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# Weierstrass-type representations for timelike surfaces

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**Abstract.**

In this paper we give the Weierstrass-type representation for Lorentz conformal minimal surfaces in Minkowski 3-space that was derived by Konderak, and a new one for Lorentz conformal constant mean curvature 1 surfaces in anti de Sitter 3-space, using integrable systems techniques. As an application, we analyze their singularities. Finally, we describe first steps toward discretization of these timelike surfaces.

## §1. Introduction

The study of zero mean curvature (ZMC, for short) surfaces in Minkowski 3-space $\mathbb{R}^{2,1}$ is one recent topic of research on the differential geometry of surfaces. Kobayashi [11] derived a Weierstrass-type representation for conformal immersions with mean curvature identically 0 in $\mathbb{R}^{2,1}$, called conformal maximal surfaces. Magid [16] derived a Weierstrass-type representation for timelike immersions with mean curvature identically 0 and null coordinate systems, which are called *timelike minimal surfaces* in $\mathbb{R}^{2,1}$, and Inoguchi, Toda [9] derived its normalized version using a loop groups formulation, which is closely related to integrable systems techniques. Unlike the case of minimal surfaces in Euclidean 3-space $\mathbb{R}^3$, ZMC surfaces in $\mathbb{R}^{2,1}$ have certain singularities (see [6], [7], [18], [20] for example). Very recently, Akamine [1]

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analyzed behaviors of two null curves generating a timelike minimal surface and its Gaussian curvature near singular points.

Stepping away from ZMC surfaces in $\mathbb{R}^{2,1}$, here we also see timelike constant mean curvature (CMC, for short) surfaces in other Lorentzian spaceforms. As mentioned in [14], there exists a correspondence between timelike CMC surfaces in different Lorentzian spaceforms, which is called the Lawson-type correspondence. In particular, there exists a Lawson-type correspondence between timelike minimal surfaces in $\mathbb{R}^{2,1}$ and timelike CMC 1 surfaces in anti de Sitter 3-space $\mathbb{H}^{2,1}$. So, like in the case of conformal CMC 1 surfaces in $\mathbb{H}^3$ (see [3], [19] for example), it is natural to expect that there exists a Weierstrass-type representation for timelike CMC 1 surfaces in $\mathbb{H}^{2,1}$. In fact, Lee [14] derived the Weierstrass-type representation for timelike CMC 1 surfaces in $\mathbb{H}^{2,1}$ parametrized by null coordinate systems.

In the "smooth" (or, continuous) case, we can reparametrize surfaces. In particular, we can reparametrize timelike surfaces with null coordinate systems to timelike surfaces with Lorentz conformal coordinate systems (see [9] for example). However, as typified by the work in [2], it is generally difficult to reparametrize discrete surfaces (cf. [8]). For this reason, when discretizing surfaces, we would like to find suitable coordinate systems that are compatible with discretization. In [21], we describe discrete surfaces with Lorentz conformal curvature line coordinate systems in $\mathbb{R}^{2,1}$ called *discrete timelike isothermic surfaces*. In the case of isothermic surfaces, away from umbilic points, CMC (possibly, zero mean curvature) surfaces can be reparametrized to be isothermic. In contrast to that, timelike minimal and CMC surfaces are not necessarily timelike isothermic. As mentioned in [15], there are three kinds of Lorentz isothermic surfaces, so discretizing the other cases is a remaining problem.

In this paper, as preparation for [21], we derive a Weierstrass-type representation for timelike minimal surfaces in $\mathbb{R}^{2,1}$ parametrized by Lorentz conformal coordinate systems (Lorentz conformal minimal surfaces in $\mathbb{R}^{2,1}$, for short) via integrable systems techniques. Using paracomplex analysis introduced in [10], we derive the Lax pair for Lorentz conformal CMC surfaces in $\mathbb{R}^{2,1}$ and $\mathbb{H}^{2,1}$ in Propositions 3.1, 5.1. As an application, we obtain the Weierstrass-type representations for Lorentz conformal minimal surfaces in $\mathbb{R}^{2,1}$ (see Theorem 1), which was derived by Konderak [13], and for Lorentz conformal CMC 1 surfaces in $\mathbb{H}^{2,1}$ (see Theorem 2). Furthermore, we analyze the singularities of these surfaces. In the case of Lorentz conformal minimal surfaces in $\mathbb{R}^{2,1}$, Takahashi [18] gave explicit criteria for cuspidal edge singularities, swallowtail singularities and cuspidal cross cap singularities to appear (see Theorem 3 here).

Here we introduce an alternative proof of this. In addition, we give explicit criteria for the same types of singularities to appear on Lorentz conformal CMC 1 surfaces in $\mathbb{H}^{2,1}$, in Theorem 4. In the last appendix, we briefly introduce discrete timelike isothermic surfaces in $\mathbb{R}^{2,1}$. In particular, we introduce the Weierstrass-type representation for discrete timelike isothermic minimal surfaces in $\mathbb{R}^{2,1}$.

## §2. Lorentz conformal CMC surfaces in $\mathbb{R}^{2,1}$

First we consider Lorentz conformal CMC surfaces in $\mathbb{R}^{2,1}$ using Lax pairs. In this section we introduce $3\times 3$ Lax pairs for Lorentz conformal CMC surfaces in $\mathbb{R}^{2,1}$ and derive the compatibility condition for Lorentz conformal CMC surfaces in $\mathbb{R}^{2,1}$. Let

$$\mathbb{R}^{n,1} := \big(\{x = (x_1, \cdots, x_n, x_0)^t; x_i \in \mathbb{R}\}, \langle\cdot,\cdot\rangle\big)$$

be the $(n+1)$-dimensional Minkowski space with Lorentz metric $\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n - x_0y_0$, and let $\mathbb{C}'$ be the set of para-complex numbers, that is,

$$\mathbb{C}' := \{a + j'b; a, b \in \mathbb{R},\ j'\text{: non-real s.t. } (j')^2 = +1\},$$

where $j'$ is called the *para-complex imaginary unit*. Then $\mathbb{C}'$ can be identified with the Minkowski plane $\mathbb{R}^{1,1}$ by the identification

$$\begin{array}{ccc} \mathbb{R}^{1,1} & \longrightarrow & \mathbb{C}' \\ \cup & & \cup \\ (x,y) & \longmapsto & x + j'y \end{array}.$$

Let $g : \Sigma(\subset \mathbb{C}') \to \mathbb{C}'$ be a map. Take $z = x + j'y \in \Sigma$ and set $\partial_z := \frac{1}{2}\left(\partial_x + j'\partial_y\right),\ \partial_{\bar z} := \frac{1}{2}\left(\partial_x - j'\partial_y\right)$, then $g$ is a *p-holomorphic function* if $g_{\bar z} = \partial_{\bar z} g = 0$ holds, that is, the Cauchy-Riemann type equation holds (see Theorem V.1 in [10]). In this paper we abbreviate $|z|_*^2 := z\bar z$ for $z \in \mathbb{C}'$.

Let $f : \Sigma \subset \mathbb{C}' \to \mathbb{R}^{2,1}$ be a Lorentz conformal immersion satisfying

$$\langle f_z, f_z\rangle = \langle f_{\bar z}, f_{\bar z}\rangle = 0,\ \langle f_z, f_{\bar z}\rangle = 2\mathrm{e}^{2u}$$

and $N : \Sigma \to \mathbb{S}^{1,1}$ its spacelike unit normal vector field, where

$$\mathbb{S}^{1,1} := \{X \in \mathbb{R}^{2,1}; \langle X, X\rangle = 1\},$$

and $u : \Sigma \to \mathbb{R}$ is a real-valued function. In the above equation, $\langle\cdot,\cdot\rangle$ denotes the split-complex bilinear extension of the usual $\mathbb{R}^{2,1}$ Lorentz inner product (which is no longer an actual Lorentz inner product). The mean curvature $H$ of $f$ is $H = \frac{1}{2}\mathrm{e}^{-2u}\langle f_{z\bar{z}}, N\rangle$. Setting $Q := \langle f_{zz}, N\rangle$, we have

$$\begin{aligned}
&\langle f_z, f_{zz}\rangle = \langle f_z, f_{z\bar{z}}\rangle = \langle f_{\bar{z}}, f_{z\bar{z}}\rangle = \langle f_{\bar{z}}, f_{\bar{z}\bar{z}}\rangle = \langle N, N_z\rangle = \langle N, N_{\bar{z}}\rangle = 0,\\
&\langle f_{\bar{z}}, f_{zz}\rangle = 4u_z\mathrm{e}^{2u},\ \langle f_z, f_{\bar{z}\bar{z}}\rangle = 4u_{\bar{z}}\mathrm{e}^{2u},\ \langle N_z, f_z\rangle = -Q,\\
&\langle N_{\bar{z}}, f_{\bar{z}}\rangle = -\bar{Q},\ \langle N, f_{z\bar{z}}\rangle = -\langle N_z, f_{\bar{z}}\rangle = -\langle N_{\bar{z}}, f_z\rangle = 2H\mathrm{e}^{2u},
\end{aligned}$$

where $Q$ is called the (coefficient of the) Hopf differential of $f$. The Hopf differential will play an important role in distinguishing the three types of Lorentz conformal immersions in Appendix A. Then we have the following Gauss-Weingarten type equations:

$$\begin{aligned}
&f_{zz} = 2u_z f_z + QN,\ f_{z\bar{z}} = 2H\mathrm{e}^{2u}N,\\
&f_{\bar{z}\bar{z}} = 2u_{\bar{z}} f_{\bar{z}} + \bar{Q}N,\ N_z = -\frac{1}{2}(2Hf_z + Q\mathrm{e}^{-2u}f_{\bar{z}}),\\
&N_{\bar{z}} = -\frac{1}{2}(2Hf_{\bar{z}} + \bar{Q}\mathrm{e}^{-2u}f_z).
\end{aligned}$$

Here we define $e_1 := (f_z + f_{\bar{z}})/(2\mathrm{e}^u)$, $e_2 := j'(f_z - f_{\bar{z}})/(2\mathrm{e}^u)$ and then $\mathcal{F} := (N, e_1, e_2)$ is an orthogonal frame of the surface satisfying

$$\begin{aligned}
&\mathcal{F}_z = \mathcal{F}(\Theta + \Upsilon_z),\quad \mathcal{F}_{\bar{z}} = \mathcal{F}(\bar{\Theta} + \Upsilon_{\bar{z}})\quad \text{with}\\
&\Theta = \begin{pmatrix} 0 & \alpha & \beta\\ -\alpha & 0 & 0\\ \beta & 0 & 0\end{pmatrix},\ \Upsilon = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & j'u\\ 0 & j'u & 0\end{pmatrix},\ \text{where}\\
&\alpha = \frac{1}{2}(Q\mathrm{e}^{-u} + 2H\mathrm{e}^u),\ \beta = \frac{j'}{2}(Q\mathrm{e}^{-u} - 2H\mathrm{e}^u).
\end{aligned}$$

The compatibility condition $\mathcal{F}_{z\bar{z}} = \mathcal{F}_{\bar{z}z}$ gives

$$\begin{aligned}
4u_{z\bar{z}} + 4H^2\mathrm{e}^{2u} - \mathrm{e}^{-2u}Q\bar{Q} = 0 &\quad \text{(Gauss)},\\
Q_{\bar{z}} = 2H_z\mathrm{e}^{2u} &\quad \text{(Codazzi)}.
\end{aligned}$$

The Codazzi equation implies that $Q$ is $p$-holomorphic when $H$ is constant. Henceforth, we assume that $H$ is constant.

## §3. The 2 × 2 Lax pair for timelike CMC surfaces

Here we identify each element in $\mathbb{R}^{2,1}$ as follows:

$$\begin{array}{ccc}
\mathbb{R}^{2,1} & \longrightarrow & \mathfrak{su}'_2\\
\cup & & \cup\\
(x_1, x_2, x_0) & \longmapsto & \begin{pmatrix} j'x_1 & -j'x_2 - x_0\\ -j'x_2 + x_0 & -j'x_1\end{pmatrix}
\end{array},\tag{1}$$

where $\mathfrak{su}'_2$ is the Lie algebra of the Lie group

$$\mathrm{SU}'_2 := \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}; a, b \in \mathbb{C}',\ |a|_*^2 + |b|_*^2 = 1 \right\}.$$

Then the metric becomes $\langle X, Y \rangle = \dfrac{1}{2}\mathrm{trace}(XY)$, with $X$ and $Y$ identified with the matrix forms above. So the metric does not change under the adjoint action $X \mapsto FXF^{-1}$, $Y \mapsto FYF^{-1}$. Now, the adjoint action $F \begin{pmatrix} j'x_1 & -j'x_2 - x_0 \\ -j'x_2 + x_0 & -j'x_1 \end{pmatrix} F^{-1} \left( F = \begin{pmatrix} a + j'b & c + j'd \\ -c + j'd & a - j'b \end{pmatrix} \in \mathrm{SU}'_2 \right)$ induces the associated linear map

$$\begin{aligned} &(x_1, x_2, x_0)^t \mapsto A(x_1, x_2, x_0)^t, \text{ where} \\ &A = \begin{pmatrix} a^2 - b^2 - c^2 + d^2 & -2ac + 2bd & -2bc + 2ad \\ 2ac + 2bd & a^2 + b^2 - c^2 - d^2 & 2ab + 2cd \\ 2bc + 2ad & 2ab - 2cd & a^2 + b^2 + c^2 + d^2 \end{pmatrix} \\ &\in \mathrm{SO}^+_{2,1} := \left\{ M = (a_{ij}) \in \mathrm{SL}_3\mathbb{R}; MI'M^t = I', a_{33} > 0 \right\}, \end{aligned}$$

with $I' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Let $F \in \mathrm{SU}'_2$ be the matrix satisfying

$$e_1 = F \begin{pmatrix} 0 & -j' \\ -j' & 0 \end{pmatrix} F^{-1},\ e_2 = F \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} F^{-1},\ N = F \begin{pmatrix} j' & 0 \\ 0 & -j' \end{pmatrix} F^{-1}.$$

Then,

$$f_z = -2j'\mathrm{e}^u F \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} F^{-1},\ f_{\bar{z}} = -2j'\mathrm{e}^u F \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} F^{-1}.$$

Set $U := F^{-1}F_z = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$, $V := F^{-1}F_{\bar{z}} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$. Our task here is to compute $U$ and $V$ explicitly. The compatibility condition $f_{z\bar{z}} = f_{\bar{z}z}$ gives $U_{12} = -V_{21},\ u_z + U_{22} - U_{11} = 0,\ u_{\bar{z}} + V_{11} - V_{22} = 0$. And the condition $f_{z\bar{z}} = 2H\mathrm{e}^{2u}N$ implies $U_{12} = -V_{21} = -H\mathrm{e}^u$. By conditions $f_{zz} = 2u_z f_z + QN,\ f_{\bar{z}\bar{z}} = 2u_{\bar{z}} f_{\bar{z}} + \bar{Q}N$, we have $U_{21} = \frac{1}{2}Q\mathrm{e}^{-u},\ V_{12} = -\frac{1}{2}\bar{Q}\mathrm{e}^{-u}$. Finally, by the trace-free conditions of $U$ and $V$, we conclude the following proposition, which is called the $2 \times 2$ Lax pair for timelike CMC surfaces:

**Proposition 3.1.** *For some* $F_{z_0 \in \Sigma} \in \mathrm{SU}'_2$, *we have the solution* $F \in \mathrm{SU}'_2$ *by solving* $F_z = FU,\ F_{\bar{z}} = FV$, *where* $F_z = FU,\ F_{\bar{z}} = FV$, *and*

$$U = \frac{1}{2}\begin{pmatrix} u_z & -2H\mathrm{e}^u \\ Q\mathrm{e}^{-u} & -u_z \end{pmatrix}, \quad V = \frac{1}{2}\begin{pmatrix} -u_{\bar{z}} & -\bar{Q}\mathrm{e}^{-u} \\ 2H\mathrm{e}^u & u_{\bar{z}} \end{pmatrix}.$$

## §4. Weierstrass-type representation for Lorentz conformal minimal surfaces in $\mathbb{R}^{2,1}$

Applying the Lax pair for timelike CMC surfaces, we will derive a Weierstrass-type representation for Lorentz conformal minimal surfaces in $\mathbb{R}^{2,1}$. Defining functions $a,\ b : \Sigma \to \mathbb{C}'$ such that

$$F = \mathrm{e}^{-\frac{u}{2}} \begin{pmatrix} a & \bar{b} \\ -b & \bar{a} \end{pmatrix} \quad (a\bar{a} + b\bar{b} = \mathrm{e}^{u}),$$

one can compute that

$$F_{\bar{z}} = \frac{\mathrm{e}^{-u/2}}{2} \begin{pmatrix} -u_{\bar{z}}a + 2a_{\bar{z}} & -u_{\bar{z}}\bar{b} + 2\bar{b}_{\bar{z}} \\ u_{\bar{z}}b - 2b_{\bar{z}} & -u_{\bar{z}}\bar{a} + 2\bar{a}_{\bar{z}} \end{pmatrix}.$$

On the other hand, by Proposition 3.1, we have

$$FV = \frac{\mathrm{e}^{-u/2}}{2} \begin{pmatrix} -u_{\bar{z}} & j'\bar{Q}\mathrm{e}^{-u}a + u_{\bar{z}}\bar{b} \\ u_{\bar{z}}b & -j'\bar{Q}\mathrm{e}^{-u}b + u_{\bar{z}}\bar{a} \end{pmatrix}.$$

So, $a_{\bar{z}} = b_{\bar{z}} = 0$. Thus, $a$ and $b$ are $p$-holomorphic functions. Now,

$$f_z = -2j'\mathrm{e}^{u}F \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} F^{-1} = -2j' \begin{pmatrix} ab & a^2 \\ -b^2 & -ab \end{pmatrix}.$$

Then we have $f_z = (-2ab, a^2 - b^2, j'(a^2 + b^2))^t$ in the standard $\mathbb{R}^{2,1}$ coordinate via the identification (1). Here we introduce the following lemma, which immediately follows from the Cauchy-Riemann type equation (compare with Remark 3.4.2 in [5]).

**Lemma 4.1.** *For any real-valued* $\phi : \Sigma \to \mathbb{R}$ *and p-holomorphic function* $\Psi : \Sigma \to \mathbb{C}'$,

$$\psi_z = \Psi_z \Leftrightarrow \psi = \mathrm{Re}\Psi + c \quad (c:\ \textit{constant}).$$

Using Lemma 4.1, we have $\mathrm{Re} \int f_z dz = \frac{1}{2} f + \overrightarrow{c}$ for some constant $\overrightarrow{c} \in \mathbb{R}^{2,1}$. Then,

$$\begin{aligned} f &= 2\mathrm{Re} \int (-2ab, a^2 - b^2, j'(a^2 + b^2))^t dz \\ &= \mathrm{Re} \int (2g, 1 - g^2, -j'(1 + g^2))^t \omega, \end{aligned}$$

where $g = a/b$ and $\omega = -2b^2 dz = \hat{\omega} dz$ ($\hat{\omega} := -2b^2$). Note that, by holomorphicity of $a$ and $b$, $g$ is a $p$-meromorphic function and $\omega$ is a $p$-holomorphic 1-form. Thus we have the following theorem, which was already described by Konderak [13]:

**Theorem 1.** *Let $(g, \omega)$ be a pair consisting of a p-meromorphic function and a p-holomorphic 1-form. Then a Lorentz conformal minimal surface $f$ in $\mathbb{R}^{2,1}$ can be locally constructed by*

$$f = \mathrm{Re} \int (2g, 1-g^2, -j'(1+g^2))^t \omega \tag{2}$$

*with metric $(1+g\bar{g})\omega\bar{\omega}$. Conversely, any Lorentz conformal minimal surface $f$ in $\mathbb{R}^{2,1}$ is locally described in this manner.*

**Remark.** We have three remarks here:

- The coefficient $\hat{\omega}$ of $\omega$ is a map from $\Sigma$ to $\{X \in \mathbb{C}'; |X|_*^2 > 0,\ \mathrm{Re}X < 0\}$. On the other hand, even when $\hat{\omega}$ takes values in $\{X \in \mathbb{C}'; |X|_*^2 > 0,\ \mathrm{Re}X > 0\}$, $f$ is still Lorentz conformal minimal.
- When $\hat{\omega}$ takes values in $\{X \in \mathbb{C}'; |X|_*^2 < 0\}$, $f$ is still Lorentz conformal minimal, but causalities of $f_x$ and $f_y$ switch. This can be interpreted as follows: When $|\hat{\omega}|_*^2 < 0$, we set $\tilde{g} := j'g$ and $\tilde{\omega} := j'\omega$. Then Equation (2) can be rewritten as

$$f = \mathrm{Re} \int \left(2\tilde{g}, j'(1-\tilde{g}^2), -1-\tilde{g}^2\right)^t \tilde{\omega}. \tag{3}$$

  The metric of $f$ is $-(1-|\tilde{g}|_*^2)^2|\tilde{\omega}|_*^2$. That form is the same as the Weierstrass-type representation for Lorentz conformal minimal surfaces in $\mathbb{R}^{2,1}$ derived in [18]. Also in our approach, if we assume that $\langle f_z, f_{\bar{z}} \rangle = -2\mathrm{e}^{2u}$ in Section 2, we have the same form as in Equation (3).
- Let $f$ be a Lorentz conformal minimal surface given as in Theorem 1. Then a surface $f^\sharp := \mathrm{Im} \int (2g, j'(1-g^2), -1-g^2)^t \omega$ is also a Lorentz timelike minimal surface with metric $-(1+g\bar{g})\omega\bar{\omega}$. The $f^\sharp$ given in this way is called the *conjugate* Lorentz conformal minimal surface of $f$.

Here we see the Gauss map $N$ of a Lorentz conformal minimal surface $f$ described by Equation (2). $N$ is described as follows:

$$N = F \begin{pmatrix} j' & 0 \\ 0 & -j' \end{pmatrix} F^{-1} = j'\mathrm{e}^{-u} \begin{pmatrix} |a|_*^2 - |b|_*^2 & -2a\bar{b} \\ -2\bar{a}b & -|a|_*^2 + |b|_*^2 \end{pmatrix}.$$

Thus we have $N = \dfrac{1}{1+g\bar{g}}(-1+g\bar{g}, 2\mathrm{Re}(g), 2\mathrm{Im}(g))^t$. This tells us that the Gauss maps of Lorentz conformal timelike minimal surfaces can be expressed via the inverse of stereographic projection of $g$.

## §5. Lorentz conformal CMC surfaces in $\mathbb{H}^{2,1}$

Here we consider Lorentz conformal CMC surfaces in $\mathbb{H}^{2,1}$. Let $\mathbb{R}^{2,2} := (\{x = (x_1, x_2, x_3, x_4)^t; x_i \in \mathbb{R}\}, \langle\cdot,\cdot\rangle_*)$ be the 4-dimensional Lorentz space with metric $\langle x, y\rangle_* = x_1y_1 + x_2y_2 - x_3y_3 - x_4y_4$. Then

$$\mathbb{H}^{2,1} := \{x \in \mathbb{R}^{2,2}; \langle x, x\rangle_* = -1\}$$

denotes the 3-dimensional anti de Sitter space. Let $f : \Sigma \to \mathbb{H}^{2,1}$ be a Lorentz conformal immersion satisfying $\langle f_z, f_z\rangle_* = \langle f_{\bar z}, f_{\bar z}\rangle_* = 0$, $\langle f_z, f_{\bar z}\rangle_* = 2\mathrm{e}^{2u}$. Here $N : \Sigma \to \mathbb{S}^{1,2}$ denotes the unit normal vector field satisfying $\langle f, N\rangle_* = \langle f_z, N\rangle_* = \langle f_{\bar z}, N\rangle_* = 0$, where $\mathbb{S}^{1,2} := \{x \in \mathbb{R}^{2,2}; \langle x, x\rangle_* = 1\}$. By a similar computation as in Section 2, we have

$$\begin{aligned}
&f_{zz} = 2u_z f_z + QN,\ f_{z\bar z} = -2\mathrm{e}^{2u} f - 2H\mathrm{e}^{2u}N,\\
&f_{\bar z\bar z} := 2u_{\bar z} f_{\bar z} + \bar QN,\ N_z = -\frac{1}{2}(2Hf_z + Q\mathrm{e}^{-2u} f_{\bar z}),\\
&N_{\bar z} = -\frac{1}{2}(2Hf_{\bar z} + \bar Q\mathrm{e}^{-2u} f_z).
\end{aligned}$$

Setting $\mathcal{F} := (f, f_z, f_{\bar z}, N)$, we have $\mathcal{F}_z = \mathcal{F}\mathcal{U}$, $\mathcal{F}_{\bar z} = \mathcal{F}\mathcal{V}$, where

$$\mathcal{U} = \begin{pmatrix} 0 & 0 & 2\mathrm{e}^{2u} & 0\\ 1 & 2u_z & 0 & -H\\ 0 & 0 & 0 & -\frac{1}{2}Q\mathrm{e}^{-2u}\\ 0 & Q & 2H\mathrm{e}^{2u} & 0 \end{pmatrix},$$
$$\mathcal{V} = \begin{pmatrix} 0 & 2\mathrm{e}^{2u} & 0 & 0\\ 0 & 0 & 0 & -\frac{1}{2}\bar Q\mathrm{e}^{-2u}\\ 1 & 0 & 2u_{\bar z} & -H\\ 0 & 2H\mathrm{e}^{2u} & \bar Q & 0 \end{pmatrix}.$$

The compatibility condition $\mathcal{F}_{z\bar z} = \mathcal{F}_{\bar z z}$ gives

$$\begin{aligned}
4u_{z\bar z} + 4\mathrm{e}^{2u}(H^2 - 1) - Q\bar Q\mathrm{e}^{-2u} = 0 \quad &\text{(Gauss equation)},\\
Q_{\bar z} = 2H_z\mathrm{e}^{2u} \quad &\text{(Codazzi equation)}.
\end{aligned}$$

From here, we consider $2\times 2$ Lax pairs for Lorentz conformal CMC surfaces in $\mathbb{H}^{2,1}$. First we identify

$$\begin{array}{ccc}
\mathbb{R}^{2,2} & \longrightarrow & \left\{\begin{pmatrix} x_1 + x_4 & x_3 - j'x_2\\ x_3 + j'x_2 & x_1 - x_4 \end{pmatrix}; x_i \in \mathbb{R}\right\}\\
\cup & & \cup\\
x = (x_1, x_2, x_3, x_4)^t & \longmapsto & X = \begin{pmatrix} x_1 + x_4 & x_3 - j'x_2\\ x_3 + j'x_2 & x_1 - x_4 \end{pmatrix},
\end{array}$$

and the Lorentz metric becomes

$$\langle X, Y\rangle_{\mathbb{R}^{2,2}} = -\frac{1}{2}\text{trace}\left(X\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}Y\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right)$$

for $X$, $Y$ considered in matrix form.

Here we give a description of rigid motions of $\mathbb{R}^{2,2}$. For a given point $X \in \mathbb{R}^{2,2}$, consider the adjoint action $X \mapsto A \cdot X \cdot \bar{A}^t$, where $A = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathrm{SL}_2\mathbb{C}'$ for $a = a_1 + j'a_2$, $b = b_1 + j'b_2$, $c = c_1 + j'c_2$, $d = d_1 + j'd_2$ with $a_i, b_i, c_i, d_i \in \mathbb{R}$. By a straightforward computation, one can show that the adjoint action induces the following matrix $R$:

**Lemma 5.1.** *Set* $\mathrm{SO}_{2,2} := \{G \in \mathrm{SL}_4\mathbb{R}; GI''G^t = I''\}$, *where*

$$I'' = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}.$$

*Then, by an adjoint action* $X \mapsto A \cdot X \cdot \bar{A}^t$, $x$, *the vector form of* $X$, *maps to* $R \cdot x$, *where*

$$R = \begin{pmatrix}\alpha_1 & \mathrm{Im}(\bar{a}b + \bar{c}d) & \mathrm{Re}(\bar{a}b + \bar{c}d) & \alpha_2\\ \mathrm{Im}(\bar{a}c + \bar{b}d) & \mathrm{Re}(\bar{a}d - \bar{b}c) & \mathrm{Im}(\bar{a}d + \bar{b}c) & \mathrm{Im}(\bar{a}c - \bar{b}d)\\ \mathrm{Re}(\bar{a}c + \bar{b}d) & \mathrm{Im}(\bar{a}d - \bar{b}c) & \mathrm{Re}(\bar{a}d + \bar{b}c) & \mathrm{Re}(\bar{a}c - \bar{b}d)\\ \alpha_3 & \mathrm{Im}(\bar{a}b - \bar{c}d) & \mathrm{Re}(\bar{a}b - \bar{c}d) & \alpha_4\end{pmatrix} \in \mathrm{SO}_{2,2}$$

*with*

$$\alpha_1 = \frac{1}{2}(a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d}),\ \alpha_2 = \frac{1}{2}(a\bar{a} - b\bar{b} + c\bar{c} - d\bar{d}),$$
$$\alpha_3 = \frac{1}{2}(a\bar{a} + b\bar{b} - c\bar{c} - d\bar{d}),\ \alpha_4 = \frac{1}{2}(a\bar{a} - b\bar{b} - c\bar{c} + d\bar{d}).$$

Using the matrix form, $\mathbb{H}^{2,1}$ is expressed as

$$\mathbb{H}^{2,1} = \{X | X = \bar{X}^t,\ \langle X, X\rangle_{\mathbb{R}^{2,2}} = -1\}.$$

Moreover, we have another expression of $\mathbb{H}^{2,1}$ as in Lemma 5.2. The proof of Lemma 5.2 is almost the same as the proof of Lemma 5.2.1 in [5], so here we mention only the result.

**Lemma 5.2.** $\mathbb{H}^{2,1}$ *can be written as* $\left\{A\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\bar{A}^t; A \in \mathrm{SL}_2\mathbb{C}'\right\}$.

In this setting, there exists an $F \in \mathrm{SL}_2\mathbb{C}'$ such that

$$f = F\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\bar{F}^t,\ e_1 = F\begin{pmatrix}0 & -j'\\ j' & 0\end{pmatrix}\bar{F}^t,$$
$$e_2 = F\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\bar{F}^t,\ N = F\bar{F}^t,$$

where $e_1 := \dfrac{f_u}{2\mathrm{e}^u},\ e_2 := \dfrac{f_v}{2\mathrm{e}^u}$. Our task is to determine such $F = F(z,\bar{z}) \in \mathrm{SL}_2\mathbb{C}'$. Defining $U := F^{-1}F_z = \begin{pmatrix}U_{11} & U_{12}\\ U_{21} & U_{22}\end{pmatrix},\ V := F^{-1}F_{\bar{z}} = \begin{pmatrix}V_{11} & V_{12}\\ V_{21} & V_{22}\end{pmatrix}$, we can write

$$f_u = 2j'\mathrm{e}^u F\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\bar{F}^t,\ f_v = 2\mathrm{e}^u F\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\bar{F}^t,$$

and we have

$$f_z = 2j'\mathrm{e}^u F\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\bar{F}^t,\ f_{\bar{z}} = -2j'\mathrm{e}^u F\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\bar{F}^t.$$

The compatibility condition $f_{z\bar{z}} = f_{\bar{z}z}$ of $f$ implies that $V_{12} = -\overline{V_{12}},\ u_{\bar{z}} + V_{22} + \overline{U_{11}} = 0,\ U_{21} = -\overline{U_{21}}$. By the condition $f_{z\bar{z}} = 2\mathrm{e}^{2u}f + 2H\mathrm{e}^{2u}N$, we have $V_{12} = j'\mathrm{e}^u(1+H),\ U_{21} = j'\mathrm{e}^u(1-H)$, and the condition $f_{zz} = 2u_zf_z + QN$ implies $U_{12} = \frac{j'}{2}Q\mathrm{e}^{-u},\ u_z - U_{22} - \overline{V_{11}} = 0,\ V_{21} = -\frac{j'}{2}\bar{Q}\mathrm{e}^{-u}$, and also the condition $N_z = -Hf_z - \frac{1}{2}Q\mathrm{e}^{-2u}f_{\bar{z}}$ gives $U_{11} = -\overline{V_{11}},\ U_{22} = -\overline{V_{22}}$. In conclusion, we have the following proposition.

**Proposition 5.1.** *For some $F_{z_0\in\Sigma} \in \mathrm{SL}_2\mathbb{C}'$, we obtain the solution $F \in \mathrm{SL}_2\mathbb{C}'$ by solving $F_z = FU,\ F_{\bar{z}} = FV$, where*

$$U = \frac{1}{2}\begin{pmatrix}-u_z & j'Q\mathrm{e}^{-u}\\ 2j'\mathrm{e}^u(1-H) & u_z\end{pmatrix},$$
$$V = \frac{1}{2}\begin{pmatrix}u_{\bar{z}} & 2j'\mathrm{e}^u(1+H)\\ -j'\bar{Q}\mathrm{e}^{-u} & -u_{\bar{z}}\end{pmatrix}.$$

Proposition 5.1 implies that we obtain any CMC $H$ surface in $\mathbb{H}^{2,1}$ by solving the equations in Proposition 5.1 and inserting the solution into $f = F\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\bar{F}^t$. We should remark that, for any $B \in \mathrm{SU}'_{1,1} := \left\{\begin{pmatrix}p & q\\ \bar{q} & \bar{p}\end{pmatrix}\middle|\, p,q \in \mathbb{C}',\ |p|_*^2 - |q|_*^2 = 1\right\}$, replacing $F$ in $f = F\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\bar{F}^t$

with $FB$ does not change the resulting surface. Like in the case of conformal CMC 1 surfaces in $\mathbb{H}^3$, we would like to determine the $B \in \mathrm{SU}'_{1,1}$ so that $FB$ is anti $p$-holomorphic i.e. $(FB)_z = 0$. The condition $(FB)_z = 0$ implies $B_z = -UB$. Setting $W := B_{\bar{z}}B^{-1}$, we have the following expression

$$B_x = (W-U)B,\ B_y = -j'(W+U)B.$$

Then we must choose two matrices $W-U,\ -j'(W+U) \in \mathrm{su}'_{1,1}$ so that $B \in \mathrm{SU}'_{1,1}$, where $\mathrm{su}'_{1,1}$ is the Lie algebra of $\mathrm{SU}'_{1,1}$. By a simple computation, we can show that

$$W-U,\ -j'(W+U) \in \mathrm{su}'_{1,1} \Leftrightarrow W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bar{U}^t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Here we assume that $H \equiv 1$. Then the compatibility condition $U_{\bar{z}} + W_z + [U,W] = 0$ for $B$ does hold. Setting $B = \begin{pmatrix} p & q \\ \bar{q} & \bar{p} \end{pmatrix} \in \mathrm{SU}'_{1,1}$, we have

$$\begin{aligned} (FB)^{-1}(FB)_{\bar{z}} &= B^{-1}\left\{V + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bar{U}^t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\} B \\ &= 2j'\mathrm{e}^u \begin{pmatrix} \bar{p}\bar{q} & \bar{p}^2 \\ -\bar{q}^2 & -\bar{p}\bar{q} \end{pmatrix}. \end{aligned}$$

So $(\bar{F}\bar{B})^{-1}(\bar{F}\bar{B})_z = -2j'\mathrm{e}^u \begin{pmatrix} pq & p^2 \\ -q^2 & -pq \end{pmatrix} = 2\mathrm{e}^u p^2 \begin{pmatrix} -\frac{j'q}{p} & -j' \\ j'\left(\frac{j'q}{p}\right)^2 & \frac{j'q}{p} \end{pmatrix}$.

Setting $g = \dfrac{j'q}{p}$, $\hat{\omega} = 2\mathrm{e}^u p^2$, $\hat{F} = \bar{F}\bar{B}$, and we have

$$\hat{F}_z = \hat{F} \begin{pmatrix} -g & -j' \\ j'g^2 & g \end{pmatrix} \hat{\omega}.$$

Setting $\tilde{F} := (\hat{F}^{-1})^t$, we have $\tilde{F}_z = \tilde{F} \begin{pmatrix} g & -j'g^2 \\ j' & -g \end{pmatrix} \hat{\omega}$. Note that, writing $\hat{f} = \hat{F} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(\overline{\hat{F}}\right)^t = (x_1, x_2, x_3, x_4)^t$ in the vector form, we have $\tilde{f} = \tilde{F} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(\overline{\tilde{F}}\right)^t = (-x_1, -x_2, x_3, x_4)^t$. So $\hat{f}$ and $\tilde{f}$ coincide, up to a rigid motion of $\mathbb{R}^{2,2}$. In conclusion, replacing $\tilde{F}$ with $F$, we have the following theorem, which we call the Weierstrass-type representation for Lorentz conformal CMC 1 surfaces in $\mathbb{H}^{2,1}$.

**Theorem 2.** *Any Lorentz conformal CMC 1 surface in $\mathbb{H}^{2,1}$ can be locally constructed in the following way:*

(1) *Solve*

$$F_z = F \begin{pmatrix} g & -j'g^2 \\ j' & -g \end{pmatrix} \hat{\omega}$$

*with some initial condition* $F_{z_0 \in \Sigma} \in \mathrm{SL}_2\mathbb{C}'$.

(2) *Substitute* $F$ *in* (1) *into* $f = F \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bar{F}^t$.

*Furthermore, the metric of* $f$ *becomes* $df^2 = (1 + |g|_*^2)^2 |\omega|_*^2 dz d\bar{z}$.

Like in the case of Lorentz conformal minimal surfaces in $\mathbb{R}^{2,1}$, replacing $g$ and $\hat{\omega}$ with $j'g$ and $j'\hat{\omega}$, we have another expression for the Weierstrass-type representation for Lorentz conformal CMC 1 surfaces in $\mathbb{H}^{2,1}$, as follows:

**Proposition 5.2.** *Any Lorentz conformal CMC 1 surface in* $\mathbb{H}^{2,1}$ *can be locally constructed in the following way*:

(1) *Solve*

$$F_z = F \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \hat{\omega}$$

*with some initial condition* $F_{z_0 \in \Sigma} \in \mathrm{SL}_2\mathbb{C}'$.

(2) *Substitute* $F$ *in* (1) *into* $f = F \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bar{F}^t$.

*Furthermore, the metric of* $f$ *becomes* $df^2 = -(1 - |g|_*^2)^2 |\hat{\omega}|_*^2 dz d\bar{z}$.

Like in the case of conformal CMC 1 surfaces in 3-dimensional de Sitter space, surfaces described by Theorem 2 (or Proposition 5.2) generally have singularities. Their singularities are analyzed in Section 7.

## §6. Singularities of Lorentz conformal minimal surfaces in $\mathbb{R}^{2,1}$

Here we analyze singularities of Lorentz conformal minimal surfaces in $\mathbb{R}^{2,1}$. Note that surfaces described by Equation (2) are locally Lorentz conformal minimal immersions in $\mathbb{R}^{2,1}$, but this is not the case globally. In fact, the following proposition holds.

**Lemma 6.1.** *Let* $f$ *be a surface in* $\mathbb{R}^{2,1}$ *given by Equation* (2). *Then, away from* $|\omega|_*^2 = 0$, $f$ *has singularities if and only if* $|g|_*^2 = -1$.

*Proof.* By a direct computation, we have

$$f_x = \left( g\omega + \bar{g}\bar{\omega}, \frac{1}{2}(\omega + \bar{\omega} - g^2\omega - \bar{g}^2\bar{\omega}), -\frac{j'}{2}(\omega - \bar{\omega} + g^2\omega - \bar{g}^2\bar{\omega}) \right)^t,$$

$$f_y = j' \left( g\omega - \bar{g}\bar{\omega}, \frac{1}{2}(\omega - \bar{\omega} - g^2\omega + \bar{g}^2\bar{\omega}), -\frac{j'}{2}(\omega + \bar{\omega} + g^2\omega + \bar{g}^2\bar{\omega}) \right)^t.$$

Now we regard $f_x$ and $f_y$ as vectors in $\mathbb{R}^3$ and we take the vector product $f_x \times_{\mathbb{R}^3} f_y$ of $f_x$ and $f_y$ in $\mathbb{R}^3$. Then we have

$$f_x \times_{\mathbb{R}^3} f_y = (1 + |g|_*^2)|\omega|_*^2(-1 + |g|_*^2, 2\mathrm{Re}(g), -2\mathrm{Im}(g))^t.$$

Since we assume that $|\omega|_*^2 \neq 0$, $f$ has singularities when $|g|_*^2 = -1$. Q.E.D.

**Remark.** By the proof of Lemma 6.1, we can take the unit normal vector field $\nu : D \to \mathbb{S}^2$ of $f$, as a surface in $\mathbb{R}^3$, as

$$\nu = \frac{(-1 + |g|_*^2, 2\mathrm{Re}(g), -2\mathrm{Im}(g))^t}{\sqrt{(|g|_*^2 - 1)^2 + 4(\mathrm{Re}(g)^2 + \mathrm{Im}(g)^2)}}. \tag{4}$$

Here we introduce useful criteria for an image of a singular point to be $\mathcal{A}$-equivalent to a cuspidal edge, swallowtail, or cuspidal cross cap, which was shown in [7], [12] (see also [18], [20]).

**Proposition 6.1** ([7], [12]). *Let $p = \gamma(0) \in D \subset \mathbb{R}^2$ be a non-degenerate singular point of a front $f : D \to \mathbb{R}^3$, let $\gamma(t)$ be a singular curve around $p$, and let $\eta(t)$ be a vector field of null directions along $\gamma(t)$. Then we have the following*:

(1) *The image $f(p)$ is $\mathcal{A}$-equivalent to a cuspidal edge if and only if $\eta(0)$ is not proportional to $\dot{\gamma}$, where $\dot{\gamma} = d\gamma/dt$.*

(2) *The image $f(p)$ is $\mathcal{A}$-equivalent to a swallowtail if and only if $\eta(0)$ is proportional to $\dot{\gamma}$, and*

$$\frac{d}{dt}\det(\dot{\gamma}, \eta(t))\Big|_{t=0} \neq 0.$$

(3) *Let $f : D \to \mathbb{R}^3$ be a frontal with normal vector field $\nu$, and let $\gamma(t)$ be a singular curve on $D$ passing through a non-degenerate singular point $p = \gamma(0)$. Then the image $f(p)$ is $\mathcal{A}$-equivalent to a cuspidal cross cap if and only if*
- $\det(\dot{\gamma}(0), \eta(0)) \neq 0$,
- $\det(df(\dot{\gamma}(0)), \nu(0), d\nu(\eta(0))) = 0$,
- $\dfrac{d}{dt}\det(df(\dot{\gamma}(0)), \nu(0), d\nu(\eta(0)))\Big|_{t=0} \neq 0.$

Applying these useful criteria for such singularities, we give explicit conditions for cuspidal edge, swallowtail and cuspidal cross cap singularities of Lorentz conformal minimal surfaces to appear, which was obtained by Takahashi [18]:

**Theorem 3.** *Let $f : \Sigma \to \mathbb{R}^{2,1}$ be a surface given by Equation* (2), *and let $p \in \Sigma$ be a singular point of $f$. Then*

(1) *$f(p)$ is $\mathcal{A}$-equivalent to a cuspidal edge if and only if*

$$\mathrm{Re}\left(\frac{g'}{g^2\hat{\omega}}\right) \neq 0, \quad \mathrm{Im}\left(\frac{g'}{g^2\hat{\omega}}\right) \neq 0,$$

(2) *$f(p)$ is $\mathcal{A}$-equivalent to a swallowtail if and only if*

$$\frac{g'}{g^2\hat{\omega}} \in \mathbb{R} \setminus \{0\}, \quad \mathrm{Re}\left\{\frac{g}{g'}\left(\frac{g'}{g^2\hat{\omega}}\right)'\right\} \neq 0,$$

(3) *$f(p)$ is $\mathcal{A}$-equivalent to a cuspidal cross cap if and only if*

$$\frac{g'}{g^2\hat{\omega}} \in j'\mathbb{R} \setminus \{0\}, \quad \mathrm{Im}\left\{\frac{g}{g'}\left(\frac{g'}{g^2\hat{\omega}}\right)'\right\} \neq 0,$$

*where $' = d/dz$.*

In [18], Takahashi substituted the Weierstrass-type representation (3) for another expression as in Theorem 4.3 in [16] (in [18], this expression is called the real representation for timelike minimal surfaces in $\mathbb{R}^{2,1}$). After that, he derived explicit criteria for singularities of timelike minimal surfaces [16] in $\mathbb{R}^{2,1}$. As a corollary, he showed Theorem 3 here. In this paper we give Theorem 3 more directly. In order to show Theorem 3, we give the following lemma.

**Lemma 6.2.** *Let $f$ be a surface described by Equation* (2) *and let $p$ be a point in $\Sigma$. Then $f$ is a front on a neighborhood of $p$, and $p$ is a non-degenerate singular point if and only if* $\mathrm{Re}\left(\dfrac{g'}{g^2\hat{\omega}}\right) \neq 0$.

*Proof.* Here we assume that $|g|_*^2 = -1$ and $|\hat{\omega}|_*^2 \neq 0$. Then

$$\begin{aligned} df &= \frac{1}{2}\left(2, \frac{1}{g} - g, -j\left(\frac{1}{g} + g\right)\right)^t g\omega + \frac{1}{2}\left(2, \frac{1}{\bar{g}} - \bar{g}, j\left(\frac{1}{\bar{g}} + \bar{g}\right)\right)^t \bar{g}\bar{\omega} \\ &= \frac{1}{2}\left(2, \frac{1}{g} - g, -j\left(\frac{1}{g} + g\right)\right)^t (g\omega + \bar{g}\bar{\omega}) \\ &= (1, -\mathrm{Re}(g), -\mathrm{Im}(g))^t (g\omega + \bar{g}\bar{\omega}). \end{aligned}$$

In particular, $\eta = \dfrac{j}{g\hat{\omega}}$ gives the null direction at $p$ with the following identification:

$$(a, b) \in \mathbb{R}^2 \leftrightarrow z := a + j'b \in \mathbb{C}' \leftrightarrow a\partial_x + b\partial_y \leftrightarrow z\partial_z + \bar{z}\partial_{\bar{z}}.$$

Take $\nu$ as in Equation (4), and we have

$$d\nu = \operatorname{sgn}(\operatorname{Im}(g))\frac{j}{2\sqrt{2}}\left(\frac{dg}{g} - \frac{d\bar{g}}{\bar{g}}\right)\frac{1}{(\operatorname{Im}(g))^2}(\operatorname{Re}(g), 1, 0)^t.$$

If $dg(p) = 0$, the pair $(f, \nu) : \Sigma \to \mathbb{R}^3 \times \mathbb{S}^2$ is not immersed. Assume that $dg \neq 0$, and we have the null direction of $d\nu$ at $p$ which is proportional to $\mu := \overline{\left(\frac{g'}{g}\right)}$. On the other hand, $f$ is a front on a neighborhood of $p$ if and only if $\eta$ and $\mu$ are linearly independent, implying

$$\det(\mu, \eta) = \operatorname{Im}(\bar{\mu}\eta) = \operatorname{Im}\left(\frac{g'}{g} \cdot \frac{j'}{g\hat{\omega}}\right) = \operatorname{Re}\left(\frac{g'}{g^2\hat{\omega}}\right) \neq 0.$$

Define the signed area density as follows:

$$\lambda := (f_x \times f_y) \cdot \nu = (1 + |g|_*^2)|\omega|_*^2\sqrt{(|g|_*^2 - 1)^2 + 4(\operatorname{Re}(g)^2 + \operatorname{Im}(g)^2)},$$

where $\cdot$ in the above equation denotes the ordinary Euclidean inner product. When $p$ is a singular point, since $|g(p)|_*^2 = -1$, we have

$$d\lambda = -2\sqrt{2}|\operatorname{Im}(g)||\hat{\omega}|_*^2\left(\frac{dg}{g} + \frac{d\bar{g}}{\bar{g}}\right).$$

Thus we have that $d\lambda(p) \neq 0$ if and only if $dg(p) \neq 0$. Therefore, if $\operatorname{Re}\left(\frac{g'}{g^2\hat{\omega}}\right)$ holds at $p$, $p$ is non-degenerate, since $dg(p) \neq 0$. Q.E.D.

Here we go back to the proof of Theorem 3. First we assume that $\operatorname{Re}\left(g'/(g^2\hat{\omega})\right) \neq 0$ at a singular point $p$. This condition implies that $f$ is a front and $p$ is a non-degenerate singular point. Since the set of singular points must satisfy $|g|_*^2 = -1$, the singular curve $\gamma(t)$ with $\gamma(0) = p$ satisfies $g(\gamma(t))\overline{g(\gamma(t))} = -1$. Differentiating this equation with respect to $t$ implies $\operatorname{Re}\left(\frac{g'}{g}\dot{\gamma}\right) = 0$, where $\dot{\gamma} := \frac{d\gamma}{dt}$. This implies

$$\dot{\gamma} \perp \overline{\left(\frac{g'}{g}\right)} \Rightarrow \dot{\gamma} \parallel j'\overline{\left(\frac{g'}{g}\right)}.$$

So we can parametrize $\gamma$ as $\dot{\gamma}(t) = j'\overline{\left(\frac{g'}{g}\right)}(\gamma(t))$. Applying item (1) in Proposition 6.1, we have

$$\det(\dot{\gamma}, \eta) = \operatorname{Im}(\overline{\dot{\gamma}}\eta) = -\operatorname{Im}\left(\frac{g'}{g^2\hat{\omega}}\right) \neq 0.$$

Thus we have proven item (1) in Theorem 3. Next we assume that $\mathrm{Im}\left(\frac{g'}{g^2\hat{\omega}}\right)\Big|_{t=0} = 0$. Then

$$\begin{aligned}
&\frac{d}{dt}\det(\dot{\gamma},\eta)\Big|_{t=0} = -\mathrm{Im}\left\{\left(\frac{g'}{g^2\hat{\omega}}\right)'\frac{d\gamma}{dt}\right\} \\
&= -\mathrm{Im}\left\{j'\left(\frac{g'}{g^2\hat{\omega}}\right)'\overline{\left(\frac{g'}{g}\right)}\right\} = -\mathrm{Re}\left\{\left(\frac{g'}{g^2\hat{\omega}}\right)'\overline{\left(\frac{g'}{g}\right)}\right\} \\
&= -\left|\frac{g'}{g}\right|_*^2 \mathrm{Re}\left\{\frac{g}{g'}\left(\frac{g'}{g^2\hat{\omega}}\right)'\right\} \neq 0.
\end{aligned}$$

Applying item (2) in Proposition 6.1, we have the condition of item (2) in Theorem 3. Finally we show item (3) in Theorem 3. First we have

$$\det(\dot{\gamma}(0),\eta(0)) = \mathrm{Im}(\overline{\dot{\gamma}},\eta) = -\mathrm{Im}\left(\frac{g'}{g^2\hat{\omega}}\right) \neq 0,$$

$$\det(df(\dot{\gamma}),\nu,d\nu(\eta)) = \mathrm{Re}\left(\frac{g'}{g^2\hat{\omega}}\right)\cdot\psi_0 = 0,$$

where $\psi_0$ is a smooth function on a neighborhood of $p$ satisfying $\psi_0(p) \neq 0$. Thus we have the first condition of item (3) in Theorem 3. By the last condition of item (3) in Proposition 6.1, we have

$$\begin{aligned}
&\frac{d}{dt}\det(df(\dot{\gamma}),\nu,d\nu(\eta))\Big|_{t=0} = \frac{d}{dt}\mathrm{Re}\left(\frac{g'}{g^2\hat{\omega}}\right)\Big|_{t=0} \\
&= \left|\frac{g'}{g}\right|_*^2 \mathrm{Im}\left\{\frac{g}{g'}\left(\frac{g'}{g^2\hat{\omega}}\right)'\right\} \neq 0.
\end{aligned}$$

This completes the proof of Theorem 3.

## §7. Singularities of Lorentz conformal CMC 1 surfaces in $\mathbb{H}^{2,1}$

Here we introduce our second result about criteria for Lorentz conformal CMC 1 surfaces in $\mathbb{H}^{2,1}$. Again we assume that $|\hat{\omega}|_*^2 \neq 0$. We can use the Weierstrass-type representation in Proposition 5.2 to prove Theorem 4, but we will omit the complete proof, as the following result is analogous to that of Theorem 3.

**Theorem 4.** *Let $f : \Sigma \to \mathbb{H}^{2,1}$ be a surface in $\mathbb{H}^{2,1}$ described by Proposition 5.2 and let $p$ be a non-degenerate singular point of $f$. Then $f$ has singularities if and only if $|g(p)|_*^2 = 1$, and*

(1) *$f(p)$ is $\mathcal{A}$-equivalent to a cuspidal edge if and only if*

$$\mathrm{Re}\left(\frac{g'}{g^2\hat{\omega}}\right) \neq 0, \quad \mathrm{Im}\left(\frac{g'}{g^2\hat{\omega}}\right) \neq 0,$$

(2) *$f(p)$ is $\mathcal{A}$-equivalent to a swallowtail if and only if*

$$\frac{g'}{g^2\hat{\omega}} \in \mathbb{R} \setminus \{0\}, \quad \mathrm{Re}\left\{\frac{g}{g'}\left(\frac{g'}{g^2\hat{\omega}}\right)'\right\} \neq 0,$$

(3) *$f(p)$ is $\mathcal{A}$-equivalent to a cuspidal cross cap if and only if*

$$\frac{g'}{g^2\hat{\omega}} \in j'\mathbb{R} \setminus \{0\}, \quad \mathrm{Im}\left\{\frac{g}{g'}\left(\frac{g'}{g^2\hat{\omega}}\right)'\right\} \neq 0.$$

Due to the replacement of $\hat{F}$ with $\tilde{F}$ in Section 5 (see the above argument for Theorem 2), the proof of Theorem 4 is almost the same as of Theorem 3.4 in [7], so here we remark on only the differences.

**Lemma 7.1.** *Let $f$ be a surface in $\mathbb{H}^{2,1}$ described by Proposition 5.2. Away from $|\hat{\omega}|_*^2 \neq 0$, $f$ has singular points if and only if $|g(p)|_*^2 = 1$.*

*Proof.* Define $\xi := F\bar{F}^t$ and a 3-form $\Omega$ on $\mathbb{H}^{2,1}$ by

$$\Omega(X_1, X_2, X_3) := \det(f, X_1, X_2, X_3)$$

for arbitrary vector fields $X_1, X_2, X_3$ of $\mathbb{H}^{2,1}$, where $f$ denotes the position vector in $\mathbb{H}^{2,1}$. Then $\Omega$ gives a volume element on $\mathbb{H}^{2,1}$, since

$$\begin{aligned}
\Omega(f_x, f_y, \xi) &= \det(f, f_x, f_y, \xi) \\
&= \det\begin{pmatrix} 0 & 2\mathrm{Re}(g\hat{\omega}) & 2\mathrm{Im}(g\hat{\omega}) & 1 \\ 0 & \mathrm{Im}((1-g^2)\hat{\omega}) & \mathrm{Re}((1-g^2)\hat{\omega}) & 0 \\ 0 & \mathrm{Re}((1+g^2)\hat{\omega}) & \mathrm{Im}((1+g^2)\hat{\omega}) & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\
&= (1-|g|_*^2)(1+|g|_*^2)|\hat{\omega}|_*^2.
\end{aligned}$$

Note that, unlike the case of CMC 1 faces in $\mathbb{S}^{2,1}$, $f$ might have singularities if $|g|_*^2 = -1$. On the other hand, we can confirm that $f$ does not have singularities in that case, proving the lemma. Q.E.D.

Except for that point, if we admit the same $\eta$ and $\mu$ as in the case of Lorentz conformal minimal surfaces in $\mathbb{R}^{2,1}$ (see the proof of Lemma 6.2), the proof of Theorem 4 is almost the same as the one of Theorem 3.4 in [7]. Thus we omit the proof here.

Finally, we see duality of singularities for Lorentz conformal minimal surfaces in $\mathbb{R}^{2,1}$. As a direct consequence of Theorem 3, we immediately have the following property.

**Corollary 7.1** ([18]). *Let $f$ be a surface in $\mathbb{R}^{2,1}$ described by Equation* (2), *let $f^\sharp$ be its conjugate surface and let $p$ be a singular point of $f$ and $f^\sharp$. Then $f(p)$ is $\mathcal{A}$-equivalent to a swallowtail if and only if $f^\sharp(p)$ is $\mathcal{A}$-equivalent to a cuspidal cross cap.*

**Remark.** Since the criteria for the three types of singularities in Theorems 3, 4 are exactly the same, the same duality holds in the case of Lorentz conformal CMC 1 faces in $\mathbb{H}^{2,1}$.

## §Appendix A. Introduction to discrete timelike isothermic surfaces in $\mathbb{R}^{2,1}$

In this appendix we briefly introduce discrete timelike isothermic surfaces in $\mathbb{R}^{2,1}$. First we define two kinds of smooth timelike surfaces in $\mathbb{R}^{2,1}$ and $\mathbb{H}^{2,1}$.

**Definition A.1.** *Let $f$ be a timelike immersion into $\mathbb{R}^{2,1}$. Then $f$ is timelike isothermic* (*resp. anti isothermic*) *if $f$ admits Lorentz conformal curvature line coordinates* (*resp. Lorentz conformal asymptotic coordinates*).

Magid [15] considered Lorentz isothermic surfaces in $\mathbb{R}^{n-j,j}$. As mentioned in [15], there are three kinds of Lorentz isothermic surfaces in $\mathbb{R}^{n-j,j}$. Here we only consider the case $n = 2,\ j = 1$. In this case we can characterize such Lorentz isothermic surfaces in $\mathbb{R}^{2,1}$ using the notion of Hopf differential $Q$ (several terminologies can be found in [15]).

**Proposition A.1.** *Let $f : D(\subset \mathbb{C}') \to \mathbb{R}^{2,1}$ be a Lorentz conformal immersion parametrized by para-complex coordinates $z = x + j'y$ with spacelike unit normal vector field $\nu$. Then $f$ has an umbilic point at $p \in D$ if and only if $Q = 0$ at $p$. Moreover, $f$ is timelike isothermic* (*resp. anti isothermic*) *if and only if $Q$ defined in Section 2 is a non-zero real function* (*resp. pure para-imaginary unit times non-zero real function*).

**Remark.** As mentioned in [15], there is another kind of timelike isothermic surface, which has not yet been named. Here we refer to these types of surfaces as timelike surfaces of the third kind. A Lorentz conformal surface is a timelike surface of the third kind if and only if $Q$ is $(1 \pm j')$ times a non-zero real function.

When $f$ is an umbilic-free timelike isothermic surface, we can reparametrize $f$ so that $Q \equiv 1$. In particular, when we consider timelike isothermic minimal surfaces in $\mathbb{R}^{2,1}$, we can reparametrize Lorentz conformal minimal surfaces $f$ described by Equation (2) so that $Q = g'\hat{\omega} \equiv 1$. So the Weierstrass-type representation for timelike isothermic minimal surfaces in $\mathbb{R}^{2,1}$ can be written as

$$f = \mathrm{Re} \int \left( \frac{2g}{g'}, \frac{1-g^2}{g'}, \frac{-j'(1+g^2)}{g'} \right)^t dz.$$

Our first main result in [21] is a discrete analogue of the Weierstrass-type representation for discrete timelike isothermic minimal surfaces (see Theorem 5 here).

We now briefly introduce discrete timelike isothermic surfaces in $\mathbb{R}^{2,1}$. In particular, we introduce discrete timelike (isothermic) minimal surfaces in $\mathbb{R}^{2,1}$. As in Section 3, each element in $\mathbb{R}^{2,1}$ is identified with the matrix as in Equation (1), and we denote $p = (m, n)$, $q = (m+1, n)$, $r = (m+1, n+1)$, $s = (m, n+1)$. Then we define discrete timelike isothermic surfaces in $\mathbb{R}^{2,1}$ as follows (the reason why we define them in this way can be found in [21]):

**Definition A.2.** *Let $F : \mathbb{Z}^2 \to \mathbb{R}^{2,1}$ be a discrete surface. Then*

- *$F$ is called a* discrete timelike isothermic surface *if*
  - *each quadrilateral $(F_p, F_q, F_r, F_s)$ with vertices $F_p, F_q, F_r, F_s$ lies in a timelike plane,*
  - *all quadrilaterals $(F_p, F_q, F_r, F_s)$ are convex,*
  - *all quadrilaterals satisfy $cr(F_p, F_q, F_r, F_s) = 1$.*
- *A discrete timelike isothermic surface $g : \mathbb{Z}^2 \to \mathbb{R}^{1,1} \cong \mathbb{C}'$ is called a* discrete $p$-holomorphic function.

Roughly speaking, in Definition A.2, the first two conditions are the discrete counterpart of an immersion condition for a smooth timelike surface.

We have a Weierstrass-type representation for discrete timelike isothermic minimal surfaces in $\mathbb{R}^{2,1}$. Details can be found in [21].

**Theorem 5.** *A discrete timelike minimal surface $F : \mathbb{Z}^2 \to \mathbb{R}^{2,1}$ can be locally constructed using a discrete p-holomorphic function $g : \mathbb{Z}^2 \to \mathbb{C}'$ by solving*

$$F_q - F_p = \frac{1}{2}\mathrm{Re} \left( \frac{g_p + g_q}{g_q - g_p}, \frac{1 - g_p g_q}{g_q - g_p}, -\frac{j'(1 + g_p g_q)}{g_q - g_p} \right)^t,$$

$$F_s - F_p = \frac{1}{2}\mathrm{Re} \left( \frac{g_p + g_s}{g_s - g_p}, \frac{1 - g_p g_s}{g_s - g_p}, -\frac{j'(1 + g_p g_s)}{g_s - g_p} \right)^t.$$

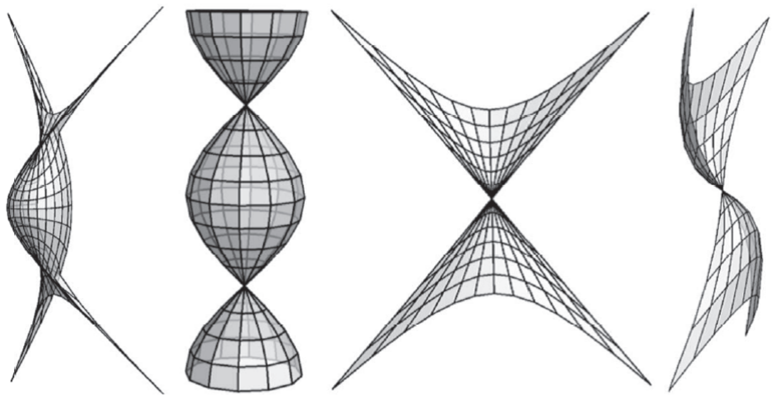

Fig. 1. Examples of discrete timelike isothermic minimal surfaces

*Conversely, any discrete timelike minimal surface locally satisfies the above equations for some discrete p-holomorphic function g.*

In Figure 1, we show several examples of discrete timelike isothermic minimal surfaces in $\mathbb{R}^{2,1}$. These pictures obviously have certain configurations of singularities. In [21], their singularities are analyzed.

Finally, we introduce several open problems related to the topics in this paper.

- How can we describe discrete timelike isothermic non-zero CMC surfaces in $\mathbb{R}^{2,1}$? If we can describe such discrete surfaces, is there any construction of discrete timelike isothermic CMC surfaces like in [17]?
- Is there a Weierstrass-type representation for discrete timelike isothermic CMC 1 surfaces in $\mathbb{H}^{2,1}$? If yes, do discrete timelike isothermic CMC 1 surfaces in $\mathbb{H}^{2,1}$ have singularities?
- How can we describe discrete anti-isothermic surfaces?

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