

Singularity analysis of lightlike hypersurfaces of partially null curves

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Abstract.

We have gotten singularity classifications of lightlike hypersurfaces of a pseudo null curve in \mathbb{R}_2^4 [6]. This paper is to characterize singularities of lightlike hypersurfaces of a partially null curve in the same space and give an example of such curves.

§1. Introduction

The notions of partially and pseudo null curves are derived from null curves, also called lightlike curves. There widely exist null curves in Minkowski spacetime. About half a century ago, null curves were researched from the view point of differential geometry [3]. In 1985, W. B. Bonnor further investigated curves with lightlike normals [4]. Until 1995, J. Walrave gave the definitions of partially and pseudo null curves [24].

A pseudo null curve is not a lightlike curve, but its tangent curve is a lightlike curve. A partially null curve is not a lightlike curve, nor is its tangent curve. Normally, partially null curves are curves with lightlike binormals [21].

Additionally, M. Petrović-Torgašev, K. İlarıslan, and E. Nešović ([21], 2005) give the Frenet equations of pseudo null and partially null curves in \mathbb{R}_2^4 and classify all such curves with constant curvatures. Thereafter, pseudo and partially null curves have been widely concerned and many good results have been obtained from the view point of differential geometry [1, 8, 9, 10, 11, 20, 23, 25]. Pseudo null Bertrand curves, pseudo

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null Mannheim curves, the inextensible flows and the position vector of partially null curves are considered, respectively in [8], [9], [23] and [25]. And the relations are gotten in [10] between pseudo and partially null rectifying curves and centrodes (Darboux vectors), which play some important roles in mechanics, kinematics as well as in differential geometry. Moreover, the involute-evolute of the pseudo null curve is studied in [20], and they prove that there is no involute of pseudo null curves in Minkowski 3-space. On the other hand, the research about submanifolds in semi-Riemannian spaces have been hot issues in recent years from the view point of singularity theory and differential geometry. There appeared many good achievements [2, 12, 13, 14, 15, 16, 17, 18, 19, 22].

We have researched pseudo null timelike curves with lightlike frames given in [21], that are curves with lightlike principal normal vectors, i.e. $\|\gamma''\| = 0$. In this paper, we focus on partially null timelike curves, that are curves with lightlike binormals. However, we find it difficult to construct the lightlike frame in [21]. For example if we take γ''' as a lightlike binormal vector, then γ'' is also lightlike. Therefore, we construct a frame without lightlike vectors and naturally extend our research to the case of $\|\gamma''\| \neq 0$. Take $\mathbf{n}_1 = \gamma''/\|\gamma''\|$ as the unit principal normal vector. When \mathbf{n}_1 is spacelike, γ has two lightlike binormal vectors which is a partially null curve. We also consider the case that \mathbf{n}_1 is timelike. The current study is inspired by the report of S. Izumiya and T. Sato [18]. We focus on the singularity analysis of lightlike hypersurfaces of partially null curves.

The paper is organized as follows: Section 2 summarizes the required formalism of the basic notions concerning the semi-Euclidean 4-space with index 2 and gives the main results about geometric information of singularities of lightlike hypersurfaces, which can measure the the order of the contact between a partially null curve and a lightcone. Section 3 introduces the one parameter family of lightcone Gaussian indicatrices named lightcone Gaussian surfaces from the view point of differential geometry. Section 4 constructs Lorentz distance-squared functions to characterize the contact relations between partially null curves and the lightcone. Section 5 gives the proof of the main result, i.e. Theorem 1, through the methods of the classical unfolding theory in singularity theory. Finally, in Section 6 we give an example to illustrate the singularities of lightlike hypersurfaces and some properties of the lightcone Gaussian surfaces.

We assume throughout the paper that all manifolds and maps are C^∞ unless explicitly stated otherwise.

§2. The basic concepts and Main Results

The semi-Euclidean four space with index two $(\mathbb{R}_2^4, \langle, \rangle)$ is the vector space \mathbb{R}^4 endowed with the metric induced by the pseudo-scalar product $\langle \mathbf{x}, \mathbf{y} \rangle = -x^0y^0 - x^1y^1 + x^2y^2 + x^3y^3$, for any vectors $\mathbf{x} = (x^0, x^1, x^2, x^3)$, $\mathbf{y} = (y^0, y^1, y^2, y^3)$ in \mathbb{R}^4 . The non-zero vector $\mathbf{x} \in \mathbb{R}_2^4$ is called *spacelike*, *lightlike* or *timelike* if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ or $\langle \mathbf{x}, \mathbf{x} \rangle < 0$ respectively. The *norm* of the vector $\mathbf{x} \in \mathbb{R}_2^4$ is defined as $\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$. The signature of a vector $\mathbf{x} \in \mathbb{R}_2^4 \setminus \{\mathbf{0}\}$ is defined as

$$\text{sign}(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \text{ is spacelike} \\ 0 & \mathbf{x} \text{ is lightlike} \\ -1 & \mathbf{x} \text{ is timelike.} \end{cases}$$

We call $NC_\alpha = \{\mathbf{x} = (x^0, x^1, x^2, x^3) \in \mathbb{R}_2^4 | \langle \mathbf{x} - \alpha, \mathbf{x} - \alpha \rangle = 0\}$ a *lightcone* with vertex α , and denote $NC^* = NC_0 \setminus \{\mathbf{0}\}$.

For any $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}_2^4$, we define the vector $\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3$ as

$$\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 = \begin{vmatrix} -\mathbf{e}_0 & -\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_1^0 & x_1^1 & x_1^2 & x_1^3 \\ x_2^0 & x_2^1 & x_2^2 & x_2^3 \\ x_3^0 & x_3^1 & x_3^2 & x_3^3 \end{vmatrix},$$

where $\mathbf{x}_i = (x_i^0, x_i^1, x_i^2, x_i^3)$ and $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the canonical basis of \mathbb{R}_2^4 . Obviously,

$$\langle \mathbf{x}, \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \rangle = \det(\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3),$$

so that $\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3$ is pseudo orthogonal to any $\mathbf{x}_i (i = 1, 2, 3)$.

Let $\gamma : I \rightarrow \mathbb{R}_2^4$ be a smooth regular curve (i.e., $\gamma'(t) \neq \mathbf{0}$), where I is an open interval. For any $t \in I$, the curve γ is called *spacelike*, *lightlike* or *timelike* if the velocity of the curve is $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle > 0$, $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle = 0$ or $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle < 0$ respectively.

Let $\gamma : I \rightarrow \mathbb{R}_2^4$ be a unit speed timelike curve, parameterized by the arclength parameter s , i.e. $\langle \gamma'(s), \gamma'(s) \rangle = -1$. If γ'' is a spacelike vector, we can choose two lightlike binormal vectors such that γ is a partially null curve. In [21], the authors have given a frame of partially null curve, which contains two transversal lightlike vectors. The tangent and the principal normal vector fields are defined respectively by

$$T(s) = \gamma'(s), \quad N(s) = \frac{\gamma''(s)}{\|\gamma''(s)\|}.$$

The first and second binormal vector fields are taken from the subspace $\{T, N\}^\perp$, denoted respectively by B_1 and B_2 . Then the lightlike frame

$\{T, N, B_1, B_2\}^{[21]}$ associated with the partially null curve satisfies

$$\begin{aligned} \langle T, T \rangle = -\langle N, N \rangle = -1, \quad \langle B_1, B_1 \rangle = \langle B_2, B_2 \rangle = 0, \quad \langle B_1, B_2 \rangle = -1, \\ \langle N, B_1 \rangle = \langle N, B_2 \rangle = \langle T, N \rangle = \langle T, B_1 \rangle = \langle T, B_2 \rangle = 0. \end{aligned}$$

The two transversal lightlike vectors can be substituted by a spacelike vector and a timelike vector

$$\frac{B_1 + B_2}{\sqrt{2}} \text{ and } \frac{B_1 - B_2}{\sqrt{2}}.$$

Therefore, for a general situation, if $\|\gamma''(s)\| \neq 0$ (i.e. γ'' is a spacelike vector or a timelike vector), we can construct, without loss of generality, a pseudo-orthogonal frame without lightlike vectors. Denote $\mathbf{t}(s) = T(s)$, $\mathbf{n}_1(s) = N(s)$. Take $k_1(s) = \|\gamma''(s)\|$ as a curvature function. As γ is not a pseudo null curve, $k_1(s) \neq 0$. Then take $\mathbf{n}_2(s) = (\delta_1 k_1(s)\mathbf{t}(s) - \mathbf{n}'_1(s))/k_2(s)$, where $k_2(s) = \|\delta_1 k_1(s)\mathbf{t}(s) - \mathbf{n}'_1(s)\|$ and $\delta_i = \text{sign}(\mathbf{n}_i(s))$ ($i = 1, 2, 3$). $\mathbf{n}_3(s)$ is defined as

$$\mathbf{n}_3(s) = \frac{\mathbf{t}(s) \wedge \mathbf{n}_1(s) \wedge \mathbf{n}_2(s)}{\|\mathbf{t}(s) \wedge \mathbf{n}_1(s) \wedge \mathbf{n}_2(s)\|}.$$

So we define a pseudo-orthogonal frame $F = \{\mathbf{t}(s), \mathbf{n}_1(s), \mathbf{n}_2(s), \mathbf{n}_3(s)\}$ of \mathbb{R}_2^4 which is a positively oriented 4-tuple of vectors satisfying

$$(1) \quad \begin{aligned} \langle \mathbf{t}, \mathbf{t} \rangle = -1, \quad \langle \mathbf{n}_i, \mathbf{n}_i \rangle = \delta_i, \\ \langle \mathbf{t}, \mathbf{n}_1 \rangle = \langle \mathbf{t}, \mathbf{n}_2 \rangle = \langle \mathbf{t}, \mathbf{n}_3 \rangle = \langle \mathbf{n}_1, \mathbf{n}_2 \rangle = \langle \mathbf{n}_1, \mathbf{n}_3 \rangle = \langle \mathbf{n}_2, \mathbf{n}_3 \rangle = 0, \end{aligned}$$

where $\delta_1\delta_2\delta_3 = -1$ and $\delta_1 + \delta_2 + \delta_3 = 1$.

The Frenet formula of γ with respect to the frame F is as follows

$$(2) \quad \begin{cases} \mathbf{t}'(s) = k_1(s)\mathbf{n}_1(s) \\ \mathbf{n}'_1(s) = \delta_1 k_1(s)\mathbf{t}(s) - k_2(s)\mathbf{n}_2(s) \\ \mathbf{n}'_2(s) = -\delta_3 k_2(s)\mathbf{n}_1(s) - k_3(s)\mathbf{n}_3(s) \\ \mathbf{n}'_3(s) = -\delta_1 k_3(s)\mathbf{n}_2(s), \end{cases}$$

where $k_2(s) = -\delta_2 \langle \mathbf{n}'_1(s), \mathbf{n}_2(s) \rangle$, $k_3(s) = -\delta_3 \langle \mathbf{n}'_2(s), \mathbf{n}_3(s) \rangle$.

Remark 1. γ is a partially null curve when $\delta_1 = 1$. We can take $\mathbf{n}_2 \pm \mathbf{n}_3$ as the two lightlike binormal vectors. For the sake of completeness and unification, we take the pseudo-orthogonal frame $F = \{\mathbf{t}, \mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ without lightlike vectors, and naturally extend our results to the case of $\|\gamma''\| \neq 0$.

Remark 2. We focus on $k_2(s) \neq 0$. Otherwise,

$$\mathbf{n}'_1(s) \equiv \delta_1 k_1(s) \mathbf{t}(s).$$

It means that γ is locally a plane curve.

We define $\mathbf{ng}_\gamma : U \rightarrow NC^*$, where $U = I \times [0, 2\pi)$, by

$$\begin{aligned} \mathbf{ng}_\gamma(s, \theta) &= \left(\frac{1 - \delta_1}{2} + \frac{1 - \delta_2}{2} \cos \theta + \frac{1 - \delta_3}{2} \cos \theta \right) \mathbf{n}_1(s) \\ (3) \quad &+ \left(\frac{1 - \delta_1}{2} \cos \theta + \frac{1 - \delta_2}{2} + \frac{1 - \delta_3}{2} \sin \theta \right) \mathbf{n}_2(s) \\ &+ \left(\frac{1 - \delta_1}{2} \sin \theta + \frac{1 - \delta_2}{2} \sin \theta + \frac{1 - \delta_3}{2} \right) \mathbf{n}_3(s), \end{aligned}$$

it is called the *lightcone Gaussian surface* of γ .

Remark 3. If $\delta_1 = \delta_3 = 1$, $\delta_2 = -1$, $\mathbf{ng}_\gamma(s, \theta) = \cos \theta \mathbf{n}_1(s) + \mathbf{n}_2(s) + \sin \theta \mathbf{n}_3(s)$, that is a surface on the lightcone.

We define the *lightlike hypersurface along γ*

$$\mathbf{nh}_\gamma : U \times \mathbb{R} \rightarrow \mathbb{R}_2^4$$

by $\mathbf{nh}_\gamma(s, \theta, t) = \gamma(s) + t \mathbf{ng}_\gamma(s, \theta)$. If we fix θ_0 , the lightlike hypersurface is just a lightlike ruled surface along γ .

We also define a new important function of the timelike curve in \mathbb{R}_2^4 by

$$\begin{aligned} (4) \quad \eta(s) &= \left(k_1 k_2 (k_1'' + \delta_3 k_1 k_2^2) - k_1' (2k_1' k_2 + k_1 k_2') \right. \\ &\quad \left. \mp k_1 k_2 k_3 \sqrt{\delta_2 k_1^2 k_2^2 + \delta_1 (k_1')^2} \right) (s). \end{aligned}$$

Let $F : \mathbb{R}_2^4 \rightarrow \mathbb{R}$ be a submersion and $\gamma : I \rightarrow \mathbb{R}_2^4$ be a timelike curve. We say that γ and $F^{-1}(0)$ have *k-point contact* for $t = t_0$ if the function $h(t) = F \circ \gamma(t)$ satisfies $h(t_0) = h'(t_0) = \dots = h^{(k-1)}(t_0) = 0$, $h^{(k)}(t_0) \neq 0$. We also say that γ and $F^{-1}(0)$ have *at least k-point contact* for $t = t_0$ if the function $h(t) = F \circ \gamma(t)$ satisfies $h(t_0) = h'(t_0) = \dots = h^{(k-1)}(t_0) = 0$. For any fixed $\mathbf{v}_0 \in \mathbb{R}_2^4$, we have a model surface $NC_{\mathbf{v}_0}$. It is a lightcone with vertex \mathbf{v}_0 . We now consider the following conditions

(A 1) The number of points p of $\gamma(I)$ where the model surface at p having five-point contact with the curve γ is finite.

(A 2) There is no point p of $\gamma(I)$ where the model surface at p having greater than or equal to six-point contact with the curve γ .

Here, we present the main results in this paper.

Theorem 1. Let $\gamma : I \rightarrow \mathbb{R}_2^4$ be a timelike curve with $\|\gamma''(s)\| \neq 0$. Let $\mathbf{v}_0 = \mathbf{nh}_\gamma(s_0, \theta_0, t_0)$, we have the following:

- (1) NC_{v_0} and γ have at least 2-point contact at s_0 .
- (2) NC_{v_0} and γ have 3-point contact at s_0 if and only if there exists $\theta_0 \in [0, 2\pi)$ such that

$$\varphi(s_0, \theta_0) \neq 0 \text{ and } \gamma(s_0) - \mathbf{v}_0 = \frac{1}{k_1(s_0)\left(\frac{\delta_1-1}{2} + \frac{\delta_1+1}{2} \cos \theta_0\right)} \mathbf{ng}_\gamma(s_0, \theta_0),$$

where $\varphi(s, \theta) = k_1'(s)\left(\frac{\delta_1-1}{2} + \frac{\delta_1+1}{2} \cos \theta\right) - k_1(s)k_2(s)\left(\frac{1-\delta_1}{2} \cos \theta + \frac{\delta_2-1}{2} + \frac{1-\delta_3}{2} \sin \theta\right)$. Under this condition, the lightlike hypersurface \mathbf{nh}_γ at \mathbf{v}_0 is locally diffeomorphic to $C(2, 3) \times \mathbb{R}^2$ and the lightlike focal set \mathbf{nf}_γ is non-singular.

- (3) NC_{v_0} and γ have 4-point contact at s_0 if and only if there exists $\theta_0 = \theta(s_0) \in [0, 2\pi)$ such that $\varphi(s_0, \theta(s_0)) = 0$, $\eta(s_0) \neq 0$ and

$$\gamma(s_0) - \mathbf{v}_0 = \frac{1}{k_1(s_0)\left(\frac{\delta_1-1}{2} + \frac{\delta_1+1}{2} \cos \theta_0\right)} \mathbf{ng}_\gamma(s_0, \theta(s_0)).$$

Under this condition, the lightlike hypersurface \mathbf{nh}_γ at \mathbf{v}_0 is locally diffeomorphic to $SW \times \mathbb{R}$, the lightlike focal set \mathbf{nf}_γ is locally diffeomorphic to $C(2, 3, 4) \times \mathbb{R}$ and the singular value set of \mathbf{nf}_γ is a regular curve.

- (4) NC_{v_0} and γ have 5-point contact at s_0 if and only if there exists $\theta(s_0) \in [0, 2\pi)$ such that $\varphi(s_0, \theta(s_0)) = \eta(s_0) = 0$, $\eta'(s_0) \neq 0$ and

$$\gamma(s_0) - \mathbf{v}_0 = \frac{1}{k_1(s_0)\left(\frac{\delta_1-1}{2} + \frac{\delta_1+1}{2} \cos \theta_0\right)} \mathbf{ng}_\gamma(s_0, \theta(s_0)).$$

Under this condition, the lightlike hypersurface \mathbf{nh}_γ at \mathbf{v}_0 is locally diffeomorphic to BF , the lightlike focal set \mathbf{nf}_γ is locally diffeomorphic to $C(BF) \times \mathbb{R}$ and the singular value set of \mathbf{nf}_γ is locally diffeomorphic to the $C(2, 3, 4, 5)$ -cusp.

We respectively call

$$C(2, 3) = \{(x^1, x^2) | x^1 = u^2, x^2 = u^3\},$$

$$C(2, 3, 4) = \{(x^1, x^2, x^3) | x^1 = u^2, x^2 = u^3, x^3 = u^4\},$$

$$C(2, 3, 4, 5) = \{(x^1, x^2, x^3, x^4) | x^1 = u^2, x^2 = u^3, x^3 = u^4, x^4 = u^5\}$$

$(2, 3)$ -cusp, $(2, 3, 4)$ -cusp, $(2, 3, 4, 5)$ -cusp.

And we respectively call $SW = \{(x^1, x^2, x^3) | x^1 = 3u^4 + u^2v, x^2 = 4u^3 + 2uv, x^3 = v\}$, $BF = \{(x^1, x^2, x^3, x^4) | x^1 = 5u^4 + 3vu^2 + 2wu, x^2 = 4u^5 + 2vu^3 + wu^2, x^3 = u, x^4 = v\}$, $C(BF) = \{(x^1, x^2, x^3, x^4) | x^1 =$

$6u^5 + u^3v, x^2 = 25u^4 + 9u^2v, x^3 = 10u^3 + 3uv, x^4 = v$ swallowtail, butterfly, *c-butterfly* (i.e., the singular value set of the butterflies). One can see Figure 1, Figure 2 and Figure 3. We will give the proof of Theorem 1 in §5.

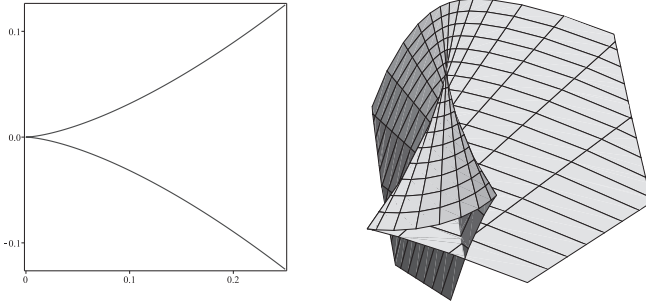


Fig. 1. (2,3)-cusp and Swallowtail.

§3. Lightcone Gaussian Surface

In this section, we analyse the lightcone Gaussian surface from the view point of differential geometry. And we obtain that a lightcone Gaussian surface is locally either a regular Lorentz surface, or a 1-lightlike surface.

Let $\mathbf{x} : U \rightarrow NC^*$ be an embedding of an open subset $U \subset \mathbb{R}^2$. We denote $M = \mathbf{x}(U)$ and identify M and U through the embedding \mathbf{x} . Denote TM and T_pM the tangent bundle and the tangent space of M at $p \in M$. M is called a *Lorentz surface* if T_pM is a Lorentz plane for any point $p \in M$.

If $\langle \cdot, \cdot \rangle$ is degenerate on TM , we say that M is a *lightlike submanifold* of NC^* . Next, we introduce some basic notions about lightlike submanifolds (see [7]). Denote by $\mathcal{F}(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $\mathcal{F}(M)$ module of smooth sections of a vector bundle E (same notation for any other vector bundle) over M . For a degenerate tensor field $\langle \cdot, \cdot \rangle$ on M , there exists locally a vector field $\xi \in \Gamma(TM)$ such that $\langle \xi, \mathbf{X} \rangle = 0$ for any $\mathbf{X} \in \Gamma(TM)$. Then for each tangent space T_pM we have $T_pM^\perp = \{\mathbf{u} \in T_pNC^* | \langle \mathbf{u}, \mathbf{v} \rangle = 0 \forall \mathbf{v} \in T_pM\}$, which is a degenerate 1-dimensional subspace of T_pNC^* . The *radical subspace* of T_pM (denoted as $\text{Rad}T_pM$) is defined by

$$\text{Rad}T_pM = \{\xi_p \in T_pM | \langle \xi_p, \mathbf{X} \rangle = 0 \forall \mathbf{X} \in T_pM\}.$$

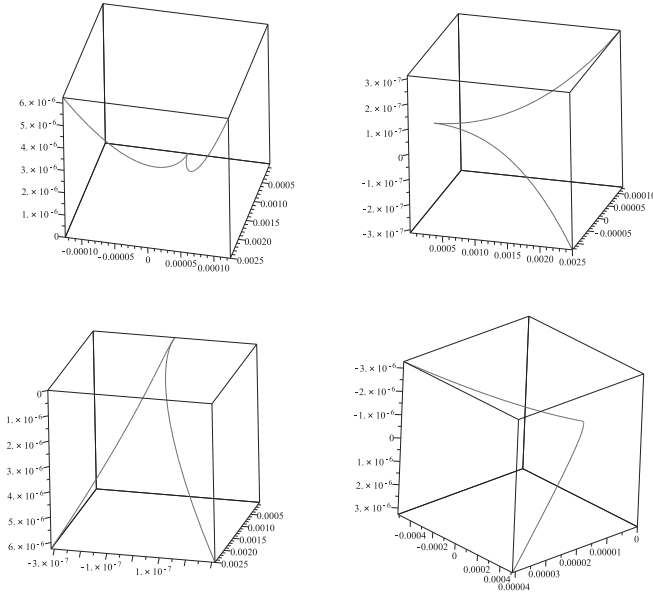


Fig. 2. Projection of a (2,3,4,5)-cusp respectively on $x^1x^2x^3$ -space, $x^1x^2x^4$ -space, $x^1x^3x^4$ -space, $x^2x^3x^4$ -space.

The dimension of $\text{Rad}T_pM = T_pM \cap T_pM^\perp$ depends on $p \in M$. The submanifold M of NC^* is said to be a 1-lightlike submanifold if the mapping

$$\begin{aligned} \text{Rad}TM : M &\longrightarrow TM \\ p &\longmapsto \text{Rad}T_pM \end{aligned}$$

defines a smooth distribution of rank 1 on M .

For the lightcone Gaussian surface \mathbf{ng}_γ , we have the following results.

Proposition 1. *Let \mathbf{ng}_γ be the lightcone Gaussian surface of γ with $\|\gamma''(s)\| \neq 0$.*

- (1) *If \mathbf{n}_1 is a timelike vector, \mathbf{ng}_γ is a regular surface.*
- (2) *If \mathbf{n}_1 is a spacelike vector, the singular set of \mathbf{ng}_γ is*

$$\{(\mathbf{n}_2 + \mathbf{n}_3)(s_0), \pm(\mathbf{n}_2 - \mathbf{n}_3)(s_0) | k_3(s_0) = 0\}.$$

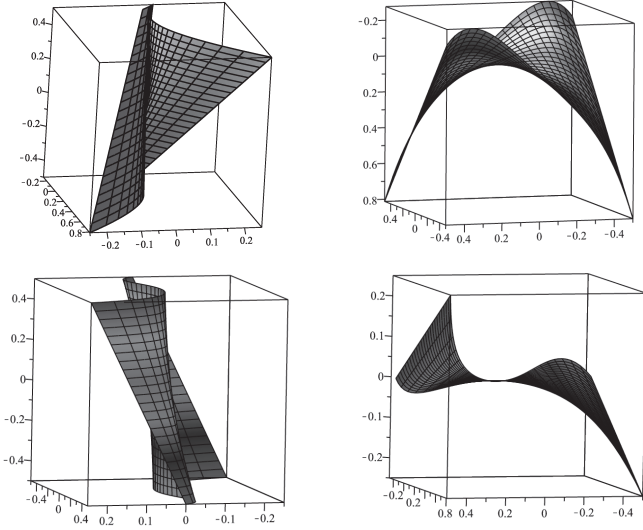


Fig. 3. When $v = 0$, projection of a butterfly respectively on $x^1x^2x^3$ -space, x^1x^2w -space, x^1x^3w -space, x^2x^3w -space.

Proof. If $\delta_1 = -1$,

$$\frac{\partial \mathbf{ng}_\gamma}{\partial \theta} = -\sin \theta \mathbf{n}_2 + \cos \theta \mathbf{n}_3,$$

$$\frac{\partial \mathbf{ng}_\gamma}{\partial s} = -k_1 \mathbf{t} - k_2 \cos \theta \mathbf{n}_1 + (k_3 \sin \theta - k_2) \mathbf{n}_2 - k_3 \cos \theta \mathbf{n}_3.$$

As $k_1(s) \neq 0$, the above two vectors are definitely linearly independent. It means \mathbf{ng}_γ is a regular surface. If $\delta_1 \neq -1$,

$$\begin{aligned} \frac{\partial \mathbf{ng}_\gamma}{\partial \theta} &= -\sin \theta \mathbf{n}_1 + \frac{1 - \delta_3}{2} \cos \theta \mathbf{n}_2 + \frac{1 + \delta_3}{2} \cos \theta \mathbf{n}_3, \\ \frac{\partial \mathbf{ng}_\gamma}{\partial s} &= \frac{1 + \delta_1}{2} k_1 \cos \theta \mathbf{t} + \left(-\frac{1 - \delta_2}{2} k_2 + \frac{1 - \delta_3}{2} k_2 \sin \theta \right) \mathbf{n}_1 \\ &\quad + \left(\frac{\delta_2 - 1}{2} k_3 \sin \theta - k_2 \cos \theta - \frac{1 - \delta_3}{2} k_3 \right) \mathbf{n}_2 \\ &\quad + \left(-\frac{1 - \delta_2}{2} k_3 - \frac{1 - \delta_3}{2} k_3 \sin \theta \right) \mathbf{n}_3. \end{aligned}$$

$\partial \mathbf{ng}_\gamma / \partial \theta$ and $\partial \mathbf{ng}_\gamma / \partial s$ are linearly dependent if and only if $\cos \theta_0 = 0$ and $k_3(s_0) = 0$. Therefore, the singular set of \mathbf{ng}_γ is given by

$$\{(\mathbf{n}_2 + \mathbf{n}_3)(s_0), \pm(\mathbf{n}_2 - \mathbf{n}_3)(s_0) | k_3(s_0) = 0\}.$$

Q.E.D.

Proposition 2. *Let \mathbf{ng}_γ be the lightcone Gaussian surface of γ with $\|\gamma''(s)\| \neq 0$.*

- (1) *If \mathbf{n}_1 is a timelike vector, \mathbf{ng}_γ is a Lorentz surface.*
- (2) *If \mathbf{n}_1 is a spacelike vector, except the singular parts \mathbf{ng}_γ is a Lorentz surface in the local neighborhood of (s, θ_0) , where $\theta_0 \neq \pi/2, 3\pi/2$. Otherwise, if $\theta_0 = \pi/2$ or $3\pi/2$, it is a 1-lightlike surface.*

Proof. At regular parts, \mathbf{ng}_γ can be locally generated by this two vectors $\partial \mathbf{ng}_\gamma / \partial \theta$ and $\partial \mathbf{ng}_\gamma / \partial s$. Obviously, $\partial \mathbf{ng}_\gamma / \partial \theta$ is a spacelike vector. Let

$$\boldsymbol{\iota} = P - \left\langle P, \frac{\partial \mathbf{ng}_\gamma}{\partial \theta} \right\rangle \frac{\partial \mathbf{ng}_\gamma}{\partial \theta},$$

where

$$P = \left(\frac{1 + \delta_3}{2} k_3 + \frac{1 - \delta_3}{2} k_2 \right) \frac{\partial \mathbf{ng}_\gamma}{\partial \theta} + \left(\frac{1 + \delta_2}{2} + \frac{1 - \delta_2}{2} \cos \theta \right) \frac{\partial \mathbf{ng}_\gamma}{\partial s}.$$

Then $\langle \boldsymbol{\iota}, \partial \mathbf{ng}_\gamma / \partial \theta \rangle = 0$ and

$$\langle \boldsymbol{\iota}, \boldsymbol{\iota} \rangle = \frac{1 - \delta_1}{2} (-k_1^2) + \frac{1 - \delta_2}{2} (-k_1^2 \cos^4 \theta) + \frac{1 - \delta_3}{2} (-k_1^2 \cos^2 \theta).$$

Thus, \mathbf{ng}_γ can also be locally generated by $\partial \mathbf{ng}_\gamma / \partial \theta$ and $\boldsymbol{\iota}$ at regular parts.

When $\delta_1 = -1$, $\langle \boldsymbol{\iota}, \boldsymbol{\iota} \rangle = -k_1^2 < 0$. It means \mathbf{ng}_γ is a Lorentz surface. When $\delta_1 \neq -1$, $\langle \boldsymbol{\iota}, \boldsymbol{\iota} \rangle \leq 0$. It means \mathbf{ng}_γ is a Lorentz surface (1-lightlike surface) in the local neighborhood of (s, θ_0) , where

$$\theta_0 \neq \frac{\pi}{2}, \frac{3}{2}\pi \left(\theta_0 = \frac{\pi}{2} \text{ or } \frac{3}{2}\pi \right).$$

This completes the proof.

Q.E.D.

§4. A Family of Lorentz Distance-Squared Functions

In this section we introduce one very useful family of functions on a partially null curve. For a partially null curve γ , we define the function

$$G : I \times \mathbb{R}_2^4 \longrightarrow \mathbb{R}, \quad G(s, \mathbf{v}) = \langle \gamma(s) - \mathbf{v}, \gamma(s) - \mathbf{v} \rangle.$$

This function is called the *Lorentz distance-squared function* of γ . We use the notation $g_v(s) = G(s, \mathbf{v})$ for any fixed vector \mathbf{v} in \mathbb{R}_2^4 . They describe the contact between $\gamma(s)$ and a lightcone. As we study this family of functions, it will become clear how singularities and the corresponding catastrophes arise.

By using Eqs. (2) and by making tedious calculations, we can state Proposition 3.

Proposition 3. *Let $\gamma : I \rightarrow \mathbb{R}_2^4$ be a timelike curve with $\|\gamma''(s)\| \neq 0$. Suppose that $\gamma(s_0) \neq \mathbf{v}_0$, then we have the following.*

(1) $g_{\mathbf{v}_0}(s_0) = g'_{\mathbf{v}_0}(s_0) = 0$ if and only if there exist $\theta_0 \in [0, 2\pi)$ and $\mu \in \mathbb{R} \setminus \{0\}$ such that $\gamma(s_0) - \mathbf{v}_0 = \mu \mathbf{ng}_\gamma(s_0, \theta_0)$.

(2) $g_{\mathbf{v}_0}(s_0) = g'_{\mathbf{v}_0}(s_0) = g''_{\mathbf{v}_0}(s_0) = 0$ if and only if there exists $\theta_0 \in [0, 2\pi)$ such that

$$\gamma(s_0) - \mathbf{v}_0 = \frac{1}{k_1(s_0)\left(\frac{\delta_1-1}{2} + \frac{\delta_1+1}{2} \cos \theta_0\right)} \mathbf{ng}_\gamma(s_0, \theta(s_0)).$$

(3) $g_{\mathbf{v}_0}(s_0) = g'_{\mathbf{v}_0}(s_0) = g''_{\mathbf{v}_0}(s_0) = g'''_{\mathbf{v}_0}(s_0) = 0$ if and only if there exists $\theta_0 \in [0, 2\pi)$ such that

$$\gamma(s_0) - \mathbf{v}_0 = \frac{1}{k_1(s_0)\left(\frac{\delta_1-1}{2} + \frac{\delta_1+1}{2} \cos \theta_0\right)} \mathbf{ng}_\gamma(s_0, \theta(s_0))$$

and $\varphi(s_0, \theta_0) = 0$, where $\varphi(s, \theta) = k'_1(s)\left(\frac{\delta_1-1}{2} + \frac{\delta_1+1}{2} \cos \theta\right) - k_1(s)k_2(s)\left(\frac{1-\delta_1}{2} \cos \theta + \frac{\delta_2-1}{2} + \frac{1-\delta_3}{2} \sin \theta\right)$. So we can write $\theta_0 = \theta(s_0)$.

(4) $g_{\mathbf{v}_0}(s_0) = g'_{\mathbf{v}_0}(s_0) = g''_{\mathbf{v}_0}(s_0) = g'''_{\mathbf{v}_0}(s_0) = g_{\mathbf{v}_0}^{(4)}(s_0) = 0$ if and only if there exists $\theta(s_0) \in [0, 2\pi)$ such that

$$\gamma(s_0) - \mathbf{v}_0 = \frac{1}{k_1(s_0)\left(\frac{\delta_1-1}{2} + \frac{\delta_1+1}{2} \cos \theta_0\right)} \mathbf{ng}_\gamma(s_0, \theta(s_0))$$

and $\varphi(s_0, \theta(s_0)) = \eta(s_0) = 0$.

(5) $g_{\mathbf{v}_0}(s_0) = g'_{\mathbf{v}_0}(s_0) = g''_{\mathbf{v}_0}(s_0) = g'''_{\mathbf{v}_0}(s_0) = g_{\mathbf{v}_0}^{(4)}(s_0) = g_{\mathbf{v}_0}^{(5)}(s_0) = 0$ if and only if there exists $\theta(s_0) \in [0, 2\pi)$ such that

$$\gamma(s_0) - \mathbf{v}_0 = \frac{1}{k_1(s_0)\left(\frac{\delta_1-1}{2} + \frac{\delta_1+1}{2} \cos \theta_0\right)} \mathbf{ng}_\gamma(s_0, \theta(s_0))$$

and $\varphi(s_0, \theta(s_0)) = \eta(s_0) = \eta'(s_0) = 0$.

The above proposition also states that the discriminant set of the Lorentz distance-squared function G is given by

$$D_G = \mathbf{nh}_\gamma(U \times \mathbb{R}) = \{\mathbf{v} = \gamma(s) + \mu \mathbf{ng}_\gamma(s, \theta) \mid (s, \theta) \in U, \mu \in \mathbb{R}\},$$

which is the image of the lightlike hypersurface along γ . Therefore, a singular point of the lightlike hypersurface is the point $\mathbf{v}_0 = \gamma(s_0) + \mu_0 \mathbf{ng}_\gamma(s_0, \theta_0)$, where $\mu_0 = (k_1(s_0)(\frac{\delta_1-1}{2} + \frac{\delta_1+1}{2} \cos \theta_0))^{-1}$.

We define $\mathbf{nf}_\gamma : U \rightarrow \mathbb{R}_2^4$ as

$$\mathbf{nf}_\gamma(s, \theta) = \gamma(s) + \mu_0 \mathbf{ng}_\gamma(s, \theta),$$

we call it the *lightlike focal set of γ* . By definition, the lightlike focal set is the singular value set of the lightlike hypersurface \mathbf{nh}_γ .

§5. Proof of the Main Results

In this section we classify singularities of the lightlike hypersurface along γ as an application of the unfolding theory of functions. Detailed descriptions could be found in [5]. Let

$$F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \rightarrow \mathbb{R}$$

be a function germ. We call F an *r-parameter unfolding of f* , if $f(s) = F_{x_0}(s, \mathbf{x}_0)$. We say f has A_k -singularity at s_0 , if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$ and $f^{(k+1)}(s_0) \neq 0$. Let F be an r -parameter unfolding of f , where f has A_k -singularity ($k \geq 1$) at s_0 . We denote the $(k - 1)$ -jet of the partial derivative $\partial F / \partial x^i$ at s_0 as

$$j^{(k-1)} \left(\frac{\partial F}{\partial x^i}(s, \mathbf{x}_0) \right) (s_0) = \sum_{j=1}^{k-1} \alpha_{ji} (s - s_0)^j, \quad (i = 1, \dots, r).$$

If the rank of $k \times r$ matrix $(\alpha_{0i}, \alpha_{ji})$ is k ($k \leq r$), then F is called a *versal unfolding of f* , where $\alpha_{0i} = \partial F / \partial x^i(s_0, \mathbf{x}_0)$.

Inspired by the proposition in the previous section, we have:

$$D_F^l = \left\{ \mathbf{x} \in \mathbb{R}^r \mid \exists s \in \mathbb{R}, F(s, \mathbf{x}) = \frac{\partial F}{\partial s}(s, \mathbf{x}) = \dots = \frac{\partial^l F}{\partial s^l}(s, \mathbf{x}) = 0 \right\},$$

which is called a *discriminant set with order l* . Therefore, we have the following proposition.

Proposition 4. *For a timelike curve γ with $\|\gamma''(s)\| \neq 0$,*

$$D_G = D_G^1 = \mathbf{nh}_\gamma(U \times \mathbb{R}), \quad D_G^2 = \mathbf{nf}_\gamma(U)$$

and

$$D_G^3 \text{ is the singular value set of } \mathbf{nf}_\gamma.$$

Then we have the following classification theorem as Corollary 7.7 in [18].

Theorem 2. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \rightarrow \mathbb{R}$ be an r -parameter unfolding of f with A_k -singularity at s_0 . Suppose F is a versal unfolding of f , then we have the following assertions:

- (a) If $k = 1$, then D_F is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$ and $D_F^2 = \emptyset$.
- (b) If $k = 2$, then D_F is locally diffeomorphic to $C(2, 3) \times \mathbb{R}^{r-2}$, D_F^2 is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-2}$ and $D_F^3 = \emptyset$.
- (c) If $k = 3$, then D_F is locally diffeomorphic to $SW \times \mathbb{R}^{r-3}$, D_F^2 is locally diffeomorphic to $C(2, 3, 4) \times \mathbb{R}^{r-3}$, D_F^3 is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-3}$ and $D_F^4 = \emptyset$.
- (d) If $k = 4$, then D_F is locally diffeomorphic to $BF \times \mathbb{R}^{r-4}$, D_F^2 is locally diffeomorphic to $C(BF) \times \mathbb{R}^{r-4}$, D_F^3 is locally diffeomorphic to $C(2, 3, 4, 5) \times \mathbb{R}^{r-4}$, D_F^4 is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-4}$ and $D_F^5 = \emptyset$.

For the proof of Theorem 1 we have the following fundamental proposition in this paper.

Proposition 5. If $g(s)$ has A_k -singularity ($k = 1, 2, 3, 4$) at s_0 , then G is a versal unfolding of g .

Proof. By definition,

$$G(s, \mathbf{v}) = -(x^0(s) - v^0)^2 - (x^1(s) - v^1)^2 + (x^2(s) - v^2)^2 + (x^3(s) - v^3)^2,$$

where $\boldsymbol{\gamma}(s) = (x^0(s), x^1(s), x^2(s), x^3(s))$ and $\mathbf{v} = (v^0, v^1, v^2, v^3)$. For a fixed $\mathbf{v}_0 = (v_0^0, v_0^1, v_0^2, v_0^3)$, the 3-jet of $\partial G / \partial v^i(s, \mathbf{v}_0)$ at s_0 is

$$j^{(3)} \frac{\partial G}{\partial v^i}(s_0) = \begin{cases} 2(x^i)'(s - s_0) + (x^i)''(s - s_0)^2 + \frac{(x^i)'''}{3}(s - s_0)^3 & i = 0, 1 \\ -2(x^i)'(s - s_0) - (x^i)''(s - s_0)^2 - \frac{(x^i)'''}{3}(s - s_0)^3 & i = 2, 3 \end{cases}.$$

The condition for versatility can be checked as follows.

- (1) When g has A_1 -singularity at s_0 , we require the 1×4 matrix

$$A_1 = (2(x^0 - v^0), 2(x^1 - v^1), -2(x^2 - v^2), -2(x^3 - v^3))$$

to have rank 1, which it always does since $\mathbf{v}_0 \neq \boldsymbol{\gamma}(s_0)$.

- (2) When g has A_2 -singularity at s_0 , we require 2×4 matrix

$$A_2 = \begin{pmatrix} 2(x^0 - v^0) & 2(x^1 - v^1) & -2(x^2 - v^2) & -2(x^3 - v^3) \\ 2(x^0)' & 2(x^1)' & -2(x^2)' & -2(x^3)' \end{pmatrix}$$

to have rank 2. Otherwise, if $\text{rank} A_2 = 1$, it means that $\boldsymbol{\gamma}(s_0) - \mathbf{v}_0$ and $\mathbf{t}(s_0)$ are linearly dependent. This contradicts with the fact that $\{\mathbf{t}(s), \mathbf{n}_1(s), \mathbf{n}_2(s), \mathbf{n}_3(s)\}$ is the pseudo-orthogonal frame of $\boldsymbol{\gamma}$.

(3) When g has A_3 -singularity at s_0 , we require 3×4 matrix

$$A_3 = \begin{pmatrix} 2(x^0 - v^0) & 2(x^1 - v^1) & -2(x^2 - v^2) & -2(x^3 - v^3) \\ 2(x^0)' & 2(x^1)' & -2(x^2)' & -2(x^3)' \\ (x^0)'' & (x^1)'' & -(x^2)'' & -(x^3)'' \end{pmatrix}$$

to have rank 3. Otherwise, if $\text{rank}A_3 = 2$, it means that $\gamma''(s_0)$ can be generated by $\gamma(s_0) - \mathbf{v}_0$ and $\mathbf{t}(s_0)$. Through a straightforward calculation, we can easily show that it is a contradiction.

(4) When g has A_4 -singularity at s_0 , we require 4×4 matrix

$$A_4 = \begin{pmatrix} 2(x^0 - v^0) & 2(x^1 - v^1) & -2(x^2 - v^2) & -2(x^3 - v^3) \\ 2(x^0)' & 2(x^1)' & -2(x^2)' & -2(x^3)' \\ (x^0)'' & (x^1)'' & -(x^2)'' & -(x^3)'' \\ \frac{1}{3}(x^0)''' & \frac{1}{3}(x^1)''' & -\frac{1}{3}(x^2)''' & -\frac{1}{3}(x^3)''' \end{pmatrix}$$

to have rank 4.

In fact

$$\begin{aligned} \det A_4 &= \frac{4}{3} \det(\gamma(s_0) - \mathbf{v}_0, \gamma'(s_0), \gamma''(s_0), \gamma'''(s_0)) \\ &= -\frac{4k_1(s_0)k_2(s_0)}{3\left(\frac{\delta_1-1}{2} + \frac{\delta_1+1}{2} \cos \theta_0\right)} \left(\frac{\delta_3 + 1}{2} \sin \theta_0 + \frac{\delta_3 - 1}{2}\right). \end{aligned}$$

When $\delta_3 = -1$

$$\det A_4 = \frac{4k_1(s_0)k_2(s_0)}{3\left(\frac{\delta_1-1}{2} + \frac{\delta_1+1}{2} \cos \theta_0\right)} \neq 0.$$

When $\delta_3 = 1$ and $\delta_1 = -1$, we have $\det A_4 \neq 0$ under the condition that $k_1'(s_0) \neq \pm(k_1k_2)(s_0)$. If $k_1'(s_0) = \pm(k_1k_2)(s_0)$, then $k_1''(s_0) = k_1(s_0)k_2^2(s_0) \pm k_1(s_0)k_2'(s_0)$ because $\eta(s_0) = 0$. Then $\eta'(s_0) = 0$. This contradicts with the assumption that g has A_4 -singularity at s_0 . When $\delta_3 = 1$ and $\delta_2 = -1$, the proof is the same. Here, it is omitted. Therefore, $\text{rank}A_4 = 4$.

In summary, G is a versal unfolding of g . This completes the proof. Q.E.D.

We now give the proofs of Theorem 1.

Proof of Theorem 1 Let $\gamma : I \rightarrow \mathbb{R}_2^4$ be a timelike regular curve with $\|\gamma''(s)\| \neq 0$. As $\mathbf{v}_0 = \mathbf{nh}_\gamma(s_0, \theta_0, t_0)$, we give a function $\mathfrak{G} : \mathbb{R}_2^4 \rightarrow \mathbb{R}$, by $\mathfrak{G}(\mathbf{u}) = \langle \mathbf{u} - \mathbf{v}_0, \mathbf{u} - \mathbf{v}_0 \rangle$, then we assume that $g_{v_0}(s) = \mathfrak{G}(\gamma(s))$. Because $\mathfrak{G}^{-1}(0) = NC_{v_0}$ and 0 is a regular value of \mathfrak{G} , γ and NC_{v_0} have $(k + 1)$ -point contact for s_0 if and only if $g_{v_0}(s)$ has A_k -singularity at s_0 .

Thus γ and NC_{v_0} have at least 2-point contact for s_0 if and only if $g_{v_0}(s_0) = g'_{v_0}(s_0) = 0$. By Proposition 4, we have

$$D_G^1 = \mathfrak{nh}_\gamma(U \times \mathbb{R}), \quad D_G^2 = \mathfrak{nf}_\gamma(U)$$

and

$$D_G^3 \text{ is the singular value set of } \mathfrak{nf}_\gamma.$$

By combining Proposition 3, Theorem 2, and Proposition 5, we get the results. Q.E.D.

For the proof of the generic properties, one can see [6] that are omitted here.

§6. Example

As an application and an illustration of the main result (Theorem 1), we give an example of a partially null curve in this section.

Example 1. Let γ be a unit speed timelike curve of \mathbb{R}_2^4 defined by $\gamma(s) = (\sqrt{2}e^s, s, e^s \cos s, e^s \sin s)$ with respect to arclength parameter s and satisfying $\|\gamma''(s)\| \neq 0$. The tangent vector $\mathbf{t}(s)$ is given by

$$\mathbf{t}(s) = (\sqrt{2}e^s, 1, e^s \cos s - e^s \sin s, e^s \sin s + e^s \cos s).$$

And

$$\begin{aligned} \mathbf{n}_1(s) &= (1, 0, -\sqrt{2} \sin s, \sqrt{2} \cos s), \\ \mathbf{n}_2(s) &= \frac{1}{\sqrt{e^s + 1}} (\sqrt{2}e^{2s}, e^s, e^{2s} \cos s - e^{2s} \sin s + \cos s, \\ &\quad e^{2s} \cos s + e^{2s} \sin s + \sin s), \\ \mathbf{n}_3(s) &= \frac{1}{\sqrt{e^{2s} + 1}} (\sqrt{2}, -e^s, -\sin s, \cos s), \\ k_1(s) &= \sqrt{2}e^s, \\ k_2(s) &= \sqrt{2e^{2s} + 2}, \\ k_3(s) &= \frac{-e^{3s} \cos s + e^{3s} \sin s + 2e^{2s} - e^s \cos s + 1}{\sqrt{e^s + 1}(e^{2s} + 1)^{3/2}}. \end{aligned}$$

Obviously, \mathbf{n}_2 and \mathbf{n}_3 can be substituted by two transversal lightlike vectors $\mathbf{n}_2 + \mathbf{n}_3$ and $\mathbf{n}_2 - \mathbf{n}_3$. Accordingly, γ is a partially null curve.

In this example $\mathfrak{ng}_\gamma(s, \theta) = \cos \theta \mathbf{n}_1(s) + \sin \theta \mathbf{n}_2(s) + \mathbf{n}_3(s)$. By maple, $k_3(s) \neq 0$ for any $s \in \mathbb{R}$. Thus it is a regular surface. Moreover, \mathfrak{ng}_γ is a Lorentz surface in the local neighborhood of (s, θ_0) , where $\theta_0 \neq \pi/2, 3\pi/2$. Otherwise, if $\theta_0 = \pi/2$ or $3\pi/2$, it is a 1-lightlike surface.

The important functions associated with γ are as follows

$$\begin{aligned} \varphi(s, \theta) &= \sqrt{2}e^s \cos \theta - 2e^s \sqrt{e^{2s} + 1} \sin \theta, \\ \eta(s) &= -2e^{2s} \sqrt{2e^{2s} + 2}(1 + 2e^{2s}) - \frac{4e^{2s}(3e^{2s} + 2)}{\sqrt{2e^{2s} + 2}} \\ &\quad + \frac{2e^s \sqrt{2e^{2s}(2e^{2s} + 3)}(-e^{3s} \cos s + e^{3s} \sin s + 2e^{2s} - e^s \cos s + 1)}{\sqrt{e^s + 1}(e^{2s} + 1)}. \end{aligned}$$

By maple, we find $\eta(s) \neq 0$ for any $s \in \mathbb{R}$ and $\varphi(s, \theta) = 0$ if and only if $\tan \theta = 1/\sqrt{2e^{2s} + 2}$.

Furthermore, the vector parametric equations of the lightlike hypersurface \mathbf{nh}_γ are given by

$$\{\mathbf{nh}_{\gamma 1}, \mathbf{nh}_{\gamma 2}, \mathbf{nh}_{\gamma 3}, \mathbf{nh}_{\gamma 4}\},$$

where

$$\begin{aligned} \mathbf{nh}_{\gamma 1}(s, \theta, t) &= \sqrt{2}e^s + t \left(\cos \theta + \frac{\sqrt{2} + \sqrt{2}e^{2s} \sin \theta}{\sqrt{1 + e^{2s}}} \right), \\ \mathbf{nh}_{\gamma 2}(s, \theta, t) &= s + \frac{te^s(\sin \theta - 1)}{\sqrt{1 + e^{2s}}}, \\ \mathbf{nh}_{\gamma 3}(s, \theta, t) &= e^s \cos s + t \left(-\sqrt{2} \sin s \cos \theta \right. \\ &\quad \left. + \frac{\sin \theta(e^{2s} \cos s - e^{2s} \sin s + \cos s) - \sin s}{\sqrt{e^{2s} + 1}} \right), \\ \mathbf{nh}_{\gamma 4}(s, \theta, t) &= e^s \sin s + t \left(\sqrt{2} \cos s \cos \theta \right. \\ &\quad \left. + \frac{\sin \theta(e^{2s} \sin s + e^{2s} \cos s + \sin s) + \cos s}{\sqrt{e^{2s} + 1}} \right). \end{aligned}$$

We take $\mathbf{v}_0 = \mathbf{nh}_\gamma(s_0, \theta_0, t_0)$, where $s_0 = 0$, $\theta_0 = \arctan(1/\sqrt{4})$ and $t_0 = -(\sqrt{2} \cos(\arctan(1/\sqrt{4})))^{-1}$. So $\varphi(s_0, \theta_0) = 0$ and $\eta(s_0) \neq 0$. By Theorem 1, \mathbf{nh}_γ at \mathbf{v}_0 is locally diffeomorphic to $SW \times \mathbb{R}$, see Figure 1. In general, for any

$$\mathbf{v} = \mathbf{nh}_\gamma(s, \arctan(1/\sqrt{2e^{2s} + 2}), -(\sqrt{2} \cos(\arctan(1/\sqrt{4})))^{-1}),$$

we have $\varphi(s, \arctan(1/\sqrt{2e^{2s} + 2})) = 0$ and $\eta(s) \neq 0$. Accordingly, \mathbf{nh}_γ is locally diffeomorphic to $SW \times \mathbb{R}$ at \mathbf{v} .

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