RANDOM MATRICES, LOG-GASES AND THE CALOGERO-SUTHERLAND MODEL

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CONTENTS

0. Introduction

Random matrices, log-gases and the Calogero-Sutherland model are inter-related topics in Mathematical Physics. They are also topics of independent interest and significance. In particular, random matrices were first studied in Mathematical Statistics as long ago as the 1920's because of their relevance to the theory of error distributions in the theory of multicomponent data sampling (see e.g. [1]). In the 1950's E.P. Wigner introduced random symmetric matrices as a model of the Hamiltonian of complex nuclei: it was hypothesized that the eigenvalues of these random matrices have the same statistical properties as the energy spectra of complex nuclei. This was soon confirmed experimentally by comparing a histogram of the spacing between consecutive energy levels in complex nuclei with the theoretical p.d.f. for this statistic given by Wigner (see e.g. [2]).

In the early 1960's F.J. Dyson undertook a detailed study of random matrices as they apply to the statistical properties of the energy levels of complex systems (see e.g. [3]). One of the many new ideas to come out of this work was the log-gas analogy. For the random matrices under consideration, Dyson found that the eigenvalue p.d.f. could be expressed in the form

$$\frac{1}{C_\beta} \exp\Big[-\beta\Big(\sum_{j=1}^{N} U_1(x_j) + \sum_{j<k} U_2(x_j, x_k)\Big)\Big] \tag{0.1}$$

with the parameter β related to the symmetry of the matrix (e.g. $\beta = 1$ for symmetric matrices). Now this can be interpreted as the Boltzmann factor for a classical gas with potential energy

$$\sum_{j=1}^{N} U_1(x_j) + \sum_{j<k} U_2(x_j, x_k),$$

in equilibrium at inverse temperature β. Furthermore, by a suitable change of variables $y_j = f(x_j)$, it turns out that the two-body potential U_2 is such that

$$U_2(y_j, y_k) = -\log|y_j - y_k|.$$

The classical gas is therefore the log-gas. This is a special potential because it is the potential between two charges in two-dimensional electrostatics. All the charges in this log-gas are of the same magnitude and sign. For the cases in which the domain is unbounded (i.e. the possible values of the eigenvalues is unbounded), the one-body potential U_1 acts so as to stop the particles repelling to infinity by attracting them towards the origin.

The p.d.f. (0.1) also occurs as the square of the ground state wavefunction for some quantum many body systems. The Schrödinger operators for these systems contain one and two body potentials only (see (1.1) below). The coupling constant for the two-body potential is a function of the quantity β in (0.1) and can take any positive real value. Furthermore, in some asymptotic limit at least, the two-body potential is proportional to $1/|x_j - x_k|^2$. Quantum many body systems with these properties are said to belong to the Calogero-Sutherland type.

In Section 1 four random matrix ensembles will be considered: Gaussian, circular, Laguerre (Wishart) and Jacobi. Matrices from these ensembles are defined in terms of the p.d.f. for the elements of the matrix itself, or some decomposition of the matrix. Furthermore, for each of the ensembles there are three symmetry classes: orthogonal ($\beta = 1$), unitary ($\beta = 2$) and symplectic ($\beta = 4$) to consider.

We address the problem of calculating the eigenvalue p.d.f. from knowledge of the p.d.f. for the elements. The main technical task is to calculate the Jacobian for the change of variables from the elements of the matrix to its eigenvalues and variables associated with the eigenvectors. In all cases the Jacobian factorizes into a function of the eigenvalues, which can be given explicitly, and a function of the variables associated with the eigenvectors. Since the eigenvalue p.d.f. is obtained by integrating over these latter variables, their contribution can be absorbed into the normalization constant.

In all four cases the eigenvalue p.d.f. is found to be of the form (0.1), and this p.d.f. can be identified both with the Boltzmann factor of a log-gas and the absolute value squared of the ground state of a Calogero-Sutherland type quantum system. Since the analogies are at the level of the p.d.f., it follows that the distribution functions – which are integrals over the p.d.f. (see (1.36) below) – are identical for the three systems. Thus the distribution functions for the eigenvalues of a particular random matrix ensemble is the same as for the corresponding log-gas, which in turn is the same as the static ground state distribution of the corresponding quantum system. This holds true for $\beta = 1, 2$ and 4. For all $\beta \geq 0$ the distribution functions for the log-gas are identical to the static ground state distributions for the corresponding quantum system. A significant feature of the log-gas system is that the electrostatic origin of the two-body potential allows predictions to be made for the behaviour of the one and two body distribution functions in certain long-wavelength 'macroscopic limits'. Thus in Section 1.8 we note some analytic

predictions for the behaviour of these distributions, which due to the analogies must hold true for the corresponding random matrix and quantum system also.

Dyson noted not only an analogy between the eigenvalue p.d.f.'s of random matrices and the Boltzmann factor of log-gas systems, but also an analogy between dynamical theories of random matrices and the log-gas. A dynamical theory of random matrices is constructed by introducing a parameter in the definition of the ensembles, which plays the role of a time-like variable. Section 2 begins by considering parameter-dependent Gaussian random matrices. As the parameter τ tends to infinity, these random matrices coincide with the Gaussian random matrices of Section 1. At $\tau = 0$ the random matrix is prescribed – its eigenvalue p.d.f. is the initial condition. Thus in parameter-dependent random matrices there is an evolution from the prescribed eigenvalue p.d.f. to the eigenvalue p.d.f. for the Gaussian random matrices of Section 1 (the equilibrium state) as τ increases. Parameter-dependent Gaussian random matrices are defined so that the joint distribution of their elements satisfies the Fokker-Planck equation for overdamped Brownian motion in a harmonic potential. By changing variables from the elements of the matrix to the eigenvalues and variables associated with the eigenvectors, the eigenvalue p.d.f. can also be specified as the solution of a Fokker-Planck equation for overdamped Brownian motion. However, in addition to the one-body harmonic term, the Fokker-Planck equation also contains a two-body logarithmic potential which is precisely U_2 in (0.1). Thus, in the sense that parameter-dependent Gaussian random matrices represents Brownian dynamics of the Gaussian random matrices of Section 1, the analogy between Gaussian random matrices and the log-gas extends beyond the equilibrium description to the full Brownian dynamics.

Parameter-dependent theories can also be formulated for the other random matrix ensembles introduced in Section 1. In each case the eigenvalue p.d.f. satisfies a Fokker-Planck equation which can also be interpreted as describing the overdamped Brownian dynamics of the corresponding classical log-gas. The dynamical analogy also extends to the quantum Calogero-Sutherland systems. In general the Fokker-Planck operator (see (2.2) below) can be transformed into a Schrödinger operator and the Fokker-Planck equation becomes the Schrödinger equation in imaginary time. For the particular Fokker-Planck operators describing the Brownian dynamics of random matrices, the corresponding Schrödinger operators are precisely those noted in Section 1, for which the square of the ground state is given by (0.1).

Correlations fundamental to many-body dynamical problems are the dynamic density-density correlation, and the current-current correlation. The dynamical density-density correlation for the Brownian motion problem is defined in terms of the Green function for the Fokker-Planck equation by (2.17)-(2.19) below. The Green function solution of the Fokker-Planck equation is simply related to the Green function solution of the Schrödinger equation for the corresponding Calogero-Sutherland model (see (2.11)). This implies that the dynamical density-density correlation in the log-gas and the Calogero-Sutherland model are also related: in fact they are identical if the initial distribution in the Brownian motion problem is chosen to be the equilibrium distribution.

One use of the dynamical density-density correlation $S(x,t)$ in the theory of parameter-dependent random matrices relates to linear statistics [4]:

$$A(x,t) = \sum_{j=1}^{N} a(x_j(t)).$$

Properties of the distribution of such quantities can be empirically determined from experimental data, one example being the integrated correlator

$$\xi_a = \int_0^\infty \langle (A(x_1,t) - \langle A \rangle)(A(x_2,t) - \langle A \rangle) \rangle \, dt$$

Here the average over positions $\langle\,\rangle$ is directly related to $S(x,t)$:

$$\langle (A(x_1,t) - \langle A \rangle)(A(x_2,t) - \langle A \rangle) \rangle = \int_{-\infty}^{\infty} dx_1 a(x_1) \int_{-\infty}^{\infty} dx_2 a(x_2)\, S(x_1 - x_2, t).$$

In the Calogero-Sutherland quantum many body systems, $S(x,t)$ is of interest because it probes the excitation spectrum. Indeed $S(x,t)$ can be rewritten as an infinite sum involving the eigenstates (see (4.13)).

The dynamical density-density correlation and the current-current correlation are related by (2.22) as a consequence of the continuity equation (2.25) which relates the microscopic density and current for both classical and quantum systems. We will show that the density-density correlation must satisfy a generalization of the f-sum rule (Proposition 2.3), which constrains its small-time behaviour. Also, for Fokker-Planck operators with only one and two body potentials, we will show that the static density-density distribution and current-current distribution are simply related (Proposition 2.4).

The dynamical density-density correlation can be expanded as an infinite series involving the eigenfunctions of the appropriate Schrödinger operator (or equivalently, the Fokker-Planck operator). In Section 3 the mathematical theory of these eigenfunctions is outlined. For the four Schrödinger operators corresponding to the four random matrix ensembles introduced in Section 1, the eigenfunctions can be written as a product of the ground state and an N-variable symmetric polynomial. In the circular case, these polynomials are known as Jack polynomials, and their theory is developed to a stage which allows the dynamical density-density function in this case to be computed exactly. As is seen from the formula (4.13), this requires developing a theory of integration formulas involving the Jack polynomials and the square of the ground state wavefunction.

The Jack polynomials are a fairly new construction in Pure Mathematics, with the first substantial works being due to Macdonald [5] and Stanley [6] less than ten years ago. One of the reasons that their properties can be developed to such a detailed level is that they can be defined not only as the eigenfunctions of the transformed Schrödinger operator, but also as orthogonal polynomials associated with a power sum inner product (see (3.14a)). This is a generalization of an inner product well known from the theory of Schur polynomials (the Jack polynomials

reduce to Schur polynomials for the coupling $\beta = 2$, which corresponds to the free fermion system in the language of the Calogero-Sutherland model).

The polynomial eigenfunctions in the Hermite, Laguerre and Jacobi cases are all multi-dimensional generalizations of their classical, single-variable counterparts. They are most naturally written as finite series of Jack polynomials. In the Hermite and Laguerre cases multi-dimensional generalizations of the classical generating functions are presented. The method of derivation suggests a family of N independent differential operators which have the generalized Hermite (Laguerre) polynomials as eigenfunctions. These operators have the feature that their corresponding eigenvalues can also be specified and that they remove the degeneracy of the spectrum.

The exact calculation of correlation functions is presented in Section 4. The first correlation considered is the static one-body density in the Jacobi ensemble. For even values of β this density, which is by definition an N-dimensional integral, is written as a β-dimensional integral. The large-N asymptotics of the β-dimensional integral can be calculated and the electrostatic prediction (1.38b) established. A similar study is presented for the static two-particle distribution function in the circular ensemble. Again a β-dimensional integral formula is obtained, and the large-N asymptotics computed to verify the prediction (1.43). In the thermodynamic limit the β-dimensional integral formula remains well defined. The large-distance asymptotics have the expected leading order non-oscillatory behaviour (1.40).

The main application of Jack polynomial theory relates to the calculation of the dynamical density-density correlation $\rho_2^{(T)}((0,0),(x,\tau))$ for the Dyson Brownian motion model of the log-gas on a circle, when the initial state is equal to the equilibrium state (or equivalently, for the ground state of the Calogero-Sutherland model on a circle). In the thermodynamic limit it is possible to calculate $\rho_{(2)}^T((0,0),(x,\tau))$ for all rational values of the coupling. Thus in Proposition 4.3, for $\beta/2 = p/q$ (p and q relatively prime integers), $\rho_{(2)}^T$ is expressed as a multiple integral over p variables x_i $(0 \le x_i < \infty)$ and q variables y_j $(0 \le y_j \le 1)$. These variables can be interpreted as being associated with quasi-particles and quasi-holes respectively, and the particular coupling between the variables is consistent with an interpretation of fractional statistics associated with the hole excitations. It is pointed out that the integrand is of a functional form first obtained by Dotsenko and Fateev in conformal field theory. This knowledge is put to use in the calculation of the small-k behaviour of the structure factor $\tilde{S}(k,\tau)$, where a limiting case of the Dotsenko-Fateev integral is used in the verification of the expected behaviour (2.36).

In the final Section, the Calogero-Sutherland model with exchange terms [7] is introduced. For certain prescribed symmetries of the allowed states the absolute value squared of the ground state is identical to the Boltzmann factor for a generalized plasma system. The Calogero-Sutherland model with exchange terms provides the natural setting for the study of the integrability of the original Calogero-Sutherland models. Furthermore, it opens the way for future research opportunities

in the study of the polynomial part of the eigenfunctions and the calculation of correlation functions.

In this article the central theme will be the Calogero-Sutherland quantum many body problem, with the pathway to these systems provided by parameter-dependent random matrices. In a monograph in preparation, the author is producing a work in which the central theme is the log-gas. Although a small amount of repetition is necessary to make both this work and the monograph self contained, their emphasis and main body of content is distinct.

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1. Classical Random Matrix Ensembles

1.1. Calogero-Sutherland model and the log-gas analogy. The Calogero-Sutherland model refers to a class of exactly solvable quantum many body systems in one dimension. In this exposition we will be concerned with the subclass of such models which can be characterized as the Schrödinger operators in one dimension with one and two body potentials only:

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{j=1}^{N} V_1(x_j) + \sum_{1\le j<k\le N} V_2(x_j, x_k) \tag{1.1}$$

which possess exact BDJ–type ground states:

$$\psi_0 = \prod_{j=1}^{N} f_1(x_j) \prod_{1\le j<k\le N} f_2(x_j, x_k). \tag{1.2}$$

Two particular Schrödinger operators of this type are

$$H^{(H)} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \frac{\beta^2}{4}\sum_{j=1}^{N} x_j^2 + \beta(\beta/2-1) \sum_{1\le j<k\le N} \frac{1}{(x_j - x_k)^2} \tag{1.3}$$

and

$$H^{(C)} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \beta(\beta/2-1) \sum_{1\le j<k\le N} \left(\frac{\pi}{L}\right)^2 \frac{1}{\sin^2 \pi(x_j - x_k)/L} \tag{1.4}$$

where in (1.4) $0 \leq x_j \leq L$. Here the superscript (H) refers to Hermite (there is a connection with the Hermite polynomials, while (C) refers to circular. Notice that in both cases the pair potentials are proportional to $1/r^2$ (in the circular case the distance function is the chord length on a circle of circumference length L). It turns out that all Schrödinger operators of the Calogero-Sutherland type are such that $V_2(x_j, x_k)$ is proportional to $1/(x_k - x_j)^2$, in some asymptotic limit at least.

A direct calculation shows that both (1.3) and (1.4) have eigenfunctions of the form $Ae^{-\beta W/2}$ (A is fixed for each ordering of the particles, but may differ by a phase for different orderings) with

$$W^{(H)} = \frac{1}{2}\sum_{j=1}^{N} x_j^2 - \sum_{1 \leq j < k \leq N} \log|x_k - x_j| \tag{1.5}$$

and

$$W^{(C)} = -\sum_{1 \leq j < k \leq N} \log|e^{2\pi i x_k/L} - e^{2\pi i x_j/L}| \tag{1.6}$$

respectively. The verifications are straightforward provided one is aware of the identities

$$\frac{1}{xy} + \frac{1}{xz} + \frac{1}{yz} = 0 \tag{1.7}$$

and

$$\cot x \cot y + \cot x \cot z + \cot y \cot z = 1 \tag{1.8}$$

for $x + y + z = 0$. Since these eigenfunctions do not change sign for any particular ordering of the particles, they correspond to the ground state wave function (i.e. they are the eigenfunctions with the most negative eigenvalue E_0). Notice that in both cases ψ_0 is indeed of the type (1.2).

Already an analogy with another class of physical systems is evident: $|\psi_0|^2 = |A|^2 e^{-\beta W}$ is proportional to the Boltzmann factor for the classical gas with total potential energy $W + C$, where C is a constant. The pair potential for the classical gases are logarithmic, which is very significant for in two dimensions

$$\phi(\vec{r}, \vec{r}') = -\log\left(|\vec{r} - \vec{r}'|/l\right) \tag{1.9}$$

(l is some length scale which we can set equal to unity) is the solution of the Poisson equation

$$\nabla^2_{\vec{r}}\,\phi(\vec{r}, \vec{r}') = -2\pi\delta(\vec{r} - \vec{r}') \quad \textit{where} \quad \nabla^2_{\vec{r}} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{1.10}$$

Thus $\phi(\vec{r}, \vec{r}')$ represents the Coulombic pair potential for two-dimensional unit charges. The log-potential Coulomb systems with energies (up to an additive constant) given by (1.5) and (1.6) have the particles confined to a line and circle respectively. In (1.5) there is a one-body harmonic potential. This attracts the particles towards the origin and stops them repelling to infinity. We will make use of the log-potential Coulomb system (log-gas) analogy to make analytic predictions for the behaviour of certain correlation functions later.

For now we will redirect attention to an analogy between the ground state and random matrices, which for the Schrödinger operators (1.3) and (1.4) was noticed in one of the original papers on the subject by Sutherland [1].

1.2. Gaussian random matrices as a model of a quantum Hamiltonian. Random matrices have been studied in physics since the late 1950's when E. P. Wigner applied them to the study of the statistical properties of the energy levels of complex nuclei (see e.g. [2]). There are now four known random matrix ensembles which have analogies with the Calogero-Sutherland model. The first of these, studied in some detail by F.J. Dyson [3], is referred to as the Gaussian random matrix ensemble. Gaussian random matrices are used to model the Hamiltonian of the quantum system under consideration.

The general hypothesis is that for a sufficiently complicated quantum system, the individual matrix elements of the Hamiltonian are unimportant for determining the statistical properties of the energy levels, rather these properties are determined by the global time reversal symmetry. (This statement refers to the eigenvalues within a block of specific angular momentum, parity etc..) Furthermore, since the discrete portion of the spectrum is of interest, a reasonable approximation is to use an $N \times N$ Hermitian matrix $\boldsymbol{X}$ in place of the in general infinite dimensional Hamiltonian operator. The detail of the matrix elements being unimportant, they may be chosen as completely random, subject only to the constraint imposed by time reversal symmetry.

As with all the random matrix ensembles we will consider, there are three subclasses of Gaussian random matrices, called the Gaussian Orthogonal Ensemble (GOE), Gaussian Unitary Ensemble (GUE) and Gaussian Symplectic Ensemble (GSE). They are used to model systems with time reversal symmetry and an even number of spin-$\frac{1}{2}$ particles, no time reversal symmetry, and time reversal symmetry with an odd number of spin-$\frac{1}{2}$ particles respectively.

These ensembles are defined by the statement that the joint probability density function (p.d.f.) for the elements, $P(\boldsymbol{X})$ say, for a member of the ensemble $\boldsymbol{X}$, is given by

$$P(\boldsymbol{X}) = A_{\beta,N} \exp\Big(- \beta Tr(\boldsymbol{X}^2)/2\Big) \tag{1.11}$$

where $\beta = 1, 2$ and 4 for the GOE, GUE and GSE respectively, and $A_{\beta,N}$ is the normalization. For the GOE $\boldsymbol{X}$ is real symmetric while for the GUE $\boldsymbol{X}$ is Hermitian and for the GSE $\boldsymbol{X}$ is self-dual real quaternion. A self dual real quaternion matrix is an Hermitian matrix which has each element itself a 2×2 matrix of the form

$$\begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix} \tag{1.12}$$

(this is the matrix representation of a real quaternion). The p.d.f. (1.11) is clearly unchanged if $\boldsymbol{X}$ undergoes the similarity transformation

$$\boldsymbol{X} \to \boldsymbol{U}^{-1}\boldsymbol{X}\boldsymbol{U}. \tag{1.13}$$

This is a fundamental requirement of the model since a general property of quantum mechanics is that two operators related by a similarity transformation of unitary operators are equally valid physical descriptions.

An equivalent way to specify the p.d.f. (1.11) is in terms of the p.d.f. of each element.

Definition 1.1

(1) A random real symmetric $N \times N$ matrix X is said to belong to the GOE if the diagonal and upper triangular elements are independently chosen with p.d.f.'s

$$\frac{1}{\sqrt{2\pi}} e^{-x_{jj}^2/2} \quad \textit{and} \quad \frac{1}{\sqrt{\pi}} e^{-x_{jk}^2}$$

respectively.

(2) A random Hermitian $N \times N$ matrix X is said to belong to the GUE if the diagonal elements (which must be real) and the upper triangular elements $x_{jk} = u_{jk} + iv_{jk}$ are independently chosen with p.d.f.'s

$$\frac{1}{\sqrt{\pi}} e^{-x_{jj}^2} \quad \textit{and} \quad \frac{2}{\pi} e^{-2(u_{jk}^2 + v_{jk}^2)} = \frac{2}{\pi} e^{-2|x_{jk}|^2}$$

(3) A random Hermitian $N \times N$ matrix $\boldsymbol{X}$ with real quaternion elements is said to belong to the GSE if the elements z_{jj} of each diagonal quaternion real block, which must be real, are independently chosen with p.d.f.

$$\sqrt{\frac{2}{\pi}} e^{-2z_{jj}^2},$$

while the upper triangular off diagonal elements $z_{jk} = u_{jk} + iv_{jk}$ and $w_{jk} = u'_{jk} + iv'_{jk}$ are independently chosen with real and imaginary parts having p.d.f.

$$\frac{4}{\pi} e^{-4|z_{jk}|^2} \qquad \text{and} \qquad \frac{4}{\pi} e^{-4|w_{jk}|^2}.$$

The advantage of using Definition 1.1 is that it is a simple matter to construct computer generated matrices belonging to any of the particular Gaussian ensembles. Note from Definition 1.1 that $P(\boldsymbol{X})$ has the factorization property

$$P(\boldsymbol{X}) = A_{\beta,N} \prod_{\nu} \prod_{j,k=1}^{N} f(x_{jk}^{(\nu)}) \quad \textit{with} \quad f(x) = e^{-\beta x^2/2} \tag{1.14}$$

where the sum over ν applies for the off-diagonal elements and is from 1 up to β, corresponding to the independent real and imaginary parts of the element in position (jk). A well known result (see e.g. [4]) is that (1.11) is the unique p.d.f. which has the factorization property (1.14) and invariance under the transformation (1.13).

It remains to explain the origin of the constraints on the random matrices for a system with time reversal symmetry: real symmetric for a system with an even number (or no) spin-$\frac{1}{2}$ particles and self dual real quaternion for a system with an odd number of spin-$\frac{1}{2}$ particles, and their relationship to real orthogonal and

symplectic matrices. For this purpose we recall that a quantum system has a time reversal symmetry if the Hamiltonian commutes with an anti-unitary operator i.e. a unitary operator, T say, which has the property

$$T(a|\psi_1\rangle + b|\psi_2\rangle) = a^*T|\psi_1\rangle + b^*T|\psi_2\rangle. \tag{1.15}$$

Study of time reversal operators in the context of physical systems (see e.g. [5,6]) further restricts their form. For systems with an even number of (or no) spin-$\frac{1}{2}$ particles it is required that $T^2 = 1$, while for a finite dimensional system with an odd number of spin-$\frac{1}{2}$ particles one must have $T^2 = -1$ with $T = \boldsymbol{Z}K$, where $\boldsymbol{Z}$ is a $2N \times 2N$ block diagonal matrix, with 2×2 blocks

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1.16}$$

and K is the complex conjugation operator.

The case $T^2 = 1$. For any quantum Hamiltonian H which is invariant with respect to a time reversal symmetry T, where T has the additional property $T^2 = 1$, $\boldsymbol{H}$ can always be given a T–invariant basis, and with respect to this basis the (in general infinite) matrix representation of $\boldsymbol{H}$ is real. To see this, let $\{\phi'_n\}$ be a complete set. Let $\psi_1 = \alpha_1\phi_1 + T(\alpha_1\phi_1)$ where $\phi_1 = \phi'_1$. Construct from ϕ'_1 and ϕ'_2 a vector orthogonal to ψ_1, ϕ_2 say, and let $\psi_2 = \alpha_2\phi_2 + T(\alpha_2\phi_2)$. Then ψ_2 is T invariant and

$$\langle\psi_2|\psi_1\rangle = \alpha_2^*\langle\phi_2|\psi_1\rangle + \alpha_2\langle T\phi_2|\psi_1\rangle = 0 + \alpha_2\langle T^2\phi_2|\psi_1\rangle^* = \alpha_2\langle\phi_2|\psi_1\rangle^* = 0$$

Here we have used the anti-unitary property of T which implies in general that $\langle u|Tv\rangle = \langle T^\dagger u|v\rangle^*$ so that with $T^2 = 1$, $\langle u|v\rangle = \langle Tu|Tv\rangle^*$. Proceeding in this fashion gives a T–invariant orthogonal basis $\{\psi_n\}$ (the constants $\alpha_1, \alpha_2, \ldots$ can be chosen to normalize the basis). One then sees that

$$\langle\psi_m|H\psi_n\rangle = \langle T\psi_m|T(H\psi_n)\rangle^* = \langle T\psi_m|HT\psi_n\rangle^* = \langle\psi_m|H\psi_n\rangle^*$$

which says the matrix elements are real, as required.

Furthermore, the subgroup of unitary matrices for which the similarity transformation (1.13) maps real symmetric matrices to real symmetric matrices is the real orthogonal group of matrices (or i times these matrices). To verify this statement, let $\boldsymbol{X}' = \boldsymbol{U}^{-1}\boldsymbol{X}\boldsymbol{U}$ where $\boldsymbol{U}$ is unitary and $\boldsymbol{X}, \boldsymbol{X}'$ are real symmetric, and suppose that the only symmetry of $\boldsymbol{X}$ and $\boldsymbol{X}'$ in general (other than some constant times the identity) is the time reversal operator T with $T^2 = 1$. Now, by assumption of the time reversal symmetry $\boldsymbol{X}' = T^{-1}\boldsymbol{X}'T$ and $\boldsymbol{X} = T^{-1}\boldsymbol{X}T$. Substituting into the equation relating $\boldsymbol{X}$ and $\boldsymbol{X}'$ we see that

$$\boldsymbol{T}\boldsymbol{U}T^{-1}\boldsymbol{U}^{-1}\boldsymbol{X}\boldsymbol{U} = \boldsymbol{X}\boldsymbol{T}\boldsymbol{U}T^{-1} \tag{1.17}$$

and so $\boldsymbol{T}\boldsymbol{U}T^{-1}\boldsymbol{U}^{-1}$ commutes with $\boldsymbol{X}$. But by assumption, the only operators which commute with $\boldsymbol{X}$ are T and some multiple of the identity. The first possibility

implies $T = 1$, which is not allowed since T is then not anti-unitary, so we must have

$$TU = cUT \tag{1.18}$$

for some constant c. Taking the inverse of this equation implies $c = 1/c$ and thus $c = \pm 1$. Using this property, we see that the elements of U with respect to the T invariant basis $\{\psi_n\}$ obey the equation

$$\langle \psi_n | U \psi_m \rangle = \langle T\psi_n | TU\psi_m \rangle^* = c \langle \psi_n | U \psi_m \rangle^*.$$

Hence the elements of U are either all real ($c = 1$) or all pure imaginary ($c = -1$), which is the required result.

The case $T^2 = -1$. In this case $T = ZK$, and T commutes with the $2N \times 2N$ matrix X approximating the Hamiltonian. Thus

$$X = TXT^{-1} = ZKXK^{-1}Z^{-1} = ZKXKZ^{-1} = ZX^*Z^{-1}$$

which from the definition of Z implies X consists of 2×2 blocks of the form (1.12). The subgroup of unitary matrices which maps self dual real quaternion matrices into themselves under the similarity transformation (1.13) is determined in the same way as for the symmetric matrices case detailed above. We find that the matrices must have the property $Z = \pm UZU^T$. In the case of the plus sign, this is the condition for U to be symplectic.

Another feature of a self-dual real quaternion matrix X is that its eigenvalues are doubly degenerate. To see this we use the fact that X commutes with $T = ZK$, so that if $|\phi\rangle$ is an eigenvector of X with eigenvalue λ, then $T|\phi\rangle$ is also. It remains to check that $|\phi\rangle$ and $T|\phi\rangle$ are linearly independent. We have

$$\langle \phi | T\phi \rangle = \langle T\phi | T^2\phi \rangle^* = -\langle T\phi | \phi \rangle^* = -\langle \phi | T\phi \rangle$$

and thus $\langle \phi | T\phi \rangle = 0$, which establishes the linearly independence.

1.3. Calculation of the eigenvalue p.d.f. The p.d.f. for the elements of the three Gaussian ensembles is given by (1.11). We seek to change variables in terms of the eigenvalues of X, $\lambda_1, \dots, \lambda_N$ say, and variables associated with the eigenvectors, denoted $p_1, \dots, p_{\beta N(N-1)/2}$ say:

$$\exp^{-\beta Tr(X)^2} \prod_{j=1}^{N} dx_{jj} \prod_{\nu=1}^{\beta} \prod_{1 \le j < k \le N} dx_{jk}^{(\nu)} = \exp^{-\beta \sum_{l=1}^{N} \lambda_l^2/2} |J| \prod_{i=1}^{N} d\lambda_i \prod_{j=1}^{\frac{\beta N(N-1)}{2}} dp_j$$

where the Jacobian is given by

$$J := \det \begin{pmatrix} \frac{\partial x_{11}}{\partial \lambda_1} & \frac{\partial x_{12}}{\partial \lambda_1} & \cdots & \frac{\partial x_{NN}}{\partial \lambda_1} \\ \frac{\partial x_{11}}{\partial \lambda_2} & \frac{\partial x_{12}}{\partial \lambda_2} & \cdots & \frac{\partial x_{NN}}{\partial \lambda_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_{11}}{\partial p_{N(N-1)/2}} & \frac{\partial x_{12}}{\partial p_{N(N-1)/2}} & \cdots & \frac{\partial x_{NN}}{\partial p_{N(N-1)/2}} \end{pmatrix}$$

Thus we must evaluate the Jacobian and then integrate over the variables $p_1, \ldots, p_{\beta N(N-1)/2}$ to obtain the eigenvalue p.d.f..

In fact, as we will show below explicitly in the Hermitian case, for each ensemble J factorizes:

$$J = \prod_{1 \le j < k \le N} (\lambda_k - \lambda_j)^\beta \, f(p_1, \ldots, p_{\beta N(N-1)/2}) \tag{1.19}$$

so the integration over the variables $p_1, \ldots, p_{\beta N(N-1)/2}$ only alters the normalization constant. Hence the final expression for the eigenvalue p.d.f. in the Gaussian ensembles is

$$\frac{1}{C_{\beta N}} e^{-\beta \sum_{j=1}^N \lambda_j^2/2} \prod_{1 \le j < k \le N} |\lambda_k - \lambda_j|^\beta \tag{1.20}$$

where $C_{\beta N}$ is the normalization constant. With $\lambda_j = x_j$ this expression is precisely the absolute value squared of the ground state wave function of the Schrödinger operator (1.3).

Consider now the derivation of (1.19). The calculation of J in the Hermitian case was first given by Goodman [7], in the context of multivariable statistics. We will adapt the more recent treatment of Muirhead [8], who derives J in the real symmetric case, also in the context of studies in multivariate statistics. This is based on exterior products.

Definition 1.2 Let the differential du_j of the function $u_j = u_j(v_1, \ldots, v_m)$ be given by

$$du_j := \frac{\partial u_j}{\partial v_1} dv_1 + \cdots + \frac{\partial u_j}{\partial v_m} dv_m$$

and define the exterior (also known as wedge) product operator $\wedge$ by

$$du_j \wedge du_k = du_j du_k \; j < k \quad du_j \wedge du_j = 0 \quad \textit{and} \quad du_j \wedge du_k = -du_k \wedge du_j, \; j > k$$

with these operations extended by linearity, where the product $du_j du_k$ is interpreted as in the theory of integration.

The possibility of using exterior products to calculate Jacobians is apparent from the formula

$$\bigwedge_{j=1}^m du_j := du_1 \wedge du_2 \wedge \cdots \wedge du_m = \det \left(\frac{\partial u_j}{\partial v_k} \right)_{j,k=1,\ldots,m} dv_1 \wedge dv_2 \cdots \wedge dv_m \tag{1.21}$$

which is a direct consequence of the above definition and the definition of a determinant. Two further definitions help streamline the working in the calculation.

Definition 1.3 For any $N \times N$ matrix $\boldsymbol{X} = [x_{jk}^{(1)} + ix_{jk}^{(2)}]$, the matrix of differentials is defined as

$$d\boldsymbol{X} = \begin{pmatrix} dx_{11}^{(1)} + idx_{11}^{(2)} & dx_{12}^{(1)} + idx_{12}^{(2)} & \dots & dx_{1N}^{(1)} + idv_{1N}^{(2)} \\ dx_{21}^{(1)} + idx_{21}^{(2)} & dx_{22}^{(1)} + idx_{22}^{(2)} & \dots & dx_{2N}^{(1)} + idx_{2N}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ dx_{N1}^{(1)} + idx_{N1}^{(2)} & dx_{N2}^{(1)} + idx_{N2}^{(2)} & \dots & dx_{NN}^{(1)} + idx_{NN}^{(2)} \end{pmatrix}$$

With this definition the usual product rule for differentiation holds:

$$d(\boldsymbol{XY}) = d\boldsymbol{X}\,\boldsymbol{Y} + \boldsymbol{X}\,d\boldsymbol{Y}. \tag{1.22}$$

Definition 1.4 The symbol $(d\boldsymbol{X})$ denotes the exterior product of all the independent elements of $d\boldsymbol{X}$. In particular, if $\boldsymbol{X}$ is an Hermitian matrix with off diagonal elements $x_{jk} = x_{jk}^{(1)} + ix_{jk}^{(2)}$, then

$$(d\boldsymbol{X}) = \bigwedge_{i=1}^{N} dx_{jj}^{(1)} \bigwedge_{1 \le j < k \le N} dx_{jk}^{(1)} dx_{jk}^{(2)}. \tag{1.23}$$

Calculation of J in the Hermitian case. From Definition 1.4 and (1.21) we see that for $\boldsymbol{X}$ Hermitian

$$J \prod_{i=1}^{N} d\lambda_i \prod_{j=1}^{N(N-1)} dp_j = (d\boldsymbol{X})$$

To calculate $(d\boldsymbol{X})$ in terms of the eigenvalues and eigenvectors we use the formula

$$\boldsymbol{X} = \boldsymbol{U}\boldsymbol{L}\boldsymbol{U}^{-1}$$

where $\boldsymbol{L}$ is a diagonal matrix consisting of the N eigenvalues of $\boldsymbol{X}$ and the columns of the unitary matrix $\boldsymbol{U}$ consist of the corresponding normalized eigenvectors. By using this formula we are implicitly assuming that $\boldsymbol{X}$ has no repeated eigenvalues. This is justified because such an event occurs with zero probability, because then $\det \boldsymbol{X}$ must take on a specific value (zero). Using the notation of Definition 1.3, the product rule for differentiation (1.22) gives

$$d\boldsymbol{X} = d\boldsymbol{U}\,\boldsymbol{L}\boldsymbol{U}^{-1} + \boldsymbol{U}\,d\boldsymbol{L}\boldsymbol{U}^{-1} + \boldsymbol{U}\boldsymbol{L}d\boldsymbol{U}^{-1}.$$

Rather than take the exterior product of both sides of this equation, it is simpler to first premultiply by $\boldsymbol{U}^{-1}$ and postmultiply by $\boldsymbol{U}$ to obtain

$$\begin{aligned} \boldsymbol{U}^{-1}d\boldsymbol{X}\boldsymbol{U} &= \boldsymbol{U}^{-1}d\boldsymbol{U}\boldsymbol{L} + \boldsymbol{L}d\boldsymbol{U}^{-1}\boldsymbol{U} + d\boldsymbol{L} \\ &= \boldsymbol{U}^{-1}d\boldsymbol{U}\boldsymbol{L} - \boldsymbol{L}\boldsymbol{U}^{-1}d\boldsymbol{U} + d\boldsymbol{L}, \end{aligned} \tag{1.24}$$

where to obtain the last line the formula $d\boldsymbol{U}^{-1}\boldsymbol{U} = -\boldsymbol{U}^{-1}d\boldsymbol{U}$ has been used (this follows from $\boldsymbol{U}^{-1}\boldsymbol{U} = 1$).

According to Definition 1.4, the exterior product of the l.h.s. of (1.24) is related to $(d\boldsymbol{X})$ by

$$(\boldsymbol{U}^{-1}d\boldsymbol{X}\boldsymbol{U}) = p(\boldsymbol{U},\boldsymbol{U}^*)(d\boldsymbol{X})$$

for some polynomial p in the elements of $\boldsymbol{U}$ and $\boldsymbol{U}^*$. In fact this polynomial is identically equal to 1. To derive this result, first note that for a general matrix $\boldsymbol{A}$, $(d\boldsymbol{A})$ must be real according to its definition, and so $p(\boldsymbol{U},\boldsymbol{U}^*)$ must be real. Furthermore, we have

$$((\boldsymbol{U}_1\boldsymbol{U}_2)^\dagger d\boldsymbol{X}\,\boldsymbol{U}_1\boldsymbol{U}_2) = p(\boldsymbol{U}_1\boldsymbol{U}_2,\boldsymbol{U}_1^*\boldsymbol{U}_2^*)(d\boldsymbol{X}).$$

On the other hand

$$((\boldsymbol{U}_1\boldsymbol{U}_2)^\dagger d\boldsymbol{X}\,\boldsymbol{U}_1\boldsymbol{U}_2) = \boldsymbol{U}_2^\dagger\boldsymbol{U}_1^\dagger d\boldsymbol{X}\,\boldsymbol{U}_1\boldsymbol{U}_2 = p(\boldsymbol{U}_1,\boldsymbol{U}_1^*)p(\boldsymbol{U}_2,\boldsymbol{U}_2^*)$$

so that

$$p(\boldsymbol{U}_1\boldsymbol{U}_2,\boldsymbol{U}_1^*\boldsymbol{U}_2^*) = p(\boldsymbol{U}_1,\boldsymbol{U}_1^*)p(\boldsymbol{U}_2,\boldsymbol{U}_2^*).$$

But it is known [9] that the only polynomial in the elements satisfying this equation is

$$p(\boldsymbol{U},\boldsymbol{U}^*) = (\det\boldsymbol{U})^{k_1}(\det\boldsymbol{U}^*)^{k_2},$$

and p is real so we must have $k_1 = k_2$. Recalling $\boldsymbol{U}$ is unitary, the result now follows.

A small amount of additional working establishes the sought result.

Proposition 1.1. *For $\boldsymbol{X}$ Hermitian we have*

$$(d\boldsymbol{X}) = \prod_{1\le j<k\le N}(\lambda_k-\lambda_j)^2 \bigwedge_{j=1}^{N} d\lambda_j\,(\boldsymbol{U}^\dagger d\boldsymbol{U}).$$

Proof. From the working so far, $(d\boldsymbol{X})$ is given by the wedge product of the r.h.s. of (1.24). To compute this wedge product, we expand out the matrix products on the r.h.s. of (1.24) and simplify using the fact that $\vec{u}_j^\dagger\cdot d\vec{u}_k = -\vec{u}_k\cdot d\vec{u}_j^\dagger$ (which follows from $\boldsymbol{U}\boldsymbol{U}^\dagger = 1$, and where we have set $\boldsymbol{U} = [u_{jk}]$ and $\vec{u}_j = [u_{jk}]_{k=1,\dots,N}$) to obtain

$$\begin{pmatrix} d\lambda_1 & (\lambda_2-\lambda_1)\vec{u}_1^\dagger\cdot d\vec{u}_2 & \dots & (\lambda_N-\lambda_1)\vec{u}_1^\dagger\cdot d\vec{u}_N \\ (\lambda_2-\lambda_1)\vec{u}_1\cdot d\vec{u}_2^\dagger & d\lambda_2 & \dots & (\lambda_N-\lambda_2)\vec{u}_2^\dagger\cdot d\vec{u}_N \\ \vdots & \vdots & \ddots & \vdots \\ (\lambda_N-\lambda_1)\vec{u}_1\cdot d\vec{u}_N^\dagger & (\lambda_N-\lambda_2)\vec{u}_2\cdot d\vec{u}_N^\dagger & \dots & d\lambda_N \end{pmatrix}$$

The facts that each $\vec{u}_j^\dagger\cdot d\vec{u}_k$ has an independent real and imaginary part, and that only the elements on and above the diagonal are independent we see that taking the wedge product gives the stated result. □

Substituting Proposition 1.1 in (1.22) shows that J factorizes according to (1.19) and thus establishes (1.20) in the Hermitian case.

1.4. Random unitary matrices and Floquet operators. In the study of time dependent problems in quantum mechanics, a fundamental quantity is the evolution operator

$$U(t) := \mathcal{T} \exp\left(-\frac{i}{\hbar}\int_0^t dt'\, H(t')\right) \tag{1.25}$$

where $\mathcal{T}$ denotes the usual time ordering. When the Hamiltonian is periodic of period τ (i.e. $H(t) = H(t+\tau)$) we have $U(n\tau) = F^n$ $(n \in Z)$ where $F := U(\tau)$. The operator F is called the Floquet operator, and is unitary since U is unitary. For classical chaotic time dependent systems, analogous to the situation with Hamiltonians of complex systems, one hypothesizes that the feature determining the statistical properties of the eigenvalues of F is the global time reversal symmetry. This suggests modelling F by a random unitary matrix.

Let us now investigate the constraint on F imposed by a time reversal symmetry. In the static case (Section 1.2), the system was said to have a time reversal symmetry T if T was antiunitary and commuted with the Hamiltonian. For a time dependent Hamiltonian an appropriate definition of a time reversal symmetry T is that T is antiunitary, and

$$T^{-1}H(t)T = H(-t) \tag{1.26}$$

(see e.g. [5]). From (1.25) it is not hard to deduce that (1.26) holds with $H(t)$ replaced by $U(t)$ – one approach is to use the integral equation

$$U(t) = 1 + \frac{1}{i\hbar}\int_0^t dt'\, H(t')U(t')$$

which is equivalent to (1.25). We thus have

$$T^{-1}FT = F^{-1} = F^\dagger, \tag{1.27}$$

where the final equality follows since F is unitary.

Consider first the case $T^2 = 1$, and let $\{\psi_n\}$ be a T invariant basis. Then

$$\langle \psi_m | F\psi_n\rangle = \langle T\psi_m | TF\psi_n\rangle^* = \langle T\psi_m | F^\dagger T\psi_n\rangle^*$$
$$= \langle \psi_m | F^\dagger \psi_n\rangle^* = \langle \psi_n | F\psi_m\rangle.$$

Thus in this case F is a symmetric unitary matrix.

Consider next the case $T^2 = -1$ with $T = \boldsymbol{Z}K$. Then (1.27) gives

$$F = TF^\dagger T^{-1} = \boldsymbol{Z}KF^\dagger K^{-1}\boldsymbol{Z}^{-1} = \boldsymbol{Z}KF^\dagger K\boldsymbol{Z}^{-1} = \boldsymbol{Z}F^T\boldsymbol{Z}^{-1}$$

This constraint is said to specify F as a self dual quaternion matrix. In terms of 2×2 blocks in position (jk) and (kj) of F it says

$$\begin{pmatrix} z_{jk}^{(1)} & z_{jk}^{(2)} \\ z_{jk}^{(3)} & z_{jk}^{(4)} \end{pmatrix} = \begin{pmatrix} z_{kj}^{(4)} & -z_{kj}^{(2)} \\ -z_{kj}^{(3)} & z_{kj}^{(1)} \end{pmatrix}$$

Consideration of Floquet operators thus indicates that a theory of random symmetric unitary matrices, random unitary matrices, and random self-dual quaternion

unitary matrices is required. These three classes of random unitary matrices have been defined and studied by Dyson [3]. We will content ourselves with presenting the definition, and calculating the eigenvalue p.d.f., in the symmetric case. The random symmetric unitary matrices are said to form the Circular Orthogonal Ensemble (COE) (as in the Gaussian case, the term orthogonal refers to the symmetry of the matrices: random symmetric unitary matrices map into themselves under similarity transformations with real orthogonal matrices).

Definition 1.5 Consider an arbitrary symmetric unitary matrix $\boldsymbol{S}$, and write $\boldsymbol{S}$ as

$$\boldsymbol{S} = \boldsymbol{U}^T\boldsymbol{U}$$

where $\boldsymbol{U}$ is unitary (e.g. diagonalize $\boldsymbol{S}$ by $\boldsymbol{S} = \boldsymbol{R}\Theta\boldsymbol{R}^T$ where $\boldsymbol{R}$ is real orthogonal and Θ is diagonal and take $\boldsymbol{U} = \Theta^{1/2}\boldsymbol{R}^T$). Let $d\boldsymbol{M}$ denote a real symmetric matrix of differentials dM_{jk} and define an infinitesimal neighbourhood of $\boldsymbol{S}$ by

$$\boldsymbol{S} + d\boldsymbol{S} = \boldsymbol{U}^T(\mathbf{1} + id\boldsymbol{M})\boldsymbol{U}$$

(note that $\boldsymbol{S} + d\boldsymbol{S}$ is unitary at $O(d\boldsymbol{M})$). Suppose furthermore that the elements dM_{jk}, $j \leq k$, vary independently in some intervals of length $d\mu_{jk}$. With these notations, the probability that a matrix from the circular orthogonal ensemble (COE) lies between $\boldsymbol{S}$ and $\boldsymbol{S} + d\boldsymbol{S}$ is defined to be proportional to

$$\mu_1(d\boldsymbol{S}) := \prod_{j \leq k} d\mu_{jk}$$

i.e. all matrices $\boldsymbol{S}$ are required to be equally probable.

The decomposition $\boldsymbol{S} = \boldsymbol{U}^T\boldsymbol{U}$ is arbitrary in that $\boldsymbol{U}$ can be replaced by $\boldsymbol{V} = \boldsymbol{R}'\boldsymbol{U}$ where $\boldsymbol{R}'$ is any real orthogonal matrix. An essential feature of the above definition of the measure is that it is independent of $\boldsymbol{R}'$. Since with $\boldsymbol{U}$ replaced by $\boldsymbol{R}'\boldsymbol{U}$

$$\boldsymbol{S} + d\boldsymbol{S} = \boldsymbol{U}^T(1 + \boldsymbol{R}'^T d\boldsymbol{M}\boldsymbol{R}')\boldsymbol{U}$$

the measure will be unchanged if the absolute value of the Jacobian of the change of variables $d\boldsymbol{M} \mapsto \boldsymbol{R}'^T d\boldsymbol{M}\boldsymbol{R}'$ is unity. The Jacobian can be computed by taking exterior products. We have

$$(\boldsymbol{R}'^T d\boldsymbol{M}\boldsymbol{R}') = J(d\boldsymbol{M}).$$

But from Definition 1.5, $(\boldsymbol{R}'^T d\boldsymbol{M}\boldsymbol{R}') = p(\boldsymbol{R})(d\boldsymbol{M})$ for some polynomial in the elements of $\boldsymbol{R}'$. Arguing as in Section 1.3, $p(\boldsymbol{R}')$ must be some power of $\det \boldsymbol{R}'$ and so equal to ± 1, thus giving the required result. Note that this implies the measure is invariant under the transformation $\boldsymbol{S} \mapsto \boldsymbol{R}^T\boldsymbol{S}\boldsymbol{R}$.

The Circular Unitary Ensemble (CUE) and Circular Symplectic Ensemble (CSE) are defined analogously to the COE – in particular, all matrices are required to be equally probable. The analogy with the ground state of the Schrödinger operator (1.4) is for the eigenvalue p.d.f.'s of the circular ensembles. Since we are dealing

with unitary matrices, the eigenvalues λ_j lie on the unit circle in the complex plane, and so can be written $\lambda_j = e^{i\theta_j}$. We have the following result.

Proposition 1.2. *The eigenvalue p.d.f. of the circular ensembles is given by*

$$\frac{1}{C_{\beta N}} \prod_{1 \le j < k \le N} |e^{i\theta_k} - e^{i\theta_j}|^\beta,$$

where $C_{\beta N}$ is the normalization, with $\beta = 1, 2$ and 4 for the COE, CUE and CSE respectively.

Proof. Only the COE will be considered here. The starting point is to express the measure of Definition 1.5 for the CUE in terms of the eigenvalues of $\boldsymbol{S}$ in the diagonalization formula $\boldsymbol{S} = \boldsymbol{R}\Theta\boldsymbol{R}^T$, where Θ is a diagonal matrix consisting of the eigenvalues $e^{i\theta_j}$ $(j = 1, \dots, N)$ of $\boldsymbol{S}$, and the columns of $\boldsymbol{R}$ consist of the corresponding normalized eigenvectors which are real so that $\boldsymbol{R}$ is real orthogonal. Analogous to (1.3), differentiation and minor manipulation gives

$$\boldsymbol{R}^T d\boldsymbol{S}\boldsymbol{R} = \boldsymbol{R}^T d\boldsymbol{R}\,\Theta - \Theta\boldsymbol{R}^T d\boldsymbol{R} + i\Theta d\theta,$$

where θ is the diagonal matrix with entries θ_j $(j = 1, \dots, N)$. But from Definition 1.5

$$\boldsymbol{R}^T d\boldsymbol{S}\boldsymbol{R} = i(\boldsymbol{U}\boldsymbol{R})^T d\boldsymbol{M}(\boldsymbol{U}\boldsymbol{R})$$

where $\boldsymbol{U}$ is any unitary matrix such that $\boldsymbol{S} = \boldsymbol{U}^T\boldsymbol{U}$. In particular, with $\boldsymbol{U} = \Theta^{1/2}\boldsymbol{R}^T$ where $\Theta^{1/2}$ is the diagonal matrix with entries $e^{i\theta_j/2}$ $(j = 1, \dots, N)$, comparison of the two equations for $\boldsymbol{R}^T d\boldsymbol{S}\,\boldsymbol{R}$ gives

$$\begin{aligned} d\boldsymbol{M} &= \Theta d\theta - i\Theta^{-1/2}\boldsymbol{R}^T d\boldsymbol{R}\,\Theta^{1/2} + i\Theta^{1/2}\boldsymbol{R}^T d\boldsymbol{R}\,\Theta^{-1/2} \\ &= \begin{pmatrix} d\theta_1 & 2\sin(\theta_2-\theta_1)/2\,\vec{r}_1 \cdot d\vec{r}_2 \dots & \dots & 2\sin(\theta_N-\theta_1)/2\,\vec{r}_1 \cdot d\vec{r}_N \\ * & d\theta_2 & \dots & 2\sin(\theta_N-\theta_2)/2\,\vec{r}_2 \cdot d\vec{r}_N \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & d\theta_N \end{pmatrix} \end{aligned}$$

where the elements * are chosen so that the matrix is symmetric. Taking the wedge product of the independent elements, and noting that

$$2\sin(\theta_k - \theta_j)/2 = |e^{i\theta_k} - e^{i\theta_j}| \qquad \text{for} \qquad \theta_k > \theta_j$$

gives the result of the proposition for the COE. □

The p.d.f. in Proposition 1.2 is precisely the absolute value squared of the ground state wave function of the Schrödinger operator (1.4).

The definitions of the circular ensembles presented above give no information on the construction of the matrix elements. Nonetheless, an explicit construction in the unitary case is possible [10], and the method is in fact implicit in a classical result of Hurwitz [11]. Having generated a member of the CUE, it follows from the definition above that a member of the COE can then be generated (the same is true for a member of the CSE).

Hurwitz's result is the following.

Proposition 1.3. *Almost all $N \times N$ unitary matrices $\boldsymbol{U}$ have the unique decomposition*

$$\boldsymbol{U} = e^{i\alpha_0} \boldsymbol{E}_1 \boldsymbol{E}_2 \dots \boldsymbol{E}_{N-1}$$

with

$$\begin{aligned} \boldsymbol{E}_{k-1} &= \boldsymbol{E}^{(k-1,k)}(\phi_{k-1,k}, \psi_{k-1,k}, 0)\boldsymbol{E}^{(k-2,k)}(\phi_{k-2,k}, \psi_{k-2,k}, 0) \\ &\quad \times \cdots \times \boldsymbol{E}^{(2,k)}(\phi_{2,k}, \psi_{2,k}, 0)\boldsymbol{E}^{(1,k)}(\phi_{1,k}, \psi_{1,k}, \alpha_k), \end{aligned}$$

$$-\pi \le \alpha_k < \pi, \qquad -\pi \le \psi_{jk} < \pi, \qquad 0 \le \phi_{jk} \le \pi/2,$$

where the $N \times N$ unimodular matrices $\boldsymbol{E}^{(j,k)}(\phi, \psi, \alpha)$ are defined so that all diagonal elements are 1 excepts for the jth and kth which are equal to $\cos\phi\, e^{i\psi}$ and $\cos\phi\, e^{-i\psi}$ respectively, and all off diagonal elements are zero except for the element in the jth row and kth column which is equal to $\sin\phi\, e^{i\alpha}$, and the element in the kth row and jth column which is equal to $-\sin\phi\, e^{-i\alpha}$ (the quantities ϕ and ψ are referred to as Euler angles). Furthermore

$$\mu_2(d\boldsymbol{U}) = C \prod_{1 \le j < k \le N} d[(\sin\phi_{jk})^{2j}] d\psi_{jk} \prod_{j=1}^{N} d\alpha_j$$

From Proposition 1.3, we see that with $\phi_{jk} = (arsin\xi_{jk}^{1/2j})$, $0 \le \xi_{jk} \le 1$, $(d\boldsymbol{U})$ gives the uniform measure (in α_j, ψ_{jk} and ξ_{jk}) so that all matrices are equally probable, which is what is required for the CUE. Thus with α_j, ψ_{jk} and ξ_{jk} chosen at random with uniform density from their respective intervals, the unitary matrix $\boldsymbol{U}$ formed according to the decomposition in Proposition 1.3 will be a member of the CUE.

1.5. Wishart matrices. We have seen how the modelling of a quantum Hamiltonian and Floquet operator leads to the Gaussian and circular ensembles respectively. In this section we will show how the modelling of quantum transport problems leads to Wishart matrices or equivalently the Laguerre ensemble of random matrices.

For the physical setting (following Stone et. al. [12]) we consider a quasi one-dimensional conductor containing scattering impurities and having N available scattering channels (i.e. allowed plane wave states). A current is passed through the wire and we are interested in the statistical properties of the corresponding conductance.

For the theoretical description, one introduces the electron fluxes at the left and right hand edges of the conductor, which are specified by the $2N$ component vectors

$$\begin{pmatrix} \vec{I} \\ \vec{O} \end{pmatrix} \qquad \textit{and} \qquad \begin{pmatrix} \vec{I}' \\ \vec{O}' \end{pmatrix},$$

where $\vec{I}\,(\vec{O})$ and $\vec{I}'\,(\vec{O}')$ denote the amplitude of the plane wave states travelling into (out of) the left and right sides of the conductor respectively. The flux conservation condition is

$$|\vec{I}|^2 + |\vec{I}'|^2 = |\vec{O}|^2 + |\vec{O}'|^2.$$

By definition, the $2N \times 2N$ scattering matrix $\boldsymbol{S}$ relates the flux travelling into the conductor to that travelling out:

$$\boldsymbol{S}\begin{pmatrix} \vec{I} \\ \vec{I}' \end{pmatrix} := \begin{pmatrix} \vec{O} \\ \vec{O}' \end{pmatrix}. \tag{1.28a}$$

It follows from the flux conservation condition that $\boldsymbol{S}$ is unitary:

$$\left\langle \boldsymbol{S}\begin{pmatrix} \vec{I} \\ \vec{I}' \end{pmatrix}, \boldsymbol{S}\begin{pmatrix} \vec{I} \\ \vec{I}' \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} \vec{O} \\ \vec{O}' \end{pmatrix}, \begin{pmatrix} \vec{O} \\ \vec{O}' \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} \vec{I} \\ \vec{I}' \end{pmatrix}, \begin{pmatrix} \vec{I} \\ \vec{I}' \end{pmatrix} \right\rangle.$$

The scattering matrix is further decomposed in terms of $N \times N$ reflection and transmission matrices by

$$\boldsymbol{S} = \begin{pmatrix} \boldsymbol{r} & \boldsymbol{t}' \\ \boldsymbol{t} & \boldsymbol{r}' \end{pmatrix}. \tag{1.28b}$$

The so called two-probe Landauer formula expresses the conductance G in terms of the transmission matrix $\boldsymbol{t}$ (or $\boldsymbol{t}'$) by

$$G/G_0 = Tr(\boldsymbol{t}^\dagger \boldsymbol{t}) = Tr(\boldsymbol{t}'^\dagger \boldsymbol{t}')$$

where $G_0 = 2e^2/h$ is twice the fundamental quantum unit of conductance.

The $2N \times 2N$ transfer matrix $\boldsymbol{M}$ is defined to relate the fluxes at each end:

$$\boldsymbol{M}\begin{pmatrix} \vec{I} \\ \vec{O} \end{pmatrix} := \begin{pmatrix} \vec{O}' \\ \vec{I}' \end{pmatrix}$$

and is decomposed into $N \times N$ blocks by

$$\boldsymbol{M} = \begin{pmatrix} \boldsymbol{m}_1 & \boldsymbol{m}_2 \\ \boldsymbol{m}_3 & \boldsymbol{m}_4 \end{pmatrix}.$$

A similar argument used to show that $\boldsymbol{S}$ is unitary gives

$$\boldsymbol{M}^\dagger \boldsymbol{\Sigma}_z \boldsymbol{M} = \boldsymbol{\Sigma}_z \tag{1.28c}$$

where

$$\boldsymbol{\Sigma}_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

From this formula, and the unitarity of $\boldsymbol{S}$, we find

$$\boldsymbol{t}^\dagger = \boldsymbol{m}_1^{-1},\ \boldsymbol{m}_1^\dagger \boldsymbol{m}_1 = 1 + \boldsymbol{m}_3^\dagger \boldsymbol{m}_3 \quad \textit{and} \quad \boldsymbol{t}'^\dagger = \boldsymbol{m}_4^{-1},\ \boldsymbol{m}_4^\dagger \boldsymbol{m}_4 = 1 + \boldsymbol{m}_2^\dagger \boldsymbol{m}_2$$

Thus the two-probe Landauer formula can be rewritten as

$$G/G_0 = Tr(1 + \boldsymbol{m}_3^\dagger \boldsymbol{m}_3)^{-1} = Tr(1 + \boldsymbol{m}_2^\dagger \boldsymbol{m}_2)^{-1} \tag{1.29}$$

which is our final formula for G.

We now introduce a random matrix hypothesis for the products $\boldsymbol{m}_3^\dagger \boldsymbol{m}_3$ and $\boldsymbol{m}_2^\dagger \boldsymbol{m}_2$. Such products of random matrices first occurred in mathematical statistics, and can be defined in an analogous way to the Gaussian ensemble. They are known as Wishart matrices.

Definition 1.6

(1) Let $\boldsymbol{X}$ be a random $n \times m$ $(n \geq m)$ matrix of real elements x_{jk} with each element independently distributed according to the Gaussian density

$$\frac{1}{\sqrt{2\pi}} e^{-x_{jk}^2/2}.$$

The random non-negative $m \times m$ matrix $\boldsymbol{A} := \boldsymbol{X}^T \boldsymbol{X}$ is called a real Wishart matrix.

(2) Let $\boldsymbol{X}$ be a $n \times m$ $(n \geq m)$ matrix of complex elements z_{jk} with each element independent and distributed according to the Gaussian density $\frac{1}{\pi} e^{-|z_{jk}|^2}$. The random non-negative $m \times m$ matrix $\boldsymbol{A} := \boldsymbol{X}^\dagger \boldsymbol{X}$ is called a complex Wishart matrix.

(3) Let $\boldsymbol{X}$ be a random $n \times m$ $(n \geq m)$ matrix of real quaternion elements (as an ordinary matrix $\boldsymbol{X}$ is thus $2n \times 2m$)

$$\begin{pmatrix} z_{jk} & w_{jk} \\ -w_{jk}^* & z_{jk}^* \end{pmatrix},$$

where z_{jk} and w_{jk} are complex numbers distributed according to the Gaussian densities

$$\frac{2}{\pi} e^{-2|z_{jk}|^2} \qquad \textit{and} \qquad \frac{2}{\pi} e^{-2|w_{jk}|^2}.$$

The random non-negative $m \times m$ matrix $\boldsymbol{A} = \boldsymbol{X}^\dagger \boldsymbol{X}$ of real quaternion elements is said to be a real quaternion Wishart matrix.

Real Wishart matrices apply when there is a time reversal symmetry with $T^2 = 1$, while real quaternion Wishart matrices apply when $T^2 = -1$ and $T = \boldsymbol{Z}K$. When there is no time reversal symmetry there is no constraint on the elements of $\boldsymbol{m}_3$ and $\boldsymbol{m}_2$ and so complex Wishart matrices are applicable.

Let us show how time reversal symmetry with $T^2 = -1$ and $T = \boldsymbol{Z}K$ implies that the elements of $\boldsymbol{m}_2$ and $\boldsymbol{m}_3$ must be real quaternion. Here it is appropriate to take $\boldsymbol{Z}$ as having dimension $4N \times 4N$. Reading off from (1.28), this time reversal symmetry implies $\boldsymbol{S}$ is a $4N \times 4N$ symplectic matrix:

$$\boldsymbol{S} = \boldsymbol{Z}\boldsymbol{S}^T \boldsymbol{Z}^{-1}.$$

Using this relation and the definition of $\boldsymbol{S}$ we see that

$$\boldsymbol{S}\boldsymbol{Z}\begin{pmatrix}\vec{O}^* \\ \vec{O'}^*\end{pmatrix} = \boldsymbol{Z}\begin{pmatrix}\vec{I}^* \\ \vec{I'}^*\end{pmatrix}.$$

Comparing with the original definition of $\boldsymbol{S}$, we see that the time reversal symmetry implies the states

$$\begin{pmatrix}\vec{I} \\ \vec{I'}\end{pmatrix} \qquad and \qquad \begin{pmatrix}\vec{O} \\ \vec{O'}\end{pmatrix}$$

can be replaced by

$$\boldsymbol{Z}\begin{pmatrix}\vec{O}^* \\ \vec{O'}^*\end{pmatrix} \qquad and \qquad \boldsymbol{Z}\begin{pmatrix}\vec{I}^* \\ \vec{I'}^*\end{pmatrix}$$

respectively. Hence, from the defining property, in addition to (1.28c) $\boldsymbol{M}$ must satisfy

$$\boldsymbol{M}\boldsymbol{Z}\begin{pmatrix}\vec{O}^* \\ \vec{I'}^*\end{pmatrix} = \boldsymbol{Z}\begin{pmatrix}\vec{I}^* \\ \vec{O'}^*\end{pmatrix},$$

or equivalently, again using the defining property

$$\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\boldsymbol{Z}^{-1}\boldsymbol{M}^*\boldsymbol{Z}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix} = \boldsymbol{M}.$$

Writing the elements of $\boldsymbol{m}_j$ $(j = 1, \ldots 4)$ as

$$\begin{pmatrix} m_j^{(1)} & m_j^{(2)} \\ m_j^{(3)} & m_j^{(4)} \end{pmatrix}_{kk'}$$

this equation says

$$\begin{pmatrix} m_2^{(1)} & m_2^{(2)} \\ m_2^{(3)} & m_2^{(4)} \end{pmatrix}_{kk'} = \begin{pmatrix} m_3^{(4)} & -m_3^{(3)} \\ -m_3^{(2)} & m_3^{(1)} \end{pmatrix}_{kk'}$$

Since we are modelling the elements of $\boldsymbol{m}_2$ and $\boldsymbol{m}_3$ by independent Gaussian distributions, we interpret this equation as relating the p.d.f.'s of each element. It therefore says that the elements of $\boldsymbol{m}_2$ and $\boldsymbol{m}_3$ must be of the form

$$\begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix}$$

and thus real quaternion.

Wishart matrices have also found recent application as model of Dirac operators in studies of quantum chromodynamics (qcd) by Verbaarschot [13]. This theory is analogous to the random matrix theroy of non-relativistic quantum Hamiltonians discussed in Section 1.2.

We recall that an essential feature of the Dirac operator (see e.g [5] is that it is Hermitian and the eigenvalues occur in pairs $\pm\lambda$. Furthermore the application to

qcd requires that the Dirac operator has a given number, ν say, of zero eigenvalues. A matrix structure consistent with these requirements is given by

$$\boldsymbol{X} = \begin{pmatrix} 0 & \boldsymbol{D} \\ \boldsymbol{D}^\dagger & 0 \end{pmatrix},$$

where $\boldsymbol{D}$ is an $n \times m$ $(n \geq m)$ matrix. Thus $\boldsymbol{X}$ has in general $n-m$ zero eigenvalues and the remaining eigenvalues given by $\pm$ the positive square roots of the eigenvalues of $\boldsymbol{D}^\dagger \boldsymbol{D}$ (this result can be deduced from the singular value decomposition of $\boldsymbol{D}$). Since this result holds independent of the details of $\boldsymbol{D}$ a random matrix hypothesis can be made. Choosing the elements of $\boldsymbol{D}$ as independent complex numbers with distribution as in Definition 1.6 part (2), we see that the distribution of the positive eigenvalues of $\boldsymbol{X}$ above is precisely that of the eigenvalues λ_j of a complex Wishart matrix, with λ_j replaced by λ_j^2.

The choice of independent complex elements for $\boldsymbol{D}$ is only consistent in the absence of time reversal symmetry. In a system with a time reversal symmetry, the matrix $\boldsymbol{X}$ modelling the Dirac operator is subject to the same constraints as those of a non-relativistic Hamiltonian. Thus, from Section 1.2, if the Dirac operator has a time reversal symmetry with $T^2 = 1$, then $\boldsymbol{X}$ can be chosen to have real elements, while if the time reversal symmetry is such that $T^2 = -1$ with $T = \boldsymbol{Z}K$, then $\boldsymbol{X}$ and thus $\boldsymbol{D}$ must be real quaternion. If in these cases the distribution of the elements of $\boldsymbol{D}$ are chosen as in the Definitions 1.6 of real and real quaternion Wishart matrices, then the corresponding p.d.f. for the positive eigenvalues of $\boldsymbol{X}$ is precisely that of the eigenvalues λ_j for real and quaternion real Wishart matrices, with λ replaced by λ_j^2.

1.6. Eigenvalue p.d.f.'s of the Wishart matrices and analogy with Calogero Sutherland model. We will provide some of the details of the derivation of the eigenvalue p.d.f. of the complex Wishart matrices, as specified by Definition 1.6 part (2), by following the method detailed by Muirhead [8] to calculate the p.d.f. of the real Wishart matrices. The method uses exterior products, as in the calculation of the eigenvalue p.d.f. for the Gaussian and circular ensembles.

Let's first outline the overall strategy. According to Definition 1.6, we are given that the joint probability distribution of the elements of the $n \times m$ complex matrix $\boldsymbol{X}$ is

$$\frac{1}{\pi^{nm}} \prod_{j=1}^{n} \prod_{k=1}^{m} e^{-|z_{jk}|^2} (d\boldsymbol{X}).$$

With $\boldsymbol{A} := \boldsymbol{X}^\dagger \boldsymbol{X}$ our first task is to express $(d\boldsymbol{A})$ in terms of $(d\boldsymbol{X})$ and thus to obtain the p.d.f. of the matrix $\boldsymbol{A}$. To do this we use the Gram-Schmidt orthogonalization procedure to write

$$\boldsymbol{X} = \boldsymbol{U}_1 \boldsymbol{T} \tag{1.30}$$

where $\boldsymbol{U}$ is an $n \times m$ matrix such that

$$\boldsymbol{U}_1^\dagger \boldsymbol{U}_1 = 1$$

and $\boldsymbol{T}$ is a $m \times m$ upper triangular matrix with diagonal entries real and positive. From (1.30) we calculate $(d\boldsymbol{X})$ in terms of $(d\boldsymbol{U}_1)$ and $(d\boldsymbol{T})$. Noting from (1.30) and the definition of $\boldsymbol{A}$ that

$$\boldsymbol{A} = \boldsymbol{T}^\dagger \boldsymbol{T}$$

we then calculate $(d\boldsymbol{T})$ in terms of $(d\boldsymbol{A})$. Substituting the result of this second calculation into the first gives $(d\boldsymbol{X})$ in terms of $(d\boldsymbol{A})$ as required.

Having calculated the p.d.f of $\boldsymbol{A}$, we then use the results of Section 1.4 to express this p.d.f. in terms of the eigenvalues of $\boldsymbol{A}$.

Proceeding as in the above outline, we must relate $(d\boldsymbol{T})$ and $(d\boldsymbol{A})$. This requires the fact that the Jacobian of the transformation $\vec{z} = \boldsymbol{A}\vec{w}$,, where $\vec{w}, \vec{z}$ and $\boldsymbol{A}$ have complex entries, is $|\det \boldsymbol{A}|^2$. To see this, let $\vec{z} = [x_j + iy_j]_{j=1,\dots,N}$, $\vec{w} = [u_j + iv_j]_{j=1,\dots,N}$ and $\boldsymbol{A} = [a_{jk} + ib_{jk}]_{j,k=1,\dots,N}$. Then the equation $d\vec{z} = \boldsymbol{A}d\vec{w}$ can be rewritten as the real matrix equation

$$\begin{bmatrix} (dx_j)_{j=1,\dots,N} \\ (dy_j)_{j=1,\dots,N} \end{bmatrix} = \begin{bmatrix} (a_{jk})_{j,k=1,\dots,N} & -(b_{jk})_{j,k=1,\dots,N} \\ (b_{jk})_{j,k=1,\dots,N} & (a_{jk})_{j,k=1,\dots,N} \end{bmatrix} \begin{bmatrix} (du_j)_{j=1,\dots,N} \\ (dv_j)_{j=1,\dots,N} \end{bmatrix}$$

The determinant of the $2N \times 2N$ matrix, and thus the Jacobian, can be evaluated by adding i times the blocks in the bottom half to the blocks in the top half, and then subtracting i times the blocks in the left half to the blocks in the right half so that the top right block is now the zero matrix. This result is used in the proof of the following result.

Proposition 1.4. *With $\boldsymbol{U}_1$ and $\boldsymbol{T}$ defined by (1.30) we have*

$$(d\boldsymbol{X}) = \prod_{j=1}^{m} t_{jj}^{2(n-j)+1} (d\boldsymbol{T})(\boldsymbol{U}_1^\dagger d\boldsymbol{U}_1),$$

where the t_{jj} are the diagonal entries of the matrix $\boldsymbol{T}$.

Proof. (Outline) Since $\boldsymbol{X} = \boldsymbol{U}_1\boldsymbol{T}$ we have

$$d\boldsymbol{X} = d\boldsymbol{U}_1\,\boldsymbol{T} + \boldsymbol{U}_1 d\boldsymbol{T}.$$

Now extend the number of columns of $\boldsymbol{U}_1$ from m to n by defining an $(n-m) \times n$ matrix $\boldsymbol{U}_2$ such that

$$\boldsymbol{U} = [\boldsymbol{U}_1 \boldsymbol{U}_2]$$

and

$$\boldsymbol{U}^\dagger \boldsymbol{U} = 1.$$

Then

$$\boldsymbol{U}^\dagger d\boldsymbol{Z} = \begin{bmatrix} \boldsymbol{U}_1^\dagger \\ \boldsymbol{U}_2^\dagger \end{bmatrix} d\boldsymbol{X} = \begin{bmatrix} \boldsymbol{U}_1^\dagger (d\boldsymbol{U}_1\,\boldsymbol{T} + \boldsymbol{U}_1 d\boldsymbol{T}) \\ \boldsymbol{U}_2^\dagger (d\boldsymbol{U}_1\,\boldsymbol{T} + \boldsymbol{U}_1 d\boldsymbol{T}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{U}_1^\dagger d\boldsymbol{U}_1\,\boldsymbol{T} + d\boldsymbol{T} \\ \boldsymbol{U}_2^\dagger \boldsymbol{U}_1\,\boldsymbol{T} \end{bmatrix},$$

since $\boldsymbol{U}_1^\dagger \boldsymbol{U}_1 = 1$ and $\boldsymbol{U}_2^\dagger \boldsymbol{U}_1 = 0$.

Consider the above equation. On the l.h.s., from the preliminary result we have

$$(\boldsymbol{U}^\dagger d\boldsymbol{Z}) = |\det \boldsymbol{U}^\dagger|^{2m}(d\boldsymbol{X}) = (d\boldsymbol{X}),$$

which is the left hand side of the assertion. To take the wedge product of the independent elements on the r.h.s. is more involved. We merely remark that again the preliminary result is used, and that the r.h.s. of the assertion results. □

The next result is used to express $(d\boldsymbol{A})$ in terms of $(d\boldsymbol{T})$. It is proved by a straightforward computation of wedge products.

Proposition 1.5. *Let the $m \times m$ matrix $\boldsymbol{T}$ be as in (1.30), so that $\boldsymbol{A} = \boldsymbol{T}^\dagger \boldsymbol{T}$. We have*

$$(d\boldsymbol{A}) = 2^m \prod_{j=1}^{m} t_{jj}^{2m+1-2j} (d\boldsymbol{T}).$$

Combining Propositions 1.4 and 1.5, and noting that

$$\prod_{j=1}^{m} t_{jj}^2 = (\det \boldsymbol{T})^2 = \det \boldsymbol{A} \quad \textit{and} \quad \sum_{j,k=1}^{m} |x_{jk}|^2 = Tr\boldsymbol{A}$$

we can write down the p.d.f. of $\boldsymbol{A}$. This result was first given by Goodman [7], and is the analogue of the classical result of Wishart for the p.d.f. or what are now termed real Wishart matrices.

Proposition 1.6. *Let $\boldsymbol{A}$ be a complex Wishart matrix as defined by Definition 1.6. Then the p.d.f. of $\boldsymbol{A}$ is*

$$\frac{1}{\hat{C}_{2\,N}} e^{-Tr(A)} (\det \boldsymbol{A})^{n-m},$$

where $\hat{C}_{2\,N}$ is a normalization constant.

Finally, we use the results of Section 1.3 to change variables from the elements of $\boldsymbol{A}$ to the eigenvalues of $\boldsymbol{A}$ (and some further variables depending on the eigenvectors). Integrating out the further variables gives the desired eigenvalue p.d.f. of the complex Wishart matrix $\boldsymbol{A}$. The final result, together with that for real and real quaternion Wishart matrices is as follows.

Proposition 1.7. *With $\beta = 1, 2$ and 4, $\chi_\beta = -1, 0, 1/2$ respectively (or equivalently $\chi_\beta = 1 - 2/\beta$), the eigenvalue p.d.f. for real, complex, and real quaternion Wishart matrices is given by*

$$\frac{1}{W_{a\beta m}} \prod_{j=1}^{m} \lambda_j^{(\beta/2)a} e^{-\beta\lambda_j/2} \prod_{1 \le j < k \le m} |\lambda_k - \lambda_j|^\beta, \quad \lambda_j \ge 0$$

where $a = n - m + \chi_\beta$ (due to the weightings of the form $\lambda_j^{\beta a/2} e^{-\beta\lambda_j/2}$, this p.d.f. is said to define the Laguerre ensemble of random matrices).

1.7. Relationship to Calogero-Sutherland model. We have seen that the eigenvalue p.d.f. for the Gaussian and Circular ensembles is related to the Calogero-Sutherland model as the modulus squared of the ground state wave function. The same is also true of the eigenvalue p.d.f. for the Laguerre ensemble, provided we first change variables to the positive square roots $x_j^2 = \lambda_j$ of the eigenvalues of the Wishart matrices. In terms of these variables the eigenvalue p.d.f. is proportional to

$$\prod_{j=1}^{m} x_j^{\beta a'} e^{-\beta x_j^2/2} \prod_{1 \le j < k \le m} |x_j^2 - x_k^2|^\beta \tag{1.31}$$

where $a' = a + 1/\beta$ with a as specified in Proposition 1.7. This is of the form $e^{-\beta W^{(L)}}$ (the superscript (L) denotes Laguerre) where

$$W^{(L)} = \frac{1}{2}\sum_{j=1}^{m} x_j^2 - \frac{a'}{2}\sum_{j=1}^{m} \log x_j^2 - \sum_{1 \le j < k \le m} \log |x_k^2 - x_j^2| \tag{1.32a}$$

One can check by explicit calculation that $e^{-\beta W^{(L)}/2}$ is the ground state wave function of the Schrödinger operator

$$H^{(L)} = -\sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2} + \sum_{j=1}^{m} \left(\frac{\beta a'}{2}\left(\frac{\beta a'}{2} - 1\right)\frac{1}{x_j^2} + \frac{\beta^2}{4} x_j^2 \right) + 2\beta(\beta/2 - 1) \sum_{\substack{j,k=1 \\ j \ne k}}^{m} \frac{x_j^2}{(x_k^2 - x_j^2)^2} \tag{1.32b}$$

Since the ground state has the factorization property (1.2), this is the Schrödinger operator for a particular Calogero-Sutherland model. Note also that the double sum in (1.32b) can be rewritten as

$$\frac{1}{2}\sum_{j<k} \left(\frac{1}{(x_k - x_j)^2} + \frac{1}{(x_k + x_j)^2} \right). \tag{1.32c}$$

With the substitution (1.32c) $H^{(L)}$ is closely related to the B type reflection group, and for this reason is referred to as the B-type Calogero-Sutherland model in a harmonic potential.

1.8. The Jacobi ensemble. The Jacobi random matrix ensemble refers to the eigenvalue p.d.f.

$$C_{ab\beta N}^{-1} \prod_{j=1}^{N} x_j^{a\beta/2} (1 - x_j)^{b\beta/2} \prod_{1 \le j < k \le N} |x_k - x_j|^\beta \qquad x_j \in [0, 1]. \tag{1.33}$$

It turns out [14] that the eigenvalues of $\boldsymbol{r}^\dagger \boldsymbol{r}$ in the scattering matrix (1.28b) contribute a factor of the form (1.33) to the Jacobian implied by the singular value decomposition of $\boldsymbol{r}$. Furthermore, we have the following theorem, well known in mathematical statistics in the real case [8].

Proposition 1.8. *Let $\boldsymbol{A} = \boldsymbol{a}^\dagger \boldsymbol{a}$, $\boldsymbol{B} = \boldsymbol{b}^\dagger \boldsymbol{b}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are $n_1 \times m$ and $n_2 \times m$ random real ($\beta = 1$), complex ($\beta = 2$) or real quaternion ($\beta = 4$) Gaussian random matrices. Then the eigenvalues of $\boldsymbol{A}(\boldsymbol{A}+\boldsymbol{B})^{-1}$ are given by (1.33) with $N = m$, $a = n_1 - m + \chi_\beta$ and $b = n_2 - m + \chi_\beta$.*

To relate (1.33) to the ground state wave function of a Calogero-Sutherland system, we write $x_j = \sin^2 \phi_j$, $0 \le \phi_j \le \pi/2$. Now writing (1.33) in the form $Ce^{-\beta W}$ we have

$$W = W^{(J)} = -\frac{a'}{2}\sum_{j=1}^{N} \log \sin^2 \phi_j - \frac{b'}{2}\sum_{j=1}^{N} \log \cos^2 \phi_j - \sum_{1 \le j < k \le N} \log |\sin^2 \phi_j - \sin^2 \phi_k| \tag{1.34}$$

where $a' = a + 1/\beta$, $b' = b + 1/\beta$ (the superscript (J) denotes Jacobi). By explicit calculation we can show that $e^{-\beta W^{(J)}/2}$ is the ground state wave function of the Schrödinger operator

$$\begin{aligned} H^{(J)} \;=\; & -\sum_{j=1}^{N} \frac{\partial^2}{\partial \phi_j^2} + \sum_{j=1}^{N} \left(\frac{a'\beta}{2}\left(\frac{a'\beta}{2} - 1\right)\frac{1}{\sin^2 \phi_j} + \frac{b'\beta}{2}\left(\frac{b'\beta}{2} - 1\right)\frac{1}{\cos^2 \phi_j} \right) \\ & + 2\beta(\beta/2 - 1) \sum_{\substack{j,k=1 \\ j \ne k}}^{N} \frac{\sin^2 \phi_j \cos^2 \phi_j}{(\sin^2 \phi_j - \sin^2 \phi_k)^2} \end{aligned} \tag{1.35}$$

We note that the double sum can be rewritten as

$$\frac{1}{2}\sum_{j<k} \left(\frac{1}{\sin^2(\phi_j - \phi_k)} + \frac{1}{\sin^2(\phi_j + \phi_k)} \right).$$

The Hamiltonian (1.35) is then referred to as the BC-type Calogero-Sutherland model, due to its relation to the reflection group of the same name.

In the next Section, the relationship between random matrices and the Calogero-Sutherland model is further developed, as are the properties of the corresponding Schrödinger operators. In the course of these studies, a systematic way of deducing (1.32b) from (1.32a), and (1.35) from (1.34), will be presented. Before proceeding to this topic, let us define the n-particle distribution functions, which characterize the statistical properties of the ground state, and note some of their expected asymptotic properties.

1.9. The static n-particle distribution functions. For a general classical gas of N indistinguishable particles with Boltzmann factor $e^{-\beta W}$, the n-particle distribution is given by

$$\rho_{(n)}(x_1, \ldots, x_n) = \frac{N(N-1)\ldots(N-n)}{\hat{Z}_N} \int_I dx_{n+1} \cdots \int_I dx_N \, e^{-\beta W} \tag{1.36}$$

where

$$\hat{Z}_N = \int_I dx_1 \cdots \int_I dx_N \, e^{-\beta W}.$$

We note that $\rho_{(n)}(x_1,\dots,x_n)/\rho_{(n-1)}(x_1,\dots,x_{n-1})$ can be interpreted as the density at the point x_n given that there are particles at the points $x_1,\dots,x_{n-1}$. With $e^{-\beta W}$ interpreted as the modulus squared of the ground state wave function, (1.36) also holds for the n-particle distribution of the quantum system in the ground state.

In the $n = 1$ case the distribution (1.36) corresponds to the particle density. For W given by (1.6) and $I = [0, L]$, the system can be interpreted as being confined to a circle of circumference length L. The particle density must then be uniform and so given by $\rho_1(x) == N/L$. However for W given by (1.5), (1.32a) and (1.34) the particle density is not uniform. In these cases, predictions can be made for the limiting behaviour of the density by making use of the fact that the classical gas is a log-gas, and interpreting the one-body terms in W as potentials in two-dimensional electrostatics (see e.g. [15]).

For $W = W^{(H)}$ as given by (1.5), this calculation predicts that

$$\lim_{N\to\infty} \sqrt{\frac{2}{N}}\rho_{(1)}(\sqrt{2N}x) = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases} \tag{1.37}$$

independent of β. This limit gives the so-called global density, and the result is known as the Wigner semi-circle law. For $W = W^{(L)}$ as given by (1.32) and with $x_j^2 = y_j$, the electrostatic calculation gives

$$\lim_{N\to\infty} \rho_{(1)}(4Ny) = \begin{cases} \frac{1}{2\pi y^{1/2}}\sqrt{1-y}, & 0 < y < 1 \\ 0, & y \geq 1 \end{cases} \tag{1.38a}$$

independent of β and a, while for $W = W^{(J)}$ as given by (1.33) the electrostatic calculation gives

$$\lim_{N\to\infty} \rho_{(1)}(x) = \frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}, \tag{1.38b}$$

$0 < x < 1$, again independent of all the parameters. The log-gas interpretation also gives predictions for the asymptotic behaviour of the truncated two-particle distribution

$$\rho_{(2)}^T(x_1,x_2) := \rho_{(2)}(x_1,x_2) - \rho_{(1)}(x_1)\rho_{(1)}(x_2). \tag{1.39}$$

Firstly, suppose the thermodynamic limit has been taken and the log-gas is of infinite extent. The system is then translationally invariant so that $\rho_{(2)}^T(x_1,x_2) = \rho_{(2)}^T(x)$, where $x := |x_1 - x_2|$. For this situation, macroscopic electrostatic arguments, together with linear response theory [16] predict that

$$\rho_{(2)}^T(x) \sim -\frac{1}{\beta\pi^2 x^2} \tag{1.40}$$

for the leading order large-x behaviour of the non-oscillatory term. For the log-gas confined to the half-line $x > 0$ the same type of argument predicts [17]

$$\rho_{(2)}^T(x_1,x_2) \sim -\frac{1}{\beta 2\pi^2\sqrt{x_1 x_2}}\frac{x_1+x_2}{(x_1-x_2)^2} \tag{1.41}$$

for the leading non-oscillatory behaviour. (In (1.40) and (1.41) $\rho_{(2)}^T$ refers to the quantity (1.39) after the thermodynamic limit has been computed.)

In addition to the properties of infinite and semi-infinite states, states in which the particle density is taken to infinity but the interval containing the particles remains fixed are also of interest (the limiting procedure in (1.37) and (1.38) produces examples of such states). If the log-gas is confined to the interval (a, b), then we expect [18-21]

$$\begin{aligned} &\lim_{N\to\infty} \frac{N}{b-a} \int_x^{x'+(b-a)/N} ds\, \rho_{(2)}^T(x,s) \\ &= -\frac{1}{\beta\pi^2(x-x')^2} \frac{(a+b)(x+x')/2 - ab - xx'}{[(x-a)(b-x)(x'-a)(b-x']^{1/2}}, \; x \neq x' \end{aligned} \tag{1.42}$$

while for a log-gas confined to a circle of unit radius we expect [20,21]

$$\lim_{N\to\infty} \frac{N}{2\pi} \int_\theta^{\theta'+2\pi/N} d\phi\, \rho_{(2)}^T(\theta,\phi) = -\frac{1}{\beta(2\pi)^2 \sin^2((\theta-\theta')/2)}, \theta \neq \theta'. \tag{1.43}$$

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2. Parameter-dependent random matrices

2.1. Parameter-dependent Gaussian random matrices. The connection between random matrices and the Calogero-Sutherland model goes much deeper than the correspondence between the eigenvalue p.d.f.'s and the ground state wave functions. In fact by generalizing the joint p.d.f. for the elements defining Gaussian and Wishart random matrices to include a parameter, we find that the corresponding eigenvalue p.d.f.'s satisfy a Fokker-Planck equation, which is just a transformed version of the Schrödinger equation for the Calogero-Sutherland models with Schrödinger operators (1.3) and (1.32b).

Let us first consider Gaussian random matrices and reveal the connection with the Fokker-Planck equation. The parameter-dependent random matrices generalize (1.11) (or equivalently Definition 1.1) in such a way that the Gaussian p.d.f.'s of the individual elements can have a general mean and standard deviation.

Definition 2.1

For $\boldsymbol{H}$ real orthogonal, Hermitian or self-dual real quaternion, and $\beta = 1, 2$ and 4 respectively, the parameter-dependent Gaussian ensembles are defined so that their joint p.d.f. for the elements is

$$\begin{aligned} P(\boldsymbol{H}^{(0)}; \boldsymbol{H}; \tau) &= A_{\beta,\tau} \prod_{j,k=1}^{N} \exp\Big(-\beta |H_{jk} - e^{-\tau} H_{jk}^{(0)}|^2 / 2|1 - e^{-2\tau}| \Big) \\ &= A_{\beta,\tau} \exp\Big(-\beta Tr\Big\{ (\boldsymbol{H} - e^{-\tau}\boldsymbol{H}^{(0)})^2 \Big\} / 2|1 - e^{-2\tau}| \Big), \end{aligned}$$

where $A_{\beta,\tau}$ is the normalization.

The precise choice of the mean and standard deviation in Definition 2.1 has been made so that the joint p.d.f. satisfies the Fokker-Planck equation describing Brownian motion in a harmonic potential:

$$\frac{\partial P}{\partial \tau} = \sum_{\mu} \Big(\frac{\partial}{\partial H_\mu} (H_\mu P) + \frac{1}{\beta} D_\mu \frac{\partial^2 P}{\partial H_\mu^2} \Big), \tag{2.1}$$

where the label μ ranges over the independent elements, including both the real and imaginary parts of the off diagonal elements if they are complex, and $D_\mu = 1$ for the diagonal elements and $D_\mu = 1/2$ for the off diagonal elements.

By averaging over the distribution of $\boldsymbol{H}^{(0)}$, we find that the eigenvalue p.d.f. of $\boldsymbol{H}$ satisfies a certain Fokker-Planck equation. In general this is given by

$$P(\boldsymbol{H}; \tau) = \int d\boldsymbol{H}^{(0)} \, P(\boldsymbol{H}^{(0)}) P(\boldsymbol{H}^{(0)}; \boldsymbol{H}; \tau)$$

As a specific example, let $\beta = 2$ and suppose the joint p.d.f. at $\tau = 0$ is that of the GOE:

$$P(\boldsymbol{H}^{(0)}) = \prod_{j=1}^{N} \left(\frac{1}{2\pi}\right)^{1/2} e^{-H_{jj}^{(0)2}/2} \prod_{j<k} \frac{1}{\pi^{1/2}} e^{-H_{jk}^{(0)2}}$$

We then have

$$\begin{aligned} P(\boldsymbol{H};\tau) \quad = \quad & \prod_{j=1}^{N} \frac{e^{-H_{jj}^2/(1+e^{-2\tau})}}{\sqrt{\pi(1+e^{-2\tau})}} \prod_{j<k} \frac{2}{\pi\sqrt{1-e^{-4\tau}}} \\ & \times e^{-2(Re\, H_{jk})^2/(1+e^{-2\tau})-2(Im\, H_{jk})^2/(1-e^{-2\tau})} \end{aligned}$$

Thus the standard deviation of the real part of each element differs from the standard deviation of the imaginary part. Returning now to the eigenvalue p.d.f., we have the following result [1].

Proposition 2.1. *Let*

$$\boldsymbol{L}^{(0)} = diag[\nu_j] = \boldsymbol{U}^{(0)-1}\boldsymbol{H}^{(0)}\boldsymbol{U}^{(0)} \qquad \text{and} \qquad \boldsymbol{L} = diag[\lambda_j] = \boldsymbol{U}^{-1}\boldsymbol{H}\boldsymbol{U}$$

be the diagonalization of $\boldsymbol{H}^{(0)}$ and $\boldsymbol{H}$ respectively, and suppose the eigenvalue p.d.f. of $\boldsymbol{H}^{(0)}$ is given by $f(\nu_1, \ldots, \nu_N)$. The eigenvalue p.d.f. of $\boldsymbol{H}$ after averaging over f and $\boldsymbol{U}^{(0)}$, F say, is independent of $\boldsymbol{U}$ and is given by

$$F(\lambda_1, \ldots, \lambda_N; \tau) := \frac{1}{\hat{G}_{N\beta}} \prod_{1 \le j < k \le N} |\lambda_k - \lambda_j|^{\beta} F_1(\lambda_1, \ldots, \lambda_N; \tau)$$

where $\beta = 1, 2$ or 4 as in Definition 4.1, $\hat{G}_{N\beta}$ is a normalization and

$$\begin{aligned} F_1(\lambda_1, \ldots, \lambda_N; \tau) \quad := \quad & \int d\nu_1 \ldots d\nu_N \, f(\nu_1, \ldots, \nu_N) \\ & \times \int [\boldsymbol{U}^{(0)\dagger} d\boldsymbol{U}^{(0)}] P(\boldsymbol{U}^{(0)} \boldsymbol{L}^{(0)} \boldsymbol{U}^{(0)-1}; \boldsymbol{U}\boldsymbol{L}\boldsymbol{U}^{-1}; \tau), \end{aligned}$$

($[\boldsymbol{U}^{(0)\dagger} d\boldsymbol{U}^{(0)}]$ denotes the normalized measure associated with the eigenvectors of $\boldsymbol{H}^{(0)}$; recall Section 1.4). Furthermore F satisfies the p.d.e.

$$\frac{\partial F}{\partial \tau} = \mathcal{L}F \qquad \text{where} \qquad \mathcal{L} := \frac{1}{\beta} \sum_{j=1}^{N} \frac{\partial^2}{\partial \lambda_j^2} + \sum_{j=1}^{N} \frac{\partial}{\partial \lambda_j} \left(\lambda_j - \sum_{\substack{k=1 \\ k \ne j}}^{N} \frac{1}{\lambda_j - \lambda_k} \right),$$

subject to the initial condition

$$F(\lambda_1, \ldots, \lambda_N; 0) = f(\lambda_1, \ldots, \lambda_N).$$

We won't give the proof of this proposition here, however in the next section we will give the proof of the analogous result for parameter dependent Wishart matrices. Proposition 2.1 can be proved in a similar way.

The Fokker-Planck equation in Proposition 2.1 also describes the overdamped Brownian motion dynamics of the classical log-gas with potential energy (1.5). This follows because for any interacting N particle system with a general potential energy W, executing overdamped Brownian motion in a fictitious viscous fluid with friction coefficient γ at temperature β^{-1}, the evolution of the p.d.f. $p(x_1, \dots, x_N; \tau)$ for the location of the N particles at the points $x_1, \dots, x_N$ is given by the Fokker-Planck equation

$$\gamma \frac{\partial p}{\partial \tau} = \mathcal{L} p \qquad where \quad \mathcal{L} = \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \left(\frac{\partial W}{\partial x_j} + \beta^{-1} \frac{\partial}{\partial x_j} \right). \tag{2.2}$$

The Fokker-Planck operator in Proposition 2.1 is of this form with W given by (1.5).

2.2. Parameter-dependent Wishart matrices. With parameter-dependent Gaussian random matrices of dimension $n \times m$, defined according to the second formula in Definition 2.1, let us take up the problem of calculating the eigenvalues of the corresponding Wishart matrices $\boldsymbol{A} = \boldsymbol{X}^\dagger \boldsymbol{X}$.

To do this we decompose $\boldsymbol{X}$ as in Section 1.6, and thus write

$$\boldsymbol{X} = \boldsymbol{U}_1 \boldsymbol{T} \quad and \quad \boldsymbol{T} = \boldsymbol{V} \boldsymbol{L} \boldsymbol{V}^\dagger, \tag{2.3a}$$

($\boldsymbol{L} = diag(x_1, \dots, x_m)$) so that

$$(d\boldsymbol{X}) = J (d\boldsymbol{L}) (\boldsymbol{U}_1^\dagger \boldsymbol{U}_1) (\boldsymbol{V}^\dagger \boldsymbol{V}) \tag{2.3b}$$

where

$$J = \prod_{j=1}^{m} x_j^{\beta a + 1} \prod_{1 \le j < k \le m} |x_k^2 - x_j^2|^\beta \tag{2.3c}$$

Analogous formulas hold for the decomposition of $\boldsymbol{X}^{(0)}$: we can simply replace all the variables above by the same variables with a superscript (0).

Now suppose that the p.d.f. of the positive square roots $x_1^{(0)}, \dots, x_m^{(0)}$ of the eigenvalues of $\boldsymbol{X}^{(0)\dagger} \boldsymbol{X}^{(0)}$ is given by $f(x_1, \dots, x_m)$, and let us average the p.d.f.

$$P(\boldsymbol{X}^{(0)}; \boldsymbol{X}; \tau)$$

over f and the variables associated with $\boldsymbol{U}_1^{(0)}$ and $\boldsymbol{V}^{(0)}$. From (2.3) we obtain

$$F(x_1, \dots, x_m; \tau) = J F_1 \tag{2.4a}$$

where

$$F_1 := C\int_0^\infty dx_1^{(0)}\cdots\int_0^\infty dx_m^{(0)}\, f(x_1^{(0)},\dots,x_m^{(0)}) \times \int [\boldsymbol{U}_1^{(0)\dagger} d\boldsymbol{U}_1^{(0)}][\boldsymbol{V}^{(0)\dagger} d\boldsymbol{V}^{(0)}] P(\boldsymbol{X}^{(0)};\boldsymbol{X};\tau) \tag{2.4b}$$

Integrating over the variables associated with the measures $[\boldsymbol{U}_1^\dagger d\boldsymbol{U}_1]$ and $[\boldsymbol{V}^\dagger d\boldsymbol{V}]$ gives the p.d.f. for the positive square roots of the eigenvalues of $\boldsymbol{A}$. In fact (2.4b) is independent of $\boldsymbol{U}_1$ and $\boldsymbol{V}$. This is seen by changing variables

$$\boldsymbol{U}_1^{(0)} \mapsto \boldsymbol{U}_1\boldsymbol{V}\boldsymbol{U}_1^{(0)} \quad \boldsymbol{V}^{(0)} \mapsto \boldsymbol{V}\boldsymbol{V}^{(0)},$$

for then, due to the cyclic property of the trace, P becomes $P(\boldsymbol{X}^{(0)};\boldsymbol{L};\tau)$ while the measures $[\boldsymbol{U}_1^{(0)\dagger} d\boldsymbol{U}_1^{(0)}]$ and $[\boldsymbol{V}^{(0)\dagger} d\boldsymbol{V}^{(0)}]$ are unaltered.

Thus the p.d.f. for the positive square roots of the eigenvalues of a parameter-dependent Wishart matrix $\boldsymbol{A} = \boldsymbol{X}^\dagger\boldsymbol{X}$, after averaging over the positive square roots of the eigenvalues of $\boldsymbol{X}^{(0)\dagger}\boldsymbol{X}^{(0)}$ $(=\boldsymbol{X}^\dagger\boldsymbol{X}|_{\tau=0})$, and the other quantities associated with the change of variables (2.3), is equal to (2.4a). Analogous to the result of Proposition 2.1, this p.d.f. can be characterised as the solution of a certain Fokker-Planck equation.

Proposition 2.2. *The p.d.f. (2.4a) satisfies the Fokker-Planck equation (2.2) with $N = m$ and W given by (1.32a), subject to the initial condition*

$$F(x_1,\dots,x_m;\tau)\Big|_{\tau=0} = f(x_1,\dots,x_m).$$

Proof. Since $P(\boldsymbol{X}^{(0)};\boldsymbol{X};\tau)$ satisfies the Fokker-Planck equation (2.2), it follows immediately from (2.4b) that F_1 satisfies the same equation. We want to change variables from the elements of $\boldsymbol{X}$ to the variables in the decomposition (2.3a), which can be accomplished by using the formulas from tensor calculus (the first formula gives the so called Laplace-Beltrami operator for the respective space of matrices)

$$\sum_\mu D_\mu \frac{\partial^2}{\partial X_\mu^2} = \frac{1}{J}\sum_{j=1}^m \frac{\partial}{\partial x_j}\Big(J\frac{\partial}{\partial x_j}\Big) + O_{\boldsymbol{U}_1,\boldsymbol{V}},$$

and

$$\sum_\mu X_\mu \frac{\partial}{\partial X_\mu} = \sum_{j=1}^m x_j \frac{\partial}{\partial x_j} + O'_{\boldsymbol{U}_1,\boldsymbol{V}}$$

where J is given by (2.3c), and the operators $O_{\boldsymbol{U}_1,\boldsymbol{V}}$ and $O'_{\boldsymbol{U}_1,\boldsymbol{V}}$ involve derivatives with respect to the variables associated with $\boldsymbol{U}_1$ and $\boldsymbol{V}$. But F_1 is independent of $\boldsymbol{U}_1$ and $\boldsymbol{V}$, and furthermore

$$\sum_\mu \frac{\partial}{\partial X_\mu}(X_\mu P) = \sum_\mu X_\mu \frac{\partial P}{\partial X_\mu} + P\sum_{j=1}^m \frac{\partial}{\partial x_j} x_j + P(\beta m(m-1) + \beta m a' + m)$$

($\beta m(m-1) + \beta m a' + m$ is the total number of independent variables associated with $\boldsymbol{U}_1$ and $\boldsymbol{V}$). Thus (2.1) becomes

$$\frac{\partial F_1}{\partial \tau} = \mathcal{L}_0 F_1 + \Big(\beta m(m-1) + \beta m a' + m\Big) F_1$$

where

$$\mathcal{L}_0 := \frac{1}{\beta J} \sum_{j=1}^{m} \frac{\partial}{\partial x_j} \left(J \frac{\partial}{\partial x_j} \right) + \sum_{j=1}^{m} \frac{\partial}{\partial x_j} x_j .$$

Substituting $F_1 = J^{-1} F$ and expanding the derivatives using the product rule gives

$$\frac{\partial F}{\partial \tau} = \frac{1}{\beta} \sum_{j=1}^{m} \left(\frac{\partial^2 F}{\partial x_j^2} + F \frac{\partial J}{\partial x_j} \frac{\partial J^{-1}}{\partial x_j} + J \frac{\partial F}{\partial x_j} \frac{\partial J^{-1}}{\partial x_j} + F J \frac{\partial^2 J^{-1}}{\partial x_j^2} \right)$$

$$+ J F \sum_{j=1}^{m} \lambda_j \frac{\partial}{\partial x_j} J^{-1} + \sum_{j=1}^{m} \frac{\partial}{\partial x_j} x_j F + \Big(\beta m(m-1) + \beta m a' + m\Big) F.$$

This can be simplified by noting

$$J \frac{\partial J^{-1}}{\partial x_j} = -\frac{\partial}{\partial x_j} \log J$$

and

$$\sum_{j=1}^{m} x_j \frac{\partial}{\partial x_j} \log J = \beta \sum_{\substack{j,k=1 \\ j \neq k}}^{m} \frac{2x_j^2}{x_j^2 - x_k^2} + (\beta a' + 1) m = \beta m(m-1) + \beta m a' + m.$$

Thus

$$\frac{\partial F}{\partial \tau} = \frac{1}{\beta} \sum_{j=1}^{N} \left(\frac{\partial^2 F}{\partial x_j^2} - F \frac{\partial^2}{\partial x_j^2} \log J - \frac{\partial F}{\partial x_j} \frac{\partial}{\partial x_j} \log J + \frac{\partial}{\partial x_j} x_j F \right)$$

which, after computation of $\partial \log J / \partial x_j$, is seen to be precisely the Fokker-Planck equation (2.2) with W given by (1.32a). □

Analogous to the Gaussian case, we can interpret the Fokker-Planck equation specifying the p.d.f. for the square root of the eigenvalues of the parameter-dependent Wishart matrices as describing the Brownian evolution of the classical gas with potential energy (1.32a).

2.3. Relationship to the Calogero-Sutherland model. In general the Fokker-Planck operator (2.2) can be rewritten as [2]

$$\mathcal{L} = \frac{1}{\beta} \sum_{j=1}^{N} \frac{\partial}{\partial x_j} e^{-\beta W} \frac{\partial}{\partial x_j} e^{\beta W} \tag{2.5}$$

From this formula, and the general property of adjoints $(AB)^\dagger = B^\dagger A^\dagger$, we see that $e^{\beta W/2} \mathcal{L} e^{-\beta W/2}$ is an Hermitian operator. Explicitly, we have

$$e^{\beta W/2} \mathcal{L} e^{-\beta W/2} = \sum_{j=1}^{N} \Big(\frac{1}{\beta} \frac{\partial^2}{\partial x_j^2} - \frac{\beta}{4} \Big(\frac{\partial W}{\partial x_j} \Big)^2 + \frac{1}{2} \frac{\partial^2 W}{\partial x_j^2} \Big). \tag{2.6}$$

which can be interpreted as $-\beta^{-1}$ times a Schrödinger operator.

For W given by (1.5), and thus the Fokker-Planck equation of Proposition 2.1 we find (after using the identity (1.7))

$$-e^{\beta W/2} \mathcal{L} e^{-\beta W/2} = (H - E_0)/\beta \tag{2.7}$$

where H is given by (1.3) and $E_0 = N\beta/2 + \beta^2 N(N-1)/4$. For W given by (1.32a), and $N = m$, which corresponds to the Fokker-Planck equation of Proposition 2.2, the operator identity (2.7) holds with H now given by (1.32b) (the precise value of E_0 is unimportant). In the study of the circular ensembles we encounted the potential (1.6). Dyson [3] has developed an abstract theory of parameter-dependent unitary random matrices from the circular ensemble (the theory is abstract in the sense that there is no known explicit construction of the matrices). The eigenvalue p.d.f. is specified by a Fokker-Planck equation with W given by (1.6). By substituting (1.6) in (2.6), and using the identity (1.8), we find that the operator identity (2.7) holds with H now given by (1.4). Also, a similar theory of parameter-dependent random matrices from the Jacobi ensemble has been given [4], in which the eigenvalue p.d.f. is specified by the Fokker-Planck equation (2.2) with x_j replaced by ϕ_j and W given by (1.34).

In general, writing $\tau = i\beta\gamma t$ and $p = e^{iE_0 t} e^{-\beta W/2} \psi$, the operator identity (2.7) shows that the Fokker-Planck equation in (2.2) transforms to the Schrödinger equation

$$i \frac{\partial}{\partial t} \psi(\{x_j\}; t) = H \psi(\{x_j\}; t) \tag{2.8}$$

Thus the study of the eigenvalue p.d.f.'s of the parameter-dependent random matrices as characterized by Fokker-Planck equations, leads us to consider the full Schrödinger equation for the corresponding Calogero-Sutherland models.

There are some simple general properties of the Fokker-Planck operator (2.2) with $\gamma = 1$ of relevance to the Calogero-Sutherland model. In particular, it is evident from (2.5) that $\mathcal{L} e^{-\beta W} = 0$. Thus the Boltzmann factor is the equilibrium state of the Fokker-Planck equation, and so from (2.7) $e^{-\beta W/2}$ is an eigenstate of

H with eigenvalue E_0. In fact $e^{-\beta W/2}$ is the ground state of H. To see this we note that [2]

$$e^{\beta W/2}\mathcal{L}e^{-\beta W/2} = -\beta^{-1}\sum_{j=1}^{N}\Pi_j^\dagger\Pi_j \quad \textit{where} \quad \Pi_j := \frac{1}{i}\frac{\partial}{\partial\lambda_j} - \frac{i\beta}{2}\frac{\partial W}{\partial\lambda_j} \tag{2.9}$$

and thus the eigenvalues of $\mathcal{L}$ are ≤ 0, so from (2.7) the eigenvalues of $H - E_0$ are ≥ 0.

2.4. The Green function. A key quantity in the description of dynamical correlations is the Green function solution of the Fokker-Planck and Schrödinger equations. Consider first the Schrödinger equation (2.8), and write $t = \tau/i\beta$ to give the so-called imaginary time Schrödinger equation

$$-\beta\frac{\partial}{\partial\tau}\psi(\{x_j\};\tau) = H\psi(\{x_j\};\tau) \tag{2.10a}$$

We say that $\psi = \tilde{G}(x_1^{(0)},\dots,x_N^{(0)};x_1,\dots,x_N;\tau)$ is the Green function solution of this equation if it is the solution which satisfies the initial condition

$$\psi(x_1,\dots,x_N;\tau)\big|_{\tau=0} = \prod_{l=1}^{N}\delta(x_l - x_l^{(0)}), \qquad (x_1^{(0)} < \cdots < x_N^{(0)}), \tag{2.10b}$$

It follows from the formula (2.7) that

$$\begin{aligned} &G(x_1^{(0)},\dots,x_N^{(0)};x_1,\dots,x_N;\tau) \\ &= e^{\tau E_0/\beta}\frac{e^{-\beta W(x_1,\dots,x_N)/2}}{e^{-\beta W(x_1^{(0)},\dots,x_N^{(0)})/2}}\tilde{G}(x_1^{(0)},\dots,x_N^{(0)};x_1,\dots,x_N;\tau) \end{aligned} \tag{2.11}$$

is the Green function solution of the Fokker-Planck equation (2.2) with $\gamma = 1$ (i.e. that G satisfies this equation subject to the initial condition that G is given by the r.h.s. of (2.10b) for $\tau = 0$).

In general the Green function may be written in terms of the eigenvalues and eigenfunctions of the corresponding operator. Consider in particular the Schrödinger operator H. Suppose that $\{\psi_\kappa\}_\kappa$ is a complete set of orthogonal eigenfunctions with corresponding eigenvalues $\{E_\kappa\}_\kappa$ (for an N-particle system the label κ will typically be an N-tuple of integers). Then the Green function solution is given by

$$\tilde{G}(x_1^{(0)},\dots,x_N^{(0)};x_1,\dots,x_N;\tau) = \sum_\kappa \frac{\psi_\kappa(x_1^{(0)},\dots,x_N^{(0)})\psi_\kappa(x_1,\dots,x_N)}{\langle\psi_\kappa|\psi_\kappa\rangle}e^{-\tau E_\kappa/\beta} \tag{2.12}$$

with $\langle\psi_\kappa|\psi_\kappa\rangle := \int_I dx_1\cdots\int_I dx_N\,|\psi_\kappa|^2$.

As an example of the computation of a Green function, let us consider the Fokker-Planck operator in (2.2) with W given by (1.32a). According to (2.7) we

can first consider the corresponding Schrödinger operator, which is given by (1.32b) and in the $m = 1$, $\beta = 2$ case reads

$$H^{(L)} = -\frac{d^2}{dx^2} + \frac{a'(a'-1)}{x^2} + x^2 \tag{2.13}$$

Using the fact [5] that $z = e^{-x/2}x^{\alpha/2}L_n^\alpha(x)$, where $L_n^\alpha(x)$ denotes the Laguerre polynomial, satisfies the equation

$$\Big[(x\frac{d}{dx})' + (n + \frac{\alpha+1}{2} - \frac{x}{4} - \frac{a^2}{4x})\Big]z = 0,$$

we can readily check that the eigenvalues and eigenfunctions of (2.13) are given by

$$E_n = 4(n + \frac{a' + 1/2}{2}), \quad \psi_n = e^{-x^2/2}|x|^{a'}L_n^{a'-1/2}(x^2)$$

Using (2.11) this implies that in terms of the variable $y = x^2$ the Green function for the Fokker-Planck operator in Proposition 2.2 is

$$\begin{aligned} G(y^{(0)}, y; \tau) &= y^a e^{-y} \sum_{n=0}^{\infty} \frac{n! L_n^a(y^{(0)}) L_n^a(y)}{\Gamma(a+n+1)} e^{-2n\tau} \\ &= \Big(\frac{y}{y^{(0)}}\Big)^{a/2} e^{-y} \frac{e^{\tau a}}{1-e^{-2\tau}} \exp\Big(-\frac{y+y^{(0)}}{e^{2\tau}-1}\Big) I_a\Big(\frac{(y^{(0)}y)^{1/2}}{\sinh\tau}\Big) \end{aligned} \tag{2.14}$$

where we have put $a := a' - 1/2$ and to obtain the first line we have used the fact that

$$\int_0^\infty e^{-y}y^\alpha \Big(L_n^\alpha(y)\Big)^2 dy = \frac{\Gamma(\alpha+n+1)}{n!}$$

and to obtain the second line the summation formula [5]

$$\sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+\alpha+1)} L_n^\alpha(x) L_n^\alpha(y) z^n = \frac{(xyz)^{-\alpha/2}}{1-z} \exp\Big(-z\frac{x+y}{1-z}\Big) I_\alpha\Big(2\frac{(xyz)^{1/2}}{1-z}\Big) \tag{2.15}$$

($I_\alpha(x)$ denotes the Bessel function of pure imaginary argument) has been used. Note from Section 2.2 that we can interpret (2.14) as giving the p.d.f. for the distribution of $y = \sum_{j=1}^{a+1} X_j^2 + Y_j^2$ where each X_j is chosen with p.d.f.

$$\Big(\frac{1}{\pi(1-e^{-2\tau})}\Big)^{1/2} \exp\Big[-(X - e^{-\tau}X^{(0)})^2/(1-e^{-2\tau})\Big]$$

and similarly for the p.d.f. of each Y_j, except that $X^{(0)}$ is to be replaced by $Y^{(0)}$. The quantity y_0 in (2.14) is given by $y^{(0)} = X^{(0)2} + Y^{(0)2}$.

2.5. The dynamical correlations. In this section, we study the density-density correlation function and the current-current distribution.

The density-density correlation. The dynamical density-density correlation $\rho^T_{(2)}((x_a,0),(x_b,\tau_b))$ for a system described by the Fokker-Planck equation (2.2) measures the correlation between the density at the point x_a initially, and the density at the point x_b after time τ_b. It is specified in terms of the Green function G and the initial position p.d.f. f by

$$\rho^T_{(2)}((x_a,0),(x_b,\tau_b)) = \rho_{(2)}((x_a,\tau_a),(x_b,\tau_b)) - \rho_{(1)}(x_a,0)\rho_{(1)}(x_b,\tau_b) \tag{2.16}$$

where

$$\rho((x_a,0),(x_b,\tau_b)) := \int_I dx_1^{(0)} \cdots \int_I dx_N^{(0)} f(x_1^{(0)},\ldots,x_N^{(0)}) \sum_{l=1}^N \delta(x_l^{(0)} - x_a)$$
$$\times \int_I dx_1 \cdots \int_I dx_N \sum_{l=1}^N \delta(x_l - x_b) G(x_1^{(0)},\ldots,x_N^{(0)};x_1,\ldots,x_N;\tau_b) \tag{2.17}$$

$$\rho_{(1)}(x_a,0) := \int_I dx_1^{(0)} \cdots \int_I dx_N^{(0)} f(x_1^{(0)},\ldots,x_N^{(0)}) \sum_{l=1}^N \delta(x_l^{(0)} - x_a) \tag{2.18}$$

$$\rho_{(1)}(x_b,\tau_b) := \int_I dx_1^{(0)} \cdots \int_I dx_N^{(0)} f(x_1^{(0)},\ldots,x_N^{(0)})$$
$$\times \int_I dx_1 \cdots \int_I dx_N \sum_{l=1}^N \delta(x_l - x_b) G(x_1^{(0)},\ldots,x_N^{(0)};x_1,\ldots,x_N;\tau_b) \tag{2.19}$$

On the other hand, for a quantum mechanical system described by the imaginary-time Schrödinger equation (2.10a), the ground-state dynamical density-density correlation is defined in terms of the density operator $n(x) := \sum_{l=1}^N \delta(x_l - x)$, the evolution operator $e^{-\tau_b(H-E_0)/\beta}$ and the ground state ψ_0 by

$$\rho^T_{(2)}((x_a,0),(x_b,\tau_b) = \langle\psi_0|n(x_b)e^{-\tau_b(H-E_0)/\beta}n(x_a)|\psi_0\rangle - \langle\psi_0|n(x_a)|\psi_0\rangle\langle\psi_0|n(x_b)|\psi_0\rangle \tag{2.20}$$

In fact the quantum mechanical formula (2.20) is the special case

$$f = e^{-\beta W} \Big/ \int_I dx_1 \cdots \int_I dx_N e^{-\beta W}$$

of the classical formula (2.16)-(2.19), which corresponds to the situation when the initial state equals the final state [6].

To see this, we note from (2.11) and (2.12) that

$$fG = \frac{\psi_0(x_1^{(0)},\dots,x_N^{(0)})\psi_0(x_1,\dots,x_N)}{\langle\psi_0|\psi_0\rangle} \times \sum_\kappa \frac{\psi_\kappa(x_1^{(0)},\dots,x_N^{(0)})\psi_\kappa(x_1,\dots,x_N)}{\langle\psi_\kappa|\psi_\kappa\rangle} e^{-\tau(E_\kappa-E_0)/\beta}$$

which when substituted in (2.17)-(2.19) gives

$$\begin{aligned}\rho_{(2)}((x_a,0),(x_b,\tau_b)) &= \sum_\kappa \frac{\langle\psi_0|n(x_b)|\psi_\kappa\rangle\langle\psi_\kappa|n(x_a)|\psi_0\rangle}{\langle\psi_0|\psi_0\rangle\langle\psi_\kappa|\psi_\kappa\rangle} e^{-\tau(E_\kappa-E_0)/\beta} \\ &= \langle\psi_0|n(x_b)e^{-\tau_b(H-E_0)/\beta}n(x_a)|\psi_0\rangle\end{aligned}$$

and

$$\rho_{(1)}(x_a,0) = \langle\psi_0|n(x_a)|\psi_0\rangle, \quad \rho_{(1)}(x_b,\tau_b) = \langle\psi_0|n(x_b)|\psi_0\rangle$$

as required.

The current-current distribution. In general the current-current distribution $C((x_a,\tau_a),(x_b,\tau_b))$ for a classical one-dimensional dynamical model measures the correlation between velocities at two points (x_a,τ_a) and (x_b,τ_b). For systems described by the Fokker-Planck equation, this definition is only formal as the velocities of the individual particles are not defined.

For the formal definition, define the average $\langle\,\rangle$ by

$$\begin{aligned}\langle u_a(x_1^{(1)},\dots,x_N^{(1)})u_b(x_1^{(2)},\dots,x_N^{(2)})\rangle &= \int_I dx_1^{(0)}\cdots\int_I dx_N^{(0)}\, f(x_1^{(0)},\dots,x_N^{(0)}) \\ &\times \int_I dx_1^{(1)}\cdots\int_I dx_N^{(1)}\, u_a(x_1^{(1)},\dots,x_N^{(1)})G(x_1^{(0)},\dots,x_N^{(0)};x_1^{(1)},\dots,x_N^{(1)};\tau_a) \\ &\times \int_I dx_1^{(2)}\cdots\int_I dx_N^{(2)}\, u_b(x_1^{(2)},\dots,x_N^{(2)})G(x_1^{(1)},\dots,x_N^{(1)};x_1^{(2)},\dots,x_N^{(2)};\tau_b-\tau_a)\end{aligned}$$

The current-current distribution is defined in terms of this average by

$$C((x_a,\tau_a),(x_b,\tau_b)) := \Big\langle \sum_{j=1}^N \frac{dx_j^{(1)}}{d\tau_a}\delta(x_a-x_j^{(1)}) \sum_{j=1}^N \frac{dx_j^{(2)}}{d\tau_b}\delta(x_b-x_j^{(2)})\Big\rangle. \tag{2.21}$$

The current-current and density-density distribution functions are related by a type of continuity equation, which for translationally invariant systems allows the current-current distribution to be calculated in the Brownian motion model. To

derive this equation, consider the partial derivatives of the current-current distribution with respect to x_a and x_b:

$$\begin{aligned}
&\frac{\partial^2}{\partial x_a \partial x_b} C((x_a,\tau_a),(x_b,\tau_b)) \\
&= \Big\langle \sum_{j=1}^{N} \frac{dx_j^{(1)}}{d\tau_a} \frac{\partial}{\partial x_a} \delta(x_a - x_j^{(1)}) \sum_{j=1}^{N} \frac{dx_j^{(2)}}{d\tau_b} \frac{\partial}{\partial x_b} \delta(x_b - x_j^{(2)}) \Big\rangle \\
&= \Big\langle \sum_{j=1}^{N} \frac{dx_j^{(1)}}{d\tau_a} \frac{\partial}{\partial x_j^{(1)}} \delta(x_a - x_j^{(1)}) \sum_{j=1}^{N} \frac{dx_j^{(2)}}{d\tau_b} \frac{\partial}{\partial x_j^{(2)}} \delta(x_b - x_j^{(2)}) \Big\rangle \\
&= \frac{\partial^2}{\partial \tau_a \partial \tau_b} \Big\langle \sum_{j=1}^{N} \delta(x_a - x_j^{(1)}) \sum_{j=1}^{N} \delta(x_b - x_j^{(2)}) \Big\rangle
\end{aligned}$$

since in general $\partial/\partial\tau = (\partial x/\partial\tau)(\partial/\partial x)$. Thus we have

$$\frac{\partial^2}{\partial \tau_a \partial \tau_b} \rho_{(2)}((x_a,\tau_a),(x_b,\tau_b)) = \frac{\partial^2}{\partial x_a \partial x_b} C((x_a,\tau_a),(x_b,\tau_b)). \tag{2.22}$$

which is the desired equation. This further simplifies in translationally invariant systems, since then

$$\rho_{(2)}((x_a,\tau_a),(x_b,\tau_b)) = \rho_{(2)}(x_b - x_a;\tau_a,\tau_b)$$

and

$$C((x_a,\tau_a),(x_b,\tau_b)) = C(x_b - x_a;\tau_a,\tau_b)$$

and so by taking the Fourier transform and integrating by parts we obtain

$$\tilde{C}(k;\tau_a,\tau_b) = \frac{1}{k^2} \frac{\partial^2}{\partial \tau_a \partial \tau_b} \tilde{\rho}_{(2)}(k;\tau_a,\tau_b), \tag{2.23}$$

where the tilde refers to the Fourier transform:

$$\tilde{F}(k;\tau) := \int_{-\infty}^{\infty} F(x;\tau) e^{ikx} dx.$$

For a one-dimensional quantum system the current operator is defined by

$$j(x) = \sum_{j=1}^{N} \Big(\frac{1}{i} \frac{\partial}{\partial x_j} \delta(x - x_j) + \delta(x - x_j) \frac{1}{i} \frac{\partial}{\partial x_j} \Big) \tag{2.24}$$

and the corresponding dynamical operator is $j(x,t) = e^{itH} j(x) e^{-itH}$ (here we have set $\hbar = 1$). The quantity $j(x,t)$ is related to the dynamical density operator

$$\rho(x,t) = e^{itH} \sum_{j=1}^{N} \delta(x - x_j) e^{-itH}$$

by the continuity equation

$$\frac{\partial}{\partial t}\rho(x,t) = -\frac{\partial}{\partial x}j(x,t) \tag{2.25}$$

To see this, we can use Heisenberg's equation of motion

$$\frac{d}{dt}(e^{itH}Ae^{-itH}) = -ie^{itH}[A,H]e^{-itH}.$$

to show that

$$\frac{d}{dt}(e^{itH}\rho_k e^{-itH}) = -ike^{itH}j_k e^{-itH}$$

where ρ_k denotes the Fourier transform of $n(x) := \sum_{j=1}^{N}\delta(x-x_j)$ and j_k denotes the Fourier transform of $j(x)$. Taking the inverse transform gives (2.25). From (2.25) we have

$$\frac{\partial^2}{\partial t_a \partial t_b}\langle\psi_0|\rho(x_a,t_a)\rho(x_b,t_b)|\psi_0\rangle = \frac{\partial^2}{\partial x_a \partial x_b}\langle\psi_0|j(x_a,t_a)j(x_b,t_b)|\psi_0\rangle \tag{2.26}$$

which is the quantum mechanical version of (2.22).

We know that the density-density distribution in (2.26) is identical to that of the corresponding Fokker-Planck system with the initial distribution f given by the equilibrium distribution. It can also be shown directly [7] that the current-current distribution for the quantum system is the same as that in the Fokker-Planck system (after multiplication of the former by $(1/\beta i)^2$ to account for going from t to τ), as comparison of (2.22) and (2.26) suggests.

2.6. Sum rules and asymptotic behaviour. We will show that the density-density correlation satisfies a f-sum rule. We also show that the static density-density distribution and current-current distribution are related.

The f-sum rule. When the initial state f corresponds to the equilibrium state, and thus the quantum mechanical formula (2.20) holds, a number of special formulas for the density-density and current-current correlations hold [7]. The first such formula we shall present is a generalization of the so-called f-sum rule [8].

Proposition 2.3. *Let* $S((x_a,0),(x_b,\tau)) := \rho_{(2)}^T((x_a,0),(x_b,\tau)$ *as given by (2.20), and define*

$$\tilde{S}((x_a,0),(k,\tau)) = \int_{-\infty}^{\infty} S((x_a,0),(x_b,\tau))e^{ikx_b}\,dx_b.$$

We have

$$\frac{\partial}{\partial\tau}\tilde{S}((x_a,0),(k,\tau))\Big|_{\tau=0} = \frac{ik}{\beta}\frac{\partial}{\partial x_a}\Big(e^{ikx_a}\rho(x_a)\Big).$$

Proof. From (2.20) and (2.25) with $t = \tau/i\beta$ we have

$$\frac{\partial}{\partial \tau} S((x_a,0),(x_b,\tau))\Big|_{\tau=0} = \frac{i}{\beta}\langle\psi_0|\frac{\partial}{\partial x_b} j(x_b,0)\rho(x_a,0)|\psi_0\rangle.$$

where $j(x_b,0)$ is defined by (2.24). Taking the Fourier transform of both sides with respect to x_b this reads

$$\frac{\partial}{\partial \tau} \tilde{S}((x_a,0),(k,\tau))\Big|_{\tau=0} = \frac{k}{\beta}\langle\psi_0|\tilde{j}(k,0)\rho(x_a,0)|\psi_0\rangle. \tag{2.27}$$

Now from (2.24)

$$\tilde{j}(k,0) = \sum_{j=1}^{N}\Big(\frac{1}{i}\frac{\partial}{\partial x_j}e^{ikx_j} + e^{ikx_j}\frac{1}{i}\frac{\partial}{\partial x_j}\Big)$$

which after use of the definition (2.9) of Π_j can be written

$$\tilde{j}(k,0) = \sum_{j=1}^{N}\Big(\Pi_j^\dagger e^{ikx_j} + e^{ikx_j}\Pi_j\Big). \tag{2.28}$$

Substituting (2.28) in (2.27) and using the fact that $\langle\psi_0|\Pi_j^\dagger = 0$ gives

$$\frac{\partial}{\partial \tau} \tilde{S}((x_a,0),(k,\tau))\Big|_{\tau=0} = \frac{k}{\beta}\sum_{j,l=1}^{N}\langle\psi_0|e^{ikx_l}\Pi_l\delta(x_a - x_j)|\psi_0\rangle \tag{2.29}$$

Note that for $j \neq l$, $\Pi_l\delta(x_a - x_j) = \delta(x_a - x_j)\Pi_l$, which since

$$\Pi_l|\psi_0\rangle = 0 \tag{2.30}$$

shows that only the diagonal term in (2.29) is non-zero. Furthermore, since ψ_0 is proportional to $e^{-\beta W/2}$ we have

$$\langle\psi_0|e^{ikx_l}\Pi_l = -\langle\psi_0|(k + i\beta\frac{\partial W}{\partial x_l})e^{ikx_l} \tag{2.31}$$

Substituting (2.31) in the diagonal term of (2.29) and noting that

$$\frac{\partial W}{\partial x_l}\psi_0^2 = -\frac{1}{\beta}\frac{\partial}{\partial x_l}\psi_0^2,$$

the stated result follows after minor manipulation. □

In the case of a uniform background density $\rho(x) = \rho$ the formula of Proposition 2.3 gives

$$\frac{\partial}{\partial \tau} \tilde{S}((0,0),(k,\tau))\Big|_{\tau=0} = -\frac{k^2\rho}{\beta} \tag{2.32}$$

which is equivalent to the f-sum rule for quantum fluids [8].

Static current-current distribution. The static current-current distribution can be written in terms of the static density-density distribution according

to a result [7] which was first obtained for the Calogero-Sutherland system with Schrödinger operator (1.3) by Taniguchi et al. [9].

Proposition 2.4. *For $x_a \neq x_b$ we have*

$$\langle \psi_0 | j(x_a,0) j(x_b,0) | \psi_0 \rangle = \beta \Big(\frac{\partial W}{\partial x_1 \partial x_2} \Big) \Big|_{\substack{x_1 = x_a \\ x_2 = x_b}} \langle \psi_0 | \rho(x_a,0) \rho(x_b,0) | \psi_0 \rangle .$$

Proof. Analogous to (2.28) we have

$$j(x,0) = \sum_{j=1}^{N} \Big(\Pi_j^\dagger \delta(x - x_j) + \delta(x - x_j) \Pi_j \Big) .$$

From this formula and (2.30) we have

$$\langle \psi_0 | j(x_a,0) j(x_b,0) | \psi_0 \rangle = \sum_{j,k=1}^{N} \langle \psi_0 | \delta(x_b - x_j) \Pi_j \Pi_k^\dagger \delta(x - x_k) | \psi_0 \rangle \tag{2.33}$$

Assuming $x_a \neq x_b$, we see that the $j = k$ term vanishes. For $j \neq k$ we have

$$[\Pi_j, \Pi_k^\dagger] = \beta \frac{\partial W}{\partial x_j \partial x_k} ,$$

which together with the fact that $\Pi_j \delta(x - x_k) = \delta(x - x_k) \Pi_j$ for $j \neq k$ and the use of (2.30) allows the r.h.s. of (2.33) to be written in the required form. □

Compressibility sum rule. For translationally invariant systems $\rho_{(2)}^T((x_a,0),(x_b,\tau_b))$ depends on $x_b - x_a$ so we can define the Fourier transform by

$$\tilde{S}(k,\tau) := \int_{-\infty}^{\infty} \rho_{(2)}^T((0,0),(x,\tau)) e^{ikx} \, dx$$

(here and throughout this section we are referring to the quantities in the thermodynamic limit). In general for a quantum system the value of

$$\lim_{k \to 0} \int_0^\infty \tilde{S}(k,\tau) \, d\tau = \int_{-\infty}^{\infty} dx \int_0^\infty d\tau \, \rho_{(2)}^T((0,0),(x,\tau) \tag{2.34}$$

is known to be related to the ground state compressibility [9]. For the particular quantum system (1.4) the value of the integral (2.34) can be deduced from the expected small-k behaviour [6]

$$\tilde{S}(k,\tau) \sim \frac{|k|}{\pi\beta} e^{-\pi\rho|k|\tau} \tag{2.35}$$

valid for τ fixed. This formula is deduced from a hydrodynamic approximation to the Fokker-Planck equation, first formulated by Dyson [3]. Integrating (2.35) with respect to τ gives

$$\lim_{k \to 0} \int_0^\infty \tilde{S}(k,\tau) \, d\tau = \frac{1}{\pi^2 \beta \rho} .$$

For the quantum system (1.4) the next term as a function of k in the expansion (2.35) has also been predicted [7]:

$$\tilde{S}(k,\tau) \sim \frac{|k|}{\pi\beta}\Big(1+\frac{1}{2\pi\rho}\Big(\frac{\beta-2}{\beta}\Big)|k|\Big)\exp\Big\{-\frac{\pi\rho|k|\tau}{\gamma(1+(\beta-2)|k|/(2\pi\rho\beta)}\Big\} \quad (2.36)$$

This follows by modifying the hydrodynamic approximation to include a force due to the pressure gradient, and so this extra term is related to the compressibility in the classical log-gas.

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3. Polynomial eigenfunctions

3.1. The case $\beta = 2$. From (2.12) we know that the Green function can be expanded in terms of eigenfunctions of the corresponding Fokker-Planck or Schrödinger operator. For the Schrödinger operators (1.3), (1.4), (1.32b) and (1.35) the calculation of the eigenfunctions is a simple task at $\beta = 2$, since then the coefficient of the two-body term vanishes and the system corresponds to free fermions in an external field (the free fermion condition means that we seek anti-symmetric eigenfunctions). The eigenfunctions are Slater determinants constructed from the solution of the single particle problem. Thus, in general the anti-symmetric eigenfunctions of the Schrödinger operator

$$H_N = -\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + \sum_{j=1}^{N} V(x_j)$$

are given by

$$\psi_{(n_1,\dots,n_N)}(x_1,\dots,x_N) = \det[\phi_{n_k}(x_j)]_{j,k=1,\dots,N} \quad (3.1)$$

where $\{\phi_k\}_{k\in Z}$ is the complete set of eigenfunctions of H_1 and $n_1 > n_2 > \cdots > n_N$.

For example, consider the Schrödinger operator (1.3). We seek the eigenfunctions of the $N = 1$ operator which are periodic for N odd and antiperiodic for N even. These are given by

$$\{e^{2\pi inx/L}\}_{n\in Z} \quad and \quad \{e^{2\pi i(n-1/2)x/L}\}_{n\in Z}$$

respectively. From (3.1) we see that instead of labelling the N-particle eigenstates by $(n_1, \dots, n_N)$ we can introduce a single non-negative integer l together with a partition $\kappa = (\kappa_1, \dots, \kappa_N)$, $\kappa_1 \geq \kappa_2 \cdots \geq \kappa_N \geq 0$, and write

$$\psi_{l,\kappa} = \prod_{j=1}^{N} z_j^{-l-(N-1)/2} \det[z_j^{k-1+\kappa_{N-k+1}}]_{j,k=1,\dots,N} \tag{3.2}$$

where $z_j = e^{2\pi i x_j/L}$. The general eigenstate (3.2) has the property that it can be factored into a product of the simple factor $\prod_{j=1}^{N} z_j^{-l}$, the ground state, and the Schur polynomial, which is a symmetric polynomial defined by

$$s_\kappa(z_1, \dots, z_N) := \frac{\det[z_j^{k-1+\kappa_{N-k+1}}]_{j,k=1,\dots,N}}{\det[z_j^{k-1}]_{j,k=1,\dots,N}} \tag{3.3}$$

This follows from the Vandermonde determinant identity

$$\det[z_j^{k-1}]_{j,k=1,\dots,N} = \prod_{1\leq j<k\leq N} (z_k - z_j) \tag{3.4}$$

and the fact that

$$\prod_{j=1}^{N} z_j^{-(N-1)/2} \prod_{1\leq j<k\leq N} (z_k - z_j) = \prod_{1\leq j<k\leq N} 2i \sin \pi(x_k - x_j)/L = C\psi_0$$

where ψ_0 is the ground state.

More generally, if we use (2.6) to factor the ground state in the operator identity (2.7) we obtain

$$-e^{\beta W/2}(H - E_0)e^{-\beta W/2} = \beta e^{\beta W} \mathcal{L} e^{-\beta W} = \sum_{j=1}^{N} \Big(\frac{\partial^2}{\partial x_j^2} - \beta \frac{\partial W}{\partial x_j} \frac{\partial}{\partial x_j} \Big). \tag{3.5}$$

For each of the choices of W (1.5), (1.6), (1.32a) and (1.34), the operator (3.5) has a complete set of polynomial eigenfunctions. We will first consider the eigenfunctions of (3.5) for W given by (1.6).

3.2. The Jack polynomials. In the case of W given by (1.6), by making the change of variables $z_j = e^{2\pi i x_j/L}$ and performing some straightforward manipulation, we find from (3.5) that

$$\Big(\frac{L}{2\pi}\Big)^2 e^{\beta W} \mathcal{L} e^{-\beta W} = D_2(2/\beta) + (-\beta(N-1)/2 + 1)E_1 \tag{3.6a}$$

where

$$D_2(\alpha) := \sum_{j=1}^N z_j^2 \frac{\partial^2}{\partial z_j^2} + \frac{2}{\alpha} \sum_{\substack{j,k=1 \\ j\neq k}}^N \frac{z_j^2}{z_j - z_k} \frac{\partial}{\partial z_j} \quad \textit{and} \quad E_1 := \sum_{j=1}^N z_j \frac{\partial}{\partial z_j} \tag{3.6b}$$

From the result of the above section we know that for $\beta = 2$ the symmetric polynomial eigenfunctions of (3.6) are the Schur polynomials. For general β, the symmetric polynomial eigenfunctions of (3.6) are known as the Jack polynomials. Although the Schur polynomials are classical, going back to Jacobi at least, the Jack polynomials have not been studied in any detail until the past decade [1-5]. To revise some of the properties of the Jack polynomials, we need some notation and definitions associated with partitions.

The modulus of the partition, denoted $|\kappa|$, is defined as $|\kappa| := \sum_{j=1}^N \kappa_j$ and the number of non-zero parts is called the length of the partition and is denoted $l(\kappa)$. A partition κ which has f_j parts equal to j is sometimes written $\kappa = (\kappa_1^{f_{\kappa_1}}(\kappa_1 - 1)^{f_{\kappa_1 - 1}} \dots 2^{f_2} 1^{f_1})$. We adopt reverse lexicographical (i.e. reverse dictionary) ordering of partitions of the same modulus, so that assuming $|\kappa| = |\mu|$ we write

$$\kappa >^R \mu \quad \textit{if} \quad \kappa_j = \mu_j \, (j = 1, \dots, p-1) \quad \textit{and} \quad \kappa_p > \mu_p \quad \text{for some} \quad p = 1, \dots, N.$$

For example, $(4) >^R (31) >^R (2^2) >^R (21^2) >^R (1^4)$. If $|\kappa| = |\mu|$ ($|\kappa| \neq |\mu|$) and $\sum_{j=1}^p \kappa_j \geq \sum_{j=1}^p \mu_j$ for all $p = 1, \dots, N$ we remove the superscript R and write $\kappa > \mu$. This is called the natural or dominance ordering (unlike reverse lexicographical ordering, it is only a partial ordering; for example (411) and (33) are incomparable using the ordering $>$).

Sometimes we refer to the diagram of a partition:

$$\kappa = \{(i,j) : 1 \leq i \leq l(\kappa), 1 \leq j \leq \kappa_i\}$$

(the points (i,j) are conventionally ordered as in the entries of a matrix and represented as dots or squares). The conjugate partition κ' is defined as a diagram by interchanging the rows and columns of the diagram of κ. Furthermore, let $n(\kappa) := \sum_{i=1}^N (i-1)\kappa_i = \sum_{i=1}^N \kappa_i'(\kappa_i' - 1)/2$.

We are now in a position to discuss some of the properties of the eigenfunctions of (3.6). A direct calculation shows

$$D_2(\alpha) m_\kappa = e(\kappa, \alpha) m_\kappa + \sum_{\nu < \kappa} a_{\nu\kappa} m_\nu \tag{3.7a}$$

where the $a_{\nu\kappa}$ are independent of $z_1, \ldots, z_N$, m_κ denotes the monomial symmetric function in the variables $z_1, \ldots, z_N$ indexed by the partition κ and

$$e(\kappa, \alpha) = 2\Big(n(\kappa') - \frac{1}{\alpha} n(\kappa)\Big) + \frac{2}{\alpha}(N-1)|\kappa|. \tag{3.7b}$$

It follows immediately that in the Taylor polynomial basis of monomial symmetric functions $\{m_\kappa : |\kappa| = k, k = 0, 1, \ldots\}$, ordered with reverse lexicographical ordering, the operator $D_2(\alpha)$ has polynomial eigenfunctions of the form

$$J_\kappa^{(\alpha)}(z_1, \ldots, z_N) := b_{\kappa\kappa} m_\kappa + \sum_{\mu < \kappa} b_{\mu\kappa} m_\mu \tag{3.8}$$

with eigenvalue $e(\kappa, \alpha)$. These polynomials are unique up to normalization. We adopt the normalization of Stanley, which is to fix $b_{1^{|\kappa|}\kappa} = |\kappa|!$.

Although the Jack polynomials form a Taylor polynomial basis we require that the complete set of eigenfunctions form a Laurent polynomial basis (positive and negative powers). This is achieved by noting that, analogous to (3.2),

$$\prod_{j=1}^{N} z_j^{-l} J_\kappa^{(\alpha)}(z_1, \ldots, z_N), \quad l > 0$$

is an eigenfunction of $D_2(\alpha)$ with eigenvalue $e(\kappa - l, \alpha)$. A complete set of eigenfunctions with respect to the space of Laurent expandable functions for the operators (3.6) is therefore given by [6]

$$\Big\{ \prod_{j=1}^{N} z_j^{-l} J_\kappa^{(\alpha)}(z_1, \ldots, z_N) \Big\}_{l \geq 0, \kappa} \tag{3.9}$$

(to prevent double counting, for $l > 0$ we require $\kappa_N = 0$).

Furthermore we have that

$$\Big\{ \prod_{1 \leq j < k \leq N} |z_k - z_j|^{\beta/2} \prod_{j=1}^{N} z_j^{-l} J_\kappa^{(2/\beta)}(z_1, \ldots, z_N) \Big\}_{l \geq 0, \kappa} \tag{3.10}$$

is an orthogonal set with respect to the inner product

$$\langle f | g \rangle_I := \prod_{l=1}^{N} \int_0^L dx_l \, f^* g, \tag{3.11a}$$

or equivalently the set (3.9) is an orthogonal set with respect to the inner product

$$\langle f | g \rangle_I' := \prod_{l=1}^{N} \int_0^L dx_l \prod_{1 \leq j < k \leq N} |z_k - z_j|^\beta f^* g, \tag{3.11b}$$

in all cases except possibly when the two members of the sets (3.9) or (3.10), labelled by (p, κ) and (q, σ) say, are such that $|\kappa| = |\sigma|$, $p = q$ and $e(\kappa) = e(\sigma)$ (e.g. $\kappa = (31^3)$ and $\sigma = 2^3$). This follows because for $\kappa \neq \sigma$ the integrand

is homogeneous of a non-zero degree and thus must integrate to zero, while for $|\kappa| = |\sigma|$, $e(\kappa) \neq e(\sigma)$ we note that the set (3.9) gives the eigenfunctions of the Hermitian operator $e^{-\beta W/2}\mathcal{L}e^{-\beta W/2}$, and it is a general property of Hermitian operators that eigenfunctions with distinct eigenvalues are orthogonal.

To establish the orthogonality in the case $|\kappa| = |\sigma|$, $e(\kappa) = e(\sigma)$ (it suffices to assume $p = p' = 0$) is a difficult task. It follows according to the following result of Macdonald [2].

Proposition 3.1. *With* $\Delta_+ := \prod_{1\le j<k\le N}(z_k - z_j)$ *and* $z_j := e^{2\pi i x_j/L}$, *define the family of operators* $\{D_N^p\}_{p=1,\dots,N}$ *depending on the parameter* $2/\beta$ *by*

$$D_N^p := \sum_{l=0}^{p}(2/\beta)^{p-l} \sum_{1\le i_1<i_2<\cdots<i_l} \frac{1}{\Delta_+}\Big(z_{i_1}\frac{\partial}{\partial z_{i_1}}\cdots z_{i_l}\frac{\partial}{\partial z_{i_l}}\Big)\Delta_+ \times \sum_{\substack{1\le i_{l+1}<\cdots<i_p\le N \\ \neq i_1\cdots\neq i_l}} \Big(z_{i_{l+1}}\frac{\partial}{\partial z_{i_{l+1}}}\cdots z_{i_p}\frac{\partial}{\partial z_{i_p}}\Big),$$

and define the corresponding generating function by

$$D_N(X;2/\beta) := \sum_{k=0}^{N} X^{N-k} D_N^k$$

We have that the operator $e^{-\beta W/2}D_N(X;2/\beta)e^{\beta W/2}$ *is Hermitian with respect to the inner product (3.11a) and the Jack polynomials* $J_\kappa^{(2/\beta)}$ *are eigenfunctions of* $D_N(X;2/\beta)$ *(and thus each* D_N^p *separately) with corresponding eigenvalue*

$$e(\kappa, 2/\beta; X) := \prod_{j=1}^{N}(X + N - j + (2/\beta)\kappa_j). \tag{3.12}$$

Since $e(\kappa, 2/\beta; X) \neq e(\sigma, 2/\beta; X)$ for $\kappa \neq \sigma$ the operators $\{D_N^p\}$, which are called the Sekiguchi-Debiard operators, resolve the degeneracy in the spectrum. The orthogonality of the set (3.10) in general then follows from the fact that $e^{-\beta W/2}D_N^p e^{\beta W/2}$ is Hermitian.

An understanding of the operator $D_N(X;2/\beta)$ is provided by the work of Bernard et al. [7], who introduced a decomposition of (3.6a) in terms of a commuting set of operators known as the Cherednik operators. Using these operators, an operator equivalent to $D_N(X;2/\beta)$ can be constructed (see e.g. [8] and Section 5 below).

A fundamental property of the Jack polynomials is the so called Cauchy formula

$$\prod_{j,k=1}^{N}(1 - x_j y_k)^{-1/\alpha} = \sum_{\kappa} j_\kappa^{-1} J_\kappa^{(\alpha)}(x_1,\dots,x_N)J_\kappa^{(\alpha)}(y_1,\dots,y_N). \tag{3.13a}$$

It is straightforward to show [2] that this identity is equivalent to the statement that

$$\langle J_\kappa^{(\alpha)}, J_\sigma^{(\alpha)}\rangle_p = j_\kappa \delta_{\kappa,\sigma} \tag{3.13b}$$

where j_κ is a normalization and $\langle\ ,\ \rangle_p$ is the power sum inner product

$$\langle p_\kappa, p_\sigma\rangle_p = \alpha^{\sum_{j=1}^{\kappa_1} f_j} \prod_{k=1}^{\kappa_1} k^{f_k} f_k! \tag{3.14a}$$

with

$$p_\kappa := p_{\kappa_1} p_{\kappa_2} \dots p_{\kappa_N} = p_1^{f_1} p_2^{f_2} \dots p_N^{f_N} \quad \textit{where} \quad p_j := \sum_{k=1}^{N} z_k^j \tag{3.14b}$$

The coefficients of the Jack polynomials are independent of N (this can be shown directly from the eigenoperator $D_2(\alpha)$ in (3.6)) so they can be given a unique decomposition in terms of power sums and the value of the inner product is thus unambiguous. Macdonald [2] has proved the validity of (3.13) (the proof involves showing that the family of operators D_N^p are Hermitian with respect to the inner product (3.14a)).

An alternative formulation of (3.13a) is possible. Define the operator ω_α by

$$\omega_\alpha p_\kappa = \alpha^{l(\kappa)} p_\kappa \tag{3.15}$$

Macdonald [2] has proved that

$$\omega_{-\alpha} J_{\kappa'}^{(\alpha)} = (-1)^{|\kappa|} \alpha^{-|\kappa|} J_\kappa^{(1/\alpha)} \tag{3.16}$$

Furthermore, it is straightforward to show that

$$\omega_{-\alpha} \prod_{j,k=1}^{N} (1 - x_j y_k)^{-1/\alpha} = \prod_{j,k=1}^{N} (1 - x_j y_k) \tag{3.17}$$

where for definiteness we suppose $\omega_{-\alpha}$ acts on the variables $\{x_j\}$. Hence, applying $\omega_{-\alpha}$ to the Cauchy formula (3.13a) gives

$$\prod_{j,k=1}^{N} (1 - x_j y_k) = \sum_\kappa j_\kappa^{-1} (-1)^\kappa \alpha^{-|\kappa|} J_{\kappa'}^{(1/\alpha)}(x_1, \dots, x_N) J_\kappa^{(\alpha)}(y_1, \dots, y_N) \tag{3.18}$$

From the Cauchy formula (3.13a) a useful formula for $p_1^n := (\sum_{j=1}^N x_j)^n$ can be derived. Thus we consider the coefficient of $y_1 \dots y_n$, $n \le N$, on both sides of (3.13a). On the l.h.s. the coefficient is

$$\alpha^{-n}(x_1 + \dots + x_N)^n \tag{3.19}$$

On the r.h.s., recalling that $J_\kappa^{(\alpha)}$ is homogeneous of order $|\kappa|$, the only terms involving $y_1 \dots y_n$ must have $|\kappa| = n$. Furthermore, the normalization of $J_\kappa^{(\alpha)}(y_1, \dots, y_n)$, $|\kappa| = n$, is such that the coefficient of $y_1 \dots y_n$ in $J_\kappa^{(\alpha)}(y_1, \dots, y_n)$ is $|\kappa|!$, so that the total coefficient is

$$\sum_{|\kappa|=n} |\kappa|! j_\kappa^{-1} J_\kappa^{(\alpha)}(x_1, \dots, x_n) \tag{3.20}$$

Equating (3.19) and (3.20) gives

$$(x_1 + \cdots + x_N)^n = \sum_{|\kappa|=n} \alpha^{|\kappa|} |\kappa|! j_\kappa^{-1} J_\kappa^{(\alpha)}(x_1, \ldots, x_N) \tag{3.21}$$

It is convenient to define a Jack polynomial $C_\kappa^{(\alpha)}$ with normalization absorbing the prefactors of $J_\kappa^{(\alpha)}$ in (3.21):

$$C_\kappa^{(\alpha)}(x_1, \ldots, x_N) = \alpha^{|\kappa|} |\kappa|! j_\kappa^{-1} J_\kappa^{(\alpha)}(x_1, \ldots, x_N). \tag{3.22}$$

From (3.21) we then have

$$(x_1 + \cdots + x_N)^n = \sum_{|\kappa|=n} C_\kappa^{(\alpha)}(x_1, \ldots, x_N) \tag{3.23}$$

which implies the useful identity

$$\exp(x_1 + \cdots + x_N) = \sum_{n=0}^{\infty} \sum_{|\kappa|=n} C_\kappa(x_1, \ldots, x_N)/n!. \tag{3.24}$$

We will conclude this section by stating two formulas for quantities associated with Jack polynomials. These formulas are for the coefficient $b_{\kappa\kappa}$ in (3.8) and the normalization j_κ in (3.13). They read [1]

$$b_{\kappa\kappa} = \prod_{(i,j)\in\kappa} h_*^\kappa(i,j) \quad \textit{and} \quad j_\kappa = \prod_{(i,j)\in\kappa} h_*^\kappa(i,j) h_\kappa^*(i,j) \tag{3.25}$$

where

$$h_*^\kappa(i,j) = \kappa_j' - i + 1 + \alpha(\kappa_i - j) \quad \textit{and} \quad h_\kappa^*(i,j) = \kappa_j' - i + \alpha(\kappa_i - j + 1).$$

An equivalent, more convenient form for the above products follows from the work of Kadell [5]:

$$\prod_{(i,j)\in\kappa} h_*^\kappa(i,j) = \frac{J_\kappa^{(\alpha)}(1^n)}{f_n^{1/\alpha}(\kappa)} \tag{3.26a}$$

where

$$f_n^\lambda(\kappa) := \prod_{1\le i<j\le n} \frac{((j-i)\lambda + \kappa_i - \kappa_j)_\lambda}{((j-i)\lambda)_\lambda}$$

and

$$\prod_{(i,j)\in\kappa} h_\kappa^*(i,j) = \alpha^{|\kappa|} \frac{[(n-1)/\alpha + 1]_\kappa^{(\alpha)}}{\bar{f}_n^{1/\alpha}(\kappa)} \tag{3.26b}$$

where

$$\bar{f}_n^\lambda(\kappa) := \prod_{1\le i<j\le n} \frac{(1 - \lambda + (j-i)\lambda + \kappa_i - \kappa_j)_\lambda}{(1 - \lambda + (j-i)\lambda)_\lambda},$$

(the quantity $[(n-1)/\alpha + 1]_\kappa^{(\alpha)}$ is defined by (3.32b) below and $(x)_\lambda := \Gamma(x + \lambda)/\Gamma(x)$).

3.3. Generalized Jacobi polynomials. Further development in the theory of Jack polynomials can be made by considering eigenfunctions associated with the operator (3.5) in the case W given by (1.35) and with x_j replaced by ϕ_j. Changing variables $\sin^2 \phi_j = x_j$ we find that

$$-\frac{1}{4} e^{\beta W^{(J)}/2} (H^{(J)} - E_0) e^{-\beta W^{(J)}/2} := \tilde{H}^{(J)}$$
$$= \sum_{j=1}^{N} \Big(x_j(1-x_j) \frac{\partial^2}{\partial x_j^2} + [\tilde{a} + 1 - x_j(\tilde{a} + \tilde{b} + 2)] \frac{\partial}{\partial x_j} + \beta \sum_{\substack{k=1 \\ k \neq j}}^{N} \frac{x_j(1-x_j)}{x_j - x_k} \frac{\partial}{\partial x_j} \Big) \tag{3.27}$$

where $\tilde{a} := (\beta a' - 1)/2$, $\tilde{b} := (\beta b' - 1)/2$. In the $N = 1$ case the unique polynomial eigenfunctions of this operator are the Jacobi polynomials $P_n^{(\tilde{b},\tilde{a})}(2x - 1)$. For general N, by direct substitution, it is easy to see that for each partition κ there is a unique symmetric polynomial eigenfunction with heighest weight m_κ. In fact all other monomials in the expansion of the eigenfunction are of the form m_σ with $\sigma < \kappa$ (dominance ordering) or $|\sigma| < |\kappa|$. For the partitions $\kappa = (p^N)$ $(p = 1, 2, \dots)$ two different types of formulas for the eigenfunctions, termed generalized Jacobi polynomials and to be denoted

$$G_\kappa^{(\tilde{a},\tilde{b})}(x_1, \dots, x_N; 2/\beta),$$

have been obtained by Kaneko [4].

To motivate the first formula, which is an integral representation, consider the problem of evaluating the density (1.36) (for convenience in a system of $N + 1$ particles), with the Boltzmann factor given by the eigenvalue p.d.f. (1.33) for the Jacobi ensemble (with N replaced by $N + 1$). The density is given by

$$\rho_1(y) = \frac{N+1}{Z_{N+1}} y^{\beta a/2} (1-y)^{\beta b/2} \prod_{l=1}^{N} \int_0^1 dy_l |y - y_l|^\beta \, y_l^{\beta a/2} (1-y_l)^{\beta b/2} \prod_{1 \le j < k \le N} |y_k - y_j|^\beta \tag{3.28a}$$

where

$$Z_{N+1} := \prod_{l=1}^{N+1} \int_0^1 dy_l \, y_l^{\beta a/2} (1-y_l)^{\beta b/2} \prod_{1 \le j < k \le N+1} |y_k - y_j|^\beta. \tag{3.28b}$$

Suppose we now introduce the integrals

$$S_{n,m}(\lambda_1, \lambda_2, \lambda; t_1, \dots, t_m) := \int_{[0,1]^n} dx_1 \dots dx_n \prod_{j=1}^{n} \prod_{k=1}^{m} (x_j - t_k) D_{\lambda_1,\lambda_2,\lambda}(x_1, \dots, x_n) \tag{3.29}$$

where

$$D_{\lambda_1,\lambda_2,\lambda}(x_1,\dots,x_n) = \prod_{j=1}^{n} x_j^{\lambda_1}(1-x_j)^{\lambda_2} \prod_{1\le j<k\le n} |x_k - x_j|^{2\lambda} \tag{3.30}$$

Then for β even $\rho_{(1)}(y)$ can be written in terms of these integrals according to

$$\rho_1(y) = \frac{N+1}{Z_{N+1}} y^{\beta a/2}(1-y)^{\beta b/2} S_{N,\beta}(\beta a/2, \beta b/2, \beta/2; t_1, \dots, t_\beta)|_{t_1=\cdots=t_\beta=y}. \tag{3.31}$$

Kaneko [4] has shown that the integrals (3.29) are eigenfunctions of the operator (3.27) and are thus examples of generalized Jacobi polynomials.

To derive this result, we first require some equations satisfied by integrals related to $S_{n,m}$.

Proposition 3.2. *Let*

$$S_{n,m}[f] := \int_{[0,1]^n} dx_1 \dots dx_n \, f \prod_{j=1}^{n}\prod_{k=1}^{m}(x_j - t_k) D_{\lambda_1,\lambda_2,\lambda}(x_1,\dots,x_n),$$

where f may be an operator acting on all terms to the right. For $\lambda_1, \lambda_2 > 0$ the following equations hold:

$$0 = \lambda_1 S_{n,m}\Big[\sum_{j=1}^{n}\frac{1}{x_j}\Big] - \lambda_2 S_{n,m}\Big[\sum_{j=1}^{n}\frac{1}{1-x_j}\Big] + S_{n,m}\Big[\sum_{j=1}^{n}\sum_{k=1}^{m}\frac{1}{x_j - t_k}\Big]$$

$$0 = n(1+\lambda_1+\lambda_2+m+(n-1)\lambda)S_{n,m}[1] - \lambda_2 S_{n,m}\Big[\sum_{j=1}^{n}\frac{1}{1-x_j}\Big] + S_{n,m}\Big[\sum_{j=1}^{n}\sum_{k=1}^{m}\frac{t_k}{x_j - t_k}\Big]$$

and

$$0 = -2\lambda S_{n,m}\Big[\sum_{1\le j<k\le n}\frac{1}{(x_j - t_p)(x_k - t_p)}\Big] + \frac{\lambda_1}{t_p} S_{n,m}\Big[\sum_{j=1}^{n}\Big(\frac{1}{x_j - t_p} - \frac{1}{x_j}\Big)\Big]$$

$$-\frac{\lambda_2}{1-t_p} S_{n,m}\Big[\sum_{j=1}^{n}\Big(\frac{1}{x_j - t_p} + \frac{1}{1-x_j}\Big)\Big] + \sum_{\substack{l=1\\ l\ne p}}^{m}\frac{1}{t_p - t_l} S_{n,m}\Big[\sum_{j=1}^{n}\Big(\frac{1}{x_j - t_p} - \frac{1}{x_j - t_l}\Big)\Big]$$

for each $p = 1, \dots, m$.

Proof. These equations are derived from the three equations

$$0 = S_{n,m}\Big[\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\Big],\quad 0 = S_{n,m}\Big[\sum_{j=1}^{n}\frac{\partial}{\partial x_j}x_j\Big],\quad 0 = S_{n,m}\Big[\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\frac{1}{x_j - t_p}\Big]$$

($p = 1, \dots, m$) respectively, which follow from the fundamental theorem of calculus (the integrand vanishes at both endpoints). The explicit form of the r.h.s.'s follow by explicitly calculating the partial derivatives, and using the symmetry properties of the integrand to further simplify the resulting expressions. □

The main result regarding these integrals can now be derived.

Proposition 3.3. *$S_{n,m}$ satisfies the partial differential equations*

$$t_p(1-t_p)\frac{\partial^2 F}{\partial t_p^2}+[c-\frac{1}{\alpha}(m-1)-(a+b+1-\frac{1}{\alpha}(m-1))t_p]\frac{\partial F}{\partial t_p}-abF$$

$$+\frac{1}{\alpha}\sum_{\substack{j=1\\ j\neq p}}^{m}\frac{1}{t_p-t_j}\left(t_p(1-t_p)\frac{\partial F}{\partial t_p}-t_j(1-t_j)\frac{\partial F}{\partial t_j}\right)=0, \qquad p=1,\dots,m,$$

with

$$\alpha=\lambda, \quad a=-n, \quad b=\frac{1}{\lambda}(\lambda_1+\lambda_2+m+1)+n-1, \quad c=\frac{1}{\lambda}(\lambda_1+m)$$

Proof. Regarding

$$\lambda_1 S_{n,m}\Big[\sum_{j=1}^{n}\frac{1}{x_j}\Big] \quad \textit{and} \quad \lambda_2 S_{n,m}\Big[\sum_{j=1}^{n}\frac{1}{1-x_j}\Big]$$

as unknowns in the first two equations of Proposition 3.2 and solving for them gives

$$\lambda_1 S_{n,m}\Big[\sum_{j=1}^{n}\frac{1}{x_j}\Big]=n(1+\lambda_1+\lambda_2+m+(n-1)\lambda)S_{n,m}[1]-S_{n,m}\Big[\sum_{j=1}^{n}\sum_{k=1}^{m}\frac{1-t_k}{x_j-t_k}\Big]$$

and

$$\lambda_2 S_{n,m}\Big[\sum_{j=1}^{n}\frac{1}{1-x_j}\Big]=n(1+\lambda_1+\lambda_2+m+(n-1)\lambda)S_{n,m}[1]+S_{n,m}\Big[\sum_{j=1}^{n}\sum_{k=1}^{m}\frac{t_k}{x_j-t_k}\Big].$$

Substituting these equations into the third equation of Proposition 3.2 gives

$$0=-2\lambda S_{n,m}\Big[\sum_{1\le j<k\le n}\frac{1}{(x_j-t_p)(x_k-t_p)}\Big]+\left(\frac{\lambda_1}{t_p}-\frac{\lambda_2}{1-t_p}\right)S_{n,m}\Big[\sum_{j=1}^{n}\frac{1}{x_j-t_p}\Big]$$

$$-\frac{1}{t_p}\Big(n(1+\lambda_1+\lambda_2+m+(n-1)\lambda)S_{n,m}[1]-S_{n,m}\Big[\sum_{j=1}^{n}\sum_{k=1}^{m}\frac{1-t_k}{x_j-t_k}\Big]\Big)$$

$$-\frac{1}{1-t_p}\Big(n(1+\lambda_1+\lambda_2+m+(n-1)\lambda)S_{n,m}[1]+S_{n,m}\Big[\sum_{j=1}^{n}\sum_{k=1}^{m}\frac{t_k}{x_j-t_k}\Big]\Big)$$

$$+S_{n,m}\Big[\sum_{\substack{l=1\\ l\neq p}}^{m}\frac{1}{t_p-t_l}\sum_{j=1}^{n}\frac{1}{x_j-t_p}-\frac{1}{x_j-t_l}\Big]$$

But from the definition (3.29) of $S_{n,m}$ we see

$$\frac{\partial S_{n,m}}{\partial t_p}=-S_{n,m}\Big[\sum_{j=1}^{n}\frac{1}{x_j-t_p}\Big]$$

and

$$\frac{\partial^2 S_{n,m}}{\partial t_p^2} = 2S_{n,m}\Big[\sum_{1\le j<k\le n}\frac{1}{(x_j-t_p)(x_k-t_p)}\Big]$$

These equations allow all terms involving $x_1,\dots,x_n$ to be eliminated. The partial differential equations of the proposition result (to obtain the very last term the manipulation $t_j(1-t_j)=t_p(1-t_p)+(t_j-t_p)$ $\times(1-t_j-t_p)$ is required). □

Summing over p in the equation of Proposition 3.3 gives the eigenvalue equation for the eigenoperator (3.27), with

$$N=m,\quad \beta=\frac{\lambda}{2},\quad \tilde{a}+1=\frac{1}{\lambda}(\lambda_1+1)-1\quad \tilde{b}+1=\frac{1}{\lambda}(\lambda_2+1)-1$$

and eigenvalue $-nm[(\lambda_1+\lambda_2+m+1)/\lambda+n-1]$. Hence $S_{n,m}(\lambda_1,\lambda_2,\lambda;t_1,\dots,t_m)$ is proportional to

$$G_{(n^m)}^{((\lambda_1+1)/\lambda-1,(\lambda_2+1)/\lambda-1)}(t_1,\dots,t_m;\lambda)$$

On the other hand, the summed up form of the equations in Proposition 3.3 permit a solution as a series of Jack polynomials. In fact the polynomial solution of the summed up equations is given by the generalized hypergeometric function

$${}_2F_1^{(\alpha)}(a,b;c;t_1,\dots,t_m)=\sum_{d=0}^{\infty}\frac{1}{d!}\sum_{|\kappa|=d}\frac{[a]_\kappa^{(\alpha)}[b]_\kappa^{(\alpha)}}{[c]_\kappa^{(\alpha)}}C_\kappa^{(\alpha)}(t_1,\dots,t_m),\qquad (3.32a)$$

with $a=-n$, where the generalized factorial function is defined by

$$\begin{aligned}[u]_\kappa^{(\alpha)} &:= \prod_{j=1}^{l(\kappa)}(u-\tfrac{1}{\alpha}(j-1))_{\kappa_j}\qquad \textit{with}\quad (x)_n:=x(x+1)\dots(x+n-1)\\ &= \prod_{j=1}^{m}\frac{\Gamma(u-\frac{1}{\alpha}(j-1)+\kappa_j)}{\Gamma(u-\frac{1}{\alpha}(j-1))}\end{aligned}\qquad (3.32b)$$

and $C_\kappa^{(\alpha)}(t_1,\dots,t_m)$ is the Jack polynomial (3.22).

To show this fact, it is convenient to introduce the operators

$$\begin{aligned}E_k &= \sum_{i=1}^{m}x_i^k\frac{\partial}{\partial x_i}\\ D_k(\alpha) &= \sum_{i=1}^{m}x_i^k\frac{\partial^2}{\partial x_i^2}+\frac{2}{\alpha}\sum_{i\ne j}\frac{x_i^k}{x_i-x_j}\frac{\partial}{\partial x_i}\end{aligned}$$

The summed up equations can then be written as

$$\Big(D_2(\alpha)-D_1(\alpha)+(c-\frac{1}{\alpha}(m-1))E_0-(a+b+1-\frac{1}{\alpha}(m-1))E_1\Big)F=mabF\qquad (3.33)$$

It is therefore necessary to compute the action of the operators $E_0, E_1, D_1(\alpha), D_2(\alpha)$ on $C_\kappa^{(\alpha)}$. From Section 3.2 we know that

$$E_1 C_\kappa^{(\alpha)} = |\kappa| C_\kappa^{(\alpha)} \tag{3.34a}$$

and

$$D_2(\alpha) C_\kappa^{(\alpha)} = \sum_{i=1}^{m} \Big(\kappa_i(\kappa_i - 1)/2 - \frac{1}{\alpha}(i-1)\kappa_i \Big) C_\kappa^{(\alpha)} \tag{3.34b}$$

To compute the action of E_0, it is necessary to introduce the generalized binomial coefficients $\binom{\kappa}{\sigma}$ defined by the expansion

$$\frac{C_\kappa^{(\alpha)}(1+t_1, \ldots, 1+t_m)}{C_\kappa^{(\alpha)}(1^m)} = \sum_{s=0}^{|\kappa|} \sum_{|\sigma|=s} \binom{\kappa}{\sigma} \frac{C_\sigma^{(\alpha)}(t_1, \ldots, t_m)}{C_\sigma^{(\alpha)}(1^m)} \tag{3.35}$$

where $C_\kappa^{(\alpha)}(1^m) := C_\kappa^{(\alpha)}(t_1, \ldots, t_m)\Big|_{t_1 = \cdots = t_m = 1}$. Now

$$\begin{aligned} E_0 \frac{C_\kappa^{(\alpha)}(t_1, \ldots, t_m)}{C_\kappa^{(\alpha)}(1^m)} &= \lim_{\epsilon \to 0} \frac{C_\kappa^{(\alpha)}(\epsilon + t_1, \ldots, \epsilon + t_m) - C_\kappa^{(\alpha)}(t_1, \ldots, t_m)}{\epsilon C_\kappa^{(\alpha)}(1^m)} \\ &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \sum_{s=0}^{|\kappa|} \epsilon^{|\kappa|-s} \sum_{|\sigma|=s} \binom{\kappa}{\sigma} \frac{C_\sigma^{(\alpha)}(t_1, \ldots, t_m)}{C_\sigma^{(\alpha)}(1^m)} \\ &= \sum_i \binom{\kappa}{\kappa_{(i)}} \frac{C_{\kappa_{(i)}}^{(\alpha)}(t_1, \ldots, t_m)}{C_{\kappa_{(i)}}^{(\alpha)}(1^m)} \end{aligned} \tag{3.36a}$$

where $\kappa_{(i)} := (\kappa_1, \ldots, \kappa_{i-1}, \kappa_i - 1, \kappa_{i+1}, \ldots, \kappa_N)$ and it is assumed $\kappa_{(i)}$ forms a partition. For the action of D_1 we observe that $D_1 = \frac{1}{2}[E_0, D_2]$, which gives

$$D_1 \frac{C_\kappa^{(\alpha)}(t_1, \ldots, t_m)}{C_\kappa^{(\alpha)}(1^m)} = \sum_i \binom{\kappa}{\kappa_{(i)}} (\kappa_i - 1 + \frac{1}{\alpha}(m-i)) \frac{C_{\kappa_{(i)}}^{(\alpha)}(t_1, \ldots, t_m)}{C_{\kappa_{(i)}}^{(\alpha)}(1^m)} \tag{3.36b}$$

We can now substitute the series (3.32) in the summed up equations of Proposition 3.3 and equate coefficients of $C_\kappa^{(\alpha)}$ by using (3.34) to (3.36). We find that the series satisfies the equation provided

$$\begin{aligned} &\sum_{i=1}^{m}{}' \binom{\kappa^{(i)}}{\kappa} \Big(a + \kappa_i - \alpha(i-1)\Big)\Big(b + \kappa_i - \alpha(i-1)\Big) C_{\kappa^{(i)}}^{(\alpha)}(1^m) \\ &\quad = (|\kappa| + 1)\Big(mab + 2\alpha\rho_\kappa + (a+b)|\kappa| + \alpha(m+1)|\kappa|\Big) C_\kappa^{(\alpha)}(1^m), \end{aligned} \tag{3.37}$$

where

$$\kappa^{(i)} := (\kappa_1, \ldots, \kappa_{i-1}, \kappa_i + 1, \kappa_{i+1} \ldots, \kappa_m), \quad \rho_\kappa := \sum_{j=1}^{m} \Big(\kappa_j^2 - \frac{2}{\alpha} j \kappa_j \Big)$$

To verify the equation (3.37), summation formulas for

$$\sum_{i=1}^{m} \Big(\kappa_i - \alpha(i-1)\Big)^s \binom{\kappa_{(i)}}{\kappa} C_{\kappa_{(i)}}(1^m), \qquad s = 0, 1 \text{ and } 2$$

are required. These summation formulas can be derived from some generalizations of the identity (3.24), which give the explicit Jack polynomial expansions of

$$p_1^{s_1} p_2^{s_2} \exp(t_1 + \cdots + t_m), \qquad s_1, s_2 = 0, 1.$$

The result is that

$$\begin{aligned} &\sum_{i=1}^{m} \Big(\kappa_i - \alpha(i-1)\Big)^s \binom{\kappa_{(i)}}{\kappa} C_{\kappa_{(i)}}(1^m) \\ &= \begin{cases} m(|\kappa|+1)C_\kappa(1^m) & \text{if } s = 0; \\ |\kappa|(|\kappa|+1)C_\kappa(1^m) & \text{if } s = 1; \\ 2\alpha(|\kappa|+1)(\rho_\kappa + \frac{1}{2}(m+1)|\kappa|)C_\kappa(1^m) & \text{if } s = 2. \end{cases} \end{aligned} \tag{3.38}$$

The equation (3.37) can be immediately verified by using the summations (3.38). In the case $a = -n$, the series (3.32) is a polynomial with leading weight monomial $m_{(n^m)}$. Hence for a suitable choice of b and c it is proportional to $S_{n,m}$ and thus, according to (3.31), provides an expression for the particle density. Explicitly, we have

$$\begin{aligned} \rho_{(1)}(y) &= (N+1)\frac{S_N(\beta a/2 + \beta N, \beta b/2, \beta/2)}{Z_{N+1}} y^{\beta a/2}(1-y)^{\beta b/2} \\ &\quad \times {}_2F_1^{(\beta/2)}(-N, a+b+1+\frac{2}{\beta}+N; a+2; t_1, \ldots, t_\beta)\Big|_{t_1=\ldots t_\beta=1} \end{aligned} \tag{3.39}$$

where

$$S_N(\lambda_1, \lambda_2, \lambda) := \prod_{l=1}^{N} \int_0^1 dt_l \, D_{\lambda_1,\lambda_2,\lambda}(t_1, \ldots, t_N) \tag{3.40}$$

We recall that the integral (3.40) has been evaluated by Selberg (see e.g. [7]) as

$$S_N(\lambda_1, \lambda_2, \lambda) = \prod_{j=0}^{N-1} \frac{\Gamma(\lambda_1 + 1 + j\lambda)\Gamma(\lambda_2 + 1 + j\lambda)\Gamma(1 + (j+1)\lambda)}{\Gamma(\lambda_1 + \lambda_2 + 2 + (N+j-1)\lambda)\Gamma(1+\lambda)}. \tag{3.41}$$

3.4. An integration formula. By comparing the integral and series polynomial solutions of the summed up p.d.e.'s in Proposition 3.3, a fundamental integration formula for the Jack polynomials can be deduced, which is of application in the

calculation of correlation functions. The comparison will be done by using the Cauchy expansion (3.18). To use this expansion, we consider

$$\tilde{S}_{n,m}(\lambda_1,\lambda_2,\lambda;t_1,\dots,t_m) := (-1)^{nm\mu}\prod_{j=1}^{m} t_j^{n\mu} S_{n,m}(\lambda_1,\lambda_2,\lambda;1/t_1,\dots,1/t_m)$$

$$= \int_{[0,1]^n} dx_1\dots dx_n \prod_{j=1}^{n}\prod_{k=1}^{m}(1-x_j t_k)^{\mu} D_{\lambda_1,\lambda_2,\lambda}(x_1,\dots,x_n) \tag{3.42}$$

Now, we can check by direct substitution that in general by writing the solution of the p.d.e.'s in Proposition 3.3 as $F = \prod_{j=1}^{m} t_j^{-a}\tilde{F}(1/t_1,\dots,1/t_m)$, $\tilde{F}(t_1,\dots,t_m)$ satisfies the same p.d.e.'s with $b \mapsto a-c+1+(m-1)/\alpha$, $c \mapsto a-b+1+(m-1)/\alpha$. Hence $\tilde{S}_{n,m}$ satisfies the p.d.e.'s of Proposition 3.3 with

$$\alpha = \lambda,\ a = -n,\ b = -n - \frac{1}{\lambda}(\lambda_1+1)+1,\ c = -\frac{1}{\lambda}(\lambda_1+\lambda_2+2) - 2n + 2, \tag{3.43}$$

and so

$$\frac{\tilde{S}_{n,m}(\lambda_1,\lambda_2,\lambda;t_1,\dots,t_m)}{S_n(\lambda_1,\lambda_2,\lambda)}$$
$$= {}_2F_1^{(\lambda)}(-n,-n-\frac{1}{\lambda}(\lambda_1+1)+1;-\frac{1}{\lambda}(\lambda_1+\lambda_2+2)-2n+2;t_1,\dots,t_m) \tag{3.44}$$

From (3.44) we can derive the sought integration formula. Thus by using (3.18) with $\alpha = \lambda$, $\{y_j\} = \{t_j\}$ in the definition (3.42) of $\tilde{S}_{n,m}$ and equating coefficients of $J_\kappa^{(\lambda)}(t_1,\dots,t_n)$ we obtain

$$\int_{[0,1]^n} dx_1\dots dx_n\, J_{\kappa'}^{(1/\lambda)}(x_1,\dots,x_n) D_{\lambda_1,\lambda_2,\lambda}(x_1,\dots,x_n)$$
$$= (-1)^{|\kappa|} S_n(\lambda_1,\lambda_2,\lambda)\frac{[-n]_\kappa^{(\lambda)}[-n-\frac{1}{\lambda}(\lambda_1)+1]_\kappa^{(\lambda)}}{[-2(n-1)-\frac{1}{\lambda}(\lambda_1+\lambda_2+2)]_\kappa^{(\lambda)}} \tag{3.45}$$

In general $[u]_{\kappa'}^{(\alpha)} = (-1)^{|\kappa|}\alpha^{-|\kappa|}[-\alpha u]_\kappa^{(1/\alpha)}$, so by replacing κ' by κ this can be rewritten as

$$\int_{[0,1]^n} dx_1\dots dx_n\, J_\kappa^{(1/\lambda)}(x_1,\dots,x_n) D_{\lambda_1,\lambda_2,\lambda}(x_1,\dots,x_n)$$
$$= S_n(\lambda_1,\lambda_2,\lambda)(1/\lambda)^{|\kappa|}\frac{[\lambda n]_\kappa^{(1/\lambda)}[\lambda(n-1)+\lambda_1+1]_\kappa^{(1/\lambda)}}{[2\lambda(n-1)+\lambda_1+\lambda_2+2]_\kappa^{(1/\lambda)}} \tag{3.46}$$

Note that by changing variables $x_j \mapsto 1 - x_j/L$ in (3.46), setting $\lambda_1 = L$, then taking the $L \to \infty$ limit we obtain

$$J_\kappa^{(1/\lambda)}(1^n) = \lambda^{-|\kappa|}[\lambda n]_\kappa^{(1/\lambda)} \tag{3.47}$$

Thus (3.46) can be written as

$$\int_{[0,1]^n} dx_1 \dots dx_n \frac{C_\kappa^{(1/\lambda)}(x_1,\dots,x_n)}{C_\kappa^{(1/\lambda)}(1^n)} D_{\lambda_1,\lambda_2,\lambda}(x_1,\dots,x_n)$$

$$= S_n(\lambda_1,\lambda_2,\lambda)\frac{[\lambda(n-1)+\lambda_1+1]_\kappa^{(1/\lambda)}}{[2\lambda(n-1)+\lambda_1+\lambda_2+2]_\kappa^{(1/\lambda)}} \tag{3.48}$$

The integration formula (3.48) implies an integral representation for ${}_2F_1^{(1/\lambda)}$, distinct from that implied by (3.44). By multiplying both sides by

$$[a']_\kappa^{(1/\lambda)} C_\kappa^{(1/\lambda)}(t_1,\dots,t_n)/|\kappa|!$$

and summing over $|\kappa|$ we obtain the formula [18]

$$\begin{aligned} &{}_2F_1^{(1/\lambda)}(a',\lambda(n-1)+\lambda_1+1;2\lambda(n-1)+\lambda_1+\lambda_2+2;t_1,\dots,t_n) \\ &\quad = \frac{1}{S_n(\lambda_1,\lambda_2,\lambda)}\int_{[0,1]^n} dx_1\dots dx_n \\ &\quad \times {}_1\mathcal{F}_0^{(1/\lambda)}(a';\lambda t_1/2,\dots,\lambda t_n/2;x_1,\dots,x_n) D_{\lambda_1,\lambda_2,\lambda}(x_1,\dots,x_n), \end{aligned} \tag{3.49}$$

where

$${}_1\mathcal{F}_0^{(\alpha)}(a';r_1,\dots,r_n;s_1,\dots,s_n) := \sum_{d=0}^{\infty}\frac{1}{d!}\sum_{|\kappa|=d}[a']_\kappa^{(\alpha)}\frac{C_\kappa^{(\alpha)}(r_1,\dots,r_n)C_\kappa^{(\alpha)}(s_1,\dots,s_n)}{C_\kappa^{(\alpha)}(1^n)}. \tag{3.50}$$

The function ${}_1\mathcal{F}_0^{(1/\lambda)}$ can be written explicitly in terms of an elementary function in the case $t_1 = \dots = t_n = t$. From the definition (3.50), and the fact that $C_\kappa^{(\alpha)}$ is homogeneous of order $|\kappa|$ we have

$${}_1\mathcal{F}_0^{(\alpha)}(a';r_1,\dots,r_n;s_1,\dots,s_n)\Big|_{r_1=\dots=r_n=r} = {}_1F_0^{(\alpha)}(a';rs_1,\dots,rs_n). \tag{3.51}$$

On the other hand, by the Cauchy identity (3.18)

$$\prod_{j=1}^n (1-rs_j)^p = \sum_\kappa \frac{(-1)^{|\kappa|}}{|\kappa|!} r^p J_{\kappa'}^{(1/\alpha)}(1^p) C_\kappa^{(\alpha)}(s_1,\dots,s_n) \tag{3.52a}$$

for any integer $p \in Z_{\geq 0}$. Using (3.47) and the formula below (3.45) we can identify the r.h.s. of (3.52) as

$${}_1F_0^{(\alpha)}(-p;rs_1,\dots,rs_n), \tag{3.52b}$$

so we have the explicit formula

$${}_1\mathcal{F}_0^{(\alpha)}(a';r_1,\dots,r_n;s_1,\dots,s_n)\Big|_{r_1=\dots=r_n=r} = \prod_{j=1}^n (1-rs_j)^{-a'} \tag{3.53}$$

Substituting this in (3.49) we obtain [9]

$$
{}_2F_1^{(1/\lambda)}(a', \lambda(n-1)+\lambda_1+1; 2\lambda(n-1)+\lambda_1+\lambda_2+2; t_1, \dots, t_n)\Big|_{t_j=t}
$$

$$
= \frac{1}{S_n(\lambda_1, \lambda_2, \lambda)} \int_{[0,1]^n} dx_1 \dots dx_n \prod_{j=1}^{n} (1-tx_j)^{-a'} D_{\lambda_1, \lambda_2, \lambda}(x_1, \dots, x_n). \tag{3.54}
$$

An immediate application of (3.54) is that it provides a β-dimensional integral formula for ${}_2F_1^{(\beta/2)}$ in (3.39), and thus for $\rho^{(1)}(y)$. To obtain the generalized hypergeometric function in (3.39) we must choose

$$
\lambda = 2/\beta, \quad a' = -N, \lambda_1 = \frac{4}{\beta} + a + b + N - 2, \quad \lambda_2 = -2 - b - n. \tag{3.55}
$$

But we see that λ_2 is then negative so that the integral (3.54) is not defined. However we can readily analytically continue the integral (3.54) so that it is valid for λ_2 negative by following the procedure detailed in [10]. Thus we deform the contours $[0,1]^m$ to the contours $\mathcal{C}^m$, where $\mathcal{C}$ is any simple closed contour which starts at the origin and encircles the point $x = 1$ (this is first done under the assumption that λ_2 is not an integer, and λ is an integer; it is extended to all λ_2 by analytic continuation and to all λ by noting that the r.h.s. is analytic in λ when it is defined, while the l.h.s. is a rational function of λ in the case of interest ($a' = -N$). Thus we can replace ${}_2F_1^{(\beta/2)}$ in (3.39) by the integral

$$
\frac{1}{C} \int_{\mathcal{C}^\beta} dx_1 \dots dx_\beta \prod_{l=1}^{\beta} (1-yx_l)^N x_l^{4/\beta+a+b+N-2} (1-x_l)^{-2-b-N} \prod_{1 \le j < k \le \beta} |x_k - x_j|^{4/\beta} \tag{3.56}
$$

where C is chosen so that at $y = 0$, the expression is unity.

3.5. A trigonometric version of the integration formula. In general, integrals of the form (3.46) can be transformed into trigonometric integrals according to the following result [7], which follows by term-by-term integration.

Proposition 3.4. *Let $f(t_1, \dots, t_N; \{p\})$ be a Laurent polynomial in $t_1, \dots, t_N$, with $\{p\}$ as parameters. For $Re(\epsilon)$ large enough so that the r.h.s. exists*

$$
\left(\frac{\pi}{\sin \pi\epsilon}\right)^N \prod_{l=1}^{N} \int_{-1/2}^{1/2} d\theta_l \, e^{2\pi i \theta_l \epsilon} f(-e^{2\pi i\theta_1}, \dots, -e^{2\pi i \theta_N}; \{p\})
$$

$$
= \prod_{l=1}^{N} \int_0^1 dt_l \, t_l^{-1+\epsilon} f(t_1, \dots, t_N; \{p\}).
$$

For notational simplicity we set

$$\iota_n(\theta;\lambda) = \prod_{1\le j<k\le n} |e^{2\pi i\theta_k} - e^{2\pi i\theta_j}|^{2\lambda}.$$

By putting $-1-b-\lambda(N-1) = \lambda_1$ and $a+b = \lambda_2$ in (3.46), we can use this result with $\epsilon = (a-b)/2$ to deduce the equivalent trigonometric integral [5,6]

$$\begin{aligned}\left(\prod_{l=1}^n \int_{-1/2}^{1/2} d\theta_l\, e^{\pi i\theta_l(a-b)}|1+e^{2\pi i\theta_l}|^{a+b}\right) \iota_n(\theta;\lambda) J_\kappa^{(1/\lambda)}(e^{2\pi i\theta_1},\dots,e^{2\pi i\theta_n}) \\ = M_n(a,b,\lambda)(-1)^{|\kappa|} J_\kappa^{(1/\lambda)}(1^n) \frac{[-b]_\kappa^{(1/\lambda)}}{[\lambda(n-1)+a+1]_\kappa^{(1/\lambda)}} \\ = J_\kappa^{(1/\lambda)}(1^n) \prod_{j=1}^n \frac{\Gamma(1+a+b+\lambda(j-1))\Gamma(\lambda j+1)}{\Gamma(1+a+\lambda(n-j)+\kappa_j)\Gamma(1+b+\lambda(j-1)-\kappa_j)\Gamma(1+\lambda)}\end{aligned} \tag{3.57}$$

where

$$\begin{aligned} M_n(a,b,\lambda) &:= \left(\prod_{l=1}^n \int_{-1/2}^{1/2} d\theta_l\, e^{\pi i\theta_l(a-b)}|1+e^{2\pi i\theta_l}|^{a+b}\right) \iota_n(\theta;\lambda) \\ &= \prod_{j=0}^{n-1} \frac{\Gamma(\lambda j+a+b+1)\Gamma(\lambda(j+1)+1)}{\Gamma(\lambda j+a+1)\Gamma(\lambda j+b+1)\Gamma(1+\lambda)} \end{aligned} \tag{3.58}$$

This latter equality follows by applying Proposition 3.4 to the Selberg integral (3.40); the resulting trigonometric integral written as a constant term identity was first considered by Morris [11]. The second equality in (3.57) follows from the expression (3.58) for M_n, and the functional property of the gamma function.

Although the use of Proposition 3.4 in the derivation of (3.57) requires $a+b, \lambda \in Z_{\ge 0}$, these restrictions can be relaxed to $Re(a+b)$, $Re(\lambda) \ge 0$ (at least) by applying Carlson's theorem [15]. Suppose $f(z) = g(z)$ for $z \in Z_{>0}$ and $f(z)$, $g(z)$ are analytic and bounded by $e^{(\mu|z|)}$ $(\mu < \pi)$ for $Re(z) \ge 0$, then Carlson's theorem says that $f(z) = g(z)$ identically.

For $a, b \in Z_{\ge 0}$, the l.h.s. of (3.57) can be rewritten as

$$(-1)^{|\kappa|}\left(\prod_{l=1}^n \int_0^1 d\theta_l\,(1-e^{2\pi i\theta_l})^a(1-e^{-2\pi i\theta_l})^b\right) \iota_n(\theta;\lambda) J_\kappa^{(1/\lambda)}(e^{2\pi i\theta_1},\dots,e^{2\pi i\theta_n}). \tag{3.59}$$

In the case $a = 0$, this can be interpreted as the Fourier coefficient in the eigenfunction expansion of the symmetric polynomial $\prod_{l=1}^n (1-z_l)^b$ in terms of Jack polynomials. Thus, since the set (3.9) is a complete set for symmetric functions

and is orthogonal with respect to the inner product (3.11b), we have that

$$\prod_{l=1}^{n}(1-z_l)^b = \sum_{\kappa} \frac{L^N c_\kappa}{\langle J_\kappa^{(1/\lambda)} | J_\kappa^{(1/\lambda)} \rangle_I'} J_\kappa^{(1/\lambda)}(z_1,\ldots,z_n) \tag{3.60a}$$

where

$$c_\kappa = \left(\prod_{l=1}^{n} \int_0^1 d\theta_l \, (1-e^{-2\pi i \theta_l})^b \right) \iota_n(\theta;\lambda) J_\kappa^{(1/\lambda)}(e^{2\pi i\theta_1},\ldots,e^{2\pi i\theta_n}). \tag{3.60b}$$

On the other hand we know from the result (3.52) that

$$\prod_{l=1}^{n}(1-z_l)^b = \sum_{\kappa} \frac{[-b]_\kappa^{(1/\lambda)}}{\lambda^{|\kappa|} j_\kappa} J_\kappa^{(1/\lambda)}(z_1,\ldots,z_n) \tag{3.61}$$

Since (3.59) and (3.57) give the value of c_κ in (3.60), comparison of (3.60a) and (3.61) gives the value of the normalization integral in (3.60a) as

$$\begin{aligned} \langle \psi_0 J_\kappa^{1/\lambda} | \psi_0 J_\kappa^{1/\lambda} \rangle_I &= L^n \frac{\Gamma(\lambda n+1)}{(\Gamma(\lambda+1))^n} \frac{j_\kappa J_\kappa^{(1/\lambda)}(1^n)}{[\lambda(n-1)+1]_\kappa^{(1/\lambda)}} \\ &= L^n \frac{\Gamma(\lambda n+1)}{(\Gamma(\lambda+1))^n} \frac{(J_\kappa^{(1/\lambda)}(1^n))^2}{\tilde{f}_n^\lambda(\kappa) f_n^\lambda(\kappa)} \end{aligned} \tag{3.62}$$

where the last equality follows upon using (3.26).

3.6. Generalized Laguerre polynomials. Consider now the eigenfunctions associated with the operator (3.5) in the case W given by (1.32b). Making the change of variables $y = \beta x^2/2$ gives

$$\begin{aligned} \tilde{H}^{(L)} &:= -\frac{1}{2\beta} e^{\beta W^{(L)}/2} (H^{(L)} - E_0) e^{-\beta W^{(L)}/2} \\ &= \sum_{j=1}^{N} \left(y_j \frac{\partial^2}{\partial y_j^2} + (a - y_j + 1)\frac{\partial}{\partial y_j} + \beta \sum_{\substack{k=1 \\ \neq j}} \frac{y_j}{y_j - y_k} \frac{\partial}{\partial y_j} \right), \end{aligned} \tag{3.63}$$

where

$$a := (\beta a' - 1)/2. \tag{3.64}$$

In the case $N = 1$ the unique (up to normalization) polynomial eigenfunction of degree n of $\tilde{H}^{(L)}$ is the Laguerre polynomial

$$L_n^a(y) := (a+1)_n \sum_{j=0}^{n} \binom{n}{j} \frac{(-y)^j}{(a+1)_j}. \tag{3.65}$$

For general N, direct computation shows that $\tilde{H}^{(L)}$ acting on the monomial symmetric function m_κ gives a series of the form

$$-2|\kappa|m_\kappa + \sum_{|\sigma|<|\kappa|} u_{\kappa\sigma} m_\sigma .$$

Thus for each m_κ we see that by constructing the corresponding eigenfunction in order $|\kappa| = 0, 1, \dots$, there is a unique symmetric polynomial eigenfunction of the form

$$L_\kappa^a(x_1, \dots, x_N; \alpha) = v_{\kappa\kappa} C_\kappa^{(\alpha)}(y_1, \dots, y_N) + \sum_{|\sigma|<|\kappa|} v_{\kappa\sigma} C_\kappa^{(\sigma)}(y_1, \dots, y_N) \tag{3.66}$$

where $\alpha := 2/\beta$, which is termed the generalized Laguerre polynomial (for consistency with (3.65) the normalization is chosen so that $v_{\kappa\kappa} = (-1)^\kappa / C_\kappa^{(\alpha)}(1^N)|\kappa|!)$.

By comparing the operators $\tilde{H}^{(J)}$ and $\tilde{H}^{(L)}$, and using the fact that $G_\kappa^{(a,b)}$ has an expansion in terms of Jack polynomials with leading term as in (3.66), we see that the generalized Laguerre polynomials are a limiting case of the generalized Jacobi polynomials:

$$\lim_{b\to\infty} ((-1)^{|\kappa|} b^{|\kappa|} / C_\kappa^{(\alpha)}(1^N)|\kappa|!) \tilde{G}_\kappa^{(a,b)}(y_1/b, \dots, y_N/b; \alpha) = L_\kappa^a(x_1, \dots, x_N; \alpha) \tag{3.67}$$

where $\tilde{G}_\kappa^{(a,b)}$ is normalized so that the coefficient of $C_\kappa^{(\alpha)}$ is unity. Since it has been proved [16] that the generalized Jacobi polynomials are orthogonal with respect to the inner product

$$\langle f|g\rangle^{(J)} := \prod_{l=1}^N \int_0^1 dy_l\, y_l^a (1-y_l)^b \prod_{1\le j<k\le N} |y_k - y_j|^{2/\alpha} f(y_1, \dots, y_N) g(y_1, \dots, y_N) \tag{3.68}$$

it follows that the Laguerre polynomials are orthogonal with respect to

$$\langle f|g\rangle^{(L)} := \prod_{l=1}^N \int_0^\infty dy_l\, y_l^a e^{-y_l} \prod_{1\le j<k\le N} |y_k - y_j|^{2/\alpha} f(y_1, \dots, y_N) g(y_1, \dots, y_N) \tag{3.69}$$

It is possible to derive a family of independent mutually commuting differential operators which include $\tilde{H}^{(L)}$, which all have L_κ^a as eigenfunctions, and for which the eigenvalues can explicitly be given [17]. This is done by first characterising the polynomials L_κ^a in terms of a generating function. This in turn requires a preliminary result, which is derived from the formulas (3.34) and (3.36) for the action of the operators E_0, D_0 etc. on $C_\kappa^{(\alpha)}$, as well as the formula [4]

$$(x_1 + \cdots + x_N) C_\kappa^{(\alpha)}(x_1, \dots, x_N) = \frac{1}{1+|\kappa|} \sum_{i=1}^N \binom{\kappa^{(i)}}{\kappa} C_{\kappa^{(i)}}^{(\alpha)}(x_1, \dots, x_N) \tag{3.70}$$

For ease of notation we will present the results using the abbreviations $x := (x_1, \dots, x_N)$ (and similarly the meaning of y, z) and p_j as defined in (3.14b).

Proposition 3.5. *We have*

$$(-D_1^{(y)} - (a+1)E_0^{(y)})_0\mathcal{F}_1^{(\alpha)}(a+q; y; -z) = p_1(z)\,{}_0\mathcal{F}_1(a+q; y; -z)$$

Now we can establish the generating function.

Proposition 3.6. *We have*

$$e^{p_1(z)}{}_0\mathcal{F}_1^{(\alpha)}(a+q; x; -z) = \sum_{k=0}^{\infty} \sum_{|\kappa|=k} \frac{L_\kappa^a(x;\alpha) C_\kappa^{(\alpha)}(z)}{[a+q]_\kappa^{(\alpha)}}$$

where $q := 1 + (N-1)/\alpha$ *and*

$$L_\kappa^a(x;\alpha) = \frac{[a+q]_\kappa^{(\alpha)}}{|\kappa|!} \sum_{s=0}^{|\kappa|} \sum_{|\sigma|=s} \binom{\kappa}{\sigma} \frac{(-1)^{|\sigma|} C_\sigma^{(\alpha)}(t)}{[a+q]_\sigma^{(\alpha)} C_\sigma^{(\alpha)}(1^N)}$$

Proof. We need to check that L_κ^a as defined by this formula is an eigenfunction of $\tilde{H}^{(L)}$ with eigenvalue $-|\kappa|$. The second stated formula, which is a consequence of the first, establishes that L_κ^a is of the form (3.66) with the correct normalization.

Now

$$E_1^{(z)}{}_0\mathcal{F}_1^{(\alpha)}(a+q; x; -z)e^{p_1(z)} = e^{p_1(z)}\left(E_1^{(z)} + p_1(z)\right){}_0\mathcal{F}_1^{(\alpha)}(a+q; x; -z) \quad (3.71)$$

Using Proposition 3.5, and the fact that $E_1^{(z)}$ is an eigenoperator of the Jack polynomials so that its action on ${}_0\mathcal{F}_1^{(\alpha)}$ is the same as the action of $E_1^{(y)}$, the r.h.s. can be rewritten as

$$\left(E_1^{(y)} - D_1^{(y)} - (a+1)E_0^{(y)}\right){}_0\mathcal{F}_1^{(\alpha)}(a+q; x; -z)e^{p_1(z)} \quad (3.72)$$

The required eigenvalue equation now follows by substituting the generating function on the l.h.s. and computing the action of $E_1^{(z)}$ on the Jack polynomials.

The second stated formula follows from the generating function by applying the identity [4]

$$e^{z_1+\cdots+z_N} C_\lambda^{(\alpha)}(z_1, \dots, z_N) = \sum_\mu \binom{\mu}{\lambda} \frac{|\lambda|!}{|\mu|!} C_\mu^{(\alpha)}(z_1, \dots, z_N)$$

and equating coefficients of $C_\kappa^{(\alpha)}(z_1, \dots, z_N)$. □

The operator E_1 is related to the Macdonald operator D_N^1 (recall Proposition 3.1) by $D_N^1 = \alpha E_1 + N(N-1)/2$. By considering the analogue of (3.71) with $E_1^{(z)}$ replaced by D_N^p $(p = 1, 2, \dots, N)$, a family of differential operators which have the

generalized Laguerre polynomials as eigenfunctions can be given [17]. The r.h.s. of (3.71) is then computed according to the Baker-Campbell-Hausdorff formula

$$D_N^p\, f\, e^{p_1(z)} = e^{p_1(z)} \Big(D_N^p + [D_N^p, p_1(z)] + \frac{1}{2!}\left[[D_N^p, p_1(z)], p_1(z)\right] + \cdots \frac{1}{p!}\left[\cdots [D_N^p, p_1(z)], \cdots, p_1(z)\right] \Big) f$$

where $p_1(z) := z_1 + \cdots + z_N$ and $f = {}_0\mathcal{F}_1^{(\alpha)}$ with arguments as in (3.71). Note that the sum on the RHS terminates after the p-th nested commutator since the highest derivative in D_N^p has degree p.

We now want to rewrite the r.h.s. as in (3.72). To do this, we note that in general, if $A^{(y)}F = \hat{A}^{(z)}F$ and $B^{(y)}F = \hat{B}^{(z)}F$, where the superscript denotes the set of variables the operator is acting on, then

$$A^{(y)}B^{(y)}F = A^{(y)}\hat{B}^{(z)}F = \hat{B}^{(z)}A^{(y)}F = \hat{B}^{(z)}\hat{A}^{(z)}F$$

where the second equality follows because operators acting on different sets of variables always commute.

Analogous to the derivation of the eigenoperator given in the proof of Proposition 3.6, we can now deduce a family of eigenoperators of the L_κ^a, together the corresponding eigenvalues.

Proposition 3.7. *Let*

$$\tilde{H}_p^{(L)(y)} := \Big(D_N^{p(y)} - \Big[D_1^{(y)} + (a+1)E_0^{(y)}, D_N^{p(y)} \Big] + \cdots + \frac{(-1)^p}{p!} \Big[D_1^{(y)} + (a+1)E_0^{(y)}, \Big[\cdots \Big[D_1^{(y)} + (a+1)E_0^{(y)}, D_N^{p(y)} \Big] \cdots \Big] \Big] \Big)$$

Then, $L_\kappa^a(y_1, \ldots, y_N; \alpha)$ is an eigenfunction of $\tilde{H}_p^{(L)(y)}$ for each $p = 1, \ldots, N$, with eigenvalue $e_p(\kappa; \alpha)$ given by the coefficient of X^{N-p} in the equation for $e(\kappa, \alpha; X)$ given by (3.12).

Although the operators $\tilde{H}_p^{(L)(y)}$ separate the eigenvalues, there does not appear to be any direct way to establish that the operators are self-adjoint with respect to the inner product (3.69), and thus the orthogonality of $\{L_\kappa^a\}$. Instead, these features can be deduced [9] by developing a theory of non-symmetric Laguerre polynomials and associated operators, similar to the theory of the Cherednik operators used in the study of Jack polynomials (recall the second paragraph below Proposition 3.1).

3.7. Generalized Hermite polynomials. A similar development to that just given is possible for the polynomial eigenfunctions of the operator (3.5) with W given by (1.5). The change of variables $y_j := \sqrt{\beta/2}x_j$ gives

$$\tilde{H}^{(H)} := -\frac{2}{\beta}e^{\beta W^{(H)}/2}(H^{(H)} - E_0)e^{-\beta W^{(H)}/2}$$

$$= \sum_{j=1}^{N}\Big(y_j\frac{\partial^2}{\partial y_j^2} - 2y_j\frac{\partial}{\partial y_j} + \frac{2}{\alpha}\sum_{\substack{k=1\\ \neq j}}\frac{1}{y_j - y_k}\frac{\partial}{\partial y_j}\Big), \tag{3.73}$$

($\alpha := 2/\beta$ as in the previous section), which for $N = 1$ has the Hermite polynomials as its eigenfunctions. Notice that this operator can be derived as a limit of $\tilde{H}^{(L)}$ as given by (3.63) [15]. We have

$$\lim_{a\to\infty}\tilde{H}^{(L)}\Big|_{y_j\mapsto a+\sqrt{2a}y_j} = \tilde{H}^{(H)} \tag{3.74}$$

Also, computation of the action of the operator (3.73) on the monomial symmetric function m_κ shows that the eigenfunctions can be expanded in the form of the r.h.s. of (3.66). Denoting this eigenfunction by $H_\kappa(y_1,\dots,y_n;\alpha)$, which is known as a generalized Hermite polynomial (see [14] and references therein), we we therefore have from (3.74) that

$$\lim_{a\to\infty}(-1)^{|\kappa|}|\kappa|!2^{|\kappa|/2}a^{-|\kappa|/2}L_\kappa^a(a+\sqrt{2a}x_1,\dots,a+\sqrt{2a}x_N;\alpha) = H_\kappa(x_1,\dots,x_n;\alpha) \tag{3.75}$$

where H_κ is normalized so that the coefficient of C_κ is $2^{|\kappa|}/C_\kappa^{(\alpha)}(1^N)$, which is consistent with the $N = 1$ case. Taking this limit in the inner product (3.69) gives that the generalized Hermite polynomials are orthogonal with respect to the inner product

$$\langle f|g\rangle^{(H)} := \prod_{l=1}^{N}\int_{-\infty}^{\infty}dy_l\,e^{-y_l^2}\prod_{1\le j<k\le N}|y_k - y_j|^{2/\alpha}f(y_1,\dots,y_N)g(y_1,\dots,y_N)$$

Analogous to Proposition 3.6 for the generalized Laguerre polynomials, the generalized Hermite polynomials can be specified by a generating function [17].

Proposition 3.8. *We have*

$$e^{-p_2(z)}{}_0\mathcal{F}_0^{(\alpha)}(2y;z) = \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{|\kappa|=k}H_\kappa(y;\alpha)C_\kappa^{(\alpha)}(z)$$

Proof. Note that in terms of the operators E_k, $D_k(\alpha)$ defined above (3.33) we have $\tilde{H}_\kappa^{(H)} = D_0(\alpha) - 2E_1$. To compute the action of $D_0(\alpha)$ on ${}_0\mathcal{F}_0^{(\alpha)}$, we first compute the action of $E_0^{(y)}$ and $D_1^{(y)}$. Now it follows from (3.36a) and (3.70) that

$$E_0^{(y)}{}_0\mathcal{F}_0^{(\alpha)}(2y;z) = 2p_1(z){}_0\mathcal{F}_0^{(\alpha)}(2y;z) \tag{3.76}$$

Since $D_1^{(y)} = \frac{1}{2}[E_0^{(y)}, D_2^{(y)}]$, using (3.76), the fact that $D_2^{(y)}$ is an eigenoperator for the Jack polynomials, and (3.73) gives

$$\begin{aligned} D_1^{(y)} {}_0\mathcal{F}_0^{(\alpha)}(2y;z) &= [D_2^{(z)}, p_1(z)]_0\mathcal{F}_0^{(\alpha)}(2y;z) \\ &= \Big(\frac{2}{\alpha}(N-1)p_1(z) + 2E_2^{(z)}\Big)_0\mathcal{F}_0^{(\alpha)}(2y;z) \end{aligned} \tag{3.77}$$

where the second equality follows by computing the commutator. Finally, since $D_0^{(y)} = [E_0^{(y)}, D_1^{(y)}]$, from (3.76), (3.77) and (3.73) we have

$$\begin{aligned} D_0^{(y)} {}_0\mathcal{F}_0^{(\alpha)}(2y;z) &= [\frac{2}{\alpha}(N-1)p_1(z) + 2E_2^{(z)}, 2p_1(z)]_0\mathcal{F}_0^{(\alpha)}(2y;z) \\ &= 4p_2(z)_0\mathcal{F}_0^{(\alpha)}(2y;z) \end{aligned} \tag{3.78}$$

To compute the action of $E_1^{(y)}$ on ${}_0\mathcal{F}_0^{(\alpha)}$ we note that $E_1^{(y)}$ is an eigenoperator of the Jack polynomials, so that

$$E_1^{(y)} {}_0\mathcal{F}_0^{(\alpha)}(2y;z) = E_1^{(z)} {}_0\mathcal{F}_0^{(\alpha)}(2y;z) \tag{3.79}$$

We will now use (3.78) and (3.79) to show that H_κ as defined by the generating function is an eigenfunction of $\tilde{H}^{(H)}$ with eigenvalue $-2|\kappa|$. The method is analogous to the proof of Proposition 3.6. Using (3.78) and (3.79) we have

$$\begin{aligned} E_1^{(z)} {}_0\mathcal{F}_0^{(\alpha)}(2y;z)e^{-p_2(z)} &= e^{-p_2(z)}E_1^{(z)} {}_0\mathcal{F}_0^{(\alpha)}(2y;z) - 2p_2(z)_0\mathcal{F}_0^{(\alpha)}(2y;z)e^{-p_2(z)} \\ &= e^{-p_2(z)}E_1^{(y)} {}_0\mathcal{F}_0^{(\alpha)}(2y;z) - \frac{1}{2}e^{-p_2(z)}D_0^{(y)} {}_0\mathcal{F}_0^{(\alpha)}(2y;z) \end{aligned}$$

The required eigenvalue equation now follows from the action of $E_1^{(z)}$ on the l.h.s. of the generating function.

Furthermore, H_κ has an expansion of the form of the r.h.s. of (3.66) with $v_{\kappa\kappa} = 2^{|\kappa|}/C_\kappa^{(\alpha)}(1^N)$. This follows from the expansion

$${}_0\mathcal{F}_0^{(\alpha)}(2y;z) = \sum_{k=0}^{\infty} \frac{2^k}{k!} \sum_{|\sigma|=k} \frac{C_\sigma^{(\alpha)}(y)C_\sigma^{(\alpha)}(z)}{C_\kappa^{(\alpha)}(1^N)} \tag{3.80}$$

and the fact that to compute the coefficient of $C_\kappa^{(\alpha)}(z)$ in ${}_0\mathcal{F}_0^{(\alpha)}(2y;z)e^{-p_2(z)}$ the sum over k in (3.80) can be terminated at $k = |\kappa|$. □

The method used to show that the polynomials H_κ defined by the generating function are eigenfunctions of (3.73) can be used in conjunction with the Baker-Campbell-Hausdorff formula of the previous section, and the Macdonald operators, to deduce the family of eigenoperators for the H_κ which are the analogues of Proposition 3.7.

Proposition 3.9. *Let*

$$\tilde{H}_j^{(H)} := D_N^j - \frac{1}{4}\left[D_0, D_N^j\right] + \frac{1}{4^2\,2!}\left[D_0, \left[D_0, D_N^j\right]\right] - \cdots + \frac{(-1)^j}{4^j\,j!}\left[D_0, \left[D_0, \cdots, \left[D_0, D_N^j\right]\right]\cdots\right]$$

We have that $H_\kappa(y_1, \ldots, y_N; \alpha)$ is an eigenfunction of $\tilde{H}_j^{(H)}$ for each $j = 1, \ldots, N$, with eigenvalue $e_j(\kappa; \alpha)$ given by the coefficient of X^{N-j} in (3.12).

In the next Section we will discuss the calculation of correlation functions. For the Hermite case this requires an integral representation of H_κ, which can be deduced from the generating function and the orthogonality of the generalized Hermite polynomials with respect to the inner product (3.76) [17].

Proposition 3.10. *We have*

$$e^{-p_2(z)} H_\kappa(z; \alpha) = \frac{2^{|\kappa|}}{\mathcal{N}_0^{(H)} C_\kappa^{(\alpha)}(1^N)} \int_{(-\infty,\infty)^N} {}_0\mathcal{F}_0^{(\alpha)}(2y; -iz) C_\kappa^{(\alpha)}(iy)\, d\mu^{(H)}(y)$$

where

$$d\mu^{(H)}(y) := \prod_{j=1}^N e^{-y_j^2} \prod_{1 \le j < k \le N} |y_j - y_k|^{2/\alpha}\, dy_1 \ldots dy_N$$

and

$$\mathcal{N}_0^{(H)} = 2^{-N(N-1)/2\alpha} \pi^{N/2} \prod_{j=0}^{N-1} \frac{\Gamma(1 + (j+1)/\alpha)}{\Gamma(1 + 1/\alpha)}.$$

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4. Correlation functions

4.1. The density. The density is defined by (1.36) with $n = 1$. For $e^{-\beta W}$ corresponding to the eigenvalue probability density function of the classical random matrix ensembles (we exclude the circular ensemble where the density is just the constant N/L) and β even, it is possible to express the density as a β-dimensional integral [1]. Furthermore, asymptotic analysis of the integrals gives the global limit as that predicted by electrostatics (recall (1.37), (1.38a) and (1.38b)). We will present the details in the Jacobi case (in [1] the details were presented in the Hermite case, where the integral representation follows from (3.56)).

From the discussion of Section 3.4 (in particular equations (3.39) and (3.56)) we know that in the Jacobi case the β-dimensional integral representation of the density is given by

$$\begin{aligned}\rho_{(1)}(y) &= A_N(\beta,a,b)y^{\beta a/2}(1-y)^{\beta b/2}\int_{C^\beta} dx_1\dots dx_\beta \prod_{l=1}^{\beta}(1-yx_l)^N \\ &\quad \times x_l^{4/\beta+a+b+N-2}(1-x_l)^{-2-b-N}\prod_{1\le j<k\le\beta}|x_k-x_j|^{4/\beta} \qquad (4.1)\end{aligned}$$

where C is any simple closed contour which starts at the origin and encircles the point $x = 1$, and the coefficient A_N is such that

$$\int_0^1 \rho_1(y)\,dy = N+1. \qquad (4.2)$$

For each integration variable x_l, the N-dependent terms in the integrand are

$$(1-yx_l)^N x_l^N (1-x_l)^{-N} = e^{N(\log(1-yx_l)+\log x_l-\log(1-x_l))}. \qquad (4.3)$$

The exponent in (4.3) has a stationary point when

$$x_l = 1 \pm i\Big(\frac{1}{y}(1-y)\Big)^{1/2} =: x_\pm, \qquad (4.4)$$

so according to the saddle point method of asymptotic analysis we should deform each of the contours of integration in (4.1) to pass through x_+ and x_-.

With the contours of integration so deformed, we must expand the integrand in the neighbourhood of the saddle points. Due to the factor $\prod_{1\le j<k\le\beta}|x_k -$

$x_j|^{4/\beta}$ the maximum contribution will be obtained by expanding $\beta/2$ integration variables $(x_1, \dots, x_{\beta/2}$ say) about x_+ and the remaining $\beta/2$ integration variables $(x_{\beta/2+1}, \dots, x_\beta)$ about x_-. This specific choice is only one of the $\binom{\beta}{\beta/2}$ equivalent ways of dividing the integration variables into these two classes, so after expanding the variables with the specific choice we must multiply by the combinatorial factor.

In the neighbourhood of the saddle points we have

$$e^{N(\log(1-yx_l)+\log x_l-\log(1-x_l))} \sim \exp\Big[N\Big(\log(1-yx_\pm)+\log x_\pm-\log(1-x_\pm)\Big)\Big]$$

$$\times \exp\Big[-\frac{N}{2}(x_l-x_\pm)^2\Big(\frac{y^2}{(1-yx_\pm)^2}+\frac{1}{x_\pm^2}-\frac{1}{(1-x_\pm)^2}\Big)\Big]$$

where on the r.h.s. x_+ is to be taken for $j = 1, \dots, \beta/2$ while x_- is to be taken for $j = \beta/2+1, \dots, \beta$. Also

$$\prod_{1\le j<k\le\beta}|x_k-x_j|^{4/\beta} \sim |x_+-x_-|^\beta \prod_{1\le j<k\le\beta/2}|x_k-x_j|^{4/\beta} \prod_{\beta/2+1\le j<k\le\beta}|x_k-x_j|^{4/\beta}$$

Thus we have

$$\begin{aligned}\rho_{N+1}(y) \sim{}& A_N(\beta,a,b)\binom{\beta}{\beta/2}y^{\beta a/2}(1-y)^{\beta b/2}\\ &\times|x_+-x_-|^\beta|x_+|^{\beta(4/\beta+a+b-2)}|1-x_+|^{\beta(-2-b)}e^{N(\beta/2)\log(|1-yx_+|^2|x_+|^2/|1-x_+|^2)}\\ &\times\Big|\int_{-\infty}^{\infty}dx_1\cdots\int_{-\infty}^{\infty}dx_{\beta/2}\prod_{l=1}^{\beta/2}\exp\Big[-\frac{N}{2}x_l^2\Big(\frac{y^2}{(1-yx_+)^2}+\frac{1}{x_+^2}-\frac{1}{(1-x_+)^2}\Big)\Big]\\ &\times\prod_{1\le j<k\le\beta/2}|x_k-x_j|^{4/\beta}\Big|^2,\end{aligned} \tag{4.5}$$

where the fact that $x_+ = x_-*$ has been used.

To simplify (4.5) we note that a simple change of variables gives that the y-dependence in the absolute value squared of the integral is proportional to

$$\Big|\frac{y^2}{(1-yx_+)^2}+\frac{1}{x_+^2}-\frac{1}{(1-x_+)^2}\Big|^{-(\beta-1)}$$

Also, from the explicit formula (4.4),

$$\Big|\frac{y^2}{(1-yx_+)^2}+\frac{1}{x_+^2}-\frac{1}{(1-x_+)^2}\Big| = \frac{2y^{3/2}}{(1-y)^{1/2}}, \quad \log(|1-yx_+|^2|x_+|^2/|1-x_+|^2) = 0$$

$$|x_+-x_-| = 2\frac{(1-y)^{1/2}}{y^{1/2}}, \quad |x_+|^2 = \frac{1}{y}, \quad |1-x_+|^2 = \frac{1-y}{y}.$$

The calculation is now complete, since making these substitutions in (4.5) shows that in the limit $N \to \infty$ $\rho_{(1)}(y)$ is proportional to $(y(1-y))^{-1/2}$, which is the electrostatic prediction (1.38b) (the proportionality constant is uniquely determined by the condition (4.2)).

4.2. Two-particle distribution for β even in the circular ensemble. In a system of $N+2$ particles in the circular ensemble, the two-particle distribution function is given by (1.36) with N replaced by $N+2$, $e^{-\beta W}$ given by the product of differences of Proposition 1.2, and Z_{N+2} evaluated by (3.58). Thus we have

$$\rho_{(2)}(r_1, r_2) = \frac{(N+2)(N+1)}{L^2} \frac{((\beta/2)!)^{N+2}}{(\beta(N+2)/2)!} |e^{2\pi i r_1/L} - e^{2\pi i r_2/L}|^\beta I_N(\beta; r_1, r_2) \tag{4.6a}$$

where

$$I_N(\beta; r_1, r_2) := \prod_{i=1}^{N} \int_0^1 dx_i \prod_{j=1}^{N} \prod_{k=1}^{2} |1 - e^{2\pi i (x_j - r_k/L)}|^\beta \prod_{1 \le j < k \le N} |e^{2\pi i x_k} - e^{2\pi i x_j}|^\beta. \tag{4.6b}$$

For β even this integral is a special case of the integrals $S_{n,m}$ defined by (3.29), written in trigonometric form by use of Proposition 3.4. But we know that $S_{n,m}$ can be written in terms of the generalized hypergeometric function ${}_2F_1^{(\lambda)}$, so it is possible to write $\rho_{(2)}(r_1, r_2)$ in terms of this function [2].

Proposition 4.1. *With $r := r_2 - r_1$ we have*

$$\rho_{(2)}(r_1, r_2) := \rho_{(2)}(r) = \frac{(N+2)(N+1)}{L^2} \frac{((\beta/2)!)^{N+2}}{(\beta(N+2)/2)!} |1 - e^{2\pi i r/L}|^\beta M_N(\beta, \beta, \beta/2)$$

$$\times e^{\pi i N \beta r/L} {}_2F_1^{(\beta/2)}(-N, 2; 4; 1 - t_1, \ldots, 1 - t_\beta)\Big|_{t_j = e^{-2\pi i r/L}\ (j=1,\ldots,\beta)},$$

where M_N is specified by (3.58).

Proof. Consider $S_{n,m}$ defined by (3.29) with

$$n = N, \quad m = \beta, \quad \lambda_2 = \beta, \quad \lambda = \beta/2, \quad t_1 = \cdots = t_m = e^{-2\pi i r/L}. \tag{4.7}$$

Writing

$$\prod_{1 \le j < k \le N} |x_k - x_j|^\beta = (-1)^{\beta N(N-1)/2} \prod_{j=1}^{N} x_j^{\beta(N-1)/2} \prod_{1 \le j < k \le N} (1 - \frac{x_k}{x_j})^{\beta/2} (1 - \frac{x_j}{x_k})^{\beta/2},$$

$$\prod_{j=1}^{N} |1 - x_j|^\beta = (-1)^{N\beta/2} \prod_{j=1}^{N} x_j^{\beta/2} (1 - x_j)^{\beta/2} (1 - 1/x_j)^{\beta/2}$$

and

$$\prod_{j=1}^{N}(1-x_j e^{-2\pi i r/L})^{\beta} = \left(-e^{-2\pi i r/L}\right)^{\frac{N\beta}{2}} \prod_{j=1}^{N}\left(x_j(1-x_j e^{-2\pi i r/L})\left(1-\frac{e^{2\pi i r/L}}{x_j}\right)\right)^{\frac{\beta}{2}}$$

we see from Proposition 3.4 with $\lambda_1 = -1-\beta(N+1)/2$ that $S_{n,m}$ is proportional to

$$e^{-\pi i N\beta r/L} I_N(\beta; r_1, r_2)$$

(the change of variables $x_j \to x_j + r_1$ puts $I_N(\beta; r_1, r_2)$ in the sought form). But from Proposition 3.3 and (3.32), $S_{n,m}$ with parameters (4.2) is proportional to

$${}_2F_1^{(\beta/2)}(-N, 2; 1-2/\beta-N; t_1, \dots, t_\beta).$$

The stated result now follows by using the transformation formula [2]

$${}_2F_1^{(\alpha)}(a,b;c;t_1,\dots,t_m) = \frac{{}_2F_1^{(\alpha)}(a,b;a+b+1+(m-1)/\alpha-c;1-t_1,\dots,1-t_m)}{{}_2F_1^{(\alpha)}(a,b;a+b+1+(m-1)/\alpha-c;1,\dots,1)}.$$

valid provided $a-(i-1)/\alpha \in Z_{\le 0}$ for some $1 \le i \le m$, which is the criterion for the series (3.32a) defining ${}_2F_1^{(\alpha)}$ to terminate. □

From (3.49) we can write ${}_2F_1^{(\beta/2)}$ in Proposition 4.1 as a β-dimensional integral, thus obtaining a β-dimensional integral representation of $\rho_{(2)}(r)$. We have [2]

$$\rho_{(2)}(r) = \frac{(N+2)(N+1)}{L^2}\frac{(\beta N/2)!((\beta/2)!)^2}{(\beta(N+2)/2)!}\frac{M_N(\beta,\beta,\beta/2)}{S_\beta(2/\beta,2/\beta,4/\beta)}$$
$$\times \int_{[0,1]^\beta} du_1 \dots du_\beta \prod_{j=1}^{\beta}(1-(1-e^{2\pi i r/L}u_j)^N D_{2/\beta,2/\beta,4/\beta}(u_1,\dots,u_\beta). \quad (4.8)$$

In the thermodynamic limit ($N, L \to \infty, N/L = \rho$), $\rho_{(2)}(r)$ can be written as the generalized hypergeometric ${}_1F_1^{(\alpha)}$, which is a confluent version of (3.32):

$$\begin{aligned} {}_1F_1^{(\alpha)}(b;c;t_1,\dots,t_m) &:= \sum_{d=0}^{\infty}\frac{1}{d!}\sum_{|\kappa|=d}\frac{[b]_\kappa^{(\alpha)}}{[c]_\kappa^{(\alpha)}}C_\kappa^{(\alpha)}(t_1,\dots,t_m) \\ &= \lim_{|a|\to\infty} {}_2F_1^{(\alpha)}(a,b;c;t_1/a,\dots,t_m/a) \end{aligned}$$

Thus use of this formula and Stirling's formula in Proposition 4.1 shows [3]

$$\lim_{\substack{N,L\to\infty \\ N/L=\rho}} \rho_{(2)}(r) = C\,{}_1F_1^{(\beta/2)}(2,4;-2\pi i\rho r_1,\dots,-2\pi i\rho r_\beta)\Big|_{r_j=r\,(j=1,\dots,\beta)} \quad (4.9)$$

while taking the thermodynamic limit in the integral representation (4.8) gives

$$\lim_{\substack{N,L\to\infty \\ N/L=\rho}} \rho_{(2)}(r) = C' \int_{[0,1]^\beta} du_1 \dots du_\beta \prod_{j=1}^{\beta} e^{-2\pi i \rho r u_j} D_{2/\beta,2/\beta,4/\beta}(u_1,\dots,u_\beta). \tag{4.10}$$

In these formulas, the constants C and C' are given by

$$C = \rho^2 \frac{((\beta/2)!)^3}{\beta!(3\beta/2)!}(2\pi\rho|r|)^\beta e^{\pi i \rho \beta r}, \quad C' = \frac{C}{S_\beta(2/\beta,2/\beta,4/\beta)}.$$

4.3. Asymptotic expansions. According to (1.40) and (1.43), the exact solution for $\rho^T_{(2)} := \rho_{(2)} - \rho^2$ should exhibit some specific asymptotic properties. The asymptotics can be studied from the integral representations (4.8) and (4.10).

Consider first (4.10). In the $r \to \infty$ limit, the major contribution to the integral comes from the vicinity of the endpoints $u_j = 0, 1$. By expanding the integrand in the neighbourhood of these points it was shown in [4] that

$$\rho^T_{(2)} \sim \rho^2 \Big[-\frac{1}{\beta(\pi\rho r)^2} + \frac{a^{(4)}}{(\rho r)^4} + \dots + \sum_{n=1}^{\beta/2} \frac{\cos 2\pi\rho r n}{(\rho r)^{4n^2/\beta}} \Big(b_n^{(0)} + \frac{b_n^{(2)}}{(\rho r)^2} + \dots \Big)$$

$$+ \sum_{n=1}^{\beta/2} \frac{\sin 2\pi\rho r n}{(\rho r)^{4n^2/\beta+1}} \Big(c_n^{(0)} + \frac{c_n^{(2)}}{(\rho r)^2} + \dots \Big) \tag{4.11}$$

for some coefficients $a^{(2j)}, b_n^{(2j)}, c_n^{(2j)}$, which are given explicitly in terms of multiple integrals. We see that the leading non-oscillatory term is indeed of the form (1.40).

To test (1.43), we require the large-N asymptotics of (4.8) with $L = 2\pi$. The N-dependent terms in the integrand take on their maximum values at the endpoints $u_j = 0$ and $u_j = 1$, where for $u_j \sim 0$

$$[1 - (1 - e^{ir})u_j]^N \sim e^{-N(1-e^{ir})u_j}$$

while for $u_j \sim 1$

$$[1 - (1 - e^{ir})u_j]^N \sim e^{iNr} e^{-N(1-e^{-ir})(1-u_j)}.$$

These formulas show that the problem of computing the large-r asymptotics of (4.8) is identical to that of computing the large-N asymptotics of (4.10) (with $L = 2\pi$) provided we make the replacements

$$\pi\rho r \mapsto 2N \sin r/2, \qquad \rho \mapsto N/2\pi.$$

Substituting these replacements in (4.11), and substituting the resulting expansion in the l.h.s. of (1.43) we see that the oscillatory terms all integrate to zero, and all but the first non-oscillatory term vanish in the $N \to \infty$ limit. The first non-oscillatory term gives the r.h.s. of (1.43) [5].

4.4. The dynamical density-density correlation. Consider the Dyson Brownian motion model of the one-component log-gas on a circle as discussed in Section 2, and suppose that the initial state is chosen to be the equilibrium state. From the theory of Section 2.5 we know that the dynamical density-density correlation can be written as

$$\rho^T_{(2)}((x_a,0),(x_b,\tau_b)) = \sum_{\substack{l,\kappa \\ (l,\kappa)\neq(0,0)}} \frac{\langle\psi_0|n(x_b)|\psi_{l,\kappa}\rangle\langle\psi_{l,\kappa}|n(x_a)|\psi_0\rangle}{\langle\psi_0|\psi_0\rangle\langle\psi_{l,\kappa}|\psi_{l,\kappa}\rangle} e^{-\tau(E_{l,\kappa}-E_0)/\beta}$$

where $|\psi_\kappa\rangle$ denotes the eigenstates of the Schrödinger operator (1.4), and are therefore given by (3.9) (with $z_j := e^{2\pi i x_j/L}$) multiplied by $e^{-\beta W/2}$ (with W given by (1.6)), and $E_{l,\kappa}$ denotes the corresponding energies which are given by (recall (3.6) and (3.7))

$$\begin{aligned} E_{l,\kappa} - E_0 &= \left(\frac{2\pi}{L}\right)^2 \left(e(\kappa - l, 2/\beta) + \left(\frac{\beta}{2}(N-1)+1\right)|\kappa - l| \right) \\ &= \frac{2\pi^2}{L^2}\left(\sum_{j=1}^N (\kappa_j - l)^2 + \frac{\beta}{2}\sum_{j=1}^N (\kappa_j - l)(N-2j+1) \right). \end{aligned} \tag{4.12}$$

The inner products in the numerator can be simplified by using the periodicity of the eigenstates and the fact that they are homogeneous of degree $|\kappa|$. This allows us to write

$$\rho^T_{(2)}((x_a,0),(x_b,\tau)) = \sum_{\substack{l,\kappa \\ (l,\kappa)\neq 0}} \frac{|\langle\psi_0|n(0)|\psi_{l,\kappa}\rangle|^2}{\langle\psi_0|\psi_0\rangle\langle\psi_{l,\kappa}|\psi_{l,\kappa}\rangle} e^{2\pi i(x_b-x_a)|\kappa-l|/L} e^{-\tau(E_{l,\kappa}-E_0)/\beta} \tag{4.13}$$

The values of the normalizations $\langle\psi_0|\psi_0\rangle$ and $\langle\psi_{l,\kappa}|\psi_{l,\kappa}\rangle$ are given by (3.62), so the task is to compute $\langle\psi_0|n(0)|\psi_{l,\kappa}\rangle$.

One way to do this (see [6] and [7] for alternative approaches) is to use the integration formula (3.57) in conjuction with following lemma [8].

Proposition 4.2. *Suppose $f(z_1,\dots,z_N)$ $(z_j := e^{2\pi i x_j/L})$ is symmetric, homogeneous of order κ and periodic in x_j of period L. We have*

$$\begin{aligned} &\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{-L/2}^{L/2} dx_1 \cdots \int_{-L/2}^{L/2} dx_N \prod_{l=1}^N z_l^\epsilon f(z_1,\dots,z_N) \\ &= \frac{(-1)^\kappa N}{|\kappa|}\int_0^L dx_2 \cdots \int_0^L dx_N f(1,z_2,\dots,z_N) \end{aligned}$$

Proof. This follows by first making the expansion

$$\prod_{l=1}^N z_l^\epsilon = 1 + \frac{2\pi i\epsilon}{L}\sum_{l=1}^N x_l + O(\epsilon^2)$$

in the integrand, and using the symmetry of the integrand to replace $\sum_{l=1}^N x_l$ by Nx_1. Next the periodicity in $x_2, \dots, x_N$ is used to change variables $z_j \mapsto z_1 z_j$ ($j = 2, \dots, N$), which allows the dependence on z_1 to be factored since f is homogeneous. Integrating over z_1 gives the required result. □

Now

$$\begin{aligned}\langle \psi_0 | n(0) | \psi_{l,\kappa} \rangle &= NL^{N-1} \prod_{l=1}^N \int_0^1 d\theta_l \, |1 - e^{2\pi i \theta_l}|^\beta e^{-2\pi i l \theta_l} \\ &\quad \times \prod_{2 \le j < k \le N} |e^{2\pi i \theta_k} - e^{2\pi i \theta_j}|^\beta J_\kappa^{(\beta/2)}(1, e^{2\pi i \theta_2}, \dots, e^{2\pi i \theta_N})\end{aligned}$$

so applying Proposition 4.2 to (3.57) with $b = -a$, $a = -l + \epsilon$ gives

$$\langle \psi_0 | n(0) | \psi_{l,\kappa} \rangle = L^{N-1} |\kappa - l| J_\kappa^{(\beta/2)}(1^N) \frac{\lim_{\epsilon \to 0} \frac{1}{\epsilon} [-l + \epsilon]_\kappa^{(\beta/2)}}{[\frac{2}{\beta}(N-1) + 1 - l]_\kappa^{(\beta/2)}} \tag{4.14}$$

Consider first the case $l = 0$. Then from (3.32b)

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} [-l + \epsilon]_\kappa^{(\beta/2)} := [0]_\kappa'^{(\beta/2)} = (\beta/2)^{\kappa_1 - 1} (\kappa_1 - 1)! \prod_{j=2}^{l(\kappa)} \Big(-\frac{\beta}{2}(j-1) \Big)_{\kappa_j}. \tag{4.15}$$

Inspection of this expression shows that for rational values of β such that

$$\beta/2 = p/q \tag{4.16}$$

where p and q are relatively prime positive integers, it vanishes whenever the diagram of the partition κ (recall Section 3.2) contains the point (q, p). Thus we have the crucial result that for rational values of the β (4.16), the inner product (4.15) with $l = 0$ is non-zero only if the partition κ is on the form

$$\kappa = (\alpha_1, \dots, \alpha_q, \underbrace{p, \dots, p}_{\beta_1 \, p's}, \dots, \underbrace{1, \dots, 1}_{\beta_p \, 1's}, 0, \dots, 0) \tag{4.17}$$

In the quantum mechanical interpretation of $\rho_{(2)}^T$ as the ground state density-density correlation for the Calogero-Sutherland system on a circle, this result is saying that only states with p particle-like excitations and q hole-like excitations give a non-zero contribution to the matrix element (4.15).

For $l > 0$, inspection of (4.15) shows that it always vanishes unless $\kappa_1 \le l$. Assuming this condition, if we write $\mu_j := l - \kappa_{N-j+1}$ (note that $\mu_1 = l$ and $\mu_N \ge 0$) we have

$$\frac{\langle \psi_0 | n(0) | \psi_{l,\kappa} \rangle}{\langle \psi_{l,\kappa} | \psi_{l,\kappa} \rangle} = \frac{\langle \psi_0 | n(0) | \psi_\mu \rangle}{\langle \psi_\mu | \psi_\mu \rangle}. \tag{4.18}$$

This follows by taking the complex conjugate in the integral defining the inner product in (4.14) and using the fact that

$$\prod_{j=1}^{N} z_j^l J_\kappa^{(\alpha)}\Big(\frac{1}{z_1},\dots,\frac{1}{z_N}\Big)$$

is proportional to $J_\mu^{(\alpha)}(z_1,\dots,z_N)$, which can be verified by checking that both polynomials have the same highest weight monomial and are eigenfunctions of the operator $D_2(\alpha)$.

From (4.18), and the further facts that

$$|\kappa - l| = -|\mu| \qquad \textit{and} \qquad E_{l,\kappa} - E_0 = E_\mu$$

we have that

$$\rho_{(2)}^T((x_a,0),(x_b,\tau)) = \frac{L^{2(N-1)}}{\langle\psi_0|\psi_0\rangle}\sum_\kappa \frac{|\kappa|^2}{\langle\psi_\kappa|\psi_\kappa\rangle}\Big(\frac{J_\kappa^{(\beta/2)}(1^N)[0]_\kappa'^{(\beta/2)}}{[\frac{2}{\beta}(N-1)+1]_\kappa^{(\beta/2)}}\Big)^2$$
$$\times e^{-\tau(E_{l,\kappa}-E_0)/\beta} 2\cos(2\pi(x_b-x_a)|\kappa|/L) \tag{4.19}$$

where the inner products giving the normalizations are specified by (3.62). Moreover, in the case of rational β (4.16), the summation over the partitions κ can be restricted to partitions of the form (4.17).

The next task is to take the thermodynamic limit in (4.19). This is relatedious but presents no essential problems. It is necessary to manipulate the products of gamma functions which make up the summand into a form suitable for the application of Stirling's formula. The partitions (4.17) are regarded as functions of $\alpha_1,\dots,\alpha_q$, and the quantities $\gamma_j := \sum_{j=1}^k \beta_j$ $(k=1,\dots,q)$ which are the first q parts of the partition κ' conjugate to κ. For large N, L one finds that (4.19) is the Riemann approximation to a multidimensional integral in the variables $x_j := \alpha_j/N$ $(x_j \geq 0)$ and $(y_j := \gamma_j/N)$ $(1 \geq y_j \geq 0)$. Taking the thermodynamic limit gives the following result [6].

Proposition 4.3. *For $\beta/2 := \lambda = p/q$ (p and q relatively prime) we have*

$$\lim_{\substack{N,V\to\infty \\ N/V=\rho}} \rho_{(2)}^T((x_a,0),(x_b,\tau)) = C_{p,q}(\lambda)\prod_{i=1}^{q}\int_0^\infty dx_i \prod_{j=1}^{p}\int_0^1 dy_j$$
$$\times Q_{p,q}^2 F(q,p,\lambda|\{x_i,y_j\})\cos Q_{p,q}(x_b-x_a)\exp(-E_{p,q}\tau/2\lambda)$$

where the momentum Q and the energy E variables are given by

$$Q_{p,q} := 2\pi\rho\Big(\sum_{i=1}^{q} x_i + \sum_{j=1}^{p} y_j\Big), \quad E_{p,q} := (2\pi\rho)^2\Big(\sum_{i=1}^{q}\epsilon_P(x_j) + \sum_{j=1}^{p}\epsilon_H(y_j)\Big)$$

with $\epsilon_P(x) = x(x+\lambda)$ *and* $\epsilon_H(y) = \lambda y(1-y)$, *the so called form factor* F *is given by*

$$F(q,p,\lambda|\{x_i,y_j\}) = \prod_{i=1}^{q}\prod_{j=1}^{p}(x_i+\lambda y_j)^{-2}\frac{\prod_{i<i'}|x_i-x_{i'}|^{2\lambda}\prod_{j<j'}|y_j-y_{j'}|^{2/\lambda}}{\prod_{i=1}^{q}(\epsilon_P(x_i))^{1-\lambda}\prod_{j=1}^{p}(\epsilon_H(y_j))^{1-1/\lambda}}$$

and the normalization is given by

$$C_{p,q}(\lambda) = \frac{\lambda^{2p(q-1)}\Gamma^2(p)}{2\pi^2 p!q!}\frac{\Gamma^q(\lambda)\Gamma^p(1/\lambda)}{\prod_{i=1}^{q}\Gamma^2(p-\lambda(i-1))\prod_{j=1}^{p}\Gamma^2(1-(j-1)/\lambda)}.$$

From the viewpoint of condensed matter physics the structure of this integral formula is significant. The p integration variables $\{y_j\}$ are associated with hole-like excitations and the q integration variables $\{x_j\}$ are associated with particle-like excitations. Furthermore the exponent for the coupling between the particle-like excitation variables is twice $\beta/2$ while the coupling between the hole-like excitation variables in F is twice $2/\beta$ (i.e. the reciprocal of $\beta/2$). This feature is significant in the interpretation of the fractional statistics associated with the hole excitations [6]. Also, F is precisely of the functional form obtained by Dotsenko and Fateev [9] for the form factor in their calculation of correlation functions in conformal field theory. We will make use of this observation as we proceed to verify the expected asymptotic behaviour (2.35). But before doing so, note from the equation (2.22) relating the current-current and particle-particle distributions that it is possible to write down from Proposition 4.3 the corresponding current-current distribution. Thus in the thermodynamic limit we must have

$$\lim_{\substack{N,V\to\infty \\ N/V=\rho}} C((0,0),(x,\tau)) = -\Big(\frac{1}{2\lambda}\Big)^2 C_{p,q}(\lambda)$$
$$\times\prod_{i=1}^{q}\int_0^\infty dx_i\prod_{j=1}^{p}\int_0^1 dy_j E_{p,q}^2 F(q,p,\lambda|\{x_i,y_j\})\cos Q_{p,q}x\,\exp(-E_{p,q}\tau/2\lambda\gamma) \tag{4.20}$$

4.5. Asymptotics. To test the prediction (2.35) [10] we require the small-k expansion of the Fourier transform of $\rho_{(2)}^T$ as given in Proposition 4.3. First we note that since $\rho_{(2)}^T$ is even in $x_b - x_a$, the Fourier transform with respect to $x_b - x_a$ will be even in k, so it suffices to consider the case $k > 0$. Taking the Fourier transform then replaces $\cos Q_{p,q}x$ by $\pi\delta(k - Q_{p,q})$ so we have

$$\begin{aligned}\tilde{S}(k,\tau) &= \pi C_{p,q}(\lambda)\prod_{i=1}^{q}\int_0^\infty dx_i\prod_{j=1}^{p}\int_0^1 dy_j Q_{p,q}^2 F(q,p,\lambda|\{x_i,y_j\}) \\ &\quad\times\delta(k-Q_{p,q})\exp(-E_{p,q}\tau/2\lambda).\end{aligned}$$

The change of variables $x_i \mapsto kx_i$, $y_j \mapsto ky_j$ shows that to leading order in k, for $k \to 0$,

$$\tilde{S}(k,\tau) \sim A_{p,q}(\lambda)|k|e^{-|k|\pi\rho\tau} \tag{4.21a}$$

where

$$\begin{aligned} A_{p,q}(\lambda) &= \pi C_{p,q}(\lambda) \prod_{i=1}^{q} \int_0^\infty dx_i \prod_{j=1}^{p} \int_0^\infty dy_j Q_{p,q}^2 \prod_{i=1}^{q}\prod_{j=1}^{p}(x_i+\lambda y_j)^{-2} \\ &\quad \times \frac{\prod_{i<i'}|x_i-x_{i'}|^{2\lambda}\prod_{j<j'}|y_j-y_{j'}|^{2/\lambda}}{\prod_{i=1}^{q}x_i^{1-\lambda}\prod_{j=1}^{p}y_j^{1-1/\lambda}}\delta(1-Q_{p,q}) \end{aligned} \tag{4.21b}$$

We require the evaluation of $A_{p,q}(\lambda)$. First we write

$$\delta(1-Q_{p,q}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} du\, e^{iu}e^{-iQ_{p,q}u},$$

and introduce a convergence factor $e^{-\epsilon Q_{p,q}}$ $(\epsilon > 0)$ into the integrand. Changing variables $x_i \to x_i/(2\pi\rho(\epsilon+iu))$ and $y_j \to y_j/(2\pi\rho(\epsilon+iu))$ shows that

$$\begin{aligned} A_{p,q}(\lambda) &= \frac{1}{2}C_{p,q}(\lambda)\Big(\lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty} du\,\frac{e^{iu}}{(\epsilon+iu)^2}\Big)\prod_{i=1}^{q}\int_0^\infty dx_i\, e^{-x_i}\prod_{j=1}^{p}\int_0^\infty dy_j\, e^{-y_j} \\ &\quad \times\Big(\sum_i x_i+\sum_j y_j\Big)^2\frac{\prod_{i<i'}|x_i-x_{i'}|^{2\lambda}\prod_{j<j'}|y_j-y_{j'}|^{2/\lambda}}{\prod_{i=1}^{q}x_i^{1-\lambda}\prod_{j=1}^{p}y_j^{1-1/\lambda}}. \end{aligned} \tag{4.22}$$

The integral over u equals 2π, while the evaluation of the multiple integral can be obtained as a limiting case of the Dotsenko-Fateev integral [9].

To obtain the desired limiting case, we require that the Dotsenko-Fateev integral be written in the form [8]

$$J_{nm}(\alpha,\beta,\rho) := \prod_{i=1}^{n}\int_1^\infty dt_i \prod_{j=1}^{m}\int_0^1 d\tau_j f_{nm}(\{t_i\},\{\tau_j\},\alpha,\beta;\rho) \tag{4.23a}$$

where

$$\begin{aligned} &f_{nm}(\{t_i\},\{\tau_j\},\alpha,\beta;\rho) \\ &:= \Big|\prod_{i=1}^{n}t_i^{\alpha'}(1-t_i)^{\beta'}\prod_{j=1}^{m}\tau_j^{\alpha}(1-\tau_j)^{\beta}\frac{\prod_{i<i'}(t_i-t_{i'})^{2\rho'}\prod_{j<j'}(\tau_j-\tau_{j'})^{2\rho}}{\prod_{i=1}^{n}\prod_{j=1}^{m}(\tau_j-t_i)^2}\Big|, \end{aligned} \tag{4.23b}$$

and the parameters are subject to the relations

$$\alpha' = -\rho'\alpha \qquad \beta' = -\rho'\beta \qquad \rho' = 1/\rho. \tag{4.23c}$$

For notational symplicity, set

$$D_{n,m}(\rho) = \prod_{l=1}^{n} \frac{\Gamma(l\rho')}{\Gamma(\rho')} \prod_{j=1}^{m} \frac{\Gamma(j\rho - n)}{\Gamma(\rho)}.$$

Note that f_{nm} reproduces the form factor $F_{p,q}$ of Proposition 4.3 by setting $\tau_j = 1-y_j$, $t_i = 1+x_i/\lambda$, $\rho = 1/\lambda$, $\alpha = \beta = 1/\lambda - 1$, $m = p$, $n = q$. The Dotsenko-Fateev integral generalizes the Selberg integral (3.40), and like the Selberg integral has a product of gamma function evaluation:

$$J_{nm}(\alpha,\beta,\rho) = \rho^{2nm} D_{n,m}(\rho) \prod_{l=0}^{n-1} \Gamma_l \prod_{j=0}^{m-1} \frac{\Gamma(1-n+\alpha+j\rho)\Gamma(1-n+\beta+j\rho)}{\Gamma(2-n+\alpha+\beta+(m-1+j)\rho)}, \tag{4.24}$$

where

$$\Gamma_l = \frac{\Gamma(1+\beta'+l\rho')\Gamma(-1+2m-\alpha'-\beta'-(n-1+l)\rho')}{\Gamma(-\alpha'-l\rho')}.$$

The sought limiting form is obtained by making the change of variables $t_i = \exp(s_i/\alpha)$ and $\tau_j = \exp(-\xi_j/\alpha)$, and computing the leading large-α asymptotics (which is a simple exercise). Comparing this with the leading large-α asymptotics of (4.24), which is obtained using Stirling's formula, gives the integral identity [11]

$$\begin{aligned} I_{nm}(z,\beta,\rho) &= \prod_{i=1}^{n} \int_0^\infty dr_i\, r_i^{-\beta/\rho} e^{-zr_i} \prod_{j=1}^{m} \int_0^\infty d\eta_j\, \eta_j^{\beta} e^{-z\eta_j} \\ &\quad \times \frac{|\prod_{i<i'}(r_i - r_{i'})^{2/\rho} \prod_{j<j'}(\eta_{j'} - \eta_j)^{2\rho}|}{\prod_{i=1}^{n}\prod_{j=1}^{m}(\eta_j + \rho r_i)^2} \\ &= m!n! z^{-c_{n\,m}(\beta,\rho)} D_{n,m}(\rho) \prod_{l=0}^{n-1} \Gamma(1-\beta/\rho+l/\rho) \prod_{j=0}^{m-1} \Gamma(1-n+\alpha+j\rho) \end{aligned} \tag{4.25}$$

where

$$c_{n\,m}(\beta,\rho) := m+n+m\beta - n\beta/\rho + m(m-1)\rho + n(n-1)/\rho - 2mn$$

(the parameter z, which can be removed by a simple change of variables, has been included for its utility in evaluating (4.22)). Now comparing the definition of I_{mn} with the multiple integral in (4.22), $\mathcal{A}_{p,q}(\lambda)$ say, shows that

$$\mathcal{A}_{p,q}(\lambda = p/q) = \lim_{\epsilon \to 0} \frac{\partial^2}{\partial z^2} I_{pq}(z, \rho - 1, \rho)\Big|_{z=1} \tag{4.26}$$

where $\rho = (p+\epsilon)/q$. We must introduce the parameter ϵ because, as is seen from the final equality in (4.25), $I_{pq}(z, \lambda - 1, \lambda)$ is singular; only after the differentiations can the required parameter values be set.

Using the final equality in (4.25) to evaluate (4.26) gives

$$\begin{aligned} \mathcal{A}_{p,q}(p/q) &= \frac{q!p!}{(\Gamma(p/q))^q}\frac{q}{p}\prod_{l=1}^{p}\frac{\Gamma^2(ql/p)}{\Gamma(p/q)}\prod_{j=1}^{q-1}\Gamma^2(jp/q-p) \\ &= \frac{1}{2\pi^2\lambda}\frac{1}{C_{p,q}(\lambda)} \end{aligned} \tag{4.27}$$

(to obtain the last equality requires some manipulation of the expression for $C_{p,q}$ in Proposition 4.3). Substituting in (4.22), with the first integral therein replaced by 2π, and then substituting the result in (4.21a) reclaims (2.35).

The term of order k^2 is obtained by changing variables as before in the integral above (4.21a), and expanding the integrand to the next order. For $\tau = 0$ this procedure gives

$$\begin{aligned} &\frac{1}{2\pi\rho}(1-1/\lambda)k^2\pi C_{p,q}(\lambda)\prod_{i=1}^{q}\int_0^\infty dx_i\prod_{j=1}^{p}\int_0^1 dy_j Q_{p,q}^3\prod_{i=1}^{q}\prod_{j=1}^{p}(x_i+\lambda y_j)^{-2} \\ &\times\frac{\prod_{i<i'}|x_i-x_{i'}|^{2\lambda}\prod_{j<j'}|y_j-y_{j'}|^{2/\lambda}}{\prod_{i=1}^{q}x_i^{1-\lambda}\prod_{j=1}^{p}y_j^{1-1/\lambda}}\delta(1-Q_{p,q}) \\ = \;& \frac{1}{2\pi\rho}(1-1/\lambda)k^2A_{p,q}(\lambda) \\ = \;& \frac{1}{2\pi\rho}\Big(\frac{\beta-2}{\beta}\Big)k^2\frac{\pi}{\beta} \end{aligned} \tag{4.28}$$

where to obtain the first equality we have used the fact that the integral with $Q_{p,q}^3$ is the same as the integral with $Q_{p,q}^2$ due to the factor $\delta(1-Q_{p,q})$ in the integrand. The term obtained in (4.28) is in agreement with the prediction (2.36) for $\tau = 0$. For non-zero τ the task is to compute the integral in (4.28) with $Q_{p,q}^3$ replaced by

$$-\tau Q_{p,q}^2\frac{(2\pi\rho)^2}{2\lambda}\Big(\sum_{i=1}^{q}x_i^2-\lambda\sum_{j=1}^{p}y_j^2\Big).$$

No direct method of evaluation is known. However, since this term is linear in τ, its value is constrained by the f-sum rule (2.32) and knowledge of the static term result (4.28) to be as given by (2.36).

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5. The Calogero-Sutherland model with exchange terms

All the Schrödinger operators for the Calogero-Sutherland models introduced in Section 1 can be generalized to include an exchange operator. For the Schrödinger operator (1.4) this generalization reads [1]

$$H^{(C,Ex)} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \beta\Big(\frac{\pi}{L}\Big)^2 \sum_{1\le j<k\le N} \frac{(\beta/2 - M_{jk})}{\sin^2 \pi(x_j - x_k)/L} \tag{5.1}$$

where the operator M_{jk} exchanges coordinates x_j and x_k. Note that when acting on functions symmetric in $x_1, \dots, x_N$ (5.1) reduces to (1.4). The operator (5.1) is fundamental to the construction of the Sekiguchi-Debiard operators of Proposition 3.1, and also leads to a set of polynomials more fundamental than the symmetric Jack polynomials introduced in Section 3. Furthermore, there is an analogy between the absolute value squared of certain eigenfunctions of (5.1) and the Boltzmann factor for a type of log-potential system known as the generalized plasma. The calculation of the correlation functions for (5.1) and the generalized plasma offer future research opportunities.

5.1. Non-symmetric Jack polynomials and the Cherednik operators. Conjugation of (5.1) with $e^{-\beta W^{(C)}/2}$ gives the transformed operator

$$\begin{aligned} \tilde{H}^{(C,Ex)} &:= \Big(\frac{L}{2\pi}\Big)^2 e^{\beta W^{(C)}/2}(H^{(C,Ex)} - E_0^{(C)})e^{-\beta W^{(C)}/2} \\ &= \sum_{j=1}^{N}\Big(z_j\frac{\partial}{\partial z_j}\Big)^2 + \frac{N-1}{\alpha}\sum_{j=1}^{N} z_j\frac{\partial}{\partial z_j} + \frac{2}{\alpha}\sum_{1\le j<k\le N}\frac{z_j z_k}{z_j - z_k} \\ &\quad \times\Big[\Big(\frac{\partial}{\partial z_j} - \frac{\partial}{\partial z_k}\Big) - \frac{1 - M_{jk}}{z_j - z_k}\Big] \end{aligned} \tag{5.2}$$

Direct calculation shows that this operator permits non-symmetric eigenfunctions of the form

$$E_\eta(z,\alpha) = z^\eta + \sum_{\nu<\eta} b_{\nu\eta} z^\nu \tag{5.3}$$

(z^η will be referred to as the leading term), where η and ν are compositions of non-negative integers and the $b_{\nu\eta}$ are coefficients. The ordering $<$ is a partial ordering defined by the statement that $\nu < \eta$ if $\nu^+ < \eta^+$ (the superscript $+$ denotes the partition corresponding to the composition) with dominance ordering, or in the case $\nu^+ = \eta^+$, if the last non-vanishing difference of $\eta - \nu$ is negative.

The eigenfunctions (5.3) are referred to as non-symmetric Jack polynomials, and have corresponding eigenvalue

$$\epsilon_\eta = \sum_{j=1}^{N} \Big((\eta_j^+)^2 + \frac{1}{\alpha}(N+1-2j)\eta_j^+ \Big) \tag{5.4}$$

Since $P\eta$ gives the same η^+ for any permutation P we see that

$$\sum_{P \in S_N} a_P E_{P^{-1}\eta^+}(z,\alpha), \tag{5.5}$$

where S_N denotes the symmetric group on N symbols and the a_P are arbitrary, is also an eigenfunction of (5.2) with eigenvalue (5.4). In particular, choosing the a_P so that the leading term of (5.5) is the monomial symmetric function m_{η^+} we see that the symmetric Jack polynomials can be written in terms of the non-symmetric Jack polynomials in a series of this form.

The operator (5.2) can also be decomposed in terms of more fundamental operators. Now the type A Dunkl operators [2] are defined by

$$T_j := \frac{\partial}{\partial z_j} + \frac{1}{\alpha} \sum_{\substack{k=1 \\ k \neq j}}^{N} \frac{1}{z_j - z_k}(1 - M_{jk}) \tag{5.6}$$

and the Cherednik operators [3] by

$$\begin{aligned} \hat{D}_j &:= z_j T_j + \frac{1}{\alpha} \sum_{k=1}^{j-1} M_{jk} \\ &= z_j \frac{\partial}{\partial z_j} + \frac{1}{\alpha} \Big(\sum_{l<j} \frac{z_l}{z_j - z_l}(1 - M_{lj}) + \sum_{l>j} \frac{z_j}{z_j - z_l}(1 - M_{lj}) \Big) + \frac{(j-1)}{\alpha}, \end{aligned} \tag{5.7}$$

$(j = 1, \ldots, N)$. A direct calculation using the fact that

$$M_{jk} M_{kl} = M_{kl} M_{lj} = M_{lj} M_{jk}$$

and the identity (1.7) shows that $[T_j, T_k] = 0$, and this in turn implies that $\{\hat{D}_j\}$ commute. A further direct calculation shows that the Cherednik operators are related to the operator (5.2) by [4,5]

$$\tilde{H}^{(C,Ex)} = \sum_{j=1}^{N} \Big(\hat{D}_j - \frac{N-1}{2\alpha} \Big)^2 - \Big(\frac{L}{2\pi} \Big)^2 E_0^{(C)}. \tag{5.8}$$

A direct calculation also shows that each $\hat{D}_j$ has a unique eigenfunction of the form of the r.h.s. of (5.3) with eigenvalue

$$e_{j,\eta} = \eta_j + \frac{1}{\alpha}\Big(-\sum_{l<j} h(\eta_l - \eta_j) + \sum_{l>j} h(\eta_j - \eta_l)\Big) + \frac{(j-1)}{\alpha} \tag{5.9}$$

where

$$h(x) = \begin{cases} 1, & x > 0 \\ 0, & \textit{otherwise.} \end{cases}$$

From the fact that $\{\hat{D}_j\}$ commute it follows that the same eigenfunction with leading term z^η is simultaneously an eigenfunction of each $\hat{D}_j$ $(j = 1, \dots, N)$, and the decomposition (5.8) implies that this eigenfunction is precisely the non-symmetric Jack polynomial.

The decomposition (5.8) can also be used to construct the Sekiguchi-Debiard operators. First, we note from (5.9) that for a composition η' related to η by interchanging η_i and $\eta_{i'}$ the corresponding eigenvalues of $\{\hat{D}_j\}$ are related by $e_{i,\eta'} = e_{i',\eta}$, $e_{i',\eta'} = e_{i,\eta}$ and $e_{j,\eta'} = e_{j,\eta}$ $(j \neq i', i)$. This says that $\{e_{j,\eta}\}_{j=1,\dots,N}$ with $\eta = P^{-1}\eta^+$ is independent of the permutation P. Choosing the permutation $P(j) = N + 1 - j$ $(j = 1, \dots, N)$ shows that $\{e_{j,\eta}\}_{j=1,\dots,N} = \{\eta^+_{N+1-j} + (j-1)/\alpha\}_{j=1,\dots,N}$. This allows an eigenoperator of the $E_{P^{-1}\eta^+}$ to be constructed for which the eigenvalues are independent of P:

$$\Big(1 + u(\hat{D}_1 - (N-1)/2\alpha)\Big) \dots \Big(1 + u(\hat{D}_N - (N-1)/2\alpha)\Big) E_{P^{-1}\eta^+}$$
$$= \prod_{j=1}^{N} \Big(1 + u(\eta^+_j + (N+1-2j)/2\alpha)\Big) E_{P^{-1}\eta^+} \tag{5.10}$$

(the constants $-(N-1)/2\alpha$ are not essential and could have been omitted). Since the symmetric Jack polynomial $J^{(\alpha)}_{\eta^+}$ can be written in the form (5.5) it follows immediately that $J^{(\alpha)}_{\eta^+}$ satisfies the eigenvalue equation (5.10). Comparison with Proposition 3.1 and the fact that $\{J^{(\alpha)}_{\eta^+}\}$ form a basis for symmetric functions then shows that when acting on symmetric functions

$$\prod_{j=1}^{N} (X + \alpha\hat{D}_j) = D_N(X; \alpha). \tag{5.11}$$

The essential point is that the operators given by the coefficient of X^p on the l.h.s. of (5.11) can be systematically constructed. They separate the eigenvalues of $J^{(\alpha)}_{\eta^+}$ and are self adjoint with respect to the inner product (3.11b) (indeed a simple direct calculation shows that each D_j is self adjoint with respect to (3.11b)), and so provide a natural setting for the Sekiguchi-Debiard operators.

5.2. Jack polynomials with prescribed symmetry. Jack polynomials with prescribed symmetry have been introduced in ref. [6]. These polynomials, to be denoted $S_\eta(z,\alpha)$ are linear combinations of the form (5.5) of the non-symmetric Jack polynomials, and are thus eigenfunctions of (5.2) with eigenvalue (5.4). The coefficients a_P in (5.5) are chosen so that $S_\eta(z,\alpha)$ is symmetric with respect to the interchange of certain sets of variables, and antisymmetric with respect to the interchange of other sets of variables.

To be more precise, let us rewrite the coordinates $\{z_j\}_{j=1,\dots,N}$ as

$$\Big(\bigcup_{\alpha=1}^{q}\{w_j^{(\alpha)}\}_{j=1,\dots,N_\alpha^{(w)}}\Big)\bigcup\Big(\bigcup_{\gamma=1}^{p}\{z_j^{(\gamma)}\}_{j=1,\dots,N_\gamma^{(z)}}\Big)$$

taken in order so that $w_1^{(1)}=z_1,\dots,z_{N_p}^{(p)}=z_N$ and $N=\sum_{\mu=1}^{q}N_\mu^{(w)}+\sum_{\gamma=1}^{p}N_\gamma^{(z)}$. The polynomials $S_\eta(z,\alpha)$ are the linear combination (5.5) which is symmetric in $\{w_j^{(\alpha)}\}_{j=1,\dots,N_\alpha^{(w)}}$ and antisymmetric in $\{z_j^{(\gamma)}\}_{j=1,\dots,N_\gamma^{(z)}}$. They can be written as

$$S_\eta(z,\alpha)=\mathcal{O}\Big(E_{P^{-1}\kappa}(z,\alpha)\Big) \tag{5.12}$$

where $\mathcal{O}$ denotes the operation of symmetrization in $\{w_j^{(\alpha)}\}_{j=1,\dots,N_\alpha^{(w)}}$, antisymmetrization in $\{z_j^{(\gamma)}\}_{j=1,\dots,N_\gamma^{(z)}}$ and normalization such that the coefficient of z^η is unity. Due to the operation $\mathcal{O}$ the label η in S_η can be replaced by $q+p$ partitions $(\rho,\mu):=(\rho^{(1)},\dots\rho^{(q)},\mu^{(1)},\dots,\mu^{(p)})$ where $\rho^{(\alpha)}$ consists of $N_\alpha^{(w)}$ parts $(\alpha=1,\dots,q)$ and $\mu^{(\gamma)}$ consists of $N_\gamma^{(z)}$ parts $(\alpha=1,\dots,q)$ and $\mu^{(\gamma)}$ consists of $N_\gamma^{(z)}$ parts. For the composition η any rearrangements of

$$\{\eta_j\}_{j=1,\dots,N_1^{(w)}},\ \{\eta_{N_1^{(w)}+j}\}_{j=1,\dots,N_2^{(w)}},\dots,\{\eta_{N^{(w)}+\sum_{\gamma=1}^{p-1}N_\gamma^{(z)}+j}\}_{j=1,\dots,N_p^{(z)}}, \tag{5.13}$$

where $N^{(w)}:=\sum_{\alpha=1}^{q}N_\alpha^{(w)}$, give the same partitions (ρ,μ) and thus the same polynomial with prescribed symmetry.

To relate the formula (5.12) to the linear combination (5.5) we know from ref. [7] that with $s_i:=M_{i\,i+1}$ and $\delta_i:=\bar\eta_i-\bar\eta_{i+1}$, where

$$\bar\eta_j=\alpha\eta_j-\Big(\sum_{l<j}h(\eta_l+1-\eta_j)+\sum_{l>j}h(\eta_l-\eta_j)\Big),$$

we have

$$s_iE_\eta=\begin{cases}\frac{1}{\delta_i}E_\eta+(1-\frac{1}{\delta_i^2})E_{s_i\eta}, & \eta_i>\eta_{i+1}\\ E_\eta, & \eta_i=\eta_{i+1}\\ \frac{1}{\delta_i}E_\eta+E_{s_i\eta}, & \eta_i<\eta_{i+1}\end{cases}$$

(in this equation the E_η are the eigenfunctions of the $\hat D_j$ with the arguments $z_1,\dots,z_N$ reversed). Since each permutation which makes up the operation $\mathcal{O}$ can

be written as a product of elementary transpositions s_i, we conclude that

$$S_\eta(z,\alpha) = \sum_{rearrangements} b_\nu E_\nu(z,\alpha) \tag{5.14}$$

where the sum is over rearrangements ν of η obtained by permuting within the sets (5.13), and the b_ν are some (unknown) coefficients.

Now two distinct sequences of partitions (ρ,μ) and $(\hat\rho,\hat\mu)$ as defined below (5.12) cannot have any rearrangements of (5.13) in common, as they wouldn't then be distinct. Hence the expansion (5.14) for distinct $S_{(\rho,\mu)}$ and $S_{(\hat\rho,\hat\mu)}$ does not contain any common E_η. It follows immediately from the orthogonality of $\{E_\eta\}$ with respect to (3.11b) that $\{S_{(\rho,\mu)}\}$ are also orthogonal with respect to (3.11b).

5.3. The generalized plasma. Consider the $q = p = 1$ case in (5.13), so there is only one symmetric and one antisymmetric component. Any antisymmetric polynomial in $z_1^{(1)},\dots,z_{N_1^{(z)}}^{(1)}$ can be written in the form

$$\Delta(z_1^{(1)},\dots,z_{N_1^{(z)}}^{(1)})\, p((z_1^{(1)},\dots,z_{N_1^{(z)}}^{(1)})$$

where $\Delta(z_1^{(1)},\dots,z_{N_1^{(z)}}^{(1)}) = \prod_{1\le j<k\le N_1^{(z)}}(z_k^{(1)} - z_j^{(1)})$ and p is symmetric in $\{z_j^{(1)}\}$. It follows that the lowest degree antisymmetric polynomial is $\Delta(z_1^{(1)},\dots,z_{N_1^{(z)}}^{(1)})$, and that the corresponding ground state of (5.1) with this symmetry has absolute value squared proportional to

$$\prod_{1\le j<k\le N} |z_k - z_j|^\beta \, |\Delta(z_1^{(1)},\dots,z_{N_1^{(z)}}^{(1)})|^2 \tag{5.15}$$

(recall that $z_{N_0+j} := z_j^{(1)}$, $j = 1,\dots,N_1^{(z)}$).

Now a general two-component classical log-gas system with complex exponential coordinates $\{w_1,\dots,w_{N_0}\}$ and $\{z_1,\dots,z_{N_1}$ for the respective components, and confined to a line with periodic boundary conditions has potential energy

$$U = -g_0 \sum_{1\le j<k\le N_0} \log|w_j - w_k| - g_1 \sum_{1\le\alpha<\gamma\le N_1} \log|z_\alpha - z_\gamma| - g_{01} \sum_{j=1}^{N_0}\sum_{\alpha=1}^{N_1} \log|w_j - z_\alpha| \tag{5.16}$$

(there may also be an additive constant in (5.16)). In the case

$$g_0 g_1 = (g_{01})^2 \tag{5.17}$$

(5.16) is the potential energy for a plasma system obeying the laws of two dimensional electrostatics. For general couplings the system log-potential system with potential energy (5.16) is referred to as generalized plasma [8]. We see that for

$$g_0 = g_{01} = 1,\ g_1 = \frac{\beta+2}{\beta},$$

which since (5.17) is not obeyed is an example of a generalized plasma, the Boltzmann factor $e^{-\beta U}$ is identical to (5.15).

An open problem is to develop the theory of Jack polynomials with prescribed symmetry to a stage which allows the correlation functions for this two-component log-gas system to be computed. More generally, one would like to use the theory of Jack polynomials with prescribed symmetry to compute the ground state dynamical correlations of the Calogero-Sutherland model with exchange terms, restricted to states with a prescribed symmetry.

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