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The theory of graph-like Legendrian unfoldings : Equivalence relations

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Abstract.

This is a half survey article on the recent development of the theory of graph-like Legendrian unfoldings which is the sequel to the previous surveys. The notion of big Legendrian submanifolds was introduced by Zakalyukin for describing the wave front propagations. Graph-like Legendrian unfoldings belong to a special class of big Legendrian submanifolds. In particular, natural equivalence relations among graph-like Legendrian unfoldings are introduced and geometric properties of these equivalence relations are investigated. Although this is a survey article, some new original results and proofs for some implicitly known results are given.

§1. Introduction

The notion of graph-like Legendrian unfoldings was introduced in [14]. It belongs to a special class of the big Legendrian submanifolds which Zakalyukin introduced in [38, 39]. There have been some developments on this theory during past two decades [14, 15, 19, 20, 21]. This is a sequel to a survey article on the theory of graph-like Legendrian unfoldings and its applications [24]. The results in the first half part of this paper have been already presented, implicitly or explicitly, in the above articles. The later half of this paper focuses on natural equivalence relations among big Legendrian submanifolds and graph-like Legendrian unfoldings as a special case. Some of the results here explain how the theory of graph-like Legendrian unfoldings is useful for applying to many situations related to the theory of Lagrangian singularities (caustics).

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The caustic is described as the set of critical values of the projection of a Lagrangian submanifold from the phase space onto the configuration space. Moreover, it has been known that caustics equivalence (i.e., diffeomorphic caustics) does not imply Lagrangian equivalence. This is one of the main differences from the theory of Legendrian singularities. In the theory of Legendrian singularities, wave fronts equivalence (i.e., diffeomorphic wave fronts) implies Legendrian equivalence generically.

On the other hand, in the real world, the caustics given by refracted rays are visible. However, the wave front propagations are not visible. We give a picture of the caustic generated by the rays through a wine glass (cf. Fig.1).

Fig.1: The caustic generated by the ray through a wine glass. The picture was taken at A-TABLE in Sapporo.

We can observe the caustic but cannot observe the wave front propagation of the rays. However, if we draw the pictures of the parallels (cf. Fig.2) and the normal lines of a parabola (cf. Fig.3) respectively, we can observe the caustic (i.e. the evolute), the wave front propagation (i.e. the parallels) and the family of the rays (i.e. the lines) respectively. Therefore, we can say that there are hidden structures (i.e., wave front propagations and the family of the rays) on the picture of caustics in the real world.

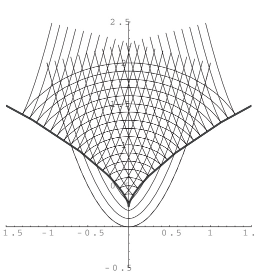


Fig.2: The parallels and the evolute of a parabola.

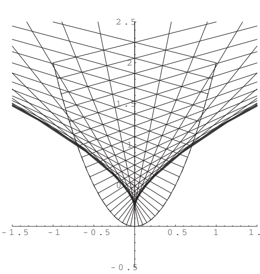


Fig.3: The normal lines and the evolute of a parabola.

In fact, caustics are a subject of classical physics (i.e. optics). The corresponding Lagrangian submanifold is, however, deeply related to the *semi-classical approximation* of quantum mechanics (cf. [9, 12, 29]). Moreover, it was believed around 1989 that the correct framework to describe the parallels of a curve is the theory of big wave fronts [2]. But it was pointed out that A_1 and A_2 bifurcations do not occur as the parallels of curves [3, 7]. Therefore, the framework of the theory of big wave fronts is too wide for describing the parallels of curves. The theory of the graph-like Legendrian unfoldings was introduced to construct the correct framework for the parallels of a curve in [14]. In order to understand such geometric properties, we introduce natural five equivalence relations among big Legendrian submanifolds. In particular, $S.P^+$-Legendrian equivalence is a key notion to understand the relations between the wave front propagation and the caustics, which was introduced in [15, 40] independently for different purposes. One of the main results in the theory of graph-like Legendrian unfoldings is Theorem 6.1 which reveals the relation between caustics and wave front propagations by using $S.P^+$-Legendrian equivalence. In Example 6.11 we give examples of Lagrangian submanifolds with diffeomorphic caustics but are not Lagrangian equivalent. Those examples are famous examples. However, we clarify the reason why these have diffeomorphic caustics but are not Lagrangian equivalent geometrically. Moreover, those examples explain the situation that even if the phase portraits of wave front propagations are different but those are Lagrangian equivalent (cf. Fig.7 and Fig.10). Here, the phase portrait means that the picture of the arrangement of both the caustics and the family of momentary fronts.

We give two examples of applications of the theory of wave front propagations in §8. One of the examples is a brief explanation of the results on the stability of caustics formulated in the framework of Hamiltonian systems by Jänich [27] and Wassermann [35]. They adopted the notion of universal unfoldings with respect to $\mathcal{A}$-equivalence (i.e. right-left equivalence). However, it is known that the Lagrangian stability of the caustic is equivalent to the universality of unfoldings with respect to $\mathcal{R}^+$-equivalence (i.e. right equivalence) [1, 9, 12, 37]. Therefore, their stability of caustics is not corresponding to the Lagrangian stability. In this paper we show that their stability for caustics is equivalent to the s-P-Legendrian stability for the corresponding graph-like Legendrian unfolding (cf. Theorems 7.9 and 8.11).

Another example is a survey on the caustics of world hyper-sheets in the Lorentz-Minkowski space-time [22, 23]. A *world hyper-sheet* in the Lorentz-Minkowski space-time is a timelike hypersurface formed by a one-parameter family of spacelike submanifolds of codimension two in the ambient space. Each spacelike submanifold in the world hyper-sheet is called a *momentary space*. We consider the family of lightlike hypersurfaces along monetary spaces in the world hyper-sheet. In [4, 5] Bousso and Randall considered that the locus of the singularities (the lightlike focal sets) of lightlike hypersurfaces along momentary spaces form a caustic in the Lorentz-Minkowski space-time. This construction is originally from the theoretical physics (the string theory, the brane world scenario, the cosmology, and so on). We call it a *BR-caustic* of the world hyper-sheet. We have no notion of the time constant in the relativity theory. Hence everything that is moving depends on the time. Therefore, we have to consider world hyper-sheets in the relativity theory. Even if we consider a fixed light source (i.e. a shining surface) in the Euclidean 3-space, it must be a world hyper-sheet in the 4-dimensional Lorentz-Minkowski space-time. So the caustic in the Euclidean 3-space is a slice of the BR-caustic of the world hyper-sheet with a spacelike hyperplane. Since the parameter of a world hyper-sheet is intrinsically given, we really need the theory of graph-like Legendrian unfoldings for the study of BR-caustics.

§2. Lagrangian and Legendrian singularities

We give a brief review of the local theory of Lagrangian and Legendrian singularities. We have already written surveys on these theory in several articles [17, 24, 26]. However, it is better to explain the basic results in those theories here again.

Firstly, we consider the cotangent bundle $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$ over $\mathbb{R}^n$. Let $(x, p) = (x_1, \dots, x_n, p_1, \dots, p_n)$ be the canonical coordinates on $T^*\mathbb{R}^n$. Then the canonical symplectic structure on $T^*\mathbb{R}^n$ is given by the *canonical two form* $\omega = \sum_{i=1}^n dp_i \wedge dx_i$. Let $i : L \subset T^*\mathbb{R}^n$ be a submanifold. We say that i is a *Lagrangian submanifold* if $\dim L = n$ and $i^*\omega = 0$. In this case, the set of critical values of $\pi \circ i$ is called the *caustic* of $i : L \subset T^*\mathbb{R}^n$, which is denoted by C_L. One of the main results in the theory of Lagrangian singularities is the description of Lagrangian submanifold germs by using families of function germs. Let $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an n-parameter unfolding of a function germ $f = F|_{\mathbb{R}^k \times \{0\}} : (\mathbb{R}^k, 0) \to (\mathbb{R}, 0)$. We say that F is a *Morse family of functions* if the map germ

$$\Delta F = \left(\frac{\partial F}{\partial q_1}, \dots, \frac{\partial F}{\partial q_k} \right) : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$$

is non-singular, where $(q, x) = (q_1, \dots, q_k, x_1, \dots, x_n) \in (\mathbb{R}^k \times \mathbb{R}^n, 0)$. In this case, we have a smooth n-dimensional submanifold germ $C(F) =$

$(\Delta F)^{-1}(0) \subset (\mathbb{R}^k \times \mathbb{R}^n, 0)$ and a map germ $L(F) : (C(F), 0) \to T^*\mathbb{R}^n$ defined by

$$L(F)(q,x) = \left(x, \frac{\partial F}{\partial x_1}(q,x), \ldots, \frac{\partial F}{\partial x_n}(q,x) \right).$$

We can show that $L(F)(C(F))$ is a Lagrangian submanifold germ. It is known ([1], page 300) that all Lagrangian submanifold germs in $T^*\mathbb{R}^n$ are constructed by the above method. A Morse family of functions $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ is said to be a *generating family* of $L(F)(C(F))$.

We now define a natural equivalence relation among Lagrangian submanifold germs. Let $i : (L, p) \subset (T^*\mathbb{R}^n, p)$ and $i' : (L', p') \subset (T^*\mathbb{R}^n, p')$ be Lagrangian submanifold germs. Then we say that i and i' are *Lagrangian equivalent* if there exist a symplectic diffeomorphism germ $\hat{\tau} : (T^*\mathbb{R}^n, p) \to (T^*\mathbb{R}^n, p')$ and a diffeomorphism germ $\tau : (\mathbb{R}^n, \pi(p)) \to (\mathbb{R}^n, \pi(p'))$ such that $(\hat{\tau}(L), p') = (L', p')$ as set germs and $\pi \circ \hat{\tau} = \tau \circ \pi$, where $\pi : (T^*\mathbb{R}^n, p) \to (\mathbb{R}^n, \pi(p))$ is the canonical projection. Here $\hat{\tau}$ is said to be a *symplectic diffeomorphism germ* if it is a diffeomorphism germ such that $\hat{\tau}^*\omega = \omega$. Then the caustic C_L is diffeomorphic to the caustic $C_{L'}$ by the diffeomorphism germ τ. We say that L and L' are *caustics equivalent* if there is a diffeomorphism germ $\tau : (\mathbb{R}^n, \pi(p)) \to (\mathbb{R}^n, \pi(p'))$ such that $(\tau(C_L), \pi(p')) = (C_{L'}, \pi(p'))$ as set germs. It is known that caustic equivalence does not imply Lagrangian equivalence even generically (cf. Example 6.11).

We can interpret the Lagrangian equivalence by using the notion of generating families. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be function germs. We say that F and G are *P-$\mathcal{R}^+$-equivalent* if there exist a diffeomorphism germ $\Phi : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}^k \times \mathbb{R}^n, 0)$ of the form $\Phi(q,x) = (\phi_1(q,x), \phi_2(x))$ and a function germ $h : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ such that $G(q,x) = F(\Phi(q,x)) + h(x)$. For any $F_1 : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ and $F_2 : (\mathbb{R}^{k'} \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, F_1 and F_2 are said to be *stably P-$\mathcal{R}^+$-equivalent* if they become P-$\mathcal{R}^+$-equivalent after the addition to the arguments q_i of new arguments q'_i and to the functions F_i of non-degenerate quadratic forms Q_i in the new arguments, i.e., $F_1 + Q_1$ and $F_2 + Q_2$ are P-$\mathcal{R}^+$-equivalent.

Let $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a Morse family of functions and $\mathcal{E}_k$ the local ring of function germs of $q = (q_1, \ldots, q_k)$ variables at the origin with the unique maximal ideal $\mathfrak{M}_k = \{h \in \mathcal{E}_k \mid h(0) = 0\}$. We say that F is an *infinitesimally $\mathcal{R}^+$-versal unfolding* of $f = F|_{\mathbb{R}^k \times \{0\}}$ (cf. [6]) if

$$\mathcal{E}_k = J_f + \left\langle \frac{\partial F}{\partial x_1}|_{\mathbb{R}^k \times \{0\}}, \ldots, \frac{\partial F}{\partial x_n}|_{\mathbb{R}^k \times \{0\}} \right\rangle_{\mathbb{R}} + \langle 1 \rangle_{\mathbb{R}},$$

where $f = F|_{\mathbb{R}^k \times \{0\}}$ and

$$J_f = \left\langle \frac{\partial f}{\partial q_1}(q), \dots, \frac{\partial f}{\partial q_k}(q) \right\rangle_{\mathcal{E}_k}.$$

Remark 2.1. There is a definition of *Lagrangian stability* (cf. [1, §21.1]) of a Lagrangian submanifold germ. In this paper we do not need the original definition of Lagrangian stability, so that we omit to give the definition.

Then we have the following fundamental theorem of the theory of Lagrangian singularities (cf. [1]):

Theorem 2.2. *Let* $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ *and* $G : (\mathbb{R}^{k'} \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ *be Morse families of functions. Then we have the following*:
(1) $L(F)(C(F))$ *and* $L(G)(C(G))$ *are Lagrangian equivalent if and only if* F *and* G *are stably* P-$\mathcal{R}^+$*-equivalent.*
(2) $L(F)(C(F))$ *is Lagrangian stable if and only if* F *is an infinitesimally* $\mathcal{R}^+$*-versal unfolding of* $f = F|_{\mathbb{R}^k \times \{0\}}$.

On the other hand, we give a brief review on the theory of Legendrian singularities. Let $\overline{\pi} : PT^*(\mathbb{R}^m) \to \mathbb{R}^m$ be the projective cotangent bundle over $\mathbb{R}^m$. This fibration can be considered as a Legendrian fibration with the canonical contact structure K on $PT^*(\mathbb{R}^m)$. We now review geometric properties of this space. Consider the tangent bundle $\tau : TPT^*(\mathbb{R}^m) \to PT^*(\mathbb{R}^m)$ and the differential map $d\overline{\pi} : TPT^*(\mathbb{R}^m) \to T\mathbb{R}^m$ of $\overline{\pi}$. For any $X \in TPT^*(\mathbb{R}^m)$, there exists an element $\alpha \in T^*(\mathbb{R}^m)$ such that $\tau(X) = [\alpha]$. For an element $V \in T_x(\mathbb{R}^m)$, the property $\alpha(V) = \mathbf{0}$ does not depend on the choice of representative of the class $[\alpha]$. Thus we can define the canonical contact structure on $PT^*(\mathbb{R}^m)$ by $K = \{X \in TPT^*(\mathbb{R}^m) | \tau(X)(d\overline{\pi}(X)) = 0\}$. We have the trivialization $PT^*(\mathbb{R}^m) \cong \mathbb{R}^m \times P(\mathbb{R}^{m*})$ and we call $(x, [\xi])$ *homogeneous coordinates*, where $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ and $[\xi] = [\xi_1 : \cdots : \xi_m]$ are homogeneous coordinates of the dual projective space $P(\mathbb{R}^{m*})$. It is easy to show that $X \in K_{(x,[\xi])}$ if and only if $\sum_{i=1}^m \mu_i \xi_i = 0$, where $d\overline{\pi}(X) = \sum_{i=1}^n \mu_i \frac{\partial}{\partial x_i}$. Let $\Phi : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ be a diffeomorphism germ. Then we have a unique contact diffeomorphism germ $\widehat{\Phi} : PT^*\mathbb{R}^m \to PT^*\mathbb{R}^m$ defined by $\widehat{\Phi}(x, [\xi]) = (\Phi(x), [\xi \circ d_{\Phi(x)}(\Phi^{-1})])$. We call $\widehat{\Phi}$ the *contact lift* of Φ.

A submanifold $i : \mathscr{L} \subset PT^*(\mathbb{R}^m)$ is said to be a *Legendrian submanifold* if $\dim \mathscr{L} = m - 1$ and $di_p(T_p\mathscr{L}) \subset K_{i(p)}$ for any $p \in \mathscr{L}$. We also call $\overline{\pi} \circ i = \overline{\pi}|_{\mathscr{L}} : \mathscr{L} \to \mathbb{R}^m$ a *Legendrian map* and $W(\mathscr{L}) = \overline{\pi}(\mathscr{L})$ a *wave front* of $i : \mathscr{L} \subset PT^*(\mathbb{R}^m)$. We say that a point $p \in \mathscr{L}$ is a

Legendrian singular point if $\operatorname{rank} d(\overline{\pi}|_{\mathscr{L}})_p < m-1$. In this case $\overline{\pi}(p)$ is the singular point of $W(\mathscr{L})$.

The main tool of the theory of Legendrian singularities is also the notion of generating families. Let $F : (\mathbb{R}^k \times \mathbb{R}^m, 0) \to (\mathbb{R}, 0)$ be a function germ. We say that F is a *Morse family of hypersurfaces* if the map germ

$$\Delta^* F = \left(F, \frac{\partial F}{\partial q_1}, \dots, \frac{\partial F}{\partial q_k} \right) : (\mathbb{R}^k \times \mathbb{R}^m, 0) \to (\mathbb{R} \times \mathbb{R}^k, 0)$$

is non-singular, where $(q, x) = (q_1, \dots, q_k, x_1, \dots, x_m) \in (\mathbb{R}^k \times \mathbb{R}^m, 0)$. In this case we have a smooth $(m-1)$-dimensional submanifold germ

$$\Sigma_*(F) = \left\{ (q, x) \in (\mathbb{R}^k \times \mathbb{R}^m, 0) \;\middle|\; F(q, x) = \frac{\partial F}{\partial q_i}(q, x) = 0, i = 1, \dots, k \right\}$$

and a map germ $\mathscr{L}_F : (\Sigma_*(F), 0) \to PT^*\mathbb{R}^m$ defined by

$$\mathscr{L}_F(q, x) = \left(x, \left[\frac{\partial F}{\partial x_1}(q, x) : \cdots : \frac{\partial F}{\partial x_m}(q, x) \right] \right).$$

We can show that $\mathscr{L}_F(\Sigma_*(F)) \subset PT^*(\mathbb{R}^m)$ is a Legendrian submanifold germ. It is known ([1, page 320]) that all Legendrian submanifold germs in $PT^*(\mathbb{R}^m)$ are constructed by the above method. We call F a *generating family* of $\mathscr{L}_F(\Sigma_*(F))$. Therefore the wave front is given by

$$W(\mathscr{L}_F(\Sigma_*(F)) = \left\{ x \in \mathbb{R}^m \;|\exists q \in \mathbb{R}^k \text{ s.t } (q, x) \in \Sigma_*(F) \right\}.$$

Since the Legendrian submanifold germ $i : (\mathscr{L}, p) \subset (PT^*\mathbb{R}^n, p)$ is uniquely determined on the regular part of the wave front $W(\mathscr{L})$, we have the following simple but significant property of Legendrian submanifold germs [39].

Proposition 2.3 (Zakalyukin)**.** *Let $i : (\mathscr{L}, p) \subset (PT^*\mathbb{R}^m, p)$ and $i' : (\mathscr{L}', p') \subset (PT^*\mathbb{R}^m, p')$ be Legendrian submanifold germs such that $\overline{\pi} \circ i, \overline{\pi} \circ i'$ are proper map germs and the sets singularities of these map germs are nowhere dense respectively. Then $(\mathscr{L}, p) = (\mathscr{L}', p')$ if and only if $(W(\mathscr{L}), \overline{\pi}(p)) = (W(\mathscr{L}'), \overline{\pi}(p'))$.*

In order to understand the ambiguity of generating families for a fixed Legendrian submanifold germ we introduce the following equivalence relation among Morse families of hypersurfaces. For function germs $F, G : (\mathbb{R}^k \times \mathbb{R}^m, 0) \to (\mathbb{R}, 0)$, we say that F and G are *strictly parametrized $\mathcal{K}$-equivalent* (briefly, *S.P-$\mathcal{K}$-equivalent*) if there exists a diffeomorphism germ $\Psi : (\mathbb{R}^k \times \mathbb{R}^m, 0) \to (\mathbb{R}^k \times \mathbb{R}^m, 0)$ of the form $\Psi(q, x) = (\psi_1(q, x), x)$ for $(q, x) \in (\mathbb{R}^k \times \mathbb{R}^m, 0)$ such that $\Psi^*(\langle F \rangle_{\mathcal{E}_{k+m}}) = \langle G \rangle_{\mathcal{E}_{k+m}}$.

Here $\Psi^* : \mathcal{E}_{k+m} \to \mathcal{E}_{k+m}$ is the pull back $\mathbb{R}$-algebra isomorphism defined by $\Psi^*(h) = h \circ \Psi$. The definition of *stably S.P-$\mathcal{K}$-equivalence* among Morse families of hypersurfaces is similar to the definition of stably P-$\mathcal{R}^+$-equivalence among Morse families of functions. The following is the key lemma of the theory of Legendrian singularities (cf. [1, 10, 37]).

Lemma 2.4 (Zakalyukin)**.** *Let $F : (\mathbb{R}^k \times \mathbb{R}^m, 0) \to (\mathbb{R}, 0)$ and $G : (\mathbb{R}^{k'} \times \mathbb{R}^m, 0) \to (\mathbb{R}, 0)$ be Morse families of hypersurfaces. Then $(\mathscr{L}_F(\Sigma_*(F)), p) = (\mathscr{L}_G(\Sigma_*(G)), p)$ if and only if F and G are stably S.P-$\mathcal{K}$-equivalent.*

Let $F : (\mathbb{R}^k \times \mathbb{R}^m, 0) \to (\mathbb{R}, 0)$ be a Morse family of hypersurfaces and $\Phi : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ a diffeomorphism germ. We define $\Phi^* F : (\mathbb{R}^k \times \mathbb{R}^m, 0) \to (\mathbb{R}, 0)$ by $\Phi^* F(q, x) = F(q, \Phi(x))$. Then we have $(1_{\mathbb{R}^q} \times \Phi)(\Sigma_*(\Phi^* F)) = \Sigma_*(F)$ and

$$\mathscr{L}_{\Phi^* F}(\Sigma_*(\Phi^* F)) = \left\{ \left(x, \left[\left(\frac{\partial F}{\partial x}(q, \Phi(x)) \right) \circ d\Phi_x \right] \right) \,\middle|\, (q, \Phi(x)) \in \Sigma_*(F) \right\},$$

so that $\widehat{\Phi}(\mathscr{L}_{\Phi^* F}(\Sigma_*(\Phi^* F))) = \mathscr{L}_F(\Sigma_*(F))$ as set germs.

Proposition 2.5. *Let $F : (\mathbb{R}^k \times \mathbb{R}^m, 0) \to (\mathbb{R}, 0)$ and $G : (\mathbb{R}^{k'} \times \mathbb{R}^m, 0) \to (\mathbb{R}, 0)$ be Morse families of hypersurfaces. For a diffeomorphism germ $\Phi : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$, $\widehat{\Phi}(\mathscr{L}_G(\Sigma_*(G))) = \mathscr{L}_F(\Sigma_*(F))$ if and only if $\Phi^* F$ and G are stably S.P-$\mathcal{K}$-equivalent.*

Proof. Since $\widehat{\Phi}(\mathscr{L}_{\Phi^* F}(\Sigma_*(\Phi^* F))) = \mathscr{L}_F(\Sigma_*(F))$, we have

$$\mathscr{L}_{\Phi^* F}(\Sigma_*(\Phi^* F)) = \mathscr{L}_G(\Sigma_*(G)).$$

By Lemma 2.4, the assertion holds. □

We say that $\mathscr{L}_F(\Sigma_*(F))$ and $\mathscr{L}_G(\Sigma_*(G))$ are *Legendrian equivalent* if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ such that the condition in the above proposition holds. By Proposition 2.3, with the generic condition on F and G, $\Phi(W(\mathscr{L}_G(\Sigma_*(G)))) = W(\mathscr{L}_F(\Sigma_*(F)))$ if and only if $\widehat{\Phi}(\mathscr{L}_G(\Sigma_*(G))) = \mathscr{L}_F(\Sigma_*(F))$ for a diffeomorphism germ $\Phi : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$.

For function germs $F, G : (\mathbb{R}^k \times \mathbb{R}^m, 0) \to (\mathbb{R}, 0)$, we say that F and G are *parametrized $\mathcal{K}$-equivalent* (briefly, *P-$\mathcal{K}$-equivalent*) if there exists a diffeomorphism germ $\Psi : (\mathbb{R}^k \times \mathbb{R}^m, 0) \to (\mathbb{R}^k \times \mathbb{R}^m, 0)$ of the form $\Psi(q, x) = (\psi_1(q, x), \psi_2(x))$ for $(q, x) \in (\mathbb{R}^k \times \mathbb{R}^m, 0)$ such that $\Psi^*(\langle F \rangle_{\mathcal{E}_{k+m}}) = \langle G \rangle_{\mathcal{E}_{k+m}}$. We also say that F is an *infinitesimally $\mathcal{K}$-versal unfolding* of $f = F|_{\mathbb{R}^k \times \{0\}}$ if

$$\mathcal{E}_k = T_e(\mathcal{K})(f) + \left\langle \frac{\partial F}{\partial x_1}|_{\mathbb{R}^k \times \{0\}}, \dots, \frac{\partial F}{\partial x_n}|_{\mathbb{R}^k \times \{0\}} \right\rangle_{\mathbb{R}},$$

where $f = F|_{\mathbb{R}^k \times \{0\}}$ and

$$T_e(\mathcal{K})(f) = J_f + \langle f \rangle_{\mathcal{E}_k} = \left\langle \frac{\partial f}{\partial q_1}(q), \ldots, \frac{\partial f}{\partial q_k}(q), f(q) \right\rangle_{\mathcal{E}_k}.$$

Remark 2.6. There is a definition of *Legendrian stability* (cf. [1, §21.1]) of a Legendrian submanifold germ. In this paper we do not need the original definition of Legendrian stability, so that we omit to give the definition.

Then we have the following fundamental theorem of the theory of Legendrian singularities (cf. [1, 37]):

Theorem 2.7. *Let* $F : (\mathbb{R}^k \times \mathbb{R}^m, 0) \to (\mathbb{R}, 0)$, $G : (\mathbb{R}^{k'} \times \mathbb{R}^m, 0) \to (\mathbb{R}, 0)$ *be Morse families of hypersurfaces. Then we have the following*:
(1) $\mathscr{L}_F(\Sigma_*(F))$ *and* $\mathscr{L}_G(\Sigma_*(G))$ *are Legendrian equivalent if and only if* F *and* G *are stably* P-$\mathcal{K}$-*equivalent.*
(2) $\mathscr{L}_F(\Sigma_*(F))$ *is Legendrian stable if and only if* F *is an infinitesimally* $\mathcal{K}$-*versal unfolding of* $f = F|_{\mathbb{R}^k \times \{0\}}$.

We have the following classification theorem as a corollary of Proposition 2.3 and Theorem 2.7 (cf. [16, Proposition A.4]).

Theorem 2.8. *Let* $F : (\mathbb{R}^k \times \mathbb{R}^m, 0) \to (\mathbb{R}, 0)$, $G : (\mathbb{R}^{k'} \times \mathbb{R}^m, 0) \to (\mathbb{R}, 0)$ *be Morse families of hypersurfaces. Suppose that* $\mathscr{L}_F(\Sigma_*(F))$ *and* $\mathscr{L}_G(\Sigma_*(G))$ *are Legendrian stable. Then the following conditions are equivalent*:
(1) $\mathscr{L}_F(\Sigma_*(F))$ *and* $\mathscr{L}_G(\Sigma_*(G))$ *are Legendrian equivalent,*
(2) $f = F|_{\mathbb{R}^k \times \{0\}}$ *and* $g = G|_{\mathbb{R}^{k'} \times \{0\}}$ *are stably* $\mathcal{K}$-*equivalent,*
(3) $W(\mathscr{L}_F(\Sigma_*(F)))$ *and* $W(\mathscr{L}_G(\Sigma_*(G)))$ *are diffeomorphic as set germs.*

Remark 2.9. We say that $f, g : (\mathbb{R}^k, 0) \to (\mathbb{R}, 0)$ are $\mathcal{K}$-*equivalent* if there exists a diffeomorphism germ $\phi : (\mathbb{R}^k, 0) \to (\mathbb{R}^k, 0)$ such that $\phi^*(\langle f \rangle_{\mathcal{E}_k}) = \langle g \rangle_{\mathcal{E}_k}$. We also say that f is r-*determined relative to* $\mathcal{K}$ if f and g are $\mathcal{K}$-equivalent for any $g \in \mathfrak{M}_k$ with $f - g \in \mathfrak{M}_k^{r+1}$. Suppose that f and g are r-determined relative to $\mathcal{K}$. Then it is known that f and g are $\mathcal{K}$-equivalent if and only if $Q_r(f)$ and $Q_r(g)$ are isomorphic as $\mathbb{R}$-algebras (cf. [30]), where $Q_r(f) = \mathcal{E}_k / (\langle g \rangle_{\mathcal{E}_k} + \mathfrak{M}_k^{r+1})$. Moreover, it is known that $f = F|_{\mathbb{R}^k \times \{0\}}$ is $m + 1$-determined if $F : (\mathbb{R}^k \times \mathbb{R}^m, 0) \to (\mathbb{R}, 0)$ is an infinitesimally $\mathcal{K}$-versal unfolding of f (cf. [28]). Therefore, condition (2) in the above theorem can be replace by the following condition:
(2′) $Q_{m+2}(f)$ and $Q_{m+2}(g)$ are isomorphism as $\mathbb{R}$-algebras.

§3. Theory of the wave front propagations

In this section we give a brief survey of the theory of wave front propagations (for details, see [1, 14, 39, 40], etc). We consider one parameter families of wave fronts and their bifurcations. The principal idea is that a one parameter family of wave fronts is considered to be a wave front whose dimension is one dimension higher than each member of the family. This is called a *big wave front*. We consider the case when $m = n+1$ and distinguish space and time coordinates, so that we denote that $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ and coordinates are denoted by $(x,t) = (x_1,\dots,x_n,t) \in \mathbb{R}^n \times \mathbb{R}$. Then we consider the projective cotangent bundle $\overline{\pi} : PT^*(\mathbb{R}^n \times \mathbb{R}) \to \mathbb{R}^n \times \mathbb{R}$. Because of the trivialization $PT^*(\mathbb{R}^n \times \mathbb{R}) \cong (\mathbb{R}^n \times \mathbb{R}) \times P((\mathbb{R}^n \times \mathbb{R})^*)$, we have homogeneous coordinates $((x_1,\dots,x_n,t),[\xi_1 : \cdots : \xi_n : \tau])$. We remark that $PT^*(\mathbb{R}^n \times \mathbb{R})$ is a fiber-wise compactification of the 1-jet space as follows: We consider an affine open subset $U_\tau = \{((x,t),[\xi : \tau]) | \tau \neq 0\}$ of $PT^*(\mathbb{R}^n \times \mathbb{R})$. For any $((x,t),[\xi : \tau]) \in U_\tau$, we have

$$((x_1,\dots,x_n,t),\,[\xi_1 : \cdots : \xi_n : \tau]) \\ = ((x_1,\dots,x_n,t),[-(\xi_1/\tau) : \cdots : -(\xi_n/\tau) : -1]),$$

so that we may adopt the corresponding *affine coordinates*

$$((x_1,\dots,x_n,t),(p_1,\dots,p_n)),$$

where $p_i = -\xi_i/\tau$. On U_τ we can easily show that $\theta^{-1}(0) = K|_{U_\tau}$, where $\theta = dt - \sum_{i=1}^n p_i dx_i$. This means that U_τ can be identified with the 1-jet space which is denoted by $J^1_{GA}(\mathbb{R}^n, \mathbb{R}) \subset PT^*(\mathbb{R}^n \times \mathbb{R})$. We call the above coordinates *a system of graph-like affine coordinates*. Throughout this paper, we use this identification.

For a Legendrian submanifold $i : \mathscr{L} \subset PT^*(\mathbb{R}^n \times \mathbb{R})$, the corresponding wave front $\overline{\pi} \circ i(\mathscr{L}) = W(\mathscr{L})$ is called a *big wave front*. We call

$$W_t(\mathscr{L}) = \pi_1(\pi_2^{-1}(t) \cap W(\mathscr{L})) \quad (t \in \mathbb{R})$$

a *momentary front* (or, a *small front*) for each $t \in (\mathbb{R}, 0)$, where $\pi_1 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $\pi_2 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ are the canonical projections defined by $\pi_1(x,t) = x$ and $\pi_2(x,t) = t$ respectively. In this sense, we call $\mathscr{L}$ a *big Legendrian submanifold*. We say that a point $p \in \mathscr{L}$ is a *space-singular point* if $\operatorname{rank} d(\pi_1 \circ \overline{\pi}|_{\mathscr{L}})_p < n$ and a *time-singular point* if $\operatorname{rank} d(\pi_2 \circ \overline{\pi}|_{\mathscr{L}})_p = 0$, respectively. By definition, if $p \in \mathscr{L}$ is a Legendrian singular point, then it is a space-singular point of $\mathscr{L}$. Even if we have no Legendrian singular points, we have space-singular points. In this case we have the following lemma.

Lemma 3.1 ([24]). *Let $i : \mathscr{L} \subset PT^*(\mathbb{R}^n \times \mathbb{R})$ be a big Legendrian submanifold without Legendrian singular points. If $p \in \mathscr{L}$ is a space-singular point of $\mathscr{L}$, then p is not a time-singular point of $\mathscr{L}$.*

The *discriminant of the family* $\{W_t(\mathscr{L})\}_{t\in(\mathbb{R},0)}$ is defined as the image of singular points of $\pi_1|_{W(\mathscr{L})}$. In the general case, the discriminant consists of three components: *the caustic* $C_{\mathscr{L}} = \pi_1(\Sigma(W(\mathscr{L}))$, where $\Sigma(W(\mathscr{L}))$ is the set of singular points of $W(\mathscr{L})$ (i.e, the critical value set of the Legendrian mapping $\overline{\pi}|_{\mathscr{L}}$), *the Maxwell stratified set* $M_{\mathscr{L}}$, the projection of the closure of the self intersection set of $W(\mathscr{L})$; and also of the critical value set $\Delta_{\mathscr{L}}$ of $\pi_1|_{W(\mathscr{L})\setminus\Sigma(W(\mathscr{L}))}$. In [19, 40], it has been stated that $\Delta_{\mathscr{L}}$ is the *envelope of the family of momentary fronts*. However, we remark that $\Delta_{\mathscr{L}}$ is not necessarily the envelope of the family of the projection of smooth momentary fronts $\overline{\pi}(W_t(\mathscr{L}))$. It can be happened that $\pi_2^{-1}(t)\cap W(\mathscr{L})$ is non-singular but $\pi_1|_{\pi_2^{-1}(t)\cap W(\mathscr{L})}$ has singularities, so that $\Delta_{\mathscr{L}}$ is the set of critical values of the family of mappings $\pi_1|_{\pi_2^{-1}(t)\cap W(\mathscr{L})}$ for smooth $\pi_2^{-1}(t)\cap W(\mathscr{L})$ (cf. [24, §5]).

For any Legendrian submanifold germ $i : (\mathscr{L}, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$, there exists a generating family. Let $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be a Morse family of hypersurfaces. In this case, we call $\mathcal{F}$ a *big Morse family of hypersurfaces.* Then $\Sigma_*(\mathcal{F}) = \Delta^*(\mathcal{F})^{-1}(0)$ is a smooth n-dimensional submanifold germ. By the previous arguments, we have a big Legendrian submanifold $\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ where

$$\mathscr{L}_{\mathcal{F}}(q,x,t) = \left(x, t, \left[\frac{\partial\mathcal{F}}{\partial x}(q,x,t) : \frac{\partial\mathcal{F}}{\partial t}(q,x,t)\right]\right),$$

and

$$\left[\frac{\partial\mathcal{F}}{\partial x}(q,x,t) : \frac{\partial\mathcal{F}}{\partial t}(q,x,t)\right] = \left[\frac{\partial\mathcal{F}}{\partial x_1}(q,x,t) : \cdots : \frac{\partial\mathcal{F}}{\partial x_n}(q,x,t) : \frac{\partial\mathcal{F}}{\partial t}(q,x,t)\right].$$

§4. Equivalence relations

We now consider five equivalence relations among big Legendrian submanifolds. Let $i : (\mathscr{L}, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ and $i' : (\mathscr{L}', p'_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p'_0)$ be big Legendrian submanifold germs. Then we respectively say that $i : (\mathscr{L}, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ and $i' : (\mathscr{L}', p'_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p'_0)$ are

(1) *strictly parametrized Legendrian equivalent* (or, briefly *S.P-Legendrian equivalent*) if there exists a diffeomorphism germ

$\Phi : (\mathbb{R}^n \times \mathbb{R}, \overline{\pi}(p_0)) \to (\mathbb{R}^n \times \mathbb{R}, \overline{\pi}(p'_0))$ of the form $\Phi(x,t) = (\phi_1(x), t)$ such that $\widehat{\Phi}(\mathscr{L}) = \mathscr{L}'$ as set germs,

(2) *space-time Legendrian equivalent* (briefly, (s,t)-*Legendrian equivalent*) if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^n \times \mathbb{R}, \overline{\pi}(p_0)) \to (\mathbb{R}^n \times \mathbb{R}, \overline{\pi}(p'_0))$ of the form $\Phi(x,t) = (\phi_1(x), \phi_2(t))$ such that $\widehat{\Phi}(\mathscr{L}) = \mathscr{L}'$ as set germs,

(3) i and i' are *strictly parametrized*$^+$ *Legendrian equivalent* (briefly, $S.P^+$-*Legendrian equivalent*) if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^n \times \mathbb{R}, \overline{\pi}(p_0)) \to (\mathbb{R}^n \times \mathbb{R}, \overline{\pi}(p'_0))$ of the form $\Phi(x,t) = (\phi_1(x), t + \alpha(x))$ such that $\widehat{\Phi}(\mathscr{L}) = \mathscr{L}'$ as set germs,

(4) i and i' are *time parametrized Legendrian equivalent* (briefly, t-P-*Legendrian equivalent*) if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^n \times \mathbb{R}, \overline{\pi}(p_0)) \to (\mathbb{R}^n \times \mathbb{R}, \overline{\pi}(p'_0))$ of the form $\Phi(x,t) = (\phi_1(x,t), \phi_2(t))$ such that $\widehat{\Phi}(\mathscr{L}) = \mathscr{L}'$ as set germs,

(5) i and i' are *space parametrized Legendrian equivalent* (briefly, s-P-*Legendrian equivalent*) if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^n \times \mathbb{R}, \overline{\pi}(p_0)) \to (\mathbb{R}^n \times \mathbb{R}, \overline{\pi}(p'_0))$ of the form $\Phi(x,t) = (\phi_1(x), \phi_2(x,t))$ such that $\widehat{\Phi}(\mathscr{L}) = \mathscr{L}'$ as set germs,

where $\widehat{\Phi} : (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0) \to (PT^*(\mathbb{R}^n \times \mathbb{R}), p'_0)$ is the unique contact lift of Φ. We remark that (s,t)-Legendrian equivalence looks a natural equivalence relation among big Legendrian submanifold germs. It induces, however, the isomorphisms among divergent diagrams $\mathbb{R} \leftarrow \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ on the base space which is not a geometric subgroup of $\mathcal{A}$ or $\mathcal{K}$ in the sense of Damon [8]. Although $S.P$-Legendrian equivalence relation gets rid of the difficulty for the (s,t)-Legendrian equivalence relation, there appear function moduli for generic classifications in very low dimensions (cf. [11], [24, §5]). In order to avoid the function moduli, we define the $S.P^+$-Legendrian equivalence among big Legendrian submanifolds, which has been independently introduced in [15, 40] for different purposes. If we have a generic classification of big Legendrian submanifold germs by $S.P^+$-Legendrian equivalence, then we have a classification by the $S.P$-Legendrian equivalence modulo function moduli. See [15, 40] for details. This equivalence relation plays an important role in the theory of graph-like Legendrian unfoldings. By definition, (1) implies (2) and (2) implies (4). Moreover, (1) implies (3) and (3) implies (5). The weakest equivalence (5) preserves the diffeomorphism type of $C_{\mathscr{L}} \cup M_{\mathscr{L}} \cup \Delta_{\mathscr{L}}$. Moreover (4) preserve the bifurcations of monetary fronts, which was deeply investigated and given a generic classification by Zakalyukin [39]. We used s-P-Legendrian equivalence among big Legendrian submanifolds for applying to the geometry of world sheets in in Lorentz-Minkowski space [22]. We can also define the notion of stability of Legendrian submanifold germs with respect to all the above equivalence relations which are analogous to the stability of Legendrian submanifold germs with respect to Legendrian equivalence (cf. [1, Part III]).

On the other hand, the assumption in Proposition 2.3 is a generic condition for i, i'. Here, we denote that $\mathcal{G} = S.P,\ (s,t),\ S.P^+,\ t\text{-}P$ or $s\text{-}P$. Concerning the discriminant and the bifurcation of momentary fronts, we define the following equivalence relation among big wave front germs. Let $i : (\mathscr{L}, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ and $i' : (\mathscr{L}', p'_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p'_0)$ be big Legendrian submanifold germs. We say that $W(\mathscr{L})$ and $W(\mathscr{L}')$ are *$\mathcal{G}$-diffeomorphic* if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^n \times \mathbb{R}, \overline{\pi}(p_0)) \to (\mathbb{R}^n \times \mathbb{R}, \overline{\pi}(p'_0))$ of the form corresponding to the diffeomorphism germ in the list of $\mathcal{G}$-Legendrian equivalence such that $\Phi(W(\mathscr{L})) = W(\mathscr{L}')$ as set germs. We also call Φ a *$\mathcal{G}$-diffeomorphism germ*. We remark that a $\mathcal{G}$-diffeomorphism among big wave front germs preserves the diffeomorphism types of $C_{\mathscr{L}} \cup M_{\mathscr{L}} \cup \Delta_{\mathscr{L}}$. By Proposition 2.3, we have the following proposition.

Proposition 4.1. *Let $i : (\mathscr{L}, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ and $i' : (\mathscr{L}', p'_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p'_0)$ be big Legendrian submanifold germs such that $\overline{\pi} \circ i, \overline{\pi} \circ i'$ are proper map germs and the sets of critical points of these map germs are nowhere dense respectively. Then i and i' are $\mathcal{G}$-Legendrian equivalent if and only if $(W(\mathscr{L}), \overline{\pi}(p_0))$ and $(W(\mathscr{L}'), \overline{\pi}(p'_0))$ are $\mathcal{G}$-diffeomorphic.*

Proof. By definition, if i and i' are $\mathcal{G}$-Legendrian equivalent, then $(W(\mathscr{L}), \overline{\pi}(p_0))$ and $(W(\mathscr{L}'), \overline{\pi}(p'_0))$ are $\mathcal{G}$-diffeomorphic. For the converse, suppose that there exists a $\mathcal{G}$-diffeomorphism germ $\Phi : (\mathbb{R}^n \times \mathbb{R}, \overline{\pi}(p_0)) \to (\mathbb{R}^n \times \mathbb{R}, \overline{\pi}(p'_0))$ such that $\Phi(W(\mathscr{L})) = W(\mathscr{L}')$ as set germs. Then $\widehat{\Phi}(\mathscr{L})$ is a big Legendrian submanifold such that $W(\widehat{\Phi}(\mathscr{L})) = \Phi(W(\mathscr{L})) = W(\mathscr{L}')$ as set germs. By Proposition 2.3, we have $\widehat{\Phi}(\mathscr{L}) = \mathscr{L}'$. This completes the proof. □

We explain s-P-Legendrian equivalence and $S.P^+$-Legendrian equivalence can be investigated by using the notion of generating families of Legendrian submanifold germs. For t-P-Legendrian equivalence, [39] is a good survey, so that we omit the detail here.

Let $\overline{f}, \overline{g} : (\mathbb{R}^k \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be function germs. We say that $\overline{f}$ and $\overline{g}$ are *P-$\mathcal{K}$-equivalent* if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^k \times \mathbb{R}, 0) \to (\mathbb{R}^k \times \mathbb{R}, 0)$ of the form $\Phi(q,t) = (\phi_1(q,t), \phi_2(t))$ such that $\langle \overline{f} \circ \Phi \rangle_{\mathcal{E}_{k+1}} = \langle \overline{g} \rangle_{\mathcal{E}_{k+1}}$. We also say that $\overline{f}$ and $\overline{g}$ are *$S.P$-$\mathcal{K}$-equivalent* if these are P-$\mathcal{K}$-equivalent by the diffeomorphism Φ of the form $\Phi(q,t) = (\phi_1(q,t), t)$. Let $\mathcal{F}, \mathcal{G} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$

be function germs. We say that $\mathcal{F}$ and $\mathcal{G}$ are *space-P-$\mathcal{K}$-equivalent* (or, briefly, *s-P-$\mathcal{K}$-equivalent*) if there exists a diffeomorphism germ $\Psi : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0)$ of the form $\Psi(q, x, t) = (\phi(q, x, t), \phi_1(x), \phi_2(x, t)))$ such that $\langle \mathcal{F} \circ \Psi \rangle_{\mathcal{E}_{k+n+1}} = \langle \mathcal{G} \rangle_{\mathcal{E}_{k+n+1}}$. We also say that $\mathcal{F}$ and $\mathcal{G}$ are *space-$S.P^+$-$\mathcal{K}$-equivalent* (or, briefly, *s-$S.P^+$-$\mathcal{K}$-equivalent*) if these are s-P-$\mathcal{K}$-equivalent by the diffeomorphism germ Ψ of the form $\Psi(q, x, t) = (\phi(q, x, t), \phi_1(x), t + \alpha(x))$. The notion of P-$\mathcal{K}$-versal unfoldings and $S.P^+$-$\mathcal{K}$-versal unfoldings play important roles for our purpose. We define the extended tangent spaces of $\overline{f} : (\mathbb{R}^k \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ relative to P-$\mathcal{K}$ by

$$T_e(P\text{-}\mathcal{K})(\overline{f}) = \left\langle \frac{\partial \overline{f}}{\partial q_1}, \dots, \frac{\partial \overline{f}}{\partial q_k}, \overline{f} \right\rangle_{\mathcal{E}_{k+1}} + \left\langle \frac{\partial \overline{f}}{\partial t} \right\rangle_{\mathcal{E}_1}$$

and the extended tangent spaces of $\overline{f}$ relative to $S.P^+$-$\mathcal{K}$ by

$$T_e(S.P^+\text{-}\mathcal{K})(\overline{f}) = \left\langle \frac{\partial \overline{f}}{\partial q_1}, \dots, \frac{\partial \overline{f}}{\partial q_k}, \overline{f} \right\rangle_{\mathcal{E}_{k+1}} + \left\langle \frac{\partial \overline{f}}{\partial t} \right\rangle_{\mathbb{R}},$$

respectively. Then we say that $\mathcal{F}$ an *infinitesimally P-$\mathcal{K}$-versal* unfolding of $\overline{f} = \mathcal{F}|_{\mathbb{R}^k \times \{0\} \times \mathbb{R}}$ if it satisfies

$$\mathcal{E}_{k+1} = T_e(P\text{-}\mathcal{K})(\overline{f}) + \left\langle \frac{\partial \mathcal{F}}{\partial x_1}|_{\mathbb{R}^k \times \{0\} \times \mathbb{R}}, \dots, \frac{\partial \mathcal{F}}{\partial x_n}|_{\mathbb{R}^k \times \{0\} \times \mathbb{R}} \right\rangle_{\mathbb{R}}$$

and it is an *infinitesimally $S.P^+$-$\mathcal{K}$-versal* unfolding of $\overline{f} = \mathcal{F}|_{\mathbb{R}^k \times \{0\} \times \mathbb{R}}$ if it satisfies

$$\mathcal{E}_{k+1} = T_e(S.P^+\text{-}\mathcal{K})(\overline{f}) + \left\langle \frac{\partial \mathcal{F}}{\partial x_1}|_{\mathbb{R}^k \times \{0\} \times \mathbb{R}}, \dots, \frac{\partial \mathcal{F}}{\partial x_n}|_{\mathbb{R}^k \times \{0\} \times \mathbb{R}} \right\rangle_{\mathbb{R}},$$

respectively. We can show the following theorem analogous to those in [15, 39, 40]. We only remark here that the proof is analogous to the proof of [1, Theorem in §21.4].

Theorem 4.2 ([15, 24, 40]). *Let $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ and $\mathcal{G} : (\mathbb{R}^{k'} \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be big Morse families of hypersurfaces. Then*
(1) *$\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ and $\mathscr{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G}))$ are s-P-Legendrian equivalent if and only if $\mathcal{F}$ and $\mathcal{G}$ are stably s-P-$\mathcal{K}$-equivalent.*
(2) *$\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ is s-P-Legendre stable if and only if $\mathcal{F}$ is an infinitesimally P-$\mathcal{K}$-versal unfolding of $\overline{f} = \mathcal{F}|_{\mathbb{R}^k \times \{0\} \times \mathbb{R}}$.*
(3) *$\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ and $\mathscr{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G}))$ are $S.P^+$-Legendrian equivalent if and only if $\mathcal{F}$ and $\mathcal{G}$ are stably s-$S.P^+$-$\mathcal{K}$-equivalent.*

(4) $\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ *is* $S.P^+$*-Legendre stable if and only if* $\mathcal{F}$ *is an infinitesimally* $S.P^+$*-*$\mathcal{K}$*-versal unfolding of* $\overline{f} = \mathcal{F}|_{\mathbb{R}^k \times \{0\} \times \mathbb{R}}$.

§5. Graph-like Legendrian unfoldings

In this section we explain the theory of graph-like Legendrian unfoldings. Graph-like Legendrian unfoldings belong to a special class of big Legendrian submanifolds. A big Legendrian submanifold $i : \mathscr{L} \subset PT^*(\mathbb{R}^n \times \mathbb{R})$ is said to be a *graph-like Legendrian unfolding* if $\mathscr{L} \subset J^1_{GA}(\mathbb{R}^n, \mathbb{R})$. We call $W(\mathscr{L}) = \overline{\pi}(\mathscr{L})$ a *graph-like wave front* of $\mathscr{L}$, where $\overline{\pi} : J^1_{GA}(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}^n \times \mathbb{R}$ is the canonical projection. We define a mapping $\Pi : J^1_{GA}(\mathbb{R}^n, \mathbb{R}) \to T^*\mathbb{R}^n$ by $\Pi(x, t, p) = (x, p)$, where $(x, t, p) = (x_1, \dots, x_n, t, p_1, \dots, p_n)$ and the canonical contact form on $J^1_{GA}(\mathbb{R}^n, \mathbb{R})$ is given by $\theta = dt - \sum_{i=1}^n p_i dx_i$. Here, $T^*\mathbb{R}^n$ is a symplectic manifold with the canonical symplectic structure $\omega = \sum_{i=1}^n dp_i \wedge dx_i$ (cf. [1]). Then we have the following proposition.

Proposition 5.1 ([19])**.** *For a graph-like Legendrian unfolding* $\mathscr{L} \subset J^1_{GA}(\mathbb{R}^n, \mathbb{R})$, $z \in \mathscr{L}$ *is a singular point of* $\overline{\pi}|_{\mathscr{L}} : \mathscr{L} \to \mathbb{R}^n \times \mathbb{R}$ *if and only if it is a singular point of* $\pi_1 \circ \overline{\pi}|_{\mathscr{L}} : \mathscr{L} \to \mathbb{R}^n$*. Moreover,* $\Pi|_{\mathscr{L}} : \mathscr{L} \to T^*\mathbb{R}^n$ *is immersive, so that* $\Pi(\mathscr{L})$ *is a Lagrangian submanifold in* $T^*\mathbb{R}^n$.

We have the following corollary of Proposition 5.1.

Corollary 5.2 ([19])**.** *For a graph-like Legendrian unfolding* $\mathscr{L} \subset J^1_{GA}(\mathbb{R}^n, \mathbb{R})$, $\Delta_{\mathscr{L}}$ *is the empty set, so that the discriminant of the family of momentary fronts is* $C_{\mathscr{L}} \cup M_{\mathscr{L}}$.

Since $\mathscr{L}$ is a big Legendrian submanifold in $PT^*(\mathbb{R}^n \times \mathbb{R})$, it has a generating family $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ at least locally. Since $\mathscr{L} \subset J^1_{GA}(\mathbb{R}^n, \mathbb{R}) = U_\tau \subset PT^*(\mathbb{R}^n \times \mathbb{R})$, it satisfies the condition $(\partial \mathcal{F}/\partial t)(0) \neq 0$. Let $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be a big Morse family of hypersurfaces. We say that $\mathcal{F}$ is a *graph-like Morse family of hypersurfaces* if $(\partial \mathcal{F}/\partial t)(0) \neq 0$. It is easy to show that the corresponding big Legendrian submanifold germ is a graph-like Legendrian unfolding. Of course, all graph-like Legendrian unfolding germs can be constructed by the above way. We say that $\mathcal{F}$ is a *graph-like generating family* of $\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$. We remark that the notion of graph-like Legendrian unfoldings and corresponding generating families have been introduced in [14] to describe the perestroikas of wave fronts given as the solutions for general eikonal equations. In this case, there is an additional condition. We say that $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ is *non-degenerate* if $\mathcal{F}$ satisfies the conditions $(\partial \mathcal{F}/\partial t)(0) \neq 0$ and $\Delta^* \mathcal{F}|_{\mathbb{R}^k \times \mathbb{R}^n \times \{0\}}$ is a submersion germ. In this case we call $\mathcal{F}$ a *non-degenerate graph-like generating family.* We have the following proposition.

Proposition 5.3 ([24]). *Let $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be a graph-like Morse family of hypersurfaces. Then $\mathcal{F}$ is non-degenerate if and only if $\pi_2 \circ \overline{\pi}|_{\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))}$ is submersive.*

We say that a graph-like Legendrian unfolding $\mathscr{L} \subset J^1_{GA}(\mathbb{R}^n, \mathbb{R})$ is *non-degenerate* if $\pi_2 \circ \overline{\pi}|\mathscr{L}$ is submersive. Non-degeneracy was assumed for general graph-like Legendrian unfoldings when the notion of graph-like Legendrian unfoldings was introduced in [14]. However, during the last two decades, we have clarified the situation and non-degeneracy is now defined as above.

We can consider the following more restrictive class of graph-like generating families: Let $\mathcal{F}$ be a graph-like Morse family of hypersurfaces. By the implicit function theorem, there exists a function $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ such that

$$\langle \mathcal{F}(q, x, t) \rangle_{\mathcal{E}_{k+n+1}} = \langle F(q, x) - t \rangle_{\mathcal{E}_{k+n+1}}.$$

Then we have the following proposition.

Proposition 5.4 ([24]). *Let $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ and $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be function germs such that $\langle \mathcal{F}(q, x, t) \rangle_{\mathcal{E}_{k+n+1}} = \langle F(q, x) - t \rangle_{\mathcal{E}_{k+n+1}}$. Then $\mathcal{F}$ is a graph-like Morse family of hypersurfaces if and only if F is a Morse family of functions.*

We now consider the case $\mathcal{F}(q, x, t) = \lambda(q, x, t)(F(q, x) - t)$. In this case,

$$\Sigma_*(\mathcal{F}) = \{(q, x, F(q, x)) \in (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \mid (q, x) \in C(F)\},$$

where $C(F) = \Delta F^{-1}(0)$. Moreover, we have the Lagrangian submanifold germ $L(F)(C(F)) \subset T^*\mathbb{R}^n$, where

$$L(F)(q, x) = \left(x, \frac{\partial F}{\partial x_1}(q, x), \dots, \frac{\partial F}{\partial x_n}(q, x) \right).$$

Since $\mathcal{F}$ is a graph-like Morse family of hypersurfaces, we have a big Legendrian submanifold germ $\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F})) \subset J^1_{GA}(\mathbb{R}^n, \mathbb{R})$, where $\mathscr{L}_{\mathcal{F}} : (\Sigma_*(\mathcal{F}), 0) \to J^1_{GA}(\mathbb{R}^n, \mathbb{R}) \cong T^*\mathbb{R}^n \times \mathbb{R}$ is defined by

$$\mathscr{L}_{\mathcal{F}}(q, x, t) = \left(x, t, -\frac{\dfrac{\partial \mathcal{F}}{\partial x_1}(q, x, t)}{\dfrac{\partial \mathcal{F}}{\partial t}(q, x, t)}, \dots, -\frac{\dfrac{\partial \mathcal{F}}{\partial x_n}(q, x, t)}{\dfrac{\partial \mathcal{F}}{\partial t}(q, x, t)}, \right).$$

We also define a map germ $\mathfrak{L}_F : (C(F), 0) \to J^1_{GA}(\mathbb{R}^n, \mathbb{R})$ by

$$\mathfrak{L}_F(q,x) = \left(x, F(q,x), \frac{\partial F}{\partial x_1}(q,x), \ldots, \frac{\partial F}{\partial x_n}(q,x)\right).$$

Since $\partial\mathcal{F}/\partial x_i = \partial\lambda/\partial x_i(F-t)+\lambda\partial F/\partial x_i$ and $\partial\mathcal{F}/\partial t = \partial\lambda/\partial t(F-t)-\lambda$, we have $\partial\mathcal{F}/\partial x_i(q,x,t) = \lambda(q,x,t)\partial F/\partial x_i(q,x,t)$ and $\partial\mathcal{F}/\partial t(q,x,t) = -\lambda(q,x,t)$ for $(q,x,t) \in \Sigma_*(\mathcal{F})$. It follows that $\mathfrak{L}_F(C(F)) = \mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$. By definition, we have $\Pi(\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))) = \Pi(\mathfrak{L}_F(C(F))) = L(F)(C(F))$. The graph-like wave front of $\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F})) = \mathfrak{L}_F(C(F))$ is the graph of $F|_{C(F)}$. This is the reason why we call it a graph-like Legendrian unfolding. For a non-degenerate graph-like Morse family of hypersurfaces, we have the following proposition.

Proposition 5.5 ([24])**.** *With the same notations as Proposition 5.4, $\mathcal{F}$ is a non-degenerate graph-like Morse family of hypersurfaces if and only if F is a Morse family of hypersurfaces. In this case, F is also a Morse family of functions such that*

$$\left(\frac{\partial F}{\partial x_1}(0), \ldots, \frac{\partial F}{\partial x_n}(0)\right) \neq \mathbf{0}.$$

The momentary front for a fixed $t \in (\mathbb{R}, 0)$ is $W_t(\mathscr{L}) = \pi_1(\pi_2^{-1}(t) \cap W(\mathscr{L}))$. We define $\mathscr{L}_t = \mathscr{L} \cap (\pi_2 \circ \overline{\pi})^{-1}(t) = \mathscr{L} \cap (T^*\mathbb{R}^n \times \{t\})$ under the canonical identification $J^1_{GA}(\mathbb{R}^n, \mathbb{R}) \cong T^*\mathbb{R}^n \times \mathbb{R}$. Then $\Pi(\mathscr{L}) \subset T^*\mathbb{R}^n$ and $\widetilde{\pi} \circ \Pi(\mathscr{L}_t) \subset PT^*\mathbb{R}^n$, where $\widetilde{\pi} : T^*\mathbb{R}^n \to PT^*(\mathbb{R}^n)$ is the canonical projection. We also have the canonical projections $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$ and $\varpi : PT^*\mathbb{R}^n \to \mathbb{R}^n$ such that $\pi_1 \circ \overline{\pi} = \pi \circ \Pi$ and $\varpi \circ \widetilde{\pi} = \pi$. Then we have the following proposition.

Proposition 5.6 ([24])**.** *Let $\mathscr{L} \subset J^1_{GA}(\mathbb{R}^n, \mathbb{R})$ be a non-degenerate graph-like Legendrian unfolding. Then $\widetilde{\pi} \circ \Pi(\mathscr{L}_t)$ is a Legendrian submanifold in $PT^*(\mathbb{R}^n)$.*

The momentary front $W_t(\mathscr{L})$ of a big Legendrian submanifold $\mathscr{L} \subset PT^*(\mathbb{R}^n \times \mathbb{R})$ is not necessarily a wave front of a Legendrian submanifold in the ordinary sense, generally. However, for a non-degenerate Legendrian unfolding in $J^1_{GA}(\mathbb{R}^n, \mathbb{R})$, we have the following corollary.

Corollary 5.7. [24] *Let $\mathscr{L} \subset J^1_{GA}(\mathbb{R}^n, \mathbb{R})$ be a non-degenerate graph-like Legendrian unfolding. Then the momentary front $W_t(\mathscr{L})$ is the wave front set of the Legendrian submanifold $\widetilde{\pi} \circ \Pi(\mathscr{L}_t) \subset PT^*(\mathbb{R}^n)$. Moreover, the caustic C_L is the caustic of the Lagrangian submanifold $\Pi(L) \subset T^*\mathbb{R}^n$. In other words, $W_t(\mathscr{L}) = \varpi(\widetilde{\pi} \circ \Pi(\mathscr{L}_t))$ and $C_{\mathscr{L}}$ is the singular value set of $\pi|\Pi(\mathscr{L})$.*

§6. $S.P^+$-Legendrian equivalence among graph-like Legendrian unfoldings

In this section we describe the properties of $S.P^+$-Legendrian equivalence among graph-like Legendrian unfoldings. For a graph-like Morse family of hypersurfaces $\mathcal{F}(q,x,t) = \lambda(q,x,t)(F(q,x)-t)$, $\mathcal{F}(q,x,t)$ and $\overline{F}(q,x,t) = F(q,x) - t$ are s-$S.P^+$-$\mathcal{K}$-equivalent, so that we consider $\overline{F}(q,x,t) = F(q,x) - t$ as a graph-like Morse family. Moreover, by Proposition 5.4, $F(q,x)$ is a Morse family of functions. We now suppose that $F(q,x)$ is a Morse family of functions. Consider the graph-like Morse family of hypersurfaces $\overline{F}(q,x,t) = F(q,x)-t$ which is not necessarily non-degenerate. Then we have $\mathscr{L}_{\overline{F}}(\Sigma_*(\overline{F})) = \mathfrak{L}_F(C(F))$. We also denote that $\overline{f}(q,t) = f(q) - t$ for any $f \in \mathfrak{M}_k$. We can represent the extended tangent space of $\overline{f} : (\mathbb{R}^k \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ relative to $S.P^+$-$\mathcal{K}$ by

$$T_e(S.P^+\text{-}\mathcal{K})(\overline{f}) = \left\langle \frac{\partial f}{\partial q_1}(q), \dots, \frac{\partial f}{\partial q_k}(q), f(q) - t \right\rangle_{\mathcal{E}_{k+1}} + \langle 1 \rangle_{\mathbb{R}}.$$

For a unfolding $\overline{F} : (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ of $\overline{f}$, $\overline{F}$ is infinitesimally $S.P^+$-$\mathcal{K}$-versal unfolding of $\overline{f}$ if and only if

$$\mathcal{E}_{k+1} = T_e(S.P^+\text{-}\mathcal{K})(\overline{f}) + \left\langle \frac{\partial F}{\partial x_1}|_{\mathbb{R}^k\times\mathbb{R}\times\{0\}}, \dots, \frac{\partial F}{\partial x_n}|_{\mathbb{R}^k\times\mathbb{R}\times\{0\}} \right\rangle_{\mathbb{R}}.$$

We compare the equivalence relations between Lagrangian submanifold germs and induced graph-like Legendrian unfoldings. As a consequence, we give a relationship between caustics and graph-like wave fronts.

Theorem 6.1 ([19, 25])**.** *Let $\mathcal{F} : (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ and $\mathcal{G} : (\mathbb{R}^{k'} \times \mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be graph-like Morse families of hypersurfaces of the forms $\mathcal{F}(q,x,t) = \lambda(q,x,t)(F(q,x)-t)$ and $\mathcal{G}(q',x,t) = \mu(q',x,t)(G(q',x)-t)$. Then Lagrangian submanifold germs $L(F)(C(F))$ and $L(G)(C(G))$ are Lagrangian equivalent if and only if the graph-like Legendrian unfoldings $\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ and $\mathscr{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G}))$ are $S.P^+$-Legendrian equivalent.*

Proof. By Theorem 2.2, if $L(F)(C(F))$ and $L(G)(C(G))$ are Lagrangian equivalent, then F and G are stably P-$\mathcal{R}^+$-equivalent. In this case we may assume that $k = k'$ and F and G are P-$\mathcal{R}^+$-equivalent, so that there exist a diffeomorphism germ $\Phi : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}^k \times \mathbb{R}^n, 0)$ of the form $\Phi(q,x) = (\phi_1(q,x), \phi(x))$ and a function $\alpha : (\mathbb{R}^n, 0) \to \mathbb{R}$ such that $G(q,x) = F \circ \Phi(q,x) + \alpha(x)$. Then we define a diffeomorphism germ

$\widetilde{\Phi} : (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}, 0)$ by $\widetilde{\Phi}(q, x, t) = (\phi_1(q, x), \phi(x), t - \alpha(x))$. It follows that $G(q, x) - t = F \circ \Phi(q, x) + t - \alpha(x)$. This means that $\mathcal{G}$ and $\mathcal{F}$ are s-$S.P^+$-$\mathcal{K}$-equivalent. By Theorem 4.2, $\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ and $\mathscr{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G}))$ are $S.P^+$-Legendrian equivalent. For the converse assertion, we assume that $\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ and $\mathscr{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G}))$ are $S.P^+$-Legendrian equivalent. By the assumption, there exists a diffeomorphism germ $\Phi : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ of the form $\Phi(x, t) = (\phi_1(x), t + \alpha(x))$ such that $\widehat{\Phi}(\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))) = \mathscr{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G}))$. Then we have $\Phi^{-1}(x, t) = (\phi_1^{-1}(x), t - \alpha(\phi_1^{-1}(x)))$, so that the Jacobi matrix is

$$J_{\Phi(x,t)}\Phi^{-1} = \begin{pmatrix} \dfrac{\partial \phi_1^{-1}}{\partial x}(\phi_1(x)) & 0 \\ -\dfrac{\partial \alpha \circ \phi_1^{-1}}{\partial x}(\phi_1(x)) & 1 \end{pmatrix}.$$

It follows that

$$\widehat{\Phi}((x, t), [\xi : \tau]) = \left(\Phi(x, t), \left[\xi \cdot \frac{\partial \phi_1^{-1}}{\partial x}(\phi_1(x)) - \tau \frac{\partial \alpha \circ \phi_1^{-1}}{\partial x}(\phi_1(x)) : \tau \right] \right).$$

Since $\tau \neq 0$,

$$\begin{aligned} &\left[\xi \cdot \frac{\partial \phi_1^{-1}}{\partial x}(\phi_1(x)) - \frac{\partial \alpha \circ \phi_1^{-1}}{\partial x}(\phi_1(x)) : \tau \right] \\ &\qquad = \left[-\frac{\xi}{\tau} \cdot \frac{\partial \phi_1^{-1}}{\partial x}(\phi_1(x)) + \frac{\partial \alpha \circ \phi_1^{-1}}{\partial x}(\phi_1(x)) : -1 \right]. \end{aligned}$$

We consider the graph-like affine coordinates $((x, t), p) \in J^1_{GA}(\mathbb{R}^n, \mathbb{R})$, where $p = -\dfrac{\xi}{\tau}$. Then we have $\widehat{\Phi}(J^1_{GA}(\mathbb{R}^n, \mathbb{R})) = J^1_{GA}(\mathbb{R}^n, \mathbb{R})$ and

$$\widehat{\Phi}((x, t), p) = \left(\phi_1(x), t + \alpha(x), p \cdot \frac{\partial \phi_1^{-1}}{\partial x}(\phi_1(x)) + \frac{\partial \alpha \circ \phi_1^{-1}}{\partial x}(\phi_1(x)) \right).$$

We now define a map $\widetilde{\phi_1} : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ by

$$\widetilde{\phi_1}(x, p) = \left(\phi_1(x), p \cdot \frac{\partial \phi_1^{-1}}{\partial x}(\phi_1(x)) + \frac{\partial \alpha \circ \phi_1^{-1}}{\partial x}(\phi_1(x)) \right).$$

Since $\widehat{\Phi}$ is a contact diffeomorphism germ, there exists a function germ $\mu : J^1_{GA}(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$ with $\mu(x, t, p) \neq 0$ such that $\widehat{\Phi}^*\theta = \mu\theta$. Therefore, we have

$$dt + d\alpha - \widetilde{\phi_1}^*(p \cdot dx) = \mu(dt - p \cdot dx) = \mu dt - \mu(p \cdot dx),$$

so that $\mu \equiv 1$. It follows that $-p \cdot dx = d\alpha - \widetilde{\phi_1}^*(p \cdot dx)$. Thus we have

$$\widetilde{\phi_1}^*(\omega) = \widetilde{\phi_1}^*(d(p \cdot dx)) = d\widetilde{\phi_1}^*(p \cdot dx) = d(p \cdot dx) = \omega.$$

This means that $\widetilde{\phi_1}$ is a symplectic diffeomorphism germ (i.e. Lagrangian diffeomorphism germ). Since $\Pi \circ \widehat{\Phi}|J^1_{GA}(\mathbb{R}^n, \mathbb{R}) = \widetilde{\phi_1} \circ \Pi|J^1_{GA}(\mathbb{R}^n, \mathbb{R})$, we have

$$\begin{aligned} L(G)(C(G)) &= \Pi(\mathscr{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G}))) = \Pi \circ \widehat{\Phi}(\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))) \\ &= \widetilde{\phi_1}(\Pi(\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))) = \widetilde{\phi_1}(L(F)(C(F))). \end{aligned}$$

This completes the proof. □

By definition, the set of Legendrian singular points of a graph-like Legendrian unfolding $\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ coincides with the set of singular points of $\pi \circ L(F)$. Therefore the singularities of graph-like wave fronts of $\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ lie on the caustics of $L(F)$. It follows that we can apply Proposition 4.1 to $S.P^+$-Legendrian equivalence.

Corollary 6.2 ([25]). *Suppose that $\overline{\pi}|_{\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))}$, $\overline{\pi}|_{\mathscr{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G}))}$ are proper map germs and the both sets of Legendrian singular points of graph-like Legendrian unfoldings $\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$, $\mathscr{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G}))$ are no-where dense respectively. Then the following conditions are equivalent*:
(1) *Lagrangian submanifold germs $L(F)(C(F))$, $L(G)(C(G))$ are Lagrangian equivalent*,
(2) *graph-like wave fronts $W(\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F})))$, $W(\mathscr{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G})))$ are $S.P^+$-diffeomorphic.*

Moreover, we have the following direct corollary of Theorem 6.1.

Corollary 6.3 ([21]). *Suppose that $\mathcal{F}(q, x, t) = \lambda(q, x, t)(F(q, x) - t)$ is a graph-like Morse family of hypersurfaces. Then $\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ is $S.P^+$-Legendrian stable if and only if $L(F)(C(F))$ is Lagrangian stable.*

If a Lagrangian submanifold germ $L(F)(C(F))$ is Lagrangian stable, then $\overline{\pi}|_{\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))}$ is a proper map germ and the regular set of this map germ is dense. Hence we can apply Proposition 4.1 to our situation and obtain the following theorem on the relations among graph-like Legendrian unfoldings and Lagrangian singularities.

Theorem 6.4 ([24]). *Let $\mathcal{F} : (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ and $\mathcal{G} : (\mathbb{R}^{k'} \times \mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be graph-like Morse families of hypersurfaces of the forms $\mathcal{F}(q, x, t) = \lambda(q, x, t)(F(q, x) - t)$ and $\mathcal{G}(q', x, t) = \mu(q', x, t)(G(q', x) - t)$ such that $\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ and $\mathscr{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G}))$ are $S.P^+$-Legendrian stable. Then the following conditions are equivalent*:

(1) $\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ *and* $\mathscr{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G}))$ *are* $S.P^+$*-Legendrian equivalent,*
(2) $\mathcal{F}$ *and* $\mathcal{G}$ *are stably* s*-*$S.P^+$*-*$\mathcal{K}$*-equivalent,*
(3) $\overline{f}(q,t) = F(q,0) - t$ *and* $\overline{g}(q',t) = G(q',0) - t$ *are stably* $S.P$*-*$\mathcal{K}$*-equivalent,*
(4) $f(q) = F(q,0)$ *and* $g(q') = G(q',0)$ *are stably* $\mathcal{R}$*-equivalent,*
(5) $F(q,x)$ *and* $G(q',x)$ *are stably* P*-*$\mathcal{R}^+$*-equivalent,*
(6) $L(F)(C(F))$ *and* $L(G)(C(G))$ *are Lagrangian equivalent,*
(7) $W(\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F})))$ *and* $W(\mathscr{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G})))$ *are* $S.P^+$*-diffeomorphic.*

Remark 6.5. (i) The above theorem was shown in [24].
(ii) By Corollary 6.3, the assumption of the above theorem is equivalent to the condition that $L(F)(C(F))$ and $L(G)(C(G))$ are Lagrangian stable.
(iii) If $k = k'$ and $q = q'$ in the above theorem, we can remove the word "stably" in conditions (2), (3), (4) and (5).
(iv) By Theorem 6.1, conditions (1), (2), (5) and (6) are always equivalent without any assumptions. This fact was not known when I wrote the survey paper [24]. After I wrote the paper Theorem 6.1 has been shown in [25]. Therefore, this assertion is a new result.
(v) Conditions (3) and (4) are equivalent without any assumptions.
(vi) Equivalency for (2) and (3) (respectively, (4) and (5)), we need the assumption that the $S.P^+$-Legendrian stability (respectively, the Lagrangian stability). Fortunately, these two stability are equivalent by Corollary 6.3.
(vii) The $S.P^+$-Legendrian stability of $\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ is a generic condition for $n \leq 5$.
(viii) By Proposition 4.1 and Corollary 6.2, the conditions (1), (6) and (7) are equivalent generically for an arbitrary dimension n without the assumption on the $S.P^+$-Legendrian stability.

On the other hand, we consider another geometric condition on the generating families. For a function germ $f : (\mathbb{R}^k, 0) \to (\mathbb{R}, 0)$, the *level set foliation germ* of f is defined to be $\mathscr{F}_f = \{f^{-1}(c) \mid c \in (\mathbb{R}, 0)\}$. For function germs $f, g : (\mathbb{R}^k, 0) \to (\mathbb{R}, 0)$, we say that the level set foliation germs $\mathscr{F}_f$ and $\mathscr{F}_g$ are *strictly diffeomorphic* if there exists a diffeomorphism germ $\psi : (\mathbb{R}^k, 0) \to (\mathbb{R}^k, 0)$ such that $\psi(f^{-1}(c)) = g^{-1}(c)$ as a set germ for any $c \in (\mathbb{R}, 0)$. Then we have the following proposition.

Proposition 6.6. *For function germs* $f, g : (\mathbb{R}^k, 0) \to (\mathbb{R}, 0)$, *the level set foliation germs* $\mathscr{F}_f$ *and* $\mathscr{F}_g$ *are strictly diffeomorphic if and only if* f *and* g *are* $\mathcal{R}$*-equivalent.*

Proof. By definition, if f and g are $\mathcal{R}$-equivalent, then $\mathscr{F}_f$ and $\mathscr{F}_g$ are strictly diffeomorphic. If $\mathscr{F}_f$ and $\mathscr{F}_g$ are strictly diffeomorphic, then there exists a diffeomorphism germ $\psi : (\mathbb{R}^k, 0) \to (\mathbb{R}^k, 0)$ such that $\psi(f^{-1}(c)) = g^{-1}(c)$ as a set germ for any $c \in (\mathbb{R}, 0)$. We consider $\psi \times 1_{\mathbb{R}} : (\mathbb{R}^k \times \mathbb{R}, 0) \to (\mathbb{R}^k \times \mathbb{R}, 0)$ which is a diffeomorphism germ. For any $(q, f(q)) \in \overline{f}^{-1}(0)$, we have $(\psi \times 1_{\mathbb{R}})(q, f(q)) = (\psi(q), f(q))$. If we set $c = f(q)$, then $\psi(q) \in g^{-1}(c) = \overline{g}^{-1}(0) \cap (\mathbb{R}^k \times \{c\})$, so that $(\psi \times 1_{\mathbb{R}})(q, f(q)) \in \overline{g}^{-1}(0)$. Therefore $(\psi \times 1_{\mathbb{R}})(\overline{f}^{-1}(0)) = \overline{g}^{-1}(0)$ as set germs. For any $q \in (\mathbb{R}^k, 0)$, we have $(q, f(q)) \in \overline{f}^{-1}(0)$. Then we have

$$0 = \overline{g}((\psi \times 1_{\mathbb{R}})(q, f(q)) = \overline{g}(\psi(q), f(q)) = g \circ \psi(q) - f(q),$$

so that $g \circ \psi = f$. This completes the proof. □

For function germs $f : (\mathbb{R}^k, 0) \to (\mathbb{R}, 0)$ and $g : (\mathbb{R}^{k'}, 0) \to (\mathbb{R}, 0)$, we say that the level set foliation germs $\mathscr{F}_f$ and $\mathscr{F}_g$ are *stably strictly diffeomorphic* if they become strictly diffeomorphic after the addition to the arguments q_i of new arguments q'_i and to functions f, g of non-degenerate quadratic forms. Thus, we have the following proposition.

Proposition 6.7. *For function germs $f : (\mathbb{R}^k, 0) \to (\mathbb{R}, 0)$ and $g : (\mathbb{R}^{k'}, 0) \to (\mathbb{R}, 0)$, the following conditions are equivalent*:
(1) *f and g are stably $\mathcal{R}$-equivalent*,
(2) *$\mathscr{F}_f$ and $\mathscr{F}_g$ are stably strictly diffeomorphic*,
(3) *$\overline{f}$ and $\overline{g}$ are stably S.P-$\mathcal{K}$-equivalent.*

As a corollary of Theorem 6.4 and Proposition 6.7, we have the following theorem.

Theorem 6.8. *With the same assumptions as those in Theorem 6.4, the following condition is equivalent to* (1) $\sim$ (7) *in Theorem 6.4*:
(8) *$\mathscr{F}_f$ and $\mathscr{F}_g$ are stably strictly diffeomorphic.*

We consider another geometric property of graph-like Legendrian unfoldings. Let $(\mathscr{L}, p)$ be a graph-like Legendrian unfolding germ. We consider a representative $\widetilde{\mathscr{L}}$ of $(\mathscr{L}, p)$ on $\overline{\pi}^{-1}(W)$, where $W \subset \mathbb{R}^n \times \mathbb{R}$ is an open neighborhood of $\overline{\pi}(p) \in \mathbb{R}^n \times \mathbb{R}$. We now show that $W(\widetilde{\mathscr{L}}) \cap W \cap (\{\pi_1 \circ \overline{\pi}(p)\} \times \mathbb{R})$ is a discrete set. Suppose that there exists a sequence of points $\{u_i\}_{i=1}^{\infty} \subset U$ such that $\lim_{i\to\infty} u_i = u_0$ and $\overline{\pi}(\mathcal{L}(u_i)) \in W(\widetilde{\mathscr{L}}) \cap W \cap (\{\pi_1 \circ \overline{\pi}(p)\} \times \mathbb{R})$ for any $i \in \mathbb{N}$. Then $\overrightarrow{\overline{\pi}(p)\overline{\pi}(\mathcal{L}(u_i))}$ is parallel to the vector $\partial/\partial t$. If necessary we can choose a subsequence of $\{u_i\}_{i=1}^{\infty}$, we may suppose that

$$\lim_{i\to\infty} \frac{\overrightarrow{\overline{\pi}(p)\overline{\pi}(\mathcal{L}(u_i))}}{\|\overrightarrow{\overline{\pi}(p)\overline{\pi}(\mathcal{L}(u_i))}\|}$$

exists. Therefore $\partial/\partial t$ and $\nu(u_0)$ are orthogonal. This contradicts to the fact that $\nu(u)$ is given by $(p_1(u),\dots,p_n(u),-1)$ (i.e., $W(\mathscr{L})$ is a graph-like wave front). It follows that $W(\widetilde{\mathscr{L}})\cap W\cap(\{x\}\times\mathbb{R})$ is a finite set for $x\in\pi_1(W)$ for sufficiently small neighborhood W of $\overline{\pi}(p)$. We now define

$$\mathrm{Max}(W(\widetilde{\mathscr{L}})\cap W)=\bigcup_{x\in\pi_1(W)}\{\max(W(\widetilde{\mathscr{L}})\cap(\{x\}\times\mathbb{R}))\},$$
$$\mathrm{mini}(W(\widetilde{\mathscr{L}})\cap W)=\bigcup_{x\in\pi_1(W)}\{\mathrm{mini}(W(\widetilde{\mathscr{L}})\cap(\{x\}\times\mathbb{R}))\}.$$

We denote that the germs of the above sets as $(\mathrm{Max}(W(\mathscr{L})),p)$ and $(\min(W(\mathscr{L})),p)$ respectively. We call $(\mathrm{Max}(W(\mathscr{L})),p)$ a *local maximum graph* and $(\min(W(\mathscr{L})),p)$ a *local minimum graph* of the graph-like wave front $W(\mathscr{L})$ respectively. Let $\Phi:(\mathbb{R}^n\times\mathbb{R},0)\to(\mathbb{R}^n\times\mathbb{R},0)$ be an $S.P^+$-diffeomorphism defined by $\Phi(x,t)=(\phi_1(x),t+\alpha(x))$. Then there exist neighborhoods $U_1,U_2\subset\mathbb{R}^n\times\mathbb{R}$ of the origin and a diffeomorphism $\widetilde{\Phi}:U_1\to U_2$ of the form $\widetilde{\Phi}(x,t)=(\widetilde{\phi}_1(x),t+\widetilde{\alpha}(x))$, which is a representative of the map germ Φ. If $t_1\geq t_2$, then $t_1+\widetilde{\alpha}(x)\geq t_2+\widetilde{\alpha}(x)$ for any $x\in\pi_1(U_1)$. Therefore we have the following lemma.

Lemma 6.9. *Let $\Phi:(\mathbb{R}^n\times\mathbb{R},q_1)\to(\mathbb{R}^n\times\mathbb{R},q_2)$ be an $S.P^+$-diffeomorphism. Then we have*

$$\Phi(\mathrm{Max}(W(\mathscr{L})))=\mathrm{Max}(\Phi(W(\mathscr{L}))),\ \Phi(\min(W(\mathscr{L})))=\min(\Phi(W(\mathscr{L})))$$

as set germs.

We have the following corollary of Theorem 6.1 and Lemma 6.9.

Corollary 6.10. *Let $(\mathscr{L}_1,p_1)$ and $(\mathscr{L}_2,p_2)$ be graph-like Legendrian unfolding germs. If $(\Pi(\mathscr{L}_1),\Pi(p_1))$ and $(\Pi(\mathscr{L}_2),\Pi(p_2))$ are Lagrangian equivalent, then there exists a diffeomorphism germ $\Phi:(\mathbb{R}^n\times\mathbb{R},\overline{\pi}(p_1))\to(\mathbb{R}^n\times\mathbb{R},\overline{\pi}(p_2))$ of the form $\Phi(x,t)=(\phi_1(x),t+\alpha(x))$ such that $\Phi(\mathrm{Max}(W(\mathscr{L}_1)))=\mathrm{Max}(W(\mathscr{L}_2))$ and $\Phi(\min(W(\mathscr{L}_1)))=\min(W(\mathscr{L}_2))$ as set germs.*

The following standard examples clarify the difference between the equivalence relations among graph-like Legendrian unfoldings.

Example 6.11. It is known that one of the germs in the list of 2-parameter $\mathcal{R}^+$-versal unfoldings is the cusp (cf. [6]). The normal form is given by

$$F(q,x_1,x_2)=\mp q^4\mp x_2q^2-x_1q.$$

It is P-$\mathcal{R}^+$-equivalent to $F_1(q,x_1,x_2) = \mp q^4 \mp x_2(q^2+1) - x_1 q$. Then we have

$$C(F_1) = \{(q, \mp(4q^3 + 2qx_2), x_2) \in \mathbb{R}^3 \mid (q, x_2) \in (\mathbb{R}^2, 0)\}.$$

Since $\partial F_1/\partial x_1 = -q$ and $\partial F_1/\partial x_2 = \mp(q^2+1)$,

$$L(F_1)(C(F_1)) = \{(\mp(4q^3 + 2qx_2), x_2, -q, \mp(q^2+1)) \mid (q, x_2) \in (\mathbb{R}^2, 0)\}$$

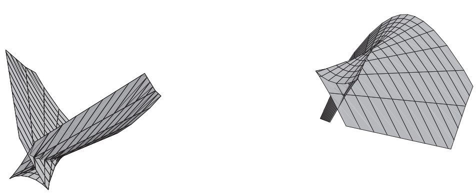
are Lagrangian submanifold germs of $T^*\mathbb{R}^2$. If we set $u = \mp q$ and $v = x_2$, then we have Lagrangian embeddings $\mathcal{L}_1^{\pm} : U \to T^*\mathbb{R}^2 \equiv \mathbb{R}^2 \times (\mathbb{R}^2)^*$ defined by

$$\mathcal{L}_1^{\pm}(u,v) = ((4u^3 + 2uv, v), (\pm u, \mp(u^2+1))),$$

where $U \subset \mathbb{R}^2$ is an open subset. Therefore, $L_1^{\pm} = \mathcal{L}_1^{\pm}(U)$ are Lagrangian submanifolds in $T^*\mathbb{R}^2$. Moreover, if we consider graph-like Morse families of hypersurfaces defined by $\overline{F}(q,x_1,x_2,t) = \mp q^4 \mp x_2(q^2+1) - x_1 q - t$, then the corresponding graph-like Legendrian unfoldings are given by mappings $\mathfrak{L}_1^{\pm} : U \to J^1_{GA}(\mathbb{R}^2, \mathbb{R})$ where

$$\mathfrak{L}_1^{\pm}(u,v) = ((4u^3 + 2uv, v), \pm(3u^4 + u^2 v - v), (\pm u, \mp(u^2+1))).$$

Then $\mathscr{L}_1^{\pm} = \mathfrak{L}_1^{\pm}(U)$ are graph-like Legendrian unfoldings such that $\Pi(\mathscr{L}_1^{\pm}) = L_1^{\pm}$.



$W(\mathscr{L}_1^+)$ $W(\mathscr{L}_1^-)$

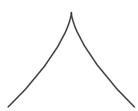
Fig.4: Graph-like wave fronts.

We remark that both the graph-like wave fronts are swallowtails (cf. Fig. 4) at $(u,v) = (0,0)$. We observe that $(\mathrm{Max}(W(\mathscr{L}_1^+)), 0)$ is a graph of continuous function but $(\mathrm{Max}(W(\mathscr{L}_1^-)), 0)$ is not (cf. Fig.5), so that these are not diffeomorphic as set germs.

Fig.5: $\mathrm{Max}\, W(\mathscr{L}_1^+)$ $\mathrm{Max}\, W(\mathscr{L}_1^-)$

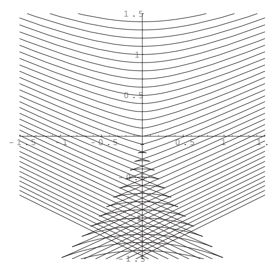
By Corollary 6.10, the germs of L^+ and L^- at the origin are not Lagrangian equivalent. Since both the caustics of L^+ and L^- at the origin are the ordinary cusp, these are diffeomorphic as set germs (cf. Fig.6).

On the other hand, we consider the bifurcation of the family of momentary fronts for graph-like Legendrian unfoldings. If we consider a diffeomorphism germ $\Phi : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ defined by $\Phi(x, t) = (x, -t)$, then we can show $\Phi(W(\mathscr{L}_1^+)) = W(\mathscr{L}_1^-)$. Thus, $W(\mathscr{L}_1^+)$ and $W(\mathscr{L}_1^-)$ are (s, t)-diffeomorphic but not $S.P^+$-diffeomorphic. As we mentioned above, L_1^+ and L_1^- are not Lagrangian equivalent. The bifurcations of $\{W_t(\mathscr{L}_1^\pm)\}_{t\in(\mathbb{R},0)}$ are depicted in Fig.7. We can observe that



$C_{L_1^\pm}$

Fig.6: The caustic.



$\{W_t(\mathscr{L}_1^\pm)\}_{t\in(\mathbb{R},0)}$

Fig.7: The bifurcations of momentary fronts.

both the caustics are ordinary cusps. Therefore, these are examples of Lagrangian submanifold germs such that those caustics are diffeomorphic but these are not Lagrangian equivalent.

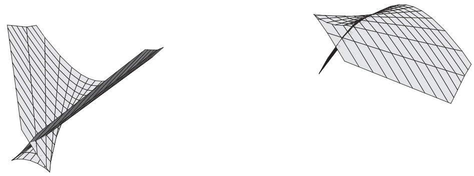
We also consider $\mathcal{R}^+$-versal unfoldings $F_2(q, x_1, x_2) = \mp q^4 \mp x_2(q^2 - 1) - x_1 q$ which are P-$\mathcal{R}^+$-equivalent to $F(q, x_1, x_2)$. By the calculation similar to the above, we have embeddings $\mathcal{L}_2^\pm : U \to T^*\mathbb{R}^2 \equiv \mathbb{R}^2 \times (\mathbb{R}^2)^*$ defined by

$$\mathcal{L}_2^\pm(u, v) = ((4u^3 + 2uv, v), (\pm u, \mp(u^2 - 1))),$$

where $U \subset \mathbb{R}^2$ is an open subset. Then $L_2^\pm = \mathcal{L}_2^\pm(U)$ are Lagrangian submanifolds. Moreover, we have the corresponding graph-like Legendrian unfoldings defined by mappings $\mathfrak{L}_2^\pm : U \to J^1_{GA}(\mathbb{R}^2, \mathbb{R})$ where

$$\mathfrak{L}_2^\pm(u, v) = ((4u^3 + 2uv, v), \pm(3u^4 + u^2 v + v), (\pm u, \mp(u^2 - 1))).$$

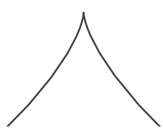
By the same reasons as the above case, L_2^+ and L_2^- are not Lagrangian equivalent (cf. Fig.8). However, if we consider diffeomorphism germ $\Phi^\pm : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ defined by $\Phi^\pm(x_1, x_2, t) = (x_1, x_2, t \pm 2x_2)$, then we have $\Phi^\pm(W(\mathscr{L}_1^\pm)) = W(\mathscr{L}_2^\pm)$ as set germs. By Corollary 6.2,



$W(\mathscr{L}_2^+)$ $W(\mathscr{L}_2^-)$

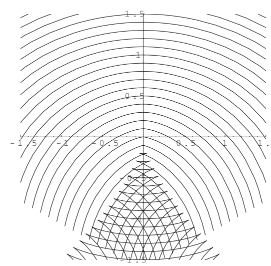
Fig.8: Graph-like wave fronts.

L_1^+ and L_2^+ (respectively, L_1^- and L_2^-) are Lagrangian equivalent. The pictures of $W(\mathscr{L}_2^\pm)$ are similar to those of $W(\mathscr{L}_1^\pm)$. Moreover, the caustics are the same as the above (Fig.9). However, the bifurcations of momentary fronts $\{W_t(\mathscr{L}_2^\pm)\}_{t\in(\mathbb{R},0)}$ are different from $\{W_t(\mathscr{L}_1^\pm)\}_{t\in(\mathbb{R},0)}$ (cf. Fig.7 and Fig.10). Actually, we can apply the criterion in [11] and show that $W(\mathscr{L}_1^\pm)$ and $W(\mathscr{L}_2^\pm)$ are not $S.P$-diffeomorphic as set germs.



$C_{L_2^\pm}$

Fig.9: The caustic.



$\{W_t(\mathscr{L}_2^\pm)\}_{t\in(\mathbb{R},0)}$

Fig.10: The bifurcation of momentary fronts.

§7. *s*-*P*-Legendrian equivalence among graph-like Legendrian unfoldings

In §6 we have given a brief survey on $S.P^+$-Legendrian equivalence among graph-like Legendrian unfoldings. One of the main consequences is that $S.P^+$-Legendrian equivalence among graph-like Legendrian unfoldings is equivalent to Lagrangian equivalence among induced Lagrangian submanifolds. This fact can be considered as a geometric interpretation of Lagrangian equivalence. On the other hand, s-P-Legendrian equivalence is weaker than $S.P^+$-Legendrian equivalence among graph-like Legendrian unfoldings. Therefore, Lagrangian equivalence is stronger than s-P-Legendrian equivalence. In this section we explain detailed properties of s-P-Legendrian equivalence among graph-like Legendrian unfoldings as an application of the results in [13, 35], which might be new results. We also use a graph-like Morse family of hypersurfaces of the form $\mathcal{F}(q,x,t) = \lambda(q,x,t)(F(q,x)-t)$. Since $\mathcal{F}(q,x,t)$ and $\overline{F}(q,x,t) = F(q,x)-t$ are s-$S.P$-$\mathcal{K}$-equivalent, we consider $\overline{F}(q,x,t) = F(q,x)-t$ as a graph-like Morse family. Moreover, by Proposition 5.4, $F(q,x)$ is a Morse family of functions. We now suppose that $F(q,x)$ is a Morse family of functions. Consider the graph-like Morse family of hypersurfaces $\overline{F}(q,x,t) = F(q,x)-t$ which is not necessarily non-degenerate. Then we have $\mathscr{L}_{\overline{F}}(\Sigma_*(\overline{F})) = \mathfrak{L}_F(C(F))$. We also denote that $\overline{f}(q,t) = f(q)-t$ for any $f \in \mathfrak{M}_k$. For function germs $f,g : (\mathbb{R}^k,0) \to (\mathbb{R},0)$, we say that f,g are *$\mathcal{A}$-equivalent* if there exist diffeomorphism germs $\phi : (\mathbb{R}^k,0) \to (\mathbb{R}^k,0)$ and $\psi : (\mathbb{R},0) \to (\mathbb{R},0)$ such that $\psi \circ f = g \circ \phi$. Moreover, let $F,G : (\mathbb{R}^k \times \mathbb{R}^n,0) \to (\mathbb{R},0)$ be function germs, we say that F,G are *P-$\mathcal{A}$-equivalent* if there exist diffeomorphism germs $\Phi : (\mathbb{R}^k \times \mathbb{R}^n,0) \to (\mathbb{R}^k \times \mathbb{R}^n,0)$ of the form $\Phi(q,x) = (\phi_1(q,x),\phi_2(x))$ and $\Psi : (\mathbb{R}\times\mathbb{R}^n,0) \to (\mathbb{R}\times\mathbb{R}^n,0)$ of the form $\Psi(t,x) = (\psi(t,x),x)$ such that

$$\Psi(F(q,x),x) = (G \circ \Phi(q,x),x)$$

for any $(q,x) \in (\mathbb{R}^k \times \mathbb{R}^n,0)$. We remark that if F,G are P-$\mathcal{A}$-equivalent, then $f = F|_{\mathbb{R}^k\times\{0\}}, g = G|_{\mathbb{R}^k\times\{0\}}$ are $\mathcal{A}$-equivalent. Then we have the following proposition.

Proposition 7.1. *Let $F,G : (\mathbb{R}^k \times \mathbb{R}^n,0) \to (\mathbb{R},0)$ be function germs. Then $\overline{F}(q,x,t) = F(q,x)-t, \overline{G}(q,x,t) = G(q,x)-t$ are s-P-$\mathcal{K}$-equivalent if and only if F,G are P-$\mathcal{A}$-equivalent.*

Proof. Suppose that $\overline{F},\overline{G}$ are s-P-$\mathcal{K}$-equivalent. Then there exists a diffeomorphism germ $\overline{\Psi} : (\mathbb{R}^k \times (\mathbb{R}\times\mathbb{R}^n),0) \to (\mathbb{R}^k \times (\mathbb{R}\times\mathbb{R}^n),0)$ of the form $\overline{\Psi}(q,t,x) = (\overline{\psi}(q,t,x),\psi_1(t,x),\psi_2(x))$ such that $\langle \overline{F}\circ\overline{\Psi}\rangle_{\mathcal{E}_{k+1+n}} = \langle \overline{G}\rangle_{\mathcal{E}_{k+1+n}}$. It follows that $\overline{\Psi}(G^{-1}(0)) = F^{-1}(0)$. By definition, we have

$$\overline{F}^{-1}(0) = \{(q,F(q,x),x) \mid (q,x) \in (\mathbb{R}^k \times \mathbb{R}^n,0)\},$$
$$\overline{G}^{-1}(0) = \{(q,G(q,x),x) \mid (q,x) \in (\mathbb{R}^k \times \mathbb{R}^n,0)\}.$$

Therefore, we have

$$\overline{\Psi}(q,G(q,x),x) = (\overline{\psi}(q,G(q,x),x),\psi_1(G(q,x),x),\psi_2(x)) = (\overline{q},F(\overline{q},\overline{x}),\overline{x}).$$

Hence, we have $\overline{q} = (\overline{\psi}(q,G(q,x),x), \overline{x} = \psi_2(x)$ and

$$\psi_1(G(q,x),x) = F(\overline{q},\overline{x}) = F(\overline{\psi}(q,G(q,x),x),\psi_2(x)).$$

If we define $\Phi : (\mathbb{R}^k \times \mathbb{R}^n,0) \to (\mathbb{R}^k \times \mathbb{R}^n,0)$ by

$$\Phi(q,x) = \overline{\psi}(q,G(q,x),x),\psi_2(x)),$$

then Φ is a diffeomorphism germ. Moreover, we define $\Psi : (\mathbb{R}\times\mathbb{R}^n, 0) \to (\mathbb{R}\times\mathbb{R}^n, 0)$ by $\Psi(t, x) = (\psi_1(t, x), x)$. Then the above equality means that $\Psi(G(q, x), x) = (F \circ \Phi(q, x), x)$, so that F, G are P-$\mathcal{A}$-equivalent.

Suppose that there exist diffeomorphism germs $\Phi : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}^k\times\mathbb{R}^n, 0)$ of the form $\Phi(q, x) = (\phi_1(q, x), \phi_2(x))$ and $\Psi : (\mathbb{R}\times\mathbb{R}^n, 0) \to (\mathbb{R} \times \mathbb{R}^n, 0)$ of the form $\Psi(t, x) = (\psi(t, x), x)$ such that

$$\Psi(F(q, x), x) = (G \circ \Phi(q, x), x)$$

for any $(q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, 0)$. We define $\overline{\Psi} : (\mathbb{R}^k \times (\mathbb{R} \times \mathbb{R}^n), 0) \to (\mathbb{R}^k \times (\mathbb{R} \times \mathbb{R}^n), 0)$ by $\overline{\Psi}(q, t, x) = (\phi_1(q, x), \psi(t, x), \phi_2(x))$. Then $\overline{\Psi}$ is a diffeomorphism germ. Since $\psi(F(q, x), x) = G(\phi_1(q, x), \phi_2(x))$, we have

$$\begin{aligned}\overline{\Psi}(q, F(q, x), x) &= (\phi_1(q, x), \psi(F(q, x), x), \phi_2(x))\\ &= (\phi_1(q, x), G(\phi_1(q, x), \phi_2(x)), \phi_2(x)),\end{aligned}$$

so that $\overline{\Psi}(\overline{F}^{-1}(0)) = \overline{G}^{-1}(0)$ as set germs. Thus $\overline{F}^{-1}(0) = (\overline{G}\circ\overline{\Psi})^{-1}(0)$. Since $\overline{F}, \overline{G}$ are submersion germs, we have $\langle\overline{F}\rangle_{\mathcal{E}_{k+1+n}} = \langle\overline{G} \circ \overline{\Psi}\rangle_{\mathcal{E}_{k+1+n}}$. This completes the proof. □

We have the following simple corollary.

Corollary 7.2. *For function germs $f, g : (\mathbb{R}^k, 0) \to (\mathbb{R}, 0)$, f, g are $\mathcal{A}$-equivalent if and only if $\overline{f}(q, t) = f(q) - t, \overline{g}(q, t) = g(q) - t$ are P-$\mathcal{K}$-equivalent.*

For $(q, t, x) \in (\mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^n, 0)$ and $(q', t, x) \in (\mathbb{R}^{k'} \times \mathbb{R} \times \mathbb{R}^n, 0)$, let $\mathcal{F}(q, x, t) = \lambda(q, x, t)(F(q, x)-t)$ and $\mathcal{G}(q', x, t) = \mu(q', x, t)(G(q', x)-t)$ be graph-like Morse families of hypersurfaces. By Theorem 4.2 and Corollary 7.2, we have the following theorem.

Theorem 7.3. *The graph-like Legendrian unfoldings $\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$, $\mathscr{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G}))$ are s-P-Legendrian equivalent if and only if $F(q, x), G(q', x)$ are stably P-$\mathcal{A}$-equivalent.*

The definition of *stably P-$\mathcal{A}$-equivalence* is similar to the definition of stably P-$\mathcal{R}^+$-equivalence, so that we omit to give the definition here.

We now consider the stability of graph-like Legendrian unfoldings relative to s-P-Legendrian equivalence. Theorem 4.2 asserts that the graph-like Legendrian unfolding $\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ is s-P-Legendrian stable if and only if $\mathcal{F}$ is an infinitesimally P-$\mathcal{K}$-versal unfolding of $\mathcal{F}|_{\mathbb{R}^k\times\{0\}\times\mathbb{R}}$. Here, we have $\mathcal{F}(q, x, t) = \lambda(q, x, t)(F(q, x) - t)$. We can represent the extended tangent space of $\overline{f} : (\mathbb{R}^k \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ relative to P-$\mathcal{K}$ by

$$T_e(P\text{-}\mathcal{K})(\overline{f}) = \left\langle \frac{\partial f}{\partial q_1}(q), \dots, \frac{\partial f}{\partial q_k}(q), f(q) - t \right\rangle_{\mathcal{E}_{k+1}} + \langle 1\rangle_{\mathcal{E}_1}.$$

In this case the unfolding $\overline{F}(q,x,t)=F(q,x)-t$ of $\overline{f}(q,t)$ is an infinitesimally P-$\mathcal{K}$-versal unfolding of $\overline{f}(q,t)$ if and only if

$$\mathcal{E}_{k+1}=T_e(P\text{-}\mathcal{K})(\overline{f})+\left\langle \frac{\partial F}{\partial x_1}|_{\mathbb{R}^k\times\{0\}\times\mathbb{R}},\dots,\frac{\partial F}{\partial x_n}|_{\mathbb{R}^k\times\{0\}\times\mathbb{R}}\right\rangle_{\mathbb{R}}.$$

Moreover, we now define P-$\mathcal{K}$-versal unfolding of $\overline{f}$ (cf. [13]) as follows: For a map germ $\psi:(\mathbb{R}^m,0)\to(\mathbb{R}^n,0)$, we define an m-dimensional unfolding $\psi^*F:(\mathbb{R}^k\times\mathbb{R}^m\times\mathbb{R},0)\to(\mathbb{R},0)$ of $\overline{f}$ by $\psi^*\overline{F}(q,y,t)=\overline{F}(q,\psi(y),t)$, which we call an *induced unfolding* of $\overline{F}$ by ψ. We say that $\overline{F}$ is a *P-$\mathcal{K}$-versal unfolding* of $\overline{f}$ if for any unfolding $\overline{G}:(\mathbb{R}^k\times\mathbb{R}^m\times\mathbb{R},0)\to(\mathbb{R},0)$ of $\overline{f}$, there exists a P-$\mathcal{K}$-morphism from $\overline{G}$ to $\overline{F}$. Here, a *P-$\mathcal{K}$-morphism* from $\overline{G}$ to $\overline{F}$ is $(\psi,\widetilde{\Phi},\widetilde{\phi},\lambda)$ where $\psi:(\mathbb{R}^m,0)\to(\mathbb{R}^n,0)$ is a map-germ, $\widetilde{\Phi}:(\mathbb{R}^k\times\mathbb{R}^m\times\mathbb{R},0)\to(\mathbb{R}^k\times\mathbb{R}^m\times\mathbb{R},0)$ is a diffeomorphism germ of the form $\widetilde{\Phi}(q,u,t)=(\phi_1(q,u,t),u,\widetilde{\phi}(u,t))$ and $\lambda(q,u,t)\in\mathcal{E}_{k+m+1}$ is a function germ such that $\widetilde{\Phi}(q,0,t)=(q,0,t)$, $\lambda(q,0,t)=1$ and $\psi^*\overline{F}(q,u,t)=\lambda(q,u,t)\overline{G}\circ\widetilde{\Phi}(q,u,t)$. We have the following theorem (cf. [8, 13]).

Theorem 7.4. *An unfolding $\overline{F}:(\mathbb{R}^k\times\mathbb{R}^n\times\mathbb{R},0)\to(\mathbb{R},0)$ is a P-$\mathcal{K}$-versal unfolding of $\overline{f}:(\mathbb{R}^k\times\mathbb{R},0)\to(\mathbb{R},0)$ if and only if it is an infinitesimally P-$\mathcal{K}$-versal unfolding of $\overline{f}$.*

We consider the stability of the unfolding $\overline{F}$ of $\overline{f}$ relative to P-$\mathcal{K}$-equivalence. We say that $\overline{F}$ is *homotopically P-$\mathcal{K}$-stable* if for any one-parameter family of functions $\mathcal{F}:(\mathbb{R}^k\times(\mathbb{R}^n\times\mathbb{R})\times\mathbb{R},0)\to(\mathbb{R},0)$ with $\mathcal{F}(q,x,0,t)=\overline{F}(q,x,t)$, there is a a P-$\mathcal{K}$-morphism from $\mathcal{F}$ to $\overline{F}$ as unfoldings of $\overline{f}$. Here, we remark that $\mathcal{F}(q,x,s,t)$ can be regarded as an unfolding of $\overline{f}$ with the parameter $(x,s)\in\mathbb{R}^n\times\mathbb{R}$. By definition, if $\overline{F}$ is a P-$\mathcal{K}$-versal unfolding of $\overline{f}$, then it is homotopically P-$\mathcal{K}$-stable. Moreover, suppose that $\overline{F}$ is homotopically P-$\mathcal{K}$-stable. For any $h(q,t)\in\mathcal{E}_{k+1}$, we consider a one-parameter family of function germ $\mathcal{F}(q,x,s,t)=\overline{F}(q,x,t)+sh(q,t)$. Then there exist a map-germ $\psi:(\mathbb{R}^n\times\mathbb{R},0)\to(\mathbb{R}^n,0)$, a diffeomorphism germ $\widetilde{\Phi}:(\mathbb{R}^k\times(\mathbb{R}^n\times\mathbb{R})\times\mathbb{R},0)\to(\mathbb{R}^k\times(\mathbb{R}^n\times\mathbb{R})\times\mathbb{R},0)$ of the form $\widetilde{\Phi}(q,x,s,t)=(\phi_1(q,x,s,t),x,s,\widetilde{\phi}(x,s,t))$ and $\lambda(q,x,s,t)\in\mathcal{E}_{k+n+1+1}$ is a function germ such that $\widetilde{\Phi}(q,0,0,t)=(q,0,0,t)$, $\lambda(q,0,0,t)=1$ and

$$\begin{aligned}\overline{F}(q,\psi(x,s),t)&=\lambda(q,x,s,t)\mathcal{F}\circ\widetilde{\Phi}(q,x,s,t)\\&=\lambda(q,x,s,t)(\overline{F}(\phi_1(q,x,s,t),x,\widetilde{\phi}(x,s,t))\\&\qquad+sh(\phi_1(q,x,s,t),\widetilde{\phi}(x,s,t)).\end{aligned}$$

Differentiating with respect to s at $(x,s)=(0,0)$, we have

$$\sum_{i=1}^{n}\frac{\partial\overline{F}}{\partial x_i}(q,0,t)\frac{\partial\psi_i}{\partial s}(0,0)=\frac{\partial\lambda}{\partial s}(q,0,0,t)\overline{F}(q,0,t)$$
$$+\sum_{j=1}^{k}\frac{\partial\overline{F}}{\partial q_j}(q,0,t)\frac{\partial(\phi_1)_j}{\partial s}(q,0,0,t)+\frac{\partial\overline{F}}{\partial t}(q,0,t)\frac{\partial\widetilde{\phi}}{\partial s}(0,0,t)+h(q,t).$$

Therefore, we have

$$h(q,t)\in T_e(P\text{-}\mathcal{K})(\overline{f})+\left\langle\frac{\partial F}{\partial x_1}|_{\mathbb{R}^k\times\{0\}\times\mathbb{R}},\dots,\frac{\partial F}{\partial x_n}|_{\mathbb{R}^k\times\{0\}\times\mathbb{R}}\right\rangle_{\mathbb{R}},$$

so that

$$\mathcal{E}_{k+1}=T_e(P\text{-}\mathcal{K})(\overline{f})+\left\langle\frac{\partial F}{\partial x_1}|_{\mathbb{R}^k\times\{0\}\times\mathbb{R}},\dots,\frac{\partial F}{\partial x_n}|_{\mathbb{R}^k\times\{0\}\times\mathbb{R}}\right\rangle_{\mathbb{R}}.$$

By Theorem 7.4, we have shown the following proposition.

Proposition 7.5. *An unfolding* $\overline{F}:(\mathbb{R}^k\times\mathbb{R}^n\times\mathbb{R},0)\to(\mathbb{R},0)$ *of* $\overline{f}:(\mathbb{R}^k\times\mathbb{R},0)\to(\mathbb{R},0)$ *is homotopically* P-$\mathcal{K}$*-stable if and only if it is a* P-$\mathcal{K}$*-versal unfolding of* $\overline{f}$.

On the other hand, Wassermann [35] investigated stability and versality of unfoldings of function germs relative to $\mathcal{A}$-equivalence. We say that $F(q,x)$ is an *infinitesimally* $\mathcal{A}$*-versal unfolding* of $f(q)$ if

$$\mathcal{E}_k=T_e(\mathcal{A})(f)+\left\langle\frac{\partial F}{\partial x_1}|_{\mathbb{R}^k\times\{0\}},\dots,\frac{\partial F}{\partial x_n}|_{\mathbb{R}^k\times\{0\}}\right\rangle_{\mathbb{R}},$$

where

$$T_e(\mathcal{A})(f)=J_f+f^*(\mathcal{E}_1)\text{ and }f^*(\mathcal{E}_1)=\{h\circ f\in\mathcal{E}_k\mid h\in\mathcal{E}_1\}.$$

We also define $\mathcal{A}$-versal unfoldings. An unfolding $F:(\mathbb{R}^k\times\mathbb{R}^n,0)\to(\mathbb{R},0)$ of $f:(\mathbb{R}^k,0)\to(\mathbb{R},0)$ is an $\mathcal{A}$*-versal unfolding* if for any unfolding $G:(\mathbb{R}^k\times\mathbb{R}^m,0)\to(\mathbb{R},0)$ of f, there exists an $\mathcal{A}$-morphism from G to F. Here, an $\mathcal{A}$*-morphism* from G to F is (ψ,Φ,ϕ) where $\psi:(\mathbb{R}^m,0)\to(\mathbb{R}^n,0)$ is a map-germ, $\Phi:(\mathbb{R}^k\times\mathbb{R}^m,0)\to(\mathbb{R}^k\times\mathbb{R}^m,0)$ is a diffeomorphism germ of the form $\Phi(q,u)=(\phi_1(q,u),u)$ and $\phi:(\mathbb{R}\times\mathbb{R}^m,0)\to(\mathbb{R},0)$ is a function germ such that $\Phi(q,0)=(q,0)$, $\phi(y,0)=y$ and $\phi(\psi^*F(q,u),u)=G\circ\Phi(q,u)$. We have the following theorem.

Theorem 7.6 ([35]). *An unfolding $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ is an $\mathcal{A}$-versal unfolding of $f : (\mathbb{R}^k, 0) \to (\mathbb{R}, 0)$ if and only if it is an infinitesimally $\mathcal{A}$-versal unfolding of f.*

We say that F is *homotopically $\mathcal{A}$-stable* if for any one-parameter family of functions $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ with $\mathcal{F}(q, x, 0) = F(q, x)$, there is a an $\mathcal{A}$-morphism from $\mathcal{F}$ to F as unfoldings of f. Here, we remark that $\mathcal{F}(q, x, s)$ can be regarded as an unfolding of f with the parameter $(x, s) \in \mathbb{R}^n \times \mathbb{R}$. By definition, if F is an $\mathcal{A}$-versal unfolding of f, then it is homotopically $\mathcal{A}$-stable. Moreover, suppose that F is homotopically $\mathcal{A}$-stable. For any $h(q) \in \mathcal{E}_k$, we consider a one-parameter family of function germ $\mathcal{F}(q, x, s) = F(q, x) + sh(q)$. Then there exist a map-germ $\psi : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n, 0)$, a diffeomorphism germ $\Phi : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0)$ of the form $\Phi(q, x, s) = (\phi_1(q, x, s), x, s)$ and a function germ $\phi(y, x, s) \in \mathfrak{M}_{1+n+1}$ such that $\Phi(q, 0, 0) = (q, 0, 0)$, $\phi(y, 0, 0) = y$ and

$$\phi(F(q, \psi(x, s)), x, s) = \mathcal{F} \circ \Phi(q, x, s) = F(\phi_1(q, x, s), x) + sh(\phi_1(q, x, s)).$$

Differentiating with respect to s at $(x, s) = (0, 0)$, we have

$$\begin{aligned}\sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(q, 0) \frac{\partial \psi_i}{\partial s}(0, 0) + \frac{\partial \phi}{\partial s}(f(q), 0, 0) \\ = \sum_{j=1}^{k} \frac{\partial F}{\partial q_j}(q, 0, t) \frac{\partial (\phi_1)_j}{\partial s}(q, 0, 0) + h(q, t).\end{aligned}$$

Therefore, we have

$$h(q) \in T_e(\mathcal{A})(f) + \left\langle \frac{\partial F}{\partial x_1}|_{\mathbb{R}^k \times \{0\} \times \mathbb{R}}, \dots, \frac{\partial F}{\partial x_n}|_{\mathbb{R}^k \times \{0\} \times \mathbb{R}} \right\rangle_{\mathbb{R}},$$

so that

$$\mathcal{E}_k = T_e(\mathcal{A})(f) + \left\langle \frac{\partial F}{\partial x_1}|_{\mathbb{R}^k \times \{0\} \times \mathbb{R}}, \dots, \frac{\partial F}{\partial x_n}|_{\mathbb{R}^k \times \{0\} \times \mathbb{R}} \right\rangle_{\mathbb{R}}.$$

By Theorem 7.6, we have shown the following proposition.

Proposition 7.7 ([35]). *An unfolding $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ of $f : (\mathbb{R}^k, 0) \to (\mathbb{R}, 0)$ is homotopically $\mathcal{A}$-stable if and only if it is a $\mathcal{A}$-versal unfolding of f.*

We also have the following proposition.

Proposition 7.8. *For an unfolding* $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ *of* $f : (\mathbb{R}^k, 0) \to (\mathbb{R}, 0)$, *the following conditions are equivalent*:
(1) $\overline{F} : (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ *is homotopically* P-$\mathcal{K}$-*stable*,
(2) $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ *is homotopically* $\mathcal{A}$-*stable.*

Proof. Suppose that F is homotopically $\mathcal{A}$-stable. For any one-parameter family of functions $\mathcal{G} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ with $\mathcal{G}(q, x, 0, t) = \overline{F}(q, x, t)$, we have $\partial\mathcal{G}/\partial t(0) \neq 0$, so that there exists a function germ $G(q, x, s)$ and $\mu(q, x, s, t)$ such that

$$\mu(0) \neq 0 \text{ and } \mathcal{G}(q, x, s, t) = \mu(q, x, s, t)(G(q, x, s) - t).$$

It follows that $F(q, x) - t = \mathcal{G}(q, x, t, 0) = \mu(q, x, 0, t)(G(q, x, 0) - t)$. By Proposition 7.1, $F(q, x)$ and $G(q, x, 0)$ are P-$\mathcal{A}$-equivalent. Thus $G(q, x, 0)$ is a homotopically $\mathcal{A}$-stable unfolding. Then there exist a map-germ $\psi : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n, 0)$, a diffeomorphism germ $\Phi : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0)$ of the form $\Phi(q, x, s) = (\phi_1(q, x, s), x, s)$ and a function germ $\phi(t, x, s) \in \mathfrak{M}_{1+n+1}$ such that $\Phi(q, 0, 0) = (q, 0, 0)$, $\phi(t, 0, 0) = t$ and

$$\phi(G(q, \psi(x, s), 0), x, s) = G(\Phi(q, x, s)) = G(\phi_1(q, x, s), x, s).$$

We now define a diffeomorphism germ $\widetilde{\Phi} : (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}, 0) \to (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}, 0)$ by $\widetilde{\Phi}(q, x, s, t) = (\phi_1(q, x, s), x, s, \phi(t, x, s))$. The above equality means that

$$\widetilde{\Phi}(q, x, s, G(q, \psi(x, s), 0)) = (\phi_1(q, x, s), x, s, G(\phi_1(q, x, s), x, s)).$$

If we denote that $G(q, x, 0) = G_0(q, x)$, then $\psi^* G_0(q, x, s) = G(q, \psi(x, s), 0)$, so that we have $\widetilde{\Phi}(\overline{\psi^* G_0}^{-1}(0)) = \overline{G}^{-1}(0)$. Therefore, there exists a function germ $\lambda(q, x, s, t) \in \mathcal{E}_{k+n+1+1}$ with $\lambda(0) \neq 0$ such that

$$\overline{G} \circ \widetilde{\Phi}(q, x, s, t) = \lambda(q, x, s, t)\overline{\psi^* G_0}(q, x, s, t).$$

Here, $\widetilde{\Phi}(q, 0, 0, t) = (q, 0, 0, t)$ and

$$\begin{aligned} G_0(q, 0) - t &= \overline{G} \circ \widetilde{\Phi}(q, 0, 0, t) \\ &= \lambda(q, 0, 0, t)\overline{\psi^* G_0}(q, 0, 0, t) = \lambda(q, 0, 0, t)(G_0(q, 0) - t), \end{aligned}$$

so that $\lambda(q, 0, 0, t) = 1$. This means that $\overline{G}(q, x, t)$ is homotopically P-$\mathcal{K}$-stable. Therefore, $\overline{F}$ is homotopically P-$\mathcal{K}$-stable.

For the converse assertion, suppose that $\overline{F}$ is homotopically P-$\mathcal{K}$-stable. Let $G(q, x, s) \in \mathfrak{M}_{k+n+1}$ be a function germ with $G(q, x, 0) =$

$F(q,x)$. Then we have $\overline{G}(q,x,s,t) = G(q,x,s) - t$. Since $\overline{F}$ is homotopically P-$\mathcal{K}$-stable, there exist a map-germ $\psi : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n, 0)$, a diffeomorphism germ $\widetilde{\Phi} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}, 0) \to (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}, 0)$ of the form $\widetilde{\Phi}(q,x,s,t) = (\phi_1(q,x,s,t), x, s, \widetilde{\phi}(x,s,t))$ and $\lambda(q,x,s,t) \in \mathcal{E}_{k+n+1+1}$ is a function germ such that $\widetilde{\Phi}(q,0,0,t) = (q,0,0,t)$, $\lambda(q,0,0,t) = 1$ and

$$\begin{aligned} \overline{F}(q,\psi(x,s),t) &= \lambda(q,x,s,t)\overline{G} \circ \widetilde{\Phi}(q,x,s,t) \\ &= \lambda(q,x,s,t)(G(\phi_1(q,x,s,t),x,s) - \widetilde{\phi}(x,s,t)). \end{aligned}$$

It follows that $\overline{\psi^*F}^{-1}(0) = \widetilde{\Phi}^{-1}(\overline{G}^{-1}(0))$. Here, we have

$$\begin{aligned} \overline{\psi^*F}^{-1}(0) &= \{(q,x,s,\psi^*F(q,x,s)) \mid (q,x,s) \in \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}\}, \\ \overline{G}^{-1}(0) &= \{(q,x,s,G(q,x,s)) \mid (q,x,s) \in \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}\}. \end{aligned}$$

By definition, we have

$$\begin{aligned} &\widetilde{\Phi}(q,x,s,\psi^*F(q,x,s)) \\ &\quad = (\phi_1(q,x,s,\psi^*F(q,x,s)), x, s, \widetilde{\phi}(x,s,\psi^*F(q,x,s))). \end{aligned}$$

Therefore, if we put $\overline{q} = \phi_1(q,x,s,\psi^*F(q,x,s))$, $\overline{x} = x, \overline{s} = s$, then

$$\widetilde{\phi}(x,s,\psi^*F(q,x,s)) = G(\overline{q},\overline{x},\overline{s}) = G(\phi_1(q,x,s,\psi^*F(q,x,s)),x,s).$$

We define $\phi : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0)$ by

$$\phi(t,x,s) = (\widetilde{\phi}(x,s,t), x, s).$$

Then the above equality means that

$$\phi(\psi^*F(q,x,s),x,s) = G(\phi_1(q,x,s,\psi^*F(q,x,s)),x,s).$$

Since $\widetilde{\Phi}(q,0,0,t) = (q,0,0,t)$, we have $\phi_1(q,0,0,\psi^*F(q,0,0)) = q$ and $\phi(t,0,0) = \widetilde{\phi}(0,0,t) = t$, so that F is homotopically $\mathcal{A}$-stable. This completes the proof. □

As a consequence of the above arguments, we have the following theorem.

Theorem 7.9. *Let $\mathcal{F}(q,x,t) = \lambda(q,x,t)(F(q,x) - t)$ be a graph-like Morse family of hyper surfaces. Then the following conditions are equivalent*:
(1) *The graph-like Legendrian unfolding $\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ is s-P-Legendrian stable,*

(2) $\overline{F}(q,x,t) = F(q,x) - t$ *is a* P-$\mathcal{K}$*-versal unfolding of* $\overline{f}(q) = \overline{F}(q,0) = F(q,0) - t$,
(3) $F(q,x)$ *is an* $\mathcal{A}$*-versal unfolding of* $f(q) = F(q,0)$.

Proof. By Theorem 4.2, (1) and (2) are equivalent. By Propositions 7.5, 7.7 and 7.8, (2) and (3) are equivalent. □

By the above arguments, $\mathcal{A}$-equivalence among function germs is an important notion for the study of s-P-Legendrian equivalence among graph-like Legendrian unfoldings. We consider geometric characterization for $\mathcal{A}$-equivalence among function germs. For function germs $f, g : (\mathbb{R}^k, 0) \to (\mathbb{R}, 0)$, we say that the level set foliation germs $\mathscr{F}_f$ and $\mathscr{F}_g$ are *diffeomorphic* if there exist diffeomorphism germs $\psi : (\mathbb{R}^k, 0) \to (\mathbb{R}^k, 0)$ and $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $\psi(f^{-1}(c)) = g^{-1}(\phi(c))$ as a set germ for any $c \in (\mathbb{R}, 0)$. Then we have the following proposition.

Proposition 7.10. *For function germs* $f, g : (\mathbb{R}^k, 0) \to (\mathbb{R}, 0)$, *the level set foliation germs* $\mathscr{F}_f, \mathscr{F}_g$ *are diffeomorphic if and only if* f, g *are* $\mathcal{A}$*-equivalent.*

Proof. By definition, if f and g are $\mathcal{A}$-equivalent, then $\mathscr{F}_f$ and $\mathscr{F}_g$ are diffeomorphic. If $\mathscr{F}_f$ and $\mathscr{F}_g$ are strictly diffeomorphic, then there exist diffeomorphism germs $\psi : (\mathbb{R}^k, 0) \to (\mathbb{R}^k, 0)$ and $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $\psi(f^{-1}(c)) = g^{-1}(\phi(c)) = (\phi^{-1} \circ g)^{-1}(c)$ as a set germ for any $c \in (\mathbb{R}, 0)$. This means that $\mathscr{F}_f$ and $\mathscr{F}_{\phi^{-1} \circ g}$ are strictly diffeomorphic. By Proposition 6.6, f and $\phi^{-1} \circ g$ are $\mathcal{R}$-equivalent, so that f and g are $\mathcal{A}$-equivalent. This completes the proof. □

For function germs $f : (\mathbb{R}^k, 0) \to (\mathbb{R}, 0)$ and $g : (\mathbb{R}^{k'}, 0) \to (\mathbb{R}, 0)$, we say that the level set foliation germs $\mathscr{F}_f$ and $\mathscr{F}_g$ are *stably diffeomorphic* if they become strictly diffeomorphic after the addition to the arguments q_i of new arguments q'_i and to functions f, g of non-degenerate quadratic forms. Then we have the following classification theorem.

Theorem 7.11. *Let* $\mathcal{F} : (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ *and* $\mathcal{G} : (\mathbb{R}^{k'} \times \mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ *be graph-like Morse families of hypersurfaces of the forms* $\mathcal{F}(q,x,t) = \lambda(q,x,t)(F(q,x) - t)$ *and* $\mathcal{G}(q',x,t) = \mu(q',x,t)(G(q',x) - t)$ *such that* $\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ *and* $\mathscr{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G}))$ *are s-P-Legendrian stable. Then the following conditions are equivalent*:
(1) $\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ *and* $\mathscr{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G}))$ *are s-P-Legendrian equivalent,*
(2) $\mathcal{F}$ *and* $\mathcal{G}$ *are stably s-*P*-*$\mathcal{K}$*-equivalent,*
(3) $\overline{f}(q,t) = F(q,0) - t$ *and* $\overline{g}(q',t) = G(q',0) - t$ *are stably* P*-*$\mathcal{K}$*-equivalent,*
(4) $f(q) = F(q,0)$ *and* $g(q') = G(q',0)$ *are stably* $\mathcal{A}$*-equivalent,*
(5) $F(q,x)$ *and* $G(q',x)$ *are stably* P*-*$\mathcal{A}$*-equivalent,*

(6) $W(\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F})))$ *and* $W(\mathscr{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G})))$ *are s-P-diffeomorphic,*
(7) $\mathscr{F}_f$ *and* $\mathscr{F}_g$ *are stably diffeomorphic.*

Proof. By Theorem 4.2, (1) and (2) are equivalent. By Proposition 7.1, (2) and (5) are equivalent. By Corollary 7.2, (3) and (4) are equivalent. By Proposition 7.10, (4) and (7) are equivalent. By definition, (1) implies (6). By Proposition 4.1, (6) implies (1). By definition, (2) implies (3). By the uniqueness of P-$\mathcal{K}$-versal unfoldings, (3) implies (2). This completes the proof. □

Remark 7.12. (i) If $k = k'$ and $q = q'$ in the above theorem, we can remove the word "stably" in conditions (2), (3), (4), (5) and (7).
(ii) By Theorem 4.2 and Proposition 7.1, conditions (1), (2) and (5) are always equivalent without any assumptions.
(iii) By Corollary 7.2 and Proposition 7.10, conditions (3), (4) and (7) are equivalent without any assumptions.
(iv) By Proposition 4.1, conditions (1) and (6) are equivalent generically for an arbitrary dimension n without the assumption on s-P-Legendrian stability.

§8. Applications

In this section we explain some applications of the theory of wave front propagations. In [24] we explained some applications on both of general theory of wave front propagations and the theory of graph-like Legendrian unfoldings. Here, we only give two important cases that the notion of graph-like Legendrian unfoldings are essentially needed.

8.1. Stability of Caustics due to Jänich and Wassermann

Following Thom, Jänich [27] and Wassermann [36] considered the propagation of wave fronts on a manifold depending on the choice of a Hamiltonian on the contangent bundle of the manifold. Let $H : T^*\mathbb{R}^n \setminus 0 \to \mathbb{R}$ be a smooth function, which is called a *Hamiltonian function*, where 0 is the zero-section of $T^*\mathbb{R}^n$. We suppose H to be everywhere positive and positively homogeneous of degree one (i.e. $H(x, \lambda\xi) = \lambda H(x, \xi)$ of any $\lambda > 0$, $(x, \xi) \in T^*\mathbb{R}^n \setminus 0$). If we adopt the canonical coordinates $x_1, \dots, x_n, p_1, \dots p_n$ of $T^*\mathbb{R}^n \cong \mathbb{R}^n \times (\mathbb{R}^*)^n$, then $(x, \xi) = (x_1, \dots, x_n, p_1, \dots, p_n)$. We have a vector field X_H on $T^*\mathbb{R}^n \setminus 0$ associate to H defined by

$$X_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial}{\partial p_i} \right).$$

The vector field X_H is determined by the relation $\omega(X_H, Y) = dH(Y)$ for $Y \in TT^*\mathbb{R}^n$, where $\omega = \sum_{i=1}^n dp_i \wedge dx_i$. Since H is positive and positively homogeneous one, we have

$$\sum_{i=1}^n p_i \frac{\partial H}{\partial p_i} = H.$$

Therefore, for any point $(x, \xi) \in H^{-1}(1)$,

$$\sum_{i=1}^n p_i \frac{\partial H}{\partial p_i}(x, \xi) = H(x, \xi) = 1,$$

so that $\mathrm{grad}_p H(x, \xi) \neq \mathbf{0}$. This means that $H^{-1}(1)$ is a regular hypersurface in $T^*\mathbb{R}^n$. Since $dH(X_H) = \omega(X_H, X_H) = 0$, $X_H|_{H^{-1}(1)}$ is tangent to the hypersurface $H^{-1}(1)$. Let $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$ be the projection of the cotangent bundle. Since $\mathrm{grad}_p H(x, p) \neq \mathbf{0}$ on $H^{-1}(1)$, $\pi|_{H^{-1}(1)} : H^{-1}(1) \to \mathbb{R}^n$ is also a fibre bundle whose fibre is diffeomorphic to an $(n-1)$-sphere. The image under π of the flow lines of $X_H|_{H^{-1}(1)}$ are called the *ray* of H. The flow of the vector field $X_H|_{H^{-1}(1)}$ on $H^{-1}(1)$ induces a map $\rho : \mathbb{R}_+ \times H^{-1}(1) \to H^{-1}(1)$ on at least a neighborhood of $\{0\} \times H^{-1}(1) \subset \mathbb{R}_+ \times H^{-1}(1)$. Then we define a map $\exp : \mathbb{R}_+ \times H^{-1}(1) \to \mathbb{R}^n$ by $\exp = \pi \circ \rho$, which is also defined on at least a neighborhood of $\{0\} \times H^{-1}(1)$. Let V_0 be a co-oriented hypersurface in $\mathbb{R}^n$. We consider V_0 as an *initial wave front.* At any $x \in X_0$ the co-oriented tangent space of V_0 at x defines an element $\overline{\xi}(x) \in T_x^*\mathbb{R}^n \setminus 0$ such that $\mathrm{Ker}\, \overline{\xi}(x) = T_x V_0$ and the direction is compatible with the co-orientation of V_0 (i.e. the positive co-normal vector of V_0 at x). ince H is positive and positively homogeneous degree one, we have $H(x, \overline{\xi}(x)) = \eta(x) > 0$ and $H(x, \overline{\xi}(x)/\eta(x)) = H(x, \overline{\xi}(x))/\eta(x) = 1$. If we put $\xi(x) = \overline{\xi}(x)/\eta(x)$, then $H(x, \xi(x)) = 1$. Thus we have an $(n-1)$-dimensional submanifold $\ell(V_0) = \{(x, \xi(x)) \mid x \in V_0\} \subset H^{-1}(1)$. We now consider the Liouville form $\alpha = \sum_{i=1}^n p_i dx_i$ on $T^*\mathbb{R}^n$. Then $\omega = d\alpha$. Since $\mathrm{Ker}\, \xi(x) = T_x V_0$, we have $\alpha|_{\ell(V_0)} = \sum_{i=1}^n p_i dx_i|_{\ell(V_0)} = 0$, so that $\omega|_{\ell(V_0)} = 0$. This means that $\ell(V_0)$ is an isotropic submanifold of the symplectic structure ω. Moreover, if X_H is tangent to $\ell(V_0)$, then we have $0 = \sum_{i=1}^n p_i dx_i(X_H) = \sum_{i=1}^n p_i(\partial H/\partial p_i) = H(x, \xi(x)) = 1$. This is a contradiction, so that X_H is not tangent to $\ell(V_0)$. If $\exp(t, \xi(x))$ is defined for all $x \in V_0$ for some fixed $t > 0$, we call $V_t = \{\exp(t, \xi(x)) \mid x \in V_0\}$ the *wave front* at time t. We also call the canonical map $V_0 \to V_t$ defined by $x \mapsto \exp(t, \xi(x))$ the *ray map* at time t.

Remark 8.1. The restriction of the Liouville form α on $H^{-1}(1)$ defines a contact structure on $H^{-1}(1)$. Moreover, the projection $\pi|_{H^{-1}(1)}$:

$H^{-1}(1) \to \mathbb{R}^n$ is a Legendrian fibration. Since $\alpha|_{\ell(V_0)} = 0$, $\ell(V_0)$ is a Legendrian submanifold of $H^{-1}(1)$. By definition,

$$\ell(V_t) = \{\rho(t, (x, \xi(x))) \mid x \in V_0\}$$

is a Legendrian submanifold of $H^{-1}(1)$, so that $V_t = \pi(\ell(V_t))$ is the wave front in the sense of §2.

The singular value set of the ray map is called the *caustic points* at time t. We denote the set of caustic points at time t by C_t:

$$C_t = \{\exp(t, \xi(x)) \mid \operatorname{rank} d(\exp)_x (t, \xi(x)) < n - 1\}.$$

The set of all the caustic points at all time t in the time interval during which the propagation is considered is called the *caustic* of the propagation.

In [27, 36] Jänich and Wassermann investigated the stability of germs of such caustics. They described the caustics as bifurcation sets in the theory of unfoldings.

Let $t_0 \in \mathbb{R}_+$ and $\xi_0 \in H^{-1}(1)$ be given, such that $\exp(t_0, \xi_0)$ is defined, and such that (t_0, ξ_0) is a regular point of the map

$$(\pi, \exp) : \mathbb{R}_+ \times H^{-1}(1) \to \mathbb{R}^n \times \mathbb{R}^n; (t, \xi) \mapsto (\pi(\xi), \exp(t, \xi)).$$

We set $x_0 = \pi(\xi_0)$ and $u_0 = \exp(t_0, \xi_0)$. Under the above assumptions $(\pi, \exp)$ is a local diffeomorphism at (t_0, ξ_0). Therefore, there exists a local inverse $s : X \times U \to \mathbb{R}_+ \times H^{-1}(1)$ of $(\pi, \exp)$ such that $s(x_0, u_0) = (t_0, \xi_0)$, where X is a neighborhood of x_0 and U is a neighborhood of u_0 in $\mathbb{R}^n$ respectively. We define a function $\tau = \pi_{\mathbb{R}_+} \circ s : X \times U \to \mathbb{R}_+$, where $\pi_{\mathbb{R}_+} : \mathbb{R}_+ \times H^{-1}(1) \to \mathbb{R}_+$ is the canonical projection. τ is called a *ray length function* associated to (t_0, ξ_0). We remark that for given H the germ of τ at (x_0, u_0) depends only on (t_0, ξ_0), not on the choice of s We say that X and U are *sufficiently small* of $s(X \times U)$ which never contains both (t, ξ) and $(t, -\xi)$. Suppose that $x_0 \in V_0$ and $\xi(x_0) = \xi_0$. Given $\varepsilon > 0$, we say that V_0 and ε are *sufficiently small for s* if $(t_0 - \varepsilon, t_0 + \varepsilon) \times \ell(V_0) \subset s(X \times U)$. With these definitions and assumptions, Jänich has shown the following theorem.

Theorem 8.2 (Jänich [27]). *Let (t_0, ξ_0) be a regular point of $(\pi, \exp)$, let $s : X \times U \to \mathbb{R}_+ \times H^{-1}(1)$ be a local inverse to $(\pi, \exp)$ near (t_0, ξ_0), with associated ray-length function τ, and let $V_0 \subset \mathbb{R}^n$ be a normally oriented hypersurface such that $x_0 = \pi(\xi_0) \in V_0$ and $\xi(x_0) = \xi_0$. We define a function $F : V_0 \times U \to \mathbb{R}$ by $F = (\tau - t_0)|_{V_0 \times U}$. Suppose that X and U are sufficiently small, and let $\varepsilon > 0$ be given such that V_0 and*

ε are sufficiently small for s. Then for $t_0 - \varepsilon < t < t_0 + \varepsilon$ we have the following:
(a) $V_t = \{u \in U \mid \exists x \in V_0$ *with* $F(x,u) = t - t_0$ *and* $d_xF(x,u) = 0\}$ *and*
(b) $C_t = \{u \in U \mid \exists x \in V_0$ *with* $F(x,u) = t - t_0,\ d_xF(x,u) = 0$ *and* $d_x^2F(x,u)$ *degenerate*$\}$,
where d_xF is the differential of F with respect to the first variable (u fixed) and d_x^2F is the Hessian of F with respect to the first variable.

Remark 8.3. Since F can take values outside the interval $(-\varepsilon, \varepsilon)$, the *full caustic set* (briefly, *full caustic*) $C = \bigcup_{t_0-\varepsilon<t<t_0+\varepsilon} C_t$ is not necessarily be equal to the full bifurcation set

$$B_F = \{u \in U \mid \exists x \in V_0 \text{ with } d_xF(x,u) = 0 \text{ and } d_x^2F(x,u) \text{ degenerate}\}$$

of F generally. However, for sufficiently small representative F' of the germ F, we have $B_{F'} \subset C \subset B_F$. Moreover, for sufficiently small representative of the germ of V_0 and sufficiently small time interval about t_0, the caustic C' satisfies $C' \subset B_{F'} \subset C$. Therefore, a knowledge of the germ F gives us all information about the local generation of the full caustic.

By assertion (a) in Theorem 8.2, for $t_0 - \varepsilon < t < t_0 + \varepsilon$, we have

$$\ell(V_t) = \{\pi_{H^{-1}(1)} \circ s(x,u) \in H^{-1}(1) \mid F(x,u) = t - t_0 \text{ and } d_xF(x,u) = 0\},$$

where $\pi_{H^{-1}(1)} : \mathbb{R}_+ \times H^{-1}(1) \to H^{-1}(1)$ is the canonical projection. Moreover,

$$L(V_0; (t_0, \xi_0), \varepsilon) = \bigcup_{t_0-\varepsilon<t<t_0+\varepsilon} \ell(V_t)$$

is a Lagrangian submanifold in $H^{-1}(1) \subset T^*\mathbb{R}^n$. Since

$$T_{\xi_0}L(V_0; (t_0, \xi_0), \varepsilon) = T_{\xi_0}\ell(V_t) \oplus \langle X_H \rangle_{\mathbb{R}},$$

$\operatorname{rank} d(\pi|_L)_{\xi_0} < n$ if and only if $\operatorname{rank} d(\pi|_{\ell(V_t)})_{\xi_0} = d(\exp)_{x_0} < n-1$, so that we have $C = C_{L(V_0;(t_0,\xi_0),\varepsilon)}$. Here $C_{L(V_0;(t_0,\xi_0),\varepsilon)}$ is the caustic of the Lagrangian submanifold $L(V_0; (t_0, \xi_0), \varepsilon)$ defined in §2. It follows that $\ell(V_t)$, $(t_0 - \varepsilon, t_0 + \varepsilon)$ is considered to be a momentary front of a graph-like Legendrian unfolding. We define a set germ $\mathscr{L}(V_0; (t_0, \xi_0), \varepsilon) \subset \mathbb{R}_+ \times H^{-1}(1)$ by

$$\{s(x,u) \mid t_0 - \varepsilon, t < t_0 + \varepsilon, F(x,u) = t - t_0 \text{ and } d_xF(x,u) = 0\},$$

which is a graph-like Legendrian unfolding with a generating family $\mathcal{F}(x,u,t) = F(x,u) - (t - t_0)$.

In [27] Jänich considered stability of the caustic in terms of germs of F and of τ. The caustic C is said to be an *$\mathcal{A}$-universal caustic* at time t_0 if (t_0, ξ_0) is a regular point of $(\pi, \exp)$ and if, for a ray-length function $\tau : X \times U \to \mathbb{R}$ associated to $(t_0, \xi(x_0))$, the germ at (x_0, u_0) of the function $F = (\tau - t_0)|_{(V_0 \cap X) \times U}$, considered as an unfolding of the germ of $F(x, u_0)$ at $x_0 \in V_0$ is infinitesimally $\mathcal{A}$-versal. In [27] Jänich said that C is *universal* if it is $\mathcal{A}$-universal. However, we can consider $\mathcal{R}^+$-equivalence instead of $\mathcal{A}$-equivalence here. We also say that C is an *$\mathcal{R}^+$-universal caustic* at time t_0 if the germ of $F(x, u_0)$ at $x_0 \in V_0$ is infinitesimally $\mathcal{R}^+$-versal.

Universality of the caustic is a very strong stability condition, for, roughly speaking, a universal caustic will survive small perturbations not only of the initial wave front but also of the Hamiltonian. However, Jänich conjectured that the stability of the caustic under perturbations of the initial wave front is sufficient to assure the universality of the caustic. In [36] Wassermann has shown that the conjecture of Jänich is true. He considered a general framework as follows: Let $\tau : (\mathbb{R}^m \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a function germ and let $\iota : (\mathbb{R}^k, 0) \to (\mathbb{R}^m, 0)$ be a map germ. We say that the pair (τ, ι) is *P-$\mathcal{A}$-stable under perturbations of ι* if the following holds: Given any open neighborhood X of $\mathbb{R}^n$, and any open neighborhood U of 0 in $\mathbb{R}^n$, and any representative $\tau' : X \times U \to \mathbb{R}$ of τ, and given any open neighborhood V of 0 in $\mathbb{R}^k$ and any representative $\iota' : V \to X$ of ι, there is a neighborhood N of ι' in $C^\infty(V, X)$ (in the weak C^∞-topology) such that for every $\kappa \in N$ there are a point $q_0 \in V$ and a point $x_0 \in U$ such that the germ of $\tau(\kappa(q), x)$ at (q_0, x_0) is P-$\mathcal{A}$-equivalent to the germ $\tau'(\iota'(q), x)$ at 0. Wassermann has shown the following theorem.

Theorem 8.4 ([36])**.** *Let $\tau : (\mathbb{R}^m \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a function germ such that $\tau|_{\mathbb{R}^m \times \{0\}}$ is submersive. Let $\iota : (\mathbb{R}^k, 0) \to (\mathbb{R}^m, 0)$ be a map germ and define $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ by $F(q, x) = \tau(\iota(q), x)$. Then (τ, ι) is P-$\mathcal{A}$-stable under perturbations of ι if and only if F is an infinitesimally $\mathcal{A}$-versal unfolding of $f(q) = F(q, 0)$.*

Remark 8.5. In [36] Wassermann said that the pair (τ, ι) is *r-stable for caustics* if it is P-$\mathcal{A}$-stable under perturbations of ι. However, we can change P-$\mathcal{A}$-equivalence to P-$\mathcal{R}^+$-equivalence. We say that the pair (τ, ι) is *P-$\mathcal{R}^+$-stable under perturbations of ι* if we change P-$\mathcal{A}$-equivalence to P-$\mathcal{R}^+$-equivalence in the above definition. By exactly the same arguments as in the proof of Theorem 8.4, we have the following theorem.

Theorem 8.6. *Let $\tau : (\mathbb{R}^m \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a function germ such that $\tau|\mathbb{R}^m \times \{0\}$ is submersive. Let $\iota : (\mathbb{R}^k, 0) \to (\mathbb{R}^m, 0)$ be a* *map germ and define* $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ *by* $F(q, x) = \tau(\iota(q), x)$. *Then* (τ, ι) *is* P-$\mathcal{R}^+$*-stable under perturbations of* ι *if and only if* F *is an infinitesimally* $\mathcal{R}^+$*-versal unfolding of* $f(q) = F(q, 0)$.

We now apply the definition in the above general framework to our situation. In this case ι will be the germ of an embedding. Since the space of embedding $\mathrm{Emb}\,(V, X)$ is an open subset of $C^\infty(V, X)$, we have no problems to change ι to be an embedding. Suppose that V_0 is a normally oriented hypersurface in $\mathbb{R}^n$, $x_0 \in V_0$ and $t_0 > 0$. Let $\xi_0 \in T^*_{x_0}\mathbb{R}^n$ be given by the normally oriented tangent space to V_0 at x_0. We say that V_0 at x_0 *produces an* $\mathcal{A}$*-stable caustics at* t_0 (respectively, *produces an* $\mathcal{R}^+$*-stable caustics at* t_0), if (t_0, ξ_0) is a regular point of $(\pi, \exp)$ and if for some ray length function $\tau : X \times U \to \mathbb{R}$ associated to (t_0, ξ_0), and for some open neighborhood V of 0 in $\mathbb{R}^n$ and some embedding $\iota : V \to X$ whose image is contained in V_0 and such that $\iota(0) = x_0$, and in some choice of coordinates near $x_0 \in X$ and near $u_0 = \exp\,(t_0, \xi_0) \in U$, the germ of the pair (τ, ι) at $((x_0, u_0), 0)$ is P-$\mathcal{A}$-stable (respectively, P-$\mathcal{R}^+$-stable) under perturbations of ι.

Remark 8.7. In [36] Wassermann said that V_0 *at* x_0 *produces a stable caustic at* t_0 if it produces an $\mathcal{A}$-stable caustic at time t_0. However, we also consider $\mathcal{R}^+$-equivalence, so that we have to distinguish these two cases.

In [36] the conjecture of Jänich [27] was solved affirmatively by Wassermann as a corollary of Theorem 8.4.

Theorem 8.8 ([36])**.** *With the same assumptions as the above paragraph,* V_0 *at* x_0 *produces an* $\mathcal{A}$*-universal caustic if and only if* V_0 *at* x_0 *produces an* $\mathcal{A}$*-stable caustic at* t_0.

Of course, we have the following theorem for $\mathcal{R}^+$-universal caustics as a corollary of Theorem 8.6.

Theorem 8.9. *With the same assumptions as the above paragraph,* V_0 *at* x_0 *produces an* $\mathcal{R}^+$*-universal caustic if and only if* V_0 *at* x_0 *produces an* $\mathcal{R}^+$*-stable caustic at* t_0.

Moreover, we say that V_0 at x_0 *produces a Lagrangian stable Lagrangian submanifold at time* t_0 if (t_0, ξ_0) is a regular point of $(\pi, \exp)$ and if for some ray length function $\tau : X \times U \to \mathbb{R}$ associated to (t_0, ξ_0), and for some open neighborhood V of 0 in $\mathbb{R}^n$ and some embedding $\iota : V \to X$ whose image is contained in V_0 and such that $\iota(0) = x_0$, and in some choice of coordinates near $x_0 \in X$ and near $u_0 = \exp\,(t_0, \xi_0) \in U$, there exists a neighborhood N of ι in $\mathrm{Emb}\,(V, X)$ such that for every $\kappa \in N$ there are a point $q_0 \in V$ and $x \in X$ such that the germ of the Lagrangian submanifold $L(\kappa(V);(t_0,\xi_0),\varepsilon)$ is Lagrangian equivalent to $L(\iota(V);(t_0,\xi_0),\varepsilon)$. We also say that V_0 at x_0 *produces an* $S.P^+$-*Legendrian stable graph-like Legendrian unfolding at* t_0 (respectively, *produces an* s-P-*Legendrian stable graph-like Legendrian unfolding at* t_0) if (t_0,ξ_0) is a regular point of $(\pi,\exp)$ and if for some ray length function $\tau : X\times U\to\mathbb{R}$ associated to (t_0,ξ_0), and for some open neighborhood V of 0 in $\mathbb{R}^n$ and some embedding $\iota : V\to X$ whose image is contained in V_0 and such that $\iota(0)=x_0$, and in some choice of coordinates near $x_0\in X$ and near $u_0=\exp(t_0,\xi_0)\in U$, there exists a neighborhood N of ι in $\mathrm{Emb}\,(V,X)$ such that for every $\kappa\in N$ there are a point $q_0\in V$ and $x\in X$ such that the germ of the graph-like Legendrian unfolding $\mathscr{L}(\kappa(V);(t_0,\xi_0),\varepsilon)$ is $S.P^+$-Legendrian equivalent (respectively, s-P-Legendrian equivalent) to $\mathscr{L}(\iota(V);(t_0,\xi_0),\varepsilon)$. Then we have the following theorem as a corollary of Theorems 2.2, 4.2, Corollary 6.3 and Theorem 8.9.

Theorem 8.10. *With the same assumptions as the above paragraph, the following conditions are equivalent*:
(1) V_0 *at* x_0 *produces a Lagrangian stable Lagrangian submanifold at* t_0,
(2) V_0 *at* x_0 *produces an* $\mathcal{R}^+$-*stable caustic at* t_0,
(3) V_0 *at* x_0 *produces an* $\mathcal{R}^+$-*universal caustic*,
(4) $L(V_0;(t_0,\xi_0),\varepsilon)$ *is Lagrangian stable*,
(5) $\mathscr{L}(V_0;(t_0,\xi_0),\varepsilon)$ *is* $S.P^+$-*Legendrian stable*,
(6) V_0 *at* x_0 *produces an* $S.P^+$-*Legendrian stable graph-like Legendrian unfolding at* t_0.

On the other hand, Jänich and Wassermann considered $\mathcal{A}$-versality of unfoldings instead of $\mathcal{R}^+$-versality. It has been considered that there might be no corresponding geometric equivalence to P-$\mathcal{A}$-equivalence among generating families. However, from the view point of the theory of graph-like Legendrian unfoldings, we have the following theorem as a corollary of Theorems 4.2 and 8.8.

Theorem 8.11. *With the same assumptions as the above paragraph, the following conditions are equivalent*:
(1) V_0 *at* x_0 *produces an* $\mathcal{A}$-*stable caustic at* t_0,
(2) V_0 *at* x_0 *produces an* $\mathcal{A}$-*universal caustic*,
(3) $\mathscr{L}(V_0;(t_0,\xi_0),\varepsilon)$ *is* s-P-*Legendrian stable*,
(6) V_0 *at* x_0 *produces an* s-P-*Legendrian stable graph-like Legendrian unfolding at* t_0.

Example 8.12. As a special case, we have parallels of hypersurfaces in the Euclidean space. In this case we induce the metric on

$T^*_x\mathbb{R}^n$ by $\langle dx_i, dx_j\rangle = \delta_{ij}$, therefore $T^*_x\mathbb{R}^n$ can be canonically identified to the Euclidean n-space. Thus, for $(x,\xi) \in T^*\mathbb{R}^n$, we may regard that $\xi \in \mathbb{R}^n \cong T^*_x\mathbb{R}^n$ with the above identification. In this case we consider the Hamiltonian function $H : T^*\mathbb{R}^n \setminus 0 \to \mathbb{R}$ defined by $H(x,\xi) = \sqrt{\sum_{i=1}^n p_i^2} = \|\xi\|$ for the canonical coordinates $(x,\xi) = (x_1,\dots,x_n,p_1,\dots,p_n)$. It follows that

$$\frac{\partial H}{\partial x_i} = 0,\ \frac{\partial H}{\partial p_i} = \frac{p_i}{\sqrt{\sum_{i=1}^n p_i^2}},$$

so that the corresponding system of ODE for the Hamiltonian vector field is given by

$$(*) \quad \begin{cases} \dfrac{dx_i}{dt} = \dfrac{p_i}{\sqrt{\sum_{i=1}^n p_i^2}}, \\ \dfrac{dp_i}{dt} = 0. \end{cases}$$

For $(x,\xi) \in H^{-1}(1) \cong \mathbb{R}^n \times S^{n-1}$, we solve $(*)$ with the initial data $x(0) = x$ and $\xi(0) = \xi$. Then the solution is given by

$$x(t) = t\xi + x,\ \xi(t) = \xi,$$

so that the flow map $\rho : \mathbb{R}_+ \times H^{-1}(1) \to H^{-1}(1)$ is given by $\rho(t,(x,\xi)) = (t\xi + x, \xi)$. Therefore the exponential map is

$$\exp(t,(x,\xi)) = \pi \circ \rho(t,(x,\xi)) = t\xi + x.$$

Let V_0 be an initial front. We assume that V_0 is parametrized by an embedding $\iota : U \to \mathbb{R}^n$ such that $\iota(U) = V_0$, $\iota(\mathbf{0}) = x_0$, $\iota(u) = (x_1(u),\dots,x_n(u))$ and $u = (u_1,\dots,u_{n-1})$. Since V_0 is normally oriented, we have a unit normal vector field $\boldsymbol{n}(u)$ along V_0 in $\mathbb{R}^n$. Then we choose a one-form $(\iota(u),\xi(\iota(u))) \in T^*\mathbb{R}^n$ such that $\operatorname{Ker}\xi(\iota(u)) = T_{\iota(u)}V_0$ and $H(\iota(u),\xi(\iota(u))) = 1$. This means that $\xi(\iota(u)) = \pm\boldsymbol{n}(u)$, so that we choose $\xi(\iota(u)) = \boldsymbol{n}(u)$. For a fixed $t \in \mathbb{R}_+$, the ray map $V_0 \to V_t$ is $\iota(u) \mapsto \exp(t,\xi(\iota(u)) = \iota(u) + t\xi(\iota(u)) = \iota(u) + t\boldsymbol{n}(u)$. Thus we have

$$V_t = \{\iota(u) + t\boldsymbol{n}(u) \in \mathbb{R}^n \mid u \in U\},$$

which is called a *parallel* of V_0 in the classical differential geometry (cf. [17, 26]).

Then the map $(\pi,\exp) : \mathbb{R}_+ \times H^{-1}(1) \to \mathbb{R}^n \times \mathbb{R}^n$ is given by $(\pi,\exp)(t,(x,\xi)) = (x, x+t\xi)$, so that it is a diffeomorphism onto an open set W in $\mathbb{R}^n\times\mathbb{R}^n$. Therefore, we have the inverse mapping $s : W \to \mathbb{R}_+ \times H^{-1}(1)$ and the ray length function is $\tau(x,v) = \pi_{\mathbb{R}_+} \circ s(x,v)$. If we write $s(x,v) = (t,(x,\xi))$, then we have $v = x + t\xi$, so that $t = \|x - v\|$. This means that $\tau(x,v)=\|x-v\|$. Therefore we consider a *distance function* $F:U\times\mathbb{R}^n\to\mathbb{R}$ on V_0 defined by $F(u,v)=\|\iota(u)-v\|=\tau(\iota(u),v)$. If we consider the extended distance function $\widetilde{F}(u,v,t)=F(u,v)-t$, then the discriminant set

$$D_{\widetilde{F}}=\left\{(v,t)\;\middle|\;\exists u\in U\text{ s.t. }\widetilde{F}(u,v,t)=\frac{\partial\widetilde{F}}{\partial u_i}(u,v,t)=0, i=1,\dots,u_{n-1}\right\}$$

of $\widetilde{F}$ is the graph-like wave front whose momentary fronts are $\{V_t\}_{t\in\mathbb{R}_+}$.

On the other hand, we can use the distance squared function $D:U\times\mathbb{R}^n\to\mathbb{R}$ defined by $D(u,v)=\|\iota(u)-v\|^2=\langle\iota(u)-v,\iota(u)-v\rangle$ instead of the distance function F. In this case, calculations are rather easier than the case when we adopt the distance function F. Actually the (full) bifurcation set B_D is the caustic C and it is also called the *evolute* (or, the *focal set*) of V_0 (cf. [17]). For $n=2$, V_0 is a regular curve. In this case, V_0 is parametrized by an immersion $\boldsymbol{\gamma}:I\to\mathbb{R}^2$ from an open interval I such that $\boldsymbol{t}(s)=\boldsymbol{\gamma}'(s)$ is a unit vector. Then we have the Frenet frame $\{\boldsymbol{t}(s),\boldsymbol{n}(s)\}$ along the curve $\boldsymbol{\gamma}$, where $\boldsymbol{n}(s)$ is the unit normal vector defined by the anti-clockwise $\pi/2$-rotation of $\boldsymbol{t}(s)$. Then we have the Frenet formulae:

$$\begin{cases}\boldsymbol{t}'(s)=\kappa(s)\boldsymbol{n}(s),\\ \boldsymbol{n}'(s)=-\kappa(s)\boldsymbol{t}(s),\end{cases}$$

where $\kappa(s)=\langle\boldsymbol{t}'(s),\boldsymbol{n}(s)\rangle$ is the *curvature* of $\boldsymbol{\gamma}$. In this case the parallel is

$$V_t=\{\boldsymbol{\gamma}(s)+t\boldsymbol{n}(s)\mid s\in I\}$$

and the evolute of $\boldsymbol{\gamma}$ is

$$C=\left\{\boldsymbol{\gamma}(s)+\frac{1}{\kappa(s)}\boldsymbol{n}(s)\;\middle|\;\kappa(s)\neq 0\right\}.$$

The evolute of $\boldsymbol{\gamma}$ is known to be the bifurcation set of the diastase squared function $D:I\times\mathbb{R}^2\to\mathbb{R}$. In fact, we have

$$\frac{\partial D}{\partial s}(s,\boldsymbol{v})=2\langle\boldsymbol{t}(s),\boldsymbol{\gamma}(s)-\boldsymbol{v}\rangle,\;\frac{\partial^2 D}{\partial s^2}(s,\boldsymbol{v})=2(\langle\kappa(s)\boldsymbol{n}(s),\boldsymbol{\gamma}(s)-\boldsymbol{v}\rangle+1),$$

so that $B_D=C$. Here, the equation $\langle\boldsymbol{t}(s),\boldsymbol{\gamma}(s)-\boldsymbol{v}\rangle=0$ defines a normal line of $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(s)$. Therefore, the evolute is the envelope of the normal lines along $\boldsymbol{\gamma}$. Moreover, the singular point of the evolute is $s\in I$ such that $\kappa'(s)=0$, which is called a *vertex* of $\boldsymbol{\gamma}$ in the classical differential geometry (cf. Fig.2 and Fig.3)

8.2. Caustics of world hyper-sheets in Lorentz-Minkowski space-time

In the Lorentz-Minkowski space-time, a world hyper-sheet is a timelike hypersurface formed by a one-parameter family of spacelike submanifolds. In the theory of relativity, we do not have the notion of time constant, so that everything that is moving depends on the time. Therefore, we consider world sheets. Although we have the notion of world sheets with general codimension, we stick to the case when the codimension one, that are called world hyper-sheets in the Lorentz-Minkowski space-time. Recently, Bousso and Randall introduced the notion of caustics of world hyper-sheets in order to define the notion of holographic domains in the space-time. Here, we give a mathematical framework for describing the caustics of world hyper-sheets in the Lorentz-Minkowski space-time.

We now introduce some basic notions on the $(n+1)$-dimensional Lorentz-Minkowski space-time. For basic concepts and properties, see [32]. Let $\mathbb{R}^{n+1} = \{(x_0, x_1, \dots, x_n) \mid x_i \in \mathbb{R}\ (i = 0, 1, \dots, n)\ \}$ be an $(n+1)$-dimensional cartesian space. For any $\boldsymbol{x} = (x_0, x_1, \dots, x_n)$, $\boldsymbol{y} = (y_0, y_1, \dots, y_n) \in \mathbb{R}^{n+1}$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined to be $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i$. We call $(\mathbb{R}^{n+1}, \langle , \rangle)$ the $(n+1)$-*dimensional Minkowski space-time* (or briefly, the *Lorentz-Minkowski* $(n+1)$-*space*). We write $\mathbb{R}^{n+1}_1$ instead of $(\mathbb{R}^{n+1}, \langle , \rangle)$. We say that a non-zero vector $\boldsymbol{x} \in \mathbb{R}^{n+1}_1$ is *spacelike, lightlike or timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$, respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^{n+1}_1$ is defined to be $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. We have the canonical projection $\pi : \mathbb{R}^{n+1}_1 \to \mathbb{R}^n$ defined by $\pi(x_0, x_1, \dots, x_n) = (x_1, \dots, x_n)$. Here we identify $\{\boldsymbol{0}\} \times \mathbb{R}^n$ with $\mathbb{R}^n$ and it is considered as the Euclidean n-space whose scalar product is induced by the pseudo scalar product $\langle , \rangle$. For a vector $\boldsymbol{v} \in \mathbb{R}^{n+1}_1$ and a real number c, we define a *hyperplane with pseudo normal* $\boldsymbol{v}$ by

$$HP(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c\ \}.$$

We call $HP(\boldsymbol{v}, c)$ a *spacelike hyperplane*, a *timelike hyperplane* or a *lightlike hyperplane* if $\boldsymbol{v}$ is timelike, spacelike or lightlike, respectively. We now define

$$LC(\boldsymbol{\lambda}) = \{\boldsymbol{x} = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x} - \boldsymbol{\lambda}, \boldsymbol{x} - \boldsymbol{\lambda} \rangle = 0\}$$

and we call it *the lightcone* with the vertex $\boldsymbol{\lambda} \in \mathbb{R}^{n+1}_1$. We write $LC^* = LC(\boldsymbol{0}) \setminus \{\boldsymbol{0}\}$, which is called an *open lightcone* at the origin.

For any $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n \in \mathbb{R}^{n+1}_1$, we define a vector $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_n$ by

$$\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_n = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_n \\ x_0^1 & x_1^1 & \cdots & x_n^1 \\ x_0^2 & x_1^2 & \cdots & x_n^2 \\ \vdots & \vdots & \cdots & \vdots \\ x_0^n & x_1^n & \cdots & x_n^n \end{vmatrix},$$

where $\boldsymbol{e}_i = (0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0)$ and $\boldsymbol{x}_i = (x_0^i, x_1^i, \ldots, x_n^i)$. We can easily show that $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_n$ is pseudo orthogonal to any $\boldsymbol{x}_i$ $(i = 1, \ldots, n)$.

We briefly review the basic geometrical framework for the study of world hyper-sheets in the $(n+1)$-dimensional Lorentz-Minkowski space-time in [22]. Let $\mathbb{R}^{n+1}_1$ be a time-oriented space (cf. [32]). We choose $\boldsymbol{e}_0 = (1, 0, \ldots, 0)$ as the future timelike vector field. A world hyper-sheet is defined to be a timelike hypersurface foliated by codimension one spacelike submanifolds. Here, we only investigate the local situation, so that we consider a one-parameter family of spacelike submanifolds. Let $\boldsymbol{X} : U \times I \to \mathbb{R}^{n+1}_1$ be a timelike embedding, where $U \subset \mathbb{R}^{n-1}$ is an open subset and I is an open interval. We write $W = \boldsymbol{X}(U \times I)$ and identify W and $U \times I$ through the embedding $\boldsymbol{X}$. The embedding $\boldsymbol{X}$ is said to be *timelike* if the tangent space T_pW of W is a timelike hyperplane at any point $p \in W$. We write that $\mathcal{S}_t = \boldsymbol{X}(U \times \{t\})$ for each $t \in I$. We have a foliation of W defined by $\mathcal{S} = \{\mathcal{S}_t\}_{t \in I}$. We say that $W = \boldsymbol{X}(U \times I)$ (or, $(W, \mathcal{S})$) is a *world hyper-sheet* if W is a time-orientable timelike hypersurface and each $\mathcal{S}_t$ is spacelike. Here, we say that $\mathcal{S}_t$ is *spacelike* if the tangent space $T_p\mathcal{S}_t$ consists only spacelike vectors (i.e. spacelike subspace of $\mathbb{R}^{n+1}_1$) for any point $p \in \mathcal{S}_t$. Each $\mathcal{S}_t$ is called a *momentary space* of W. For any $p = \boldsymbol{X}(\overline{u}, t) \in W \subset \mathbb{R}^{n+1}_1$, we have

$$T_pW = \langle \boldsymbol{X}_{u_1}(\overline{u}, t), \ldots, \boldsymbol{X}_{u_{n-1}}(\overline{u}, t), \boldsymbol{X}_t(\overline{u}, t) \rangle_{\mathbb{R}},$$

where we write $(\overline{u}, t) = (u_1, \ldots, u_{n-1}, t) \in U \times I$, $\boldsymbol{X}_t = \partial \boldsymbol{X}/\partial t$ and $\boldsymbol{X}_{u_j} = \partial \boldsymbol{X}/\partial u_j$. We also have

$$T_p\mathcal{S}_t = \langle \boldsymbol{X}_{u_1}(\overline{u}, t), \ldots, \boldsymbol{X}_{u_{n-1}}(\overline{u}, t) \rangle_{\mathbb{R}}.$$

Since W is time-orientable, there exists a timelike vector field $\boldsymbol{v}(\overline{u}, t)$ on W [32, Lemma 32] Moreover, we can choose that $\boldsymbol{v}$ is *future directed* which means that $\langle \boldsymbol{v}(\overline{u}, t), \boldsymbol{e}_0 \rangle < 0$. Since $\operatorname{codim} W = 1$, we have $\operatorname{codim} \mathcal{S}_t = 2$. Moreover, $\mathcal{S}_t$ is spacelike, so that we can apply the method developed in [18]. We consider the unit normal spacelike vector of W

defined by

$$\boldsymbol{n}^S(\overline{u},t) = \frac{\boldsymbol{X}_{u_1}(\overline{u},t)\wedge\cdots\wedge\boldsymbol{X}_{u_{n-1}}(\overline{u},t)\wedge\boldsymbol{X}_t(\overline{u},t)}{\|\boldsymbol{X}_{u_1}(\overline{u},t)\wedge\cdots\wedge\boldsymbol{X}_{u_{n-1}}(\overline{u},t)\wedge\boldsymbol{X}_t(\overline{u},t)\|}.$$

For any $t\in I$, Let $N_p(\mathcal{S}_t)$ be the pseudo-normal space of $\mathcal{S}_t$ at $p=\boldsymbol{X}(\overline{u},t)$ in $\mathbb{R}^{n+1}_1$. Since $\mathcal{S}_t$ is a codimension one in W, $N_p(\mathcal{S}_t)$ is a two dimensional Lorentz space. There exists a unique timelike unit vector field $\boldsymbol{n}^T(\overline{u},t)\in N_p(\mathcal{S}_t)\cap T_pW$ such that it is future directed (i.e. $\langle \boldsymbol{n}^T(\overline{u},t),\boldsymbol{e}_0\rangle<0$). We now define maps $\mathbb{LG}^{\pm}(\mathcal{S}_t):\mathcal{S}_t\to LC^*$ by $\mathbb{LG}^{\pm}(\mathcal{S}_t)(p)=\boldsymbol{n}^T(\overline{u},t)\pm\boldsymbol{n}^S(\overline{u},t)$, where $p=\boldsymbol{X}(\overline{u},t)$. We call each one of $\mathbb{LG}^{\pm}(\mathcal{S}_t)$ a *momentary lightcone Gauss map*. These maps lead us to the notion of curvatures (cf. [22]). We have linear maps $d\mathbb{LG}^{\pm}(\mathcal{S}_t)_p : T_p\mathcal{S}_t\to T_{\widetilde{p}}LC^*\subset T_{\widetilde{p}}\mathbb{R}^{n+1}_1$, where $p=\boldsymbol{X}(\overline{u},t)$ and $\widetilde{p}=\boldsymbol{n}^T(\overline{u},t)\pm\boldsymbol{n}^S(\overline{u},t)$. With the identification $T_{\widetilde{p}}\mathbb{R}^{n+1}_1\equiv\mathbb{R}^{n+1}_1\equiv T_p\mathbb{R}^{n+1}_1$, we have the canonical decomposition $T_p\mathbb{R}^{n+1}=T_p\mathcal{S}_t\oplus N_p(\mathcal{S}_t)$. Let $\Pi^t : T_p\mathbb{R}^{n+1}=T_p\mathcal{S}_t\oplus N_p(\mathcal{S}_t)\to T_p\mathcal{S}_t$ be the canonical projection. Then we have linear transformations

$$S^{\pm}_{\ell}(\mathcal{S}_t)_p=-\Pi^t\circ d\mathbb{LG}^{\pm}(\mathcal{S}_t)_p : T_p\mathcal{S}_t\to T_p\mathcal{S}_t.$$

Each one of the above mappings is called a *momentary lightcone shape operator* of $\mathcal{S}_t$ at $p=\boldsymbol{X}(\overline{u},t)$. Let $\{\kappa^{\pm}_i(\mathcal{S}_t)(p)\}_{i=1}^{n-1}$ be the set of eigenvalues of $S^{\pm}_{\ell}(\mathcal{S}_t)_p$, which are called *momentary lightcone principal curvatures* of $\mathcal{S}_t$ at $p=\boldsymbol{X}(\overline{u},t)$. Then *momentary lightcone Gauss-Kronecker curvatures* of $\mathcal{S}_t$ at $p=\boldsymbol{X}(\overline{u},t)$ are defined to be

$$K^{\pm}_{\ell}(\mathcal{S}_t)(p)=\det S^{\pm}_{\ell}(\mathcal{S}_t)_p.$$

We obtain now the lightcone Weingarten formulae. Since $\mathcal{S}_t$ is a spacelike submanifold, we have a Riemannian metric (the *first fundamental form*) on $\mathcal{S}_t$ defined by $ds^2=\sum_{i=1}^{n-1}g_{ij}du_idu_j$, where $g_{ij}(\overline{u},t)=\langle\boldsymbol{X}_{u_i}(\overline{u},t),\boldsymbol{X}_{u_j}(\overline{u},t)\rangle$ for any $(\overline{u},t)\in U\times I$. *Lightcone second fundamental invariants* are defined to be

$$h_{ij}[\pm](\overline{u},t)=\langle-(\boldsymbol{n}^T\pm\boldsymbol{n}^S)_{u_i}(\overline{u},t),\boldsymbol{X}_{u_j}(\overline{u},t)\rangle$$

for any $(\overline{u},t)\in U\times I$. The following lightcone Weingarten formulae are given as special cases of the formulae in [18]:

(a) $(\boldsymbol{n}^T\pm\boldsymbol{n}^S)_{u_i}=\langle\boldsymbol{n}^S,\boldsymbol{n}^T_{u_i}\rangle(\boldsymbol{n}^T\pm\boldsymbol{n}^S)-\sum_{j=1}^{n-1}h^j_i[\pm]\boldsymbol{X}_{u_j}$

(b) $\Pi^t\circ(\boldsymbol{n}^T+\boldsymbol{n}^S)_{u_i}=-\sum_{j=1}^{n-1}h^j_i[\pm]\boldsymbol{X}_{u_j}$.

Here $\left(h^j_i[\pm]\right)=(h_{ik}[\pm])\left(g^{kj}\right)$ and $\left(g^{kj}\right)=(g_{kj})^{-1}$.

It follows that the momentary lightcone principal curvatures are the eigenvalues of $\left(h_i^j[\pm]\right)$.

On the other hand, as an application of the theory of Legendrian unfoldings, a geometric framework for the study of caustics of world hyper-sheets in the Lorentz-Minkowski space-time has been constructed in [23]. We give a brief survey here. We define hypersurfaces $\mathbb{LH}^{\pm}_{\mathcal{S}_{t_0}} : U \times \{t_0\} \times \mathbb{R} \to \mathbb{R}^{n+1}_1$ by

$$\mathbb{LH}^{\pm}_{\mathcal{S}_{t_0}}(p, \mu) = \mathbb{LH}^{\pm}_{\mathcal{S}_{t_0}}(\overline{u}, t_0, \mu) = \boldsymbol{X}(\overline{u}, t_0) + \mu \mathbb{LG}^{\pm}(\mathcal{S}_{t_0})(\overline{u}, t_0),$$

where $p = \boldsymbol{X}(\overline{u}, t_0)$. We call $\mathbb{LH}^{\pm}_{\mathcal{S}_{t_0}}$ *light sheets* along $\mathcal{S}_{t_0}$. A hypersurface $H \subset \mathbb{R}^{n+1}_1$ is, generally, called a *lightlike hypersurface* if it is tangent to a lightcone at any point. The light sheet along $\mathcal{S}_{t_0}$ is a lightlike hypersurface. We also define $\mathbb{LH}^{\pm}_{W} : U \times I \times \mathbb{R} \to \mathbb{R}^{n+1}_1 \times I$ by

$$\mathbb{LH}^{\pm}_{W}(\overline{u}, t, \mu) = (\mathbb{LH}^{\pm}_{\mathcal{S}_t}(\overline{u}, t, \mu), t),$$

which are called *unfolded light sheets* of $(W, \mathcal{S})$.

We introduce the notion of Lorentz distance-squared functions on a world hyper-sheet, which is useful for the study of singularities of light sheets. We define a family of functions $G : W \times \mathbb{R}^{n+1}_1 \to \mathbb{R}$ on $W = \boldsymbol{X}(U \times I)$ by

$$G(p, \boldsymbol{\lambda}) = G(\overline{u}, t, \boldsymbol{\lambda}) = \langle \boldsymbol{X}(\overline{u}, t) - \boldsymbol{\lambda}, \boldsymbol{X}(\overline{u}, t) - \boldsymbol{\lambda} \rangle,$$

where $p = \boldsymbol{X}(\overline{u}, t)$. We call G a *Lorentz distance-squared function* on the world hyper-sheet $(W, \mathcal{S})$. For any fixed $(t_0, \boldsymbol{\lambda}_0) \in I \times \mathbb{R}^{n+1}_1$, we write $g(\overline{u}) = G_{(t_0, \boldsymbol{\lambda}_0)}(\overline{u}) = G(\overline{u}, t_0, \boldsymbol{\lambda}_0)$ and have the following proposition.

Proposition 8.13 ([23]). *Let $\mathcal{S}_{t_0}$ be a momentary space of $(W, \mathcal{S})$ and $G : W \times \mathbb{R}^{n+1}_1 \to \mathbb{R}$ the Lorentz distance-squared function on $(W, \mathcal{S})$. Suppose that $p_0 = \boldsymbol{X}(\overline{u}_0, t_0) \neq \boldsymbol{\lambda}_0$. Then we have the following*:
(1) *$g(\overline{u}_0) = \partial g/\partial u_i(\overline{u}_0) = 0$ $(i = 1, \dots, n-1)$ if and only if $p_0 - \boldsymbol{\lambda}_0 = \mu \mathbb{LG}^{\pm}(\mathcal{S}_{t_0})(p_0)$ for some $\mu \in \mathbb{R} \setminus \{0\}$.*
(2) *$g(\overline{u}_0) = \partial g/\partial u_i(\overline{u}_0) = \det \mathcal{H}(g)(\overline{u}_0) = 0$ $(i = 1, \dots, n-1)$ if and only if*

$$p_0 - \boldsymbol{\lambda}_0 = \mu \mathbb{LG}^{\pm}(\mathcal{S}_{t_0})(p_0)$$

for $\mu \in \mathbb{R} \setminus \{0\}$ such that $-1/\mu$ is one of the non-zero momentary lightcone principal curvatures $\{\kappa_i^{\pm}(\mathcal{S}_t)(p)\}_{i=1}^{n-1}$.
Here, $\det \mathcal{H}(g)(\overline{u}_0)$ *is the determinant of the Hessian matrix of g at $\overline{u}_0$.*

Inspired by the above result, we define a set $\mathbb{LF}^{\pm}_{\mathcal{S}_{t_0}}$ by

$$\bigcup_{i=1}^{n-1}\left\{\boldsymbol{X}(u,t_0)+\frac{1}{\kappa_i^{\pm}(\mathcal{S}_t)(p)}\mathbb{LG}^{\pm}(\mathcal{S}_{t_0})(p)\mid u\in U, p=\boldsymbol{X}(u,t_0)\right\},$$

which are called *lightlike focal sets* of $\mathcal{S}_{t_0}$. Moreover, *unfolded lightcone focal sets* of $(W,\mathcal{S})$ are defined to be

$$\mathbb{LF}^{\pm}_{(W,\mathcal{S})}=\bigcup_{t\in I}\mathbb{LF}^{\pm}_{\mathcal{S}_t}\times\{t\}\subset\mathbb{R}^{n+1}_1\times I.$$

Each one of $\mathbb{LF}^{\pm}_{(W,\mathcal{S})}$ is the critical value set of $\mathbb{LH}^{\pm}_W$, respectively.

We consider the relationship between the contact of a one parameter family of submanifolds with a submanifold and $S.P$-$\mathcal{K}$-equivalence among functions (cf. [13]). Let $U_i\subset\mathbb{R}^r$, $(i=1,2)$ be open sets and $g_i:(U_i\times I,(\overline{u}_i,t_i))\to(\mathbb{R}^n,\boldsymbol{y}_i)$ immersion germs. We define $\overline{g}_i:(U_i\times I,(\overline{u}_i,t_i))\to(\mathbb{R}^n\times I,(\boldsymbol{y}_i,t_i))$ by $\overline{g}_i(\overline{u},t)=(g_i(\overline{u}),t)$. We write that $(\overline{Y}_i,(\boldsymbol{y}_i,t_i))=(\overline{g}_i(U_i\times I),(\boldsymbol{y}_i,t_i))$. Let $f_i:(\mathbb{R}^n,\boldsymbol{y}_i)\to(\mathbb{R},0)$ be submersion germs and write that $(V(f_i),\boldsymbol{y}_i)=(f_i^{-1}(0),\boldsymbol{y}_i)$. We say that *the contact of $\overline{Y}_1$ with the trivial family of $V(f_1)$ at $(\boldsymbol{y}_1,t_1)$* is of the *same type in the strict sense* as *the contact of $\overline{Y}_2$ with the trivial family of $V(f_2)$ at $(\boldsymbol{y}_2,t_2)$* if there is a diffeomorphism germ $\Phi:(\mathbb{R}^n\times I,(\boldsymbol{y}_1,t_1))\to(\mathbb{R}^n\times I,(\boldsymbol{y}_2,t_2))$ of the form $\Phi(\boldsymbol{y},t)=(\phi_1(\boldsymbol{y},t),t+(t_2-t_1))$ such that $\Phi(\overline{Y}_1)=\overline{Y}_2$ and $\Phi(V(f_1)\times I)=V(f_2)\times I$. In this case we write $SK(\overline{Y}_1,V(f_1)\times I;(\boldsymbol{y}_1,t_1))=SK(\overline{Y}_2,V(f_2)\times I;(\boldsymbol{y}_2,t_2))$. In [23] we claimed that the following proposition holds which is analogous to Montaldi's theorem on contact between submanifolds in [31]:

Proposition 8.14. *With the same notations as in the above paragraphs,*

$$SK(\overline{Y}_1,V(f_1)\times I;(\boldsymbol{y}_1,t_1))=SK(\overline{Y}_2,V(f_2)\times I;(\boldsymbol{y}_2,t_2))$$

if and only if $f_1\circ g_1$ and $f_2\circ g_2$ are $S.P$-$\mathcal{K}$-equivalent [i.e. there exists a diffeomorphism germ $\Psi:(U_1\times I,(\overline{u}_1,t_1))\to(U_2\times I,(\overline{u}_2,t_2))$ of the form $\Psi(\overline{u},t)=(\psi_1(\overline{u},t),t-(t_2-t_1))$ and a function germ $\lambda:(U_1\times I,(\overline{u}_1,t_1))\to\mathbb{R}$ with $\lambda(\overline{u}_1,t_1)\neq 0$ such that $(f_2\circ g_2)\circ\Psi(\overline{u},t)=\lambda(\overline{u},t)f_1\circ g_1(\overline{u},t)$].

On the other hand, we also consider a little weaker version of the above definition of the contact of a one parameter family of submanifolds with a submanifold. We say that *the contact of $\overline{Y}_1$ with the trivial family of $V(f_1)$ at $(\boldsymbol{y}_1,t_1)$* is of the *same type* as *the contact of $\overline{Y}_2$ with the trivial family of $V(f_2)$ at $(\boldsymbol{y}_2,t_2)$* if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n \times I, (\boldsymbol{y}_1, t_1)) \to (\mathbb{R}^n \times I, (\boldsymbol{y}_2, t_2))$ of the form $\Phi(\boldsymbol{y}, t) = (\phi_1(\boldsymbol{y}, t), \phi_2(t))$ such that $\Phi(\overline{Y}_1) = \overline{Y}_2$ and $\Phi(V(f_1) \times I) = V(f_2) \times I$. In this case we write $PK(\overline{Y}_1, V(f_1) \times I; (\boldsymbol{y}_1, t_1)) = PK(\overline{Y}_2, V(f_2) \times I; (\boldsymbol{y}_2, t_2))$. We also claim that the following proposition holds which is analogous to Montaldi's theorem [31] and we omit to give the proof here.

Proposition 8.15. *With the same notations as in the above paragraphs,*

$$PK(\overline{Y}_1, V(f_1) \times I; (\boldsymbol{y}_1, t_1)) = PK(\overline{Y}_2, V(f_2) \times I; (\boldsymbol{y}_2, t_2))$$

if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are P-$\mathcal{K}$-equivalent [i.e. there exists a diffeomorphism germ $\Psi : (U_1 \times I, (\overline{u}_1, t_1)) \to (U_2 \times I, (\overline{u}_2, t_2))$ of the form $\Psi(\overline{u}, t) = (\psi_1(\overline{u}, t), \psi_2(t))$ and a function germ $\lambda : (U_1 \times I, (\overline{u}_1, t_1)) \to \mathbb{R}$ with $\lambda(\overline{u}_1, t_1) \neq 0$ such that $(f_2 \circ g_2) \circ \Psi(\overline{u}, t) = \lambda(\overline{u}, t) f_1 \circ g_1(\overline{u}, t)$].

We now consider a function $\mathfrak{g}_{\boldsymbol{\lambda}} : \mathbb{R}^{n+1}_1 \to \mathbb{R}$ defined by $\mathfrak{g}_{\boldsymbol{\lambda}}(\boldsymbol{x}) = \langle \boldsymbol{x} - \boldsymbol{\lambda}, \boldsymbol{x} - \boldsymbol{\lambda} \rangle$, where $\boldsymbol{\lambda} \in \mathbb{R}^{n+1}_1 \setminus W$. For any $\boldsymbol{\lambda}_0 \in \mathbb{R}^{n+1}_1$, we have a lightcone $\mathfrak{g}^{-1}_{\boldsymbol{\lambda}_0}(0) = LC(\boldsymbol{\lambda}_0)$. Moreover, we consider the lightlike vectors $\boldsymbol{\lambda}^{\pm}_0 = \mathbb{LH}^{\pm}_{\mathcal{S}_{t_0}}(p_0, \mu_0)$, where $p_0 = \boldsymbol{X}(\overline{u}_0, t_0)$. Then we have

$$\mathfrak{g}_{\boldsymbol{\lambda}^{\pm}_0} \circ \boldsymbol{X}(\overline{u}_0, t_0) = G((u_0, t_0), \mathbb{LH}^{\pm}_{\mathcal{S}_{t_0}}(p_0, \mu_0)) = 0.$$

By Proposition 8.13, we also have relations that

$$\frac{\partial \mathfrak{g}_{\boldsymbol{\lambda}^{\pm}_0} \circ \boldsymbol{X}}{\partial u_i}(\overline{u}_0, t_0) = \frac{\partial G}{\partial u_i}((u_0, t_0), \mathbb{LH}^{\pm}_{\mathcal{S}_{t_0}}(p_0, \mu_0)) = 0.$$

for $i = 1, \dots, n-1$. These relations mean that the lightcones $\mathfrak{g}^{-1}_{\boldsymbol{\lambda}^{\pm}_0}(0) = LC(\boldsymbol{\lambda}^{\pm}_0)$ are tangent to $\mathcal{S}_{t_0} = \boldsymbol{X}(U \times \{t_0\})$ at $p_0 = \boldsymbol{X}(\overline{u}_0, t_0)$. Each one of the lightcones $LC(\boldsymbol{\lambda}^{\pm}_0)$ is said to be a *tangent lightcone* of $\mathcal{S}_{t_0} = \boldsymbol{X}(U \times \{t_0\})$ at $p_0 = \boldsymbol{X}(\overline{u}_0, t_0)$, which we write $TLC(\mathcal{S}_{t_0}, \boldsymbol{\lambda}^{\pm}_0)$, where $\boldsymbol{\lambda}^{\pm}_0 = \mathbb{LH}^{\pm}_{\mathcal{S}_{t_0}}(p_0, \mu_0)$. Then we have the following simple lemma.

Lemma 8.16. *Let $\boldsymbol{X} : U \times I \to \mathbb{R}^{n+1}_1$ be a world hyper-sheet. Consider two points $p_i = \boldsymbol{X}(\overline{u}_i, t_0)$, $(i = 1, 2)$. Then*

$$\mathbb{LH}^{\pm}_{\mathcal{S}_{t_0}}(p_1, \mu_1) = \mathbb{LH}^{\pm}_{\mathcal{S}_{t_0}}(p_2, \mu_2)$$

if and only if

$$TLC(\mathcal{S}_{t_0}, \mathbb{LH}^{\pm}_{\mathcal{S}_{t_0}}(p_1, \mu_1)) = TLP(\mathcal{S}_{t_0}, \mathbb{LH}^{\pm}_{\mathcal{S}_{t_0}}(p_2, \mu_2)).$$

As a consequence, we have tools for the study of the contact between momentary spaces and families of lightcones. We write that $g_{\boldsymbol{\lambda}}(\overline{u},t) = G(\overline{u},t,\boldsymbol{\lambda})$. Then we have $g_{\boldsymbol{\lambda}}(\overline{u},t) = \mathfrak{g}_{\boldsymbol{\lambda}} \circ \boldsymbol{X}(\overline{u},t)$, so that we have the following proposition as a corollary of Proposition 8.14.

Proposition 8.17. *Let* $\boldsymbol{X}_i : (U \times I, (\overline{u}_i, t_0)) \to (\mathbb{R}^{n+1}_1, p_i)$, $(i = 1,2)$, *be world hypersheet germs and* $\boldsymbol{\lambda}^{\pm}_i = \mathbb{LH}^{\pm}_{\mathcal{S}_{t_0}}(p_i,\mu_i)$ *and* $W_i = \boldsymbol{X}_i(U \times I)$. *Then the following conditions are equivalent*:
(1) $SK(\overline{W}_1, TLC(\mathcal{S}_{t_0}, \boldsymbol{\lambda}^{\pm}_1) \times I; (p_1,t_0)) = SK(\overline{W}_2, TLC(\mathcal{S}_{t_0}, \boldsymbol{\lambda}^{\pm}_2) \times I; (p_2,t_0))$,
(2) $g_{1,\boldsymbol{\lambda}^{\pm}_1}$ *and* $g_{2,\boldsymbol{\lambda}^{\pm}_2}$ *are S.P-$\mathcal{K}$-equivalent.*
Here, $g_{i,\boldsymbol{\lambda}^{\pm}_i}(\overline{u},t) = G_i(\overline{u},t,\boldsymbol{\lambda}^{\pm}_i) = \langle \boldsymbol{X}_i(\overline{u},t) - \boldsymbol{\lambda}^{\pm}_i, \boldsymbol{X}_i(\overline{u},t) - \boldsymbol{\lambda}^{\pm}_i \rangle$, $(i = 1,2)$.

We also have the following proposition as a corollary of Proposition 8.15.

Proposition 8.18. *With the same notations as those in Proposition 8.17, the following conditions are equivalent*:
(1) $PK(\overline{W}_1, TLC(\mathcal{S}_{t_0}, \boldsymbol{\lambda}^{\pm}_1) \times I; (p_1,t_0)) = PK(\overline{W}_2, TLC(\mathcal{S}_{t_0}, \boldsymbol{\lambda}^{\pm}_2) \times I; (p_2,t_0))$,
(2) $g_{1,\boldsymbol{\lambda}^{\pm}_1}$ *and* $g_{2,\boldsymbol{\lambda}^{\pm}_2}$ *are P-$\mathcal{K}$-equivalent.*

We can investigate unfolded lightcone focal sets of world hypersheets as an application of the theory of graph-like Legendrian unfoldings. We have shown the following key-proposition in [23].

Proposition 8.19 ([23]). *Let* $G : U \times I \times (\mathbb{R}^{n+1}_1 \setminus W) \to \mathbb{R}$ *be a Lorentz distance-squared function on a world hyper-sheet* $(W, \mathcal{S})$. *For any point* $(\overline{u}_0, t_0, \boldsymbol{\lambda}_0) \in \Sigma_*(G)$, G *is a non-degenerate graph-like Morse family of hypersurfaces around* $(\overline{u}_0, t_0, \boldsymbol{\lambda}_0)$.

By Proposition 8.13, we have

$$\Sigma_*(G) = \{(\overline{u}, t, \mathbb{LH}^{\pm}_{\mathcal{S}_t}(p,\mu)) \in U \times I \times \mathbb{R}^{n+1}_1 \mid p = \boldsymbol{X}(\overline{u},t), \mu \in \mathbb{R} \setminus \{0\}\}.$$

We define a map $\mathscr{L}_G : \Sigma_*(G) \to J^1(\mathbb{R}^{n+1}_1, I)$ by

$$\mathscr{L}_G(\overline{u}, t, \mathbb{LH}^{\pm}_{\mathcal{S}_t}(p,\mu)) = \left(\mathbb{LH}^{\pm}_{\mathcal{S}_t}(p,\mu), t, \frac{2}{\langle \boldsymbol{X}_t(\overline{u},t), \boldsymbol{n}^T(\overline{u},t) \rangle} \overline{\mathbb{LG}^{\pm}(\mathcal{S}_t)(\overline{u},t)} \right),$$

where we define $\overline{\boldsymbol{x}} = (-x_0, x_1, \dots x_n)$ for $\boldsymbol{x} = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}_1$. By the construction of the graph-like Legendrian unfolding from a graph-like Morse family of hypersurfaces, $\mathscr{L}_G(\Sigma_*(G))$ is a graph-like Legendrian unfolding in $J^1(\mathbb{R}^{n+1}_1, I)$. Therefore, the graph-like wave front

$W(\mathscr{L}_G(\Sigma_*(G)))$ is equal to

$$\{(\mathbb{LH}^{\pm}_{\mathcal{S}_t}(p,\mu),t) \in \mathbb{R}^{n+1}_1 \times I \mid p = \boldsymbol{X}(\overline{u},t), (\overline{u},t) \in U \times I, \mu \in \mathbb{R} \setminus \{0\}\}.$$

This means that $W(\mathscr{L}_G(\Sigma_*(G))) = \mathbb{LH}^+_W(U \times I \times (\mathbb{R}\setminus\{0\})) \cup \mathbb{LH}^-_W(U \times I \times (\mathbb{R}\setminus\{0\}))$. By Proposition 8.13, the singular set of $W(\mathscr{L}_G(\Sigma_*(G)))$ is the union of the critical value sets of $\mathbb{LH}^{\pm}_W$ which is the union of unfolded lightcone focal sets $\mathbb{LF}^+_W \cup \mathbb{LF}^-_W$. Therefore, we have shown the following proposition.

Proposition 8.20 ([23]). *Let $(W,\mathcal{S})$ be a world hyper-sheet in $\mathbb{R}^{n+1}_1$ and $G : W \times (\mathbb{R}^{n+1}_1 \setminus W) \to \mathbb{R}$ the Lorentz distance squared function. Then we have the graph-like Legendrian unfolding $\mathscr{L}_G(\Sigma_*(G)) \subset J^1(\mathbb{R}^{n+1}_1, I)$ such that*

$$W(\mathscr{L}_G(\Sigma_*(G))) = \mathbb{LH}^+_W(U \times I \times (\mathbb{R} \setminus \{0\})) \cup \mathbb{LH}^-_W(U \times I \times (\mathbb{R} \setminus \{0\})).$$

We write $\mathbb{LH}^{\pm}_{(W,\mathcal{S})} = \mathbb{LH}^{\pm}_W(U \times I \times (\mathbb{R}\setminus\{0\}))$. We also call $\mathbb{LH}^+_{(W,\mathcal{S})} \cup \mathbb{LH}^-_{(W,\mathcal{S})}$ an *unfolded light sheet* of $(W,\mathcal{S})$. On the other hand, we have the corresponding Lagrangian submanifold $\Pi(\mathscr{L}_G(\Sigma_*(G))) \subset T^*\mathbb{R}^{n+1}_1$. We now consider the natural question what are the caustic $C_{\mathscr{L}_G(\Sigma_*(G))}$ and the Maxwell set $M_{\mathscr{L}_G(\Sigma_*(G))}$? Moreover, are there any meanings of $C_{\mathscr{L}_G(\Sigma_*(G))}$ and $M_{\mathscr{L}_G(\Sigma_*(G))}$ in physics?

In [4, 5] Bousso and Randall gave an idea of caustics of world hyper-sheets in order to define the notion of holographic domains. The family of light sheets $\{\mathbb{LH}^{\pm}_{\mathcal{S}_t}(U \times \{t\}) \times \mathbb{R}\}_{t\in J}$ sweeps out a region in $\mathbb{R}^{n+1}_1$. A *caustic* of a world hyper-sheet is the union of the sets of critical values of light sheets along momentary spaces $\{\mathcal{S}_t\}_{t\in I}$. A *holographic domain* of the world hyper-sheet is the region where the light-sheets sweep out until *caustics*. So this means that the boundary of the holographic domain consists the caustic of the world hyper-sheet. The set of critical values of the light sheet of a momentary space is the lightlike focal set of the momentary space. Therefore the notion of caustics in the sense of Bousso-Randall is formulated as follows: *Caustics of a world sheet* $(W,\mathcal{S})$ are defined to be

$$C^{\pm}(W,\mathcal{S}) = \bigcup_{t\in I} \mathbb{LF}^{\pm}_{\mathcal{S}_t} = \pi_1(\mathbb{LF}^{\pm}_{(W,\mathcal{S})}),$$

where $\pi_1 : \mathbb{R}^{n+1}_1 \times I \to \mathbb{R}^{n+1}_1$ is the canonical projection. We call $C^{\pm}(W,\mathcal{S})$ *BR-caustics* of $(W,\mathcal{S})$ (cf. [23]). We write that $C(W,\mathcal{S}) = \pi_1(\mathbb{LF}^+_W \cup \mathbb{LF}^-_W)$ and call it a *total BR-caustic* of $(W,\mathcal{S})$. By definition, we have $\Sigma(W(\mathscr{L}_G(\Sigma_*(G))) = \mathbb{LF}^+_{(W,\mathcal{S})} \cup \mathbb{LF}^-_{(W,\mathcal{S})}$, so that we have the following proposition.

Proposition 8.21 ([23]). *Let* $(W, \mathcal{S})$ *be a world hyper-sheet in* $\mathbb{R}^{n+1}_1$ *and* $G : U \times I \times (\mathbb{R}^{n+1}_1 \setminus W) \to \mathbb{R}$ *the Lorentz distance squared function. Then we have* $C(W, \mathcal{S}) = C_{\mathscr{L}_G(\Sigma_*(G))}$.

In [4, 5] the authors did not consider the Maxwell set of a world hyper-sheet. However, the notion of Maxwell sets plays an important role in the cosmology which has been called a *crease set* by Penrose (cf. [33, 34]). Actually, the topological shape of the event horizon is determined by the crease set of light sheets. Here, we write $M(W, S) = M_{\mathscr{L}_G(\Sigma_*(G))}$ and call it a *BR-Maxwell set* of the world sheet $(W, \mathcal{S})$.

Let $\boldsymbol{X}_i : (U_i \times I_i, (\overline{u}_i, t_i)) \to (\mathbb{R}^{n+1}_1, p_i)$, $(i = 1, 2)$ be germs of timelike embeddings such that $(W_i, \mathcal{S}_i)$ are world hyper-sheet germs, where $W_i = \boldsymbol{X}_i(U)$. For $\boldsymbol{\lambda}_i = \mathbb{LH}^+_{\mathcal{S}_i}(p_i, \overline{u}_i)$ or $\boldsymbol{\lambda}_i = \mathbb{LH}^-_{\mathcal{S}_i}(p_i, \overline{u}_i)$, let $G_i : (U_i \times I_i \times (\mathbb{R}^{n+1}_1 \setminus W_i), (\overline{u}_i, t_i, \boldsymbol{\lambda}_i)) \to \mathbb{R}$ be Lorentz distance squared function germs. We also write that $g_{i,\boldsymbol{\lambda}_i}(\overline{u}, t) = G_i(\overline{u}, t, \boldsymbol{\lambda}_i)$. Since

$$W(\mathscr{L}_{G_i}(\Sigma_*(G_i))) = \mathbb{LH}^+_{(W_i,\mathcal{S}_i)} \cup \mathbb{LH}^-_{(W_i,\mathcal{S}_i)},$$

we can apply Theorem 6.1 and Corollary 6.2 to our case. Then we have the following theorem.

Theorem 8.22. *Suppose that* $\overline{\pi}|_{\mathscr{L}_{G_i}(\Sigma_*(G_i))}$ *is a proper map germ and the singular set of the map germ is nowhere dense for each* $i = 1, 2$, *respectively. Then the following conditions are equivalent*:
(1) $(\mathbb{LH}^+_{(W_1,\mathcal{S}_1)} \cup \mathbb{LH}^-_{(W_1,\mathcal{S}_1)}, \boldsymbol{\lambda}_1), (\mathbb{LH}^+_{(W_2,\mathcal{S}_2)} \cup \mathbb{LH}^-_{(W_2,\mathcal{S}_2)}, \boldsymbol{\lambda}_2)$ *are* $S.P^+$*-diffeomorphic*,
(2) $\mathscr{L}_{G_1}(\Sigma_*(G_1))$, $\mathscr{L}_{G_2}(\Sigma_*(G_2))$ *are* $S.P^+$*-Legendrian equivalent*,
(3) $\Pi(\mathscr{L}_{G_1}(\Sigma_*(G_1)))$, $\Pi(\mathscr{L}_{G_2}(\Sigma_*(G_2))$ *are Lagrangian equivalent.*

We remark that conditions (2) and (3) are equivalent without any assumptions (cf. Theorem 6.1). Moreover, if we assume that $\mathscr{L}_{G_i}(\Sigma_*(G_i))$ is $S.P^+$-Legendrian stable, then we can apply Proposition 8.14 and Theorem 6.4 and show the following theorem.

Theorem 8.23. *Suppose that* $\mathscr{L}_{G_i}(\Sigma_*(G_i))$ *is* $S.P^+$*-Legendrian stable for each* $i = 1, 2$, *respectively. Then the following conditions are equivalent*:
(1) $(\mathbb{LH}^+_{(W_1,\mathcal{S}_1)} \cup \mathbb{LH}^-_{(W_1,\mathcal{S}_1)}, \boldsymbol{\lambda}_1),(\mathbb{LH}^+_{(W_2,\mathcal{S}_2)} \cup \mathbb{LH}^-_{(W_2,\mathcal{S}_2)}, \boldsymbol{\lambda}_2)$ *are* $S.P^+$*-diffeomorphic*,
(2) $\mathscr{L}_{G_1}(\Sigma_*(G_1))$, $\mathscr{L}_{G_2}(\Sigma_*(G_2))$ *are* $S.P^+$*-Legendrian equivalent*,
(3) $\Pi(\mathscr{L}_{G_1}(\Sigma_*(G_1)))$, $\Pi(\mathscr{L}_{G_2}(\Sigma_*(G_2))$ *are Lagrangian equivalent*,
(4) $g_{1,\boldsymbol{\lambda}_1}$, $g_{2,\boldsymbol{\lambda}_2}$ *are* $S.P$*-*$\mathcal{K}$*-equivalent*,
(5) $SK(\overline{W}_1, TLC(\mathcal{S}_{t_0}, \boldsymbol{\lambda}_1) \times I; (p_1, t_0))$
$= SK(\overline{W}_2, TLC(\mathcal{S}_{t_0}, \boldsymbol{\lambda}_2) \times I; (p_2, t_0)).$

For s-P-Legendrian equivalence, we have the following theorem as a corollary of Theorem 7.11 and Proposition 8.18.

Theorem 8.24. *Suppose that* $\mathscr{L}_{G_i}(\Sigma_*(G_i))$ *is* s-P-*Legendrian stable for each* $i = 1, 2$, *respectively. Then the following conditions are equivalent*:
(1) $(\mathbb{LH}^+_{(W_1,\mathcal{S}_1)} \cup \mathbb{LH}^-_{(W_1,\mathcal{S}_1)}, \boldsymbol{\lambda}_1)$,$(\mathbb{LH}^+_{(W_2,\mathcal{S}_2)} \cup \mathbb{LH}^-_{(W_2,\mathcal{S}_2)}, \boldsymbol{\lambda}_2)$ *are* s-P -*diffeomorphic*,
(2) $\mathscr{L}_{G_1}(\Sigma_*(G_1))$, $\mathscr{L}_{G_2}(\Sigma_*(G_2))$ *are* s-P-*Legendrian equivalent*,
(3) $g_{1,\boldsymbol{\lambda}_1}$, $g_{2,\boldsymbol{\lambda}_2}$ *are* P-$\mathcal{K}$-*equivalent*,
(4) $PK(\overline{W}_1, TLC(\mathcal{S}_{t_0}, \boldsymbol{\lambda}_1) \times I; (p_1, t_0))$
$$= PK(\overline{W}_2, TLC(\mathcal{S}_{t_0}, \boldsymbol{\lambda}_2) \times I; (p_2, t_0)).$$

Moreover, $S.P^+$-Legendrian equivalence among graph-like Legendrian unfoldings implies s-P-Legendrian equivalence. By Proposition 8.19, the caustic and the Maxwell set of $\mathscr{L}_G(\Sigma_*(G))$ are the BR-caustic and the BR-Mawxell set. Since s-P-Legendrian equivalence among graph-like Legendrian unfoldings preserves both the diffeomorphism types of caustics and Maxwell sets, we have the following proposition.

Proposition 8.25. *If* $\Pi(\mathscr{L}_{G_1}(\Sigma_*(G_1)))$ *and* $\Pi(\mathscr{L}_{G_2}(\Sigma_*(G_2)))$ *are Lagrangian equivalent, then* $\mathscr{L}_{G_1}(\Sigma_*(G_1))$ *and* $\mathscr{L}_{G_2}(\Sigma_*(G_2))$ *are* s-P-*Legendrian equivalent. It follows that total BR-caustics*

$$C(W_1, \mathcal{S}_1),\ C(W_2, \mathcal{S}_2)$$

and BR-Maxwell sets

$$M(W_1, \mathcal{S}_1),\ M(W_2, \mathcal{S}_2)$$

are diffeomorphic as set germs, respectively.

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