the contraction rule

($\rightarrow c$) doesn't cause always the failure of the disjunction property. For instance, consider the sequent calculus **LJ′** for the intuitionistic logic, which is obtained from **LK** by restricting ($\rightarrow \neg$), ($\rightarrow\supset$) and ($\rightarrow \forall$) to those for **LJ**. Though **LJ′** is a cut-free system with ($\rightarrow c$), it has still the disjunction property.

3.5 Subformula property revisited

As shown in the above, the cut elimination theorem doesn't hold for the sequent calculus **GS5**. On the other hand, we can show the following.

Theorem 13 *The sequent calculus* **GS5** *has the subformula property.*

To show this, we will introduce the notion of *acceptable cuts*, which sometimes are called *analytic cuts*, as follows. An application of the following cut rule is *acceptable*

$$\frac{\Gamma \rightarrow A, \Theta \quad A, \Pi \rightarrow \Delta}{\Gamma, \Pi \rightarrow \Theta, \Delta}$$

if the cut formula A is a subformula of a formula in $\Gamma, \Theta, \Pi, \Delta$.

Then, we can show the following result, from which Theorem 13 follows immediately.

Theorem 14 *For any sequent S, if S is provable in* **GS5**, *then there exists a proof of S in* **GS5** *such that every application of the cut rule in it is acceptable.*

Note that the proof of $p \rightarrow \Box\neg\Box\neg p$ given in the proof of Lemma 5 contains an application of the cut rule, but it is an acceptable one. To show the theorem, we need to eliminate each non-acceptable application of the cut rule. This can be done in the same way as the proof of the cut elimination theorem. Similar results holds also for some other modal logics.

The importance of this result lies in the fact that most consequences of the cut elimination theorem are obtained from the subformula property. Thus, we can rephrase this fact in the following way: *The most important proof-theoretic property is the subformula property, and the most convenient way of showing the subformula property is to show the cut elimination theorem.*

3.6 Notes

To get the detailed information on the proof of the cut elimination theorem and on the consequences of it, both [43] and [45] will be useful. The result of Lemma 5 was noticed by Ohnishi and Matsumoto in their joint paper 1959. Several people including S. Kanger, G. Mints, M. Sato, M. Ohnishi etc., introduced cut-free systems for **S5**.

Theorem 6 is shown in [35] and [32]. A generalized form of the mix rule was introduced in the paper [23]. On the other hand, the negative result on $\mathbf{FL_c}$ is discussed in [4]. The subformula property for **GS5** and the related systems are discussed in Takano's paper [42]. The proof is based on a deep proof-theoretic analysis. Recently, he succeeded to extend his method to some intuitionistic modal logics, i.e. modal logics based on the intuitionistic logic. Related results are obtained by Fitting [15, 16], in which semantical methods are used. Further study in this direction seems to be promising.

4 Decision Problems for the Classical and the Intuitionistic Logics

In this and the next sections, we will discuss decision problems of various logics. A given logic **L** is said to be *decidable* if there exists an algorithm that can decide whether a formula A is provable in **L** or not, for any A. It is *undecidable* if it is not decidable. The *decision problem* of a given logic **L** is the problem as to whether **L** is decidable or not. In this section, we will show the decidability of classical and intuitionistic propositional logics as a consequence of the cut elimination theorem for propositional calculi for both **LK** and **LJ** by using Gentzen's method. In the next section, we will show that most of basic substructural logics without the contraction rule are decidable even if they are predicate logics. On the other hand, we can show that a certain complication will occur in decision problems for substructural logics with the contraction rule but without the weakening rule.

4.1 Basic ideas of proving decidability

We will explain here a method of checking the provability of a given sequent in propositional **LK**, which is due to Gentzen. The decision algorithm for propositional **LJ** can be obtained similarly. Suppose that a sequent $\Gamma \to \Delta$ is given. We will try to search for a proof of this sequent. If we succeed to find one, $\Gamma \to \Delta$ is of course provable, and if we fail, then it is not provable. But, as there may be infinitely many possible proofs of it, how can we know that we have failed?

Thus, it is necessary to see whether we can restrict the range of search of proofs, hopefully to finitely many proofs, or not. Now let us consider the following restrictions on proofs.

1) *Proofs with the subformula property*, in particular *cut-free proofs*: For **LK**, this is possible, since we have the cut elimination theorem for **LK**. Thus, if $\Gamma \to \Delta$ is provable in **LK** then there must exist a (cut-free) proof such that any sequent in it consists only of formulas which are subformulas of some formulas in $\Gamma \to \Delta$.

2) *Proofs with no redundancies*: Here, we say that a proof P has *redundancies* if the same sequent appears twice in a branch of P. Every proof having some redundancies can be transformed into one with no redundancies. In fact, a redundancy of $\Sigma \to \Pi$ in the following proof

$$\begin{array}{c} \mathsf{P}_0 \\ \Sigma \to \Pi \\ \vdots \\ \Sigma \to \Pi \\ \mathsf{P}_1 \\ \Gamma \to \Delta \end{array}$$

can be eliminated as follow;

$$\begin{array}{c} \mathsf{P}_0 \\ \Sigma \to \Pi \\ \mathsf{P}_1 \\ \Gamma \to \Delta. \end{array}$$

But, the above two conditions are not enough to reduce the number of all possible proofs to finitely many. To see this, suppose that a given sequent $D, \Lambda \to \Theta$ satisfies the condition that it consists only of subformulas of some formulas in $\Gamma \to \Delta$. Then, any sequent of the form $\overbrace{D, \ldots, D}^{n}, \Lambda \to \Theta$ also satisfies this for any natural number n.

Thus we need to add an additional condition. A sequent $\Sigma \to \Pi$ is *reduced* if every formula in it occurs at most three times in the antecedent and also at most three times in the succedent. In particular, we say that a sequent $\Sigma \to \Pi$ is *1-reduced* if every formula in the antecedent (the succedent) occurs exactly once in the antecedent (the succedent, respectively).

A sequent $\Gamma^\dagger \to \Delta^\dagger$ is called a *contraction* of a sequent $\Gamma \to \Delta$ if it is obtained from $\Gamma \to \Delta$ by applying the contraction and exchange rule repeatedly. For instance, the sequent $A, C \to B$ is a contraction of a sequent $A, C, A, A \to B$. Now, we can show easily that for any given sequent $\Gamma \to \Delta$, there exists a (1-)reduced sequent $\Gamma^* \to \Delta^*$, which is a contraction of $\Gamma \to \Delta$, such that $\Gamma \to \Delta$ is provable in **LK** if and only if $\Gamma^* \to \Delta^*$ is provable in **LK**. In fact, if a formula C appears more than three times either in Γ or in Δ then we can reduce the number of occurrences of C to three (or, even to one) by applying the contraction rule (and the exchange rule, if necessary) repeatedly. In this way, we can get a (1-)reduced sequent $\Gamma^* \to \Delta^*$. Conversely, by applying the weakening rule to $\Gamma^* \to \Delta^*$ as many times as necessary, we can recover $\Gamma \to \Delta$. Thus, there exists an *effective* way of getting a (1-)reduced sequent S' for a given sequent S, whose provability is the same as that of S. So, it suffices to get an algorithm which can decide whether a given *reduced sequent* is provable or not. We have the following.

Lemma 15 *Suppose that* $\Gamma \to \Delta$ *is a sequent which is provable in* **LK** *and that* $\Gamma^* \to \Delta^*$ *is any 1-reduced contraction of* $\Gamma \to \Delta$. *Then, there exists a cut-free proof of* $\Gamma^* \to \Delta^*$ *in* **LK** *such that every sequent appearing in it is reduced.*

Proof. Let us take a cut-free proof P of $\Gamma \to \Delta$. We will prove our lemma by induction on the *length* of P, i.e. the maximum length of branches in P which start from an initial sequent and end at the endsequent of P. This is trivial when $\Gamma \to \Delta$ is an initial sequent. Suppose that $\Gamma \to \Delta$ is the lower sequent of an application of a rule I. When I has a single upper sequent, it must be of the following form:

$$\frac{\Lambda \to \Theta}{\Gamma \to \Delta}\ (I)$$

Let $\Lambda^* \to \Theta^*$ be any 1-reduced contraction of $\Lambda \to \Theta$. Then, by the hypothesis of induction, there exists a cut-free proof of $\Lambda^* \to \Theta^*$ such that every sequent appearing in it is reduced. Let $\Gamma' \to \Delta'$ be the sequent obtained by applying the rule I to $\Lambda^* \to \Theta^*$. (When I is either $(c \to)$ or $(\to c)$, we cannot apply it to $\Lambda^* \to \Theta^*$. In this case, we take $\Lambda^* \to \Theta^*$ for $\Gamma' \to \Delta'$.) By checking the form of every rule of **LK**, we can see that $\Gamma' \to \Delta'$ is reduced. So, by applying the contraction and the exchange rule if necessary, we can get any 1-reduced contraction of $\Gamma \to \Delta$. Clearly, this proof consists only of reduced sequents. The same argument holds also in the case where I has two upper sequents.

Thus, we can add the third restriction on proofs.
3) *Proofs consisting only of reduced sequents*

Here we will add an explanation of the reason why the number "three" appears in the definition of reduced sequents. Let us consider the case when I (in the above proof) is $(\supset\to)$;

$$\frac{\Gamma \to A, \Theta \quad \Pi, B, \Sigma \to \Delta}{\Pi, A \supset B, \Gamma, \Sigma \to \Delta, \Theta}\ I$$

Moreover, suppose that $A \supset B$ occurs once in Γ and once in Π, Σ. Then, the antecedent of the lower sequent will contain *three* occurreces of $A \supset B$. To make the lower sequent *reduced*, we have to admit three occurreces of the same formula.

Now, we are ready for showing the decidability of propositional **LK**. Take any sequent $\Gamma \to \Delta$. We can assume that it is reduced. We will check whether there is a proof of it satisfying all of these three restrictions or not. By 1) and 3), the proof, if it exists at all, consists only of reduced sequents such that every formula in it must be a subformula of a formula in $\Gamma \to \Delta$. Clearly, the number of such reduced sequents is finite. Moreover, by the restriction 2), the number of *possible* proofs of $\Gamma \to \Delta$ must be finite. Thus, our decision algorithm is to generate all possible proofs one by one and to check whether it is a correct proof of $\Gamma \to \Delta$ or not. If we can find one correct proof, $\Gamma \to \Delta$ is of course provable in **LK**. If we cannot find it among them, we can say that it is not provable.

To get a possible proof, we will try to construct it from the endsequent *upward.* The last inference must be either one of structural rules or one of rules for logical connectives. In the latter case, the logical connective for which a rule is applied must be an outermost logical connective of a formula in the endsequent. Therefore, there are finitely many possiblities. We will repeat this again for sequents which are obtained already in this way. Note that a possible proof will branch upward, in general, and therefore it is necessary to continue the above construction to the top of each branch. When we reach a sequent which doesn't produce any new sequent, we will check whether it is an initial sequent or not. If we can find at least one non-initial sequent among them, we fail. This procedure is sometimes called, the *proof-search algorithm.* Thus, we have the following

Theorem 16 *Both classical and intuitionistic propositional logics are decidable.*

Here are two examples of possible proofs of $\to p \vee (p \supset q)$. The left one is a failed one and the right one is a correct proof.

$$\dfrac{\dfrac{p \to q}{\to p \supset q}\,(\to\supset)}{\to p \vee (p \supset q)}\,(\to \vee 2) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \to p}{p \to p, q}\,(\to w)}{\to p, p \supset q}\,(\to\supset)}{\to p, p \vee (p \supset q)}\,(\to \vee 2)}{\to p \vee (p \supset q), p \vee (p \supset q)}\,(\to \vee 1)}{\to p \vee (p \supset q)}\,(\to c)$$

For modal logics introduced in Section 2, we can use essentially the same decision procedure as that for **LK** and therefore have the following.

Theorem 17 *Any of modal propositional logics* **K**, **KT**, **S4** *and* **S5** *is decidable.*

Note that here the subformula property, but not the cut elimination theorem, is essential in the proof of the decidability mentioned above. Thus, the decidability of **S5** follows from Theorem 13.

4.2 Digression — a simplified decision algorithm for the classical propositional logic

In the previous subsection, we have given an algorithm which decides the provability of a given sequent $\mathcal{S}$ in **LK**. Moreover, *the algorithm gives us a cut-free proof of* $\mathcal{S}$ when it is provable. But the algorithm contains a thorough search of proofs satisfying certain conditions and thus needs trial and error. For propositional **LK**, we can give a simpler decision algorithm, as shown below. This algorithm might be quite instructive, when one learns how to construct a cut-free proof of a given sequent if it is provable.

On the other hand, for practical purposes a lot of work has been done on efficient algorithms of deciding whether a given sequent is provable in **LK** or not, or equivalently whether a given formula is tautology or not.

For a given sequent $\mathcal{S}$, define *decompositions* of $\mathcal{S}$ as follows. Any decomposition is either a sequent or an expression of the form $\langle \mathcal{S}_1; \mathcal{S}_2 \rangle$ for some sequents $\mathcal{S}_1$ and $\mathcal{S}_2$.

Decompositions

1. $\langle \Gamma, \Pi \to A, \Delta; \Gamma, B, \Pi \to \Delta \rangle$ is a decomposition of $\Gamma, A \supset B, \Pi \to \Delta$,
2. $A, \Gamma \to \Delta, B, \Theta$ is a decomposition of $\Gamma \to \Delta, A \supset B, \Theta$,
3. $\Gamma, A, B, \Pi \to \Delta$ is a decomposition of $\Gamma, A \wedge B, \Pi \to \Delta$,
4. $\langle \Gamma \to \Delta, A, \Theta; \Gamma \to \Delta, B, \Theta \rangle$ is a decomposition of $\Gamma \to \Delta, A \wedge B, \Theta$,
5. $\langle \Gamma, A, \Pi \to \Delta; \Gamma, B, \Pi \to \Delta \rangle$ is a decomposition of $\Gamma, A \vee B, \Pi \to \Delta$,
6. $\Gamma \to \Delta, A, B, \Theta$ is a decomposition of $\Gamma \to \Delta, A \vee B, \Theta$,
7. $\Gamma, \Pi \to A, \Delta$ is a decomposition of $\Gamma, \neg A, \Pi \to \Delta$,
8. $A, \Gamma \to \Delta, \Theta$ is a decomposition of $\Gamma \to \Delta, \neg A, \Theta$.

Note that a sequent may have more than one decomposition. For instance, the sequent $A \supset B \to C \vee D$ has two decompositions $A \supset B \to C, D$ and $\langle \to A, C \vee D; B \to C \vee D \rangle$. When $\mathcal{S}'$ ($\langle \mathcal{S}_1; \mathcal{S}_2 \rangle$) is a decomposition of $\mathcal{S}$, we say that $\mathcal{S}$ is *decomposed into* $\mathcal{S}'$ ($\langle \mathcal{S}_1; \mathcal{S}_2 \rangle$, respectively).

Lemma 18 *Suppose that $\mathcal{S}$ is decomposed into $\mathcal{S}'$ (or $\langle \mathcal{S}_1; \mathcal{S}_2 \rangle$). Then, the following holds.*
1) The number of all occurrences of logical connectives in $\mathcal{S}'$ (or in each of $\mathcal{S}_1$ and $\mathcal{S}_2$) is smaller than that in $\mathcal{S}$.
2) If $\mathcal{S}$ is provable in **LK** *then so is $\mathcal{S}'$ (or, both of $\mathcal{S}_1$ and $\mathcal{S}_2$).*
3) If $\mathcal{S}'$ (or, both of $\mathcal{S}_1$ and $\mathcal{S}_2$) is provable in **LK** *without applying the cut rule, then so is $\mathcal{S}$.*

Proof. 1) can be shown by checking the definition of decompositions. 2) and 3) are proved by the help of the cut rule, and of both the contraction and weakening rules, respectively.

We will consider now repeated applications of decompositions. For instance, suppose that $\mathcal{S}$ is decomposed into $\langle \mathcal{S}_1; \mathcal{S}_2 \rangle$, $\mathcal{S}_1$ into $\mathcal{S}_3$, and then $\mathcal{S}_2$ into $\langle \mathcal{S}_4; \mathcal{S}_5 \rangle$. Then, we will express these decompositions by the following figure:

$$\mathcal{S} \Rightarrow \langle \mathcal{S}_1; \mathcal{S}_2 \rangle \Rightarrow \langle \mathcal{S}_3; \mathcal{S}_2 \rangle \Rightarrow \langle \mathcal{S}_3; \langle \mathcal{S}_4; \mathcal{S}_5 \rangle \rangle$$

We will call such a figure, a *decomposition figure* of $\mathcal{S}$ and each expression in a decomposition figure, like $\langle \mathcal{S}_3; \langle \mathcal{S}_4; \mathcal{S}_5 \rangle \rangle$, a *d-expression.* (It might be more instructive to draw a tree-like figure than a decomposition figure.)

The above Lemma tells us that by any application of decompositions, each sequent in a d-expression is either unchanged or simplified. Therefore, after some applications of decompositions, we will get a d-expression in which no more decomposition is applicable to any of sequents. We will call such a d-expression, a *terminal* d-expression. Moreover, we call a decomposition figure (of $\mathcal{S}$) whose last d-expression is terminal, a *complete* decomposition figure (of $\mathcal{S}$).

It is easy to see that no decomposition is applicable to a sequent $\mathcal{S}^*$ if and only if it consists only of propositional variables, and that a sequent of the form

$p_1, \ldots, p_m \to q_1, \ldots, q_n$ with propositional variables $p_1, \ldots, p_m, q_1, \ldots, q_n$ is provable in **LK** if and only if p_i is identical with q_j for some i ($\leq m$) and j ($\leq n$).

Now, take an arbitrary sequent $\mathcal{S}$ and take any one of its complete decomposition D. Then, by Lemma 18 2) and 3), $\mathcal{S}$ is provable if and only if every sequent in the terminal d-expression of D is provable. We can decide whether the latter holds or not, by checking whether every sequent has a common propositional variable in the antecedent and the succedent or not. This gives us a decision algorithm for classical propositional logic. We note here that the present algorithm doesn't contain any trial and error.

Furthermore, when every sequent in the terminal d-expression of D is provable, we can construct a cut-free proof of $\mathcal{S}$ effectively from D by supplementing some applications of the contraction and the weakening rules. Here is an example. Consider the sequent $\to ((p \supset q) \supset p) \supset p$. The following is a complete decomposition figure of it.

$$\to ((p \supset q) \supset p) \supset p \Rightarrow (p \supset q) \supset p \to p \Rightarrow \langle \to p, p \supset q; p \to p \rangle \Rightarrow \langle p \to p, q; p \to p \rangle$$

Since each of $p \to p, q$ and $p \to p$ has a common propositional variable in the antecedent and the succedent, the above sequent must be provable in **LK**. In fact, it has a cut-free proof shown below, which obtained from the above decomposition figure by supplementing some contraction and weakening rules.

$$\dfrac{\dfrac{\dfrac{\dfrac{p \to p}{p \to p, q}\ (\to w)}{\to p, p \supset q} \qquad p \to p}{\dfrac{(p \supset q) \supset p \to p, p}{(p \supset q) \supset p \to p}\ (\to c)}}{\to ((p \supset q) \supset p) \supset p}$$

The above argument will suggest us an alternative sequent calculus for the classical propositional logic, which we call **LK***. Sequents of **LK*** are expressions of the form $\Gamma \to \Delta$, where both Γ and Δ are finite *multisets* of formulas. Initial sequents of **LK*** are sequents of the form $A, \Gamma \to \Delta, A$. **LK*** has no structural rules and no cut rule. It has only the following rules for logical connectives.

Rules for logical connectives of LK*:

$$\frac{\Gamma \to \Delta, A \quad B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta}\ (\supset\to) \qquad \frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B}\ (\to\supset)$$

$$\frac{A, B, \Gamma \to \Delta}{A \wedge B, \Gamma \to \Delta}\ (\wedge \to) \qquad \frac{\Gamma \to \Delta, A \quad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \wedge B}\ (\to \wedge)$$

$$\frac{A, \Gamma \to \Delta \quad B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}\ (\vee \to) \qquad \frac{\Gamma \to \Delta, A, B}{\Gamma \to \Delta, A \vee B}\ (\to \vee)$$

$$\frac{\Gamma \to \Delta, A}{\neg A, \Gamma \to \Delta} (\neg \to) \qquad \frac{A, \Gamma \to \Delta}{\Gamma \to \Delta, \neg A} (\to \neg)$$

From the above argument, the following result follows immediately. (Here, we will neglect the difference of the definition of sequents in these two systems for brevity.)

Theorem 19 *For any sequent $\mathcal{S}$, $\mathcal{S}$ is provable in* **LK**** *if and only if it is provable in* **LK**.

4.3 Undecidability of the classical and the intuitionistic predicate logics

A. Church proved the following in 1936.

Theorem 20 *The classical predicate logic is undecidable.*

It may be helpful to point out where the decision procedure mentioned in the previous subsection breaks for the predicate **LK**. This will be understood by looking at rules ($\forall \to$) and ($\to \exists$) for quantifiers. In these rules, corresponding to a formula of the form $\forall x A$ or $\exists x A$ in the lower sequent, a formula $A[t/x]$, which is a subformula (in a broad sense) of $\forall x A$ or $\exists x A$, appears in the upper sequent for some term t. But, this t may not appear in the lower sequent, in general. Thus, it becomes necessary to search for an appropriate t.

When our language $\mathcal{L}$ contains no function symbols, a term in $\mathcal{L}$ is either an individual variable or an individual constant. Even in this case, we cannot resolve the above difficulty. For instance, a proof of a sequent of the form $\forall x A, \Sigma \to \Pi$ following the above three restrictions contain reduced sequents of the form $A[u_1/x], \ldots, A[u_n/x], \Lambda \to \Theta$ for some n, where each u_i is either an individual variable or an individual constant. Thus, the number of reduced sequents will be infinite.

Define a mapping α on the set of first-order formulas of $\mathcal{L}$ inductively, as follows.

$\alpha(A) = \neg\neg A$ if A is an atomic formula,
$\alpha(A \wedge B) = \alpha(A) \wedge \alpha(B)$, $\quad \alpha(A \supset B) = \alpha(A) \supset \alpha(B)$,
$\alpha(\neg A) = \neg \alpha(A)$, $\quad \alpha(A \vee B) = \neg(\neg \alpha(A) \wedge \neg \alpha(B))$,
$\alpha(\forall x A) = \forall x \alpha(A)$, $\quad \alpha(\exists x A) = \neg \forall x \neg \alpha(A)$.

We can show the following.

Lemma 21 *For any first-order formula A, A is provable in* **LK** *if and only if $\alpha(A)$ is provable in* **LJ**.

Proof. It is easy to see that $\alpha(A) \supset A$ is provable in **LK** for any A. Thus, the if-part of the Lemma follows. We can show that $\neg\neg\alpha(A) \supset \alpha(A)$ is provable in **LJ** for any A. Moreover by using induction, we can show that if a sequent $A_1, \ldots, A_m \to B_1, \ldots, B_n$ is provable in **LK** then $\alpha(A_1), \ldots, \alpha(A_m), \neg\alpha(B_1), \ldots, \neg\alpha(B_n) \to$ is provable in **LJ**. Thus, if A is provable in **LK** then $\neg\neg\alpha(A)$ and hence $\alpha(A)$ are provable in **LJ**.

By Theorem 20 and Lemma 21, we have the following.

Theorem 22 *The intuitionistic predicate logic is undecidable.*

Proof. Suppose otherwise. Let us take an arbitrary formula A and calculate $\alpha(A)$. (This calculation is carried out effectively.) By using the decision algorithm for the intuitionistic predicate logic, we check whether $\alpha(A)$ is provable in the intuitionistic predicate logic or not. Then, by Lemma 21 this enables us to see whether A is provable in the classical predicate logic or not. But, this contradicts Theorem 20.

4.4 Notes

The decision algorithm mentioned in 4.1 is based on the result by Gentzen [17]. See also [43]. In implementing these algorithms by computers, it becomes necessary to consider their efficiency. Thus, such an approach as one in 4.2 may be useful. As for efficient decision algorithms for intuitionistic logic, see also Notes 5.4 in the next section. As for the details of the proof of Theorem 20, see e.g. [14].

5 Decision Problems for Substructural Logics

5.1 Decidability of substructural logics without the contraction rule

Next, we will discuss the decision problems for basic substructural logics. It will be natural to start by checking whether our decision algorithm mentioned in the previous section can be applied to the present case or not. Since the cut elimination theorem holds for all basic substructural logics but $\mathbf{FL_c}$, we can limit the range of possible proofs to those with the subformula property and of course, with no redundancies. On the other hand, our discussion on reduced sequents, in particular the proof of Lemma 15, relies on the presence of both the contraction and the weakening rules. Here, some modifications or new ideas will be necessary.

First, we will consider basic substructural logics without the contraction rule. In this case, no essential difficulties will occur, and in fact the decision algorithm will be much easier than that of **LK** and **LJ**. This follows from the following observation. Let us look at every rule of inference in the propositional **LK** or **LJ** except the cut and the contraction rules. Then, we can see easily that in any of such rules (either of) its upper sequent(s) is simpler than the lower sequent. Thus, the number of sequents which can appear in a cut-free proof of a given sequent is finite, and hence the number of possible sequents is also finite. Thus, we have the decidability of basic substructural propositional logics without the contraction rule.

This argument can be easily extended to the predicate logics, when our language contains neither function symbols nor individual constants. Consider rules for universal quantifiers:

$$\frac{\Gamma, A[t/x], \Sigma \to \Delta}{\Gamma, \forall x A, \Sigma \to \Delta} (\forall \to) \qquad \frac{\Gamma \to \Lambda, A[z/x], \Theta}{\Gamma \to \Lambda, \forall x A, \Theta} (\to \forall)$$

In our proof-search algorithm, we must find a suitable upper sequent for a given lower sequent. In the case of $(\to \forall)$, for z, we will take the first individual variable not appearing in the lower sequent, assuming an enumeration of all individual variables at the beginning. In the present case, the term t in $(\forall \to)$ must be an individual variable. We can show that as a possible upper sequent, it is enough to consider the sequent of the form $\Gamma, A[t/x], \Sigma \to \Delta$ such that t is either one of free variables appearing in the lower sequent or the first individual variable not appearing in the lower sequent. (In fact, if t is a new variable, we replace it by the first new variable, which doesn't affect its provability.) Rules for existential quantifiers can be treated in the same way.

Now, consider the proof-search for a given sequent $\mathcal{S}$. Let $y_1, \ldots, y_k$ be all the free variables in $\mathcal{S}$. For the sake of convenience, we assume that $y_1, \ldots, y_k$ come as the first k individual variables in our enumeration of all individual variables. Let m be the number of all quantifiers in $\mathcal{S}$ and put $n = k + m$. Then, by the above argument, we can see that it is enough to consider sequents consisting of subformulas of formulas in $\mathcal{S}$ whose free individual variables are among the first n variables in the enumeration. This implies that the number of such sequents is finite. Therefore, we have shown the decidability of basic substructural predicate logics without the contraction rule, provided that the language contains neither function symbols nor individual constants. Though we will not go into the details, we can extend the decidability result for them, even if our language contains both function symbols and individual constants. (see [23] for details).

Theorem 23 *Any of the substructural predicate logics* $\mathbf{FL}$, $\mathbf{FL_e}$, $\mathbf{FL_w}$, $\mathbf{FL_{ew}}$, $\mathbf{CFL_e}$ *and* $\mathbf{CFL_{ew}}$ *(with function symbols and individual constants) is decidable.*

5.2 Decision problems for substructural propositional logics with the contraction rule

Next we will consider decision problems for substructural logics $\mathbf{FL_{ec}}$ and $\mathbf{CFL_{ec}}$. We will discuss only the decision problem for $\mathbf{FL_{ec}}$ in detail, as the case for $\mathbf{CFL_{ec}}$ can be treated similarly. The arguments in this and the next subsections will be more technical than the previous ones. So, readers may skip the deatails of the proofs in their first reading and may look only at the results.

First, we will start from the propositional $\mathbf{FL_{ec}}$. As we have shown in the previous subsection, the main troubles in decision problems were caused by the contraction rule. In our proof-search for a given sequent $\mathcal{S}$, we must consider the case where it is obtained from sequents, which are more complicated than $\mathcal{S}$, by applying the contraction rule many times. As we have already shown, when we have moreover the weakening rule, with the help of reduced sequents we can prove that the number of possible proofs is finite. But, we cannot use reduced sequents in general.

To overcome this difficulty, we will remove the contraction rule from the system $\mathbf{FL_{ec}}$ and instead of this, modify each rule for logical connectives into one which contains *implicit applications of the contraction rule.* We will call the system thus obtained, $\mathbf{FL_{ec}}'$. In order to avoid making the present paper too technical, we will not give the full details of the system $\mathbf{FL_{ec}}'$ here, but introduce only its implicational fragment to explain the idea. By our definition, sequents of $\mathbf{FL_{ec}}$ are expressions of the form $\Gamma \to C$ with a (possibly empty) sequence Γ of formulas and a (possibly empty) formula C. On the other hand, sequents of $\mathbf{FL_{ec}}'$ are expressions of the form $\Gamma \to C$ with a (possibly empty) *finite multiset* Γ of formulas and a (possibly empty) formula C. (This is only for the sake of brevity. In this way, we can dispense with the exchange rule.) The system $\mathbf{FL_{ec}}'$ doesn't have the contraction rule. But, except for this, it has almost the same, but slightly modified rules of inference as those of $\mathbf{FL_{ec}}$. To see the difference between them, we will now compare the implication rules in these two systems. In the following, by $\sharp_\Pi(D)$ we will denote the multiplicity of a formula D in a given multiset Π, in other words, the number of occurrences of D in Π.

Rules for the implication in $\mathbf{FL_{ec}}$

$$\frac{\Gamma \to A \quad B, \Sigma \to C}{A \supset B, \Gamma, \Sigma \to C}\ (\supset\to) \qquad\qquad \frac{A, \Gamma \to B}{\Gamma \to A \supset B}\ (\to\supset)$$

On the other hand,

Rules for the implication in $\mathbf{FL_{ec}}'$

$$\frac{\Gamma \to A \quad B, \Sigma \to C}{A \supset B, \Pi \to C}\ (\supset\to) \qquad\qquad \frac{A, \Gamma \to B}{\Gamma \to A \supset B}\ (\to\supset)$$

where $A \supset B, \Pi$ is any multiset which is a contraction of the multiset $A \supset B, \Gamma, \Sigma$ (i.e. $A \supset B, \Pi$ is any multiset obtained from the multiset $A \supset B, \Gamma, \Sigma$ by deleting some duplicated formulas in it) and which satisfies the following requirements:

1) $\sharp_\Pi(A \supset B) \geq \sharp_{(\Gamma,\Sigma)}(A \supset B) - 2$ when $A \supset B$ belongs to both Γ and Σ. Otherwise, $\sharp_\Pi(A \supset B) \geq \sharp_{(\Gamma,\Sigma)}(A \supset B) - 1$.

2) For any formula D in Γ, Σ except $A \supset B$, $\sharp_\Pi(D) \geq \sharp_{(\Gamma,\Sigma)}(D) - 1$ when D belongs to both Γ and Σ, and $\sharp_\Pi(D) = \sharp_{(\Gamma,\Sigma)}(D)$ otherwise.

Here is an example of an application of $(\supset\to)$ of $\mathbf{FL_{ec}}'$:

$$\frac{A \supset B, C, D \to A \quad B, A \supset B, C, E \to F}{A \supset B, C, D, E \to F}$$

Note that there is some freedom of the choice of Π. We can show that the cut elimination theorem holds for $\mathbf{FL_{ec}}'$ and that for any formula A, A is provable in $\mathbf{FL_{ec}}'$ if and only if it is provable in $\mathbf{FL_{ec}}$.

We have already discussed the notion of proofs with no redundancies. Here, we need a more general notion. We say that a proof P has *redundant contractions* if there exist sequents $\mathcal{S}$ and $\mathcal{S}'$ in a branch of P in such a way as the following figure shows;

$$\begin{array}{c} \vdots \\ \mathcal{S} \\ \vdots \\ \mathcal{S}' \\ \vdots \end{array}$$

such that $\mathcal{S}'$ is a contraction of $\mathcal{S}$. The following lemma, which is called *Curry's Lemma*, can be proved by using induction on the length of the proof of $\mathcal{S}$. In the following discussion, this lemma will play the same role as Lemma 15.

Lemma 24 *Suppose that a given sequent $\mathcal{S}$ has a cut-free proof of the length m in $\mathbf{FL_{ec}}'$ and that $\mathcal{S}'$ is a contraction of $\mathcal{S}$. Then, $\mathcal{S}'$ has a cut-free proof in $\mathbf{FL_{ec}}'$ whose length is not greater than m.*

By this lemma, we have the following.

Corollary 25 *For any sequent $\mathcal{S}$, if $\mathcal{S}$ is provable in $\mathbf{FL_{ec}}'$ then it has a cut-free proof of $\mathcal{S}$ in $\mathbf{FL_{ec}}'$ which has no redundant contractions.*

Proof. We can suppose that $\mathcal{S}$ has a cut-free proof P in $\mathbf{FL_{ec}}'$. Our corollary can be shown by using induction on the length of P. Let it be n. Suppose moreover that P has a redundant contraction shown in the following figure, where $\mathcal{S}_2$ is a contraction of $\mathcal{S}_1$ and the lengths of the subproofs of them (in P) are m and k, respectively. Of course, $k < m$.

$$\begin{array}{c} \vdots \\ \mathcal{S}_1 \\ \vdots \\ \mathcal{S}_2 \\ \vdots \\ \mathcal{S}. \end{array}$$

Then, by Curry's Lemma, there exists a cut-free proof Q of $\mathcal{S}_2$ with the length k' such that $k' \leq k$. So, we can get another proof P' of $\mathcal{S}$ by replacing the subproof of $\mathcal{S}_2$ in P by Q,

$$\begin{array}{c} \text{Q} \\ \mathcal{S}_2 \\ \vdots \\ \mathcal{S} \end{array}$$

with the length n' such that $n' < n$, since $k' \leq k < m$. Hence, by the hypothesis of induction we can get a proof of $\mathcal{S}$ with no redundant contractions.

5.3 Termination of the proof-search algorithm

Here, we will look over our situation again. In the present case, we cannot limit the number of sequents in possible proofs to be finite. What we can assume now is that we can restrict our attention only to *proofs with the subformula property* and what is more, to *proofs with no redundant contractions.* Then, are these two restrictions on possible proofs enough to make the total number of possible proofs finite? This is not so obvious. Let us consider the following example. Here, we will write the sequence $A, \ldots, A$ with n occurrences of A as A^n. For a given sequent $\mathcal{S}$ which is $A^3, B^2, C^5 \to D$, take three sequents $A^4, B, C^4 \to D$, $A^2, B^3, C^4 \to D$ and $A^5, B^2, C^3 \to D$. Then, $\mathcal{S}$ is not a contraction of any of them. Moreover, any one of them is not a contraction of any other. Thus, all of them may appear in the same branch of a possible proof of $\mathcal{S}$ with no redundant contractions. So, we will face the following combinatorial problem.

Suppose that formulas $A_1, \ldots, A_m, D$ are given. Consider a sequence $\langle \mathcal{S}_1, \mathcal{S}_2, \ldots \rangle$ of sequents such that each member is always of the form ${A_1}^{k_1}, \ldots, {A_m}^{k_m} \to D$, where each k_i is positive. (Any sequence of this kind is sometimes called a sequence of *cognate* sequents.) Moreover, assume that $\mathcal{S}_i$ is not a contraction of $\mathcal{S}_j$ whenever $i < j$. Our question is; *can such a sequence be of infinite length*? Or, is the length of such a sequence always finite? (Note that the replacement of the condition "whenever $i < j$" by "whenever $i \neq j$" doesn't affect the answer, since the number of sequents which are contractions of a given $\mathcal{S}_i$ is obviously finite.)

This problem is mathematically the same as the following. Let $\mathbf{N}$ be the set of all positive integers, and $\mathbf{N}^m$ be the set of all n-tuples $(k_1, \ldots, k_m)$ such that $k_i \in \mathbf{N}$ for each i. Define a binary relation $\leq^*$ on $\mathbf{N}^m$ by $(k_1, \ldots, k_m) \leq^* (h_1, \ldots, h_m)$ if and only if $k_i \leq h_i$ for each i. Clearly, $\leq^*$ is a partial order. A subset W of $\mathbf{N}^m$ is called an *antichain* if for any distinct $\mathbf{u}, \mathbf{v} \in W$ neither $\mathbf{u} \leq^* \mathbf{v}$ nor $\mathbf{v} \leq^* \mathbf{u}$ hold. Then, *are there infinite antichains of* $\mathbf{N}^m$? The answer is negative.

Lemma 26 *Any antichain of* $\mathbf{N}^m$ *is finite.*

This result can be proved by various ways. One way is to show the following lemma, from which the above lemma follows immediately. A partial order $\preceq$ on a given set Y is *well-founded* if there are no descending chains in Y, i.e. there are no elements $\{s_i\}_{i \in \mathbf{N}}$ in Y such that $s_{i+1} \prec s_i$ for each i. A partial order $\preceq$ on a given set Y is a *well partial order* if it is well-founded and moreover Y contains no infinite antichain with respect to $\preceq$.

Lemma 27 *1) The natural order on* $\mathbf{N}$ *is a well partial order.*
2) Suppose that both sets U *and* V *have well partial orders* $\leq_1$ *and* $\leq_2$*, respectively. Then, their direct product order* $\leq$ *on* $U \times V$ *is also a well partial order. Here,* $\leq$ *is defined by* $(u, v) \leq (u', v')$ *if and only if* $u \leq_1 u'$ *and* $v \leq_2 v'$.

Corollary 28 *Suppose that* $\langle \mathcal{S}_1, \mathcal{S}_2, \ldots \rangle$ *is a sequence of cognate sequents such that* $\mathcal{S}_i$ *is not a contraction of* $\mathcal{S}_j$ *whenever* $i < j$*. Then, it is finite.*

This result is known under the name of *Kripke's Lemma.* Now, let us consider the following "complete proof-search tree" of a given sequent $\mathcal{S}$. Take any sequent or any pair of sequents from which $\mathcal{S}$ follows by a single application of a rule of inference (except the cut rule, of course). Write any of them just over $\mathcal{S}$ as long as it consists only of subformulas of formulas in $\mathcal{S}$. We will regard each of them as an immediate predecessor of $\mathcal{S}$. Then, repeat this again for each immediate predecessors, and continue this as long as possible. If a sequent is initial, it has no immediate predecessors. Also, when a sequent thus generated has a contraction below it, we will not add it in the tree.

Then, it is easy to see that if $\mathcal{S}$ is provable then there exists a proof which forms a subtree of the "complete proof-search tree" of $\mathcal{S}$. We can see that the number of the branching at each "node" of the complete proof-search tree is finite, since each sequent has only finitely many, possible upper sequents. Furthermore, the length of each branch in the complete proof-search tree is finite by Kripke's Lemma. Thus, the complete proof-search tree of any $\mathcal{S}$ is finite, by the following lemma, called *König's Lemma.*

Lemma 29 *A tree is finite if and only if the number of the branching at each node of the tree is finite and the length of each branch in the tree is finite.*

Thus, it turns out that a given sequent $\mathcal{S}$ is provable if and only if there exists a proof of $\mathcal{S}$ which is a subtree of its complete proof-search tree. Since the complete proof-search tree is finite, we can check whether it contains a proof of $\mathcal{S}$ or not. Now we have the following.

Theorem 30 *Substructural propositional logics* $\mathbf{FL_{ec}}$ *and* $\mathbf{CFL_{ec}}$ *are decidable.*

On the other hand, the same difficulties as the classical predicate logic occur in the decision problems of predicate logics $\mathbf{FL_{ec}}$ and $\mathbf{CFL_{ec}}$. In fact, we can show the following.

Theorem 31 *Substructural predicate logics* $\mathbf{FL_{ec}}$ *and* $\mathbf{CFL_{ec}}$ *are undecidable.*

5.4 Notes

In 5.2, we have explained a way of incorporating the contraction rule into rules for logical connectives and as a result, of eliminating the *explicit* contraction rule. A similar idea can be applied to intuitionistic propositional logic. But, the system still contains some circularities in the proof-search, caused by the rule $(\supset\rightarrow)$. By dividing it into several rules, it is possible to remove these circularities and hence, we can get an efficient decision algorithm for intuitionistic propositional logic. This was done independently by Hudelmaier and Dyckhoff. See [21] and [13]. See also [40].

The fact that even predicate logics will be decidable when they have no contraction rules was mentioned already at the beginning of sixties by H. Wang. As for the details of proofs in 5.1, see [24] and [23]. On the other hand, the idea of proving the decidability of propositional logic with the contraction rule was first introduced by S. Kripke in 1959 and then is extended to $\mathbf{CFL_{ec}}$ by R.K. Meyer in his dissertation of 1966. See also [12] and [23]. Lemma 27 2) is attributed to G. Higman. For more information on the lemma and the related topics, see e.g. [25]. Our decision algorithm for logics with the contraction rule relies on Kripke's Lemma which is quite nonconstructive, and hence it is quite probable that the computational complexity of the algorithm will be high. Obviously, adding the distributive law will make the situation worse. In fact, in 1984 A. Urquhart showed the undecidablity of (even the positive fragment of) the relevant propositional logic $\mathbf{R}$, which is equivalent to the system $\mathbf{CFL_{ec}}$ with the distributive law. (See [46].) Restall in [39] obtained the decidability of some substructural logics using the proof theory based on display calculi. A survey of decision problems of substructural logics is given in [34].

6 Interpolation Property

W. Craig proved in 1957 the following result, now called *Craig's interpolation theorem* for classical logic. In the following, $V(D)$ denotes the set of all predicate symbols in a formula D. (Thus, $V(D)$ denotes the set of all propositional variables in D when we are concerned with propositional logic.) When Γ is a sequence of formulas $D_1, \ldots, D_m$, we define $V(\Gamma) = V(D_1) \cup \ldots \cup V(D_m)$.

Theorem 32 *If a formula $A \supset B$ is provable in classical logic then there exists a formula C such that both $A \supset C$ and $C \supset B$ are provable, and that $V(C) \subseteq V(A) \cap V(B)$.*

Any formula C satisfying the conditions in the above theorem is called, an *interpolant* of $A \supset B$. Precisely speaking, the above statement contains a certain inaccuracy as it doesn't mention the case when the set $V(A) \cap V(B)$ is empty. For, when our language $\mathcal{L}$ doesn't contain any predicate (or propositional) constant, there is no formula C with $V(C) \subseteq V(A) \cap V(B)$. As a matter of fact, the above theorem holds even for the case when $V(C) \subseteq V(A) \cap V(B)$ if $\mathcal{L}$ contains one of $\top$ and $\bot$. On the other hand, when it has neither predicate nor propositional constants at all, we have to add the following.

When $V(A) \cap V(B)$ is empty, either $\neg A$ or *B is provable.*

We will return back to this problem again later. The same result holds also for intuitionistic logic. Craig's interpolation theorem is known to be equivalent to *Beth's definability theorem* and also to *Robinson's consistency theorem.* (See e.g. [9] for the details, in which the equivalence is shown by using model-theoretic methods.)

We say that a logic **L** has the *interpolation property* if the statement in the above theorem holds for **L**. For intermediate propositional logics, i.e. propositional logics between intuitionistic logic and classical, L. Maksimova proved in 1977 the following striking result, which we mentioned already in the Introduction, by reducing it to the *amalgamation property* of varieties of Heyting algebras. Note here that there are uncountably many intermediate propositional logics.

Theorem 33 *Only seven intermediate propositional logics have the interpolation property.*

She proved also a similar theorem for modal propositional logics which are normal extensions of **S4**.

6.1 Maehara's method

There are various ways of proving the interpolation property. Craig obtained the theorem by using semantical methods. In 1960, S. Maehara showed that Craig's interpolation theorem for classical logic follows from the cut elimination theorem for **LK**. Different from semantical methods, the proof by Maehara's method will give us a concrete form of an interpolant of a formula $A \supset B$, once a cut-free proof of $A \supset B$ is given.

In the following, we will explain Maehara's method for intuitionistic logic and then for other nonclassical logics. Let $\mathbf{LJ}^\circ$ be the sequent calculus obtained from **LJ** by adding the following initial sequents for propositional constants $\top$ and $\bot$ (see subsection 2.3):

1. $\rightarrow \top$
2. $\bot \rightarrow$

It is easy to see that the cut elimination theorem holds also for $\mathbf{LJ}^\circ$ and hence that it is a conservative extension of **LJ**. For any given finite sequence Γ of formulas, we call a pair $\langle \Gamma_1 ; \Gamma_2 \rangle$ of (possibly empty) sequences of formulas Γ_1 and Γ_2, a *partition* of Γ, if the mutiset union of Γ_1 and Γ_2 is equal to Γ when regarding Γ, Γ_1 and Γ_2 as multisets of formulas. Now, we will show the following.

Lemma 34 *Suppose that a sequent $\Gamma \rightarrow \Delta$ is provable in $\mathbf{LJ}^\circ$ and that $\langle \Gamma_1 ; \Gamma_2 \rangle$ is any partition of Γ. Then, there exists a formula C such that both $\Gamma_1 \rightarrow C$ and $C, \Gamma_2 \rightarrow \Delta$ are provable in $\mathbf{LJ}^\circ$, and moreover that $V(C) \subseteq V(\Gamma_1) \cap V(\Gamma_2, \Delta)$.*

Proof. Let us also call such a formula C, an interpolant of $\Gamma \rightarrow \Delta$ (with respect to the partition $\langle \Gamma_1 ; \Gamma_2 \rangle$). Since the cut elimination theorem holds for $\mathbf{LJ}^\circ$, we can take a cut-free proof P of $\Gamma \rightarrow \Delta$. We will prove our theorem by induction on the length n of P. Suppose that $n = 1$. In this case, $\Gamma \to \Delta$ must be an initital sequent. Let it be $A \to A$. It is necessary to consider two partitions of A, i.e. $\langle A; \emptyset \rangle$ and $\langle \emptyset; A \rangle$. For the former case, we can take A itself for an interpolant. For the latter case, we can take $\top$ for an interpolant, as both sequents $\to \top$ and $\top, A \to A$ are provable. Similarly, we can show the existence of an interpolant for both $\to \top$ and $\bot \to$ with respect to any partition.

Next suppose that $n > 1$. Let I be the last rule of inference in P. By the hypothesis of induction, we can assume that there exists an interpolant of (each of) the upper sequent(s) of I with respect to an arbitrary partition. It is necessary to show that the lower sequent $\Gamma \to \Delta$ has also an interpolant with respect to any partition. We need to check this for any rule of inference of $\mathbf{LJ}^\circ$. In the following, we will show this only when I is either $(\vee \to)$ or $(\to \vee)$.

1) Suppose that Γ is $A \vee B, \Gamma'$ and that I is an application of $(\vee \to)$ as follow;

$$\frac{A, \Gamma' \to \Delta \quad B, \Gamma' \to \Delta}{A \vee B, \Gamma' \to \Delta}$$

1.1) Consider a partition $\langle A \vee B, \Gamma_1; \Gamma_2 \rangle$ of $A \vee B, \Gamma'$. Taking a partition $\langle A, \Gamma_1; \Gamma_2 \rangle$ of A, Γ' and a partition $\langle B, \Gamma_1; \Gamma_2 \rangle$ of B, Γ' and using the hypothesis of induction, we can get formulas C and D such that

(1a) both $A, \Gamma_1 \to C$ and $C, \Gamma_2 \to \Delta$ are provable
(1b) $V(C) \subseteq V(A, \Gamma_1) \cap V(\Gamma_2, \Delta)$
(2a) both $B, \Gamma_1 \to D$ and $D, \Gamma_2 \to \Delta$ are provable
(2b) $V(D) \subseteq V(B, \Gamma_1) \cap V(\Gamma_2, \Delta)$

Then, by (1a) and (2a) we have that both $A \vee B, \Gamma_1 \to C \vee D$ and $C \vee D, \Gamma_2 \to \Delta$ are provable, and moreover by (1b) and (2b) that $V(C \vee D) \subseteq V(A \vee B, \Gamma_1) \cap V(\Gamma_2, \Delta)$. Hence, the formula $C \vee D$ is an interpolant.

1.2) Next, consider a partition $\langle \Gamma_1; A \vee B, \Gamma_2 \rangle$ of $A \vee B, \Gamma'$. This time, we take a partition $\langle \Gamma_1; A, \Gamma_2 \rangle$ of A, Γ' and a partition $\langle \Gamma_1; B, \Gamma_2 \rangle$ of B, Γ'. Then, by the hypothesis of induction, we can get formulas C and D such that

(3a) both $\Gamma_1 \to C$ and $C, A, \Gamma_2 \to \Delta$ are provable
(3b) $V(C) \subseteq V(\Gamma_1) \cap V(A, \Gamma_2, \Delta)$
(4a) both $\Gamma_1 \to D$ and $D, B, \Gamma_2 \to \Delta$ are provable
(4b) $V(D) \subseteq V(\Gamma_1) \cap V(B, \Gamma_2, \Delta)$

It follows that both $\Gamma_1 \to C \wedge D$ and $C \wedge D, A \vee B, \Gamma_2 \to \Delta$ are provable, and that $V(C \wedge D) \subseteq V(\Gamma_1) \cap V(A \vee B, \Gamma_2, \Delta)$. Therefore, the formula $C \wedge D$ is an interpolant in this case.

2) Suppose next that I is an application of $(\to \vee 1)$ as follows:

$$\frac{\Gamma \to A}{\Gamma \to A \vee B}$$

Let $\langle \Gamma_1; \Gamma_2 \rangle$ be any partition of Γ. By the hypothesis of induction, there exists a formula C such that

(5a) both $\Gamma_1 \to C$ and $C, \Gamma_2 \to A$ are provable

(5b) $V(C) \subseteq V(\Gamma_1) \cap V(\Gamma_2, A)$
Then, clearly both $\Gamma_1 \to C$ and $C, \Gamma_2 \to A \vee B$ are provable, and it holds that $V(C) \subseteq V(\Gamma_1) \cap V(\Gamma_2, A \vee B)$. Hence, the formula C is an interpolant. In the same way, we can treat the case $(\to \vee 2)$.

In Lemma 34, take $A \to B$ for $\Gamma \to \Delta$ and consider the partition $\langle A; \emptyset \rangle$ of A. Then, we can show Craig's interpolation theorem for intuitionistic logic.

Theorem 35 *If a sequent $A \to B$ is provable in* $\mathbf{LJ}^\circ$ *then there exists a formula C such that both $A \to C$ and $C \to B$ are provable in* $\mathbf{LJ}^\circ$*, and moreover that $V(C) \subseteq V(A) \cap V(B)$.*

Maehara's method can be applied also to classical logic. In this case, we need to modify the definition of partitions as follows, as sequents in **LK** are of the form $\Gamma \to \Delta$ where both Γ and Δ are arbitrary finite sequences of formulas. Suppose that a sequent $\Gamma \to \Delta$ is given. Suppose moreover that the mutiset union of Γ_1 and Γ_2 (Δ_1 and Δ_2) is equal to Γ (Δ, respectively) as multisets of formulas. Then we say that $\langle(\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2)\rangle$ is a *partition* of $\Gamma \to \Delta$. Instead of Lemma 34, we show the following lemma, where the sequent calculus $\mathbf{LK}^\circ$ is an extension of **LK**, obtained in the same way as $\mathbf{LJ}^\circ$.

Lemma 36 *Suppose that a sequent $\Gamma \to \Delta$ is provable in* $\mathbf{LK}^\circ$ *and that $\langle(\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2)\rangle$ is any* partition *of $\Gamma \to \Delta$. Then, there exists a formula C such that both $\Gamma_1 \to \Delta_1, C$ and $C, \Gamma_2 \to \Delta_2$ are provable in* $\mathbf{LK}^\circ$*, and moreover that $V(C) \subseteq V(\Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$.*

Now, Craig's interpolation theorem for classical logic follows immediately from the above lemma.

Theorem 37 *If a sequent $A \to B$ is provable in* $\mathbf{LK}^\circ$ *then there exists a formula C such that both $A \to C$ and $C \to B$ are provable in* $\mathbf{LK}^\circ$*, and moreover that $V(C) \subseteq V(A) \cap V(B)$.*

We note here that the reason why we need to consider arbitrary partitions in Lemmas 34 and 36 comes from forms of rules for implication and negation. For instance, consider the following $(\to\supset)$:

$$\frac{p \wedge r, p \supset q \to q \vee s}{p \wedge r \to (p \supset q) \supset (q \vee s)}$$

In this case, though q is an interpolant of the upper sequent, it isn't an interpolant of the lower sequent. An interpolant of the lower sequent is p. Thus, if we don't consider interpolants corresponding to each partition, the proof using induction will break down here.

6.2 Eliminating constants

Consider any interpolant C given in Theorems 35 and 37. When it contains propositional constants $\top$ and $\bot$, we can simplify the formula C by using the following logical equivalence. Here, $A \sim B$ means that A is logically equivalent to B in **LJ**.

(1) $\top \supset E \sim E$, $\bot \supset E \sim \top$, $E \supset \top \sim \top$, $E \supset \bot \sim \neg E$,

(2) $\top \wedge E \sim E$, $\bot \wedge E \sim \bot$,

(3) $\top \vee E \sim \top$, $\bot \vee E \sim E$,

(4) $\neg\top \sim \bot$, $\neg\bot \sim \top$.

By repeating this simplification, C will be transformed into a formula C' which is either a formula without any propositional constant, or a propositional constant itself. In the former case, we get an interpolant with no propositional constants.

Let us consider the latter case. First, suppose that the set $V(A) \cap V(B)$ is nonempty in Theorem 35 or in Theorem 37. Then, C' is logically equivalent to either $F \supset F$ or $\neg(F \supset F)$, where F is an arbitrary formula consisting only of predicate symbols (or propositional variables) in $V(A) \cap V(B)$. Hence, we can get an interpolant with no propositional constants also in this case.

But when the set $V(A) \cap V(B)$ is empty, we cannot get such an interpolant. Let us suppose that C' is $\top$. In this case, $A \to \top$ and $\top \to B$ are provable. But, the first one is always provable, and the second is provable if and only if $\to B$ is provable. Suppose next that C' is $\bot$. Then, $A \to \bot$ and $\bot \to B$ are provable. The second sequent is always provable, and the first is provable if and only if $A \to$ is provable. Thus, we have the following.

Theorem 38 *Suppose that a formula $A \supset B$ is provable in* **LK**. *If the set $V(A) \cap V(B)$ is nonempty, there exists a formula C such that both $A \supset C$ and $C \supset B$ are provable in* **LK** *and $V(C) \subseteq V(A) \cap V(B)$. If the set $V(A) \cap V(B)$ is empty, then either $\neg A$ or B is provable in* **LK**. *This holds also for* **LJ**.

We note here that Craig's interpolation theorem for classical and intuitionistic logic doesn't necessarily imply the theorem for fragments of them. In fact, our proof of Lemma 34 shows that a conjunctive formula $C \wedge D$ is needed in the case 1.2) where the rule of inference is for the disjunction.

6.3 Digression — least and greatest interpolants

Consider Craig's interpolation theorem of the form of Theorem 38. It is easy to see that interpolants of $A \supset B$ are not always determined uniquely (up to logical equivalence). For instance, consider the following formula which is provable in **LJ**;

$$(p \wedge (p \supset (r \wedge s))) \supset (q \supset (r \vee s)).$$

We can see that all of formulas $r \wedge s$, r, s and $r \vee s$ are interpolants of the above formula.

We say that an interpolant C of a formula $A \supset B$ is *least* if $C \supset D$ is provable for any interpolant D of a formula $A \supset B$, and is *greatest* if $D \supset C$ is provable for any interpolant D of a formula $A \supset B$. In the above case, we can show that $r \wedge s$ and $r \vee s$ are least and greatest interpolants, respectively. In fact, we have the following.

Theorem 39 *Suppose that $A \supset B$ is a formula provable in the propositional* **LK** *and that the set $V(A) \cap V(B)$ is nonempty. Then, there exist both least and greatest interpolants of $A \supset B$ (in* **LK** *).*

Proof. To show our Theorem, we will use an elementary semantical method. In fact, the proof in the following will give an alternative proof of Craig's interpolation theorem for classical propositional logic. First, we recall an elementary fact that any formula $A \supset B$ is provable in the classical propositional logic if and only if it is a *tautology*, i.e. for any *valuation* f, $f(A \supset B) = \mathsf{t}$. Here, a valuation is a mapping from the set of propositional variables (occurring in the formula under consideration) to the set of truth values $\{ \mathsf{t}, \mathsf{f} \}$. Let U (and V) be the set $V(A) \backslash V(B)$ (and the set $V(B) \backslash V(A)$, respectively). Let us take an arbitrary propositional variable $s \in V(A) \cap V(B)$ and fix it.

Now, take any valuation f with the domain U. Let A_f be the formula obtained from A by replacing each variable $p \in U$ by $s \supset s$ if $f(p) = \mathsf{t}$ and by $\neg(s \supset s)$, otherwise. (Of course, you may simplify A_f by using the logical equivalence.) Define A^* to be the formula $\bigvee_f A_f$, where f runs over all valuations with the domain U. Similarly, we define the formula B_g for any valuation g with the domain V. Then, define B_* to be the formula $\bigwedge_g B_g$, where g runs over all valuations with the domain V.

We will show that A^* and B_* are least and greatest interpolants, respectively, of the formula $A \supset B$. First, it is easily seen that both $V(A^*)$ and $V(B_*)$ are subsets of $V(A) \cap V(B)$. Suppose that h is an arbitrary valuation with the domain $V(A)$ and h' is the restriction of h to the domain U. Clearly, $h(A) = h(A_{h'})$ holds. Therefore, if $h(A) = \mathsf{t}$ then $h(A_{h'}) = \mathsf{t}$ and hence $h(A^*) = \mathsf{t}$. Thus, $A \supset A^*$ is a tautology. Next, let f and j be arbitrary valuations with the domain U and the domain $V(B)$, respectively. Since U and $V(B)$ are disjoint, we can define a valuation f_j with the domain $V(A) \cup V(B)$ by $f_j(p) = f(p)$ if $p \in U$, and $f_j(p) = j(p)$ if $p \in V(B)$. Since $A \supset B$ is a tautology, $f_j(A \supset B) = \mathsf{t}$ for any such j. Since the restriction of f_j to U is f, $f_j(A) = f_j(A_f) = j(A_f)$. Thus, $f_j(A \supset B) = j(A_f \supset B)$. Hence, the formula $A_f \supset B$ is a tautology, and therefore is provable in classical logic, for any f with the domain U. Hence, $A^* \supset B$, which is $\bigvee_f A_f \supset B$ by the definition, is provable. Thus, we have shown that A^* is an interpolant. To show that it is least, assume that C is any interpolant. Obviously, $A \supset C$ is a tautology. Thus, by using the same argument as mentioned just in the above, we can infer that $A^* \supset C$ is provable. Thus, A^* is least. Similarly, we can show that B_* is greatest.

The above proof depends on the fact that a given formula is provable in classical propositional logic if and only if it is a tautology. Thus, it will be hard to apply the present method to other logics. On the other hand, A.M. Pitts succeeded in showing that the similar result holds for intuitionistic propositional logic, by using highly technical, proof-theoretic methods.

Theorem 40 *Suppose that $A \supset B$ is a formula provable in propositional* **LJ** *and that the set $V(A) \cap V(B)$ is nonempty. Then, there exist both least and greatest interpolants of $A \supset B$ (in* **LJ** *).*

6.4 Interpolation theorem for modal logics

We can apply Maehara's method to modal logics discussed in Section 3. There are two points, of which we should be careful. The first one is to eliminate propositional constants. In the modal logic **K**, $\Box\top$ is logically equivalent to $\top$, and $\bot \supset \Box\bot$ is provable. But, $\Box\bot \supset \bot$ (or equivalently, $\Diamond\top$) is not provable. Thus, **K** (with propositional constants) has the interpolation property of the form of Theorem 37, but doesn't have the interpolation property of the form of Theorem 38. On the other hand, when a modal logic has the axiom scheme T, i.e. $\Box A \supset A$, $\Box\bot$ becomes logically equivalent to $\bot$ in it, and hence we can eliminate propositional constants by using the method mentioned in 6.2.

The second point is that what is needed in applying Maehara's method is not the cut elimination theorem, but the subformula property. Thus, we can show that Craig's interpolation theorem holds for **S5**. In the following, we will explain the details. Let us consider the following *acceptable* cut rule.

$$\frac{\Gamma \to \Theta, A \quad A, \Pi \to \Delta}{\Gamma, \Pi \to \Theta, \Delta}$$

Take any partition $\langle(\Gamma_1, \Pi_1 : \Theta_1, \Delta_1); (\Gamma_2, \Pi_2 : \Theta_2, \Delta_2)\rangle$ of $\Gamma, \Pi \to \Theta, \Delta$. We will show that there exists an interpolant E such that both $\Gamma_1, \Pi_1 \to \Theta_1, \Delta_1, E$ and $E, \Gamma_2, \Pi_2 \to \Theta_2, \Delta_2$ are provable in **S5**, and $V(E) \subseteq V(\Gamma_1, \Pi_1, \Theta_1, \Delta_1) \cap V(\Gamma_2, \Pi_2, \Theta_2, \Delta_2)$.

1) Suppose that A is one of the subformulas of a formula in $\Gamma_2, \Pi_2, \Theta_2, \Delta_2$. Let us take the partition $\langle(\Gamma_1 : \Theta_1); (\Gamma_2 : \Theta_2, A)\rangle$ of $\Gamma \to \Theta, A$, and the partition $\langle(\Pi_1 : \Delta_1); (A, \Pi_2 : \Delta_2)\rangle$ of $A, \Pi \to \Delta$. Then, by the hypothesis of induction, there exist formulas C and D such that

(1a) both $\Gamma_1 \to \Theta_1, C$ and $C, \Gamma_2 \to \Theta_2, A$ are provable,
(1b) $V(C) \subseteq V(\Gamma_1, \Theta_1) \cap V(\Gamma_2, \Theta_2, A)$,
(2a) both $\Pi_1 \to \Delta_1, D$ and $A, D, \Pi_2 \to \Delta_2$ are provable,
(2b) $V(D) \subseteq V(\Pi_1, \Delta_1) \cap V(A, \Pi_2, \Delta_2)$.

By using the first sequents of (1a) and (2a), the sequent $\Gamma_1, \Pi_1 \to \Theta_1, \Delta_1, C \wedge D$ is shown to be provable. Next, by applying the cut rule to the second sequents of (1a) and (2a) (with the cut formula A), the sequent $C, D, \Gamma_2, \Pi_2 \to \Theta_2, \Delta_2$, and hence $C \wedge D, \Gamma_2, \Pi_2 \to \Theta_2, \Delta_2$ are shown to be provable. Moreover, $V(C \wedge D) \subseteq (V(C) \cup V(D)) \subseteq (V(\Gamma_1, \Pi_1, \Theta_1, \Delta_1) \cap V(\Gamma_2, \Pi_2, \Theta_2, \Delta_2))$, since $V(A) \subseteq V(\Gamma_2, \Pi_2, \Theta_2, \Delta_2)$. Thus, $C \wedge D$ is an interpolant.

2) Suppose otherwise. That is, A is not a subformula of any formula in $\Gamma_2, \Pi_2, \Theta_2, \Delta_2$. In this case, A must be a subformula of a formula in $\Gamma_1, \Pi_1, \Theta_1, \Delta_1$. Take the partition $\langle(\Gamma_1 : \Theta_1, A); (\Gamma_2 : \Theta_2)\rangle$ of $\Gamma \to \Theta, A$, and the partition $\langle(A, \Pi_1 : \Delta_1);$

$(\Pi_2 : \Delta_2))$ of $A, \Pi \to \Delta$. Then, the hypothesis of induction, there exist formulas F and G such that

(3a) both $\Gamma_1 \to \Theta_1, A, F$ and $F, \Gamma_2 \to \Theta_2$ are provable,

(3b) $V(F) \subseteq V(\Gamma_1, \Theta_1, A) \cap V(\Gamma_2, \Theta_2)$,

(4a) both $A, \Pi_1 \to \Delta_1, G$ and $G, \Pi_2 \to \Delta_2$ are provable,

(4b) $V(G) \subseteq V(A, \Pi_1, \Delta_1) \cap V(\Pi_2, \Delta_2)$.

By applying the cut rule to the first sequents of (3a) and (4a) with the cut formula A, we get the sequent $\Gamma_1, \Pi_1 \to \Theta_1, \Delta_1, F \vee G$. Also, by taking the second sequents of (3a) and (4a), and applying the weakening rules and $(\vee \to)$, we get the sequent $F \vee G, \Gamma_2, \Pi_2 \to \Theta_2, \Delta_2$. Moreover, using both (3b) and (4b), we can see that $F \vee G$ is an interpolant. Hence, we have the following.

Theorem 41 *Craig's interpolation theorem holds for any of the modal propositional logics* **K**, **KT**, **S4** *and* **S5**.

6.5 Interpolation theorem for substructural logics

Next, consider the interpolation theorem for substructural logics. We will take the formulation of sequent calculi for basic substructural logics given in Section 2. First, suppose that our language contains propositional constants. Then, similarly to the classical and the intuitionistic cases, we can show the following, by using Maehara's method.

Theorem 42 *Craig's interpolation theorem holds for substructural logics* **FL**, $\mathbf{FL_w}$, $\mathbf{FL_e}$, $\mathbf{FL_{ew}}$, $\mathbf{FL_{ec}}$, $\mathbf{CFL_e}$, $\mathbf{CFL_{ew}}$ *and* $\mathbf{CFL_{ec}}$ *(in the language with propositional constants).*

We notice that the above theorem holds for both propositional and predicate logics. To show the theorem for substructural logics without the exchange rule, like **FL** and $\mathbf{FL_w}$, we need to modify the definition of partitions in the following way. Let $\Gamma \to \Delta$ be any sequent (of **FL**, or $\mathbf{FL_w}$). Then, a triple $\langle \Gamma_1; \Gamma_2; \Gamma_3 \rangle$ is a partition of Γ, if the sequence $\Gamma_1, \Gamma_2, \Gamma_3$ is equal to Γ (without changing the order of formulas). Then, Theorem 42 follows by showing the following.

Lemma 43 *Suppose that a sequent $\Gamma \to \Delta$ is provable in* **FL** *(and* $\mathbf{FL_w}$ *) and that $\langle \Gamma_1; \Gamma_2; \Gamma_3 \rangle$ is any partition of Γ. Then, there exists a formula C such that both $\Gamma_2 \to C$ and $\Gamma_1, C, \Gamma_3 \to \Delta$ are provable in* **FL** *(and* $\mathbf{FL_w}$, *respectively), and moreover that $V(C) \subseteq V(\Gamma_2) \cap V(\Gamma_1, \Gamma_3, \Delta)$.*

As we mentioned in 2.3, it is enough to consider two propositional constants $\top$ and $\bot$ when a given logic has the weakening rule. Moreover, the following logical equivalences hold in $\mathbf{FL_w}$ for the logical connective $*$; for any formula E,

$$\top * E \sim E,\ E * \top \sim E,\ \bot * E \sim \bot,\ E * \bot \sim \bot.$$

Thus, by using the method discussed in 6.2, we can eliminate propositional constants. Let $\mathbf{FL_w}'$ be the sequent calculus in the propositional language without propositional constants, obtained from $\mathbf{FL_w}$ by deleting all initial sequents for propositional constants and rules of inference for them.

Corollary 44 *Suppose that a formula* $A \supset B$ *is provable in* $\mathbf{FL_w}'$. *If the set* $V(A) \cap V(B)$ *is nonempty, there exists a formula* C *such that both* $A \supset C$ *and* $C \supset B$ *are provable in* $\mathbf{FL_w}'$, *and* $V(C) \subseteq V(A) \cap V(B)$. *If the set* $V(A) \cap V(B)$ *is empty, then either* $\neg A$ *or* B *is provable in* $\mathbf{FL_w}'$. *This holds also for the calculi* $\mathbf{FL_{ew}}$ *and* $\mathbf{CFL_{ew}}$ *without propositional constants.*

On the other hand, we cannot eliminate propositional constants always in other logics mentioned in Theorem 42. But, as shown in Section 7, when the set $V(A) \cap V(B)$ is empty, the formula $A \supset B$ is never provable in these basic substructural (predicate) logics without the weakening rule.

6.6 Notes

For the details of a proof of Craig's interpolation theorem for classical logic by Maehara's method, see e.g. [43]. Theorem 33 and the related result on modal logics are shown by Maksimova in [27] and [28], respectively.

Theorem 40 by Pitts is given in [36], in which he took a sequent calculus for intuitionistic propositional logic, introduced independently by Hudelmaier and Dyckhoff. (See Notes 5.4.) A proof of the interpolation theorem for modal logic **S5** based on the subformula property of the sequent calculus **GS5** is mentioned in [42]. A general result on the interpolation theorem for modal logics formulated in tableau calculi is given by Rautenberg [38].

Interpolation theorems for various substructural logics are discussed in [35] and also in [32]. Recently, Bayu Surarso succeeded to extend Maehara's method and proved the interpolation theorem for many *distributive* substructural logics (see [3]).

7 Variable Sharing and Variable Separation

We will add two more examples from the study of substructural logics which show the usefulness of proof-theoretic methods. The first one is the variable sharing property which is known for some relevant logics. The second one is Maksimova's principle of variable separation. We can see that the weakening rule plays an important role in them.

7.1 Variable sharing property of substructural logics without the weakening rule

We say that a logic **L** has the *variable sharing property*, when for any formula $A \supset B$ *without propositional constants*, if $A \supset B$ is provable in **L** then $V(A) \cap V(B)$ is nonempty. It is clear by the definition that if a logic **L** has the variable sharing property and **L**′ is weaker than **L** (i.e. every fromula provable in **L**′ is also provable in **L**), then **L**′ has also the property. In [1], it is shown that the relevant propositional logic **R** has the variable sharing property.

Theorem 45 *The predicate logic* $\mathbf{CFL_{ec}}$ *has the variable sharing property, and a fortiori any of* $\mathbf{FL}$, $\mathbf{FL_e}$, $\mathbf{FL_{ec}}$ *and* $\mathbf{CFL_e}$ *has the variable sharing property.*

We will give here a proof of the above theorem for $\mathbf{FL_{ec}}$. The proof for $\mathbf{CFL_{ec}}$ can be shown essentially in the same way. Recall that every sequent in $\mathbf{FL_{ec}}$ is of the form $\Pi \to \Lambda$, where Λ contains at most one formula. To show the above theorem, it is enough to prove the following.

Theorem 46 *Suppose that* $\Gamma \to A$ *is a sequent containing no propositional constants such that* Γ *is nonempty. If* $\Gamma \to A$ *is provable in* $\mathbf{FL_{ec}}$ *then* $V(\Gamma) \cap V(A)$ *is nonempty.*

Proof. Suppose that $\Gamma \to A$ satisfies the conditions in the theorem and is provable in $\mathbf{FL_{ec}}$. Then, it has a cut-free proof P. Clearly, P contains neither initial sequents for propositional constants nor rules for propositional constants. We will show that there exists a branch in P starting from an initial sequent and ending at the sequent $\Gamma \to A$ such that every sequent $\Pi \to \Lambda$ in the branch contains at least two formulas, i.e. either both of Π and Λ are nonempty, or Λ is empty but Π contains at least two formulas. Of course, $\Gamma \to A$ contains at least two formulas. For other sequents in P, we will show this by checking that for any rule of inference of $\mathbf{FL_{ec}}$, if the lower sequent contains at least two formulas then (at least one of) the upper sequent(s) contains at least two formulas. (For rules ($\supset\to$) and ($\to *$), one of the upper sequents may not contain two formulas. Of course, the above doesn't hold in general when we have the weakening rule.)

Now, let B be any such branch. For any sequent $\Pi \to \Lambda$ containing at least two formulas, we say that $\langle \Pi_1; \Pi_2, \Lambda \rangle$ is a *partition* of $\Pi \to \Lambda$ if the multiset union of Π_1 and Π_2 is equal to Π, and both Π_1 and Π_2, Λ are nonempty. By our assumption on $\Pi \to \Lambda$, there exists at least one partition of $\Pi \to \Lambda$. We say that a partition $\langle \Pi_1; \Pi_2, \Lambda \rangle$ *shares variables* if $V(\Pi_1) \cap V(\Pi_2, \Lambda)$ is nonempty. Then, we can show the following easily by using induction on the length of the branch B.

For any sequent $\Pi \to \Lambda$ *in* B, *every partition of* $\Pi \to \Lambda$ *shares variables.*

In particular, taking $\Gamma \to A$ for $\Pi \to \Lambda$, we have that the partition $\langle \Gamma; A \rangle$ shares variables. This completes our proof.

7.2 Maksimova's principle of variable separation

L. Maksimova showed in 1976 the following theorem for some relevant logics, including $\mathbf{R}$ and $\mathbf{E}$.

Suppose that propositional formulas $A_1 \supset A_2$ *and* $B_1 \supset B_2$ *have no propositional variables in common. If a formula* $A_1 \wedge B_1 \supset A_2 \vee B_2$ *is provable, then either* $A_1 \supset A_2$ *or* $B_1 \supset B_2$ *is provable.*

When the above property holds for a given logic $\mathbf{L}$, we say that *Maksimova's principle of variable separation* (or, simply *Maksimova's principle*) holds for $\mathbf{L}$. In this subsection, we will show the following by using proof-theoretic methods.

Theorem 47 *Maksimova's principle holds for propositional* **FL**, $\mathbf{FL_e}$, $\mathbf{FL_{ec}}$, $\mathbf{FL_w}$, $\mathbf{FL_{ew}}$, $\mathbf{CFL_e}$, $\mathbf{CFL_{ec}}$ *and* $\mathbf{CFL_{ew}}$.

Consult the paper [30] for the details of the proof of this theorem. While the proof of Maksimova's principle for logics with the weakening rule in [30] needs their interpolation property, we can prove Maksimova's principle for logics without the weakening rule without using the interpolation property explicitly. So, there might be some relations between Maksimova's principle and Craig's interpolation theorem (of the form mentioned in Corollary 44) at least when a logic has the weakening rule. In this respect, it will be worthwhile to note a result by Maksimova which says that for any intermediate propositional logic **L**, if **L** has the Craig's interpolation property then Maksimova's principle holds for it.

To see this relation, let us consider Maksimova's principle for classical logic. Suppose that formulas $A_1 \supset A_2$ and $B_1 \supset B_2$ have no propositional variables in common and that the sequent $A_1 \wedge B_1 \to A_2 \vee B_2$ is provable in **LK**. Then $A_1, B_1 \to A_2, B_2$ is also provable and hence $A_1, \neg A_2 \to \neg B_1, B_2$ is provable. By using Craig's interpolation theorem for classical logic in the (generalized) form of Theorem 38, we can infer that either $A_1, \neg A_2 \to$ or $\to \neg B_1, B_2$ is provable. Hence, either $A_1 \to A_2$ or $B_1 \to B_2$ is provable in classical logic.

As an example of logics without the weakening rule, we will take $\mathbf{FL_{ec}}$ and will give a sketch of the proof of Maksimova's principle for it. For the sake of brevity, we assume that our language doesn't contain any propositional constants. In the following, for a formula D, $S(D)$ denotes the set of all subformulas of D and for a sequence $C_1, \ldots, C_m$ of formulas, $S(C_1, \ldots, C_m)$ denotes $S(C_1) \cup \ldots \cup S(C_m)$. Finally, for a sequent $\Gamma \to \Delta$, $S(\Gamma \to \Delta)$ denotes $S(\Gamma) \cup S(\Delta)$.

Lemma 48 *Suppose that formulas $A_1 \supset A_2$ and $B_1 \supset B_2$ have no propositional variables in common. If $\Pi \to \Lambda$ is a sequent satisfying the following three conditions;*
1) $S(\Pi \to \Lambda) \subseteq (S(A_1 \wedge B_1) \cup S(A_2 \vee B_2))$,
2) $S(\Pi \to \Lambda) \cap (S(A_1) \cup S(A_2)) \neq \emptyset$,
3) $S(\Pi \to \Lambda) \cap (S(B_1) \cup S(B_2)) \neq \emptyset$,
then it is not provable in $\mathbf{FL_{ec}}$.

Proof. Let us suppose that $\Pi \to \Lambda$ is provable. Then there exists a cut-free proof P of $\Pi \to \Lambda$. By checking every application of a rule in P, we can see that if the lower sequent satisfies the above three conditions then at least one of its upper sequents satisfies these three conditions. Thus, at least one of the initial sequents in P must also satisfy them. Obviously, this is a contradiction.

Corollary 49 *Suppose that $A_1 \supset A_2$ and $B_1 \supset B_2$ have no propositional variables in common. Moreover, suppose that* P *is a cut-free proof of $\Pi \to \Lambda$ in* $\mathbf{FL_{ec}}$ *such that $S(\Pi \to \Lambda) \subseteq (S(A_1 \wedge B_1) \cup S(A_2 \vee B_2))$ and $S(\Pi \to \Lambda) \cap (S(A_1) \cup S(A_2)) \neq \emptyset$. Then every sequent $\Gamma' \to \Delta'$ appearing in* P *satisfies the following condition* $(*)$:

(*) $S(\Gamma' \to \Delta') \subseteq (S(A_1 \wedge B_1) \cup S(A_2 \vee B_2))$ *and* $S(\Gamma' \to \Delta') \cap (S(A_1) \cup S(A_2)) \neq \emptyset$.

Hence, there are no applications of the following rules of inference in P*;*

$$\frac{\Gamma, B_1, \Sigma \to \Delta}{\Gamma, A_1 \wedge B_1, \Sigma \to \Delta}\ (\wedge 2 \to) \qquad \frac{\Gamma \to A_1 \quad \Gamma \to B_1}{\Gamma \to A_1 \wedge B_1}\ (\to \wedge)$$

$$\frac{\Gamma, A_2, \Sigma \to \Delta \quad \Gamma, B_2, \Sigma \to \Delta}{\Gamma, A_2 \vee B_2, \Sigma \to \Delta}\ (\vee \to) \qquad \frac{\Gamma \to B_2}{\Gamma \to A_2 \vee B_2}\ (\to \vee 2)$$

Proof. We can prove that if the lower sequent of a rule I satisfies the condition (*) then (both of) the upper sequent(s) will satisfy the condition (*), for each rule I in P. In fact, this can be shown by checking each rule of $\mathbf{FL_{ec}}$. Since $\Pi \to \Lambda$ satisfies (*), every sequent in P must satisfy (*). Next suppose that any one of the rules in our corollary is applied in P. Then, from the form of these rules it follows that at least one of the upper sequent(s), say S, contains either B_1 or B_2. On the other hand, the sequent S must satisfy also (*). Thus, S satisfies all of the three conditions in Lemma 48. Clearly, this is a contradiction since S must be provable.

Theorem 50 *Maksimova's principle holds for* $\mathbf{FL_{ec}}$*. More precisely, suppose that formulas* $A_1 \supset A_2$ *and* $B_1 \supset B_2$ *have no propositional variables in common. Then the following holds.*
1) If the sequent $A_1 \wedge B_1 \to A_2 \vee B_2$ *is provable, then either* $A_1 \to A_2$ *or* $B_1 \to B_2$ *is provable.*
2) If the sequent $A_1 \wedge B_1 \to A_2$ *is provable, then the sequent* $A_1 \to A_2$ *is provable.*
3) If the sequent $A_1 \to A_2 \vee B_2$ *is provable, then the sequent* $A_1 \to A_2$ *is provable.*

Proof. Suppose that the sequent $A_1 \wedge B_1 \to A_2 \vee B_2$ is provable in $\mathbf{FL_{ec}}$. Clearly, it is not an initial sequent. Then, it is easy to see that the lowest part of its cut-free proof P is of the following form, where I is a rule of inference other than the exchange and contraction rules.

$$\cfrac{\cfrac{\vdots}{A_1 \wedge B_1, \ldots, A_1 \wedge B_1 \to A_2 \vee B_2}\ (I)}{A_1 \wedge B_1 \to A_2 \vee B_2}\ some(c \to)(e \to)$$

Then, I must be one of the following rules of inference: $(\wedge 1 \to), (\wedge 2 \to), (\to \vee 1)$, and $(\to \vee 2)$. If it is $(\wedge 1 \to)$, then

$$\frac{A_1 \wedge B_1, \ldots, A_1, \ldots, A_1 \wedge B_1 \to A_2 \vee B_2}{A_1 \wedge B_1, \ldots, A_1 \wedge B_1, \ldots, A_1 \wedge B_1 \to A_2 \vee B_2}\ (\wedge 1 \to)$$

Here, the antecedent of the upper sequent of I contains only one A_1 and others are $A_1 \wedge B_1$. Then by Corollary 49, the proof of the upper sequent and hence P cannot contain any application of rules of inference, mentioned in Corollary 49. It means that when an occurrence of the formulas $A_1 \wedge B_1$ and $A_2 \vee B_2$ is introduced in the proof P it must be introduced only by rules of the following form:

$$\frac{\Gamma, A_1, \Sigma \to \Delta}{\Gamma, A_1 \wedge B_1, \Sigma \to \Delta} (\wedge 1 \to) \qquad \frac{\Gamma \to A_2}{\Gamma \to A_2 \vee B_2} (\to \vee 1)$$

Now, we will replace all occurrences of $A_1 \wedge B_1$ by A_1 and of $A_2 \vee B_2$ by A_2 in P and remove redundant applications which are caused by this replacement. (Note here that these $A_1 \wedge B_1$ and $A_2 \vee B_2$ may be introduced in several places in P.) This will give us a proof of $A_1 \to A_2$ in $\mathbf{FL_{ec}}$. When I is any one of other rules mentioned in the above, we will be able to get the proof of either $A_1 \to A_2$ or $B_1 \to B_2$ in the similar way.

We will omit the proof of Maksimova's principle for logics with the weakening rule, as more complicated arguments are necessary for this case.

7.3 Notes

The proof of Theorem 45 given in the above is based on the proof by H. Naruse in his dissertation for a master's degree in 1996.

Maksimova gave an example of a relevant logic for which Maksimova's principle doesn't hold, in the paper [26]. Some relationships among Maksimova's principle, the disjunction property and Halldén-completeness for intermediate propositional logics are studied in [7]. In [30], we proved that Maksimova's principle holds for many substructural logics. Theorem 47 shows one of the results in [30]. We extended our method also to relevant logics and showed Maksimova's principle for positive fragments of **R**, **RW** and **TW**. In these proofs, one may see roles of the weakening rule in Maksimova's principle.

8 A Short Remark

In the present paper, we have introduced some basic results and techniques in the proof theory of nonclassical logics. To show the usefulness of proof-theoretic methods in the study of nonclassical logics, we presented here various interesting consequences of the cut elimination theorem. Though proof-theoretic methods will be quite powerful and promising, the proof-theoretic study of nonclassical logics often tends to concentrate only in proving the cut elimination theorem and to stop there. But, we believe that the fertility of proof-theoretic methods will be probably attained by deeper investigations of structures of proofs, which are above and beyond the cut elimination theorem.

References

[1] Anderson, A.R. and N.D. Belnap, Jr., *Entailment: The Logic of Relevance and Necessity* I, Princeton University Press, 1975.

[2] Anderson, A.R., Belnap, Jr. N.D. and J.M. Dunn, *Entailment: The Logic of Relevance and Necessity* II, Princeton University Press, 1992.

[3] Bayu Surarso, Interpolation theorem for some distributive logics, to appear in *Mathematica Japonica.*

[4] Bayu Surarso and H. Ono, Cut elimination in noncommutative substructural logics, *Reports on Mathematical Logic,* 30 (1996), pp.13-29.

[5] Bull, R. and K. Segerberg, Basic modal logic, pp.1-88 in *Handbook of Philosophical Logic* II, edited by D. Gabbay and F. Guenthner, Reidel, 1984.

[6] Buss, S.R. ed, *Handbook of Proof Theory*, Elsevier, 1998.

[7] Chagrov, A. and M. Zakharyaschev, The undecidability of the disjunction property of propositional logics and other related problems, *Journal of Symbolic Logic,* 58 (1993), pp.967-1002.

[8] Chagrov, A. and M. Zakharyaschev, *Modal Logic*, Oxford Logic Guides 35, Oxford University Press, 1997.

[9] Chang, C.C. and H.J. Keisler, *Model Theory* 3rd edition, North-Holland, 1990.

[10] van Dalen, D., *Logic and Structure* 3rd edition, Springer-Verlag, 1997.

[11] Došen, K. and P. Schroeder-Heister eds, *Substructural Logics*, Oxford University Press, 1994.

[12] Dunn, J.M., Relevance logic and entailment, pp.117-224 in *Handbook of Philosophical Logic* III, edited by D. Gabbay and F. Guenthner, Reidel, 1986.

[13] Dyckhoff, R., Contraction-free sequent calculi for intuitionistic logic, *Journal of Symbolic Logic*, 57 (1992), pp.795-807.

[14] Ebbinghaus, H.-D., J. Flum and W. Thomas, *Mathematical Logic*, Springer-Verlag, 1984.

[15] Fitting, M., Subformula results in some propositional modal logics, *Studia Logica*, 37 (1978), pp.387-391.

[16] Fitting, M., *Proof Methods for Modal and Intuitionistic Logics*, Reidel, 1983.

[17] Gentzen, G., Untersuchungen über das logische Schliessen, *Mathematische Zeitschrift*, 39 (1934), pp.176-210, pp.405-413.

[18] Girard, J.-Y., Linear logic, *Theoretical Computer Science*, 50 (1987), pp.1-102.

[19] Goré, R., Tableau methods for modal and temporal logics, in *Handbook of Tableau Methods*, edited by M. D'Agostino, D. Gabbay, R. Hähnle and J. Posegga, Kluwer, to be published in 1998.

[20] Hilbert, D. and W. Ackermann, *Grundzüge der Theoretischen Logik*, Springer-Verlag, 1928.

[21] Hudelmaier, J., An $O(n \log n)$-space decision procedure for intuitionistic propositional logic, *Journal of Logic and Computation*, 3 (1993), pp.63-75.

[22] Hughes, G.E. and M.J. Cresswell, *A New Introduction to Modal Logic*, Routledge, 1996.

[23] Kiriyama, E. and H. Ono, The contraction rule and decision problems for logics without structural rules, *Studia Logica* 50 (1991), pp.299-319.

[24] Komori, Y., Predicate logics without the structure rules, *Studia Logica*, 45 (1986), pp.393-404.

[25] Kruskal, J.B., Well-quasi-ordering, the tree theorem, and Vazsonyi's conjecture, *Transactions of American Mathematical Society*, 95 (1960), pp.210-225.

[26] Maksimova, L., The principle of separation of variables in propositional logics, *Algebra i Logika,* 15 (1976), pp.168-184.

[27] Maksimova, L., Craig's theorem in superintuitionistic logics and amalgamable varieties of pseudo-Boolean algebras, *Algebra i Logika* 16 (1977), pp.643-681.

[28] Maksimova, L., Interpolation theorems in modal logics and amalgamable varieties of topological Boolean algebras, *Algebra i Logika* 18 (1979), pp.556-586.

[29] Moortgat, M., *Categorical Investigations — Logical and Linguistic Aspects of the Lambek Calculus*, Foris, 1988.

[30] Naruse, H., Bayu Surarso and H. Ono, A syntactic approach to Maksimova's principle of variable separation, *Notre Dame Journal of Formal Logic* 39 (1998), pp.94-113.

[31] Okada, M., An introduction to linear logic: expressiveness and phase semantics, in *Theories of Types and Proofs*, MSJ-Memoir 2, edited by M. Takahashi, M. Okada and M. Dezani, Mathematical Society of Japan, 1998.

[32] Ono, H., Structual rules and a logical hierarchy, pp.95-104 in *Mathematical Logic*, edited by P.P. Petkov, Plenum Press, 1990.

[33] Ono, H., Semantics for substructural logics, pp.259-291 in *Substructural Logics*, edited by K. Došen and P. Schroeder-Heister, Oxford Univ. Press, 1993.

[34] Ono, H., Decidability and finite model property of substructural logics, pp.263-274 in *The Tbilisi Symposium on Logic, Language and Computation: Selected Papers*, edited by J. Ginzburg et al, CSLI Publications, 1998.

[35] Ono, H. and Y. Komori, Logics without the contraction rule, *Journal of Symbolic Logic* 50 (1985), pp.169-201.

[36] Pitts, A.M., On an interpretation of second order quantification in first order intuitionistic propositional logic, *Journal of Symbolic Logic* 57 (1992), pp.33-52.

[37] Prawitz, D., *Natural Deduction. A Proof-theoretical Study*, Almquist and Wiksell, 1965.

[38] Rautenberg, W., Modal tableau calculi and interpolation, *Journal of Philosophical Logic*, 12 (1983), pp.403-423. See also "Corrections for modal tableau calculi and interpolation by W. Rautenberg, JPL 12 (1983)", *Journal of Philosophical Logic*, 14 (1985), p.229.

[39] Restall, G., Displaying and deciding substructural logics 1: Logics with contraposition, *Journal of Philosophical Logic* 27 (1998), pp.179-216.

[40] Schwichtenberg, H., Normalization, pp.201-235 in *Logic, Algebra and Computation*, edited by F. Brauer, Springer-Verlag, 1991.

[41] Takahashi, M., A primer to proofs and types, pp.1-44 in *Theories of Types and Proofs*, MSJ-Memoir 2, edited by M. Takahashi, M. Okada and M. Dezani, Mathematical Society of Japan, 1998.

[42] Takano, M., Subformula property as a substitute for cut-elimination in modal propositional logics, *Mathematica Japonica* 37 (1992), pp.1129-1145.

[43] Takeuti, G., *Proof Theory* 2nd edition, North-Holland, 1987.

[44] Troelstra, A.S., *Lectures on Linear Logic*, CSLI Lecture Notes 29, Stanford University, 1992.

[45] Troelstra, A.S., and H. Schwichtenberg, *Basic Proof Theory*, Cambridge Univ. Press, 1996.

[46] Urquhart, A., The undecidability of entailment and relevant implication, *Journal of Symbolic Logic* 49 (1984), pp.1059-1073.

[47] Zeman, J.J., *Modal Logic, The Lewis-Modal Systems*, Oxford Univ. Press, 1973.