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Localization, local cohomology, and the *b*-function of a *D*-module with respect to a polynomial

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Abstract.

Given a D-module M generated by a single element, and a polynomial f, one can construct several D-modules attached to M and f and can define the notion of the (generalized) b-function following M. Kashiwara. These modules are closely related to the localization and the local cohomology of M. We show that the b-function, if it exists, controls these modules and present general algorithms for computing these modules and the b-function if it exists without any further assumptions. We also give some examples of multiplicity computation of such D-modules including a possibly well-known explicit formula for the localization of the polynomial ring by a hyperplane arrangement.

Contents

§1. Introduction

Let K be an algebraically closed field of characteristic zero and $K[x] = K[x_1, \dots, x_n]$ be the polynomial ring with $x = (x_1, \dots, x_n)$. Let $D_n = K[x]\langle\partial\rangle = K[x]\langle\partial_1, \dots, \partial_n\rangle$ be the n-th Weyl algebra, i.e., the ring

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of differential operators with polynomial coefficients with respect to the variables x, where we denote $\partial = (\partial_1, \dots, \partial_x)$ with $\partial_i = \partial_{x_i} = \partial/\partial x_i$ being the derivation with respect to x_i. An arbitrary element P of D_n is written in a finite sum

$$P = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(x)\partial^\alpha \quad \text{with} \quad a_\alpha(x) \in K[x],$$

where we denote $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ for a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ with $\mathbb{N}$ being the set of non-negative integers. One can define the dimension of a finitely generated left D_n-module M; J. Bernstein [3], [4] proved that the dimension of M is not less than n unless M is the zero module. A finitely generated left D_n-module is called *holonomic* if its dimension is n or else it is the zero module.

Let M be a finitely generated left D_n-module and $f \in K[x]$ be a non-constant polynomial. Then the localization $M[f^{-1}]$ and the local cohomology groups $H^j_{(f)}(M)$ have natural structures of left D_n-module and are holonomic if so is M, as was shown by Kashiwara [13]. More generally, one can construct a left $D_n[s]$-module

$$M(u, f, s) = D_n[s](u \otimes f^s)$$

with an indeterminate s. Suppose that M is generated by u over D_n. Then the (*generalized*) *b-function* for u and f is defined to be the univariate (and monic) polynomial $b_{u,f}(s)$ of the least degree such that

$$b_{u,f}(s)(u \otimes f^s) \in D_n[s](u \otimes f^{s+1})$$

holds. The existence of $b_{u,f}(s)$ was proved by Kashiwara [13] under the assumption that M is holonomic outside of the hypersurface $f = 0$. If M is the polynomial ring $K[x]$ with $u = 1$, then $b_{u,f}(s)$ is nothing but the classical Bernstein-Sato polynomial, or simply the b-function, of f. In the same way as the Bernstein-Sato polynomial controls the localization of the polynomial ring as a D_n-module, the b-function controls the localization $M[f^{-1}]$ or its generalization $D_n(u \otimes f^\lambda)$.

On the other hand, algorithms to compute $M(u, f, s)$ and the b-function if it exists were introduced in [17] under the assumption that M is f-torsion free. These algorithms are based on various Gröbner bases over the ring of differential operators as is presented, e.g., in [23] and [18]. Torrelli [24] studied the b-function $b_{u,f}(s)$ systematically when M is the local cohomology group $H^k_{(f_1,\dots,f_k)}(K[x])$ under the assumption that $f_1, \dots, f_k, f$ define a quasi-homogeneous non-isolated singularity, together with the general property of $M(u, f, s)$ under the assumption that M is holonomic without f-torsion.

The purpose of our study on the b-function and $M(u, f, s)$ is twofold: first, we want to clarify how the b-function controls the module $M(u, f, s)$ and the localization $M[f^{-1}]$ as well as the local cohomology $H^1_{(f)}(M)$. This will be performed in Sections 2 and 5. These results should be more or less well-known under some stronger conditions. See, e.g., [24] and Chapter VI of [5], where M is assumed to be f-torsion free, or regular holonomic. The second purpose is to remove the assumption of f-saturatedness from our former algorithms in [17]. For this purpose, we reinterpret the algorithm introduced in [21] for the localization $M[f^{-1}]$ in Section 3. Our algorithms work at least if M is holonomic outside of $f = 0$ without any further assumptions.

In the latter half of this article (Sections 5 and 6), we study the multiplicity (in the sense of Bernstein [3]) and the length of a holonomic D-module, as the most fundamental numerical invariants. This can be also used to prove a relation between $b_{u,f}(s)$ and $M(u, f, \lambda)$. We also give some examples of the multiplicity computation of the localization or the local cohomology. In the last section, we present, together with an elementary proof, a possibly well-known formula on the length and the multiplicity of the localization of the polynomial ring by a polynomial f which defines a hyperplane arrangement. The result is that the length and the multiplicity of $K[x, f^{-1}]$ both coincide with $\pi(1)$, where $\pi(t)$ is what is called the Poincaré polynomial of the hyperplane arrangement.

We use computer algebra system Risa/Asir [16] for computation of Gröbner bases over the ring of differential operators, and in particular, for computation of D-module theoretic integration, which is needed in the localization algorithm.

We would like to thank the organizers of MSJ SI 2015 for the invitation both to the conference and to the proceedings. We would be pleased if we could convince the reader who is interested in D-module theory of the usefulness of Gröbner bases, which are the main theme of MSJ SI 2015, over the ring of differential operators in our case. This work was supported by JSPS Grant-in-Aid for Scientific Research (C) 26400123.

§2. The b-function for a D-module and a polynomial

Let K be an arbitrary field of characteristic zero and $X = K^n$ be the n-dimensional affine space over K. We denote by D_X the n-th Weyl algebra D_n over K. Let M be a left D_X-module and $f \in K[x]$ a non-constant polynomial. We can associate several D_X-modules with M and f by translating the definitions by Kashiwara [13] for analytic D-modules to algebraic setting. First, the localization $M[f^{-1}] := M \otimes_{K[x]} K[x, f^{-1}]$

and the local cohomology groups $H^j_{(f)}(M)$ $(j = 0, 1)$ are defined with M being regarded as a $K[x]$-module; they become again left D_X-modules.

Introducing an indeterminate s, let

$$\mathcal{L}_f := K[x, f^{-1}, s]f^s$$

be the free $K[x, f^{-1}, s]$-module with a free generator f^s. Then $\mathcal{L}_f$ has a natural structure of left $D_X[s]$-module through the action of ∂_{x_i} on $\mathcal{L}_f$ defined by

$$\partial_{x_i}(a(x, s)f^{-k}f^s) = \left(\frac{\partial a(x, s)}{\partial x_i}f^{-k} + (s - k)f_i a(x, s)f^{-k-1}\right) f^s$$

for $j = 1, \ldots, n$ with $f_i := \partial f/\partial x_i$. Sometimes $f^{-k}f^s$ is abbreviated to f^{s-k}.

The tensor product $M \otimes_{K[x]} \mathcal{L}_f$ has a natural structure of left $D_X[s]$-module induced by

$$\partial_{x_i}(u \otimes a(x, s)f^s) = (\partial_{x_i}u) \otimes a(x, s)f^s + u \otimes \partial_{x_i}(a(x, s)f^s) \quad (1 \leq i \leq n)$$

for $u \in M$ and $a(x, s) \in K[x, s]$. In what follows, we fix an arbitrary nonzero element u of M. Let

$$M(u, f, s) := D_X[s](u \otimes f^s)$$

be the left $D_X[s]$-submodule of $M \otimes_{K[x]} \mathcal{L}_f$ generated by $u \otimes f^s$. In a special case where $M = K[x]$ and $u = 1$, let us denote by

$$\mathcal{N}_f = K[x](1, f, s) = D_X[s]f^s$$

the left $D_X[s]$-submodule of $\mathcal{L}_f$ generated by f^s. Set

$$I(u, f) := \{b(s) \in K[x] \mid b(s)(u \otimes f^s) \in D_n[s](fu \otimes f^s)\}.$$

If $I(u, f) \neq \{0\}$, then the (monic) generator $b_{u,f}(s)$ of $I(u, f)$ is called the (*generalized*) *b-function* for u and f. It was defined by Kashiwara [13] with the following existence theorem.

Theorem 2.1 (Kashiwara [13]). *Let D_X be defined over an algebraically closed field K of characteristic zero. If a left D_X-module M is holonomic on $X_f = \{x \in X \mid f(x) \neq 0\}$, then one has $I(u, f) \neq \{0\}$ for any $u \in M$.*

When $M = K[x]$ and $u = 1$, the b-function $b_{1,f}(s)$ is nothing but what is called the Bernstein-Sato polynomial, or the b-function, associated with f. In fact, Kashiwara proved this theorem for a module

M over the ring of differential operators with analytic coefficients and a complex analytic function f. This corresponds to what is called the local b-function. The coincidence of the local b-functions in the algebraic setting and in the analytic setting is noticed, e.g., as Corollary 8.6 of [17]. It will turn out in what follows that the b-function 'controls' the D-modules associated with M and f.

The b-function can exist even if M is not holonomic on X_f.

Example 2.2. Set $n = 2$, $x_1 = x$, $x_2 = y$, and $P = x\partial_x^2 + \partial_y$. Then $M := D_X/D_XP = D_Xu$ with u being the residue class of 1 is not holonomic even outside of $x = 0$ (the dimension of M is three), but has the b-functions $b_{u,x}(s) = (s+1)(s+2)$ and $b_{u,y}(s) = s+1$. In fact, one has

$$
\begin{aligned}
&(-x\partial_x^2 + 2(s+1)\partial_x - \partial_y)(u \otimes x^{s+1}) = (s+1)(s+2)u \otimes x^s,\\
&P(u \otimes y^{s+1}) = (s+1)u \otimes y^s
\end{aligned}
$$

in $M \otimes_{K[x,y]} K[x,y,x^{-1}]x^s$ and in $M \otimes_{K[x,y]} K[x,y,y^{-1}]y^s$ respectively.

Definition 2.3. A left D_X-module M is said to be *f-saturated* or *f-torsion free* if the homomorphism $f : M \to M$ is injective. This is equivalent to $H^0_{(f)}(M) = 0$.

An algorithm to determine if there exists the b-function and to compute it if it exists was given in [17] under the assumption that $M = D_Xu$ is f-torsion free.

Let us define a D_X-automorphism $t : \mathcal{L}_f \to \mathcal{L}_f$ by

$$t(a(x,s)f^{-k}f^s) = a(x,s+1)f^{-k+1}f^s$$

for $a(x,s) \in K[x,s]$ and $k \in \mathbb{N}$. The inverse t^{-1} is defined by

$$t^{-1}(a(x,s)f^{-k}f^s) = a(x,s-1)f^{-k-1}f^s.$$

It induces a D_X-automorphism

$$t : M \otimes_{K[x]} \mathcal{L}_f \longrightarrow M \otimes_{K[x]} \mathcal{L}_f,$$

which also induces a D_X-endomorphism of $M(u,f,s)$. Note that the actions of t and s on $M(u,f,s)$ satisfies the commutation relation $st = t(s-1)$. It follows that $tM(u,f,s)$ is a left $D_X[s]$-module. It also follows from the definition that $b_{u,f}(s)$ is the minimal polynomial of s acting on the left D_X-module $M(u,f,s)/tM(u,f,s)$ since $P(s)(fu \otimes f^s) = t(P(s-1)(u \otimes f^s))$.

Let $\lambda \in K$ be a constant. Then specializing the parameter s to λ, we obtain left D_X-modules

$$\mathcal{L}_f(\lambda) := \mathcal{L}_f/(s-\lambda)\mathcal{L}_f, \qquad \mathcal{N}_f(\lambda) := \mathcal{N}_f/(s-\lambda)\mathcal{N}_f.$$

Let us denote by f^λ and $f^s|_{s=\lambda}$ the residue class of f^s in $\mathcal{L}_f(\lambda)$, and that of f^s in $\mathcal{N}_f(\lambda)$ respectively. In particular, $\mathcal{L}_f(\lambda) = K[x, f^{-1}]f^\lambda$ is a free $K[x, f^{-1}]$-module generated by f^λ. In the same way, we define a left D_X-module

$$M(u, f, \lambda) = M(u, f, s)/(s-\lambda)M(u, f, s)$$

and denote the residue class of $u \otimes f^s$ in $M(u, f, \lambda)$ by $(u \otimes f^s)|_{s=\lambda}$.

Kashiwara also proved the following fundamental fact, to which we shall give an elementary proof in Section 5.

Theorem 2.4 (Kashiwara [13])**.** *Let K be algebraically closed. If M is holonomic on X_f, then $M(u, f, \lambda)$ is a holonomic D_X-module for any $u \in M$ and $\lambda \in K$.*

Let us define the specialization homomorphism

$$\rho_\lambda : M \otimes_{K[x]} \mathcal{L}_f \longrightarrow M \otimes_{K[x]} \mathcal{L}_f(\lambda)$$

by

$$\rho_\lambda(v \otimes a(x, s) f^{-k} f^s) = v \otimes a(x, \lambda) f^{-k} f^\lambda$$

for $v \in M$, $a(x, s) \in K[x, s]$, and $k \in \mathbb{N}$. Then $\rho_\lambda(P(s)w) = P(\lambda)\rho_\lambda(w)$ holds for any $w \in M \otimes_{K[x]} \mathcal{L}_f$ and $P(s) \in D_X[s]$. Since any element of $(s-\lambda)M(u, f, s)$ is sent by ρ_λ to zero, ρ_λ induces a surjective D_X-homomorphism

$$\tilde{\rho}_\lambda : M(u, f, \lambda) \longrightarrow D_X(u \otimes f^\lambda) \subset M \otimes_{K[x]} \mathcal{L}_f(\lambda),$$

which sends $(u \otimes f^s)|_{s=\lambda}$ to $u \otimes f^\lambda$. It is not injective in general even if $M = K[x]$ and $u = 1$. For example, $\partial_x x^0 = 0$ holds in $\mathcal{L}_x(0)$ but $\partial_x(x^s|_{s=0})$ does not vanish in $\mathcal{N}_x(0)$.

Lemma 2.5. *Let $M = D_X u$ be a left D_X-module generated by u.*

(1) *Every element of $M \otimes_{K[x]} \mathcal{L}_f$ can be expressed as $Q(s)(u \otimes f^{s-k})$ with some $Q(s) \in D_X[s]$ and $k \in \mathbb{N}$.*

(2) *Let λ be an arbitrary element of K. Then every element of $M \otimes_{K[x]} \mathcal{L}_f(\lambda)$ can be expressed as $Q(u \otimes f^{\lambda-k})$ with some $Q \in D_X$ and $k \in \mathbb{N}$.*

Proof. From the identity

$$\partial_{x_i}(v \otimes f^{s-k}) = (\partial_{x_i} v) \otimes f^{s-k} + v \otimes (s-k) f_i f^{s-k-1}$$

for any $v \in M$ and $k \in \mathbb{Z}$, we get

$$(\partial_{x_i} v) \otimes f^{s-k} = (\partial_{x_i} f - (s-k) f_i)(v \otimes f^{s-k-1}).$$

By induction, we can show that for any multi-index $\alpha \in \mathbb{N}^n$ and $k \in \mathbb{Z}$, there exists $Q_\alpha(s) \in D_X[s]$ such that

$$(\partial_x^\alpha v) \otimes f^{s-k} = Q_\alpha(s)(v \otimes f^{s-k-|\alpha|}).$$

This proves the statement (1). The statement (2) can be proved similarly. Q.E.D.

The following proposition should be well-known; see, e.g., Propositions 7.1 and 7.4 of [17]. The case $M = K[x]$ and $f = 1$ was proved by Kashiwara [12].

Proposition 2.6. *Let M be a left D_X-module generated by $u \in M$ and assume that there exists the b-function $b_{u,f}(s)$. Let λ be an element of K and suppose that $b_{u,f}(\lambda - k) \neq 0$ for any positive integer k. Then*

(1) *The image $\rho_\lambda(M(u, f, s)) = D_X(u \otimes f^\lambda)$ coincides with $M \otimes_{K[x]} \mathcal{L}_f(\lambda)$. In other words, $M \otimes_{K[x]} \mathcal{L}_f(\lambda)$ is generated by $u \otimes f^\lambda$ over D_X.*

(2) $\ker \rho_\lambda \cap M(u, f, s)$ *coincides with* $(s - \lambda) M(u, f, s)$. *Hence* $\tilde{\rho}_\lambda : M(u, f, \lambda) \to D_X(u \otimes f^\lambda)$ *is an isomorphism of left D_X-modules.*

Proof. (1) In view of Lemma 2.5, we have only to show that $u \otimes f^{\lambda - k}$ belongs to $\rho_\lambda(M(u, f, s))$ for any $k \in \mathbb{N}$. This is obvious for $k = 0$ since $\rho_\lambda(u \otimes f^s) = u \otimes f^\lambda$.

Let us show that $u \otimes f^{\lambda-k}$ belongs to $\rho_\lambda(M(u, f, s))$. Suppose $k \geq 1$. There exists $P(s) \in D_X[s]$ such that $P(s)(u \otimes f^{s+1}) = b_{u,f}(s)(u \otimes f^s)$. Applying t^{-k}, we get

$$P(s-k)(u \otimes f^{s+1-k}) = b_{u,f}(s-k)(u \otimes f^{s-k})$$

in $M \otimes_{K[x]} \mathcal{L}_f$. Proceeding inductively, we see that there exists $\tilde{P}(s) \in D_X[s]$ such that

$$\tilde{P}(s)(u \otimes f^s) = b_{u,f}(s-1) \cdots b_{u,f}(s-k) u \otimes f^{s-k} \tag{1}$$

holds in $M \otimes_{K[x]} \mathcal{L}_f$. The homomorphism ρ_λ gives an identity

$$\tilde{P}(\lambda)(u \otimes f^\lambda) = b_{u,f}(\lambda - 1) \cdots b_{u,f}(\lambda - k) u \otimes f^{\lambda - k}$$

in $M \otimes_{K[x]} \mathcal{L}_f(\lambda)$. Since $b_{u,f}(\lambda - j) \neq 0$ for $j = 1, \dots, k$ by the assumption, it follows that

$$u \otimes f^{\lambda-k} = \frac{1}{b_{u,f}(\lambda-1)\cdots b_{u,f}(\lambda-k)} \tilde{P}(\lambda)(u \otimes f^{\lambda}).$$

The right-hand side belongs to $\rho_\lambda(M(u,f,s))$. This completes the proof of (1).

(2) Assume $\rho_\lambda(Q(s)(u \otimes f^s)) = 0$ with $Q(s) \in D_X[s]$. There exist $l \in \mathbb{N}$ and $Q_j \in D_X$ which are zero except finitely many indices j such that

$$Q(s)(u \otimes f^s) = \sum_{j \geq 0} (Q_j u) \otimes (s-\lambda)^j f^{s-l}.$$

By the assumption, $\rho_\lambda(Q(s)(u \otimes f^s)) = (Q_0 u) \otimes f^{\lambda - l}$ vanishes in $M \otimes_{K[x]} \mathcal{L}_f(\lambda)$, which means that $(Q_0 u) \otimes f^{-l}$ vanishes in $M \otimes_{K[x]} K[x, f^{-1}]$. It follows that $(Q_0 u) \otimes 1 = f^l (Q_0 u) \otimes f^{-l} = 0$ in $M \otimes_{K[x]} K[x, f^{-1}]$. Consequently, $(Q_0 u) \otimes f^s$ vanishes in $M \otimes_{K[x]} \mathcal{L}_f$. Thus we have

$$Q(s)(u \otimes f^s) = (s-\lambda) \sum_{j \geq 1} (Q_j u) \otimes (s-\lambda)^{j-1} f^{s-l} = (s-\lambda) Q'(s)(u \otimes f^{s-k})$$

with some $k \in \mathbb{N}$ and $Q'(s) \in D_X[s]$ in view of the proof of Lemma 2.5. By using (1) we obtain

$$b_{u,f}(s-1)\cdots b_{u,f}(s-k) Q(s)(u \otimes f^s) = (s-\lambda) Q'(s) \tilde{P}(s)(u \otimes f^s).$$

Hence $b_{u,f}(\lambda-1)\cdots b_{u,f}(\lambda-k) Q(s)(u \otimes f^s)$ belongs to $(s-\lambda)M(u,f,s)$. This completes the proof of (2). Q.E.D.

The following proposition extends Lemma 1.3 of Walther [27] for the case $M = K[x]$ and $u = 1$ almost verbatim.

Lemma 2.7. *Under the same assumption as in the preceding proposition, assume moreover that* $b_{u,f}(\lambda) = 0$. *Then one has*

$$D_X(fu \otimes f^\lambda) \subsetneq D_X(u \otimes f^\lambda)$$

in $M \otimes_{K[x]} \mathcal{L}_f(\lambda)$. *In particular,* $M \otimes_{K[x]} \mathcal{L}_f(\lambda)$ *is generated by* $u \otimes f^\lambda$, *but not by* $u \otimes f^{\lambda+1} = fu \otimes f^\lambda$, *over* D_X.

Proof. There exists $P(s) \in D_X[s]$ such that $P(s)(fu \otimes f^s) = b_{u,f}(s)(u \otimes f^s)$. Assume $D_X(fu \otimes f^\lambda) = D_X(u \otimes f^\lambda)$. Then there exists $A \in D_X$ such that $(1 - Af)(u \otimes f^\lambda) = 0$. By virtue of (2) of the preceding proposition, there exist $Q(s), R(s) \in D_X[s]$ such that

$$1 - Af = Q(s) + (s-\lambda)R(s), \qquad Q(s)(u \otimes f^s) = 0.$$

It follows that

$$\begin{aligned}\frac{b_{u,f}(s)}{s-\lambda}(u\otimes f^s) &= \frac{b_{u,f}(s)}{s-\lambda}Af(u\otimes f^s)+b_{u,f}(s)R(s)(u\otimes f^s)\\ &= \left(\frac{b_{u,f}(s)}{s-\lambda}A+R(s)P(s)\right)(fu\otimes f^s).\end{aligned}$$

This means that $b_{u,f}(s)/(s-\lambda)$ belongs to the ideal $I(u,f)$, which contradicts the definition of $b_{u,f}(s)$. This completes the proof. Q.E.D.

Summing up we obtain

Theorem 2.8. *Let $M = D_X u$ be a left D_X-module generated by $u \in M$ and $f \in K[x]$ be a non-constant polynomial. Assume that there exists the b-function $b_{u,f}(s)$ for u and f. Then the following conditions on $\lambda \in K$ are equivalent:*

(1) *$b_{u,f}(\lambda - k) \neq 0$ for any positive integer k.*

(2) *$M \otimes_{K[x]} \mathcal{L}_f(\lambda)$ is generated by $u \otimes f^\lambda$ over D_X.*

Proof. Assume $b_{u,f}(\lambda - k) = 0$ for some positive integer k and let k_0 be the maximum among such k. Then by (1) of Proposition 2.6 and Lemma 2.7, we have

$$\begin{aligned}\mathcal{L}_f(\lambda) = \mathcal{L}_f(\lambda-k_0) &= D_X(u\otimes f^{\lambda-k_0})\\ &\supsetneq D_X(u\otimes f^{\lambda-k_0+1}) \supset D_X(u\otimes f^\lambda).\end{aligned}$$

Hence $\mathcal{L}_f(\lambda)$ is not generated by $u \otimes f^\lambda$. Q.E.D.

The converse of the statement (2) of Proposition 2.6 will be given in Theorem 5.9 of Section 5 under the additional assumption that M is holonomic on X_f.

Let us recall local cohomology of D-modules. Let M be a finitely generated left D_X-module, and I be an ideal of $K[x]$. Then the k-th local cohomology group $H^k_I(M)$ supported by I is defined to be the k-th derived functor of the functor

$$M \longmapsto H^0_I(M) = \{u \in M \mid I^k u = 0 \text{ for some } k \in \mathbb{N}\}.$$

They have natural structure of left D_X-module, and they are holonomic if so is M as was proved by Kashiwara [13] in the analytic category.

If I is the principal ideal (f) generated by $f \in K[x]$, then there exists an exact sequence

$$0 \longrightarrow H^0_{(f)}(M) \longrightarrow M \xrightarrow{\iota} M[f^{-1}] \longrightarrow H^1_{(f)}(M) \longrightarrow 0$$

of left D_X-modules, where ι stands for the natural homomorphism such that $\iota(v) = v \otimes 1$ in $M[f^{-1}] = M \otimes_{K[x]} K[x, f^{-1}]$ for $v \in M$. Hence there is an isomorphism $H^1_{(f)}(M) \cong M[f^{-1}]/\iota(M)$ as left D_X-module.

In general, algorithms to compute $H^k_I(M)$ as left D_X-module were given in [17] for the case I is principal, and in [26] and [20] for general I, under the condition that M is holonomic. See also [25], where related topics such as associated primes and the Weyl closure are also discussed.

Corollary 2.9. *Let $M = D_X u$ be a left D_X-module generated by $u \in M$ and $f \in K[x]$ be a non-constant polynomial. Assume that there exists the b-function $b_{u,f}(s)$ for u and f. Then the following conditions are equivalent:*

(1) $b_{u,f}(j) \neq 0$ *for any integer* $j < k$.
(2) *The localization $M[f^{-1}]$ is generated by $u \otimes f^{-k}$ over D_X.*
(3) *The local cohomology group $H^1_{(f)}(M)$ is generated by the cohomology class $[u \otimes f^{-k}]$ over D_X.*

Proof. The equivalence of (1) and (2) is a special case of Theorem 2.8. In general, if $M[f^{-1}]$ is generated by $u \otimes f^{-k}$, then $H^1_{(f)}(M) = M[f^{-1}]/\iota(M)$ is generated by its residue class. Conversely, assume that $M[f^{-1}]/\iota(M)$ is generated by $[u \otimes f^{-k}]$. Then for any $w \in M[f^{-1}]$, there exist $P, Q \in D_X$ such that

$$w = P(u \otimes f^{-k}) + (Qu) \otimes 1 = (P + Qf^k)(u \otimes f^{-k}).$$

Henece $M[f^{-1}]$ is generated by $u \otimes f^{-k}$. Q.E.D.

§3. Localization algorithm revisited

Let $X = K^n$ be the n-dimensional affine space over K. Let M be a left module over $D_X = D_n$ and $f \in K[x]$ be a non-constant polynomial. Then $X_f := \{x \in X \mid f(x) \neq 0\}$ is an affine open subset of X. Our purpose is to reformulate the algorithm given in [21] for computing the localization $M[f^{-1}] := M \otimes_{K[x]} K[x, f^{-1}]$ as left D_X-module by using local cohomology, hoping to clarify the meaning of the algorithm as well as to make the canonical homomorphism $\iota : M \to M[f^{-1}]$ more explicit.

We assume in what follows, as well as in [21], that M is a holonomic D_X-module, or else it is holonomic on X_f with K being algebraically closed; i.e., $\mathrm{Char}(M) \cap \pi^{-1}(X_f)$ is an n-dimensional algebraic set of $\pi^{-1}(X_f)$, where $\mathrm{Char}(M)$ is the characteristic variety of M, which is an algebraic set of the cotangent bundle $T^*X = \{(x, \xi) \in K^n \times K^n\}$ and $\pi : T^*X \to X$ is the projection (see e.g., 2.4 of [18]).

Introducing a new variable t, set $Y = X \times K \ni (x,t)$ and

$$Z := \{(x,t) \in Y \mid tf(x) = 1\}.$$

Then Z is an affine subset of Y which is isomorphic to X_f. Let

$$B_{Z|Y} := H^1_{(tf(x)-1)}(K[x,t]) = K[x,t,(tf-1)^{-1}]/K[x,t]$$

be the first local cohomology group of $K[x,t]$ with support in Z, which we regard as a left D_Y-module. An arbitrary element of $B_{Z|Y}$ is expressed as

$$\left[\frac{a(x,t)}{(tf(x)-1)^{k+1}}\right] \qquad (k \in \mathbb{N},\ a(x,t) \in K[x,t]),$$

where the bracket denotes the residue class in $B_{Z|Y}$.

Set $f_i = \partial f/\partial x_i$ for $i = 1, \ldots, n$ and define

$$\delta^{(k,l)} := \left[\frac{f^{l+1}}{(tf-1)^{k+1}}\right]$$

for $k, l \in \mathbb{Z}$ with $l \geq -1$. Note that $\delta^{(k,l)} = 0$ by the definition if $k < 0$. As left $K[x,t]$-module, $B_{Z|Y}$ is generated by $\delta^{(k,-1)}$ with $k \in \mathbb{N}$.

We have the following identities for $k, l \geq 0$:

$$\begin{aligned}
\partial_t \delta^{(k,l)} &= -(k+1)\left[\frac{f^{l+2}}{(tf-1)^{k+2}}\right] = -(k+1)\delta^{(k+1,l+1)},\\
\partial_{x_i}\delta^{(k,l)} &= (l+1)\left[\frac{f_i f^l}{(tf-1)^{k+1}}\right] - (k+1)\left[\frac{tf_i f^{l+1}}{(tf-1)^{k+2}}\right]\\
&= (l+1)f_i\delta^{(k,l-1)} - (k+1)\left[\frac{f_i(tf-1+1)f^l}{(tf-1)^{k+2}}\right]\\
&= (l+1)f_i\delta^{(k,l-1)} - (k+1)f_i(\delta^{(k,l-1)} + \delta^{(k+1,l-1)})\\
&= (l-k)f_i\delta^{(k,l-1)} - (k+1)f_i\delta^{(k+1,l-1)},\\
t\delta^{(k,l)} &= \left[\frac{tf^{l+1}}{(tf-1)^{k+1}}\right] = \left[\frac{\{(tf-1)+1\}f^l}{(tf-1)^{k+1}}\right] = \delta^{(k-1,l-1)} + \delta^{(k,l-1)}.
\end{aligned}$$

In particular, we have

$$(\partial_t t + k)\delta^{(k,l)} = -(k+1)\delta^{(k+1,l)}, \qquad t\delta^{(0,0)} = \delta^{(0,-1)}.$$

Hence $B_{Z|Y}$ is generated by $\delta^{(0,0)} = [f(tf-1)^{-1}]$ as a left D_Y-module.

Lemma 3.1. *One has* $(tf-1)\delta^{(0,0)} = 0$ *and* $(\partial_{x_i} - f_i\partial_t t^2)\delta^{(0,0)} = 0$ *for* $i = 1, \ldots, n$.

Proof. The first equality follows immediately from the definition. The second equality follows from

$$\partial_t t^2 \delta^{(0,0)} = \partial_t t \delta^{(0,-1)} = -\delta^{(1,-1)}$$

in view of the formulae above. Q.E.D.

Let us regard $B_{Z|Y}$ as a module over the subring $K[x]$ of D_Y and consider the localization

$$\begin{aligned} B_{Z|Y}[f^{-1}] &:= B_{Z|Y} \otimes_{K[x]} K[x, f^{-1}] \\ &= K[x,t,f^{-1},(tf-1)^{-1}]/K[x,t,f^{-1}] \end{aligned}$$

with respect to f. Let us denote the residue class in $B_{Z|Y}[f^{-1}]$ by $[\bullet]'$ in order to distinguish it from the residue class in $B_{Z|Y}$ which is denoted $[\bullet]$.

Lemma 3.2. *The natural homomorphism*

$$\iota' : B_{Z|Y} \ni \left[\frac{a(x,t)}{(tf-1)^{k+1}}\right] \longmapsto \left[\frac{a(x,t)}{(tf-1)^{k+1}}\right]' \in B_{Z|Y}[f^{-1}]$$

is an isomorphism of left D_Y-modules.

Proof. Assume $\iota'([a(x,t)(tf-1)^{-k-1}]) = 0$ with $a(x,t) \in K[x,t]$. Then there exists an integer l such that $f^l a(x,t)$ is divisible by $(tf-1)^{k+1}$ in $K[x,t]$. Since f and $tf-1$ are relatively prime, $a(x,t)$ must be divisible by $(1-tf)^{k+1}$. This proves that ι' is injective.

Let us show that ι' is surjective. It suffices to show that

$$[f^{-m}(tf-1)^{-k-1}]' \in \iota'(B_{Z|Y})$$

for any $k, m \in \mathbb{N}$ by induction on $k+m$, which obviously holds for $k = m = 0$. Suppose $k+m \geq 1$. We have

$$\left[\frac{tf}{(tf-1)^{k+1}}\right] = \left[\frac{1+(tf-1)}{(tf-1)^{k+1}}\right] = \left[\frac{1}{(tf-1)^{k+1}}\right] + \left[\frac{1}{(tf-1)^{k}}\right].$$

It follows that

$$\left[\frac{f^{-m}}{(tf-1)^{k+1}}\right]' = \left[\frac{tf^{1-m}}{(tf-1)^{k+1}}\right]' - \left[\frac{f^{-m}}{(tf-1)^{k}}\right]'.$$

By the induction hypothesis, the right-hand side belongs to the image of ι'. This completes the proof. Q.E.D.

Proposition 3.3. *Let M be a finitely generated left D_X-module. Then the homomorphism*

$$B_{Z|Y} \otimes_{K[x]} M \xrightarrow{\sim} B_{Z|Y} \otimes_{K[x]} M[f^{-1}]$$

of left D_Y-modules, which is induced by the natural homomorphism $\iota : M \to M[f^{-1}]$ is an isomorphism.

Proof. We have

$$\begin{aligned} B_{Z|Y} \otimes_{K[x]} M[f^{-1}] &= B_{Z|Y} \otimes_{K[x]} (K[x, f^{-1}] \otimes_{K[x]} M) \\ &= (B_{Z|Y} \otimes_{K[x]} K[x, f^{-1}]) \otimes_{K[x]} M \\ &= B_{Z|Y}[f^{-1}] \otimes_{K[x]} M. \end{aligned}$$

Hence the isomorphism ι' induces an isomorphism

$$B_{Z|Y} \otimes_{K[x]} M \xrightarrow{\sim} B_{Z|Y}[f^{-1}] \otimes_{K[x]} M = B_{Z|Y} \otimes_{K[x]} M[f^{-1}].$$

Q.E.D.

Proposition 3.4. *Let M be a finitely generated left D_X-module. Then there exists an isomorphism*

$$B_{Z|Y} \otimes_{K[x]} M \xrightarrow{\sim} B_{Z|Y}[f^{-1}] \otimes_{K[x,f^{-1}]} M[f^{-1}]$$

of left D_Y-modules.

Proof. We have

$$\begin{aligned} &B_{Z|Y}[f^{-1}] \otimes_{K[x,f^{-1}]} M[f^{-1}] \\ &= (B_{Z|Y} \otimes_{K[x]} K[x, f^{-1}]) \otimes_{K[x,f^{-1}]} M[f^{-1}] = B_{Z|Y} \otimes_{K[x]} M[f^{-1}]. \end{aligned}$$

This completes the proof combined with Proposition 3.4. Q.E.D.

Let $D_X[f^{-1}] := K[x, f^{-1}] \otimes_{K[x]} D_X$ and $D_Y[f^{-1}] := K[x, f^{-1}] \otimes_{K[x]} D_Y$ be the localization of D_X and D_Y by f, which can be regarded as ring extensions of D_X and of D_Y respectively. Then $B_{Z|Y}[f^{-1}]$ and $B_{Z|Y}[f^{-1}] \otimes_{K[x,f^{-1}]} M[f^{-1}]$ have natural structures of left $D_Y[f^{-1}]$-module.

Definition 3.5. We set $\delta^{(j)} = \iota'(\delta^{(j,j)})$ for $j \in \mathbb{N}$. We denote $\delta = \delta^{(0)}$.

Lemma 3.6. *As an element of the left $D_Y[f^{-1}]$-module $B_{Z|Y}[f^{-1}]$, the annihilator of δ coincides with the left ideal of $D_Y[f^{-1}]$ generated by*

$$t - f^{-1}, \quad \partial_{x_i} - f_i \partial_t t^2 \quad (i = 1, \dots, n).$$

Proof. By Lemma 3.1, these operators annihilate δ. Assume $P \in D_Y[f^{-1}]$ annihilates δ. There exist elements Q_i and R of $D_Y[f^{-1}]$ such that

$$P = Q_0(t - f^{-1}) + \sum_{i=1}^{n} Q_i(\partial_{x_i} - f_i\partial_t t^2) + R$$

and that R belongs to $K[x, f^{-1}, \partial_t]$. Writing R in a finite sum

$$R = \sum_{j=0}^{l} r_j(x) f^{-k} \partial_t^j$$

with $r_j(x) \in K[x]$ and $k, l \in \mathbb{N}$, we have

$$\begin{aligned} 0 = R\delta &= \sum_{j=0}^{l} f^{-k} r_j(x) \partial_t^j \delta = \sum_{j=0}^{l} (-1)^j j! f^{-k} r_j(x) \delta^{(j)} \\ &= f^{-k} \left[\frac{\sum_{j=0}^{l} (-1)^j j! (tf-1)^{l-j} r_j(x)}{(tf-1)^{l+1}} \right]'. \end{aligned}$$

Since ι' is injective, this implies that $r_j(x) = 0$ for any $j \geq 0$, that is, $R = 0$. Q.E.D.

Proposition 3.7. *Any element of $B_{Z|Y}[f^{-1}] \otimes_{K[x,f^{-1}]} M[f^{-1}]$ is uniquely expressed as a finite sum $\sum_{j\geq 0} \delta^{(j)} \otimes v_j$ with $v_j \in M[f^{-1}]$.*

Proof. By Lemma 3.6, $B_{Z|Y}[f^{-1}]$ is isomorphic to $K[x, f^{-1}, \partial_t]$ as left $K[x, f^{-1}]$-module. Hence $\delta^{(j)} = (-1)^j (1/j!) \partial_t^j \delta$ ($j \in \mathbb{N}$) constitute a free basis of $B_{Z|Y}[f^{-1}]$ over $K[x, f^{-1}]$. This implies the assertion of the proposition. Q.E.D.

Definition 3.8. Set $\vartheta_i := \partial_{x_i} - f_i \partial_t t^2$ for $i = 1, \ldots, n$. Define a ring homomorphism τ from D_X to D_Y by

$$\tau : D_X \ni P(x, \partial_x) \longmapsto P(x, \vartheta_1, \ldots, \vartheta_n) \in D_Y.$$

Since $\vartheta_1, \ldots, \vartheta_n$ commute with each other, and $[\vartheta_i, x_j] = \delta_{ij}$, this substitution is a well-defined ring homomorphism.

Lemma 3.9. *One has*

$$\delta^{(0,0)} \otimes Pv = \tau(P)(\delta^{(0,0)} \otimes v), \qquad \delta \otimes Pv' = \tau(P)(\delta \otimes v')$$

in $B_{Z|Y} \otimes_{K[x]} M$ and in $B_{Z|Y}[f^{-1}] \otimes_{K[x,f^{-1}]} M[f^{-1}]$ respectively for any $v \in M$, $v' \in M[f^{-1}]$ and $P \in D_X$.

Proof. We have only to show the first equality. Since

$$(\partial_{x_i} - f_i\partial_t t^2)\delta^{(0,0)} = 0,$$

we have

$$\begin{aligned}\tau(\partial_{x_i})(\delta^{(0,0)} \otimes v) &= (\partial_{x_i} - f_i\partial_t t^2)\delta^{(0,0)} \otimes v + \delta^{(0,0)} \otimes \partial_{x_i} v \\ &= \delta^{(0,0)} \otimes \partial_{x_i} v.\end{aligned}$$

We can verify that

$$\tau(P)(\delta^{(0,0)} \otimes v) = \delta^{(0,0)} \otimes (Pv)$$

holds by induction on the order of P. Q.E.D.

Proposition 3.10. *Let $v \in M[f^{-1}]$, $P \in D_X$, and $k \in \mathbb{N}$. Then $P(f^{-k}v) = 0$ holds in $M[f^{-1}]$ if and only if $\tau(P)t^k(\delta \otimes v) = 0$ holds in $B_{Z|Y}[f^{-1}] \otimes_{K[x,f^{-1}]} M[f^{-1}]$.*

Proof. Since $t^k f^k \delta = (1 + t^k f^k - 1)\delta = \delta$, we have by Lemma 3.9

$$\begin{aligned}\delta \otimes P(f^{-k}v) &= \tau(P)(\delta \otimes f^{-k}v) \\ &= \tau(P)t^k f^k(\delta \otimes f^{-k}v) = \tau(P)t^k(\delta \otimes v).\end{aligned}$$

This vanishes if and only if $P(f^{-k}v) = 0$ by Proposition 3.7. Q.E.D.

Summing up we obtain

Theorem 3.11. *Let $M = D_X u$ be a left D_X-module generated by u and $I = \mathrm{Ann}_{D_X} u$ the annihilator of u so that $M = D_X/I$. Let $\iota : M \to M[f^{-1}]$ be the canonical homomorphism which sends $u \in M$ to $u \otimes 1$. Let G be a finite set of generators of I, and J be the left ideal of D_Y generated by $\{\tau(P) \mid P \in G\}$ and $tf - 1$. Then*

(1) *J coincides with the annihilator $\mathrm{Ann}_{D_Y}(\delta \otimes \iota(u))$ of $\delta \otimes \iota(u)$ in $B_{Z|Y}[f^{-1}] \otimes_{K[x,f^{-1}]} M[f^{-1}]$.*

(2) *$B_{Z|Y}[f^{-1}] \otimes_{K[x,f^{-1}]} M[f^{-1}]$ is generated by $\delta \otimes \iota(u)$ as a left D_Y-module.*

(3) *As a left D_Y-module, $B_{Z|Y} \otimes_{K[x]} M$ is isomorphic to D_Y/J.*

Proof. (1) It is obvious that J is contained in $\mathrm{Ann}_{D_Y}(\delta \otimes \iota(u))$. Suppose $P(\delta \otimes \iota(u)) = 0$ with $P \in D_Y$. There exist $R \in D_Y[f^{-1}]$ and $a_{\alpha,j}(x) \in K[x, f^{-1}]$ which are zero except finitely many $(\alpha, j) \in \mathbb{N}^n \times \mathbb{N}$

such that

$$\begin{aligned}P &= \sum_{\alpha\in\mathbb{N}^n, j\geq 0} a_{\alpha,j}(x)\partial_t^j(\partial_{x_1} - f_1\partial_t t^2)^{\alpha_1}\cdots(\partial_{x_n} - f_n\partial_t t^2)^{\alpha_n} \\ &\quad + R(t - f^{-1}) \\ &= \sum_{j\geq 0}\partial_t^j\tau(Q_j) + R(t - f^{-1})\end{aligned}$$

with $Q_j := \sum_{\alpha\in\mathbb{N}^n} a_{\alpha,j}(x)\partial_x^\alpha \in D_X[f^{-1}]$. Then we have

$$0 = P(\delta\otimes\iota(u)) = \sum_{j\geq 0}\partial_t^j\tau(Q_j)(\delta\otimes\iota(u)) = \sum_{j\geq 0}(-1)^j j!\delta^{(j)}\otimes Q_j\iota(u)$$

and consequently $Q_j\iota(u) = 0$ for each $j \geq 0$ by Proposition 3.7. This implies that $f^lQ_ju = 0$ holds in M, that is, f^lQ_j belongs to I, for some $l \in \mathbb{N}$ independent of j. We may also assume that f^lR belongs to D_Yf. Hence $f^lP = \sum_{j\geq 0}\partial_t^j\tau(f^lQ_j) + f^lR(t - f^{-1})$ belongs to J. Since $(1 - t^lf^l)^kP$ belongs to $D_Y(1 - t^lf^l)$, and hence to J, if we take $k \in \mathbb{N}$ sufficiently large, and t^lf^lP belongs to J, we conclude that P itself belongs to J.

(2) By the assumption, Lemma 2.5, and Proposition 3.7, an arbitrary element of $B_{Z|Y}[f^{-1}]\otimes_{K[x,f^{-1}]} M[f^{-1}]$ is expressed as a finite sum

$$\sum_{j\geq 0}\partial_t^j\delta\otimes P_j(u\otimes f^{-k})$$

with $P_j \in D_X$ and $k \in \mathbb{N}$. We get

$$\begin{aligned}\sum_{j\geq 0}\partial_t^j\delta\otimes P_j(u\otimes f^{-k}) &= \sum_{j\geq 0}\partial_t^j\tau(P_j)(\delta\otimes(u\otimes f^{-k})) \\ &= \sum_{j\geq 0}\partial_t^j\tau(P_j)t^kf^k(\delta\otimes(u\otimes f^{-k})) \\ &= \sum_{j\geq 0}\partial_t^j\tau(P_j)t^k(\delta\otimes\iota(u)).\end{aligned}$$

This completes the proof of (2).

(3) follows from (1), (2) and Proposition 3.4. Q.E.D.

Definition 3.12. For the sake of simplicity of the notation, let us set

$$\tilde{M} := B_{Z|Y}[f^{-1}]\otimes_{K[x,f^{-1}]} M[f^{-1}] \cong B_{Z|Y}\otimes_{K[x]} M$$

and define a homomorphism $\varphi : M[f^{-1}] \to \tilde{M}/\partial_t \tilde{M}$ by

$$\varphi(v) = \delta \otimes v \mod \partial_t \tilde{M}.$$

for $v \in M[f^{-1}]$.

Theorem 3.13. *The homomorphism $\varphi : M[f^{-1}] \to \tilde{M}/\partial_t \tilde{M}$ is an isomorphism of left $D_X[f^{-1}]$-modules, and consequently of D_X-modules.*

Proof. By Proposition 3.7 one has direct sum decompositions

$$\tilde{M} = (\delta \otimes M[f^{-1}]) \oplus (\delta^{(1)} \otimes M[f^{-1}]) \oplus (\delta^{(2)} \otimes M[f^{-1}]) \oplus \cdots ,$$
$$\partial_t \tilde{M} = (\delta^{(1)} \otimes M[f^{-1}]) \oplus (\delta^{(2)} \otimes M[f^{-1}]) \oplus \cdots$$

as $K[x, f^{-1}]$-modules. Hence φ is an isomorphism of $K[x, f^{-1}]$-modules. For $v \in M[f^{-1}]$ and $P \in D_X[f^{-1}]$, one has

$$P(\delta \otimes v) \equiv \tau(P)(\delta \otimes v) = \delta \otimes Pv \mod \partial_t \tilde{M}$$

since $P - \tau(P)$ belongs to $\partial_t D_Y$. Hence φ is an isomorphism of left $D_X[f^{-1}]$-modules. Q.E.D.

Theorem 3.14. *Assume K is algebraically closed. If M is holonomic on X_f, i.e., if $\mathrm{Char}(M) \cap \pi^{-1}(X_f)$ is an n-dimensional algebraic set, then D_Y/J is a holonomic D_Y-module.*

Proof. We may assume $M = D_X/I$. By the definition, we have

$$\begin{aligned}
\mathrm{Char}(D_Y/J) &= \{(x, t; \xi, \tau) \in K^{2n+2} \mid \\
&\quad \sigma(P)(x, \xi_1 - f_1 t^2 \tau, \dots, \xi_n - f_n t^2 \tau) \quad (\forall P \in I), \quad tf(x) = 1\} \\
&= \{(x, t; \xi, \tau) \mid (x, \xi_1 - f_1 t^2 \tau, \dots, \xi_n - f_n t^2 \tau) \in \mathrm{Char}(M),\ tf(x) = 1\}.
\end{aligned}$$

Hence there is a bijection

$$\begin{aligned}
&(\mathrm{Char}(M) \cap \pi^{-1}(X_f)) \times K \ni (x, \xi, \tau) \\
&\qquad \longmapsto (x, 1/f(x); \xi_1 + f_1 t^2 \tau, \dots, \xi_n + f_n t^2 \tau) \in \mathrm{Char}(D_Y/J).
\end{aligned}$$

This implies that $\mathrm{Char}(D_Y/J)$ is of dimension $n + 1$. Q.E.D.

The D_X-module $\tilde{M}/\partial_t \tilde{M}$ is nothing but the integral of the D_Y-module $\tilde{M}$ with respect to t, and $\tilde{M}$ is isomorphic to D_Y/J by Theorem 3.11. Suppose that M is holonomic on X_f. Then $\tilde{M} = D_Y/J$ is a holonomic D_Y-module by the theorem above. Hence $\tilde{M}/\partial_t \tilde{M}$ is also a holonomic D_X-module. In particular, there exists $k_0 \in \mathbb{N}$, or else

$k_0 = -1$, such that $\tilde{M}/\partial_t\tilde{M}$ is generated by residue classes $[t^j\delta \otimes u] = \varphi(u \otimes f^{-j})$ for $0 \le j \le k_0$.

The relations among these generators, i.e., a presentation of the D_X-module $\tilde{M}/\partial_t\tilde{M}$ can be computed by the integration algorithm for D-modules under the assumption that what is called the b-function $b(s)$ with respect to the weight vector $w = (0, ..., 0, 1; 0, ..., 0, -1)$ for $(x_1, \dots, x_n, t; \partial_{x_1}, \dots, \partial_{x_n}, \partial_t)$ exists. The integer k_0 above can be taken as the maximum integer root of $b(s)$. See [19], [23], [18] for details. This condition is fulfilled if $B_{Z|Y} \otimes_{K[x]} M = D_Y/J$ is holonomic, which is the case if M is holonomic on X_f.

In conclusion, we get

Algorithm 3.15 (localization)**.** Input: A set G_0 of generators of a left ideal $I = \mathrm{Ann}_{D_X} u$ of D_X such that $M = D_X u$.
Output: a presentation of $M[f^{-1}]$.

(1) Let J be the left ideal of D_Y generated by $\tau(G_0) \cup \{tf - 1\}$.
(2) Compute the b-function $b(s)$ of D_Y/J with respect to the weight vector $(0, ..., 0, 1; 0, ..., 0, -1)$ by e.g., Algorithm 5.6 of [18]. Quit if $b(s)$ does not exist; the computation fails.
(3) Let k_0 be the maximum integer root of $b(s)$ if any. Then $D_Y/(J + \partial_t D_Y)$, which is isomorphic to $\tilde{M}/\partial_t\tilde{M}$, is generated by the residue classes $[t^j]$ with $0 \le j \le k_0$. If $k_0 < 0$ or $b(s)$ does not have any integer root, then $M[f^{-1}] = 0$; quit.
(4) Compute a set G_1 of generators of the left D_X-submodule

$$N := \left\{ (P_0, P_1, \dots, P_{k_0}) \in (D_X)^{k_0+1} \mid \sum_{j=0}^{k_0} P_j [t^j] = 0 \right\}$$

of $(D_X)^{k_0+1}$ by using the integration algorithm (e.g., Algorithm 5.10 of [18]). Then one has isomorphisms

$$M[f^{-1}] \cong D_Y/(J + \partial_t D_Y) \cong (D_X)^{k_0+1}/N,$$

by which $u \otimes 1, \dots, u \otimes f^{-k_0}$ correspond to the residue classes $[(1, 0, \dots, 0)], \dots, [(0, \dots, 0, 1)]$.

In [21], the homomorphism τ above (denoted ϕ in [21]) is defined with ϑ_i replaced by $\partial_{x_i} - f_i t^2 \partial_t$. This induces the homomorphism $M \to M[f^{-1}]$ which sends u to $u \otimes f^{-2}$ instead of $u \otimes 1$.

The algorithm above and the isomorphisms

$$H^0_{(f)}(M) = \ker \iota \cong \mathrm{Ann}_{D_X} \iota(u)/\mathrm{Ann}_{D_X} u, \quad H^1_{(f)}(M) = M[f^{-1}]/\iota(M)$$

also provide us with algorithms to compute $H^j_{(f)}(M)$ for $j = 0, 1$, which work at least if M is holonomic on X_f. We remark that H. Tsai gave an algorithm for $H^0_{(f)}(M)$ without any assumption on M; see Algorithms 4.3 and 4.5 of [25].

Algorithm 3.16 ($\iota(M)$ and $H^j_{(f)}(M)$)**.** Input: A set G_0 of generators of a left ideal $I = \mathrm{Ann}_{D_X} u$ of D_X such that $M = D_X u$. Output: presentations of $\iota(M)$, $H^1_{(f)}(M)$, and $H^0_{(f)}(M)$ as D_X-modules.

(1) Compute the integer k_0 and a set G_1 of generators of the module N of Algorithm 3.15.

(2) Compute a set G_2 of generators of the left ideal

$$\tilde{I} := \{P \in D_X \mid (P, 0, \dots, 0) \in N\} = \mathrm{Ann}_{D_X} \iota(u)$$

of D_X by using a Gröbner basis of N with respect to a POT (position-over-term) order (see e.g., Chapter 5 of [9]). Then one has $\iota(M) \cong D_X/\tilde{I}$ with the correspondence $\iota(u) \leftrightarrow \overline{1}$.

(3) Set

$$G_1' := \{(P_1, \dots, P_{k_0}) \mid (P_0, P_1, \dots, P_{k_0}) \in G_1 \ (\exists P_0 \in D_X)\}$$

and N' be the left D_X-submodule of $(D_X)^{k_0}$ generated by G_1'. Compute a set G_3 of generators of the left ideal

$$I_1 := \{P \in D_X \mid (0, \dots, 0, P) \in N'\} = \mathrm{Ann}_{D_X}[f^{-k_0}\iota(u)]$$

of D_X by using a Gröbner basis of N' with respect to a POT order. Then one has $H^1_{(f)}(M) \cong D_X/I_1$ with the correspondence $[f^{-k_0}\iota(u)] \leftrightarrow \overline{1}$.

(4) Set $G_2 = \{P_1, \dots, P_l\}$. Compute a set of generators of the left D_X-module

$$N_0 := \left\{(Q_1, \dots, Q_l) \in (D_X)^l \mid \sum_{j=1}^{l} Q_j P_j \in I\right\}$$

through syzygies among $\{P_1, \dots, P_l\} \cup G_0$. Then one has

$$H^0_{(f)}(M) \cong \tilde{I}/I \cong (D_X)^l/N_0$$

with the correspondence

$$P_1 u \leftrightarrow (1, 0, \dots, 0) \mod N_0, \quad \dots, \quad P_l u \leftrightarrow (0, \dots, 0, 1) \mod N_0.$$

In what follows, we freely use the notation and the terminology introduced in Chapters 2, 3, 5 of [18] concerning weight vectors, Gröbner bases, and b-functions.

Example 3.17. Set $n = 1$ and $x = x_1$. It is easy to see that

$$\mathcal{N}_x = D_X x^s \cong D_X / D_X (x\partial_x - s).$$

Since the b-function of x is $s+1$, $\mathcal{N}_x(\lambda)$ is isomorphic to the submodule $D_X x^\lambda$ of $\mathcal{L}_x(\lambda)$ and hence x-saturated, if $\lambda \notin \mathbb{N}$. So let us consider $M = \mathcal{N}_f(0) = D_X/D_X x\partial_x$. Let u be the residue class of 1 in M. The left ideal J of D_Y defined in Theorem 3.11 is generated by $tx - 1$ and $\partial_x - \partial_t t^2$. A Gröbner basis of J with respect to a monomial order adapted to the weight vector $(1, 0; -1, 0)$ for $(t, x, \partial_t, \partial_x)$ is

$$tx-1, \quad x^2\partial_x - \partial_t, \quad t\partial_t - x\partial_x + 1 = \partial_t t - x\partial_x, \quad t^2\partial_t - \partial_x + 2t = \partial_t t^2 - \partial_x.$$

The b-function of J with respect to the weight vector above (see 5.2 of [18]) is divisible by $s(s-1)$ by virtue of the operator $\partial_t t^2 - \partial_x$. Hence the integration module $\tilde{M}/\partial_t \tilde{M} = D_Y/(J + \partial_t D_Y)$ is generated by the residue classes $[u]$ and $[tu]$, which correspond to $u \otimes 1$ and $u \otimes x^{-1}$ in $M[x^{-1}]$ respectively, over D_X. The fundamental relations among the generators can be read off from the Gröbner basis above as follows (see Algorithm 5.10 of [18]):

$$x[tu] - [u] = 0, \quad \partial_x[u] = 0, \quad x^2\partial_x[tu] + [u] = 0, \quad (x\partial_x + 1)[tu] = 0.$$

We translate these relations to vectors

$$(-1, x), \quad (\partial_x, 0), \quad (1, x^2\partial_x), \quad (0, x\partial_x + 1)$$

in the free module D_X^2. Let N be the left D_X-submodule of D_X^2 generated by these vectors. By using Gröbner bases of N with respect to POT orders we can confirm that $\mathrm{Ann}_{D_X}[u] = D_X\partial_x$ and $\mathrm{Ann}_{D_X}[tu] = D_X(x\partial_x + 1)$. Hence $M[x^{-1}]$ is generated by $u \otimes x^{-1}$ and isomorphic to $D_X/D_X(x\partial_x + 1) \cong K[x, x^{-1}]$ with the correspondence $u \otimes x^{-1} \leftrightarrow \overline{1}$. The image $\iota(M)$ of $\iota : M \to M[x^{-1}]$ is isomorphic to $D_X/D_X\partial_x \cong K[x]$ with the correspondence $u \otimes 1 \leftrightarrow \overline{1}$. Finally we get

$$H^0_{(x)}(M) = \ker \iota = D_X\partial_x u = K[\partial_x]u \cong D_X/D_X x,$$
$$H^1_{(x)}(M) = D_X[tu]/D_X[u] \cong D_X/D_X x.$$

The following is an example of non-holonomic M:

Example 3.18. Set $n = 2$, $x_1 = x$, $x_2 = y$, $P = x\partial_x^2 + \partial_y$, and $M = D_X/D_X P = D_X u$ as in Example 2.2. Then M is x- and y-saturated. The localizations of M are

$$M[x^{-1}] = D_X(u \otimes x^{-2}) = D_X/D_X(x^2{\partial_x}^2 + 4x\partial_x + x\partial_y + 2),$$
$$M[y^{-1}] = D_X(u \otimes y^{-1}) = D_X/D_X(xy{\partial_x}^2 + y\partial_y + 1).$$

The first local cohomology groups are

$$H^1_{(x)}(M) = D_X[u \otimes x^{-2}] = D_X/(D_X x^2 + D_X x\partial_y),$$
$$H^1_{(y)}(M) = D_X[u \otimes y^{-1}] = D_X/D_X y,$$

both of which are not holonomic.

Example 3.19. Set $n = 2$, $x_1 = x$, $x_2 = y$, and $f = x^3 - y^2$. Let us consider

$$M = D_X/(D_X\partial_x f + D_X\partial_y f) = D_X u \quad (u = \overline{1}).$$

The characteristic variety of M is

$$\mathrm{Char}(M) = \{(x, y; \xi, \eta) \in K^4 \mid x^3 - y^2 = 0\} \cup \{(x, y; 0, 0) \mid (x, y) \in X\}.$$

Hence M is holonomic on X_f but not on X. The localization $M[f^{-1}]$ is given by

$$M[f^{-1}] = D_X/(D_X(2x\partial_x + 3y\partial_y + 6) + D_X(2y\partial_x + 3x^2\partial_y))$$

with the correspondence $u \otimes 1 \leftrightarrow \overline{1}$ and is holonomic on X. Hence we have $\iota(M) = M[f^{-1}]$ and $H^1_{(f)}(M) = 0$. Algorithm 3.16 also gives a presentation of $H^0_{(f)}(M)$, which is rather complicated. We can verify that its characteristic variety is

$$\mathrm{Char}(H^0_{(f)}(M)) = \{(x, y; \xi, \eta) \in K^4 \mid x^3 - y^2 = 0\}$$

as is expected.

§4. Algorithms for $u \otimes f^s$ and the b-function

The purpose here is to give algorithms to compute $M(u, f, s)$ and $M(u, f, \lambda)$ as well as the b-function $b_{u,f}(s)$ for an arbitrary D_X-module $M = D_X u$ that is holonomic on X_f, and an arbitrary non-constant polynomial f. Algorithms for these objects were already given in [17] under the additional assumption that M is f-saturated. We remove this assumption by using the localization algorithm.

Set $X = K^n$ and $Y = K^{n+1}$ with coordinates $x = (x_1, \dots, x_n)$ of X and (x, t) of Y. Let $f = f(x) \in K[x]$ be a non-constant polynomial and let Z be the affine subset $Z = \{(x, t) \mid t = f(x)\}$ of Y. (Note that Z is different from what was defined in the previous section.) We regard the local cohomology group

$$B_{Z|Y} := H^1_Z(K[x, t]) = K[x, t, (t - f)^{-1}]/K[x, t]$$

as a left D_Y-module. For $k \in \mathbb{N}$, let

$$\delta^{(k)}(t - f(x)) = \left[\frac{(-1)^k k!}{(t - f(x))^{k+1}}\right]$$

be the residue class in $B_{Z|X}$ and denote $\delta(t - f) = \delta^{(0)}(t - f)$. Then $\delta(t - f)$ satisfies a holonomic system

$$(t - f)\delta(t - f) = (\partial_{x_i} + f_i\partial_t)\delta(t - f) = 0 \qquad (1 \leq i \leq n)$$

with $f_i = \partial f/\partial x_i$. Hence there exists an isomorphism $B_{Z|Y} \cong D_Y/J_0$ with

$$J_0 := D_Y(t - f) + D_Y(\partial_{x_1} + f_1\partial_t) + \cdots + D_Y(\partial_{x_n} + f_n\partial_t)$$

as left D_Y-module since J_0 is maximal. In particular, $\delta^{(k)}(t - f)$ $(k \in \mathbb{N})$ constitute a free basis of $B_{Z|Y}$ over $K[x]$.

Following Malgrange [15], let us give $\mathcal{L}_f = D_X[s]f^s$ a structure of left D_Y-module by

$$t(a(x, s)f^{-k}f^s) = a(x, s + 1)f^{-k+1}f^s,$$
$$\partial_t(a(x, s)f^{-k}f^s) = -sa(x, s - 1)f^{-k-1}f^s$$

for $a(x, s) \in K[x, s]$ and $k \in \mathbb{N}$. The actions of t and ∂_t on $\mathcal{L}_f$ satisfy $[\partial_t, t] = 1$, and they commute with x_i, ∂_{x_i}. Hence the definition above extends to the action of D_Y on $\mathcal{L}_f$. In particular, $\partial_t t f^s = -sf^s$ holds, which will play an important role in what follows.

With respect to this action, we can regard $B_{Z|Y}$ as a left D_Y-submodule of $\mathcal{L}_f$ by identifying $\delta^{(k)}(t - f)$ with $(-1)^k s(s - 1) \cdots (s - k + 1)f^{s-k}$ in $\mathcal{L}_f$. In fact, we have

$$J_0 = \mathrm{Ann}_{D_Y} f^s = \mathrm{Ann}_{D_Y} \delta(t - f)$$

since J_0 annihilates f^s as well as $\delta(t - f)$ and J_0 is a maximal left ideal.

For a left D_X-module $M = D_X u$, let us consider the tensor product $M \otimes_{K[x]} B_{Z|Y}$, which is a left D_Y-module.

Definition 4.1. Set $\vartheta_i' := \partial_{x_i} + f_i\partial_t$ for $i = 1, \dots, n$. Define a ring homomorphism τ' from D_X to D_Y by

$$\tau' : D_X \ni P(x, \partial_x) \longmapsto P(x, \vartheta_1', \dots, \vartheta_n') \in D_Y.$$

Since $\vartheta_1', \dots, \vartheta_n'$ commute with each other, and $[\vartheta_i', x_j] = \delta_{ij}$, this substitution is a well-defined ring homomorphism.

Since $\vartheta_i'\delta(t-f) = 0$ for $1 \le i \le n$,

$$\tau'(P)(v \otimes \delta(t-f)) = Pv \otimes \delta(t-f)$$

holds for any $v \in M$ and $P \in D_X$. Hence we have

$$Pu \otimes \partial_t^k \delta(t-f) = \partial_t^k(Pu \otimes \delta(t-f)) = \partial_t^k \tau'(P)(u \otimes \delta(t-f)).$$

This proves

Lemma 4.2. *If $M = D_X u$, then the left D_Y-module $M \otimes_{K[x]} B_{Z|Y}$ is generated by $u \otimes \delta(t-f)$.*

Theorem 4.3. *Let $M = D_X u$ be a left D_X-module generated by u and $f \in K[x]$. Let G be a finite set of generators of $I := \mathrm{Ann}_{D_X} u$ and let J be the left ideal of D_Y generated by $\{\tau'(P) \mid P \in G\} \cup \{t-f\}$. Then J coincides with the annihilator $\mathrm{Ann}_{D_Y}(u \otimes \delta(t-f))$. Moreover, if M is a holonomic D_X-module, then $M \otimes_{K[x]} B_{Z|Y}$ is a holonomic D_Y-module.*

The first part of this proposition was proved by Walther [26]. The proof is almost the same as the proof of Theorem 3.11. The last assertion can be proved in the same way as Theorem 3.14.

Thus we have an algorithm to compute $M \otimes_{K[x]} B_{Z|Y}$. The inclusion $B_{Z|X} \subset \mathcal{L}_f$ induces a homomorphism

$$\psi : M \otimes_{K[x]} B_{Z|Y} \longrightarrow M \otimes_{K[x]} \mathcal{L}_f$$

of left D_Y-modules. Our main aim is to compute the $D_X[s]$-submodule $M(u, f, s) = D_X[s](u \otimes f^s)$ of $M \otimes_{K[x]} \mathcal{L}_f$, which is the image of the submodule $D_X[s](u \otimes f^s)$ of $M \otimes_{K[x]} B_{Z|Y}$ by ψ. The following lemma was proved in [17] as Proposition 6.13.

Lemma 4.4. *The homomorphism ψ above is injective if and only if M is f-saturated.*

Proof. An arbitrary element of $M \otimes_{K[x]} B_{Z|Y}$ is expressed uniquely as

$$w = \sum_{j=0}^{k} v_j \otimes \delta^{(j)}(t-f)$$

with $k \in \mathbb{N}$ and $v_j \in M$. Then

$$\psi(w) = \sum_{j=0}^{k} (-1)^j v_j \otimes (s(s-1)\cdots(s-j+1)f^{-j}f^s)$$

vanishes if and only if each $v_j \otimes f^{-j}$ vanishes in $M \otimes_{K[x]} K[x, f^{-1}]$, which is equivalent to $v_j \in H^0_{(f)}(M)$. Q.E.D.

Lemma 4.5. *Let $M = D_X u$ be a left D_X-module generated by u and $f \in K[x]$ be a non-constant polynomial. Let $\iota : M \to M[f^{-1}]$ be the canonical homomorphism. Then ι induces isomorphisms*

$$M \otimes_{K[x]} \mathcal{L}_f \xrightarrow{\sim} \iota(M) \otimes_{K[x]} \mathcal{L}_f, \qquad M(u, f, s) \xrightarrow{\sim} \iota(M)(u, f, s)$$

of left $D_X[s]$-modules.

Proof. Since $\mathcal{L}_f$ is isomorphic to $K[x, f^{-1}, s]$ as a $K[x, s]$-module, we have only to show that the natural homomorphism

$$M[f^{-1}] \to \iota(M)[f^{-1}]$$

is an isomorphism, which is obvious by the definition. The second isomorphism follows from the first. Q.E.D.

Summing up we obtain

Algorithm 4.6 ($M(u, f, s)$ and $M(u, f, \lambda)$)**.** Input: A finite set of generators of a left ideal I of D_X so that $M = D_X u = D_X/I$, a non-constant $f \in K[x]$, and $\lambda \in K$.
Output: presentation of $M(u, f, s)$ and of $M(u, f, \lambda)$.

(1) Compute $\iota(M) = D_X/\mathrm{Ann}_{D_Y} \iota(u)$ by Algorithm 3.16.
(2) Compute $J = \mathrm{Ann}_{D_Y}(\iota(u) \otimes \delta(t - f))$ by using Theorem 4.3.
(3) Compute $J \cap D_X[s]$, which is equal to $\mathrm{Ann}_{D_X[s]}(\iota(u) \otimes f^s) = \mathrm{Ann}_{D_X[s]}(u \otimes f^s)$.
(4) The substitution $s = \lambda$ for generators of $J \cap D_X[s]$ gives a set of generators of $\mathrm{Ann}_{D_X}(u \otimes f^s)|_{s=\lambda}$.

Proposition 4.7. *Let $M = D_X u$ be a left D_X-module generated by u and $f \in K[x]$ a non-constant polynomial. Then the b-function $b_{u,f}(x)$ exists if and only if there exists a nonzero polynomial $b(s) \in K[s]$ such that*

(2)
$$b(-\partial_t t)(u \otimes \delta(t - f)) \in tD_X[t\partial_t](u \otimes \delta(t - f)) = D_X[t\partial_t]t(u \otimes \delta(t - f)).$$

If M is f-saturated and such $b(s)$ exists, then $b_{u,f}(s)$ is the monic polynomial of the minimum degree among such $b(s)$.

Proof. Assume that (2) holds. Then there exists $P(s) \in D_X[s]$ such that

$$b(-\partial_t t)(u \otimes \delta(t-f)) = P(-\partial_t t)t(u \otimes \delta(t-f)) = P(-\partial_t t)f(u \otimes \delta(t-f)).$$

Applying the homomorphism ψ we get $b(s)(u \otimes f^s) = P(s)(u \otimes f^{s+1})$. Hence $b_{u,f}(s)$ exists and divides $b(s)$.

On the other hand, assume that there exist nonzero $b(s) \in K[s]$ and $P(s) \in D_X[s]$ such that $b(s)(u \otimes f^s) = P(s)(u \otimes f^{s+1})$ holds in $M \otimes_{K[x]} \mathcal{L}_f$. Then as is seen by the proof of Lemma 4.4, there exists $k \in \mathbb{N}$ such that

$$f^k b(-\partial_t t)(u \otimes \delta(t-f)) = f^k P(-\partial_t t)f(u \otimes \delta(t-f)).$$

Since

$$f^k b(-\partial_t t)(u \otimes \delta(t-f)) = b(-\partial_t t)f^k(u \otimes \delta(t-f)) = b(-\partial_t t)t^k(u \otimes \delta(t-f))$$

holds, we get

$$\begin{aligned}&\partial_t^k b(-\partial_t t)t^k(u \otimes \delta(t-f)) \\ &\qquad = \partial_t^k f^k P(-\partial_t t)f(u \otimes \delta(t-f)) \in D_X[t\partial_t]t(u \otimes \delta(t-f)).\end{aligned}$$

This completes the proof because there exists $c(s) \in K[x]$ such that $\partial_t^k b(-\partial_t t)t^k = c(-\partial_t t)$. Q.E.D.

Now we obtain an algorithm to determine whether the b-function exists and to compute it if it does:

Algorithm 4.8 ($b_{u,f}(s)$)**.** Input: $M = D_X u = D_X/I$ with a finite set of generators of I and a non-constant $f \in K[x]$.
Output: the b-function $b_{u,f}(s)$ if it exists. 'No' if it does not exist.

(1) Compute $J' := \mathrm{Ann}_{D_Y}(u \otimes \delta(t-f))$ by using Theorem 4.3.
(2) Compute $I' = (J' + D_X[s]f) \cap K[s]$ by elimination. If $I' \neq \{0\}$, then there exists $b_{u,f}(s)$. Otherwise, the b-function does not exist; output 'No' and quit.
(3) Compute a set of generators of $J := \mathrm{Ann}_{D_X[s]}(u \otimes f^s)$ by Algorithm 4.6.
(4) Compute $I(u,f) = (J + D_X[s]f) \cap K[s]$ by elimination. Let $b_{u,f}(s)$ be the monic generator of $I(u,f)$.

Algorithm 4.9 ($D_X(u \otimes f^\lambda)$)**.** Input: $M = D_X u = D_X/I$ with a finite set of generators of I, a non-constant $f \in K[x]$, and $\lambda \in K$.
Output: presentation of $D_X(u \otimes f^\lambda)$, i.e., $\mathrm{Ann}_{D_X}(u \otimes f^\lambda)$ if $b_{u,f}(s)$ exists.

(1) Compute $M(u,f,s)$ and $b_{u,f}(s)$ by Algorithms 4.6 and 4.8. Quit if $b_{u,f}(s)$ does not exist.
(2) Let k_0 be the maximum nonzero integer, if any, such that $b_{u,f}(\lambda - k_0) = 0$. If there is no such k_0, then set $k_0 = 0$.
(3) Compute $I := \mathrm{Ann}_{D_X}(u \otimes f^{\lambda - k_0})$ from $\mathrm{Ann}_{D_X[s]}(u \otimes f^s)$ by substitution $s = \lambda - k_0$.
(4) Compute the left ideal

$$I : f^{k_0} := \{P \in D_X \mid Pf^{k_0} \in I\}$$

by an appropriate Gröbner basis. Then one has

$$I : f^{k_0} = \mathrm{Ann}_{D_X}(u \otimes f^{\lambda}).$$

Example 4.10. Set $n = 2$, $x_1 = x$, $x_2 = y$, $P = x\partial_x^2 + \partial_y$, and $M = D_X/D_X P = D_X u$ as in Example 2.2. By Algorithms 4.6 and 4.8 we obtain $b_{u,x}(s) = (s+1)(s+2)$, $b_{u,y}(s) = s+1$, and

$$\begin{aligned} M(u,x,s) &:= D_X[s](u \otimes x^s) \\ &= D_X[s]/D_X[s](x^2{\partial_x}^2 - 2sx\partial_x + x\partial_y + s^2 + s), \\ M(u,y,s) &:= D_X[s](u \otimes y^s) = D_X[s]/D_X[s](xy{\partial_x}^2 + y\partial_y - s). \end{aligned}$$

Example 4.11. Setting $n = 2$ and $x_1 = x$, $x_2 = y$, let us consider $f(x,y) = x^3 - y^2$ and

$$\mathcal{N}_f = D_X[s]f^s, \qquad M := \mathcal{N}_f\Big(\frac{1}{6}\Big) = \mathcal{N}_f \Big/ \Big(s - \frac{1}{6}\Big)\mathcal{N}_f.$$

We have $M = D_X/I = D_X u$ with $u := f^s|_{s=1/6}$ and

$$I = \mathrm{Ann}_{D_X} u = D_X(2x\partial_x + 3y\partial_y - 1) + D_X(2y\partial_x + 3x^2\partial_y).$$

Then Algorithm 3.15 shows that $M[f^{-1}]$ is generated by $u \otimes f^{-1}$ and is isomorphic to D_X/J with

$$J = \mathrm{Ann}_{D_X} u \otimes f^{-1} = D_X(2x\partial_x + 3y\partial_y + 5) + D_X(2y\partial_x + 3x^2\partial_y).$$

In fact, $M[f^{-1}]$ is isomorphic to $\mathcal{L}_f(1/6) = K[x, f^{-1}]f^{1/6} = D_X f^{-5/6}$, and $\iota(M)$ to the submodule $D_X f^{1/6}$.

By Algorithm 3.16, we have $\iota(M) = D_X/J_0$ with the left ideal J_0 generated by

$$2x\partial_x + 3y\partial_y - 1, \quad 2y\partial_x + 3x^2\partial_y, \quad 8\partial_x^3 + 27y\partial_y^3 + 9\partial_y^2.$$

Note that J_0 is strictly larger than I because of the last generator. Algorithm 3.16 also yields presentations of the local cohomology groups

$$H^0_{(f)}(M) \cong D_X/(D_X x + D_X y) \quad \text{with} \quad (8\partial_x^3 + 27y\partial_y^3 + 9\partial_y^2)u \longleftrightarrow \overline{1},$$
$$H^1_{(f)}(M) \cong D_X/(D_X x + D_X y) \quad \text{with} \quad [u \otimes f^{-1}] \longleftrightarrow \overline{1}.$$

The b-function for u and f is

$$b_{u,f}(s) = (s+1)\Big(s+\frac{4}{3}\Big)\Big(s+\frac{7}{6}\Big) = b_f\Big(s+\frac{1}{6}\Big),$$

where $b_f(s) = (s+1)(s+5/6)(s+7/6)$ is the b-function of f.

Example 4.12. Set $n = 2$ and $x_1 = x$, $x_2 = y$ and consider

$$M = H^1_{(xy)}(K[x,y]) = D_X/(D_X xy + D_X(x\partial_x + 1) + D_Y(y\partial_y + 1)) = D_X u$$

with $u = [(xy)^{-1}]$. Then $M[x^{-1}]$ is generated by $u \otimes 1$, and hence $M[^{-1}] = \iota(M)$ and $H^1_{(x)}(M) = 0$. We have

$$M[x^{-1}] = D_X/(D_X(x\partial_x + 1) + D_X y),$$
$$H^0_{(x)}(M) \cong D_X/(D_X x + D_X \partial_y)$$

with the correspondence $u \otimes 1 \leftrightarrow \overline{1}$ and $yu \leftrightarrow \overline{1}$ respectively. The b-function of $u := [(xy)^{-1}]$ and x is $b_{u,x}(s) = s$. The module $M(u,x,s)$ is

$$M(u,x,s) = D_X[s]/(D_X[s](x\partial_x - s + 1) + D_X[s]y).$$

Example 4.13. Set $n = 3$, $x_1 = x$, $x_2 = y$, $x_3 = z$, and $f = x^3 - y^2z^2$. Let us consider

$$M = H^1_{(xf)}(K[x,y]) = D_X u \qquad (u = [(xf)^{-1}]).$$

The localizations $M[x^{-1}]$ and $M[f^{-1}]$ are given by

$$\begin{aligned} M[x^{-1}] &= D_X(u \otimes 1) \\ &= D_X/(D_X f + D_X(2x\partial_x + 3z\partial_z + 8) + D_X(y\partial_y - z\partial_z), \\ M[f^{-1}] &= D_X(u \otimes 1) \\ &= D_X/(D_X x + D_X(y\partial_y + 2) + D_X(z\partial_z + 2). \end{aligned}$$

In particular, we have $H^1_{(x)}(M) = H^1_{(f)}(M) = 0$. The zeroth cohomology groups are

$$H^0_{(x)}(M) \cong D_X/(D_X x + D_X\partial_y + D_X\partial_z)$$

with the correspondence $fu \leftrightarrow \overline{1}$, and $H^0_{(f)}(M) \cong D_X/I$ with the correspondence $(y\partial_y + 2)u \leftrightarrow \overline{1}$, where I is the left ideal generated by

$$f^2, \quad y\partial_y - z\partial_z, \quad 2x\partial_x + 3z\partial_z + 8, \quad x^3\partial_y - z^3y\partial_z - 4z^2y,$$
$$f\partial_z - 4zy^2, \quad x^3\partial_y^2 - z^4\partial_z^2 - 6z^3\partial_z - 4z^2,$$
$$8z^2y\partial_x^3 + (27z^2\partial_z^2 + 81z\partial_z + 24)\partial_y.$$

We also have

$$\begin{aligned} M(u,x,s) &= D_X[s]/(D_X[s]f + D_X[s](2x\partial_x + 3y\partial_y - 2s + 8) \\ &\qquad + D_X[s](y\partial_y - z\partial_z)), \\ M(u,f,s) &= D_X[s]/(D_X[s]x + D_X[s](y\partial_y - 2s - 2) \\ &\qquad + D_X[s](y\partial_y - z\partial_z)). \end{aligned}$$

The corresponding b-functions are

$$b_{u,x}(s) = s^2\Big(s - \frac{3}{2}\Big)^2, \qquad b_{u,f}(s) = s^2\Big(s - \frac{1}{2}\Big)^2.$$

§5. Length and multiplicity of D-modules

W set $X = K^n$ as in the preceding sections. First let us recall basic facts about the length and the multiplicity of a left D_X-module following J. Bernstein ([3],[4]). Let M be a finitely generated left D_X-module. A composition series of M of length k is a sequence

$$M = M_0 \supsetneqq M_1 \supsetneqq \cdots \supsetneqq M_k = 0$$

of left D_n-submodules such that M_i/M_{i-1} is a nonzero simple left D_X-module (i.e. having no proper left D_X-submodule other than 0) for $i = 1,\dots k$. The *length* of M, which we denote by $\operatorname{length} M$, is the least length of composition series (if any) of M. If there is no composition series, the length of M is defined to be infinite. The length is additive in the sense that if

$$0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0$$

is an exact sequence of left D_X-modules of finite length, then $\operatorname{length} M = \operatorname{length} N + \operatorname{length} L$ holds.

For each integer k, set

$$F_k(D_X) = \Big\{ \sum_{|\alpha|+|\beta| \le k} a_{\alpha\beta} x^\alpha \partial^\beta \mid a_{\alpha\beta} \in K \Big\}.$$

In particular, we have $F_k(D_X) = 0$ for $k < 0$ and $F_0(D_X) = K$. The filtration $\{F_k(D_X)\}_{k\in\mathbb{Z}}$ is called the Bernstein filtration on D_X.

Let M be a finitely generated left D_X-module. A family $\{F_k(M)\}_{k\in\mathbb{Z}}$ of K-subspaces of M is called a *Bernstein filtration* on M if it satisfies

(1) $F_k(M) \subset F_{k+1}(M) \quad (\forall k \in \mathbb{Z}), \qquad \bigcup_{k\in\mathbb{Z}} F_k(M) = M$,
(2) $F_j(D_X)F_k(M) \subset F_{j+k}(M) \quad (\forall j, k \in \mathbb{Z})$.

Moreover, $\{F_k(M)\}$ is called a *good Bernstein filtration* if there exist $u_i \in F_{k_i}(M)$ $(i = 1, \dots, m)$ such that

$$F_k(M) = F_{k-k_1}(D_X)u_1 + \cdots + F_{k-k_m}(D_X)u_m \qquad (\forall k \in \mathbb{Z}).$$

If $\{F_k(M)\}$ is a good Bernstein filtration, then each $F_k(M)$ is a finite dimensional vector space over K and $F_k(M) = 0$ for $k \ll 0$ (see e.g., 2.3 of [18]).

Let $\{F_k(M)\}$ be a good Bernstein filtration on M. Then there exists a polynomial $p(T) = c_dT^d + c_{d-1}T^{d-1} + \cdots + c_0 \in \mathbb{Q}[T]$ such that

$$\dim_K F_k(M) = p(k) \quad (k \gg 0)$$

and $d!c_d$ is a positive integer. We call $p(T)$ the Hilbert polynomial of M with respect to the filtration $\{F_k(M)\}$. The leading term of $p(T)$ does not depend on the choice of a good Bernstein filtration $\{F_k(M)\}$. The degree d of the Hilbert polynomial $p(T)$ is called the *dimension* of M and denoted $\dim M$. The *multiplicity* of M, denoted $\operatorname{mult} M$ is defined to be the positive integer $d!c_d$. The dimension and the multiplicity are invariants of a finitely generated left D_X-module.

If $M \neq 0$, then the dimension of M is not less than n (Bernstein's inequality). By definition, M is holonomic if $M = 0$ or $\dim M = n$. If M is a holonomic left D_X-module, we have an inequality $\operatorname{length} M \leq \operatorname{mult} M$ and hence M is of finite length in particular. Moreover, the multiplicity is additive for holonomic left D_X-modules.

We can compute the dimension and the multiplicity of a given finitely generated (not necessarily holonomic) D_X-module by using a Gröbner basis with respect to a term order compatible with the Bernstein filtration.

Example 5.1. Let M be the D_X-module with $X = K^2$ defined in Example 4.11. We get exact sequences

$$0 \longrightarrow H^0_{(f)}(M) \longrightarrow M \longrightarrow \iota(M) \longrightarrow 0,$$
$$0 \longrightarrow H^0_{(f)}(M) \longrightarrow M \longrightarrow M[f^{-1}] \longrightarrow H^1_{(f)}(M) \longrightarrow 0$$

with $H^0_{(f)}(M) \cong H^2_{(x,y)}(K[x,y]) \cong H^1_{(f)}(M)$. We have

$$\begin{aligned}&\operatorname{mult} M = \operatorname{mult} M[f^{-1}] = 6, \quad \operatorname{mult} \iota(M) = 5,\\ &\operatorname{mult} H^0_{(f)}(M) = \operatorname{mult} H^1_{(f)}(M) = 1.\end{aligned}$$

The following two propositions are easy and should be well-known.

Proposition 5.2. *Let f be a non-constant polynomial in $x_1, \dots, x_n$. Then the multiplicity of $K[x, f^{-1}]$ is at most $(\deg f + 1)^n$, where $\deg f$ stands for the total degree of f.*

Proof. Let d be the degree of f. Then

$$F_k(K[x, f^{-1}]) := \left\{ \frac{a}{f^{k+1}} \mid a \in K[x_1, \dots, x_n], \deg a \le (d+1)k \right\}$$

for $k \in \mathbb{Z}$ constitute a (not necessarily good) Bernstein filtration on $K[x, f^{-1}]$, which is finitely generated over D_n. Since

$$\dim_K F_k(K[x, f^{-1}]) = \binom{n + (d+1)k}{n},$$

we have $\dim K[x, f^{-1}] = n$ and $\operatorname{mult} M \le (d+1)^n$. Q.E.D.

Proposition 5.3. *Let $n = 1$ and $f \in K[x] = K[x_1]$ be non-constant square free. Then one has* $\operatorname{mult} K[x, f^{-1}] = \deg f + 1$.

Proof. Set $M := H^1_{(f)}(K[x])$. Then M is isomorphic to $D_X/D_X f$ since f is square-free. Hence

$$F_k(M) := F_k(D_1)[f^{-1}] \cong F_k(D_1)/F_{k-d}(D_1)f$$

with $d := \deg f$ constitute a good Bernstein filtration on M. Since

$$\begin{aligned}\dim F_k(M) &= \dim F_k(D_1) - \dim F_{k-d}(D_1)\\ &= \binom{k+2}{2} - \binom{k-d+2}{2} = dk - \frac{1}{2}d(d-3)\end{aligned}$$

holds for $k \ge d$, the multiplicity of M is d. Q.E.D.

We shall give two examples in two variables.

Proposition 5.4. *Set $X = K^2$ and write $x_1 = x$, $x_2 = y$. Set $f = x^m + y^l$ with positive integers l, m. Then the multiplicity of $K[x, y, f^{-1}]$ equals $2\max\{l, m\}$.*

Proof. We may assume $m \leq l$. Set $M := H^1_{(f)}(K[x,y])$. Since the b-function $b_f(s)$ of f does not have any negative integer ≤ -2 as a root (see e.g., 6.4 of [14]), M is generated by $u := [f^{-1}] \in M$ over D_X. The annihilator $\mathrm{Ann}_{D_X} u$ is generated by

$$f, \quad E := lx\partial_x + my\partial_y + ml, \quad P := ly^{l-1}\partial_x - mx^{m-1}\partial_y$$

(see also 6.4 of [14]). A Gröbner basis of $\mathrm{Ann}_D[f^{-1}]$ with respect to a total-degree reverse lexicographic order $\prec$ such that $x \succ y \succ \xi \succ \eta$ is $G = \{f, E, P\}$, where ξ and η are the commutative variables corresponding to ∂_x and ∂_y respectively. In fact, in case $m < l$ the S-pairs (see Chapter 3 of [18]) are divisible by G:

$$\mathrm{sp}_\prec(f, E) = lx\partial_x f - y^l E = x^m E - my\partial_y f,$$
$$\mathrm{sp}_\prec(f, P) = l\partial_x f - yP = x^{m-1}E, \quad \mathrm{sp}_\prec(E, P) = y^{l-1}E - xP = m\partial_y f.$$

The initial monomials of the Gröbner basis G are $\mathrm{in}_\prec(f) = y^l$, $\mathrm{in}_\prec(E) = x\xi$, $\mathrm{in}_\prec(P) = y^{l-1}\xi$. Hence for $k \geq l$ we obtain

$$\begin{aligned}
&\dim_K F_k(D_X)/(\mathrm{Ann}_{D_X}[f^{-1}] \cap F_k(D_X)) \\
&= \sharp(\{x^i y^j \xi^\mu \eta^\nu \mid i + j + \mu + \nu \leq k\} \setminus \langle y^l, x\xi, y^{l-1}\xi \rangle) \\
&= \sharp\{x^i y^j \eta^\nu \mid i + j + \nu \leq k,\, j \leq l - 1\} \\
&\quad + \sharp\{y^j \xi^\mu \eta^\nu \mid j + \mu + \nu \leq k,\, j \leq l - 2,\, \mu \geq 1\} \\
&= \sum_{j=0}^{l-1} \binom{2 + k - j}{2} + \sum_{j=0}^{l-2} \binom{2 + k - j - 1}{2} = \frac{2l - 1}{2} k^2 + \cdots .
\end{aligned}$$

On the other hand, in case $m = l$ we have

$$\begin{aligned}
&\mathrm{sp}_\prec(f, E) = l\partial_x f - x^{l-1}E = yP, \\
&\mathrm{sp}_\prec(f, P) = ly^{l-1}\partial_x f - x^l P = y^l P + lx^{l-1}\partial_y f \\
&\mathrm{sp}_\prec(E, P) = y^{l-1}E - xP = l\partial_y f.
\end{aligned}$$

The initial monomials are $\mathrm{in}_\prec(f) = x^l$, $\mathrm{in}_\prec(E) = x\xi$, $\mathrm{in}_\prec(P) = y^{l-1}\xi$. (Note that $y^{l-1}\xi \succ x^{l-1}\eta$ holds.) Hence for $k \geq l$ we obtain

$$\begin{aligned}
&\dim_K F_k(D_X)/(\mathrm{Ann}_{D_X}[f^{-1}] \cap F_k(D_X)) \\
&= \sharp(\{x^i y^j \xi^\mu \eta^\nu \mid i+j+\mu+\nu \leq k\} \setminus \langle x^l, x\xi, y^{l-1}\xi\rangle) \\
&= \sharp\{x^i y^j \eta^\nu \mid i+j+\nu \leq k,\, i \leq l-1\} \\
&\quad + \sharp\{y^j \xi^\mu \eta^\nu \mid j+\mu+\nu \leq k,\, j \leq l-2,\, \mu \geq 1\} \\
&= \sum_{i=0}^{l-1} \binom{2+k-i}{2} + \sum_{j=0}^{l-2} \binom{2+k-j-1}{2} = \frac{2l-1}{2}k^2 + \cdots .
\end{aligned}$$

Hence the multiplicity of M is $2l-1$ in both cases. This proves the assertion. Q.E.D.

Proposition 5.5. *Set $X = K^2$ with $x_1 = x$ and $x_2 = y$. Set $f = x^m + y^l + 1$ with positive integers l, m. Then the multiplicity of $K[x, y, f^{-1}]$ equals $lm + |l-m| + 1$.*

Proof. We may assume $m \leq l$. Set $M := H^1_{(f)}(K[x,y])$. Since the curve $f = 0$ is non-singular, the b-function is $b_f(s) = s+1$. Hence M is generated by $u := [f^{-1}]$. The annihilator $\mathrm{Ann}_{D_X} u$ is generated by f and $P := ly^{l-1}\partial_x - mx^{m-1}\partial_y$ since $f = 0$ is non-singular.

In case $l = m$, $G = \{f, P\}$ is a Gröbner basis of $\mathrm{Ann}_{D_X}[f^{-1}]$ with respect to a total-degree reverse lexicographic order $\prec$ such that $x \succ y \succ \xi \succ \eta$. In fact, we have

$$\mathrm{sp}_\prec(f, P) = ly^{l-1}\partial_x f - x^l P = ly^{l-1}\partial_x f + x^l P.$$

Since $\mathrm{in}_\prec(f) = x^l$ and $\mathrm{in}_\prec(P) = y^{l-1}\xi$, we have for $k \geq 2l$

$$\begin{aligned}
&\dim_K F_k(D_X)/(\mathrm{Ann}_{D_X}[f^{-1}] \cap F_k(D_X)) \\
&= \sharp(\{x^i y^j \xi^\mu \eta^\nu \mid i+j+\mu+\nu \leq N\} \setminus \langle x^l, y^{l-1}\xi\rangle) \\
&= \sharp\{x^i y^j \eta^\nu \mid i+j+\nu \leq k,\, i \leq l-1\} \\
&\quad + \sharp\{x^i y^j \xi^\mu \eta^\nu \mid i+j+\mu+\nu \leq k,\, i \leq l-1,\, 0 \leq j \leq l-2,\, \mu \geq 1\} \\
&= \sum_{i=0}^{l-1} \binom{2+k-i}{2} + \sum_{i=0}^{l-1}\sum_{j=0}^{l-2} \binom{2+k-i-j-1}{2} = \frac{l^2}{2}k^2 + \cdots .
\end{aligned}$$

In case $m < l$, the Gröbner basis of $\mathrm{Ann}_D[f^{-1}]$ with respect to the same order as above is $G = \{f, P, Q\}$ with

$$Q := l(x^m+1)\partial_x + mx^{m-1}y\partial_y + mlx^{m-1}.$$

In fact, we have

$$\begin{aligned}
\mathrm{sp}_{\prec}(f,P) &= l\partial_x f - yP = Q,\\
\mathrm{sp}_{\prec}(f,Q) &= lx^m\partial_x f - y^l Q = -mx^{m-1}y\partial_y f - yP + x^m Q,\\
\mathrm{sp}_{\prec}(P,Q) &= x^m P - y^{l-1}Q = -mx^{m-1}\partial_y f - P.
\end{aligned}$$

Since $\mathrm{in}_{\prec}(f) = y^l$, $\mathrm{in}_{\prec}(P) = y^{l-1}\xi$, $\mathrm{in}_{\prec}(Q) = x^m\xi$, we have for $k \geq l+m$,

$$\begin{aligned}
&\dim_K F_k(D_X)/(\mathrm{Ann}_{D_X}[f^{-1}] \cap F_k(D_X))\\
&= \sharp(\{x^i y^j \xi^\mu \eta^\nu \mid i+j+\mu+\nu \leq k\} \setminus \langle y^l, y^{l-1}\xi, x^m\xi\rangle)\\
&= \sharp\{x^i y^j \eta^\nu \mid i+j+\nu \leq k,\, i \leq l-1\}\\
&\quad + \sharp\{x^i y^j \xi^\mu \eta^\nu \mid i+j+\mu+\nu \leq k,\, i \leq m-1,\, j \leq l-2,\, \mu \geq 1\}\\
&= \sum_{i=0}^{l-1}\binom{2+k-i}{2} + \sum_{i=0}^{m-1}\sum_{j=0}^{l-2}\binom{2+k-i-j-1}{2}\\
&= \frac{l+m(l-1)}{2}k^2 + \cdots .
\end{aligned}$$

Hence the multiplicity of M is $l + m(l-1) = ml + l - m$. Q.E.D.

Now let us resume the study on $M(u,f,s)$ for a D_X-module $M = D_X u$ and a polynomial f. As in the preceding section, set $Y = X \times K$ and $Z = \{(x,t) \in Y \mid t = f(x)\}$.

Lemma 5.6. *Let $M = D_X u$ be a left D_X-module generated by u. For any $\lambda \in K$, the endomorphism of $M(u,f,s)$ defined by $s-\lambda$ is injective. Hence the sequence*

$$0 \longrightarrow M(u,f,s) \xrightarrow{s-\lambda} M(u,f,s) \longrightarrow M(u,f,\lambda) \longrightarrow 0$$

of left D_X-modules is exact.

Proof. We may assume that M is f-saturated as was seen in the previous section. The homomorphism $\psi : M \otimes_{K[x]} B_{Z|Y} \to M \otimes_{K[x]} \mathcal{L}_f$ is injective by Lemma 4.4.

Hence we have only to show that $s - \lambda = -\partial_t t - \lambda$ is an injective endomorphism of $M \otimes_{K[x]} B_{Z|Y}$. Let

$$v = \sum_{j=0}^{k} v_j \otimes \delta^{(j)}(t-f)$$

be an arbitrary element of $M \otimes_{K[x]} B_{Z|Y}$ with $k \in \mathbb{N}$ and $v_j \in M$. Then one has

$$\begin{aligned}(s-\lambda)v &= -\sum_{j=0}^{k} v_j \otimes (t\partial_t + \lambda + 1)\delta^{(j)}(t-f)\\ &= -\sum_{j=0}^{k} v_j \otimes (f\delta^{(j+1)}(t-f) + (\lambda - j)\delta^{(j)}(t-f))\\ &= -\lambda v_0 \otimes \delta(t-f) - \sum_{j=1}^{k}(fv_{j-1} + (\lambda - j)v_j) \otimes \delta^{(j)}(t-f)\\ &\quad - fv_k \otimes \delta^{(k+1)}(t-f).\end{aligned}$$

Thus $(s-\lambda)v = 0$ is equivalent to

$$\lambda v_0 = fv_k = fv_{j-1} + (\lambda - j)v_j = 0 \quad (1 \le j \le k),$$

which implies $v_k = v_{k-1} = \cdots = v_0 = 0$ since M is f-saturated. Q.E.D.

Theorem 5.7. *Let $f \in K[x]$ be a non-constant polynomial with K being algebraically closed. Let $M = D_X u$ be a left D_X-module generated by u which is holonomic on $X_f := \{x \in X \mid f(x) \neq 0\}$. Then $M(u, f, \lambda)$ and $M(u, f, s)/tM(u, f, s)$ are holonomic D_X-modules for any $\lambda \in K$.*

Proof. Since $M(u, f, s) = \iota(M)(\iota(u), f, s)$, we may assume M to be a nonzero holonomic D_X-module and f-saturated replacing M by $\iota(M)$. Set

$$N := M \otimes_{K[x]} B_{Z|Y}, \qquad F_k(N) := F_k(D_Y)\delta(t - f(x)) \qquad (k \in \mathbb{Z}).$$

Since N is holonomic, there exists a polynomial $p(k)$ of degree $n+1$ such that $p(k) = \dim_K F_k(N)$ for any sufficiently large k. Let us define a filtration $F_k(D_X[s])$ on the ring $D_X[s]$ by

$$F_k(D_X[s]) = \Big\{\sum_{\alpha,\beta,j} a_{\alpha,\beta,j} x^\alpha \partial_x^\beta s^j \mid |\alpha| + |\beta| + 2j \le k\Big\}.$$

Set

$$F_k(M(u, f, s)) := F_k(D_X[s])(u \otimes f^s) \quad (k \in \mathbb{Z}).$$

Applying a well-known fact in commutative algebra (see e.g., Theorem 4.4.3 in [7]) to the graded module $\mathrm{gr}(M(u, f, s))$ over the graded ring $\mathrm{gr}(D_X[s])$, we can show that there exist two polynomials $q_1(k)$ and $q_2(k)$ of the same degree d such that

$$\dim_K F_{2k}(M(u, f, s)) = q_1(2k), \quad \dim_K F_{2k+1}(M(u, f, s)) = q_2(2k+1)$$

for any sufficiently large k. We have $d \leq n+1$ since $F_k(M(u,f,s))$ is contained in $\psi(F_k(N))$ and ψ is an injective homomorphism from $M \otimes_{K[x]} B_{Z|Y}$ to $M \otimes_{K[x]} \mathcal{L}_f$.

Set $\mathcal{M} = M(u,f,s)/tM(u,f,s)$ and

$$F_k(\mathcal{M}) = F_k(M(u,f,s))/(tM(u,f,s) \cap F_k(M(u,f,s)).$$

Then $\{F_k(\mathcal{M})\}$ is a Bernstein filtration on the left D_X-module $\mathcal{M}$ (i.e., the action of s being ignored) although we do not know at this stage whether $\mathcal{M}$ is finitely generated over D_X or not.

Since $t : M(u,f,s) \to M(u,f,s)$ is injective, we have

$$\begin{aligned}\dim_K F_k(\mathcal{M}) &= \dim_K F_k(M(u,f,s)) \\ &\qquad - \dim_K(tM(u,f,s) \cap F_k(M(u,f,s)) \\ &\leq \dim_K F_k(M(u,f,s)) - \dim_K t^2 F_{k-2}(M(u,f,s)) \\ &= \dim_K F_k(M(u,f,s)) - \dim_K F_{k-2}(M(u,f,s)) \\ &= \begin{cases} q_1(k) - q_1(k-2) & \text{if } k \gg 0 \text{ is even,} \\ q_2(k) - q_2(k-2) & \text{if } k \gg 0 \text{ is odd.} \end{cases}\end{aligned}$$

Since the degree of $q_i(k) - q_i(k-2)$ $(i = 1,2)$ is $d-1 \leq n$, this inequality implies that an arbitrary finitely generated D_X-submodule of $\mathcal{M}$ is holonomic and its multiplicity is bounded in terms of the leading coefficients of $q_1(k)$ and $q_2(k)$. Hence we conclude that $\mathcal{M}$ itself is holonomic.

We can prove the holonomicity of $M(u,f,\lambda)$ in the same way replacing t^2 by $s - \lambda$. This fact is a special case of Theorem 6.10 in [18]. Q.E.D.

The first statement of the following theorem is given in 6.5 of [14] for the case $M = K[x]$ and $u = 1$.

Theorem 5.8. *Let $M = D_X u$ be a D_X-module generated by u and $f \in K[x]$ be a non-constant polynomial. Assume that the b-function $b_{u,f}(s)$ exists. For $\lambda \in K$ let $\varphi_\lambda : M(u,f,\lambda+1) \to M(u,f,\lambda)$ be the D_X-homomorphism induced by t, which sends $(u \otimes f^s)|_{s=\lambda+1}$ to $(fu \otimes f^s)|_{s=\lambda}$.*

(1) *φ_λ is an isomorphism if and only if $b_{u,f}(\lambda) \neq 0$.*

(2) *Assume that M is holonomic on X_f with K being algebraically closed. Then one has*

$$\begin{aligned}\operatorname{mult} M(u,f,\lambda+k) &= \operatorname{mult} M(u,f,\lambda), \\ \operatorname{length} M(u,f,\lambda+k) &= \operatorname{length} M(u,f,\lambda)\end{aligned}$$

for any $\lambda \in K$ and any integer k. In particular, one has

$$\operatorname{mult} M[f^{-1}] = \operatorname{mult} M(u,f,k), \qquad \operatorname{length} M[f^{-1}] = \operatorname{length} M(u,f,k)$$

for any integer k.

Proof. There exists a commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[d] & \mathcal{K}_0(\lambda) \arrow[d] & \\
0 \arrow[r] & M(u,f,s) \arrow[r, "t"] \arrow[d, "s-\lambda-1"] & M(u,f,s) \arrow[r] \arrow[d, "s-\lambda"] & \mathcal{M} \arrow[r] \arrow[d, "s-\lambda"] & 0 \\
0 \arrow[r] & M(u,f,s) \arrow[r, "t"] \arrow[d] & M(u,f,s) \arrow[r] \arrow[d] & \mathcal{M} \arrow[r] \arrow[d] & 0 \\
 & M(u,f,\lambda+1) \arrow[r, "\varphi_\lambda"] \arrow[d] & M(u,f,\lambda) \arrow[r] \arrow[d] & \mathcal{K}_1(\lambda) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with $\mathcal{M} = M(u,f,s)/tM(u,f,s)$ and some left D_X-modules $\mathcal{K}_0(\lambda)$, $\mathcal{K}_1(\lambda)$, where the three vertical sequences and the upper two horizontal sequences are exact in view of Lemma 5.6. Hence by the snake lemma we obtain an exact sequence

$$0 \longrightarrow \mathcal{K}_0(\lambda) \longrightarrow M(u,f,\lambda+1) \xrightarrow{\varphi_\lambda} M(u,f,\lambda) \longrightarrow \mathcal{K}_1(\lambda) \longrightarrow 0 \tag{3}$$

of left D_X-modules.

(1) Assume $b_{u,f}(\lambda) \neq 0$. Then there exist $a(s), c(s) \in K[s]$ such that $a(s)(s-\lambda) + c(s)b_{u,f}(s) = 1$. Hence for any $Q(s) \in D_X[s]$,

$$Q(s)(u \otimes f^s) = Q(s)c(s)b_{u,f}(s)(u \otimes f^s) + (s-\lambda)Q(s)a(s)(u \otimes f^s)$$

belongs to $tM(u,f,s) + (s-\lambda)M(u,f,s)$. If $(s-\lambda)Q(s)(u \otimes f^s)$ belongs to $tM(u,f,s)$, then $Q(s)(u \otimes f^s)$ also belongs to $tM(u,f,s)$. Hence $s-\lambda$ is an automorphism of $\mathcal{M}$.

Conversely, assume that $s-\lambda$ is an automorphism of $\mathcal{M}$. Then the minimal polynomial $b_{u,f}(s)$ of s on this module cannot be a multiple of $s-\lambda$. Summing up we have shown that $b_{u,f}(\lambda) \neq 0$ if and only if $\mathcal{K}_0(\lambda) = \mathcal{K}_1(\lambda) = 0$. In view of the exact sequence (3), this is also equivalent to φ_λ being an isomorphism.

(2) We may assume that M is a holonomic D_X-module and that M is f-saturated replacing M by $\iota(M)$. Since $\mathcal{M}$ is holonomic by Theorem 5.7, the length (and the multiplicity) of $\mathcal{K}_0(\lambda)$ and the length (and the multiplicity respectively) of $\mathcal{K}_1(\lambda)$ are the same in view of the rightmost vertical exact sequence. Combined with this fact the exact sequence (3) proves the statement (2). Q.E.D.

This theorem provides us with an algorithm to compute the multiplicity of $M[f^{-1}]$ without any information on $b_{u,f}(s)$; we have only to compute a Gröbner basis, e.g., of $M(u,f,0)$ with respect to a term order compatible with the Bernstein filtration.

Theorem 5.9. *Let $M = D_X u$ be a D_X-module generated by u and $f \in K[x]$ be a non-constant polynomial with K being algebraically closed. Assume that M is holonomic on X_f. Then the homomorphism $\tilde{\rho}_\lambda$: $M(u,f,\lambda) \to D_X(u \otimes f^\lambda)$ is an isomorphism if and only if $b_{u,f}(\lambda - k) \neq 0$ for any positive integer k.*

Proof. If $b_{u,f}(\lambda - k) \neq 0$ for any positive integer k, then $\tilde{\rho}_\lambda$ is an isomorphism by virtue of Proposition 2.6. Now suppose $b_{u,f}(\lambda - k) = 0$ holds for some positive integer k and let k_0 be the maximum among such k. Then Proposition 2.6 and Lemma 2.7 imply that $\tilde{\rho}_{\lambda - k_0}$ is an isomorphism and that $D_X(u \otimes f^{\lambda - k_0 + 1}) \subsetneq D_X(u \otimes f^{\lambda - k_0})$. Hence by (2) of Theorem 5.8 we have

$$\begin{aligned} \operatorname{length} M(u,f,\lambda) &= \operatorname{length} M(u,f,\lambda - k_0) = \operatorname{length} D_X(u \otimes f^{\lambda - k_0}) \\ &> \operatorname{length} D_X(u \otimes f^{\lambda - k_0 + 1}) \geq \operatorname{length} D_X(u \otimes f^\lambda). \end{aligned}$$

Thus $\tilde{\rho}_\lambda$ is not an isomorphism. Q.E.D.

Corollary 5.10. *Under the same assumptions as in Theorem 5.9, $M(u,f,\lambda)$ is f-saturated if and only if $b_{u,f}(\lambda - k) \neq 0$ for any positive integer k. In general, $\iota(M(u,f,\lambda))$ is isomorphic to $D_X(u \otimes f^\lambda)$.*

Proof. We may assume M to be f-saturated. First note that $M \otimes_{K[x]} \mathcal{L}_f(\lambda)$ is f-saturated for any $\lambda \in K$ since it is isomorphic to $M[f^{-1}]$ as $K[x]$-module. Hence $M(u,f,\lambda) \cong D_X(u \otimes f^\lambda)$ is also f-saturated under the assumption on $b_{u,f}(s)$.

Now assume $b_{u,f}(\lambda - k) = 0$ for some positive integer k. Then $\tilde{\rho}_\lambda$ is not injective. Thus there exists $P \in D_X$ such that $P((u \otimes f^s)|_{s=\lambda}) \neq 0$ but $P(u \otimes f^\lambda) = 0$. There exist $P_j \in D_X$ and $k, l \in \mathbb{N}$ such that

$$P(u \otimes f^s) = \sum_{j=0}^{k} (P_j u) \otimes (s - \lambda)^j f^{s-l}.$$

Then the equality $P(u\otimes u^{\lambda})=0$ means $P_0u=0$. Hence by Lemma 2.5, there exist $Q(s)\in D_X[s]$ and $m\in\mathbb{N}$ such that

$$P(u\otimes f^s)=(s-\lambda)Q(s)(u\otimes f^{s-m}).$$

Take a sufficiently large $l\in\mathbb{N}$ so that $f^lQ(s)f^{-m}$ belongs to $D_X[s]$. Then we have

$$f^lP(u\otimes f^s)=(s-\lambda)(f^lQ(s)f^{-m})(u\otimes f^s),$$

and consequently $f^lP((u\otimes f^s)|_{s=\lambda})=0$. Hence $M(u,f,\lambda)$ is not f-saturated. The last statement also follows from this argument. Q.E.D.

Example 5.11. Set $n=2$ and write $x_1=x$, $x_2=y$. Let u be the residue class of 1 in $M=D_X/I$ with I being the left ideal of D_X generated by two operators

$$P_1=x(1-x){\partial_x}^2+y(1-x)\partial_x\partial_y,\quad P_2=y(1-y){\partial_y}^2+x(1-y)\partial_x\partial_y.$$

This is Appell's hypergeometric system F_1 with all parameters equal to zero. The singular locus of M is a line arrangement defined by

$$f:=x(x-1)y(y-1)(x-y)=0.$$

Let $\iota:M\to M[f^{-1}]$ be the canonical homomorphism. Then $M[f^{-1}]$ is generated by $f^{-2}\iota(u)$ and $\iota(M)$ is given by

$$\begin{aligned}\iota(M)&=D_X\iota(u)\\&=D_X/(D_X\partial_x\partial_y+D_X((1-x){\partial_x}^2-\partial_x)+D_X((1-y){\partial_y}^2-\partial_y)).\end{aligned}$$

The b-function with respect to u and f is

$$b_{u,f}(s)=(s+1)^3(s+2)^2\bigl(s+\frac{2}{3}\bigr)^2\bigl(s+\frac{4}{3}\bigr)^2\bigl(s+\frac{5}{3}\bigr).$$

As to multiplicities we have

$$\begin{aligned}&\operatorname{mult}M=10,\quad \operatorname{mult}\iota(M)=5,\quad \operatorname{mult}M[f^{-1}]=36,\\&\operatorname{mult}H^1_{(f)}(M)=31,\quad \operatorname{mult}H^0_{(f)}(M)=5.\end{aligned}$$

The characteristic varieties are

$$
\begin{aligned}
&\mathrm{Char}(M) = \{(x,y;\xi,\eta) \mid x = y = 0\} \cup \{x = y = 1\} \cup \{x = \eta = 0\} \\
&\quad \cup \{x-1 = \eta = 0\} \cup \{y = \xi = 0\} \cup \{y-1 = \xi = 0\} \\
&\quad \cup \{x-y = \xi+\eta = 0\} \cup \{\xi = \eta = 0\}, \\
&\mathrm{Char}(\iota(M)) = \{x-1 = \eta = 0\} \cup \{y-1 = \xi = 0\} \cup \{\xi = \eta = 0\}, \\
&\mathrm{Char}(M[f^{-1}]) = \{x = y = 0\} \cup \{x-1 = y = 0\} \cup \{x = y-1 = 0\} \\
&\quad \cup \{x = y = 1\} \cup \{x = \eta = 0\} \cup \{x-1 = \eta = 0\} \\
&\quad \cup \{y = \xi = 0\} \cup \{y-1 = \xi = 0\} \cup \{x-y = \xi+\eta = 0\} \\
&\quad \cup \{\xi = \eta = 0\}, \\
&\mathrm{Char}(H^1_{(f)}(M)) = \{x = y = 0\} \cup \{x-1 = y = 0\} \cup \{x = y-1 = 0\} \\
&\quad \cup \{x = y = 1\} \cup \{x = \eta = 0\} \cup \{x-1 = \eta = 0\} \\
&\quad \cup \{y = \xi = 0\} \cup \{y-1 = \xi = 0\} \cup \{x-y = \xi+\eta = 0\}, \\
&\mathrm{Char}(H^0_{(f)}(M)) = \{x = y = 0\} \cup \{x = y = 1\} \cup \{x = \eta = 0\} \\
&\quad \cup \{y = \xi = 0\} \cup \{x-y = \xi+\eta = 0\}.
\end{aligned}
$$

Example 5.12. Set $X = K^4$ and consider the A-hypergeometric system associated with the matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$, which is taken from Example 4.3.9 of [23]. It is the left D_X-module $M_A(b_1, b_2) = D_X/H_A(b_1, b_2)$ with the left ideal $H_A(b_1, b_2)$ of D_X generated by operators

$$
\begin{aligned}
&x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_4\partial_4 - b_1, \quad x_2\partial_2 + 3x_3\partial_3 + 4x_4\partial_4 - b_2, \\
&\partial_2\partial_4^2 - \partial_3^3, \quad \partial_1\partial_4 - \partial_2\partial_3, \quad \partial_2^2\partial_4 - \partial_1\partial_3^2, \quad \partial_1^2\partial_3 - \partial_2^3
\end{aligned}
$$

with parameters $b_1, b_2 \in K$. Computing a Gröbner basis of the left ideal of $D_X[b_1, b_2]$ (i.e., regarding b_1 and b_2 as indeterminates) generated by these operators with respect to a term order $\prec$ such that

$$
|\alpha| + |\beta| < |\alpha'| + |\beta'| \quad \Rightarrow \quad b_1^i b_2^j x^\alpha \partial_x^\beta \prec b_1^k b_2^l x^{\alpha'} \partial_x^{\beta'}
$$

and that $b_1^i b_2^j \preceq x^\alpha \partial_x^\beta$ for any i, j, α, β, we can verify that the multiplicity of $M_A(b_1, b_2)$ is 16 unless $b_1 = 1$ and $b_2 = 2$, while the multiplicity of $M_A(1, 2)$ is 17. A similar phenomenon with respect to the holonomic rank was shown in [23].

On the other hand, the characteristic variety of $M_A(b_1, b_2)$ does not depend on b_1, b_2 and its singular locus is the zero set of

$$g(x) = x_1x_4(256x_1^3x_4^3 + (-192x_1^2x_2x_3 - 27x_2^4)x_4^2 - 6x_1x_2^2x_3^2x_4 - 27x_1^2x_3^4 - 4x_2^3x_3^3).$$

For example, the b-functions of $u := \overline{1} \in M_A(1,2)$ and x_1 is $b_{u,x_1}(s) = s(s+1)(s+2)$, while the b-functions of $v := \overline{1} \in M_A(0,0)$ and x_1 is $b_{v,x_1}(s) = (s+1)^2$. Algorithm 3.16 ensures that $M_A(0,0)$ and $M_A(1,2)$ are x_1- and x_4-saturated. The computation of the localization with respect to g is intractable.

§6. Hyperplane arrangements

Let us prove a formula on the multiplicity and the length of the local cohomology $H^1_{(f)}(K[x])$ or the localization $K[x, f^{-1}]$ of the polynomial ring when f defines a hyperplane arrangement in the affine space $X = K^n$. We set $R = K[x]$ in what follows.

The length of such modules was studied e.g., in [1], [27]. The characteristic cycle of the local cohomology with respect to an arrangement of linear subvarieties was studied in [2]. Although not explicitly stated, Corollary 1.3 of [2] should yield main results of this section. Nero Budur informed the author that Theorem 6.4 below follows from results in Section 1.7 of [8]. We give an elementary direct proof with a hope to make both the statement and the proof more accessible.

Lemma 6.1. *Let $h_0 = h_0(x) \in K[x]$ be a linear polynomial and I be an ideal of $R = K[x]$. Let $R' := R/Rh_0$ be the affine ring associated with the hyperplane $h_0(x) = 0$ and set $I' = (I + Rh_0)/Rh_0$. Then we have*

$$\operatorname{length} H^i_{I+Rh_0}(R) = \operatorname{length} H^{i-1}_{I'}(R'),$$
$$\operatorname{mult} H^i_{I+Rh_0}(R) = \operatorname{mult} H^{i-1}_{I'}(R')$$

for any integer i.

Proof. Since $H^i_{Rh_0}(R) = 0$ for $i \neq 1$, there is an isomorphism

$$H^i_{I+Rh_0}(R) \cong H^{i-1}_{I+Rh_0}(H^1_{Rh_0}(R)).$$

We may assume by an affine coordinate transformation, which preserves the Bernstein filtration, that $h_0(x) = x_n$. Then we may regard $R' = K[x_1, \dots, x_{n-1}]$ and have an isomorphism

$$H^1_{Rh_0}(R) \cong R' \otimes_K H^1_{(x_n)}(K[x_n]),$$

where the tensor product on the right-hand side is a left module over $D_n = D_{n-1} \otimes_K D_1$ with D_1 being the ring of differential operators in the variable x_n.

Let $\{f_1, \dots, f_r\}$ be a set of generators of I. We may assume that $f_1, \dots, f_r$ belong to R'. (We replace f_i with its homomorphic image in R'.) Then for $0 \le i_1 < \cdots < i_k \le r$ with $k \in \mathbb{N}$, the localization by $f_{i_1} \cdots f_{i_k}$ yields

$$\begin{aligned}(H^1_{Rh_0}(R))_{f_{i_1}\cdots f_{i_k}} &:= H^1_{Rh_0}(R)[(f_{i_1} \cdots f_{i_k})^{-1}] \\ &= R'_{f_{i_1}\cdots f_{i_k}} \otimes_K H^1_{(x_n)}(K[x_n]).\end{aligned}$$

On the other hand, we have

$$(H^1_{Rh_0}(R))_{x_n} := H^1_{Rh_0}(R)[x_n^{-1}] = R' \otimes_K (H^1_{(x_n)}(K[x_n]))_{x_n} = 0.$$

Hence $H^{i-1}_{I+Rh_0}(H^1_{Rh_0}(R))$ is the $(i-1)$-th cohomology group of the Čech complex

$$\begin{aligned}0 \longrightarrow R' \otimes_K H^1_{(x_n)}(K[x_n]) &\longrightarrow \bigoplus_{1\le i\le r} R'_{f_i} \otimes_K H^1_{(x_n)}(K[x_n]) \\ \longrightarrow \bigoplus_{1\le i_1<i_2\le r} & R'_{f_{i_1}f_{i_2}} \otimes_K H^1_{(x_n)}(K[x_n]) \longrightarrow \\ \cdots \longrightarrow & R'_{f_1\cdots f_r} \otimes_K H^1_{(x_n)}(K[x_n]) \longrightarrow 0,\end{aligned}$$

which is isomorphic to $H^{i-1}_{I'}(R') \otimes_K H^1_{(x_n)}(K[x_n])$ (see e.g., Theorem 7.13 in [11]). This implies

$$\begin{aligned}H^i_{I+Rh_0}(R) &\cong H^{i-1}_{I'}(R') \otimes_K H^1_{(x_n)}(K[x_n]) \\ &\cong H^{i-1}_{I'}(R') \otimes_K (D_1/D_1x_n) \\ &\cong (D_n/D_nx_n) \otimes_{D_{n-1}} H^{i-1}_{I'}(R'),\end{aligned}$$

where D_n/D_nx_n is regarded as a (D_n, D_{n-1})-bimodule. The rightmost term is the D-module theoretic direct image of $H^{i-1}_{I'}(R')$ with respect to the inclusion $H_0 := \{x \in X \mid x_n = 0\} \to X$. In view of Kashiwara's equivalence in the category of algebraic D-modules (see e.g., Theorem 7.11 of [6] or Theorem 1.6.1 of [10]), there is a one-to-one correspondence between the D_{n-1}-submodules M of $H^{i-1}_{I'}(R')$ and the D_n-submodules $M \otimes_K H^1_{(x_n)}(K[x_n])$ of $H^i_{I+Rh_0}(R)$. This implies

$$\operatorname{length} H^i_{I+Rh_0}(R) = \operatorname{length} H^{i-1}_{I'}(R').$$

Next, let us show

$$\operatorname{mult} H^i_{I+Rh_0}(R) = \operatorname{mult} H^{i-1}_{I'}(R').$$

Let $\{F_k\}$ be a good Bernstein filtration on $H^{i-1}_{I'}(R')$ and set $m = \operatorname{mult} H^{i-1}_{I'}(R')$. Then there exists a polynomial $p(k)$ and $k_1 \in \mathbb{Z}$ such that

$$\dim_K F_k = p(k) = \frac{m}{(n-1)!}k^{n-1} + (\text{terms with degree} < n-1)$$

holds for $k \geq k_1$. Define a filtration $\{G_k\}$ on $H^{i-1}_{I'}(R') \otimes_K H^1_{(x_n)}(K[x_n])$ by

$$G_k := \sum_{j=0}^{k} F_j \otimes_K (K[x_n^{-1}] + \cdots + K[x_n^{-(k-j)-1}]) = \bigoplus_{j=0}^{k} F_j \otimes_K K[x_n^{-(k-j)-1}],$$

where $K[x_n^{-j}]$ denotes the K-space spanned by the residue class of x_n^{-j}. It is easy to see that $\{G_k\}$ is a good Bernstein filtration. Hence we have

$$\dim_K G_k = \sum_{j=0}^{k} \dim_K F_j = \sum_{j=0}^{k_1-1} \dim_K F_j + \sum_{j=k_1}^{k} p(j).$$

By the assumption, there exists a polynomial $q(k)$ of degree $\leq n-2$ such that

$$p(j) = \frac{m}{(n-1)!} j(j+1)\cdots(j+n-2) + q(j).$$

Since

$$\sum_{j=k_1}^{k} j(j+1)\cdots(j+n-2)$$
$$= \frac{1}{n}\{k(k+1)\cdots(k+n-1) - (k_1-1)k_1\cdots(k_1+n-2)\},$$

we have

$$\dim_K G_k = \frac{m}{n!}k^n + (\text{terms with degree} < n) \qquad (\forall k \geq k_1).$$

Thus we also have $\operatorname{mult} H^i_{I+Rx_n}(R) = m$. This completes the proof. Q.E.D.

Theorem 6.2. *Let $f \in K[x]$ be a multiple of essentially distinct linear polynomials and $\mathcal{A}$ be the hyperplane arrangement in $X = K^n$ defined by f. Let H_0 be an element of $\mathcal{A}$. Set $\mathcal{A}' := \mathcal{A} \setminus \{H_0\}$ and let f' be the product of the defining polynomials of hyperplanes belonging to $\mathcal{A}'$. Let us regard*

$$\mathcal{A}'' := \{H \cap H_0 \mid H \in \mathcal{A}',\ H \cap H_0 \neq \emptyset\}$$

as a hyperplane arrangement in the affine space H_0. Let $R' = R/Rh_0$ be the affine ring of H_0, where h_0 is a polynomial of first degree defining H_0. Let $f'' \in R'$ be the product of the defining polynomials of the elements of $\mathcal{A}''$. (Set $f'' = 1$ if $\mathcal{A}'' = \emptyset$.) Then one has

$$\begin{aligned}
&\operatorname{length} H^1_{(f)}(R) = \operatorname{length} H^1_{(f')}(R) + \operatorname{length} H^1_{(f'')}(R') + 1,\\
&\operatorname{mult} H^1_{(f)}(R) = \operatorname{mult} H^1_{(f')}(R) + \operatorname{mult} H^1_{(f'')}(R') + 1.
\end{aligned}$$

Proof. By the Mayer-Vietoris exact sequence (see e.g., Theorem 15.1 in [11]), we get an exact sequence

$$0 \longrightarrow H^1_{(f')}(R) \oplus H^1_{(h_0)}(R) \longrightarrow H^1_{(f)}(R) \longrightarrow H^2_{(f')+(h_0)}(R) \longrightarrow 0$$

of holonomic left D_n-modules because $H^1_{(f')+(h_0)}(R) = 0$. Since the length and the multiplicity of $H^1_{(h_0)}(R)$ are both one, it follows that

$$\begin{aligned}
&\operatorname{length} H^1_{(f)}(R) = \operatorname{length} H^1_{(f')}(R) + \operatorname{length} H^2_{(f')+(h_0)}(R) + 1,\\
&\operatorname{mult} H^1_{(f)}(R) = \operatorname{mult} H^1_{(f')}(R) + \operatorname{mult} H^2_{(f')+(h_0)}(R) + 1. \qquad (4)
\end{aligned}$$

Since $(f'') = R'f'' \cong (Rf' + Rh_0)/Rh_0$, Lemma 6.1 implies

$$\begin{aligned}
\operatorname{mult} H^2_{(f')+(h_0)}(R) &= \operatorname{mult} H^1_{(f'')}(R'),\\
\operatorname{length} H^2_{(f')+(h_0)}(R) &= \operatorname{length} H^1_{(f'')}(R').
\end{aligned}$$

This completes the proof in view of (4). Q.E.D.

Corollary 6.3. $\operatorname{length} H^1_{(f)}(R) = \operatorname{mult} H^1_{(f)}(R)$.

Proof. This can be easily proved by induction on $\sharp\mathcal{A}$ by using Theorem 6.2. Q.E.D.

The intersection poset $L(\mathcal{A})$ is the set of the non-empty intersections of elements of $\mathcal{A}$ including X. For $Y, Z \in L(\mathcal{A})$, the Möbius function $\mu(Y, Z)$ is defined recursively by

$$\mu(Y,Z) = \begin{cases} -\displaystyle\sum_{Z \subsetneq W \subset Y} \mu(Y,W) & \text{if } Z \subsetneq Y \\ 1 & \text{if } Z = Y \\ 0 & \text{otherwise.} \end{cases}$$

Set $\mu(Y) = \mu(X, Y)$. Then $(-1)^{\operatorname{codim} X}\mu(X)$ is positive (see e.g. Theorem 2.47 of [22]). The Poincaré polynomial of the arrangement $\mathcal{A}$ is defined by

$$\pi(\mathcal{A}, t) = \sum_{Y \in L(\mathcal{A})} \mu(Y)(-t)^{\operatorname{codim} Y}.$$

Theorem 6.4. *Let $\mathcal{A}$ be a hyperplane arrangement in $X = K^n$ defined by a polynomial $f \in R = K[x]$. Then the length of $H^1_{(f)}(R)$ is $\pi(\mathcal{A}, 1) - 1$.*

Proof. Let H_0 be an element of $\mathcal{A}$ defined by a first degree polynomial h_0. Let us prove the equality by induction on $\sharp\mathcal{A}$. Since $H_{(h_0)}(R)$ is a simple left D_n-module and $\pi(\{H_0\}, t) = t + 1$, the equality holds if $\mathcal{A} = \{H_0\}$. Let $\mathcal{A}'$, $\mathcal{A}''$ be as in the proof of Theorem 6.2.

By the induction hypothesis, we have

$$\operatorname{length} H^1_{(f')}(R) = \pi(\mathcal{A}', 1) - 1, \qquad \operatorname{length} H^1_{(f'')}(R') = \pi(\mathcal{A}'', 1) - 1.$$

Hence by Theorem 6.2 we get

$$\begin{aligned} \operatorname{length} H^1_{(f)}(R) &= \operatorname{length} H^1_{(f')}(R) + \operatorname{length} H^1_{(f'')}(R') + 1 \\ &= \pi(\mathcal{A}', 1) + \pi(\mathcal{A}'', 1) - 1. \end{aligned}$$

On the other hand, $\pi(\mathcal{A}, t) = \pi(\mathcal{A}', t) + t\pi(\mathcal{A}'', t)$ holds (see e.g., Theorem 2.56 of [22]). Thus we get

$$\operatorname{length} H^1_{(f)}(R) = \pi(\mathcal{A}', 1) + \pi(\mathcal{A}'', 1) - 1 = \pi(\mathcal{A}, 1) - 1.$$

This completes the proof. Q.E.D.

Corollary 6.5. *Let $\mathcal{A}$ be a hyperplane arrangement in $X = K^n$ defined by a polynomial $f \in R = K[x]$. Then the length of $R[f^{-1}]$ is $\pi(\mathcal{A}, 1)$.*

Actual computation can be done effectively by using the recursive formula of Theorem 6.2. For example, we have $\pi(\mathcal{A}, t) = (2t+1)(3t+1)$ and hence $\pi(\mathcal{A}, 1) = 12$ if $f = xy(x-1)(y-1)(x-y)$ with $X = K^2$. If

$$f = xyz(x+y)(x-y)(x+z)(x-z)(y+z)(y-z)$$

with $X = K^3$, then we have $\pi(\mathcal{A}, t) = (t+1)(3t+1)(5t+1)$ and hence $\pi(\mathcal{A}, 1) = 48$. For these relatively small examples, direct computation of the local cohomology group is also possible.

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