

Constant mean curvature surfaces with D_4 -singularities

Yuta Ogata and Keisuke Teramoto

Abstract.

We study D_4 -singularities of constant mean curvature (CMC) surfaces in Riemannian and semi-Riemannian spaceforms. We will give criteria for D_4^- -singularities, which are related to the Hopf differential. We also show the non-existence of (spacelike) CMC surfaces with D_4^+ -singularities.

§1. Introduction

Singularities of wave fronts can appear on surfaces via parallel transformations. It is known that generic singularities of wave fronts in 3-spaces are cuspidal edges and swallowtails. Moreover, the singularities of the bifurcations in generic one-parameter families of wave fronts in 3-spaces are cuspidal lips, cuspidal beaks, cuspidal butterflies and D_4 -singularities, in addition to the above two (see [1, 15]). There are criteria for these singularities in [16, 17, 20, 26]. On the other hand, in [11], Fukui and Hasegawa studied singularities of the parallel surfaces of regular surfaces in Euclidean 3-space \mathbb{R}^3 . They gave criteria for cuspidal edges, swallowtails, cuspidal lips, cuspidal beaks, cuspidal butterflies and D_4 -singularities by using geometric invariants of the original surfaces, for example principal curvatures.

For constant mean curvature (CMC) surfaces in Riemannian and semi-Riemannian spaceforms, there are several studies. In general, CMC surfaces in semi-Riemannian spaceforms have singularities (see [4, 10, 14, 23, 24, 29, 30], for example). In [30], criteria for cuspidal edges and swallowtails of maximal surfaces were obtained by using Weierstrass data.

Received January 18, 2016.

Revised March 28, 2017.

2010 *Mathematics Subject Classification*. Primary 53A10; Secondary 53B30, 57R45.

Key words and phrases. constant mean curvature surface, parallel transformation, singularity, wave front.

Similarly, for maximal surfaces and CMC 1 surfaces, criteria for corank one singularities were given in [10]. Umeda [29] gave criteria for cuspidal edges, swallowtails and cuspidal cross caps of extended CMC surfaces in Minkowski 3-space $\mathbb{R}^{2,1}$, and in [23, 24], the first author of this paper also studied the analogues of Umeda's criteria for these singularities of extended CMC surfaces in other semi-Riemannian spaceforms.

However, the above previous studies did not consider corank two singularities of such surfaces. It is well-known that the corank two singularities appear even on CMC surfaces in Riemannian spaceforms. Thus in this paper, we consider the criteria for the corank two singularities, especially D_4 -singularities.

For CMC $H \neq 0$ surfaces in \mathbb{R}^3 and $\mathbb{R}^{2,1}$, the following fact is known. (For the definition of \hat{f}^t , see (2.4).)

Fact 1.1 ([11, 13]). *Let f be a conformal (spacelike) CMC surface in \mathbb{R}^3 (or $\mathbb{R}^{2,1}$) with mean curvature $H > 0$, unit normal vector ν and the Hopf differential factor Q , and let p be an umbilic point of f . Then for $t = 1/H$, the parallel transform \hat{f}^t of f becomes a conformal (spacelike) CMC surface in \mathbb{R}^3 (or $\mathbb{R}^{2,1}$) with mean curvature $-H$ and the Hopf differential factor $-Q$. Moreover, \hat{f}^t has a corank two singularity at p .*

For CMC surfaces in the spherical 3-space \mathbb{S}^3 , hyperbolic 3-space \mathbb{H}^3 , de Sitter 3-space $\mathbb{S}^{2,1}$ and anti-de Sitter 3-space $\mathbb{H}^{2,1}$, the following is known. (For definitions of \hat{f}^t and \check{f}^t , see (2.5) and (2.6).)

Fact 1.2 ([6, 8]). *Let $f : U \rightarrow M^3$ be a conformal (spacelike) CMC H surface and ν its unit normal vector.*

- (1) *If $M^3 = \mathbb{H}^3$ or $\mathbb{S}^{2,1}$ with $H > 1$, then for $t = \operatorname{arccoth} H$, \hat{f}^t becomes a conformal (spacelike) CMC surface in M^3 with mean curvature $-H$.*
- (2) *If $M^3 = \mathbb{H}^3$ (resp. $\mathbb{S}^{2,1}$) with $0 < H < 1$, then for $t = \operatorname{arctanh} H$, $\check{f}^t = \hat{\nu}^t$ becomes a conformal spacelike CMC surface in $\mathbb{S}^{2,1}$ (resp. \mathbb{H}^3) with mean curvature $-H$.*
- (3) *If $M^3 = \mathbb{S}^3$ or $\mathbb{H}^{2,1}$ with mean curvature $H > 0$, then for $t = \operatorname{arccot} H$, \hat{f}^t is a conformal (spacelike) CMC surface in M^3 with mean curvature $-H$. Moreover, if f is a conformally immersed minimal (resp. maximal) surface, then ν is a conformal minimal (resp. maximal) surface in M^3 and f is a unit normal vector to ν .*

By these facts, considering parallel transforms of CMC surfaces naturally emphasizes their umbilic points, and parallel transforms play an important role in understanding relations between umbilic points and corank two singularities. Moreover, by Facts 1.1 and 1.2, we can start

to consider a regular CMC surface f with an umbilic point instead of a CMC surface \hat{f} (or \check{f}) with a D_4 -singularity via parallel transform. Thus, in this paper, we study (spacelike) CMC surfaces with D_4 -singularities in Riemannian and semi-Riemannian spaceforms, and we give criteria for D_4 -singularities in terms of the Hopf differential factors (Theorem 3.2). For minimal surfaces, we give conditions for which they have D_4 -singularities by using Weierstrass data (Theorems 3.6 and A.4).

For surfaces in \mathbb{S}^3 , \mathbb{H}^3 , $\mathbb{S}^{2,1}$ or $\mathbb{H}^{2,1}$, their unit normal vectors form surfaces. Thus we can compare curvatures of CMC surfaces with curvatures of their unit normal vectors (Proposition 4.1). Moreover, we show a kind of duality between parallel transforms of CMC surfaces and their unit normal vectors (Proposition 4.3).

§2. Preliminaries

2.1. Surfaces in spaceforms

We recall some properties of surfaces in several spaceforms. For more details, see [15, 16, 17].

Let $\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R}, 1 \leq i \leq n\}$ be an n -dimensional vector space. For any $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$, we define the *pseudo inner product with the signature $(n - k, k)$* ($0 \leq k < n$) by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n-k} x_i y_i - \sum_{j=n-k+1}^n x_j y_j.$$

We denote $\mathbb{R}^{n-k,k} = (\mathbb{R}^n, \langle \cdot, \cdot \rangle)$. We say that a vector $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ is *spacelike*, *timelike* or *lightlike* if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$, < 0 or $= 0$ respectively. We note that $\mathbb{R}^{n,0} = \mathbb{R}^n$ is the *Euclidean n -space*. If $k = 1$, we call the space $\mathbb{R}^{n-1,1}$ the *Minkowski n -space*.

Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the pseudo orthonormal basis of $\mathbb{R}^{n-k,k}$ and $\mathbf{x}^i = (x_1^i, \dots, x_n^i) \in \mathbb{R}^{n-k,k}$ ($1 \leq i \leq n-1$). Then we define the *wedge product $\mathbf{x}^1 \wedge \dots \wedge \mathbf{x}^{n-1}$* with respect to the signature $(n - k, k)$ by

$$(2.1) \quad \mathbf{x}^1 \wedge \dots \wedge \mathbf{x}^{n-1} = \begin{vmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_{n-k} & -\mathbf{e}_{n-k+1} & \cdots & -\mathbf{e}_n \\ x_1^1 & \cdots & x_{n-k}^1 & x_{n-k+1}^1 & \cdots & x_n^1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & \cdots & x_{n-k}^{n-1} & x_{n-k+1}^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}.$$

One can check that $\langle \mathbf{x}^i, \mathbf{x}^1 \wedge \dots \wedge \mathbf{x}^{n-1} \rangle = 0$ holds for $1 \leq i \leq n - 1$.

Let $n = 4$. Then we define the following spaceforms:

$$\begin{aligned} \mathbb{S}^3 &= \{\mathbf{x} \in \mathbb{R}^4 \mid \langle \mathbf{x}, \mathbf{x} \rangle = 1\}, & \mathbb{H}^3 &= \{\mathbf{x} \in \mathbb{R}^{3,1} \mid \langle \mathbf{x}, \mathbf{x} \rangle = -1\}, \\ \mathbb{S}^{2,1} &= \{\mathbf{x} \in \mathbb{R}^{3,1} \mid \langle \mathbf{x}, \mathbf{x} \rangle = 1\}, & \mathbb{H}^{2,1} &= \{\mathbf{x} \in \mathbb{R}^{2,2} \mid \langle \mathbf{x}, \mathbf{x} \rangle = -1\}. \end{aligned}$$

We call \mathbb{S}^3 , \mathbb{H}^3 , $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$ the *spherical 3-space*, the *hyperbolic 3-space*, the *de Sitter 3-space* and the *anti-de Sitter 3-space*, respectively. It is known that \mathbb{S}^3 and $\mathbb{S}^{2,1}$ (resp. \mathbb{H}^3 and $\mathbb{H}^{2,1}$) have constant sectional curvature 1 (resp. -1).

Let M^3 be a 3-dimensional spaceform one of \mathbb{R}^3 , $\mathbb{R}^{2,1}$, \mathbb{S}^3 , \mathbb{H}^3 , $\mathbb{S}^{2,1}$ or $\mathbb{H}^{2,1}$. Let $f : U \rightarrow M^3$ be a surface, where $U \subset (\mathbb{R}^2; u, v)$ is an open set. The surface f is said to be *spacelike* if the induced metric via f is positive definite on U . Then we consider the *unit normal vector* ν to f . If $M^3 = \mathbb{R}^3$ or $\mathbb{R}^{2,1}$, ν is defined as

$$\nu = \frac{f_u \wedge f_v}{|f_u \wedge f_v|} \quad (f_u = \partial f / \partial u, \quad f_v = \partial f / \partial v),$$

where $|\mathbf{x}| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$. If M^3 is one of the other spaceforms, ν can be taken as

$$\nu = \frac{f \wedge f_u \wedge f_v}{|f \wedge f_u \wedge f_v|}.$$

In these cases, we use $\langle \cdot, \cdot \rangle$ as the induced metric from the ambient space $\mathbb{R}^{4-k,k}$ ($k = 0, 1, 2$).

2.2. CMC surface theory

In this section, we explain some basical notations, as in [2] and [5]. Let M^3 be one of \mathbb{R}^3 , $\mathbb{R}^{2,1}$, \mathbb{S}^3 , \mathbb{H}^3 , $\mathbb{S}^{2,1}$ or $\mathbb{H}^{2,1}$. Let $f : U \rightarrow M^3$ be a spacelike surface, where U is a simply-connected domain in \mathbb{C} with usual complex coordinate $z = u + iv$ ($i = \sqrt{-1}$). We say that f is a *conformal surface* if there exists a *conformal coordinate system* on U , namely, $\langle f_z, f_z \rangle = \langle f_{\bar{z}}, f_{\bar{z}} \rangle = 0$ and $\langle f_z, f_{\bar{z}} \rangle = 2g^2$ hold for some function $g : U \rightarrow \mathbb{R}$, where $\partial_z = (\partial_u - i\partial_v)/2$ and $\partial_{\bar{z}} = (\partial_u + i\partial_v)/2$. For a conformal surface f , the first fundamental form of f is given as

$$ds^2 = 4g^2(du^2 + dv^2).$$

Take the unit normal vector field ν . Then the mean curvature H and the Hopf differential factor Q are given by

$$(2.2) \quad H = \frac{1}{2g^2} \langle f_{z\bar{z}}, \nu \rangle, \quad Q = \langle f_{zz}, \nu \rangle.$$

By (2.2), one can check that H and Q change to $-H$ and $-Q$, respectively, when we change ν to $-\nu$. We assume that H is constant. It is

known that the Codazzi equation implies that Q is holomorphic. Moreover, the extrinsic Gaussian curvature K is written as

$$(2.3) \quad K = -\frac{1}{4g^4}Q\bar{Q} + H^2.$$

We now define the *parallel transforms* \hat{f}^t and \check{f}^t of f . If $M^3 = \mathbb{R}^3$ or $\mathbb{R}^{2,1}$,

$$(2.4) \quad \hat{f}^t = f + t\nu$$

for some constant $t \in \mathbb{R}$. In this case, ν is also a unit normal vector to \hat{f}^t . If $M^3 = \mathbb{S}^3$ or $\mathbb{H}^{2,1}$, \hat{f}^t and $\hat{\nu}^t$ are

$$(2.5) \quad \hat{f}^t = \cos tf + \sin t\nu, \quad \hat{\nu}^t = -\sin tf + \cos t\nu$$

for some constant $t \in \mathbb{R}$. If $M^3 = \mathbb{H}^3$ or $\mathbb{S}^{2,1}$, we define

$$(2.6) \quad \hat{f}^t = \cosh tf + \sinh t\nu, \quad \check{f}^t = \hat{\nu}^t = \sinh tf + \cosh t\nu$$

for some constant $t \in \mathbb{R}$ (cf. [6] and [8]).

Remark 2.1. It is a well-known fact that if the spaceforms or the value of H change, then the integrable equation (i.e. Gauss equation) also changes. This change creates a difference in the construction method of the CMC surface f (see [8] for example). However, we will show the existence and non-existence of D_4^\pm -singularities of the CMC surface without depending on the choice of the spaceform or the value of H (see Theorem 3.2).

2.3. Wave fronts

We recall some notions of wave fronts. For details, see [1, 10, 15, 17, 27].

Let $f : U \rightarrow M^3$ be a C^∞ map, where U is a simply-connected domain in \mathbb{R}^2 and M^3 is one of \mathbb{R}^3 , $\mathbb{R}^{2,1}$, \mathbb{S}^3 , \mathbb{H}^3 , $\mathbb{S}^{2,1}$ or $\mathbb{H}^{2,1}$. We call f a *wave front* or *front* if for each point $p \in U$ there exists a unit normal vector field ν along f and the map $L = (f, \nu) : U \rightarrow T_1M^3$ gives an surface, where T_1M^3 is the unit tangent bundle over M^3 . A point $p \in U$ is called a *singular point* if f is not an surface at p . Let $S(f)$ denote the set of singular points of f . We set a function λ on U as

$$(2.7) \quad f_u \wedge f_v = \lambda\nu$$

when $M^3 = \mathbb{R}^3$ or $\mathbb{R}^{2,1}$, and

$$(2.8) \quad f \wedge f_u \wedge f_v = \lambda\nu$$

when $M^3 = \mathbb{S}^3, \mathbb{H}^3, \mathbb{S}^{2,1}$ or $\mathbb{H}^{2,1}$, where \wedge denotes the wedge product as in (2.1). We call this function λ the *signed area density function*. By definition, $\lambda^{-1}(0) = S(f)$ holds. A singular point p is called *non-degenerate* if the exterior derivative $d\lambda$ does not vanish at p . On a neighborhood of a non-degenerate singular point, there exists a smooth regular curve $\gamma(t)$ satisfying $\gamma(0) = p$ such that $\gamma(t)$ parametrizes the set of singular points. We call this curve γ a *singular curve* and the direction of $\gamma' = d\gamma/dt$ a *singular direction*. The dimension of the kernel $\text{Ker } df_{\gamma(t)}$ of the differential map $df_{\gamma(t)}$ is one and there exists a never-vanishing vector field $\eta(t)$ such that $\langle \eta(t) \rangle_{\mathbb{R}} = \text{Ker } df_{\gamma(t)}$. We call $\eta(t)$ a *null vector field* and the direction of η a *null direction*.

Definition 2.2. Let $f : (U, p) \rightarrow (\mathbb{R}^3, f(p))$ be a map-germ around p . Then f has a *cuspidal edge* at p if the map-germ f at p is \mathcal{A} -equivalent to the map-germ $(u, v) \mapsto (u, v^2, v^3)$ at $\mathbf{0}$, and f has a *swallowtail* at p if the map-germ f at p is \mathcal{A} -equivalent to the map-germ $(u, v) \mapsto (u, 3v^4 + uv^2, 4v^3 + 2uv)$ at $\mathbf{0}$, and f has a D_4^\pm -singularity at p if the map-germ f at p is \mathcal{A} -equivalent to the map-germ $(u, v) \mapsto (2uv, \pm u^2 + 3v^2, \pm 2u^2v + 2v^3)$ at $\mathbf{0}$, where the two map-germs $f, g : (\mathbb{R}^2, \mathbf{0}) \rightarrow (\mathbb{R}^3, \mathbf{0})$ are said to be \mathcal{A} -equivalent if there exist diffeomorphism-germs $\theta : (\mathbb{R}^2, \mathbf{0}) \rightarrow (\mathbb{R}^2, \mathbf{0})$ on the source and $\Theta : (\mathbb{R}^3, \mathbf{0}) \rightarrow (\mathbb{R}^3, \mathbf{0})$ on the target such that $\Theta \circ f = g \circ \theta$ holds.

We note that cuspidal edges and swallowtails are non-degenerate singular points of fronts. On the other hand, D_4 -singularities are degenerate singular points with corank two. There are well-known criteria for cuspidal edges and swallowtails (see [20, Proposition 1.3]). There is a criterion for D_4^\pm -singularities as well.

Fact 2.3 ([26, Theorem 1.1]). *Let f be a front and λ the signed area density function. A singular point p is a D_4^+ -singularity (resp. D_4^- -singularity) if and only if the following conditions hold:*

- (1) $\text{rank } df_p = 0$.
- (2) $\det \text{Hess } \lambda < 0$ (respectively, $\det \text{Hess } \lambda > 0$) at p .

§3. Constant mean curvature surfaces with D_4 -singularities

3.1. Surfaces with non-zero constant mean curvature

In this section, we consider the cases such that

$$(3.1) \quad \begin{cases} H \neq 0 \text{ if } M^3 = \mathbb{R}^3, \mathbb{R}^{2,1}, \mathbb{S}^3 \text{ or } \mathbb{H}^{2,1}, \\ H \neq 0, 1 \text{ if } M^3 = \mathbb{H}^3 \text{ or } \mathbb{S}^{2,1}. \end{cases}$$

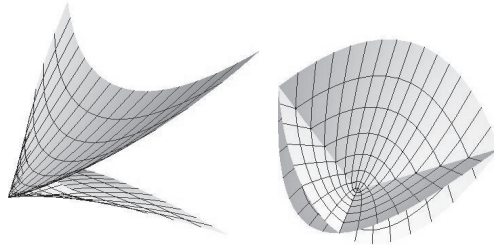


Fig. 1. The left hand side is a D_4^+ -singularity and the right hand side is a D_4^- - singularity of a wave front.

Lemma 3.1. *Let $f : (U, z) \rightarrow M^3$ be a conformal CMC H surface, ν a unit normal vector to f and p an umbilic point.*

- (1) *Suppose that $M^3 = \mathbb{S}^3$ or $\mathbb{H}^{2,1}$ and $H > 0$. Then p is a corank two singular point of \hat{f}^t if and only if $t = \operatorname{arccot} H$.*
- (2) *Suppose that $M^3 = \mathbb{H}^3$ or $\mathbb{S}^{2,1}$ and $H > 1$. Then p is a corank two singular point of \hat{f}^t if and only if $t = \operatorname{arccoth} H$.*
- (3) *Suppose that $M^3 = \mathbb{H}^3$ or $\mathbb{S}^{2,1}$ and $0 < H < 1$. Then p is a corank two singular point of \check{f}^t if and only if $t = \operatorname{arctanh} H$.*

Proof. We show (2) and (3) in the case of $M^3 = \mathbb{H}^3$. For the case of $M^3 = \mathbb{S}^{2,1}$ and (1), one can show the result in a similar way.

Let $f : U \rightarrow M^3 = \mathbb{H}^3$ be a conformal CMC H surface. Suppose that $H > 1$. Then we consider \hat{f}^t as in (2.6). Since $\nu_z = (-2Hf_z - Qg^{-2}f_{\bar{z}})/2$ and $\nu_{\bar{z}} = (-\bar{Q}g^{-2}f_z - 2Hf_{\bar{z}})/2$ by (2.2), we have

$$\begin{aligned} \hat{f}_z^t &= (\cosh t - H \sinh t)f_z - \frac{Q}{2g^2} \sinh t f_{\bar{z}}, \\ \hat{f}_{\bar{z}}^t &= -\frac{\bar{Q}}{2g^2} \sinh t f_z + (\cosh t - H \sinh t)f_{\bar{z}}. \end{aligned}$$

Since p is an umbilic point, $Q(p) = \bar{Q}(p) = 0$. Thus $\hat{f}_z^t = \hat{f}_{\bar{z}}^t = \mathbf{0}$ at p if and only if $t = \operatorname{arccoth} H$. Therefore we have the assertion (2).

Next we show (3). Assume that $0 < H < 1$. By direct computations, we see that

$$\begin{aligned} \check{f}_z^t &= (\sinh t - H \cosh t)f_z - \frac{Q}{2g^2} \cosh t f_{\bar{z}}, \\ \check{f}_{\bar{z}}^t &= -\frac{\bar{Q}}{2g^2} \cosh t f_z + (\sinh t - H \cosh t)f_{\bar{z}}. \end{aligned}$$

Hence $\check{f}_z^t = \check{f}_{\bar{z}}^t = \mathbf{0}$ at p if and only if $t = \operatorname{arctanh} H$. Q.E.D.

Theorem 3.2. *Let f be a CMC surface in M^3 with mean curvature H satisfying the condition (3.1), $Q(z)$ the Hopf differential factor and p a corank two singular point. Then f has a D_4^- -singularity at p if and only if $Q_z(p) \neq 0$. Moreover, f does not have a D_4^+ -singularity at p .*

Proof. By Facts 1.1 and 1.2, if $f : U \rightarrow M^3$ is a CMC H surface, then \hat{f}^t and \check{f}^t are CMC H surfaces on the set of regular points for suitable distance t . Therefore we consider the CMC surface f with an umbilic point p and singularities of its parallel transform \hat{f}^t at p . Since one can prove this similarly by using Lemma 3.1 in other cases, we consider just the case of $M^3 = \mathbb{R}^3$.

Let $f : U \rightarrow \mathbb{R}^3$ be a CMC surface and p an umbilic point, where $U \subset \mathbb{C}$ is a simply-connected domain with conformal coordinate $z = u + iv$. By the previous section, $\langle f_z, f_z \rangle = \langle f_{\bar{z}}, f_{\bar{z}} \rangle = 0$ and $\langle f_z, f_{\bar{z}} \rangle = 2g^2$ for some function g on U . We now consider the parallel transformation of f given by $\hat{f}^t = f + t\nu$, where $t \in \mathbb{R}$ is constant. In this case, we can take a unit normal vector $\hat{\nu}^t$ of \hat{f}^t as ν . By Fact 1.1, $\text{rank } d\hat{f}^t(p) = 0$ if and only if $t = 1/H$.

We fix $t = 1/H$. The signed area density function of \hat{f}^t is given by $\hat{\lambda}^t = \langle \hat{f}_u^t \wedge \hat{f}_v^t, \nu \rangle = -2i \langle \hat{f}_z^t \wedge \hat{f}_{\bar{z}}^t, \nu \rangle$. Using $\nu_z = (-2Hf_z - Qg^{-2}f_{\bar{z}})/2$, $\nu_{\bar{z}} = (-2Hf_{\bar{z}} - \bar{Q}g^{-2}f_z)/2$ and (2.3), the signed area density function $\hat{\lambda}^t$ is rewritten as

$$\hat{\lambda}^t = -2i(1 - 2tH + t^2K) \langle f_z \wedge f_{\bar{z}}, \nu \rangle.$$

Since $-2i \langle f_z \wedge f_{\bar{z}}, \nu \rangle \neq 0$, we may regard

$$(3.2) \quad \tilde{\lambda}^t = 1 - 2tH + t^2K$$

as the signed area density function of \hat{f}^t . By direct computations, $\tilde{\lambda}_z^t$ and $\tilde{\lambda}_{\bar{z}}^t$ are

$$\begin{aligned} \tilde{\lambda}_z^t &= -t^2 \frac{(Q_z \bar{Q} + Q \bar{Q}_z)g - 4Q \bar{Q} g_z}{4g^5}, \\ \tilde{\lambda}_{\bar{z}}^t &= -t^2 \frac{(Q_{\bar{z}} \bar{Q} + Q \bar{Q}_{\bar{z}})g - 4Q \bar{Q} g_{\bar{z}}}{4g^5}. \end{aligned}$$

Since $Q(p) = \bar{Q}(p) = 0$, $\tilde{\lambda}_z^t(p) = \tilde{\lambda}_{\bar{z}}^t(p) = 0$, that is, $d\tilde{\lambda}^t(p) = 0$ holds. We consider the Hessian of $\tilde{\lambda}^t$. The second derivative $\tilde{\lambda}_{zz}^t$ becomes

$$(3.3) \quad \tilde{\lambda}_{zz}^t = -\frac{t^2}{4g^5} \left\{ (Q_{zz}\bar{Q} + 2Q_z\bar{Q}_z + Q\bar{Q}_{zz})g + (Q_z\bar{Q} + Q\bar{Q}_z)g \right. \\ \left. - 4(Q_z\bar{Q}g_z + Q\bar{Q}_zg_z + Q\bar{Q}g_{zz}) \right\} \\ - \frac{t^2}{4g} \frac{5\{(Q_z\bar{Q} + Q\bar{Q}_z)g - 4Q\bar{Q}g_z\}g_z}{4g}.$$

Thus $\tilde{\lambda}_{zz}^t(p) = 0$ holds. Similarly, we see that $\tilde{\lambda}_{\bar{z}\bar{z}}^t(p) = 0$ holds. By direct calculation, we have

$$(3.4) \quad \tilde{\lambda}_{z\bar{z}}^t = -\frac{t^2}{4g^5} \left\{ (Q_{z\bar{z}}\bar{Q} + Q_z\bar{Q}_{\bar{z}} + Q_{\bar{z}}\bar{Q}_z + Q\bar{Q}_{z\bar{z}})g + (Q_z\bar{Q} + Q\bar{Q}_z)g_{\bar{z}} \right. \\ \left. - 4(Q_{\bar{z}}\bar{Q}g_z + Q\bar{Q}_{\bar{z}}g_z + Q\bar{Q}g_{z\bar{z}}) \right\} \\ - \frac{t^2}{4g} \frac{5\{(Q_z\bar{Q} + Q\bar{Q}_z)g - 4Q\bar{Q}g_z\}g_{\bar{z}}}{4g}.$$

By this equation, $\tilde{\lambda}_{z\bar{z}}^t = -t^2Q_z\bar{Q}_{\bar{z}}/4g^4$ holds at p . Identifying \mathbb{C} with \mathbb{R}^2 by $\mathbb{C} \ni z = u + iv \mapsto (u, v) \in \mathbb{R}^2$, we have

$$\tilde{\lambda}_{uu}^t = \tilde{\lambda}_{zz}^t + 2\tilde{\lambda}_{z\bar{z}}^t + \tilde{\lambda}_{\bar{z}\bar{z}}^t, \quad \tilde{\lambda}_{uv}^t = i(\tilde{\lambda}_{zz}^t - \tilde{\lambda}_{\bar{z}\bar{z}}^t), \quad \tilde{\lambda}_{vv}^t = -(\tilde{\lambda}_{zz}^t - 2\tilde{\lambda}_{z\bar{z}}^t + \tilde{\lambda}_{\bar{z}\bar{z}}^t).$$

By the above computations, it follows that

$$\tilde{\lambda}_{uu}^t = \tilde{\lambda}_{vv}^t = 2\tilde{\lambda}_{z\bar{z}}^t = -t^2 \frac{Q_z\bar{Q}_{\bar{z}}}{2g^4}, \quad \tilde{\lambda}_{uv}^t = 0$$

hold at p . Thus we have

$$\det \text{Hess}_{(u,v)}(\tilde{\lambda}^t)_p = \tilde{\lambda}_{uu}^t(p)\tilde{\lambda}_{vv}^t(p) - \tilde{\lambda}_{uv}^t(p)^2 = t^4 \frac{(Q_z\bar{Q}_{\bar{z}})^2}{4g^8} \geq 0.$$

This completes the proof of the case $M^3 = \mathbb{R}^3$, by Fact 2.3.

If $M^3 = \mathbb{R}^{2,1}$, we can take $\tilde{\lambda}^t$ as the same as in the case of \mathbb{R}^3 . If $M^3 = \mathbb{S}^3$ or $\mathbb{H}^{2,1}$, we have the assertion by using the signed area density for \hat{f}^t as in (2.5)

$$\tilde{\lambda}^t = \cos^2 t - 2H \cos t \sin t + K \sin^2 t.$$

If $M^3 = \mathbb{H}^3$ or $\mathbb{S}^{2,1}$, we show this by using

$$\tilde{\lambda}^t = \cosh^2 t - 2H \cosh t \sinh t + K \sinh^2 t$$

for \hat{f}^t and

$$\tilde{\lambda}^t = \sinh^2 t - 2H \cosh t \sinh t + K \cosh^2 t$$

for \check{f}^t .

Q.E.D.

Examples: Here we construct CMC surfaces with D_4^- -singularities in \mathbb{H}^3 . By Theorem 3.2, we need to choose the Hopf differential factor so that $Q_z(p) \neq 0$ at a point p . Now we fix $Q = -z$ for CMC $H > 1$ or $0 < H < 1$ surfaces and we have the 3-legged Smyth-type surfaces as in [3], [23], [24] and [28]. Applying Theorem 3.2, we get the following figures with a D_4^- -singularity at the origin $z = 0$:

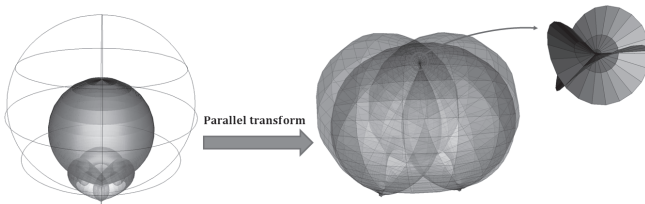


Fig. 2. 3-legged Smyth surface with $H > 1$ in \mathbb{H}^3 and its parallel transform with D_4^- -singularity.

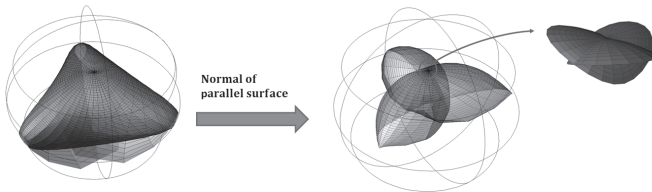


Fig. 3. 3-legged Smyth surface with $0 < H < 1$ in \mathbb{H}^3 and the normal vector of its parallel transform into $\mathbb{S}^{2,1}$ with D_4^- -singularity.

3.2. Minimal surfaces with D_4 -singularities

We now consider the condition that minimal surfaces (resp. maximal surfaces) in \mathbb{R}^3 (resp. $\mathbb{R}^{2,1}$) have D_4 -singularities. For minimal surfaces, the following representation formula is known.

Fact 3.3 ([25]). *Any simply-connected minimal surface $f : U(\subset \mathbb{C}) \rightarrow \mathbb{R}^3$ can be parametrized as*

$$(3.5) \quad f = \operatorname{Re} \int (1 - g^2, i(1 + g^2), 2g)\omega,$$

where $g : U \rightarrow \mathbb{C}$ is a meromorphic function and $\omega = \hat{\omega}dz$ and $g^2\hat{\omega}$ are holomorphic.

We call the pair (g, ω) the *Weierstrass data*. On the other hand, the representation formula for maximal surfaces is also known.

Fact 3.4 ([18]). *Any simply-connected maximal surface $f : U(\subset \mathbb{C}) \rightarrow \mathbb{R}^{2,1}$ can be parametrized as*

$$(3.6) \quad f = \operatorname{Re} \int (1 + g^2, i(1 - g^2), -2g)\omega,$$

where $g : U \rightarrow \mathbb{C}$ is a meromorphic function and $\omega = \hat{\omega}dz$ and $g^2\hat{\omega}$ are holomorphic.

We also call the pair (g, ω) the *Weierstrass data*. We should remark that there are several studies on maximal surfaces (see [7, 10, 22, 30], for example).

Here we consider a relationship between D_4 -singularities and minimal (or maximal) surfaces, using the following ansatz: the function g of Weierstrass data (g, ω) is “holomorphic” at a singular point p of f . As you can see in Facts 3.3 and 3.4, the function g is meromorphic in general, and has poles at some $z = q$. However, if a pole q of g coincides with a singular point p of f , then criteria for D_4 -singularities become more complicated. (See Appendix A for details.)

Lemma 3.5. *Let $f : U \rightarrow \mathbb{R}^3$ (resp. $\mathbb{R}^{2,1}$) be a minimal surface (resp. a maximal surface) constructed by (3.5) (resp. (3.6)) with holomorphic functions $g, \hat{\omega}$. Then $p \in U$ is a corank two singular point of f if and only if $\hat{\omega}(p) = 0$. Moreover, f is a front at p if and only if $g_z(p) \neq 0$.*

Proof. Let $f : U \rightarrow \mathbb{R}^3$ be a minimal surface with the Weierstrass data $(g, \omega = \hat{\omega}dz^2)$. The differentials of f are

$$f_z = \frac{1}{2}(1 - g^2, i(1 + g^2), 2g)\hat{\omega}, \quad f_{\bar{z}} = \frac{1}{2}(1 - \bar{g}^2, -i(1 + \bar{g}^2), 2\bar{g})\bar{\omega}.$$

Thus we have the first assertion.

Next, we show the condition of f to be a front at p . Let p be a corank two singular point of f . Then f is a front at p if and only if its unit normal vector $\nu : U \rightarrow \mathbb{S}^2$ gives an surface at p . Under the above settings, the unit normal vector ν to f is given by

$$\nu = \left(\frac{g + \bar{g}}{|g|^2 + 1}, i \frac{\bar{g} - g}{|g|^2 + 1}, \frac{|g|^2 - 1}{|g|^2 + 1} \right).$$

Since g is holomorphic at p , $g_{\bar{z}}(p) = 0$ holds. Differentiating ν , we have

$$\begin{aligned} \nu_z &= g_z \left(\frac{1 - \bar{g}^2}{(1 + |g|^2)^2}, i \frac{-1 - \bar{g}^2}{(1 + |g|^2)^2}, \frac{2\bar{g}}{(1 + |g|^2)^2} \right), \\ \nu_{\bar{z}} &= \bar{g}_z \left(\frac{1 - g^2}{(1 + |g|^2)^2}, i \frac{1 + g^2}{(1 + |g|^2)^2}, \frac{2g}{(1 + |g|^2)^2} \right) \end{aligned}$$

at p . Thus we have the conclusion.

For maximal surfaces, one can show this similarly by identifying $\mathbb{R}^{2,1}$ with \mathbb{R}^3 and using the Euclidean unit normal vector \mathbf{n}_E given as

$$\mathbf{n}_E = \frac{1}{\sqrt{(1 + |g|^2)^2 + 4|g|^2}}(g + \bar{g}, i(\bar{g} - g), 1 + |g|^2)$$

(see [30, (3.3)]).

Q.E.D.

Theorem 3.6. *Let $f : U \subset (\mathbb{C}, z) \rightarrow \mathbb{R}^3$ (resp. $f : U \subset (\mathbb{C}, z) \rightarrow \mathbb{R}^{2,1}$) be a minimal surface (resp. a maximal surface) given by holomorphic functions $g, \hat{\omega}$. Then a point $p \in U$ is a D_4^- -singularity of f if and only if $\hat{\omega}(p) = 0$ and $Q_z(p) \neq 0$ (resp. $\hat{\omega}(p) = 0$, $Q_z(p) \neq 0$ and $|g(p)| \neq 1$). Here $Q = g_z \hat{\omega}$ is the Hopf differential factor. Moreover, f does not have a D_4^+ -singularity at p .*

Proof. First, we show the case of a minimal surface. The signed area density function λ of f can be given as

$$\lambda = -2i \langle f_z \wedge f_{\bar{z}}, \nu \rangle = (1 + |g|^2)^2 |\hat{\omega}|^2.$$

Since $1 + |g|^2 \neq 0$, we may treat $\hat{\lambda} = |\hat{\omega}|^2 = \hat{\omega} \bar{\omega}$ as the signed area density function. Moreover, since $f_z(p) = f_{\bar{z}}(p) = \mathbf{0}$, p is a corank two singular point. The differentials of $\hat{\lambda}$ in $z, \bar{z} \in U$ are

$$\hat{\lambda}_z = 0, \hat{\lambda}_{\bar{z}} = 0, \hat{\lambda}_{zz} = 0, \hat{\lambda}_{z\bar{z}} = \hat{\omega}_z \bar{\omega}_{\bar{z}}, \hat{\lambda}_{\bar{z}\bar{z}} = 0$$

at p , since $\hat{\omega}(p) = \bar{\omega}(p) = \hat{\omega}_{\bar{z}}(p) = \bar{\omega}_z(p) = 0$. Identifying $z = u + iv \in \mathbb{C}$ and $(u, v) \in \mathbb{R}^2$, we see that $\hat{\lambda}_{uu}(p) = \hat{\lambda}_{vv}(p) = 2\hat{\omega}_z(p)\bar{\omega}_{\bar{z}}(p)$ and $\hat{\lambda}_{uv}(p) = 0$. Hence the Hessian of $\hat{\lambda}$ is

$$\det \text{Hess}_{(u,v)}(\hat{\lambda})_p = 4|\hat{\omega}_z(p)|^4 \geq 0.$$

By Fact 2.3 and Lemma 3.5, f has a D_4^- -singularity at p if and only if $g_z(p) \neq 0$ and $\hat{\omega}_z(p) \neq 0$. On the other hand, the derivative of the Hopf differential factor Q is

$$Q_z(p) = g_{zz}(p)\hat{\omega}(p) + g_z(p)\hat{\omega}_z(p) = g_z(p)\hat{\omega}_z(p),$$

since g_{zz} is finite at p . Thus we have the assertion.

Next, we consider a maximal surface that is not a maxface. We identify \mathbb{R}^3 with $\mathbb{R}^{2,1}$. The signed area density function is

$$\lambda = -2i \langle f_z \times f_{\bar{z}}, \mathbf{n}_E \rangle_{\text{Euc}} = (|g|^2 - 1) |\hat{\omega}|^2 \sqrt{(1 + |g|^2)^2 + 4|g|^2},$$

where \times and $\langle \cdot, \cdot \rangle_{\text{Euc}}$ mean the Euclidean vector product and the Euclidean inner product of \mathbb{R}^3 . From Lemma 3.5, we may take $\hat{\lambda} = \hat{\omega} \tilde{\omega}$. By using similar arguments, we obtain the assertion. Q.E.D.

By the Lawson correspondence, the first fundamental forms of (space-like) CMC 1 surfaces in \mathbb{H}^3 (resp. $\mathbb{S}^{2,1}$) are equal to the first fundamental forms of corresponding minimal surfaces in \mathbb{R}^3 (resp. maximal surfaces in $\mathbb{R}^{2,1}$). This means that they have the same signed area density functions. Thus we obtain the condition that (spacelike) CMC 1 surfaces have D_4^- -singularities similarly.

On the other hand, when $f : U \rightarrow M^3 = \mathbb{S}^3, \mathbb{H}^3, \mathbb{S}^{2,1}$ or $\mathbb{H}^{2,1}$ is a minimal surface, there is no Weierstrass type representation formula. However, if $p \in U$ is an umbilic point for f , then its unit normal vector ν has a corank two singularity at p . By using similar calculations as in the proof of Theorem 3.2, we see that ν has a D_4^- -singularity at p if and only if $Q_z(p) \neq 0$, where Q is the Hopf differential factor of f .

Example: Here we construct CMC 1 surfaces with D_4^- -singularities in \mathbb{H}^3 . Using Theorem 3.6, we fix the Weierstrass data $(g, \omega) = (\cot(z-1), (e^z - 1)dz)$ for a CMC 1 surface in \mathbb{H}^3 . Applying Theorem 3.6, we get the following figure with a D_4^- -singularity at the origin $z = 0$:

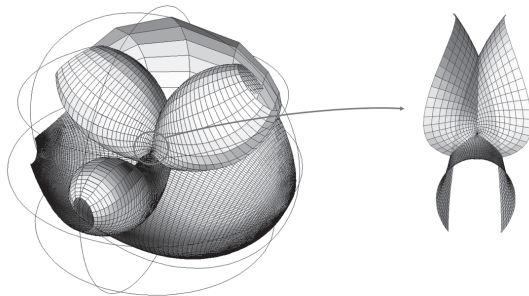


Fig. 4. CMC 1 surface with D_4^- -singularity in \mathbb{H}^3 .

§4. Curvatures of unit normal vector fields to constant mean curvature surfaces

In this section, M^3 denotes one of \mathbb{S}^3 , \mathbb{H}^3 , $\mathbb{S}^{2,1}$ or $\mathbb{H}^{2,1}$. We consider relations between a CMC surface $f : U \rightarrow M^3$ and a unit normal vector ν to f .

Proposition 4.1. *Let $f : U \rightarrow M^3$ be a (spacelike) CMC H surface, and let ν be a unit normal vector to f . Let K denote the extrinsic Gaussian curvature of f . Then the extrinsic Gaussian curvature K_ν and the mean curvature H_ν of ν are*

$$K_\nu = \frac{1}{K}, \quad H_\nu = \frac{H}{K},$$

Moreover, the unit normal vector ν has a constant harmonic mean curvature $1/2H$.

Here the harmonic mean curvature HMC is given by

$$HMC = \frac{K}{2H}.$$

Proof. We consider the case $f : U \rightarrow \mathbb{H}^3$. One can show other cases similarly. Let (u, v) be a conformal coordinate system on U . Then the determinant of the first fundamental matrix I_ν is given by

$$\det I_\nu = E_\nu G_\nu - F_\nu^2 = K^2 E^2,$$

where

$$I_\nu = \begin{pmatrix} \langle \nu_u, \nu_u \rangle & \langle \nu_u, \nu_v \rangle \\ \langle \nu_v, \nu_u \rangle & \langle \nu_v, \nu_v \rangle \end{pmatrix} = \begin{pmatrix} E_\nu & F_\nu \\ F_\nu & G_\nu \end{pmatrix},$$

and $E = \langle f_u, f_u \rangle = \langle f_v, f_v \rangle$. It follows that the coefficients of the second fundamental form of ν are the same as that of f by definition. Thus we have the assertions by straightforward calculations. Q.E.D.

For $M^3 = \mathbb{S}^3$, $\mathbb{H}^{2,1}$ and $H > 0$, or $M^3 = \mathbb{H}^3$, $\mathbb{S}^{2,1}$ and $H > 1$, the parallel transformations \hat{f}^t and $\hat{\nu}^t$ are defined as in (2.5) and (2.6). If $M^3 = \mathbb{H}^3$, $\mathbb{S}^{2,1}$ and $0 < H < 1$, the parallel transforms of f and ν are $\hat{f}^t = \hat{\nu}^t$, $\hat{\nu}^t = \hat{f}^t$, respectively. Thus it is sufficient to consider \hat{f}^t and $\hat{\nu}^t$.

Lemma 4.2. *Under the above settings, the extrinsic Gaussian curvatures K^t and K_ν^t , and the mean curvatures H^t and H_ν^t for \hat{f}^t and $\hat{\nu}^t$ are given by the following:*

$$K^t = \begin{cases} \frac{\sin^2 t + 2H \cos t \sin t + K \cos^2 t}{\cos^2 t - 2H \cos t \sin t + K \sin^2 t}, \\ \frac{\sinh^2 t + 2H \cosh t \sinh t + K \cosh^2 t}{\cosh^2 t - 2H \cosh t \sinh t + K \sinh^2 t}, \end{cases}$$

$$K_\nu^t = \begin{cases} \frac{\cos^2 t - 2H \cos t \sin t + K \sin^2 t}{\sin^2 t + 2H \cos t \sin t + K \cos^2 t}, \\ \frac{\cosh^2 t - 2H \cosh t \sinh t + K \sinh^2 t}{\sinh^2 t + 2H \cosh t \sinh t + K \cosh^2 t}, \end{cases}$$

$$H^t = \begin{cases} \frac{(1 - K) \cos t \sin t + H(\cos^2 t - \sin^2 t)}{\cos^2 t - 2H \cos t \sin t + K \sin^2 t}, \\ -\frac{(1 + K) \cosh t \sinh t - H(\cosh^2 t + \sinh^2 t)}{\cosh^2 t - 2H \cosh t \sinh t + K \sinh^2 t}, \end{cases}$$

$$H_\nu^t = \begin{cases} \frac{(1 - K) \cos t \sin t + H(\cos^2 t - \sin^2 t)}{\cos^2 t + 2H \cos t \sin t + K \sin^2 t}, \\ -\frac{(1 + K) \cosh t \sinh t - H(\cosh^2 t + \sinh^2 t)}{\cosh^2 t + 2H \cosh t \sinh t + K \sinh^2 t}. \end{cases}$$

Proof. One can show the above formulas directly by applying the same computations as in [19]. Q.E.D.

By Proposition 4.1 and Lemma 4.2, we immediately have the following:

Proposition 4.3. *Let $f : U \rightarrow M^3 = \mathbb{S}^3, \mathbb{H}^3, \mathbb{S}^{2,1}, \mathbb{H}^{2,1}$ be a CMC $H \neq 0$ surface and ν its unit normal vector. A point p is an umbilic point for f if and only if p is a non-flat umbilic point for ν . Moreover, if \hat{f}^t (resp. \check{f}^t) has a D_4^- -singularity at p , p is a flat umbilic point of $\hat{\nu}^t$ (resp. $\check{\nu}^t$) (see Figures 5, 6).*

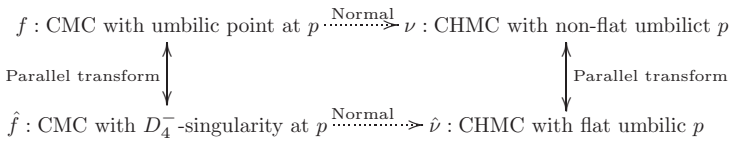


Fig. 5. The case that f is a (spacelike) CMC surface in $\mathbb{S}^3, \mathbb{H}^{2,1}$ (resp. $\mathbb{H}^3, \mathbb{S}^{2,1}$) with $H > 0$ (resp. $H > 1$)

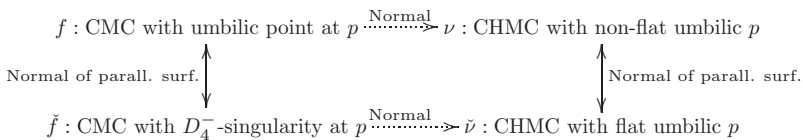


Fig. 6. The case that f is a (spacelike) CMC surface in $\mathbb{H}^3, \mathbb{S}^{2,1}$ with $0 < H < 1$

§Appendix A. The D_4 -singularity and the poles of meromorphic functions g for minimal surfaces

In Section 3.2, we considered a relationship between D_4 -singularities and minimal surfaces, and assumed that the function g of the Weierstrass data (g, ω) is “holomorphic” at a singular point p . Thus, we omitted the case that a pole of the meromorphic function g coincides with a singular point. Here we will consider this remaining case and give the criteria for D_4 -singularities of such minimal surfaces. However, for the pole of g , we no longer have the relationship between the criteria for D_4 -singularities and the Hopf differential factor Q (see Theorem A.4).

Lemma A.1. *Let f be a minimal surface with the Weierstrass data $(g, \omega = \hat{\omega}dz)$. Suppose that p is a pole of g . Then p is a corank two singular point of f if and only if $g^2\hat{\omega} = 0$ at p .*

Proof. By

$$f_z = \frac{1}{2}(1 - g^2, i(1 + g^2), 2g)\hat{\omega}, \quad f_{\bar{z}} = \frac{1}{2}(1 - \bar{g}^2, -i(1 + \bar{g}^2), 2\bar{g})\bar{\omega},$$

we see that $f_z = f_{\bar{z}} = \mathbf{0}$ at p if and only if $\hat{\omega} = g\hat{\omega} = g^2\hat{\omega} = 0$ at p . However $\hat{\omega} = g\hat{\omega} = g^2\hat{\omega} = 0$ is equivalent to $g^2\hat{\omega} = 0$ because $g(p) = \pm\infty$. Q.E.D.

Lemma A.2. *Let f be a minimal surface with the Weierstrass data $(g, \omega = \hat{\omega}dz)$. Suppose that p is a pole of g and corank two singularity of f . Then f is a front at p if and only if $\frac{|g_{\bar{z}}|^2 - |g_z|^2}{(|g|^2 + 1)^2} \neq 0$ at p .*

Proof. By $\nu = \left(\frac{g + \bar{g}}{|g|^2 + 1}, i \frac{\bar{g} - g}{|g|^2 + 1}, \frac{|g|^2 - 1}{|g|^2 + 1} \right)$, we have

$$\nu_z \wedge \nu_{\bar{z}} = \frac{2i(|g_{\bar{z}}|^2 - |g_z|^2)}{(|g|^2 + 1)^2} \nu.$$

Thus, we notice that f is a front at p if and only if $\frac{|g_{\bar{z}}|^2 - |g_z|^2}{(|g|^2 + 1)^2} \neq 0$ at p . Q.E.D.

Remark A.3. If we write $g = \frac{h_2}{h_1}$ by using holomorphic functions h_1 and h_2 such that $h_1(p) = 0$ and $h_2(p)$ is a non-zero value, then

$$\frac{|g_{\bar{z}}|^2 - |g_z|^2}{(|g|^2 + 1)^2} \neq 0 \text{ at } p \iff (h_1)_z(p) \neq 0.$$

Here we get criteria for D_4 -singularities of minimal surfaces when a point p is both a singular point of f and a pole of g .

Theorem A.4. *Let f be a minimal surface with the Weierstrass data $(g, \omega = \hat{\omega}dz)$. Suppose that p is a pole of g . Then the point p is a D_4^- -singularity of f if and only if*

$$\begin{cases} g^2\hat{\omega} = 0, \\ \frac{|g_{\bar{z}}|^2 - |g_z|^2}{(|g|^2 + 1)^2} \neq 0 \text{ and} \\ \hat{\omega}_z \neq 0 \text{ or } (g\hat{\omega})_z \neq 0 \text{ or } (g^2\hat{\omega})_z \neq 0 \text{ at } p. \end{cases}$$

Moreover, f does not have a D_4^+ -singularity at p .

Proof. The signed area density function λ can be given as

$$\lambda = (1 + |g|^2)^2|\hat{\omega}|^2 = |\hat{\omega}|^2 + 2|g\hat{\omega}|^2 + |g^2\hat{\omega}|^2.$$

Then, we have

$$\lambda_z = \hat{\omega}_z\bar{\hat{\omega}} + \hat{\omega}\bar{\hat{\omega}}_z + 2((g\hat{\omega})_z\overline{g\hat{\omega}} + g\hat{\omega}(\overline{g\hat{\omega}})_z) + (g^2\hat{\omega})_z\overline{g^2\hat{\omega}} + g^2\hat{\omega}(\overline{g^2\hat{\omega}})_z.$$

All terms appearing in the above equation are zero at p . Thus, $\lambda_z(p) = 0$. We can continue computing to get the following

$$\lambda_{\bar{z}}(p) = \lambda_{zz}(p) = \lambda_{\bar{z}\bar{z}}(p) = 0 \text{ and } \lambda_{z\bar{z}}(p) = |\hat{\omega}_z|^2 + 2|(g\hat{\omega})_z|^2 + |(g^2\hat{\omega})_z|.$$

Hence the Hessian of λ is

$$\det \text{Hess}_{(u,v)}(\lambda)_p = 4(|\hat{\omega}_z|^2 + 2|(g\hat{\omega})_z|^2 + |(g^2\hat{\omega})_z|)^2 \geq 0.$$

By Fact 2.3 and Lemma 3.5, we have the assertion. Q.E.D.

Remark A.5. For the case of maximal surfaces in $\mathbb{R}^{2,1}$, when a point p is both a singular point and a pole of g , we can apply similar computation as in Section 3.2. Thus, we omit it.

Acknowledgements. The authors would like to thank Shintaro Akamine, Wayne Rossman, Kentaro Saji and Masashi Yasumoto for valuable comments. The authors also thank the referee for many helpful suggestions.

References

- [1] V. I. Arnol'd, S. M. Gusein-Zade, and A. N. Varchenko, Singularities of differentiable maps, Vol.1, Monographs in Mathematics **82**, Birkhäuser, Boston, 1985.
- [2] A. I. Bobenko, Constant mean curvature surfaces and integrable equations, Russian Math. Surveys, 46:4, (1991), 1–45.
- [3] A. I. Bobenko and A. Its, The Painleve III equation and the Iwasawa decomposition, Manuscripta Math., **87**, (1995), 369–377.
- [4] D. Brander, Singularities of spacelike constant mean curvature surfaces in Lorentz-Minkowski space, Math. Proc. Cambridge Philos. Soc. **150** (2011), no. 3, 527–556.
- [5] D. Brander, W. Rossman and N. Schmitt, Holomorphic representation of constant mean curvature surfaces in Minkowski space: consequences of non-compactness in loop group methods, Adv. Math., **223**, (2010), no. 3, 949–986.
- [6] Q. Chen and Y. Cheng, Spectral transformation of constant mean curvature surfaces in \mathbf{H}^3 and Weierstrass representation, Sci. China Ser. A, **45**, (2002), no. 8, 1066–1075.
- [7] F. J. M. Estudillo and A. Romero, Generalized maximal surfaces in Lorentz-Minkowski space \mathbb{L}^3 , Math. Proc. Camb. Philos. Soc. **111** (1992), 515–524.
- [8] J. Dorfmeister, J. Inoguchi and S-P. Kobayashi, Constant mean curvature surfaces in hyperbolic 3-space via loop groups, J. Reine Angew. Math., **686**, (2014), 1–36. (arXiv:1108.1641.)
- [9] S. Fujimori, Spacelike CMC 1 surfaces with elliptic ends in de Sitter 3-space, Hokkaido Math. J. **35**, (2006), 289–320.
- [10] S. Fujimori, K. Saji, M. Umehara and K. Yamada, Singularities of maximal surfaces, Math. Z. **259**, (2008), 827–848.
- [11] T. Fukui and M. Hasegawa, Singularities of parallel surfaces, Tohoku Math. J. **64** (2012), 387–408.
- [12] A. Honda, Duality of singularities for spacelike CMC surfaces, to appear in Kobe J. Math.
- [13] A. Honda, Isometric surfaces with singularities between space forms of the same positive curvature, arXiv:1508.07223.
- [14] A. Honda, M. Koiso and K. Saji, Fold singularities on spacelike CMC surfaces in Lorentz-Minkowski space, to appear in Hokkaido Math. J., arXiv:1509.03050.
- [15] S. Izumiya, M. C. Romero Fuster, M. A. S. Ruas and F. Tari, Differential Geometry from a Singularity Theory Viewpoint, World Scientific, 2015.
- [16] S. Izumiya and K. Saji, The mandala of Legendrian dualities for pseudo-spheres in Lorentz-Minkowski space and “flat” spacelike surfaces. J. Singul. **2**, (2010), 92–127.
- [17] S. Izumiya, K. Saji and M. Takahashi, Horospherical flat surfaces in hyperbolic 3-space, J. Math. Soc. Japan **62** (2010), 789–849.

- [18] O. Kobayashi, Maximal surfaces in the 3-dimensional Minkowski space L^3 , Tokyo J. Math. **6** (1983), 297–309.
- [19] M. Kokubu, Surfaces and fronts with harmonic-mean curvature one in hyperbolic three-space, Tokyo J. Math. **32** (2009), no. 1, 177–200.
- [20] M. Kokubu, W. Rossman, K. Saji, M. Umehara, and K. Yamada, Singularities of flat fronts in hyperbolic space, Pacific J. Math. **221** (2005), 303–351.
- [21] M. Kokubu, W. Rossman, M. Umehara and K. Yamada, Flat fronts in hyperbolic 3-space and their caustics, J. Math. Soc. Japan **59** (2007), 265–299.
- [22] M. Kokubu and M. Umehara, Orientability of linear Weingarten surfaces, spacelike CMC-1 surfaces and maximal surfaces, Math. Nachr. **284** (2011), 1903–1918.
- [23] Y. Ogata, The DPW method for constant mean curvature surfaces in 3-dimensional Lorentzian spaceforms, with applications to Smyth type surfaces, to appear in Hokkaido J. Math.
- [24] Y. Ogata, Spacelike constant mean curvature and maximal surfaces in 3-dimensional de Sitter space via Iwasawa splitting, Tsukuba J. Math. **39**, no. 2, (2015).
- [25] R. Osserman, Global properties of minimal surfaces in E^3 and E^n , Ann. of Math. **80** (1964), 340–364.
- [26] K. Saji, Criteria for D_4 singularities of wave fronts, Tohoku Math. J. **63** (2011), 137–147.
- [27] K. Saji, M. Umehara, and K. Yamada, The geometry of fronts, Ann. of Math. **169** (2009), 491–529.
- [28] B. Smyth, A generalization of a theorem of Delaunay on constant mean curvature surfaces, Statistical thermodynamics and differential geometry of microstructured materials, IMA Vol. Math. Appl. **51**, Springer, New York, (1993), 123–130.
- [29] Y. Umeda, Constant-mean-curvature surfaces with singularities in Minkowski 3-space, Experiment. Math. **18** (2009), no. 3, 311–323.
- [30] M. Umehara and K. Yamada, Maximal surfaces with singularities in Minkowski space, Hokkaido Math. J. **35** (2006), 13–40.

*Department of Science and Technology, National Institute of Technology,
Okinawa College, 905, Henoko, Nago, Okinawa 905-2171, Japan*

*Department of Mathematics, Graduate School of Science, Kobe University,
1-1, Rokko-dai, Nada-ku, Kobe 657-8501, Japan*

E-mail address: y.ogata@okinawa-ct.ac.jp

E-mail address: teramoto@math.kobe-u.ac.jp