Differential relations for the largest root distribution of complex non-central Wishart matrices

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Abstract.

A holonomic system for the probability density function of the largest eigenvalue of a non-central complex Wishart distribution with identity covariance matrix is derived. Furthermore a new determinantal formula for the probability density function is derived (for the dimensions $m = 2, 3$) or conjectured.

§1.   Introduction

The Wishart distribution is an important higher dimensional generalization of the χ^2-distribution. In many applications the distribution of *roots* (i.e., eigenvalues) of Wishart matrices are needed (see references in Hashiguchi et al. [2]). In this paper we consider complex non-central Wishart matrices, which are important for applications to performance evaluation of wireless communication systems (Siriteanu et al. [7], [8]). The purpose of this paper is to give differential relations for the largest root distribution of complex non-central Wishart matrices based on the result of Kang and Alouini [4].

Suppose we take n random vectors $x_i \in \mathbb{C}^m$, $i = 1, \ldots, n$, independently drawn from an m-variate complex Gaussian distribution $\mathcal{CN}(v_i, \Sigma)$, with the mean vector v_i and the covariance matrix Σ. We put those vectors into $n \times m$ matrices X and V. The distribution of the random (symmetric, positive definite) $m \times m$ scatter matrices $S = X^*X$ defines the complex *Wishart distribution* $\mathcal{W}_m(\Sigma, V^*V\Sigma^{-1}, n)$ with degrees of freedom n, covariance matrix Σ and the non-centrality parameter matrix $V^*V\Sigma^{-1}$. We are interested in the distribution of largest root of S.

In the special case $m = 1$, we have the distribution of the value $|x_1|^2 + \ldots + |x_n|^2$. In the $\mathbb{R}$-valued case, this is the χ^2-distribution. The central χ^2-distribution ($V = 0$) is a special case of the gamma distribution.

The distribution of the largest root of the $\mathbb{R}$-valued central Wishart distribution is known, Muirhead [5]. The probability distribution function for the largest root is expressed in terms of a matrix hypergeometric function:

$$(1) \quad \frac{\Gamma_m\left(\frac{m+1}{2}\right)(\det \Sigma)^{-\frac{n}{2}}}{\Gamma_m\left(\frac{n+m+1}{2}\right)} \exp\left(-\frac{x}{2}\operatorname{tr}\Sigma^{-1}\right)\left(\frac{x}{2}\right)^{\frac{nm}{2}} {}_1\mathrm{F}_1\left(\begin{array}{c}\frac{m+1}{2} \\ \frac{n+m+1}{2}\end{array}\middle|\ \frac{x}{2}\Sigma^{-1}\right).$$

Here $\Gamma_m(z) = \pi^{\frac{1}{4}m(m-1)}\prod_{i=1}^{m}\Gamma\left(z - \frac{i-1}{2}\right)$ is called the multivariate Gamma function and the ${}_1\mathrm{F}_1(M)$ function is defined in terms of symmetric functions (*zonal polynomials*) of the eigenvalues of M; Constantine [1], James [3].

A holonomic system for ${}_1\mathrm{F}_1\left(\begin{array}{c}a\\c\end{array}\middle|\ M\right)$ in terms of the eigenvalues λ_i of M was derived by Muirhead [5]. For $i \in \{1, 2, \ldots, m\}$ we have

$$(2) \quad \lambda_i \frac{\partial^2}{\partial\lambda_i^2} + (c - \lambda_i)\frac{\partial}{\partial\lambda_i} + \frac{1}{2}\sum_{j=0, j\neq i}^{m}\frac{\lambda_j}{\lambda_i - \lambda_j}\left(\frac{\partial}{\partial\lambda_i} - \frac{\partial}{\partial\lambda_j}\right) - a.$$

Efficiency of the holonomic gradient method was demonstrated by Hashiguchi et al. [2].

In Section 2.2 we derive differential relations for the density function of the largest root of complex non-central Wishart matrices with the identity covariance matrix $\Sigma = \mathrm{Id}$. Additionally we assume the Gaussian distribution to be *circularly symmetric*; see [6], [11, Complex normal distribution]). A conjectural formula is given in Section 2.3. Later sections are devoted to proofs and additional results for $m \leqslant 3$.

§2. Setting and the contributions

The cumulative distribution function $\mathcal{F}_{n,m}(x)$ for the largest root x in the circularly symmetric case was derived by Kang and Alouini [4]. Let us recall the hypergeometric function

$$(3) \quad {}_0\mathrm{F}_1\left(\begin{array}{c}z\\n\end{array}\right) := {}_0\mathrm{F}_1\left(\begin{array}{c}-\\n\end{array}\middle|\ z\right) = \sum_{k=0}^{\infty}\frac{x^k}{(n)_k\, k!},$$

where $(n)_k = n(n+1)\ldots(n+k-1)$ is the Pochhammer symbol [11]. This function is related to the non-central χ^2-distribution and the modified

Bessel function [9, §9]

$$(4) \qquad I_n(z) = \frac{(z/2)^n}{n!} {}_0\mathrm{F}_1\left(\frac{z^2/4}{n+1}\right).$$

We introduce the integral

$$(5) \qquad H_n^k(x,y) = \int_0^x t^k\,\mathbf{e}^{-t}\,{}_0\mathrm{F}_1\left(\frac{ty}{n}\right)dt$$

related to the Marcum Q-function [9, §9]

$$(6) \qquad Q_n(x,y) = \frac{\mathbf{e}^{-x}}{(n-1)!} \int_y^\infty t^{n-1}\mathbf{e}^{-t}\,{}_0\mathrm{F}_1\left(\frac{xt}{n}\right)dt.$$

2.1. The distribution functions

Let $\lambda_1,\ldots,\lambda_m$ be the eigenvalues of V^*V. The Kang–Alouini distribution function for the largest root of $\mathcal{W}_m(\mathrm{Id},V^*V,n)$ is

$$(7)$$
$$\mathcal{F}_{n,m}(x,\lambda_1,\ldots,\lambda_m) = \frac{\mathbf{e}^{-\lambda_1-\ldots-\lambda_m}}{\{(n-m)!\}^m \displaystyle\prod_{i=1}^m \prod_{j>i}^m (\lambda_i-\lambda_j)}\, \det\left(H_{n-m+1}^{n-j}(x,\lambda_i)\,\rangle_{i=1}^m\right).$$

Here $\rangle_{i=1}^m$ indicates the i-th row of an $m \times m$ matrix, with the column index implicitly taken to be j.

The probability density function is

$$(8) \qquad \psi_{n,m}(x,\lambda_1,\ldots,\lambda_m) = \frac{\partial}{\partial x}\,\mathcal{F}_{n,m}(x,\lambda_1,\ldots,\lambda_m)$$

$$(9) \qquad = \frac{\mathbf{e}^{-\lambda_1-\ldots-\lambda_m}}{\{(n-m)!\}^m \displaystyle\prod_{i=1}^m \prod_{j>i}^m (\lambda_i-\lambda_j)}\, \mathcal{R}_{n,m}(x,\lambda_1,\ldots,\lambda_m),$$

where

$$(10) \qquad \mathcal{R}_{n,m}(x,\lambda_1,\ldots,\lambda_m) = \frac{\partial}{\partial x}\,\det\left(H_{n-m+1}^{n-j}(x,\lambda_i)\,\rangle_{i=1}^m\right)$$

$$(11) \qquad = \mathbf{e}^{-x}\sum_{k=1}^m {}_0\mathrm{F}_1\left(\frac{x\lambda_k}{n-m+1}\right)\det\left(\begin{matrix}H_{n-m+1}^{n-j}(x,\lambda_i)\,\rangle_{i\neq k}^{i\leqslant m}\\ x^{n-j}\qquad\rangle_{i=k}\end{matrix}\right).$$

For example, an expanded expression for $m = 3$ is

$$\mathcal{R}_{n,3}(x, \lambda_1, \lambda_2, \lambda_3) =$$

$$\mathbf{e}^{-x} {}_0\mathrm{F}_1\begin{pmatrix} x\lambda_1 \\ n-2 \end{pmatrix} \det \begin{pmatrix} x^{n-1} & x^{n-2} & x^{n-3} \\ H_{n-2}^{n-1}(x, \lambda_2) & H_{n-2}^{n-2}(x, \lambda_2) & H_{n-2}^{n-3}(x, \lambda_2) \\ H_{n-2}^{n-1}(x, \lambda_3) & H_{n-2}^{n-2}(x, \lambda_3) & H_{n-2}^{n-3}(x, \lambda_3) \end{pmatrix}$$

$$+ \mathbf{e}^{-x} {}_0\mathrm{F}_1\begin{pmatrix} x\lambda_2 \\ n-2 \end{pmatrix} \det \begin{pmatrix} H_{n-2}^{n-1}(x, \lambda_1) & H_{n-2}^{n-2}(x, \lambda_1) & H_{n-2}^{n-3}(x, \lambda_1) \\ x^{n-1} & x^{n-2} & x^{n-3} \\ H_{n-2}^{n-1}(x, \lambda_3) & H_{n-2}^{n-2}(x, \lambda_3) & H_{n-2}^{n-3}(x, \lambda_3) \end{pmatrix}$$

$$+ x^{n-3}\mathbf{e}^{-x} {}_0\mathrm{F}_1\begin{pmatrix} x\lambda_3 \\ n-2 \end{pmatrix} \det \begin{pmatrix} H_{n-2}^{n-1}(x, \lambda_1) & H_{n-2}^{n-2}(x, \lambda_1) & H_{n-2}^{n-3}(x, \lambda_1) \\ H_{n-2}^{n-1}(x, \lambda_2) & H_{n-2}^{n-2}(x, \lambda_2) & H_{n-2}^{n-3}(x, \lambda_2) \\ x^2 & x & 1 \end{pmatrix}.$$

2.2. Main results

The main result of this paper is a holonomic system of differential equations for $\mathcal{R}_{n,m}(x, \lambda_1, \ldots, \lambda_m)$, for any dimension m. It is formulated in the following two theorems. The first one introduces a holonomic system with the differentiations $\partial/\partial\lambda_i$ only. Theorem 2.2 allows to introduce or eliminate $\partial/\partial x$.

Recall [10] that a *least common left multiple* (LCLM) of several differential operators $\mathcal{L}_1, \ldots, \mathcal{L}_k$ in the Weyl algebra $\mathbb{C}(x, y)\langle \partial/\partial y \rangle$ is a differential operator $\mathcal{L}^*$ of minimal order such that $\mathcal{L}^*$ is a left multiple of any $\mathcal{L}_j$, $j \in \{1, \ldots, k\}$. An alternative defining property is that $\mathcal{L}^*Y = 0$ is a differential equation of minimal order such that all Picard-Vessiot [10] solutions of $\mathcal{L}_j Y = 0$ are solutions of $\mathcal{L}^*Y = 0$.

Theorem 2.1. *Let us define the differential operators*

$$(12) \qquad \mathcal{P}_M[y] = y\frac{\partial^2}{\partial y^2} + (M+1)\frac{\partial}{\partial y} - x,$$

$$(13) \qquad \mathcal{Q}_{N,M}[y] = y\frac{\partial^3}{\partial y^3} + (M - y + 2)\frac{\partial^2}{\partial y^2} - (x + N + 1)\frac{\partial}{\partial y} + x.$$

Let us denote $\mathcal{T}_1[y] = \mathcal{P}_{n-m}[y]$ and

$$(14) \qquad \mathcal{T}_j[y] = \mathcal{Q}_{n-m+j, n-m}[y] \qquad for\ 2 \leqslant j \leqslant m.$$

The following operators annihilate $\mathcal{R}_{n,m}(x, \lambda_1, \ldots, \lambda_m)$:

    (i)   The products $\mathcal{T}_k[\lambda_1] \cdots \mathcal{T}_k[\lambda_m]$, for $k = 1, 2, \ldots, m$.

    (ii)   The least common left multiples $\mathrm{LCLM}(\mathcal{T}_1[\lambda_k], \ldots, \mathcal{T}_m[\lambda_k])$ with $k = 1, 2, \ldots, m$.

Theorem 2.2. *This second order operator annihilates the function* $\mathcal{R}_{n,m}(x, \lambda_1, \ldots, \lambda_m)$:

$$(15) \quad x\frac{\partial}{\partial x} + \sum_{k=1}^{m}\left(\lambda_k\frac{\partial^2}{\partial \lambda_k^2} + (n - m + 1 - \lambda_k)\frac{\partial}{\partial \lambda_k} - n\right) + \frac{m(m-1)}{2} + 1.$$

The theorems are proved in §3.1 and §3.3. To get differential equations for the density function $\psi_{n,m}(x, \lambda_1, \ldots, \lambda_m)$, the presented operators must be modified by the *gauge translations*

$$(16) \qquad\qquad \frac{\partial}{\partial \lambda_i} \mapsto \frac{\partial}{\partial \lambda_i} + 1 + \sum_{j \neq i}\frac{1}{\lambda_i - \lambda_j}.$$

This is a standard technique to account for the front factor in (9).

By its determinantal form (10), the target function $\mathcal{R}_{n,m}(x, \lambda_1, \ldots, \lambda_m)$ is a non-logarithmic and anti-symmetric function. In particular, it is multiplied by the sign $(-1)^\sigma$ under a permutation σ of the variables $\lambda_1, \ldots, \lambda_m$

Theorem 2.3.   (*i*)   *The differential operators of Theorem* 2.1 *annihilating* $\mathcal{R}_{n,m}(x, \lambda_1, \ldots, \lambda_m)$ *generate a holonomic system of rank* $2m! \cdot 3^{m-1}$.

(*ii*)   *Let* $\mathcal{S}$ *denote the subspace of anti-symmetric solutions in a full solution space (of dimension* $2m! \cdot 3^{m-1}$). *Then* $\dim \mathcal{S} = 2 \cdot 3^{m-1}$.

(*iii*)   *The subspace of non-logarithmic anti-symmetric solutions has the dimension* 2^{m-1}.

(*iv*)   *There exists a holonomic system of rank* $\leqslant 3^m - 1$ *defined over* $\mathbb{Q}(x, \lambda_1, \ldots, \lambda_m)$ *and annihilating* $\mathcal{R}_{n,m}(x, \lambda_1, \ldots, \lambda_m)$.

This theorem is proved in §3.2. Our computations for $m = 2$, $m = 3$ indicate that the lower rank system has markedly more complicated equations and singularities. These computations are presented in §4.

2.3. A conjectural formula

We were led to Theorem 2.1 after computing holonomic systems for $\mathcal{R}_{n,2}(x, \lambda_1, \lambda_2)$ of rank 12 and 8, elimination of $\partial/\partial x$, $\partial/\partial\lambda_2$ and observing a differential operator in $\partial/\partial\lambda_1$ of order 5 with a simple LCLM factorization. Computations for $\mathcal{R}_{n,3}(x, \lambda_1, \lambda_2, \lambda_3)$ led to holonomic systems of rank 108 and 26 cumbersomely, but probing for a differential operator in only $\partial/\partial\lambda_1$ quickly gave one of relatively low order 8 and a remarkable LCLM factorization into operators of order 2 or 3. Theorem 2.1 establishes continuation of this pattern.

The solution space of the holonomic systems in Theorem 2.1 is highly factorizable by specificity of the presented generators. Particular solutions are

$$(17) \qquad \det \left(Y_j(x, \lambda_i) \, \rangle_{i=1}^m \right),$$

where $Y_j(x, y)$ is a solution $\mathcal{T}_j[y] \, Y_j = 0$. The LCLM operator in *(ii)* annihilates the kth row of this matrix, while the product in *(i)* annihilates the kth column. Based on obtained new expressions for $\mathcal{R}_{n,2}(x, \lambda_1, \lambda_2)$, $\mathcal{R}_{n,3}(x, \lambda_1, \lambda_2, \lambda_3)$, solutions of $\mathcal{Q}_{N,M}[y]$ and their recurrences, we conjecture that $\mathcal{R}_{n,m}(x, \lambda_1, \ldots, \lambda_m)$ has a determinantal expression (17). Here is a formulation in the transposed form.

Conjecture 2.4. *Let us define*

$$(18) \qquad G_{n,2}(x, y) = n \, {}_0\mathrm{F}_1 \begin{pmatrix} xy \\ n \end{pmatrix} + y \, {}_0\mathrm{F}_1 \begin{pmatrix} xy \\ n+1 \end{pmatrix}$$
$$+ (x - y - n + 1) \, \mathrm{e}^y \int_y^\infty \mathrm{e}^{-t} \, {}_0\mathrm{F}_1 \begin{pmatrix} xt \\ n+1 \end{pmatrix} dt.$$

For $m \geqslant 2$, we recursively define

$$(19) \quad G_{n,m+1}(x, y) = \left(-y \frac{\partial^2}{\partial y^2} - (n - m + 1) \frac{\partial}{\partial y} + x + m \right) G_{n,m}(x, y).$$

We conjecture that

$$(20) \qquad \mathcal{R}_{n,m}(x, \lambda_1, \ldots, \lambda_m) = C(x) \det \begin{pmatrix} {}_0\mathrm{F}_1 \begin{pmatrix} x\lambda_i \\ n-m+1 \end{pmatrix} \, \rangle_{j=1} \\ G_{n-m+j,j}(x, \lambda_i) \, \rangle_{j=2}^m \end{pmatrix}$$

with

$$(21) \qquad C(x) = \frac{(n - m + 1) \, x^{mn - \binom{m}{2} - 1} \, \mathrm{e}^{-mx}}{\prod_{k=1}^m (n - k + 1)^k}.$$

Note that ${}_0\mathrm{F}_1 \begin{pmatrix} x\,y \\ n-m+1 \end{pmatrix}$ is a solution of $\mathcal{P}_{n-m}[y] \, Y = 0$. As we show in §3.4, the function $G_{n,m}(x, y)$ is a solution of $\mathcal{Q}_{n-m+j,n-m}[y] = 0$ for any integers $n \geqslant m > 0$. Recurrence (19) stems from §3.4 as well.

Notably, the integral in (18) is complementary to $H_{n+1}^0(y, x)$.

The conjecture has been fully checked for $m = 2$ and $m = 3$, as described in §4.2 and §4.4. Also, the front factor (21) has been confirmed for $m = 4$. The conjecture happens to be true for $m = 1$ as well. In §4.2 we specifically prove

$$(22)$$
$$\psi_{n,2}(x, \lambda_1, \lambda_2) = \frac{x^{2n-2} \, \mathrm{e}^{-\lambda_1 - \lambda_2 - 2x}}{n! \, (n-2)! \, (\lambda_1 - \lambda_2)} \det \begin{pmatrix} {}_0\mathrm{F}_1 \begin{pmatrix} x\lambda_1 \\ n-1 \end{pmatrix} & {}_0\mathrm{F}_1 \begin{pmatrix} x\lambda_2 \\ n-1 \end{pmatrix} \\ G_{n,2}(x, \lambda_1) & G_{n,2}(x, \lambda_2) \end{pmatrix}.$$

If $\lambda_1 = \lambda_2$, application of l'Hospital's rule leads to differentiating a matrix column. For comparison, expression (8) is

(23)

$$\psi_{n,2}(x,\lambda_1,\lambda_2) = \frac{\mathbf{e}^{-\lambda_1-\lambda_2}}{\{(n-2)!\}^2}\ \frac{\dfrac{\partial}{\partial x} \det \left(\begin{array}{cc} H_{n-1}^{n-1}(x,\lambda_1) & H_{n-1}^{n-1}(x,\lambda_2) \\ H_{n-1}^{n-2}(x,\lambda_1) & H_{n-1}^{n-2}(x,\lambda_2) \end{array} \right)}{\lambda_1 - \lambda_2}.$$

Not only the differentiation $\partial/\partial x$ is avoided, but the integral in (18) is numerically preferable to the H_{n-1}^{n-1}, H_{n-1}^{n-2} functions.

Significance of the conjectured formula is that it would utilize the factorization structure of the holonomic system in Theorem 2.1. Applying the holonomic gradient method to the entries of the conjectured matrix would be more efficient than employing the same method for the large multi-variate holonomic system.

2.4. Auxiliary integrals

To get the holonomic system, we use recurrences for $H_n^k(x,y)$ in (5) and the generalization

(24)
$$H_n^{k,\ell}(x,y) = \int_0^x \mathbf{e}^{-t}\, t^k (x-t)^\ell\ _0F_1\left(\begin{array}{c} t\,y \\ n \end{array} \right) dt.$$

Surely, $H_n^{k,0}(x,y) = H_n^k(x,y)$. These differentiations are straightforward:

(25)
$$\frac{\partial}{\partial x} H_n^k(x,y) = x^k \mathbf{e}^{-x}\ _0F_1\left(\begin{array}{c} xy \\ n \end{array} \right),$$

(26)
$$\frac{\partial}{\partial y} H_n^{k,\ell}(x,y) = \frac{1}{n} H_{n+1}^{k+1,\ell}(x,y).$$

Lemma 2.5. *If $k > 0$, then*

(27)
$$H_{n-1}^k(x,y) = H_n^k(x,y) + \frac{y}{n(n-1)} H_{n+1}^{k+1}(x,y),$$

(28)
$$k\,H_n^{k-1}(x,y) = H_n^k(x,y) - \frac{y}{n} H_{n+1}^k(x,y) + x^k \mathbf{e}^{-x}\ _0F_1\left(\begin{array}{c} xy \\ n \end{array} \right),$$

(29)
$$k\,H_{n-1}^{k-1}(x,y) = \frac{n-y-1}{n-1} H_n^k(x,y) + \frac{y}{n(n-1)} H_{n+1}^{k+1}(x,y)$$
$$+ x^k \mathbf{e}^{-x}\ _0F_1\left(\begin{array}{c} xy \\ n-1 \end{array} \right).$$

Proof.   The first formula follows from the recurrence

$$(30) \qquad {}_0F_1\left(\genfrac{}{}{0pt}{}{z}{n-1}\right) = {}_0F_1\left(\genfrac{}{}{0pt}{}{z}{n}\right) + \frac{z}{n(n-1)}\,{}_0F_1\left(\genfrac{}{}{0pt}{}{z}{n+1}\right)$$

that is equivalent to the hypergeometric equation (with $a = n$)

$$(31) \qquad z\,Y''(z) + a\,Y'(z) - Y(z) = 0$$

for ${}_0F_1(z)$. The second formula follows after integration by parts

$$(32) \qquad H_n^k(x,y) = -\int_0^x t^k\, {}_0F_1\left(\genfrac{}{}{0pt}{}{ty}{n}\right) d\mathbf{e}^{-t}.$$

The last formula follows after substituting $n \mapsto n-1$ in (28) and eliminating $H_{n-1}^k(x,y)$ using (27).   Q.E.D.

Formula (29) is equivalent to the differential equation

$$(33) \quad \left(y\frac{\partial^2}{\partial y^2} + (n-y)\frac{\partial}{\partial y} - k - 1\right) H_n^k(x,y) = -x^{k+1}\mathbf{e}^{-x}\,{}_0F_1\left(\genfrac{}{}{0pt}{}{xy}{n}\right).$$

We can obtain recurrences that shift only the indices k or n, presented in the following lemma. Remarkably, both formulas lose an H-term when $k = n - 1$. The simplified formulas are readily applicable to the $j = m$, $j = m - 1$ columns in (7).

Lemma 2.6. *If $k > 0$, then*

$$n(n-1)H_{n-1}^k(x,y) = n\,(y+n-1)H_n^k(x,y) + y\,(k-n+1)H_{n+1}^k(x,y)$$

$$(34) \qquad\qquad - yx^{k+1}\mathbf{e}^{-x}\,{}_0F_1\left(\genfrac{}{}{0pt}{}{xy}{n+1}\right),$$

$$H_n^{k+1}(x,y) = (y-n+2k+2)H_n^k(x,y) + k\,(n-k-1)H_n^{k-1}(x,y)$$

$$(35) \qquad - (n-1)x^k\mathbf{e}^{-x}\,{}_0F_1\left(\genfrac{}{}{0pt}{}{xy}{n-1}\right) + (k-x)x^k\mathbf{e}^{-x}\,{}_0F_1\left(\genfrac{}{}{0pt}{}{xy}{n}\right).$$

Proof.   The first formula is obtained by eliminating $H_n^{k-1}(x,y)$, $H_{n-1}^{k-1}(x,y)$ from these 3 equations: (28), the shift $k \mapsto k-1$ of (27), and the shift $n \mapsto n-1$ of (28). For the second formula, we eliminate $H_{n+1}^k(x,y)$, $H_{n-1}^{k-1}(x,y)$, $H_{n+1}^{k+1}(x,y)$ from these 4 equations: (28), (29), the shift $k \mapsto k-1$ of (27), and the shift $k \mapsto k+1$ of (28).   Q.E.D.

Lemma 2.7. *The following recurrences with two H-terms hold, for $n > 0$:*

$$(36) \quad (n-1)H^{n-1}_{n-1}(x,y) = (y+n-1)H^{n-1}_n(x,y) - \frac{yx^n}{n}\mathbf{e}^{-x}\,_0\mathrm{F}_1\left(\frac{xy}{n+1}\right),$$

$$(37) \qquad H^n_n(x,y) = (y+n)H^{n-1}_n(x,y) - \frac{yx^n}{n}\mathbf{e}^{-x}\,_0\mathrm{F}_1\left(\frac{xy}{n+1}\right)$$

$$- x^n\mathbf{e}^{-x}\,_0\mathrm{F}_1\left(\frac{xy}{n}\right),$$

$$(38) \qquad H^n_{n+1}(x,y) = n\,H^{n-1}_n(x,y) - x^n\mathbf{e}^{-x}\,_0\mathrm{F}_1\left(\frac{xy}{n+1}\right).$$

Proof. The first two formulas constitute the special case $k = n-1$ of the previous lemma. The third formula is obtained by eliminating $H^n_n(x,y)$ from (37) and the shift $n \mapsto n+1$ of (36). Q.E.D.

Formula (38) is comparable to the recurrence for the incomplete gamma function $\gamma(a,x) = \int_0^x t^{a-1}\mathbf{e}^{-t}\,dt$:

$$(39) \qquad \gamma(a+1,z) = a\,\gamma(a,z) - x^a\mathbf{e}^{-x}.$$

The presented recurrences can be used to express all matrix entries in (7) in terms of $H^{n-m}_{n-m+1}(x,\lambda_i)$ and two $_0\mathrm{F}_1$ functions.

Proposition 2.8. *Any function $H^k_n(x,y)$ with integer $n > 1$ and $k \geqslant n-1$ can be expressed as a $\mathbb{Q}(x,y)$-linear combination of*

$$H^{N-1}_N(x,y), \quad x^N\mathbf{e}^{-x}\,_0\mathrm{F}_1\left(\frac{xy}{N}\right), \quad x^N\mathbf{e}^{-x}\,_0\mathrm{F}_1\left(\frac{xy}{N+1}\right)$$

for any $N > 1$. The same statement applies to the derivatives of $H^k_n(x,y)$ of any order (with respect to x,y).

Proof. Lemma 2.7 proves the first claim for $k = n-1$ and $k = n$. Lemma 2.6 extends the statement to larger k. Differentiation rules (25)–(26) imply the second claim. Q.E.D.

Recurrence relations $H^{k,\ell}_n(x,y)$ are obtained by a straightforward extension of the results for $H^k_n(x,y)$.

Lemma 2.9. *For $k > 0$, $\ell > 0$ we have*

$$(40) \quad x\,H^{k,\ell}_n(x,y) = H^{k+1,\ell}_n(x,y) + H^{k,\ell+1}_n(x,y),$$

$$(41) \quad H^{k,\ell}_{n-1}(x,y) = H^{k,\ell}_n(x,y) + \frac{y}{n(n-1)}\,H^{k+1,\ell}_{n+1}(x,y),$$

$$(42) \quad H^{k,\ell}_n(x,y) = k\,H^{k-1,\ell}_n(x,y) - \ell\,H^{k,\ell-1}_n(x,y) + \frac{y}{n}\,H^{k,\ell}_{n+1}(x,y).$$

Proof. The first recurrence is obtained by splitting

$$(x - t)^\ell = x(x - t)^{\ell-1} - t(x - t)^{\ell-1}$$

in the defining integral (24). The other two equations follow similarly as (27)–(28), from the three-term recurrence for the $_0F_1$ function and, respectively, by integration by parts.　　　　　　　　　　Q.E.D.

Lemma 2.10. *For $k > 0$, $\ell > 0$ we have*

$$(43) \qquad (n-1)H_{n-1}^{k-1,\ell}(x,y) = H_n^{k,\ell}(x,y) + \ell\, H_n^{k,\ell-1}(x,y)$$
$$+ (n - k - 1)H_n^{k-1,\ell}(x,y),$$

$$(44) \qquad k\, H_k^{k-1,\ell}(x, y) = H_{k+1}^{k,\ell}(x, y) + \ell\, H_{k+1}^{k,\ell-1}(x, y).$$

Proof. First we show this intermediate equation:

$$(45) \qquad H_{n-1}^{k,\ell+1}(x,y) + \frac{y}{n(n-1)}\, H_{n+1}^{k+2,\ell}(x,y) =$$
$$H_n^{k,\ell+1}(x,y) + \frac{xy}{n(n-1)}\, H_{n+1}^{k+1,\ell}(x,y).$$

It is annihilated by the relations of Lemma 2.9 as follows. The two terms with denominators are eliminated by (41) and its shift $k \mapsto k + 1$. Then elimination of $H_n^{k,\ell}(x, y)$, $H_{n-1}^{k,\ell}(x, y)$ by (40) and its shift $n \mapsto n - 1$ leaves no terms.

Now multiply equation (45) by $(n-1)$ and apply the shifts $k \mapsto k-1$, $\ell \mapsto \ell-1$. Then subtract (42) and eliminate the terms with denominators using the shifted version $n \mapsto n + 1$, $\ell \mapsto \ell - 1$ of (40). The result is (43). The second claimed recurrence is the special case $n = k + 1$ of the first one.　　　　　　　　　　Q.E.D.

§3.　Proofs and analysis

The motivation for this article was potential application of the holonomic gradient method [2] to computation of the probability density function $\psi_{n,m}(x, \lambda_1, \ldots, \lambda_m)$. Our main results are formulated in §2.2 for the function $\mathcal{R}_{n,m}(x, \lambda_1, \ldots, \lambda_m)$ as in (9). The obtained holonomic systems are more complicated than expected, in the simplest cases $m \leqslant 3$ as well. As mentioned at the end of §2.3, application of the holonomic gradient method to the entries of the conjectured matrix in (20) should be more effective than employment of multi-variate holonomic systems.

This section proves the main results presented in §2.2. Additionally, §3.2 discusses the obtained holonomic systems, and §3.4 presents explicit solutions and recurrences relevant to Conjecture 2.4.

3.1.   Proof of Theorem 2.1

The product $\mathcal{T}_1[\lambda_1]\dots\mathcal{T}_1[\lambda_m]$ annihilates $\mathcal{R}_{n,m}(x,\lambda_1,\dots,\lambda_m)$ because the operators $\mathcal{T}_1[\lambda_i] = \mathcal{P}_{n,m}[\lambda_i]$ annihilate the front factor ${}_0\mathrm{F}_1\left(\frac{x\lambda_i}{n-m+1}\right)$ of each term in (10). Remarkably, the other operators annihilate each term in (10) as well.

Consider now the action of $\mathcal{T}_m[y] = \mathcal{Q}_{n,m}[y]$. We have

$$\mathcal{T}_m[\lambda_1]\dots\mathcal{T}_m[\lambda_m]\,\mathcal{R}_{n,m}(x,\lambda_1,\dots,\lambda_m) =$$

$$(46) \qquad \mathbf{e}^{-x}\sum_{k=1}^{m}\mathcal{T}_m[\lambda_k]{}_0\mathrm{F}_1\left(\frac{x\lambda_k}{n-m+1}\right)\det\left(\begin{matrix}\mathcal{T}_m[\lambda_i]H^{n-j}_{n-m+1}(x,\lambda_i)\ \rangle^{i\leqslant m}_{i\neq k}\\ x^{n-j}\qquad\qquad\qquad \rangle_{i=k}\end{matrix}\right).$$

We claim that all m determinants are zero, because the matrices have a specific kernel vector

$$(47) \qquad\qquad \left(\ \binom{m-1}{j-1}(-x)^j\ \rangle^m_{j=1}\ \right)^T.$$

The scalar product of this vector with the $i=k$ rows

$$(48) \qquad\qquad \left(x^{n-1},x^{n-2},\dots,x^{n-m}\right)^T$$

equals 0 straightforwardly, since $\sum_{j=1}^m(-1)^j\binom{m-1}{j-1} = 0$ as well known. We want to show

$$(49) \qquad\qquad \sum_{j=1}^m\binom{m-1}{j-1}(-x)^j\ \mathcal{T}_m[y]H^{n-j}_{n-m+1}(x,y) = 0.$$

By applying the differentiation

$$(50) \qquad\qquad \mathcal{T}_m[y] = \left(\frac{\partial}{\partial y} - 1\right)\mathcal{P}_{n,m}[y] - m\frac{\partial}{\partial y}$$

and recurrence of Lemma 2.5,

$$\mathcal{T}_m[y]H^{n-j}_{n-m+1}(x,y) = x\,H^{n-j}_{n-m+1}(x,y) - \frac{x+j-1}{n-m+1}\,H^{n-j+1}_{n-m+2}(x,y)$$

$$(51) \qquad\qquad - x^{n-j+2}\mathbf{e}^{-x}{}_0\mathrm{F}_1\left(\frac{x\,y}{n-m+2}\right).$$

We ignore the last term, for linear dependency with the $i=k$ row. By permuting the summation and integration, our target is

$$(52) \qquad\qquad \int_0^x\mathbf{e}^{-t}t^{n-m}(x-t)^{m-1}P(x,t)dt = 0$$

with

$$(53)\ \ P(x,t) = (x-t)\,{}_0\mathrm{F}_1\left({y\,t \atop n-m+1}\right) - \frac{t\,(x+m-t)}{n-m+1}\,{}_0\mathrm{F}_1\left({y\,t \atop n-m+2}\right).$$

This integral is equivalent to the recurrence relation
$$(54)$$
$$(n-m+1)\,H^{n-m,\,m}_{n-m+1}(x,y) = H^{n-m+1,m}_{n-m+2}(x,y) + m\,H^{n-m+1,m-1}_{n-m+2}(x,y)$$

that is equivalent to (43). The claimed relation (49) follows.

Other products $\mathcal{T}_q[\lambda_1]\cdots\mathcal{T}_q[\lambda_m]$ with $2 \leqslant q < m$ annihilate the function $\mathcal{R}_{n,m}(x,\lambda_1,\ldots,\lambda_m)$ similarly, with the kernel vectors

$$(55)\qquad\qquad \left(\underbrace{0,\ldots,0}_{m-q},\ \left({q-1 \atop j-m+q-1}\right)(-x)^j\ \rangle^m_{j=m-q+1}\right)^T$$

of the m matrices

$$\left({\mathcal{T}_q[\lambda_i]H^{n-j}_{n-m+1}(x,\lambda_i)\ \rangle^{i\leqslant m}_{i\neq k} \atop x^{n-j}\qquad\qquad \rangle_{i=k}}\right)$$

in an expression like in (46).

For $\ell = 1,\ldots,m$, the $\mathrm{LCLM}(\mathcal{T}_1[\lambda_\ell],\ldots,\mathcal{T}_m[\lambda_\ell])$ transforms the function $\mathcal{R}_{n,m}(x,\lambda_1,\ldots,\lambda_m)$ to

$$\mathbf{e}^{-x}\sum_{k=1,\neq\ell}^m {}_0\mathrm{F}_1\left({x\lambda_k \atop n-m+1}\right)\det\left({H^{n-j}_{n-m+1}(x,\lambda_i)\quad \rangle^{i\leqslant m}_{i\neq k,\ell} \atop {\mathrm{LCLM}\,H^{n-j}_{n-m+1}(x,\lambda_i)\ \rangle_{i=\ell} \atop x^{n-j}\qquad\qquad \rangle_{i=k}}}\right).$$

The $i = \ell$ row is proportional to the $i = k$ row vector (48), because:

- For $q = 2,\ldots,m$, the operator $\mathcal{T}_q[\lambda_\ell]$ makes the $i = \ell$ row "orthogonal" to (55).
- The LCLM is a left factor of each $\mathcal{T}_q[\lambda_\ell]$, thus preserves the "orthogonality" property.
- The vector (48) is the only vector "orthogonal" to the $m-1$ independent vectors.

Hence the LCLM operators annihilate all m terms of $\mathcal{R}_{n,m}(x,\lambda_1,\ldots,\lambda_m)$.

3.2. Holonomic systems

Here we prove Theorem 2.3. To simplify technical details, we posit that differential Galois theory [10] extends straightfowardly to the considered holonomic systems.

Let $\mathcal{M}$ denote the system of differential operators in Theorem 2.1 annihilating $\mathcal{R}_{n,m}(x,\lambda_1,\ldots,\lambda_m)$. It is a holonomic system, because the

LCLM operators bound the order in each $\partial/\partial\lambda_i$. Since the rank of an LCLM operator equals $3m-1$, a straightforward upper bound for the holonomic rank is $(3m-1)^m$. After a choice of (Picard-Vessiot) solution basis for each $\mathcal{T}_k[y]$, the subsystem of LCLM operators has the following straightforward basis of solutions: $g_{j_1}(\lambda_1)\cdots g_{j_m}(\lambda_m)$, where $g_k(y)$ is a basis solution of $\mathcal{T}_k$. Let $\mathcal{B}$ denote this basis of $(3m-1)^m$ functions.

The solution space of $\mathcal{M}$ is considered inside the span of $\mathcal{B}$. The following $2m!\cdot 3^{m-1}$ functions in $\mathcal{B}$ will be solutions of the differential system $\mathcal{M}$: $g_1(\lambda_{j_1})\cdots g_m(\lambda_{j_m})$, where $g_k(y)$ is a basis solution of $\mathcal{T}_k[y]$, and $(j_1,\ldots,j_m)$ is a permutation of $(1,\ldots,m)$. Any other element of $\mathcal{B}$ is not annihilated by at least one operator in *(i)* of Theorem 2.1, and a linear combination of these elements will not be nullified by the same operator(s). The claim *(i)* of Theorem 2.3 follows.

The solution space of $\mathcal{M}$ splits into a direct sum of $2\cdot 3^{m-1}$ subspaces that are invariant under the permutations of $\lambda_1,\ldots,\lambda_m$. Each of these subspaces gives one independent anti-symmetric solution, and the claim *(ii)* follows.

Each operator $\mathcal{T}_j[y]$ has logarithmic solutions at $y=0$. A broad reason is that appearance of $_0F_1\!\left(\begin{smallmatrix}z\\n\end{smallmatrix}\right)$ functions brings the ill-determined functions $_0F_1\!\left(\begin{smallmatrix}z\\2-n\end{smallmatrix}\right)$. More precisely, logarithmic solutions appear in a limit $a\to n$ of the general solution

$$(56) \qquad C'\,_0F_1\left(\begin{matrix}z\\a\end{matrix}\right)+C''\,z^{1-a}\,_0F_1\left(\begin{matrix}z\\2-a\end{matrix}\right)$$

of the hypergeometric equation (31) with generic $a\in\mathbb{C}$. Analysis of local solutions of $\mathcal{T}_k[y]$ at the singularities $y=0$, $y=\infty$ shows that the space of non-logarithmic solutions of $\mathcal{T}_k[y]$ is one-dimensional for $k=1$ and two-dimensional for $k\geqslant 2$. Explicit instances in §3.4 demonstrate this. Similarly as above, the space of non-logarithmic solutions for $\mathcal{M}$ has the dimension $2^{m-1}m!$, and the space of non-logarithmic anti-symmetric solutions has the dimension 2^{m-1}.

Existence of a holonomic system of rank $\leqslant 3^m-1$ of claim *(iv)* follows from Proposition 2.8. It allows to express $\mathcal{R}_{n,m}(x,\lambda_1,\ldots,\lambda_m)$ and its derivatives as $\mathbb{Q}(x,\lambda_1,\ldots,\lambda_m)$-linear combinations of $f_1(\lambda_1)\cdots f_m(\lambda_m)$, where each

$$(57)\quad f_j(\lambda_j)\in\left\{H_n^{n-1}(x,\lambda_j),x^n\mathbf{e}^{-x}{}_0F_1\left(\begin{matrix}x\lambda_j\\n\end{matrix}\right),x^n\mathbf{e}^{-x}{}_0F_1\left(\begin{matrix}x\lambda_j\\n+1\end{matrix}\right)\right\}.$$

These functions generate the space of dimension 3^m, but the term

$$H_n^{n-1}(x,\lambda_1)\cdots H_n^{n-1}(x,\lambda_m)$$

does not appear, because $\mathcal{R}_{n,m}(x, \lambda_1, \ldots, \lambda_m)$ is defined after applying $\partial/\partial x$, and further differentiations will not bring this term back. Examples of these linear expressions are given in §4.3.

Remark 3.1. The determinants in (17) linearly generate the space of anti-symmetric solutions in Theorem 2.3 *(ii)*. They form a Grassmanian-like variety in this space. Taking scalar multiplication of the rows and the whole matrix into account, the dimension of this variety equals $(2-1) + (m-1)(3-1) + 1 = 2m$. Similarly, the subvariety of non-logarithmic anti-symmetric determinantal solutions has the dimension $(1-1) + (m-1) \cdot 1 + 1 = m$. For $m = 2$, comparison with the dimension count in *(iii)* of Theorem 2.3 implies that a determinantal formula like (22) is inevitable.

Remark 3.2. In the proof of Theorem 2.1 we may start with any $m-1$ independent vectors $(v_1^{(q)}, \ldots, v_m^{(q)})$ "orthogonal" to monomial vector $(x, x^2, \ldots, x^m)^T$, and take for $\mathcal{T}_2[y], \ldots, \mathcal{T}_m[y]$ the operators annihilating $\sum_{j=1}^{m} v_j^{(q)} H_{n-m+1}^{n-j}(x, y)$ up to a term proportional to (48). The alternative operators

$$\mathcal{T}_k[\lambda_1] \cdots \mathcal{T}_k[\lambda_m], \qquad \text{LCLM}(\mathcal{T}_1[\lambda_k], \ldots, \mathcal{T}_m[\lambda_k])$$

would generate a holonomic system annihilating $\mathcal{R}_{n,m}(x, \lambda_1, \ldots, \lambda_m)$ by the same reasons. They would have order 3 as well by Proposition 2.8, but they would be more complicated, with additional singularities. For example, taking $m = 3$ and the vector $(1, -x, 0)^T$ gives the differential operator

$$(58) \qquad \mathcal{Q}_{n,3}[y] + \frac{x}{xy + (n-2)(x+2)} \left(-y \frac{\partial^2}{\partial y^2} + y \frac{\partial}{\partial y} + x + 2 \right)$$

instead of $\mathcal{Q}_{n-1,2}[y]$ corresponding to $(0, 1, -x)^T$. As shown in §4.4, the LCLM operators are apparently the same as in Theorem 2.1. This powerfully demonstrates non-uniqueness of LCLM factorization in non-commutative Weyl algebras [11]. But different products in Theorem 2.1*(i)* lead to different holonomic systems, of the same rank though.

3.3.   Proof of Theorem 2.2

We have to prove that

$$(59) \qquad x \frac{\partial}{\partial x} + \sum_{k=1}^{m} \left(\lambda_k \frac{\partial^2}{\partial \lambda_k^2} + (n - m + 1 - \lambda_k) \frac{\partial}{\partial \lambda_k} \right)$$

multiplies $\mathcal{R}_{n,m}(x, \lambda_1, \ldots, \lambda_m)$ by the constant $mn - \binom{m}{2} - 1$. Note that only $\partial/\partial x$ in (59) splits the summation factors in (11) by Leibniz rule,

because the determinants do not depend on the respective λ_k in each summand of (11).

The action of (59) on the front factor $\mathbf{e}^{-x}$ in (11) is multiplication by $-x$. This is compensated by the action on the $_0F_1$ factors in (11), because:

- $x\dfrac{\partial}{\partial x} - \lambda_k \dfrac{\partial}{\partial \lambda_k}$ nullifies $_0F_1(x\lambda_k)$;
- $\lambda_k \dfrac{\partial^2}{\partial \lambda_k^2} + (n-m+1)\dfrac{\partial}{\partial \lambda_k} = \mathcal{P}_{n-m}[\lambda_k] + x.$

The action on the determinants gives

$$
(60) \qquad \mathbf{e}^{-x} \sum_{k=1}^{m} {}_0F_1\begin{pmatrix} x\lambda_k \\ n-m+1 \end{pmatrix} \det\begin{pmatrix} H_{n-m+1}^{n-j}(x,\lambda_i) \ \rangle_{i\neq k}^{i\leqslant m} \\ (n-j)\,x^{n-j} \qquad \rangle_{i=k} \end{pmatrix}
$$

$$
+\,\mathbf{e}^{-x} \sum_{k,\ell=1,\,k\neq \ell}^{m} {}_0F_1\begin{pmatrix} x\lambda_k \\ n-m+1 \end{pmatrix} \det\begin{pmatrix} H_{n-m+1}^{n-j}(x,\lambda_i) & \rangle_{i\neq k,\ell}^{i\leqslant m} \\ (n-j+1)\,H_{n-m+1}^{n-j}(x,\lambda_\ell) \ \rangle_{i=\ell} \\ x^{n-j} & \rangle_{i=k} \end{pmatrix},
$$

because:

       - Applying $\partial/\partial x$ to the rows $i \neq k$ gives linear dependence with the $i=k$ row;
       - Formula (33) implies

$$
(61) \qquad \left(\lambda_\ell \frac{\partial^2}{\partial \lambda_\ell^2} + (n-m+1+\lambda_\ell)\frac{\partial}{\partial \lambda_\ell}\right) H_{n-m+1}^{n-j}(x,\lambda_\ell)
$$

$$
= (n-j+1)\, H_{n-m+1}^{n-j}(x,\lambda_\ell) - x^{n-j+1}\mathbf{e}^{-x}\,{}_0F_1\begin{pmatrix} x\,\lambda_\ell \\ n-m+1 \end{pmatrix};
$$

       - The new terms with $_0F_1(x\lambda_\ell)$ can be ignored by linear combination with the $i=k$ row.

We split the factors $(n-j+1)$ in the rows $i=\ell$ into $(n-j)$ and $(+1)$. The $(+1)$'s aggregate to multiplication of $\mathcal{R}_{n,m}(x,\lambda_1,\ldots,\lambda_m)$ by $m(m-1)/m$. The modified summation (60) becomes a sum of m special instances $c_j = n-j$ of the following lemma:

Lemma 3.3. *For any* $m \times m$ *matrix* $\big(a_{i,j}\,\rangle_{i=1}^m\big)$ *and* m *scalars* $c_1, \ldots, c_m$ *we have*

$$
\sum_{\ell=1}^{m} \det\begin{pmatrix} a_{i,j} & \rangle_{i\neq\ell}^{i\leqslant m} \\ c_j\,a_{i,j} \ \rangle_{i=\ell} \end{pmatrix} = \Big(\sum_{j=1}^{m} c_j\Big)\det\Big(a_{i,j}\,\rangle_{i=1}^m\Big).
$$

Proof. Each of the $m!$ expanded terms of $\det\big(a_{i,j}\big)$ gets multiplied by $c_1,\ldots,c_m$ among the $m \cdot m!$ expanded terms on the left-hand side.
                                                      Q.E.D.

In conclusion, (59) multiplies $\mathcal{R}_{n,m}(x, \lambda_1, \ldots, \lambda_m)$ by

$$m - 1 + \sum_{j=1}^{m}(n-j) = m - 1 + mn - \binom{m+1}{2} = mn - \binom{m}{2} - 1.$$

3.4. Solutions and recurrences

Here we present solutions of the differential equations $\mathcal{Q}_{N,M}[y]Y = 0$ with $N - M \in \{2, 3, 4\}$.

Lemma 3.4. *Let N denote a positive integer, and let*

$$
\begin{aligned}
L_2 &= y - x + N - 1, \\
L_3 &= L_2^2 + 2x - N + 1, \\
L_4 &= L_2^3 + 3\,(2x - N + 1)\,(L_2 - 1) - N + 1.
\end{aligned}
$$

- *A solution of $\mathcal{Q}_{N,N-2}[y]\,Y = 0$ is*

$$(62) \quad Y_{N,2}(x, y) = N\,{}_0\mathrm{F}_1\binom{xy}{N} + y\,{}_0\mathrm{F}_1\binom{xy}{N+1} + L_2\,\mathrm{e}^y\,H_{N+1}^0(y, x).$$

- *A solution of $\mathcal{Q}_{N,N-3}[y]\,Y = 0$ is*

$$(63) \quad Y_{N,3}(x, y) = N\,(L_2 - 2)\,{}_0\mathrm{F}_1\binom{xy}{N} + y\,(L_2 - 1)\,{}_0\mathrm{F}_1\binom{xy}{N+1}$$
$$+ L_3\,\mathrm{e}^y\,H_{N+1}^0(y, x).$$

- *A solution of $\mathcal{Q}_{N,N-4}[y]\,Y = 0$ is*

$$(64) \quad Y_{N,4}(x, y) = N\,((L_2 - 2)^2 + 2y + 2x + 2)\,{}_0\mathrm{F}_1\binom{xy}{N}$$
$$+ y\,((L_2 - 1)^2 + y + 3x - N + 2)\,{}_0\mathrm{F}_1\binom{xy}{N+1}$$
$$+ L_4\,\mathrm{e}^y\,H_{N+1}^0(y, x).$$

Proof. The operator $\mathcal{Q}_{N,N-2}[y]$ factors as $\mathcal{K}_2\mathcal{K}_1$ in $\mathbb{R}(x, N, y)\langle\partial/\partial y\rangle$, with

$$
\begin{aligned}
\mathcal{K}_1 &= \frac{\partial}{\partial y} - 1 - \frac{1}{L_2}, \\
\mathcal{K}_2 &= y\frac{\partial^2}{\partial y^2} + \left(N + \frac{y}{L_2}\right)\frac{\partial}{\partial y} - x - 1 + \frac{y+N}{L_2} - \frac{y}{L_2^2}.
\end{aligned}
$$

The solution of $\mathcal{K}_1 Y_2 = 0$ is $Y_2(x, y) = L_2 \mathbf{e}^y$. This is a solution of $\mathcal{Q}_{N,N-2}[y] Y = \mathcal{K}_2 \mathcal{K}_1 Y = 0$ as well. A non-logarithmic solution of $\mathcal{K}_2 Y = 0$ is

$$Y_3(y) = \frac{1}{L_2} \left({}_0\mathrm{F}_1 \left(\begin{array}{c} xy \\ N \end{array} \right) + \frac{y}{N} \, {}_0\mathrm{F}_1 \left(\begin{array}{c} xy \\ N+1 \end{array} \right) \right).$$

Solving $\mathcal{Q}_{N,N-2}[y] Y = 0$ now means solving the non-homogeneous $\mathcal{K}_1 Y = Y_3$. This leads to the integration

$$\int_0^y \frac{\mathbf{e}^{-t}}{(t - x + N - 1)^2} \left({}_0\mathrm{F}_1 \left(\begin{array}{c} xt \\ N \end{array} \right) + \frac{t}{N} \, {}_0\mathrm{F}_1 \left(\begin{array}{c} xt \\ N+1 \end{array} \right) \right) dt.$$

After a step of integration by parts (and multiplication by N), we obtain (62).

The other two operators $\mathcal{Q}_{N,N-M}[y]$, $M \in \{3, 4\}$ factor similarly $\mathcal{K}_2^{(M)} \mathcal{K}_1^{(M)}$ as $\mathcal{Q}_{N,N-2}[y]$. In the same way, by solving the first order $\mathcal{K}_1^{(M)} Y_2^{(M)} = 0$ and the second order $\mathcal{K}_2^{(M)} Y_3^{(M)} = 0$, we are led to solving the non-homogeneous first order $\mathcal{K}_1^{(M)} Y = Y_3^{(M)}$. The equations $\mathcal{K}_2^{(M)} Y_3^{(M)} = 0$ have $M - 1$ apparent singularities defined by $L_M Y = 0$. Maple 18 does not solve them, but looking at (non-logarithmic) power series solutions multiplied by L_M we recognize the solutions

$$(65) \qquad Y_3^{(M)} = \frac{1}{L_M} \sum_{j=0}^{M} \binom{M}{j} \frac{y^j}{(N - M + 1)_j} \, {}_0\mathrm{F}_1 \left(\begin{array}{c} xy \\ N-M+j+1 \end{array} \right).$$

Similarly as for $M = 2$, the solutions of $\mathcal{K}_1^{(M)} Y = Y_3^{(M)}$ are integrals that can be simplified to (63) or (64) by applying integration by parts a few times.   Q.E.D.

A general non-logarithmic solution of $\mathcal{Q}_{N,N-2}[y] Y = 0$ is

$$(66) \qquad\qquad C_1 Y_{N,2}(x, y) + C_2 L_2 \mathbf{e}^y.$$

The function $G_{n,2}(x, y)$ in (18) differs from $Y_{n,2}(x, y)$ by

$$(67) \qquad\qquad C_2 = - \int_0^\infty \mathbf{e}^{-t} \, {}_0\mathrm{F}_1 \left(\begin{array}{c} xt \\ n+1 \end{array} \right) dt$$

$$(68) \qquad\qquad = - n \, x^{-n} \, \mathbf{e}^x \, \gamma(n, x).$$

The latter expression is obtained by expanding the ${}_0\mathrm{F}_1$-series, and recognizing a ${}_1\mathrm{F}_1$-series for the incomplete gamma function $\gamma(n, x)$ after the definite integration. Similarly, general non-logarithmic solutions of

$\mathcal{Q}_{N,N-3}[y]$, $\mathcal{Q}_{N,N-4}[y]$ are obtained by scalar multiplication (by C_1) and considering the integral $H^0_{N+1}(y,x)$ with an integration constant C_2.

We observe empirically, especially from differentiation relations between L_2, L_3, L_4, that

$$(69) \qquad Y_{N,M-1}(x,y) = (M-1)\left(\frac{\partial}{\partial y} - 1\right) Y_{N,M}(x,y)$$

for $M = 3, 4$. This observation indeed generalizes, leading us to lowering and raising operators on non-logarithmic solutions.

Theorem 3.5. *Let $Y_{N,M}(x,y)$ denote a solution of $\mathcal{Q}_{N,N-M}[y]$.*

(i)   *The differential operator $\dfrac{\partial}{\partial y} - 1$ transforms $Y_{N,M}(x,y)$ to a solution of $\mathcal{Q}_{N,N-M+1}[y]$.*

(ii) *The differential operator*

$$(70) \qquad y\frac{\partial^2}{\partial y^2} + (N-M+1)\frac{\partial}{\partial y} - x - M$$

transforms $Y_{N,M}(x,y)$ to a solution of $\mathcal{Q}_{N,N-M-1}[y]$.

(iii) *If $Y_{N,M-1}(x,y) = \left(\dfrac{\partial}{\partial y} - 1\right) Y_{N,M}(x,y)$*

and $Y_{N,M-2}(x,y) = \left(\dfrac{\partial}{\partial y} - 1\right)^2 Y_{N,M}(x,y)$, then

$$(71) \qquad (y - x + N - 2M + 1)Y_{N,M}(x,y)$$
$$+ (2y + N - M + 1)Y_{N,M-1}(x,y) + yY_{N,M-2}(x,y)$$

is a solution of $\mathcal{Q}_{N,N-M-1}[y]$.

Proof.   The first claim follows from the commutation relation

$$(72) \qquad \mathcal{Q}_{N,N-M+1}[y]\left(\frac{\partial}{\partial y} - 1\right) = \left(\frac{\partial}{\partial y} - 1\right)\mathcal{Q}_{N,N-M}[y].$$

The second claim follows by rewriting

$$\mathcal{Q}_{N,N-M-1}[y] = \left(y\frac{\partial^2}{\partial y^2} + (N-M+1)\frac{\partial}{\partial y} - x - M\right)\left(\frac{\partial}{\partial y} - 1\right) - M.$$

The last claim similarly follows from

$$\mathcal{Q}_{N,N-M-1}[y] = y\left(\frac{\partial}{\partial y} - 1\right)^3 + (2y + N - M + 1)\left(\frac{\partial}{\partial y} - 1\right)^2$$
$$(73) \qquad\qquad + (y - x + N - 2M + 1)\left(\frac{\partial}{\partial y} - 1\right) - M.$$

$$\text{Q.E.D.}$$

The recurrence (19) in Conjecture 2.4 is a slight modification of the differential operator (70), and the functions $G_{n,m}(x,y)$ differ from the solutions $Y_{n,m}(x,y)$ of this section by the difference (67) and the sign $(-1)^m$.

§4. Holonomic systems for $m \leqslant 3$

The results of this article originated from explicit computations for the matrix dimensions $m = 2$ and $m = 3$. The aim was holonomic systems for $\psi_{n,m}(x, \lambda_1, \ldots, \lambda_m)$, so that the holonomic gradient method [2] could be applied for numeric computation of the probability density function.

4.1. The rank 12 system

With $m = 2$, the holonomic system of Theorem 2.1 has rank 12. It is easy to obtain from standard differential equations for $_0F_1\left({x\lambda_1 \atop n-1}\right)$, $_0F_1\left({x\lambda_2 \atop n-1}\right)$ and the integrals

$$H_{n-1}^{n-1}(x, \lambda_1), \quad H_{n-1}^{n-1}(x, \lambda_2), \quad H_{n-2}^{n-1}(x, \lambda_1), \quad H_{n-2}^{n-1}(x, \lambda_2).$$

This was demonstrated by Christoph Koutschan on his Mathematica package. The singularities of the holonomic system are along

$$(74) \qquad x = 0, \quad \lambda_1 = 0, \quad \lambda_2 = 0, \quad \infty\text{-compactifations.}$$

It is generated by these three differential operators of order 2 or 3:

(75)
$$\lambda_1 \frac{\partial^2}{\partial \lambda_1^2} + \lambda_2 \frac{\partial^2}{\partial \lambda_2^2} - (\lambda_1 - n + 1)\frac{\partial}{\partial \lambda_1} - (\lambda_2 - n + 1)\frac{\partial}{\partial \lambda_2} + x\frac{\partial}{\partial x} - 2n + 2,$$

$$(76) \qquad \frac{\partial^3}{\partial x \partial \lambda_1 \partial \lambda_2} + 2\frac{\partial^2}{\partial \lambda_1 \partial \lambda_2} - \frac{\partial}{\partial \lambda_1} - \frac{\partial}{\partial \lambda_2},$$

$$(77) \quad \left(\lambda_1 \lambda_2 \frac{\partial}{\partial \lambda_1} + \lambda_1 \lambda_2 \frac{\partial}{\partial \lambda_2} + (n-1)(\lambda_1 + \lambda_2)\right)\frac{\partial^2}{\partial \lambda_1 \partial \lambda_2} + (n-1)x$$
$$+ \left(x\frac{\partial}{\partial x} + 2x - n + 2\right)\left(x\frac{\partial}{\partial x} - \lambda_1 \frac{\partial}{\partial \lambda_1} - \lambda_2 \frac{\partial}{\partial \lambda_2} + x - 2n + 2\right).$$

The first equation as in Theorem 2.2. For the sake of compactness, the last equation is expressed using non-commutative multiplication (in the last term).

Elimination of $\partial/\partial x$ and $\partial/\partial\lambda_2$ leads to the fifth order operator

$$\lambda_1^2\frac{\partial^5}{\partial\lambda_1^5} + \lambda_1(2n - \lambda_1 + 2)\frac{\partial^4}{\partial\lambda_1^4} + (n^2 + n - 2x\lambda_1 - 2n\lambda_1 - 3\lambda_1)\frac{\partial^3}{\partial\lambda_1^3}$$

(78) $$-(n^2 + 2n - 2x\lambda_1 + 2nx)\frac{\partial^2}{\partial\lambda_1^2} + x(2n + x + 1)\frac{\partial}{\partial\lambda_1} - x^2.$$

Consistent with the theory of Gröbner bases, the rank 8 system has elimination equations with the leading monomials $\partial^4/\partial\lambda_1^3\partial\lambda_2 + \ldots$ and $\partial^2/\partial\lambda_2^2 + \ldots$. The fifth order operator does not involve the variable λ_2 even. It factorizes as the LCLM of $\mathcal{P}_{n-2}[\lambda_1]$ and $\mathcal{Q}_{n,n-2}[\lambda_1]$. Correspondingly, the holonomic system factorizes nicely to a direct sum of two 6-dimensional subspaces, each of those subspaces is a tensor product of a rank 2 system in one variable λ_1 or λ_2, and rank 3 system in the other variable. The factorization corresponds nicely with the terms in the expanded determinantal formula (22), as $_0\mathrm{F}_1\binom{xy}{n-1}$ is a solution of $\mathcal{P}_{n-2}[y]$, and $G_{n,2}(x,y)$ is a solution of $\mathcal{Q}_{n,n-2}[y]$.

4.2. Proof of formula (22)

We seek to prove

(79) $$\mathcal{R}_{n,2}(x,\lambda_1,\lambda_2) = \frac{x^{2n-2}\,\mathbf{e}^{-2x}}{n(n-1)}\det\begin{pmatrix} _0\mathrm{F}_1\binom{x\lambda_1}{n-1} & _0\mathrm{F}_1\binom{x\lambda_2}{n-1} \\ G_{n,2}(x,\lambda_1) & G_{n,2}(x,\lambda_2) \end{pmatrix}.$$

With the same holonomic system of rank 12 established for both sides, it is enough to compare a few coefficients in the two series expansions in λ_1,λ_2. The subspace of non-logarithmic anti-symmetric solutions is 2-dimensional, hence it is enough to compare 2 pairs of independent coefficients.

After division by $\lambda_1 - \lambda_2$ as in (22), a proper general setting is expansion in terms of the symmetric *Schur polynomials* [11] in λ_1,λ_2. The Schur polynomials functions are defined in terms of monomial determinants. Correspondingly, we formulate the following general statement.

Lemma 4.1. *Consider m functions $f_1(y),\ldots,f_m(y)$ defined by the convergent series*

(80) $$f_k(y) = \sum_{j=0}^{\infty} c_j^{(k)} y^j.$$

Then

(81) $$\det\left(\,f_i(\lambda_j)\,\right)_{i=1}^{m} = \sum_{0\leqslant q_1 < q_2 < \ldots < q_m}\cdots\sum \det\left(c_{q_j}^{(i)}\,\right)_{j=1}^{m}\det\left(\lambda_i^{q_j}\,\right)_{j=1}^{m}.$$

Proof. Intermediate expansions are

$$(82) \qquad = \sum_{q_1=0}^{\infty} \cdots \sum_{q_m=0}^{\infty} \det\left(c_{q_j}^{(i)} \right)_{j=1}^m \lambda_1^{q_1} \cdots \lambda_m^{q_m}$$

$$(83) \qquad = \sum_{q_1=0}^{\infty} \cdots \sum_{q_m=0}^{\infty} c_{q_1}^{(1)} \cdots c_{q_m}^{(m)} \det\left(\lambda_i^{q_j} \right)_{j=1}^m.$$

The newest determinants with some $q_i = q_j$ for $i \neq j$ are zero. After collecting the terms with the same sets $\{q_1, \ldots, q_m\}$, we get the result.

Q.E.D.

We first apply this lemma to the determinant in (23) with $f_1(y) = H_{n-1}^{n-1}(x,y)$, $f_2(y) = H_{n-1}^{n-2}(x,y)$. Therefore

$$c_j^{(1)} = \frac{\gamma(n+j,x)}{(n-1)_j \, j!}, \qquad c_j^{(2)} = \frac{\gamma(n+j-1,x)}{(n-1)_j \, j!}.$$

We differentiate (81) to get $\mathcal{R}_{n,2}(x,\lambda_1,\lambda_2)$ as

$$\sum_{i=0}^{\infty} \sum_{j=i+1}^{\infty} \frac{\dfrac{d}{dx} \det\begin{pmatrix} \gamma(n+i,x) & \gamma(n+i-1,x) \\ \gamma(n+j,x) & \gamma(n+j-1,x) \end{pmatrix}}{(n-1)_i (n-1)_j \, i! j!} \det\begin{pmatrix} \lambda_1^i & \lambda_2^i \\ \lambda_1^j & \lambda_2^j \end{pmatrix}$$

$$(84) = \frac{d}{dx} \det\begin{pmatrix} \gamma(n,x) & \gamma(n-1,x) \\ \gamma(n+1,x) & \gamma(n,x) \end{pmatrix} \frac{\lambda_2 - \lambda_1}{n-1}$$

$$+ \frac{d}{dx} \det\begin{pmatrix} \gamma(n,x) & \gamma(n-1,x) \\ \gamma(n+2,x) & \gamma(n+1,x) \end{pmatrix} \frac{\lambda_2^2 - \lambda_1^2}{2n(n-1)}$$

$$+ \ldots$$

$$(85) = \left(\left(\frac{x^2}{n-1} - 2x + n \right) \gamma(n-1,x) + \frac{x-n}{n-1} x^{n-1} e^{-x} \right) x^{n-2} e^{-x} (\lambda_1 - \lambda_2)$$

$$+ \left(\left(\frac{x^3}{n(n-1)} - \frac{x^2}{n} - x + n + 1 \right) \gamma(n-1,x) \right.$$

$$\left. + \frac{(x-n-1)(x+n)}{n(n-1)} x^{n-1} e^{-x} \right) x^{n-2} e^{-x} \frac{\lambda_1^2 - \lambda_2^2}{2}$$

$$+ \ldots.$$

Surely, recurrence (39) has been used. Considering the left-hand side of (79), we set $c_j^{(1)} = x^j / (j! \, (n-1)_j)$. An expansion of (62) with $N = n$ is

$$(86) \qquad Y_{n,2}(x,y) = n + ny + \sum_{k=0}^{\infty} \left(n + \sum_{j=0}^k \frac{k+1-j}{(n+1)_j} x^j \right) \frac{y^{k+2}}{(k+2)!}.$$

We only need the first few terms. To get coefficients of $G_{n,2}(x,y)$, we add the difference $C_2(y - x + n - 1)e^y$ as in (67)–(68), thus adding

$$n\, x^{-n}\, \mathbf{e}^x\, \gamma(n,x) \sum_{k=0}^{\infty}(x - n - k + 1)\frac{y^k}{k!}.$$

Therefore

(87)
$$\begin{aligned}
c_0^{(2)} &= n + n\,(x - n + 1)\, x^{-n}\mathbf{e}^x\gamma(n,x),\\
c_1^{(2)} &= n + n\,(x - n)\, x^{-n}\mathbf{e}^x\gamma(n,x),\\
c_2^{(2)} &= \frac{n+1}{2} + \frac{n}{2}\,(x - n - 1)\, x^{-n}\mathbf{e}^x\gamma(n,x).
\end{aligned}$$

The determinant in (79) expands as

$$\det\begin{pmatrix} 1 & \frac{x}{n-1} \\ c_0^{(2)} & c_1^{(2)} \end{pmatrix}(\lambda_2 - \lambda_1) + \det\begin{pmatrix} 1 & \frac{x^2}{2n(n-1)} \\ c_0^{(2)} & c_2^{(2)} \end{pmatrix}(\lambda_2^2 - \lambda_1^2).$$

We get the same two terms as in (86).

4.3. The rank 8 system

A rank 8 holonomic system for $\mathcal{R}_{n,2}(x,\lambda_1,\lambda_2)$ is obtained by expressing this function and its derivatives as linear combinations of

$$x^n\,\mathbf{e}^{-x}{}_0\mathrm{F}_1\binom{x\lambda_2}{n-1}H_{n-1}^n(x,\lambda_1),\qquad x^n\,\mathbf{e}^{-x}{}_0\mathrm{F}_1\binom{x\lambda_2}{n}H_{n-1}^n(x,\lambda_1),$$

$$x^n\,\mathbf{e}^{-x}{}_0\mathrm{F}_1\binom{x\lambda_1}{n-1}H_{n-1}^n(x,\lambda_2),\qquad x^n\,\mathbf{e}^{-x}{}_0\mathrm{F}_1\binom{x\lambda_1}{n}H_{n-1}^n(x,\lambda_2),$$

$$x^{2n}\,\mathbf{e}^{-2x}{}_0\mathrm{F}_1\binom{x\lambda_1}{n-1}{}_0\mathrm{F}_1\binom{x\lambda_2}{n-1},\qquad x^{2n}\,\mathbf{e}^{-2x}{}_0\mathrm{F}_1\binom{x\lambda_1}{n-1}{}_0\mathrm{F}_1\binom{x\lambda_2}{n},$$

$$x^{2n}\,\mathbf{e}^{-2x}{}_0\mathrm{F}_1\binom{x\lambda_1}{n}{}_0\mathrm{F}_1\binom{x\lambda_2}{n-1},\qquad x^{2n}\,\mathbf{e}^{-2x}{}_0\mathrm{F}_1\binom{x\lambda_1}{n}{}_0\mathrm{F}_1\binom{x\lambda_2}{n}.$$

These expressions follow from Theorem 2.3 *(iv)*. For example, the expression of $\mathcal{R}_{n,2}(x,\lambda_1,\lambda_2)$ has these coefficients, respectively:

$$\frac{\lambda_1 - x}{n - 1} + 1,\ 0,\ -\frac{\lambda_2 - x}{n - 1} - 1,\ 0,\ 0,\ \frac{x}{n - 1} - 1,\ -\frac{x}{n - 1} + 1,\ 0.$$

Further, the expression of $(n - 1)\,\partial\mathcal{R}_{n,2}/\partial x$ has these coefficients:

(88)
$$\begin{aligned}
&-1,\quad \lambda_2\left(\frac{\lambda_1 - x}{n - 1} + 1\right),\quad 1,\quad -\lambda_1\left(\frac{\lambda_2 - x}{n - 1} + 1\right),\\
&0,\quad -\lambda_2 + 1,\quad \lambda_1 - 1,\quad (\lambda_1 - \lambda_2)\left(\frac{x}{n - 1} - 1\right);
\end{aligned}$$

and so on. The rank 8 system has singularities not just along (74), but additionally along $\lambda_1 = \lambda_2$ and the hypersurface $S_n(x, y) = 0$, where

$$(89) \qquad S_n(x, y) = (2x - n + 1)^2 (\lambda_1 + \lambda_2 - 2x + 2n - 2) - \frac{x}{2} (\lambda_1 - \lambda_2)^2.$$

It contains another second order operator

$$S_n(x, y)\left(x\mathcal{D}_x^2 + \frac{2\lambda_1\lambda_2}{x}\frac{\partial^2}{\partial\lambda_1\partial\lambda_2} - \mathcal{D}_\lambda(2\mathcal{D}_x + 1) + (x - n + 1)\mathcal{D}_x + \lambda_1 + \lambda_2 - 1\right)$$

$$+ (\lambda_1 + \lambda_2 - 4x + 2n - 2)\left((x - n + 1)\left(\mathcal{D}_\lambda\mathcal{D}_x - \frac{2\lambda_1\lambda_2}{x}\frac{\partial^2}{\partial\lambda_1\partial\lambda_2}\right)\right.$$

$$(90) \hspace{6cm} \left. + (2x - n)\left(\mathcal{D}_\lambda + (x - n + 1)\mathcal{D}_x - 1\right)\right)$$

$$+ 2(2x - n + 1)\left((\lambda_1 + \lambda_2)(\mathcal{D}_\lambda + 2x - n + 1) - \lambda_1^2\frac{\partial}{\partial\lambda_1^2} - \lambda_2^2\frac{\partial}{\partial\lambda_2^2} - \frac{(\lambda_1 + \lambda_2)^2}{2}\right)$$

$$+ 2\lambda_1\lambda_2\left(x\mathcal{D}_\lambda\mathcal{D}_x - \frac{\lambda_1 + \lambda_2}{2} - 2x + n\right)$$

$$+ (\lambda_1 - \lambda_2)\left(\lambda_1^2\frac{\partial^2}{\partial\lambda_1^2} - \lambda_2^2\frac{\partial^2}{\partial\lambda_2^2} - \lambda_1^2\frac{\partial}{\partial\lambda_1} + \lambda_2^2\frac{\partial}{\partial\lambda_2}\right),$$

where

$$(91) \qquad \mathcal{D}_x = \frac{\partial}{\partial x} + 1 - \frac{n - 2}{x}, \qquad \mathcal{D}_\lambda = \lambda_1\frac{\partial}{\partial\lambda_1} + \lambda_2\frac{\partial}{\partial\lambda_2}.$$

The smaller rank system appears to be more complex. It can be obtained from the rank 12 system by adjoining this 3rd order operator:

$$\left(2\lambda_1\frac{\partial}{\partial\lambda_1} + 2\lambda_2\frac{\partial}{\partial\lambda_2} - 3\lambda_1 - 3\lambda_2 + 4n - 6\right)\frac{\partial^2}{\partial\lambda_1\partial\lambda_2} + \lambda_2\frac{\partial}{\partial\lambda_1} + \lambda_1\frac{\partial}{\partial\lambda_2}$$

$$(92) \qquad - \left(x\frac{\partial^2}{\partial x^2} + (x - n + 3)\frac{\partial}{\partial x} + n\right)\left(\frac{\partial}{\partial\lambda_1} + \frac{\partial}{\partial\lambda_2}\right) + 3x\frac{\partial}{\partial x} - 2n + 6.$$

Elimination of $\partial/\partial x$ and $\partial/\partial\lambda_2$ leads to the same fifth order operator (78). The elimination equations with the leading monomials $\partial^4/\partial\lambda_1^2\partial^2\lambda_2 + \dots$ and $\partial^2/\partial\lambda_2^3 + \dots$. Factorization of the solution spaces of operator (78) is harder to follow, as the 6-dimensional subspaces intersect.

4.4.   The case $m = 3$

The holonomic system of Theorem 2.1 has rank 108 when $m = 3$. A Gröbner basis computation without $\partial/\partial x$ is fast on Maple 18 (with respect to a total degree ordering in $\partial/\partial\lambda_k$'s, on a 2.8GHz MacBook

Pro of 2014). The lowest order operator in the $\partial/\partial\lambda_k$'s is of order 5. Allowing $\partial/\partial x$, it is equivalent to

$$
(93) \qquad \frac{\partial^4}{\partial x\partial\lambda_1\partial\lambda_2\partial\lambda_3} + 3\frac{\partial^3}{\partial\lambda_1\partial\lambda_2\partial\lambda_3} - \frac{\partial^2}{\partial\lambda_1\partial\lambda_2} - \frac{\partial^2}{\partial\lambda_1\partial\lambda_3} - \frac{\partial^2}{\partial\lambda_2\partial\lambda_3}.
$$

This expression is comparable with (76). A Gröbner basis computation with $\partial/\partial x$ leads to rapid increase of memory usage, 4GB in a few minutes.

Replacing $\mathcal{T}_2[y]$ by (58) gives the same LCLM operators, but a different holonomic system of rank 108. A similar Gröbner basis computation without $\partial/\partial x$ takes about 8 minutes. The lowest order operator is

$$
\sum_{k=1}^{3} \left(\lambda_k^2 \frac{\partial^4}{\partial\lambda_k^4} + (2n-2-\lambda_k)\frac{\partial^3}{\partial\lambda_k^3} + (n^2 - 3n + 2 - (x+2n)\lambda_k)\frac{\partial^2}{\partial\lambda_k^2} \right.
$$
$$
(94) \qquad \left. + (x\lambda_k - (n-2)(x+n+1))\frac{\partial}{\partial\lambda_k} \right) + (3n-2)x.
$$

Combining both holonomic systems leads to formidable Gröbner basis computation, apparently. Computation of differential operators for the rank $\leqslant 26$ system of Theorem 2.3*(iv)* is barely viable on Singular 4 (given several hours), but further manipulation is hard.

Conjecture 2.4 was checked for $m = 4$ by expanding both sides of (20) in the determinants of

$$
\begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 \end{pmatrix}, \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 \end{pmatrix},
$$

etc., and then comparing the coefficients to these four determinants. For example, comparison of the first coefficient by Lemma 4.1 gives

$$
\frac{1}{2(n-2)^2(n-1)} \frac{\partial}{\partial x} \det \begin{pmatrix} \gamma(n,x) & \gamma(n-1,x) & \gamma(n-2,x) \\ \gamma(n+1,x) & \gamma(n,x) & \gamma(n-1,x) \\ \gamma(n+2,x) & \gamma(n+1,x) & \gamma(n,x) \end{pmatrix}
$$
$$
= C(x) \det \begin{pmatrix} 1 & \frac{x}{n-2} & \frac{x^2}{2(n-2)(n-1)} \\ \widetilde{c}_0^{(2)} & \widetilde{c}_1^{(2)} & \widetilde{c}_2^{(2)} \\ c_0^{(3)} & c_1^{(3)} & c_2^{(3)} \end{pmatrix},
$$

where $\widetilde{c}_k^{(2)}$ are the shifted $n \mapsto n-1$ versions of $c_k^{(2)}$ in (87), and $c_k^{(3)}$ are the first coefficients of the expansion of $G_{n,3}(x,y)$:

$$c_0^{(3)} = -nx + n(n-3) - n\left((x-n)^2 + 4x - 3n + 2\right)x^{-n}\mathbf{e}^x\gamma(n,x),$$

(95) $\quad c_1^{(3)} = -nx + n(n-1) - n\left((x-n)^2 + 2x - n\right)x^{-n}\mathbf{e}^x\gamma(n,x),$

$$c_2^{(3)} = \frac{-nx + n(n+1)}{2} - \frac{n}{2}\left((x-n)^2 + n\right)x^{-n}\,\mathbf{e}^x\gamma(n,x).$$

For an intermediate check, here is a quadratic expression in $A = \gamma(n-1, x)$ and $E = x^{n-1}\mathbf{e}^{-x}$ for the first coefficient:

$$\frac{x^{n-3}\mathbf{e}^{-x}}{n-2}\left(-\left((x-n+1)^2 A + (x-n)E\right)^2 + (n-1)(n-1-4x)A^2\right.$$

(96) $$\left. + 2(x^2 + 2x - n^2 + n)AE + 2(x+n)E^2\right).$$

To set up $C(x)$ in the conjecture, we compared the similar first coefficients for $m = 4$ as well.

Acknowledgement

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