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Implicit Hamiltonian systems and singular curves of distributions

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§1. Introduction

The subject of the sub-Riemannian geometry is to study a triple $(M, \mathcal{D}, g)$ of a manifold M, a distribution $\mathcal{D}$ on M and a bi-linear positive definite form g on $\mathcal{D}$, which is called a *sub-Riemannian manifold*. Here, a distribution is a sub-bundle of the tangent bundle TM of M. The object appears naturally as a collapsing Riemannian manifold and is a generalization of a Riemannian manifold. However properties of sub-Riemannian manifolds are much different from those of Riemannian manifolds. In fact, not much is known about the properties of exponential maps and even about smoothness of minimizers. For example there is a problem which is open for decades: "Are all locally minimizers smooth on any sub-Riemannian manifold?"

In this article we give a new clue to study of minimizers on sub-Riemannian manifolds.

For a distribution $\mathcal{D}$, there is an important class of curves called horizontal curves. A *horizontal curve* is an absolutely continuous curve $\gamma\colon I \to M$ such that $\dot{\gamma}(t)$ is a measurable and bounded map which satisfies $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$ for almost every $t \in I$.

According to Chow–Rashevsky's theorem, if a distribution $\mathcal{D}$ on a connected manifold M satisfies Hörmander's condition, every two points are connected by a horizontal curve. For a sub-Riemannian manifold

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which satisfies Hörmander's condition, we may define a distance

$$d_{CC}(p,q) := \inf_{\gamma}\Big\{L(\gamma) := \int_{[a,b]} \sqrt{g(\dot\gamma(t),\dot\gamma(t))}dt \mid \gamma\colon [a,b] \to M : \mathit{horizontal}, \gamma(a)=p, \gamma(b)=q\Big\}$$

which is called a *Carnot–Carathéodory* (or *sub-Riemannian*) *distance*. It is known that the topology from Carnot–Carathéodory distance agrees with the original one (ball-box theorem[10]).

A horizontal curve γ connecting p and q is called a *minimizer* if $d_{CC}(p,q) = L(\gamma)$. A horizontal curve $\gamma\colon I \to M$ is a *local minimizer* if for any $t_0 \in I$, there exists $\varepsilon > 0$ such that for all closed sub-interval J of $[t_0-\varepsilon, t_0+\varepsilon]$, $\gamma\mid_{J\cap I}$ is a minimizer between the end points. Note that any minimizer is necessarily a local minimizer.

To consider local minimizers, we classify horizontal curves on a sub-Riemannian manifold $(M,\mathcal{D},g)$ by using the end-point mapping.

For a bounded measurable curve $c\colon [0,T] \to \mathcal{D}$, if a curve $\gamma := \pi_{\mathcal{D}} \circ c\colon [0,T] \to M$ satisfies $\dot\gamma(t) = c(t)$ for almost everywhere on $[0,T]$, then γ is a horizontal curve and c is called an *admissible velocity*. Here $\pi_{\mathcal{D}}\colon \mathcal{D} \to M$ is the canonical projection.

Expressing $c(t)$ as $c(t) = u_1(t)X_1(\gamma(t)) + \cdots + u_k(t)X_k(\gamma(t))$ with respect to a local framing $\{X_1,\ldots,X_k\}$ of the distribution $\mathcal{D}$ on an open subset $O \subset M$, the condition is written as

$$\dot\gamma(t) = u_1(t)X_1(\gamma(t)) + \cdots + u_k(t)X_k(\gamma(t)),$$

with the fibre coordinates $(u_1,\ldots,u_k)$.

For any point q_0 on M, the set of admissible velocities

$$\mathcal{V}_{q_0} := \{c \mid c\colon [0,T] \to \mathcal{D} : \mathit{admissible\ velocity},\ \gamma(0) = q_0\}$$

will be a Banach manifold. The map

$$\mathrm{End}(q_0)\colon \mathcal{V}_{q_0} \to M, c \mapsto \gamma(T)$$

is called an *end-point mapping* and is differentiable by means of Fréchet derivative. A singular (resp. regular) point of the end-point mapping is called a *singular* (*resp. regular*) *velocity*. The trajectory corresponds to singular velocity is called a *singular* (*resp. regular*) *curve*, i.e. the differential map

$$d\,\mathrm{End}(q_0)_c\colon T_c\mathcal{V}_{q_0} \to T_{\gamma(T)}M$$

is not surjective (resp. surjective) at singular (resp. regular) velocity c. Since every curve is either regular or singular, every minimizer is also either a regular curve or a singular curve.

For a sub-Riemannian manifold, there is a geodesic equation given as Hamiltonian formulation, not as Lagrangian. We give a function

$$H_E(x,p) = -\frac{1}{2}\sum_{i,j} g^{ij}(x)\langle p, X_i(x)\rangle\langle p, X_j(x)\rangle$$

on T^*M, where $g_{ij} = g(X_i, X_j)$ and $(g^{ij})_{i,j}$ is the inverse matrix of $(g_{ij})_{i,j}$. We may consider a Hamiltonian vector field associated to H_E with canonical symplectic form on the cotangent bundle T^*M. An ordinary differential equation related to the Hamiltonian vector field is given by

$$\dot{x}(t) = \frac{\partial H_E}{\partial p}(x(t), p(t)), \quad \dot{p}(t) = -\frac{\partial H_E}{\partial x}(x(t), p(t))$$

with Darboux coordinates (x,p) of T^*M. A solution of this equation (which is called the geodesic equation) is called a *normal bi-extremal* and its projection to M is called a *normal extremal* (or a *normal geodesic*).

It is known that regular local minimizers are normal geodesics, and so they are smooth since they are solutions of the geodesic equation. A singular minimizer is sometimes also a normal geodesic depending on a metric g. A singular (local) minimizer which is not a normal geodesic is called a *strictly singular* (*local*) *minimizer*. Examples of singular minimizers which are not normal on Martinet distribution are given by R. Montgomery in 1994 [8].

Although the examples given by R. Montgomery guarantee the existence of strictly singular minimizers, we know few other examples. So, we are interested in finding more examples of strictly singular minimizers. As a strategy, we divide the problem into two problems; to detect singular horizontal curves which are not normal geodesics and to find local minimizers among them. In this paper we concentrate on the former one and give new examples of singular horizontal curves which are not normal geodesics.

Now, we introduce a result of foothold in the study of singular curves. Take a function $H\colon T^*M \times_M \mathcal{D} \to \mathbb{R}, H(x,p,u) := \langle p, u\rangle$ for $x \in M, p \in T^*_xM$ and $u \in \mathcal{D}_x$, here $\times_M$ is a fibre product. Let $\{X_1, \ldots, X_k\}$ be a local framing of $\mathcal{D}$ on an open neighborhood U_{x_0} of x_0 in M and $(u_1, \ldots, u_k)$ the fibre coordinates related to the local framing. Then we have locally

$$H(x,p,u) = \sum_{i=1}^{k} u_i\langle p, X_i(x)\rangle.$$

Then singular curves are characterized by a constrained Hamiltonian system;

Proposition 1.1 ([6], p.567). *A horizontal curve $x(t)$ on M with rank k distribution $\mathcal{D}$ is a singular curve if and only if there exist a positive number $\varepsilon > 0$, a curve $p(t)$ on $T^*_{x(t)}M \setminus \{0\}$ and $u(t) \in \mathcal{D}_{x(t)}$ such that the curve $(x(t), p(t), u(t))$ satisfies the following equation (which is called the constrained Hamiltonian system) for almost all $t \in [0, \varepsilon)$;*

$$\begin{cases} \dot{x}(t) = \dfrac{\partial H}{\partial p}(x(t), p(t), u(t)), \\ \dot{p}(t) = -\dfrac{\partial H}{\partial x}(x(t), p(t), u(t)), \\ \dfrac{\partial H}{\partial u_i}(x(t), p(t), u(t)) = 0 \ (1 \leq i \leq k). \end{cases}$$

A solution $(x(t), p(t))$ on T^*M of the constrained Hamiltonian system in Proposition 1.1 is called an *abnormal bi-extremal* and the projection of an abnormal bi-extremal to M is called an *abnormal extremal* (or a *singular curve*). It is known that a local minimizer is either a normal extremal or an abnormal extremal; and the two possibilities are not mutually exclusive. Abnormal extremals are not local minimizers in general.

We may consider a constrained Hamiltonian system for any distribution. However it is not known whether there exists a solution or does not, in general. To study the existence of solutions, we intend to apply the theory of solvability of implicit differential systems. The theory, which is introduced in §2, is given by T. Fukuda and S. Janeczko. We give a refinement of Fukuda–Janeczko's theory in the case of two parameters in §3. In §4, an application of the results in §3 to rank two distribution is given. Then we have the following main result in this paper.

Theorem 1. *Let $(M, \mathcal{D}, g)$ be a sub-Riemannian smooth manifold with a distribution $\mathcal{D}$ of rank two. Suppose that $\mathcal{D}_1 := \mathcal{D} + [\mathcal{D}, \mathcal{D}]$ is a sub-bundle of rank three and $\mathcal{D}_2 := \mathcal{D}_1 + [\mathcal{D}, \mathcal{D}_1]$ is a sub-bundle of rank four. Then for any point q in M, there exist an open neighborhood U_q of q in M and a C^∞ immersive singular curve $x(t)$ which is not a normal geodesic in U_q defined on a small interval.*

Theorem 1 is proved in §4 as an application of the results in §3. It is not known whether the singular curve in Theorem 1 is a minimizer or not.

§2. Preliminary

2.1. Implicit differential systems and their solvability

An implicit differential system is a generalization of a differential equation defined by a vector field. We define some basic notions. Let $\pi\colon TM \to M$ be a canonical projection and N a submanifold of M.

Definition 2.1. An *implicit differential system* on M is a subset S of tangent bundle TM.

A C^1 curve $\gamma\colon (a,b) \to N$ is called a *solution of S over N* if $(\gamma(t), \dot\gamma(t)) \in S \cap \pi^{-1}(N)$ for all $t \in (a,b)$.

A point $(x_0, \dot x_0) \in S$ is a *solvable point of S over N* if there exist a positive number $\varepsilon > 0$ and a solution $\gamma\colon (-\varepsilon, \varepsilon) \to N$ such that $(\gamma(0), \dot\gamma(0)) = (x_0, \dot x_0)$.

From now on we consider the case where S is a smooth submanifold of TM. A point $(x_0, \dot x_0) \in S$ is a *smoothly solvable point of S over N* if there exist an open neighborhood W in $S \times \mathbb{R}$ of $(x_0, \dot x_0, 0)$ and C^∞ map $\bar\gamma\colon W \to N$ such that $\gamma_{(x,\dot x)}(t) := \bar\gamma(x, \dot x, t)$ is a solution of S over N with $(\gamma(0), \dot\gamma(0)) = (x, \dot x)$ for all $(x, \dot x) \in \pi_1(W)$, where $\pi_1\colon S \times \mathbb{R} \to S$ is a natural projection.

An implicit differential system S over N is called a *smoothly solvable submanifold over N* if S consists only of smoothly solvable points of S over N.

When a submanifold N is M itself, definitions above are given in Fukuda and Janeczko's papers [4][5]. A solution of S over M is just called a *solution of S*. We say, simply, an implicit differential system S *is (smoothly) solvable* if S is (smoothly) solvable over M.

2.2. Implicit Hamiltonian systems

Let (M, ω) be a symplectic manifold. Then there is the induced symplectic structure $\dot\omega$ on the tangent bundle TM: We have an bundle isomorphism induced from interior product $\flat\colon TM \to T^*M, \flat_x(v_q) = \iota_{v_q}\omega_q$ for each point $q \in M$. The symplectic structure $\dot\omega$ is given by the pullback of the Liouville form θ on T^*M, i.e., $\dot\omega := \flat^* d\theta$. The induced symplectic structure $\dot\omega$ is locally written by

$$\dot\omega = \sum_{i=1}^{n} d\dot p_i \wedge dx_i - d\dot x_i \wedge dp_i$$

with the canonical coordinates $(x, p, \dot x, \dot p)$ of tangent bundle TM related to Darboux coordinates $(x,p) = (x_1, \dots, x_n, p_1, \dots, p_n)$ for the standard symplectic form $\omega = \sum_{i=1}^n dp_i \wedge dx_i$ of M.

We will define the notion of implicit Hamiltonian systems as Lagrangian submanifolds of $(TM, \dot{\omega})$. Then they are regarded as a generalization of Hamiltonian vector fields, i.e., of Hamiltonian dynamical systems. In what follows we set $M = \mathbb{R}^{2n}$ with the standard symplectic form ω as above.

Definition 2.2 ([4, 5]). A Lagrangian submanifold L of $(T\mathbb{R}^{2n}, \dot{\omega})$ (i.e., $\dim L = 2n$ and $\omega|_L = 0$) is called an *implicit Hamiltonian system.*

There is a well-known result that a Lagrangian submanifold is locally generated by a Morse family;

Theorem 2.3 ([1]). *Let L be a Lagrangian submanifold of $T\mathbb{R}^{2n}$ and $(q_0, \dot{q}_0) = (x_0, p_0, \dot{x}_0, \dot{p}_0) \in L$. Suppose*

$$\operatorname{corank} d(\pi\ |_L)(q_0, \dot{q}_0) = k > 0.$$

Then there exist an open neighborhood O of $(q_0, \dot{q}_0)$ in $T\mathbb{R}^{2n}$, an open neighborhood W of $(q_0, 0) \in \mathbb{R}^{2n} \times \mathbb{R}^k$ and a smooth function $F\colon W \to \mathbb{R}$ such that

$$L \cap O = \Big\{ (x_0, p_0, \dot{x}_0, \dot{p}_0) \in O \mid \exists u \in \mathbb{R}^k s.t. (x, p, u) \in W, \frac{\partial F}{\partial u_l}(x, p, u) = 0,$$
$$\dot{x}_i = \frac{\partial F}{\partial p_i}(x, p, u), \dot{p}_i = -\frac{\partial F}{\partial x_i}(x, p, u), 1 \le i \le n, 1 \le l \le k \Big\},$$

and that

$$\operatorname{rank}\left(\frac{\partial^2 F}{\partial x_i \partial u_l}(q_0, 0), \frac{\partial^2 F}{\partial p_i \partial u_l}(q_0, 0) \right)_{1 \le i \le n, 1 \le l \le k} = k,\ \frac{\partial^2 F}{\partial u_r \partial u_s}(q_0, 0) = 0$$

for $1 \le r, s \le k$.

Recall that the family of functions $F\colon \mathbb{R}^{2n} \times \mathbb{R}^k \to \mathbb{R}$ with $2n$ parameters $(x_1, \dots, x_n, p_1, \dots, p_n)$ on $(\mathbb{R}^k; u_1, \dots, u_k)$ is called a *Morse family* if $0 \in \mathbb{R}^k$ is a critical point of the map $F(q_0, u)$ and the map

$$\left(\frac{\partial F}{\partial u_1}, \dots, \frac{\partial F}{\partial u_k} \right) : \mathbb{R}^{2n} \times \mathbb{R}^k \to \mathbb{R}^k$$

is submersive at $(q_0, 0)$. We denote L_F a Lagrangian submanifold generated by Morse family $F\colon \mathbb{R}^{2n} \times \mathbb{R}^k \to \mathbb{R}$. That is, for the catastrophe set

$$C(F) = \left\{ (x, p, u) \in \mathbb{R}^{2n} \times \mathbb{R}^k \mid \frac{\partial F}{\partial u_i}(x, p, u) = 0, i = 1, \dots, k \right\}$$

of F and C^∞ map $\phi_F\colon \mathbb{R}^{2n}\times\mathbb{R}^k \to T\mathbb{R}^{2n}$ defined by

$$\phi(x,p,u) = \left(x, p, \frac{\partial F}{\partial p_i}(x,p,u), -\frac{\partial F}{\partial x_i}(x,p,u)\right),$$

we set $L_F = \phi_F(C(F))$.

The following propositions are given in [4] which are a necessary condition and a sufficient condition for L_F to be (smoothly) solvable in the sense of Definition 2.1.

Proposition 2.4 ([4]). *Let $(x,p,\dot{x},\dot{p})$ be a solvable point of L_F. We set $\phi_F(x,p,u) = (x,p,\dot{x},\dot{p})$. Then there exists a real vector $\mu = (\mu_1,\dots,\mu_k)$ in $\mathbb{R}^k$ such that*

$$\begin{pmatrix} \frac{\partial^2 F}{\partial u_1 \partial u_1}(x,p,u) & \cdots & \frac{\partial^2 F}{\partial u_1 \partial u_k}(x,p,u) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 F}{\partial u_k \partial u_1}(x,p,u) & \cdots & \frac{\partial^2 F}{\partial u_k \partial u_k}(x,p,u) \end{pmatrix} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_k \end{pmatrix} = \begin{pmatrix} \{\frac{\partial F}{\partial u_1}, F\}(x,p,u) \\ \vdots \\ \{\frac{\partial F}{\partial u_k}, F\}(x,p,u) \end{pmatrix}.$$

Here the bracket $\{,\}$ is the Poisson bracket associated to the symplectic form ω.

Proposition 2.5 ([4]). *A point $(x,p,\dot{x},\dot{p})$ in L_F is (smoothly) solvable if a linear equation*

$$\begin{pmatrix} \frac{\partial^2 F}{\partial u_1 \partial u_1}(x,p,u) & \cdots & \frac{\partial^2 F}{\partial u_1 \partial u_k}(x,p,u) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 F}{\partial u_k \partial u_1}(x,p,u) & \cdots & \frac{\partial^2 F}{\partial u_k \partial u_k}(x,p,u) \end{pmatrix} \begin{pmatrix} \mu_1(x,p,u) \\ \vdots \\ \mu_k(x,p,u) \end{pmatrix} = \begin{pmatrix} \{\frac{\partial F}{\partial u_1}, F\}(x,p,u) \\ \vdots \\ \{\frac{\partial F}{\partial u_k}, F\}(x,p,u) \end{pmatrix}$$

has a (smooth) solution on a neighborhood of $(x,p,u) = \phi_F(x,p,\dot{x},\dot{p})$ in $C(F)$.

The differences between the necessary condition and the sufficient condition appear as those of the domain and smoothness of the solution μ.

Now we consider a Morse family of particular type:

$$F\colon \mathbb{R}^{2n}\times\mathbb{R}^k \to \mathbb{R}, \quad F(x,p,u) = \sum_{j=1}^k a_j(x,p)u_j + b(x,p).$$

Note that functions $a_1,\dots,a_k$ are independent, i.e., differential one forms $da_1(x,p),\dots,da_k(x,p)$ are linearly independent in $T^*_{(x,p)}\mathbb{R}^{2n}$ at each point $(x,p)\in\mathbb{R}^{2n}$ because F is a Morse family. The catastrophe set of F is given by $C(F) = K\times\mathbb{R}^k$ with

$$K := \{(x,p)\in\mathbb{R}^{2n} \mid a_i(x,p) = 0, i = 1,\dots,k\}.$$

Applying Proposition 2.4 and Proposition 2.5, we know that L_F is smoothly solvable if and only if $\{a_i, a_j\}(x,p) = 0$, $\{b, a_i\}(x,p) = 0$ on K, $(1 \le i, j \le k)$([5]). Considering conditions in the propositions, we set

$$\widetilde{S_F} := \Big\{(x,p,u) \in C(F) \mid \sum_{j=1}^{k} \{a_i, a_j\}(x,p)u_j = \{b, a_i\}(x,p), 1 \le i \le k\Big\},$$

$$S_F := \phi_F(\widetilde{S_F}).$$

Then we see that every smoothly solvable submanifold of L_F is contained in S_F ([5]). Moreover S_F itself may be smoothly solvable:

Theorem 2.6 ([5])**.** *S_F is a smoothly solvable submanifold of L_F if*

$$\operatorname{rank}\left(\{a_i, a_j\}(x,p)\right)_{1 \le i,j \le k} = r \ (constant) \quad \text{and}$$

$$\begin{pmatrix} \{b, a_1\}(x,p) \\ \vdots \\ \{b, a_k\}(x,p) \end{pmatrix} \in \operatorname{Im}\left(\{a_i, a_j\}(x,p)\right)_{1 \le i,j \le k},$$

holds for every $(x,p) \in K = \{(x,p) \in \mathbb{R}^{2n} \mid \frac{\partial F}{\partial u_i}(x,p,u) = 0, 1 \le i \le k\}$.

§3. Main results

Now we pose a question; for which submanifold S of L_F does there exist a submanifold A of K such that S is smoothly solvable over A? In this section, we will give an answer to this question in the case $k = 2$.

Let $F\colon \mathbb{R}^{2n} \times \mathbb{R}^2 \to \mathbb{R}$ be a Morse family which is defined by

$$F(x,p,u) = a_1(x,p)u_1 + a_2(x,p)u_2.$$

We consider solvability of the Lagrangian submanifold L_F which is generated by F. Moreover we consider solvability of submanifolds of L_F. In detail, we consider a map $\phi_F\colon \mathbb{R}^{2n} \times \mathbb{R}^2 \to T(\mathbb{R}^{2n})$ which is defined by

$$\phi_F(x,p,u) = \left(x, p, \frac{\partial F}{\partial p}(x,p,u), -\frac{\partial F}{\partial x}(x,p,u)\right)$$

and the catastrophe set

$$C(F) = \{(x,p,u) \mid a_1(x,p) = a_2(x,p) = 0\} = K \times \mathbb{R}^2$$

of F for $K = \{(x,p) \mid a_1(x,p) = a_2(x,p) = 0\} \subset \mathbb{R}^{2n}$, then we define L_F by the image $\phi_F(C(F))$. According to Fukuda–Janeczko's Theorem 2.6, L_F is smoothly solvable if and only if $\{a_1, a_2\} = 0$ locally on K.

Now we consider the cases where the assumptions of Theorem 2.6 are not fulfilled. We consider the family of vector fields $X_u\colon K \to T(\mathbb{R}^{2n})$ along K with parameter u

$$X_u(x,p) = \left(x, p, \frac{\partial F}{\partial p}(x,p,u), -\frac{\partial F}{\partial x}(x,p,u)\right),$$

and detect submanifolds of K on which X_u are tangent to K. We are going to give smoothly solvable submanifolds over submanifolds of K in the following series of Propositions 3.1 – 3.6.

Let $A_0 = K$. The vector field X_u is tangent to A_0 if and only if

$$X_u(a_1) = X_u(a_2) = 0 \quad \textit{i.e. } u_1\{a_1,a_2\} = u_2\{a_1,a_2\} = 0.$$

Hence $\phi_F(A_0 \times \mathbb{R}^2)$ is smoothly solvable if and only if $\{a_1,a_2\} = 0$ on $A_0 = K$. This fact is also obtained as a corollary of Theorem 2.6.

Then we consider a submanifold A_1 of A_0 consisting of points at which X_u is tangent to A_0;

$$A_1 := \{(x,p) \mid a_1(x,p) = a_2(x,p) = \{a_1,a_2\}(x,p) = 0\}.$$

We assume that the functions $a_1, a_2, \{a_1,a_2\}$ are independent. The vector field X_u is tangent to A_1 if and only if

$$X_u(\{a_1,a_2\}) = 0 \quad \textit{i.e. } u_1\{a_1,\{a_1,a_2\}\}(x,p) + u_2\{a_2,\{a_1,a_2\}\}(x,p) = 0$$

on A_1. Note that we have $X_u(a_1) = X_u(a_2) = 0$ from the definition of A_1. Let $\mathcal{E}_{\mathbb{R}^{2n},q_0}$ be the $\mathbb{R}$ -algebra of C^∞ function germs at q_0 on $\mathbb{R}^{2n}$. We denote by $\langle a_1, a_2, \{a_1,a_2\}\rangle_{\mathcal{E}_{\mathbb{R}^{2n},q_0}}$ the $\mathcal{E}_{\mathbb{R}^{2n},q_0}$-module generated by a_1, a_2 and $\{a_1,a_2\}$. We set

$$\xi_1 := \{a_1,\{a_1,a_2\}\},\ \xi_2 := \{a_2,\{a_1,a_2\}\}.$$

Then the vector field X_u is tangent to A_1 if the functions ξ_1 and ξ_2 belong to the $\mathcal{E}_{\mathbb{R}^{2n},q_0}$-module $\langle a_1, a_2, \{a_1,a_2\}\rangle_{\mathcal{E}_{\mathbb{R}^{2n},q_0}}$ for any point $q_0 \in A_1$.

Proposition 3.1. *Assume that a_1, a_2, and $\{a_1,a_2\}$ are independent. Then $\phi_F(A_1 \times \mathbb{R}^2)$ is a smoothly solvable submanifold of L_F over A_1 if and only if $\xi_1, \xi_2 \in \langle a_1, a_2, \{a_1,a_2\}\rangle_{\mathcal{E}_{\mathbb{R}^{2n},q_0}}$ for any point q_0 in A_1.*

To find smoothly solvable submanifolds of L_F over submanifolds of A_1, we construct fiber bundles as follows. Let

$$C_{(x,p)} := \{(u_1,u_2) \mid u_1\{a_1,\{a_1,a_2\}\}(x,p) + u_2\{a_2,\{a_1,a_2\}\}(x,p) = 0\}$$

for $(x,p) \in K$ and define line bundles

$$\begin{aligned}
\overline{A_2^1}^1 &:= \{(x,p,u) \mid u \in C^1_{(x,p)}, (x,p) \in A_2^1\},\\
\overline{A_2^1}^2 &:= \{(x,p,u) \mid u \in C^2_{(x,p)}, (x,p) \in A_2^1\},\\
\overline{A_2^2}^1 &:= \{(x,p,u) \mid u \in C^1_{(x,p)}, (x,p) \in A_2^2\},\\
\overline{A_2^2}^2 &:= \{(x,p,u) \mid u \in C^2_{(x,p)}, (x,p) \in A_2^2\},\\
\overline{A_{1,1}}^2 &:= \{(x,p,u) \mid u \in C^2_{(x,p)}, (x,p) \in A_{1,1}\},\\
\overline{A_{1,2}}^1 &:= \{(x,p,u) \mid u \in C^1_{(x,p)}, (x,p) \in A_{1,2}\},\\
\overline{A_{1,(1,2)}}^{1,2} &:= \{(x,p,u) \mid u \in C^{1,2}_{(x,p)}, (x,p) \in A_{1,(1,2)}\},
\end{aligned}$$

with

$$\begin{aligned}
A_2^1 &:= A_1 \cap \{(x,p) \mid \xi_1 = 0\}, & C^1_{(x,p)} &= \{(u_1,0) \in C_{(x,p)}\},\\
A_2^2 &:= A_1 \cap \{(x,p) \mid \xi_2 = 0\}, & C^2_{(x,p)} &= \{(0,u_2) \in C_{(x,p)}\},\\
A_{1,1} &:= A_1 \cap \{(x,p) \mid \xi_1 \neq 0\}, & C^{1,2}_{(x,p)} &= C_{(x,p)} \setminus \{0\},\\
A_{1,2} &:= A_1 \cap \{(x,p) \mid \xi_2 \neq 0\}, & &\\
A_{1,(1,2)} &:= A_1 \cap \{(x,p) \mid \xi_1 \neq 0, \xi_2 \neq 0\}. & &
\end{aligned}$$

Let us consider the case that one of ξ_1 and ξ_2 belongs to the $\mathcal{E}_{\mathbb{R}^{2n},q_0}$-module $\langle a_1, a_2, \{a_1,a_2\}\rangle_{\mathcal{E}_{\mathbb{R}^{2n},q_0}}$ and the other does not.

Proposition 3.2. *Assume that a_1, a_2 and $\{a_1,a_2\}$ are independent. Assume also that*

$$\xi_2 \in \langle a_1, a_2, \{a_1,a_2\}\rangle_{\mathcal{E}_{\mathbb{R}^{2n},q_0}} \text{ and } \xi_1 \notin \langle a_1, a_2, \{a_1,a_2\}\rangle_{\mathcal{E}_{\mathbb{R}^{2n},q_0}}$$

at every point q_0 of A_1. Then the followings hold.

(1) *$\phi_F(\overline{A_{1,1}}^2)$ is a smoothly solvable submanifold of L_F over $A_{1,1}$.*

(2) *Assume, furthermore, that $\xi_1, a_1, a_2, \{a_1,a_2\}$ are independent.*

  (a) *$\phi_F(\overline{A_2^1}^2)$ is a smoothly solvable submanifold of L_F over A_2^1.*

  (b) *$\phi_F(A_2^1 \times \mathbb{R}^2)$ is a smoothly solvable submanifold of L_F over A_2^1 if $\{a_1,\xi_1\} \in \langle a_1, a_2, \{a_1,a_2\}, \xi_1\rangle_{\mathcal{E}_{\mathbb{R}^{2n},q_0}}$ for any point q_0 in A_2^1.*

Proof. (1): $A_{1,1}$ is an open submanifold of A_1 from the definition. Since there exist β_1, β_2 and $\beta_3 \in \mathcal{E}_{\mathbb{R}^{2n},q_0}$ such that $\xi_2 = \beta_1 a_1 + \beta_2 a_2 + \beta_3\{a_1,a_2\}$, the vector $X_u(x,p)$ with $u \in C^2_{(x,p)}$ is tangent to $A_{1,1}$ at each point $(x,p) \in A_{1,1}$ because

$$
\begin{aligned}
&X_u(a_1) = u_2\{a_2, a_1\} = 0,\\
&X_u(a_2) = 0 \cdot \{a_1, a_2\} = 0,\\
&X_u(\{a_1, a_2\}) = 0 \cdot \xi_1 + u_2\xi_2 = u_2\xi_2 = \beta_1 a_1 + \beta_2 a_2 + \beta_3\{a_1, a_2\} = 0
\end{aligned}
$$

on $A_{1,1}$.

(2)-(a): We check the condition that $X_u(x,p)$ is tangent to A_2^1 with $u \in C^2_{(x,p)}$ at each point $(x,p) \in A_2^1$. Note that $\{a_1, \xi_2\} = \{a_2, \xi_1\}$ from Jacobian identity:

$$
\begin{aligned}
\{a_1, \{a_2, \{a_1, a_2\}\}\} &= \{a_2, \{a_1, \{a_1, a_2\}\}\} + \{\{a_1, a_2\}, \{a_1, a_2\}\}\\
&= \{a_2, \{a_1, \{a_1, a_2\}\}\}.
\end{aligned}
$$

Then

$$
X_u(\{a_1, a_2\}) = 0 \cdot \xi_1 + u_2\xi_2 = 0,
$$

$$
\begin{aligned}
X_u(\xi_1) &= u_1\{a_1, \xi_1\} + u_2\{a_2, \xi_1\} = u_1\{a_1, \xi_1\} + u_2\{a_1, \xi_2\} \quad \cdots (\star)\\
&= u_2(\{a_1, \beta_1 a_1\} + \{a_1, \beta_2 a_2\} + \{a_1, \beta_3\{a_1, a_2\}\})\\
&= u_2(a_1\{a_1, \beta_1\} + \beta_2\{a_1, a_2\} + a_2\{a_1, \beta_2\}\\
&\qquad\qquad + \beta_3\xi_1 + \{a_1, a_2\}\{a_1, \beta_3\})\\
&= 0
\end{aligned}
$$

on A_2^1.

(2)-(b): From an equality $(\star)$ we have

$$
X_u(\xi_1) = u_1\{a_1, \xi_1\} + u_2\{a_2, \xi_1\} = u_1\{a_1, \xi_1\} + u_2\{a_1, \xi_2\}.
$$

Since $\xi_2 \in \langle a_1, a_2, \{a_1, a_2\}\rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}}$, it holds that

$$
\{a_1, \xi_2\} \in \langle a_1, a_2, \{a_1, a_2\}, \xi_2\rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}}.
$$

Consequently, by using $\{a_1, \xi_1\} \in \langle a_1, a_2, \{a_1, a_2\}, \xi_2\rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}}$, we obtain $X_u(\xi_1) = 0$ on A_2^1. Q.E.D.

In the same way we have the counterpart of Proposition 3.2.

Proposition 3.3. *Assume that a_1, a_2 and $\{a_1, a_2\}$ are independent. Assume also that*

$$
\xi_1 \in \langle a_1, a_2, \{a_1, a_2\}\rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}} \text{ and } \xi_2 \notin \langle a_1, a_2, \{a_1, a_2\}\rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}}
$$

at every point q_0 of A_1. Then the followings hold.

(1) *$\phi_F(\overline{A_{1,2}}^1)$ is a smoothly solvable submanifold of L_F over $A_{1,2}$.*

(2) *Assume, furthermore, that $\xi_2, a_1, a_2, \{a_1, a_2\}$ are independent.*

(a) *$\phi_F(\overline{A_2^2}^1)$ is a smoothly solvable submanifold of L_F over A_2^2.*

(b) *$\phi_F(A_2^2 \times \mathbb{R}^2)$ is a smoothly solvable submanifold of L_F over A_2^2 if $\{a_2, \xi_2\} \in \langle a_1, a_2, \{a_1, a_2\}, \xi_2\rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}}$ for any point q_0 in A_2^2.*

In the case $\xi_1, \xi_2 \notin \langle a_1, a_2, \{a_1, a_2\}\rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}}$ we have

Proposition 3.4. *Assume that a_1, a_2 and $\{a_1, a_2\}$ are independent. Then $\phi_F(\overline{A_{1,(1,2)}}^{1,2})$ is a smoothly solvable submanifold of L_F over $A_{1,(1,2)}$ if $\xi_1, \xi_2 \notin \langle a_1, a_2, \{a_1, a_2\}\rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}}$ for every point q_0 in $A_{1,(1,2)}$.*

Proof. $A_{1,(1,2)}$ is an open submanifold of A_1 from the definition. The vector $X_u(x,p)$ with $u \in C^{1,2}_{(x,p)}$ is tangent to $A_{1,(1,2)}$ at each point $(x,p) \in A_{1,(1,2)}$ since

$$X_u(a_1) = u_2\{a_2, a_1\} = 0,$$
$$X_u(a_2) = u_1\{a_1, a_2\} = 0,$$
$$X_u(\{a_1, a_2\}) = u_1\xi_1 + u_2\xi_2 = 0$$

on $A_{1,(1,2)}$. Q.E.D.

In Proposition 3.1-3.4, we gave sufficient conditions for existence of smoothly solvable submanifolds of L_F over $A_{1,1}$, A_2^1, $A_{1,2}$, A_2^2 and $A_{1,(1,2)}$ and examples of smoothly solvable submanifolds of L_F over them. The following two propositions give different sufficient conditions for existence of smoothly solvable submanifolds of L_F over A_2^1 and A_2^2 and examples of smoothly solvable submanifolds over them respectively.

Proposition 3.5. *Assume that a_1, a_2, $\{a_1, a_2\}$ and ξ_1 are independent. Then $\phi_F(\overline{A_2^1}^1)$ is a smoothly solvable submanifold of L_F over A_2^1 if $\{a_1, \xi_1\} \in \langle a_1, a_2, \{a_1, a_2\}, \xi_1\rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}}$ for any point q_0 in A_2^1.*

Proof. Since $a_1, a_2, \{a_1, a_2\}$ and ξ_1 are independent, A_2^1 is a submanifold of K. For the vector field X_u along A_2^1 with $(u_1, 0)$,

$$X_u(a_1) = 0 \cdot \{a_1, a_2\} = 0,$$
$$X_u(a_2) = u_1\{a_2, a_1\} = 0,$$
$$X_u(\{a_1, a_2\}) = u_1\xi_1 = 0$$

hold on A_2^1. Since $\{a_1, \xi_1\} \in \langle a_1, a_2, \{a_1, a_2\}, \xi_1 \rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}}$, there exist $\beta_1, \beta_2, \beta_3$ and $\beta_4 \in \mathcal{E}_{\mathbb{R}^{2n}, q_0}$ such that $\{a_1, \xi_1\} = \beta_1 a_1 + \beta_2 a_2 + \beta_3 \{a_1, a_2\} + \beta_4 \xi_1$. Hence we have

$$X_u(\xi_1) = u_1\{a_1, \xi_1\} = u_1(\beta_1 a_1 + \beta_2 a_2 + \beta_3\{a_1, a_2\} + \beta_4 \xi_1) = 0$$

on A_2^1. Thus the vector field X_u with $(u_1, 0)$ is tangent to A_2^1. Q.E.D.

In the same way we have

Proposition 3.6. *Assume that* a_1, a_2, $\{a_1, a_2\}$ *and* ξ_2 *are independent. Then* $\phi_F(\overline{A_2^2}^2)$ *is a smoothly solvable submanifold of* L_F *over* A_2^2 *if* $\{a_2, \xi_2\} \in \langle a_1, a_2, \{a_1, a_2\}, \xi_2 \rangle_{\mathcal{E}_{\mathbb{R}^{2n}, q_0}}$ *for any point* q_0 *in* A_2^2.

§4. Application

We apply the results obtained in §3, to study distributions and its singular curves and we prove Theorem 1. Now we recall some basic notations for the study of distributions. The *Lie flag* of a distribution $\mathcal{D}$ is the sequence $\mathcal{D}_0 \subset \mathcal{D}_1 \subset \cdots$ defined inductively by

$$\mathcal{D}_0 := \mathcal{D}, \quad \mathcal{D}_{i+1} := \mathcal{D}_i + [\mathcal{D}_0, \mathcal{D}_i], \quad i \geq 0.$$

The *small growth vector* of a distribution $\mathcal{D}$ at $q \in M$ is the sequence of the dimension of Lie flags;

$$(\dim \mathcal{D}_0(q), \dim \mathcal{D}_1(q), \dim \mathcal{D}_2(q), \ldots).$$

For example a contact distribution $\mathcal{D} \subset TM$ on a manifold M of dimension $2n+1$, has small growth vector $(2n, 2n+1)$. An Engel distribution $\mathcal{D}$ on 4 dimensional manifold M has small growth vector $(2, 3, 4)$.

In this section we consider distributions with small growth vector $(2, 3, 4, \ldots)$. The following lemma plays an essential role throughout the section.

Lemma 4.1. *Let* $\mathcal{D}$ *be a rank two distribution with small growth vector* $(2, 3, 4, \ldots)$ *at any point in an open neighborhood of* $q \in M$ *and* g *a bi-linear positive definite form on* $\mathcal{D}$. *Then there exist an open neighborhood* U_q *and local orthonormal frame* X_1, X_2 *of* $\mathcal{D}$ *on* U_q *such that*

$$X_1, X_2, [X_1, X_2], [X_1, [X_1, X_2]]$$

are linearly independent at q *and* $[X_2, [X_1, X_2]]$ *is a functional linear combination of* X_1, X_2 *and* $[X_1, X_2]$ *on* U_q.

Proof. $\mathcal{E}_{M,q}$ denotes the $\mathbb{R}$ -algebra of C^∞ function germs at q on M. Let X_1, X_2 be any local frame of $\mathcal{D}$ around q. We may suppose that $X_1, X_2, [X_1, X_2], [X_1, [X_1, X_2]]$ are linearly independent at q. From the assumption, $[X_2, [X_1, X_2]] \in \langle X_1, X_2, [X_1, X_2], [X_1, [X_1, X_2]]\rangle_{\mathcal{E}_{M,q}}$. Then there exists a $\lambda \in \mathcal{E}_{M,q}$ such that

$$[X_2, [X_1, X_2]] \equiv \lambda[X_1, [X_1, X_2]] \mod \langle X_1, X_2, [X_1, X_2]\rangle_{\mathcal{E}_{M,q}}$$

Set $\tilde{X}_2 = X_2 - \lambda X_1$. Then $(X_1, \tilde{X}_2)$ is a local frame of $\mathcal{D}$ around q. Then

$$\begin{aligned} [\tilde{X}_2, [X_1, \tilde{X}_2]] &= [\tilde{X}_2, [X_1, X_2 - \lambda X_1]] \\ &= [\tilde{X}_2, [X_1, X_2]] - [\tilde{X}_2, X_1(\lambda)X_1] \\ &= [\tilde{X}_2, [X_1, X_2]] - X_1(\lambda)[\tilde{X}_2, X_1] - \tilde{X}_2(X_1(\lambda))X_1 \\ &\equiv [\tilde{X}_2, [X_1, X_2]] = [X_2 - \lambda X_1, [X_1, X_2]] \\ &\equiv 0 \mod \langle X_1, \tilde{X}_2, [X_1, \tilde{X}_2]\rangle_{\mathcal{E}_{M,q}}. \end{aligned}$$

For functions

$$g_{11} := g(X_1, X_1),\ g_{12} := g(X_1, \tilde{X}_2) \text{ and } g_{22} := g(\tilde{X}_2, \tilde{X}_2),$$

we set

$$X_1' = \frac{\sqrt{g_{22}}}{\sqrt{g_{11}g_{22} - g_{12}^2}}\left(X_1 - \frac{g_{12}}{g_{22}}\tilde{X}_2\right),\ X_2' = \frac{1}{\sqrt{g_{22}}}\tilde{X}_2.$$

Then (X_1', X_2') is a local orthonormal basis of $\mathcal{D}$ around q i.e.,

$$g(X_1', X_1') = 1,\ g(X_1', X_2') = 0,\ g(X_2', X_2') = 1,$$

and

$$X_1', X_2', [X_1', X_2'], [X_1', [X_1', X_2']]$$

are linearly independent since X_1' and X_2' is a functional linear combination of X_1 and $\tilde{X}_2$. Moreover (X_1', X_2') satisfies

$$[X_2', [X_1', X_2']] \equiv 0 \mod \langle X_1', X_2', [X_1', X_2']\rangle_{\mathcal{E}_{M,q}}$$

because of the following for functions

$$\alpha_1 = \frac{\sqrt{g_{22}}}{\sqrt{g_{11}g_{22} - g_{12}^2}},\ \alpha_2 = -\frac{\sqrt{g_{22}}}{\sqrt{g_{11}g_{22} - g_{12}^2}}\frac{g_{12}}{g_{22}} \text{ and } \alpha_3 = \frac{1}{\sqrt{g_{22}}};$$

$$
\begin{aligned}
[X_2',[X_1',X_2']] &= [\alpha_3\tilde{X}_2,[\alpha_1 X_1,\alpha_3\tilde{X}_2]] - [\alpha_3\tilde{X}_2,[\alpha_2\tilde{X}_2,\alpha_3\tilde{X}_2]] \\
&= [\alpha_3\tilde{X}_2,\alpha_1\alpha_3[X_1,\tilde{X}_2]] + [\alpha_3\tilde{X}_2,\alpha_1 X_1(\alpha_3)\tilde{X}_2] \\
&\qquad\qquad\qquad - [\alpha_3\tilde{X}_2,\alpha_3\tilde{X}_2(\alpha_1)X_1] \\
&\equiv [\alpha_3\tilde{X}_2,\alpha_1\alpha_3[X_1,\tilde{X}_2]] \mod \langle X_1',X_2',[X_1',X_2']\rangle_{\mathcal{E}_{M,q}} \\
&\equiv \alpha_1\alpha_3^2[\tilde{X}_2,[X_1,\tilde{X}_2]] \mod \langle X_1',X_2',[X_1',X_2']\rangle_{\mathcal{E}_{M,q}} \\
&\equiv 0 \mod \langle X_1',X_2',[X_1',X_2']\rangle_{\mathcal{E}_{M,q}}
\end{aligned}
$$

Q.E.D.

Let $\{X_1,X_2\}$ be a local frame of $\mathcal{D}$ on U_q for any $q\in M$ with the property of Lemma 4.1 and define a function $H\colon T^*U_q\times_{U_q}\mathcal{D}\to\mathbb{R}$ for the distribution $\mathcal{D}$ locally by

$$H(x,p,u)=u_1\langle p,X_1(x)\rangle+u_2\langle p,X_2(x)\rangle.$$

For functions $a_1(x,p):=\langle p,X_1(x)\rangle$ and $a_2(x,p):=\langle p,X_2(x)\rangle$, Proposition 3.2-(2)-(a) can be applied and we obtain the following

Proposition 4.2. *For a rank* 2 *distribution* $\mathcal{D}$ *with small growth vector* $(2,3,4,\dots)$ *at each point* q *in* M, *there exist an open neighborhood* U_q *of* q, *a frame* $\{X_1,X_2\}$ *of* $\mathcal{D}$ *on* U_q *and an abnormal bi-extremal* $(x(t),p(t))$ *in* $T^*U_q\setminus\{o\}$ *such that*

$$\dot{x}(t)=X_2(x(t)),\quad \dot{p}(t)=-\frac{\partial\langle p,X_2(x)\rangle}{\partial x}(x(t),p(t))$$

and

$$
\begin{aligned}
\langle p(t),X_1(x(t))\rangle=0,&\qquad \langle p(t),X_2(x(t))\rangle=0,\\
\langle p(t),[X_1,X_2](x(t))\rangle=0,&\quad \langle p(t),[X_1,[X_1,X_2]](x(t))\rangle=0.
\end{aligned}
$$

Proof. From the property in Lemma 4.1 we take a local frame $\{X_1,X_2\}$ of $\mathcal{D}$ on an open neighborhood U_q of q in M such that

$$X_1,X_2,[X_1,X_2],[X_1,[X_1,X_2]]$$

are linearly independent and $[X_2,[X_1,X_2]]$ is a functional linear combination of X_1,X_2 and $[X_1,X_2]$. We take the function $H\colon T^*U_q\times_{U_q}\mathcal{D}\to\mathbb{R}$ as

$$H(x,p,u)=\langle p,X_1(x)\rangle u_1+\langle p,X_2(x)\rangle u_2$$

and set $a_1(x,p):=\langle p,X_1(x)\rangle$ and $a_2(x,p):=\langle p,X_2(x)\rangle$. Then H is a Morse family because X_1 and X_2 are linearly independent at each point.

In conformity with the property of vector fields X_1 and X_2, functions $a_1, a_2, \{a_1, a_2\}$ and $\{a_1, \{a_1, a_2\}\}$ are independent and we have

$$a_1, a_2, \{a_1, a_2\}, \{a_2, \{a_1, a_2\}\} \in \langle a_1, a_2, \{a_1, \{a_1, a_2\}\}\rangle_{\mathcal{E}_{T^*U_q,Q}}$$

for any $Q \in T^*U_q$.

According to the proof of Proposition 3.2, X_u is a tangent vector field to a submanifold

$$\begin{aligned} A_2^1 = \{(x,p) \in T^*U_q \mid a_1(x,p) = a_2(x,p) &= \{a_1, a_2\}(x,p) \\ &= \{a_1, \{a_1, a_2\}\}(x,p) = 0\} \end{aligned}$$

of T^*U_q for $u = (0,1)$. Thus there exists an integral curve $(x(t), p(t))$ of X_u starting from a point in A_2^1, that is, $(x(t), p(t))$ satisfies ordinary differential equations

$$\begin{aligned} \dot{x}(t) &= \frac{\partial H}{\partial p}(x(t), p(t), (0,1)) = \frac{\partial \langle p, X_2(x)\rangle}{\partial p}(x(t), p(t)) = X_2(x(t)) \\ \dot{p}(t) &= -\frac{\partial H}{\partial x}(x(t), p(t), (0,1)) = -\frac{\partial \langle p, X_2(x)\rangle}{\partial x}(x(t), p(t)) \end{aligned}$$

and following conditions;

$$\begin{aligned} a_1(x(t), p(t)) = \langle p(t), X_1(x(t))\rangle = 0, &\quad \langle p(t), X_2(x(t))\rangle = 0, \\ \langle p(t), [X_1, X_2](x(t))\rangle = 0, &\quad \langle p(t), [X_1, [X_1, X_2]](x(t))\rangle = 0. \end{aligned}$$

Q.E.D.

Theorem 4.3. *Let M be a smooth manifold and $\mathcal{D}$ be a rank two distribution. Suppose that the distribution $\mathcal{D}$ has small growth vector $(2, 3, 4, \ldots)$ everywhere in an open neighborhood of any point q in M. Then for any point q in M, there exist an open neighborhood U_q of q in M and a C^∞ immersive singular curve $x(t)$ in U_q which is defined on a small interval.*

Proof. From Proposition 4.2, for any point q in M, there exist an open neighborhood U_q of q and an abnormal bi-extremal $(x(t), p(t))$ on $A_2^1 \subset T^*U_q$. Therefore the projection of $(x(t), p(t))$ by canonical projection $\pi_M \colon T^*M \to M$ is a singular curve $x(t)$ with admissible velocity directed to X_2 in $\pi_M(A_2^1)$. Q.E.D.

We now prove Theorem 1.

Proof of Theorem 1. Let q be a point of M and let $x(t)$ be the C^∞ immersive singular horizontal curve in a neighborhood U_q of q obtained in Theorem 4.3. Let $(x(t), p(t))$ be the abnormal bi-extremal considered in the proof of Theorem 4.3 which is obtained in Proposition 4.2. We are going to prove that this curve $x(t)$ is not a normal extremal. From Lemma 4.1 we may take a local orthonormal frame $\{X_1, X_2\}$ of $\mathcal{D}$ on U_q. We consider the Hamiltonian function in terms of the orthonormal frame $\{X_1, X_2\}$;

$$H_E(x,p) = -\frac{1}{2}\sum_{i=1}^{2}\langle p, X_i(x)\rangle^2.$$

Suppose that $x(t)$ is a normal extremal. Then there must exist a normal bi-extremal of the form $(x(t), \tilde{p}(t))$ which satisfies the following differential equation;

$$\begin{aligned}
\dot{x}(t) &= \frac{\partial H_E}{\partial p}(x(t), \tilde{p}(t)) = -\sum_{i=1}^{2} X_i(x(t)),\\
\dot{\tilde{p}}(t) &= -\frac{\partial H_E}{\partial x}(x(t), \tilde{p}(t)) = \sum_{i=1}^{2}\frac{\partial\langle p, X_i(x)\rangle}{\partial x}(x(t), \tilde{p}(t)).
\end{aligned}$$

Since the abnormal extremal $x(t)$ satisfies $\dot{x}(t) = X_2(t)$ by Proposition 4.2,

$$X_2(x(t)) = \dot{x}(t) = -X_1(x(t)) - X_2(x(t))$$

holds. Thus $X_1(x(t)) + 2X_2(x(t)) = 0$ holds. This is a contradiction to $\{X_1, X_2\}$ being a local frame of $\mathcal{D}$. Q.E.D.

Remark 4.4 ([2], Theorem 2.8)**.** *It is known that there are no singular minimizer for a generic sub-Riemannian manifold* (*the genericity is used for the distribution as map-germs*) *with rank greater than 2.*

Remark 4.5 ([7], Proposition 11)**.** *There is a result given by Liu–Sussmann which is similar type to Theorem 1, however the method for the proof is different from ours.*

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