Triply periodic zero mean curvature surfaces in Lorentz-Minkowski 3-space

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Abstract.

We construct triply periodic zero mean curvature surfaces of mixed type in the Lorentz-Minkowski 3-space $\mathbb{L}^3$, with the same topology as the triply periodic minimal surfaces in the Euclidean 3-space $\mathbb{R}^3$, called Schwarz rPD surfaces.

§1.   Introduction

Zero mean curvature surfaces in the Lorentz-Minkowski 3-space $\mathbb{L}^3$ are smooth surfaces with mean curvature zero wherever the mean curvature is defined. Having the mean curvature defined at all points is not expected, because these surfaces can change causal type, meaning that some parts may have spacelike tangent planes and other parts may have timelike tangent planes, with lightlike tangent planes at boundary points between these parts. An interesting aspect of these surfaces is precisely that they change causal type, often resulting in interesting singular and topological behaviors, and these surfaces have been investigated in [1, 5, 6, 7, 9]. One of the main tools for the construction of such surfaces is based on the fact that fold singularities of spacelike maximal surfaces have real analytic extensions to timelike minimal surfaces [5].

In contrast to minimal surfaces in the Euclidean 3-space $\mathbb{R}^3$, the only known triply periodic zero mean curvature surfaces in $\mathbb{L}^3$ were those in a 1-parameter family constructed in [9], while there are many known triply periodic minimal surfaces in $\mathbb{R}^3$, see for example [4, 11, 12, 16].

This motivates us to broaden our knowledge of triply periodic zero mean curvature surfaces in $\mathbb{L}^3$, and in this paper we construct a new 1-parameter family of triply periodic zero mean curvature surfaces in $\mathbb{L}^3$ based on the conjugate surfaces of the triply periodic minimal surfaces in $\mathbb{R}^3$ called the Schwarz H surfaces. This is the main original result of this paper, and the family here is interesting because it exhibits both a change of causal type, and also a greater complexity than the previously known examples. The method we use to construct the family is essentially the same as in [9], but the surfaces here are less symmetric, and so the construction is more involved. It is expected that by using the method in this paper one could construct families of surfaces with more complicated topologies, based on the data of triply periodic minimal surfaces in $\mathbb{R}^3$ constructed in [11, 12]. It is also expected that the family of surfaces we construct in this paper is a prototype for the study of the moduli space of triply periodic zero mean curvature surfaces.

We remark that the surfaces constructed in this paper have the same topology and symmetry as Schwarz rPD minimal surfaces, not Schwarz H surfaces. (As for the symbols "rPD" and "H", see Remark A.4 in Appendix A.)

§2.   Preliminaries

We denote by $\mathbb{L}^3$ the Lorentz-Minkowsiki 3-space with indefinite metric $\langle\ ,\ \rangle = -dx_0^2 + dx_1^2 + dx_2^2$. Let M be a Riemann surface. A conformal immersion $f : M \to \mathbb{L}^3$ is called a *spacelike surface* if the induced metric $ds^2 = \langle df, df \rangle$ is positive definite on M. A spacelike surface $f : M \to \mathbb{L}^3$ is called *maximal* if its mean curvature vanishes identically. In [18] a notion of *maxface* was introduced as a maximal surface with certain kind of singularities. More precisely, $f : M \to \mathbb{L}^3$ is called a *maxface* if there exists an open dense subset W of M such that the restriction $f|_W$ of f to W gives a conformal maximal immersion and $df(p) \neq 0$ for all $p \in M$.

For maxfaces, a similar representation formula to Weierstrass representation for minimal surfaces in $\mathbb{R}^3$ is known.

Theorem 2.1 (Weierstrass-type representation [14, 18]). *Let* (g, η) *be a pair of a meromorphic function* g *and a holomorphic differential* η *on a Riemann surface* M *so that* $(1 + |g|^2)^2 \eta \bar{\eta}$ *gives a Riemannian metric on* M. *We set*

$$(2.1) \qquad \Phi = \begin{pmatrix} -2g\eta \\ (1 + g^2)\eta \\ i(1 - g^2)\eta \end{pmatrix},$$

where $i = \sqrt{-1}$. Then

$$(2.2) \qquad f = \mathrm{Re} \int_{z_0}^{z} \Phi : M \to \mathbb{L}^3 \qquad (z_0 \in M)$$

defines a maxface. The singular set $S(f)$ of f is given by

$$S(f) = \{p \in M \; ; \; |g(p)| = 1\}.$$

Moreover, f is single-valued on M if and only if

$$(2.3) \qquad \mathrm{Re} \oint_{\ell} \Phi = \mathbf{0}$$

for any closed curve ℓ on M. Conversely, any maxface can be obtained in this manner.

The pair (g, η) in Theorem 2.1 is called the *Weierstrass data* of f.

Remark 2.2. The first fundamental form ds^2 and the second fundamental form $\mathbb{I}$ of the surface (2.2) are given by

$$ds^2 = \left(1 - |g|^2\right)^2 \eta\bar{\eta}, \qquad \mathbb{I} = -\eta dg - \overline{\eta dg}.$$

Moreover, $g|_{M \setminus S(f)} : M \setminus S(f) \to (\mathbb{C} \cup \{\infty\}) \setminus \{|z| = 1\}$ coincides with the composition of the Gauss map

$$G|_{M \setminus S(f)} : M \setminus S(f) \to H^2 = \{x \in \mathbb{L}^3 \; ; \; \langle x, x \rangle = -1\}$$

of the maximal surface and the stereographic projection

$$\sigma : H^2 \ni (x_0, x_1, x_2) \mapsto \frac{x_1 + ix_2}{1 - x_0} \in \mathbb{C} \cup \{\infty\},$$

that is, $g|_{M \setminus S(f)} = \sigma \circ G|_{M \setminus S(f)}$. So we call g the Gauss map of the maxface.

Generic singularities of maxfaces are classified in [10]. Moreover several criteria for singular points of maxfaces by using their Weierstrass data are given in [8, 10, 18].

Definition 2.3 (Fold singular point [5]). Let $f : M \to \mathbb{L}^3$ be a maxface with Weierstrass data (g, η). We denote by $S(f)$ the singular set of f, that is, $S(f) = \{p \in M \; ; \; |g(p)| = 1\}$.

    (1)   A singular point $p \in S(f)$ of f is called *non-degenerate* if dg does not vanish at p.

(2)   A regular curve $\hat{\gamma}$ on M is called a *non-degenerate fold singu-larity* if it consists of non-degenerate singular points such that

$$\mathrm{Re}\left(\frac{dg}{g^2\eta}\right)$$

vanish identically along the curve $\hat{\gamma}$. Each point on the non-degenerate fold singularity is called a *fold singular point*.

Remark 2.4. Let Σ be a smooth 2-manifold and $f : \Sigma \to \mathbb{L}^3$ a smooth map. A singular point $p \in \Sigma$ of f is called *fold singularity* if there exist a local coordinate system $(U; \varphi)$ ccentered at $p \in \Sigma$ and a diffeomorphism ψ of $\mathbb{L}^3$ such that $(\psi \circ f \circ \varphi^{-1})(u, v) = (u, v^2, 0)$.

In [5, Lemma 2.17], it is shown that a non-degenerate fold singularity of a maxface is indeed a fold singularity.

A regular curve $\gamma : (a, b) \to \mathbb{L}^3$ is called *null* or *isotropic* if $\gamma'(t) = (d\gamma/dt)(t)$ is a lightlike vector for all $t \in (a, b)$.

Definition 2.5 (Non-degenerate null curve [5]). A null curve $\gamma : (a, b) \to \mathbb{L}^3$ is called *degenerate* or *non-degenerate* at $t = c$ if $\gamma''(c)$ is or is not proportional to the velocity vector $\gamma'(c)$, respectively. If γ is non-degenerate at each $t \in (a, b)$, it is called a *non-degenerate* null curve.

Theorem 2.6 (Analytic extension of maxface [5]). *Let $f : M \to \mathbb{L}^3$ be a maxface which has non-degenerate fold singularities along a singular curve $\hat{\gamma} : (a, b) \to M$. Then $\gamma = f \circ \hat{\gamma}$ is a non-degenerate null curve and the image of the map*

(2.4) $$f^*(u, v) = \frac{\gamma(u + v) + \gamma(u - v)}{2}$$

is real analytically connected to the image of f along γ as a timelike mini-mal immersion. Conversely, any real analytic immersion with mean cur-vature, whereever well-defined, equal to zero which changes type across a non-degenerate null curve is obtained as a real analytic extension of non-degenerate fold singularities of a maxface.

We call an immersion in $\mathbb{L}^3$ with mean curvature, whereever well-defined, equal to zero a *zero mean curvature (ZMC) surface*.

§3.   Schwarz H-type ZMC surfaces

For a constant $a \in (0, \infty)$, we set M_a a Riemann surface of genus 3 defined by the hyperelliptic curve

$$w^2 = z(z^3 + a^3)(z^3 + a^{-3}).$$

Consider the family $\{f_a\}_{0<a<\infty}$ of maxfaces

$$(3.1) \qquad f_a = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \operatorname{Re} \int \begin{pmatrix} -2g \\ 1+g^2 \\ i(1-g^2) \end{pmatrix} \eta$$

in $\mathbb{L}^3$ with the Weierstrass data

$$(3.2) \qquad g = z, \qquad \eta = i\frac{dz}{w}.$$

The singular set of f_a is $\{|z| = 1\}$. It is easy to verify that each singular point is fold singularity.

We define a $\mathbb{C}^3$-valued 1 form Φ_a on M_a by

$$\Phi_a = \begin{pmatrix} -2g \\ 1+g^2 \\ i(1-g^2) \end{pmatrix} \eta.$$

Direct computations show the following lemma.

Lemma 3.1 (Symmetries of the surface). *Define anti-holomorphic maps* $\psi_j : M_a \to M_a$ $(j=1,2,3)$ *as follows:*

$$\psi_1(z, w) = (\bar{z}, \bar{w}),$$
$$\psi_2(z, w) = (e^{2\pi i/3}\bar{z}, e^{\pi i/3}\bar{w}),$$
$$\psi_3(z, w) = \left(\frac{1}{\bar{z}}, \frac{\bar{w}}{\bar{z}} \right).$$

Then we have the following:

$$\psi_1^*\Phi_a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overline{\Phi}_a,$$

$$\psi_2^*\Phi_a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\cos(\pi/3) & \sin(\pi/3) \\ 0 & \sin(\pi/3) & \cos(\pi/3) \end{pmatrix} \overline{\Phi}_a,$$

$$\psi_3^*\Phi_a = \overline{\Phi}_a.$$

By the above lemma, we can consider

$$\Omega_a^{max} = \{ f_a(z) \; ; \; |z| \le 1, \; 0 \le \arg z \le \pi/3 \}$$

as the fundamental piece of the maxface, that is, the entire maxface consists of pieces each of which is congruent to Ω_a^{max}.

Lemma 3.2. *In Ω_a^{max}, the images of $\{0 \leq |z| \leq 1,\ \arg z = 0\}$ and $\{a \leq |z| \leq 1,\ \arg z = \pi/3\}$ by f_a are straight lines, and the image of $\{a \leq |z| \leq 1,\ \arg z = \pi/3\}$ by f_a is a curve in some timelike plane.*

Proof.   Consider the Hopf differential

$$Q = \eta dg = i\frac{dz^2}{w}$$

of f_a. If $z = t$ $(0 \leq t \leq 1)$, then $Q \in i\mathbb{R}$. Also, if $z = e^{\pi i/3}t$ $(0 \leq t \leq 1)$, then

$$Q = \frac{-dt^2}{\sqrt{t(t^3 - a^3)(t^3 - a^{-3})}} \in \begin{cases} \mathbb{R} & (0 \leq t \leq a) \\ i\mathbb{R} & (a \leq t \leq 1) \end{cases}$$

this completes the proof.   Q.E.D.

Next we consider the singular curve γ of f_a. The singular curve is the image of $z = e^{it}$ $(0 \leq t \leq \pi/3)$. Hence we can write

$$\gamma(s) = \int_0^s \begin{pmatrix} 1 \\ -\cos t \\ -\sin t \end{pmatrix} \xi(t)dt, \ \ \xi(t) = \frac{2}{\sqrt{2\cos 3t + a^3 + a^{-3}}} \ \ \left(0 \leq s \leq \frac{\pi}{3}\right)$$

by a direct computation. Thus if we set

(3.3) $$f_a^*(u, v) = \frac{1}{2}\big(\gamma(u + v) + \gamma(u - v)\big),$$

then f_a^* is a timelike minimal surface such that $\{v = 0\}$ corresponds to the fold singularities and f_a^* is the analytic extension of the maximal surface f_a.

Arguments similar to those in [9] show the following two lemmas.

Lemma 3.3 ([9, Lemma 3.1]). *$f_a^*(u, v)$ is an immersion on $(u, v) \in \mathbb{R} \times (0, \pi)$.*

Lemma 3.4 ([9, Lemma 3.2]). *$f_a^*(0, v)$ $(0 < v < \pi)$ is a straight line parallel to x_2-axis, and $f_a^*(\pi/3, v)$ $(0 < v < \pi)$ is a straight line parallel to $x_0 = x_1 + \sqrt{3}x_2 = 0$.*

Moreover, since $f_a^*(u, \pi + v) = f_a^*(u, \pi - v)$ holds, we have the following lemma.

Lemma 3.5. *$f_a^*(u, \pi)$ $(u \in \mathbb{R})$ corresponds to fold singularities.*

We set $\Omega_a^{min} = \{f_a^*(u, v)\ ;\ 0 \leq u \leq \pi/3,\ 0 \leq v \leq \pi\}$.

Remark 3.6. For the Schwarz D-type ZMC surface in [9], $f_a^*(u, \pi/2)$ is a straight line parallel to x_0-axis, but we do not have such a symmetry in this case.

We set

$$\sigma(s) = f_a^*(s, \pi) = \frac{1}{2}\big(\gamma(s + \pi) + \gamma(s - \pi)\big) \qquad (0 \le s \le \pi/3)$$

to further extend analytically from $f_a^*(u, \pi)$ to spacelike surface. Then we have

$$\sigma'(s) = \begin{pmatrix} 1 \\ \cos s \\ \sin s \end{pmatrix} \hat{\xi}(s),$$

where

$$\hat{\xi}(s) = \xi(s + \pi) = \xi(s - \pi) = \frac{2}{a^3 + a^{-3} - 2\cos 3s}.$$

A direct computation shows the following lemma.

Lemma 3.7. *The following equation*

$$\sigma'(s) = A\gamma'\left(\frac{\pi}{3} - s\right)$$

holds, where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\cos(\pi/3) & -\sin(\pi/3) \\ 0 & -\sin(\pi/3) & \cos(\pi/3) \end{pmatrix}.$$

By this lemma, we have

$$\sigma(s) = A\gamma\left(\frac{\pi}{3} - s\right) + \boldsymbol{c},$$

where

$$\boldsymbol{c} = \sigma(0) - A\gamma(\pi/3) = f_a^*(0, \pi) - Af_a^*(\pi/3, 0) \in \mathbb{L}^3.$$

Thus we have the following proposition (See Fig. 3.1).

Proposition 3.8. *We denote by $\hat{f}_a$ the spacelike extension from $\sigma(s)$. Then we have*

$$\hat{f}_a(z) = -Af_a(z) + \boldsymbol{c} \qquad (|z| \le 1, \ 0 \le \arg z \le \pi/3).$$

We set $\hat{\Omega}_a^{max} = \{\hat{f}_a(z) \ ; \ |z| \le 1, \ 0 \le \arg z \le \pi/3\}$. Then the boundary of

(3.4) $$\Omega_a^{max} \cup \Omega_a^{min} \cup \hat{\Omega}_a^{max}$$

consists of two planar curves and two straight lines. See Fig. 3.1. Now we extend this piece (3.4) by reflections with respect to planar curves, then

six copies of (3.4) look like "twisted" equilateral triangular catenoid, see Fig. 3.2. This triangular catenoid is homeomorphic and has the same symmetry to the half of rPD family (in $\mathbb{R}^3$) as in Example A.5. Therefore the ZMC surface we obtain by extending (3.4) by reflections infinitely many times is triply periodic. Though the triply periodic ZMC surface looks like embedded for any $a \in (0, 1)$, we leave the study of embeddedness of this family for another occasion. See Fig. 3.2.
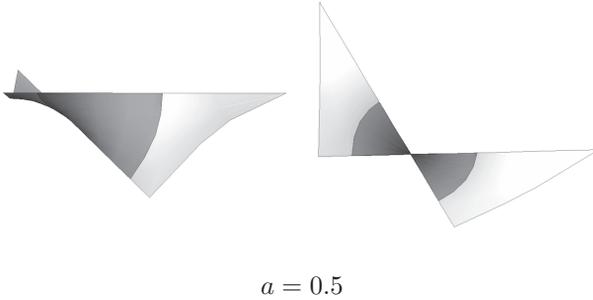

$$a = 0.5$$

Fig. 3.1. The piece $\Omega_a^{max} \cup \Omega_a^{min} \cup \hat{\Omega}_a^{max}$ in different view points. The spacelike parts are indicated by grey shades, and the timelike part is indicated by black shade.

We call the 1 parameter family of this triply periodic ZMC surface by *Schwarz H-type ZMC surfaces*.

Remark 3.9. The 1 parameter family of the conjugate surface of the maxface we have considered in this section, that is, the maxface with the Weierstrass data $g = z$, $\eta = dz/w$, have conelike singularities, and the half of the fundamental piece looks like "twisted" equilateral Lorentzian catenoid. See Fig. 3.3. Hence by extending these surfaces by reflections with respect to boundary straight lines, we have triply periodic maxfaces with conelike singularities.

§4. Limits of Schwarz H-type ZMC surfaces

In this section we consider the limits of Schwarz H-type ZMC surfaces. As $a \to 0$, the surface, with rescaled by $\sqrt{a^3 + a^{-3}}$, converges to the helicoid by the same arguments as in [9, Remark 3.6].
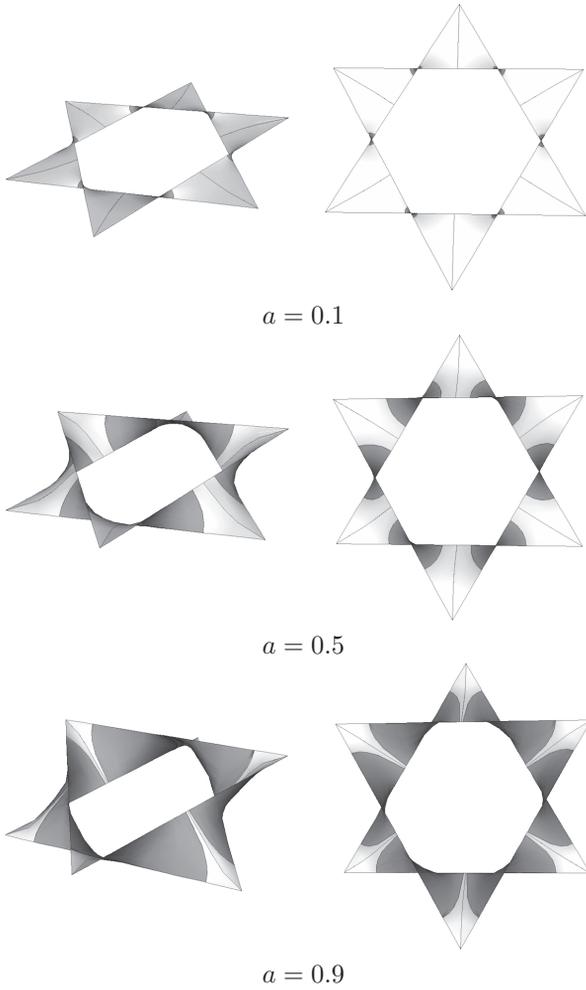
$a = 0.1$

$a = 0.5$

$a = 0.9$

Fig. 3.2. Schwarz H-type ZMC surfaces.

Next we consider the limit as $a \to 1$. Since the hyperelliptic curve $w^2 = z(z^3 + a^3)(z^3 + a^{-3})$ converges to

$$w^2 = z(z^3 + 1)^2,$$

$a = 0.1$
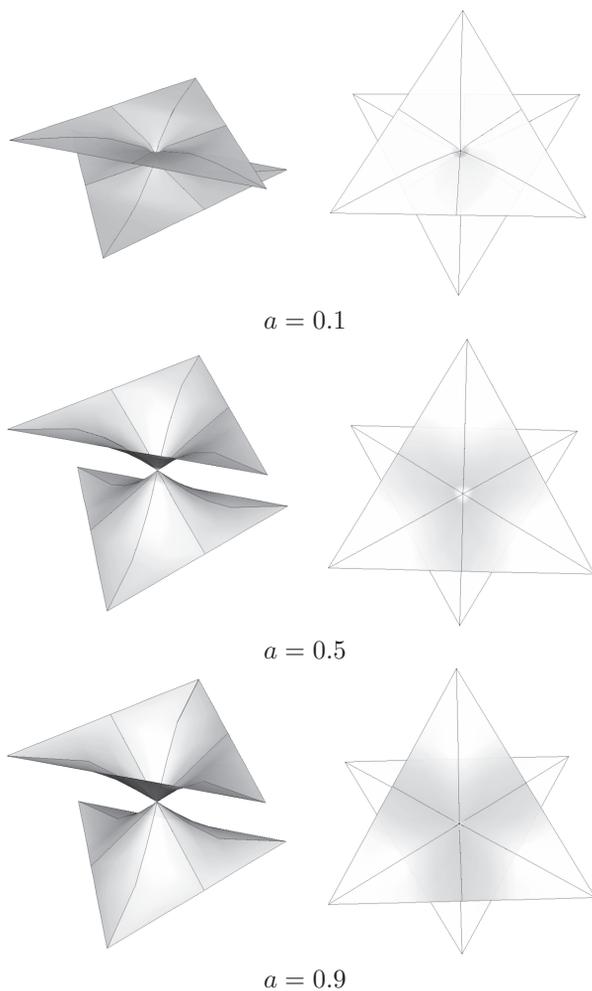

$a = 0.5$

$a = 0.9$

Fig. 3.3. The conjugate surfaces of Schwarz H-type surface.

the Riemann surface M_a converges to a Riemann surface with six nodes at

$$z = e^{\pi i/3}, \quad -1, \quad e^{-\pi i/3}$$

and two branch points at

$$z = 0, \quad \infty.$$

This Riemann surface is of genus zero with six nodal singular points. Hence the maxface f_a converges to

$$(4.1) \qquad f_a \to \pm \operatorname{Re} \int \begin{pmatrix} -2z \\ 1+z^2 \\ i(1-z^2) \end{pmatrix} \frac{i\,dz}{\sqrt{z}(z^3+1)}.$$

Let ζ be a branch of $\zeta^2 = z$. Then ζ is a coordinates of this Riemann surface, with six nodes at $\zeta = \pm e^{\pm \pi i/6}$, $\pm i$, and the right hand side of (4.1) becomes

$$\pm 2 \operatorname{Re} \int \begin{pmatrix} -2\zeta^2 \\ 1+\zeta^4 \\ i(1-\zeta^4) \end{pmatrix} \frac{i\,d\zeta}{\zeta^6+1}.$$

This surface coincides with the Karcher-type maxface with $k = 3$, which is a maxface obtained by the conjugate of the maxface with the Weierstrass data as in Example A.7. See Fig. 4.1.

Remark 4.1. In contrast to the Karcher tower in $\mathbb{R}^3$, the Karcher-type maxface is single-valued on M_k for any $k \geq 2$. Moreover, it is easy to verify that each singular point is fold singularity.

For $k = 2$, the image of the analytic extension of the maxface to ZMC surface coincides with the entire graph

$$(4.2) \qquad x_0 = \log \frac{\cosh x_1}{\cosh x_2}$$

which is called the Scherk-type ZMC surface.

§**Appendix A. Minimal surfaces in $\mathbb{R}^3$**

Here we review several examples of minimal surfaces in $\mathbb{R}^3$ which are related to ZMC surfaces we constructed in this paper. For the detail of these examples, see for example [3, 11, 12, 13].

Theorem A.1 (Weierstrass representation [15])**.** *Let (g, η) be a pair of a meromorphic function g and a holomorphic differential η on a Riemann surface M so that*

$$(A.1) \qquad (1+|g|^2)^2 \eta \bar{\eta}$$

gives a Riemannian metric on M. We set

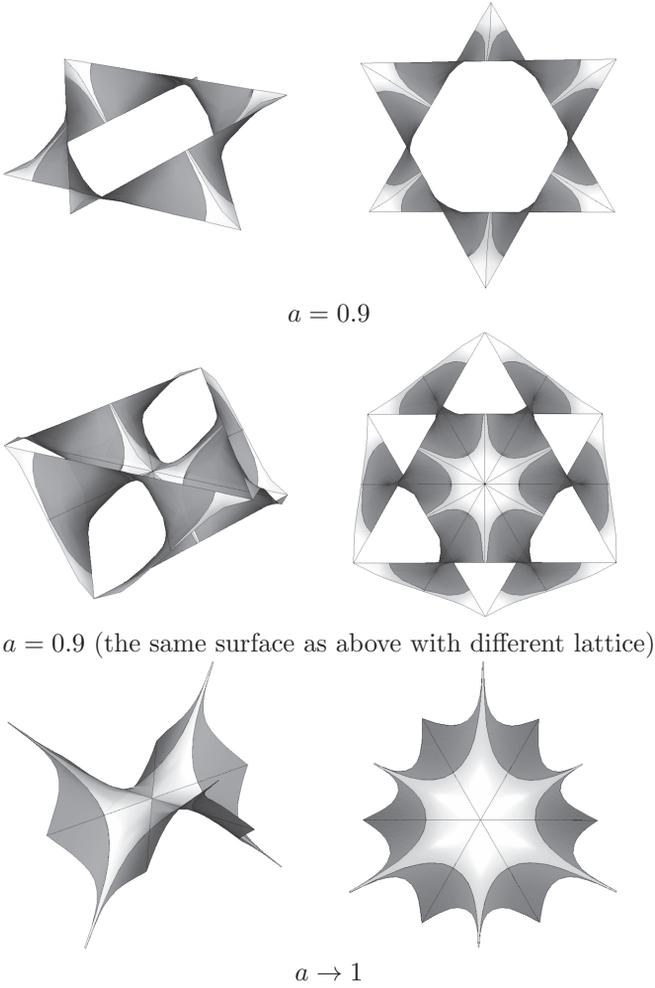$$(A.2) \qquad \Phi = \begin{pmatrix} (1-g^2)\eta \\ i(1+g^2)\eta \\ 2g\eta \end{pmatrix}.$$

$a = 0.9$

$a = 0.9$ (the same surface as above with different lattice)

$a \to 1$

Fig. 4.1. The limit of Schwarz H-type ZMC surface as $a \to 1$.

Then

(A.3) $$f = \mathrm{Re} \int_{z_0}^{z} \Phi : M \to \mathbb{R}^3 \qquad (z_0 \in M)$$

defines a conformal minimal immersion. Moreover, f is single-valued on M if and only if

(A.4) $$\mathrm{Re} \oint_{\ell} \Phi = \mathbf{0}$$

for any closed curve ℓ on M. Conversely, any minimal surface can be obtained in this manner.

The pair (g, η) in Theorem A.1 is called the *Weierstrass data* of f.

Remark A.2. To verify the periodicity of surfaces, consider the following map.

$$\text{(A.5)} \qquad \text{Per}(f) = \left\{ \text{Re} \oint_\ell \Phi \; ; \; \ell \in H_1(M, \mathbb{Z}) \right\}.$$

The periodicity can be determined in the following way:

- If $\text{Per}(f) = \{\mathbf{0}\}$, that is, f satisfies the condition (A.4) for any closed curve ℓ on M, then $f : M \to \mathbb{R}^3$ is well-defined on M, that is, f is *single-valued in $\mathbb{R}^3$*.
- If there exists only one direction $\boldsymbol{v} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ such that

  $$\text{Per}(f) \subset \Lambda_1 = \{n\boldsymbol{v} \; ; \; n \in \mathbb{Z}\},$$

  then f is *singly periodic*. In this case, f is single-valued in $\mathbb{R}^3/\Lambda_1 \approx \mathbb{R}^2 \times S^1$. (A surface invariant under *screw-motions* $\Lambda_1 + R$, where R is a rotation around an axis in the direction of Λ_1, is also singly periodic. See, for example, [2] and the references therein.)
- If there exist two linearly independent vectors $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^3$ (with span$\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ uniquely determined) such that

  $$\text{Per}(f) \subset \Lambda_2 = \left\{ \sum_{j=1}^{2} n_j \boldsymbol{v}_j \; ; \; n_j \in \mathbb{Z} \right\},$$

  then f is *doubly periodic*. In this case, f is single-valued in $\mathbb{R}^3/\Lambda_2 \approx T^2 \times \mathbb{R}$.
- If there exist three linearly independent vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \in \mathbb{R}^3$ such that

  $$\text{Per}(f) \subset \Lambda_3 = \left\{ \sum_{j=1}^{3} n_j \boldsymbol{v}_j \; ; \; n_j \in \mathbb{Z} \right\},$$

  then f is *triply periodic*. In this case, f is single-valued in $\mathbb{R}^3/\Lambda_3 \approx T^3$.

Remark A.3. The first fundamental form ds^2 and the second fundamental form $\mathrm{I\!I}$ of the surface (A.3) are given by

$$ds^2 = \left(1 + |g|^2 \right)^2 \eta\bar{\eta}, \qquad \mathrm{I\!I} = -\eta dg - \overline{\eta dg}.$$

Moreover, $g : M \rightarrow \mathbb{C} \cup \{\infty\}$ coincides with the composition of the Gauss map $G : M \rightarrow S^2$ of the minimal surface and the stereographic projection $\sigma : S^2 \rightarrow \mathbb{C} \cup \{\infty\}$, that is, $g = \sigma \circ G$. So we call g the Gauss map of the minimal surface.

Remark A.4 (A historical remark about triply periodic minimal surfaces). The first examples of triply periodic minimal surfaces in $\mathbb{R}^3$ are found by H. A. Schwarz in the 19th century [17]. Then in 1970, a NASA scientist A. Schoen found many more examples, and he named three of Schwarz' examples P surface, D surface, and H family, because they have the symmetry related to those of the *primitive* cubic lattice, *diamond* crystal structure, and *hexagonal* crystal structure, respectively [16].

In 1989, H. Karcher found a 1-parameter family of triply periodic minimal surfaces [12]. Since a half of the fundamental piece of the surface looks like *twisted* (equilateral) *trianglar* catenoid (see Fig. A.1), he named the family TT, but since the family contains both Schwarz P and D surfaces, the family is now called rPD family. See for example [4].

Example A.5 (Schwarz rPD family). For a constant $a \in (0, \infty)$, we set M_a a Riemann surface of genus 3 defined by the hyperelliptic curve

$$w^2 = z(z^3 - a^3)(z^3 + a^{-3}).$$

We define the Weierstrass data

(A.6) $$g = z, \qquad \eta = \frac{dz}{w}.$$

Then

(A.7) $$f_a = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathrm{Re} \int \begin{pmatrix} 1 - g^2 \\ i(1 + g^2) \\ 2g \end{pmatrix} \eta$$

gives a 1-parameter family $\{f_a\}_{0 < a < \infty}$ of embedded triply periodic minimal surfaces in $\mathbb{R}^3$. This family is called the *Schwarz rPD family*. When $a = 1/\sqrt{2}$, the surface coincides with Schwarz P surface, and when $a = \sqrt{2}$, the surface coincides with Schwarz D surface.

As we mentioned in Remark A.4, a half of the fundamental piece of rPD surface looks like "twisted" equilateral triangular catenoid. See Fig. A.1.

Fig. A.2 shows the relation between Schwarz P and rPD for $a = 1/\sqrt{2}$.
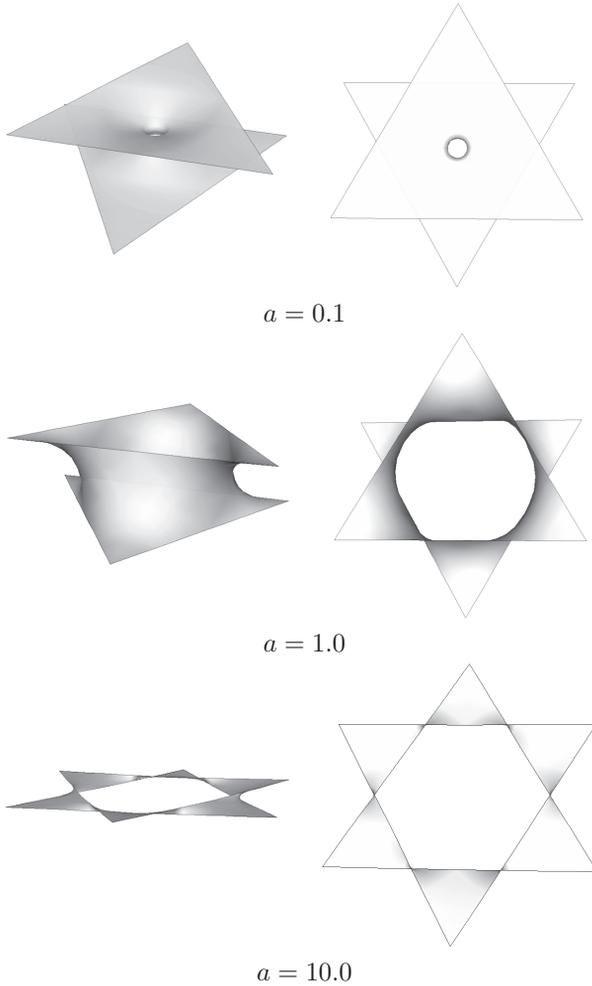
$$a = 0.1$$

$$a = 1.0$$

$$a = 10.0$$

Fig. A.1. Schwarz rPD surfaces.

Example A.6 (Schwarz H family). For a constant $a \in (0, \infty)$, we set M_a a Riemann surface of genus 3 defined by the hyperelliptic curve

$$w^2 = z(z^3 + a^3)(z^3 + a^{-3}).$$

Then the family $\{f_a\}_{0<a<1}$ of minimal surfaces (A.7) with the Weierstrass data (A.6) is a family of embedded triply periodic minimal surfaces in $\mathbb{R}^3$. This family is called the *Schwarz H family*. As $a \to 0$, f_a, with

$$a = 1/\sqrt{2}$$
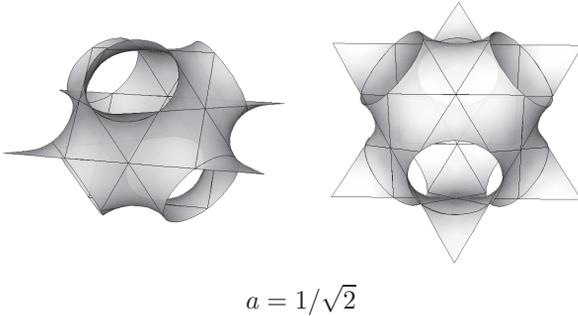
Fig. A.2. Relation between Schwarz P and rPD.

rescaled by $\sqrt{a^3 + a^{-3}}$, converges to catenoid. Also, as $a \to 1$, f_a converges to Karcher tower with $k = 3$ (see Example A.7).

A half of the fundamental piece of Schwarz H surface looks like "non-twisted " equilateral triangular catenoid. See Fig. A.3.

Fig. A.4 shows the conjugate surface of Schwarz H surface. Each vertex of the hexagon in the right hand side figure lies in the straight line parallel to x_3-axis. Hence after reflections with respect to these lines, we see that the surface has self-intersections.
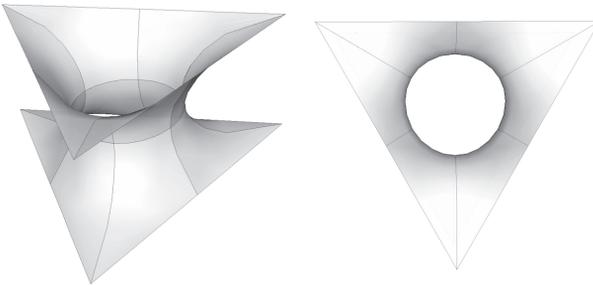

Fig. A.3. Schwarz H surface ($a = 0.5$).

Example A.7 (Karcher tower). For an integer $k \geq 2$, we define M_k by

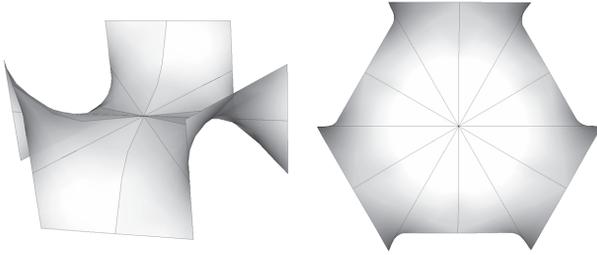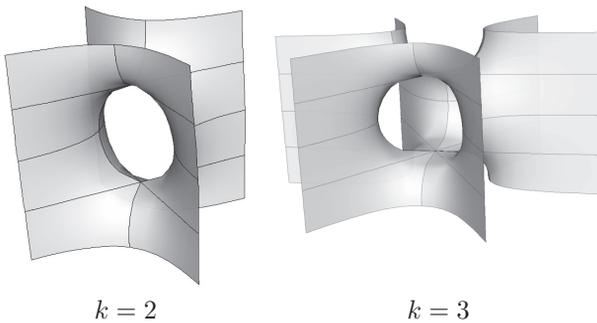(A.8) $$M_k = (\mathbb{C} \cup \{\infty\}) \setminus \{z \in \mathbb{C} \; ; \; z^{2k} = -1\}.$$

Fig. A.4. The comjugate surface of Schwarz H surface
$(a = 0.5)$.

Then the minimal surface (A.7) with the Weierstrass data

$$g = z^{k-1}, \qquad \eta = \frac{dz}{z^{2k} + 1}$$

is embedded singly periodic with $2k$ ends in $\mathbb{R}^3$. This minimal surface is called the *Karcher tower*. When $k = 2$, this surface coincides with the Scherk tower (the conjugate surface of doubly periodic Scherk surface). See Fig. A.5.

$k = 2$ $\qquad\qquad\qquad$ $k = 3$

Fig. A.5. The Karcher tower.

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