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Elliptic quantum groups $U_{q,p}(\widehat{\mathfrak{gl}}_N)$ and $E_{q,p}(\widehat{\mathfrak{gl}}_N)$

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Abstract.

We reformulate a central extension of Felder's elliptic quantum group in the FRST formulation as a topological algebra $E_{q,p}(\widehat{\mathfrak{gl}}_N)$ over the ring of formal power series in p. We then discuss the isomorphism between $E_{q,p}(\widehat{\mathfrak{gl}}_N)$ and the elliptic algebra $U_{q,p}(\widehat{\mathfrak{gl}}_N)$ of the Drinfeld realization. An evaluation H-algebra homomorphism from $U_{q,p}(\widehat{\mathfrak{gl}}_N)$ to a dynamical extension of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$ resolves the problem into the one discussed by Ding and Frenkel in the trigonometric case. We also provide some useful formulas for the elliptic quantum determinants.

§1. Introduction

An elliptic quantum algebra is an associative algebra related to an elliptic solution to the Yang-Baxter equation (YBE) or the dynamical Yang-Baxter equation (DYBE). Equipped with a co-algebra structure it is called the elliptic quantum group (EQG). Depending on YBE or DYBE, the corresponding EQG is called the vertex type or the face type, respectively[26, 37]. Through this paper we use the terminology DYBE[19] as an equation equivalent to the face type YBE or the star triangle equation (see for example [33]).

Let $\mathfrak{g}$ and $\widehat{\mathfrak{g}}$ denote a simple Lie algebra and an (untwisted) affine Lie algebra, respectively. In known quantum groups, such as the Yangian $Y(\mathfrak{g})$ (or its double $\mathcal{D}Y(\mathfrak{g})$) associated to the rational solutions to the YBE and the affine quantum group $U_q(\widehat{\mathfrak{g}})$ associated to the trigonometric solutions there are some different formulations depending on the types of the generators. In particular for $U_q(\widehat{\mathfrak{g}})$ they are the Drinfeld-Jimbo formulation[10, 32] in terms of an analogue of the Chevalley

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generators, Drinfeld's new realization[11] whose generators, called the Drinfeld generators, are natural analogues of those in the the loop algebras $\mathfrak{g}[t,t^{-1}]$, and the Faddeev-Reshetikhin-Semenov-Tian-Shansky-Takhtadjan (FRST) formulation [16, 59] in terms of the L operators satisfying the RLL relations associated with the R matrix, a solution to the YBE. The isomorphisms among these three have been discussed by several authors[11, 4, 39, 9, 30, 31].

Correspondingly there are three different formulations of EQGs: $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_N)$ and $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{g}})$[37] in terms of the Chevalley type generators, $U_{q,p}(\widehat{\mathfrak{g}})$[47, 36] and $E_{\tau,\eta}(\mathfrak{g})$[13, 14] in terms of the Drinfeld generators and $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$[24] and $E_{\tau,\eta}(\mathfrak{gl}_N)$[15, 19, 21, 41] in terms of the L operators. Here only $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_N)$ is the vertex type EQG, which is related to Baxter-Belavin's elliptic R matrix[3, 5]. The others are the face type which are related to the elliptic solutions to the face type YBE, for example [33]. These have their own co-algebra structures: the quasi-Hopf algebra structure[12] for $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_N)$, $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{g}})$[37] and $E_{\tau,\eta}(\mathfrak{sl}_2)$[14], and the Hopf algebroid structure[15, 40] for $E_{\tau,\eta}(\mathfrak{gl}_N)$[21, 41, 28] and $U_{q,p}(\widehat{\mathfrak{g}})$[50].

As like the cases in $Y(\mathfrak{g}), \mathcal{D}Y(\mathfrak{g})$ and $U_q(\widehat{\mathfrak{g}})$, each formulation has both advantages and disadvantages. The quasi-Hopf algebra formulations $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_N)$ and $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{g}})$[37] are suitable for studying formal algebraic structures such as the universal elliptic dynamical R matrices, the universal form of the dynamical RLL relations etc., but it is hard to derive concrete representations due to the complexity of the quasi-Hopf twist operation.

The Drinfeld realization $U_{q,p}(\widehat{\mathfrak{g}})$ is suitable for studying both finite and infinite dimensional representations[36, 43, 44, 46, 50, 49, 52] due to the nature of the Drinfeld generators. Recent developments include a characterization of the finite dimensional representations in terms of a theta function analogue[50] of the Drinfeld polynomials[11, 8] and a clarification of the quantum Z-algebra structures of the infinite dimensional representations[17]. An application to the algebraic analysis of the solvable lattice models[35] also have made a great success[36, 38, 46, 49, 7]. See also rather older works [54, 1, 53] whose results, in particular the vertex operators and the screening operators, are able to be reformulated by the representation theory of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$[43, 46]. In addition there are deep relationships between $U_{q,p}(\widehat{\mathfrak{g}})$ and the deformed $W(\mathfrak{g})$ algebras: the generating functions of the Drinfeld generators (the elliptic currents) $e_j(z)$ and $f_j(z)$ of $U_{q,p}(\widehat{\mathfrak{g}})$ are identified with the screening currents of the deformed $W(\mathfrak{g})$ algebras of the coset type[47, 36, 45, 17].

The FRST formulation is suitable for studying finite dimensional representations by a fusion procedure or by taking a coproduct. In this way finite dimensional representations of $E_{\tau,\eta}(\mathfrak{gl}_N)$ have been studied well [21, 41, 42] (see also [34]) and applied to the elliptic Ruijsenaars models[23, 22], the elliptic hypergeometric series[41, 42, 57], the partition function of the solvable lattice model[58, 56] and the elliptic Gaudin model[60].

On the other hand in order to formulate infinite dimensional representations of $E_{\tau,\eta}(\mathfrak{gl}_N)$ one needs it's central extension. There are two different proposals provided by [14] and [37, 36], respectively. Accordingly $E_{\tau,\eta}(\mathfrak{sl}_2)$ in [14] and $U_{q,p}(\widehat{\mathfrak{g}})$ in [36, 17, 47] have been the two proposals for their Drinfeld realizations. However the isomorphism between $E_{\tau,\eta}(\mathfrak{gl}_N)$ in the FRST formulation and neither of these two Drinfeld realizations has been discussed precisely.

The aim of this paper is to establish the isomorphism between $U_{q,p}(\widehat{\mathfrak{gl}}_N)$ and a central extension of $E_{\tau,\eta}(\mathfrak{gl}_N)$ in the FRST formulation as a Hopf algebroid. For this purpose, we first reformulate $E_{\tau,\eta}(\mathfrak{gl}_N)$ as a topological algebra over the ring of formal power series in p and at the same time we give a central extension of it according to the argument in [37, 36]. We denote the resultant algebra by $E_{q,p}(\widehat{\mathfrak{gl}}_N)$, where the generators are clear and their defining relations are well defined in the p-adic topology as in $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$[24] and $U_{q,p}(\widehat{\mathfrak{g}})$[17]. Secondly we discuss dynamical representations of $U_{q,p}(\widehat{\mathfrak{gl}}_N)$. We especially introduce an evaluation H-algebra homomorphism from $U_{q,p}(\widehat{\mathfrak{gl}}_N)$ to a dynamical extension of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$. This allows us to obtain the dynamical representations (of both finite and infinite dimensional) from any representations of $U_q(\widehat{\mathfrak{gl}}_N)$. As a result the problem resolves itself into the one discussed by Ding and Frenkel in the trigonometric case [9].

A part of the results have been reported in the workshops “Recent advances in quantum integrable systems 2012”, 10-14 September 2012, Angers, France and “Elliptic Integrable Systems and Hypergeometric Functions”, 15-19 July 2013, Lorentz Center, Leiden, the Netherlands.

This paper is organized as follows. In Section 2 preparing notations and conventions we introduce the elliptic dynamical R matrix. In Section 3 we define $U_{q,p}(\widehat{\mathfrak{gl}}_N)$ and $E_{q,p}(\widehat{\mathfrak{gl}}_N)$ as topological algebras over the ring of formal power series in p. We also give the trigonometric ($p = 0$) counter parts of them. In section 4 we show that both $U_{q,p}(\widehat{\mathfrak{gl}}_N)$ and $E_{q,p}(\widehat{\mathfrak{gl}}_N)$ are H-algebras (Proposition 4.3 and 4.4). Then we introduce an H-Hopf algebroid structure to them. In Section 5 we introduce dynamical representations of $U_{q,p}(\widehat{\mathfrak{g}})$ and give a construction of the evaluation dynamical representations from any representations of $U_q(\widehat{\mathfrak{gl}}_N)$. In Section 6 we discuss an isomorphism between $U_{q,p}(\widehat{\mathfrak{gl}}_N)$ and $E_{q,p}(\widehat{\mathfrak{gl}}_N)$

as an H-Hopf algebroid. Our arguments mainly follow those by Ding and Frenkel in the trigonometric case[9] with some additional formulas for the lower rank subalgebras of $E_{q,p}(\widehat{\mathfrak{gl}}_N)$, which make the induction process more transparent. In particular by making use of the evaluation dynamical representations in Sec. 5 our proof on the injectivity resolves itself into the results in [9]. Appendix A contains a definition of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$ which we use in Sec.5. In Appendix B we list the formulas necessary for discussing the evaluation dynamical representations. In Appendix C we summarize the formulas which identify a combination of the Gauss components of the L operator with the elliptic currents of $U_{q,p}(\widehat{\mathfrak{gl}}_N)$. In Appendix D we summarize some formulas on adding 'fractional powers in z' which clarify a connection between $U_{q,p}(\widehat{\mathfrak{g}})$ in the current paper and the previous one in [47, 36, 43, 44, 50]. Appendix E contains some formulas for the elliptic quantum determinants.

§2. The R-matrices

Let $\{\epsilon_j\ (1\le j\le N)\}$ be the orthonormal basis in $\mathbb{R}^N$ with the inner product $(\epsilon_j,\epsilon_k)=\delta_{j,k}$. Setting $\bar{\epsilon}_j=\epsilon_j-\epsilon,\ \epsilon=\dfrac{1}{N}\displaystyle\sum_{j=1}^{N}\epsilon_j$, we define the weight lattice $\mathcal{P}$ of A_{N-1} type by $\mathcal{P}=\displaystyle\sum_{j=1}^{N}\mathbb{Z}\ \bar{\epsilon}_j$. Let $I=\{1,2,\cdots,N-1\}$. We set $\alpha_j=\bar{\epsilon}_j-\bar{\epsilon}_{j+1},\ \bar{\Lambda}_j=\bar{\epsilon}_1+\cdots+\bar{\epsilon}_j\ (j\in I)$ and define $\mathcal{Q}=\mathbb{Z}\alpha_1\oplus\cdots\oplus\mathbb{Z}\alpha_{N-1}$ and $\bar{\mathfrak{h}}^*=\mathbb{C}\bar{\Lambda}_1\oplus\cdots\oplus\mathbb{C}\bar{\Lambda}_{N-1}$. We also define elements $h_{\bar{\epsilon}_j}\ (1\le j\le N)$ in the dual space $\bar{\mathfrak{h}}$ by $<\bar{\epsilon}_i,h_{\bar{\epsilon}_j}>=(\bar{\epsilon}_i,\bar{\epsilon}_j)=\delta_{j,k}-\frac{1}{N}$. Setting $h_j=h_{\bar{\epsilon}_j}-h_{\bar{\epsilon}_{j+1}}\ (j\in I)$ we have $<\bar{\Lambda}_i,h_j>=\delta_{i,j}$ so that $\bar{\mathfrak{h}}=\mathbb{C}h_1\oplus\cdots\oplus\mathbb{C}h_{N-1}$. For $\alpha=\sum_j a_j\bar{\epsilon}_j\in\bar{\mathfrak{h}}^*$, we define $h_\alpha\in\bar{\mathfrak{h}}$ by $h_\alpha=\sum_j a_jh_{\bar{\epsilon}_j}$ and $h_0=0$. We also need two more elements c and Λ_0 satisfying $<\Lambda_0,c>=1,<\Lambda_0,h_j>=0=<\bar{\Lambda}_j,c>\ \ (1\le j\le N)$. We regard $\bar{\mathfrak{h}}\oplus\bar{\mathfrak{h}}^*$ as a Heisenberg algebra by

$$[h_{\bar{\epsilon}_j},\bar{\epsilon}_k]=(\bar{\epsilon}_j,\bar{\epsilon}_k),\qquad [h_{\bar{\epsilon}_j},h_{\bar{\epsilon}_k}]=0=[\bar{\epsilon}_j,\bar{\epsilon}_k]. \tag{2.1}$$

We also introduce the dynamical parameters $P_{\bar{\epsilon}_j}\ (j\in I)$ and their duals $Q_{\bar{\epsilon}_j}$. They are the Heisenberg algebra defined by

$$[P_{\bar{\epsilon}_j},Q_{\bar{\epsilon}_k}]=(\bar{\epsilon}_j,\bar{\epsilon}_k),\qquad [P_{\bar{\epsilon}_j},P_{\bar{\epsilon}_k}]=0=[Q_{\bar{\epsilon}_j},Q_{\bar{\epsilon}_k}]. \tag{2.2}$$

We set $P_\alpha=\sum_j a_jP_{\bar{\epsilon}_j}$ for $\alpha=\sum_j a_j\bar{\epsilon}_j$ and $P_0=0$ etc. In particular we set $P_j=P_{\alpha_j}=P_{\bar{\epsilon}_j}-P_{\bar{\epsilon}_{j+1}}$ and $Q_j=Q_{\alpha_j}=Q_{\bar{\epsilon}_j}-Q_{\bar{\epsilon}_{j+1}}$.

For the abelian group $\mathcal{R}_Q = \sum_i \mathbb{Z}Q_{\bar{\epsilon}_i}$, we denote by $\mathbb{C}[\mathcal{R}_Q]$ the group algebra over $\mathbb{C}$ of $\mathcal{R}_Q$. We denote by e^{Q_α} the element of $\mathbb{C}[\mathcal{R}_Q]$ corresponding to $Q_\alpha \in \mathcal{R}_Q$. These e^{Q_α} satisfy $e^{Q_\alpha}e^{Q_\beta} = e^{Q_\alpha+Q_\beta}$ and $(e^{Q_\alpha})^{-1} = e^{-Q_\alpha}$. In particular, $e^0 = 1$ is the identity element.

Now let us introduce a dynamical extension of $\bar{\mathfrak{h}}$ and $\bar{\mathfrak{h}}^*$: $H = \bar{\mathfrak{h}} \oplus \sum_j \mathbb{C}P_{\bar{\epsilon}_j} + \mathbb{C}c = \sum_j \mathbb{C}(P+h)_{\bar{\epsilon}_j} + \sum_j \mathbb{C}P_{\bar{\epsilon}_j} + \mathbb{C}c$ and $H^* = \bar{\mathfrak{h}}^* + \sum_j \mathbb{C}Q_{\bar{\epsilon}_j} + \mathbb{C}\Lambda_0$. Through this paper we often use the abbreviation $(P+h)_{\bar{\epsilon}_j}$ for $P_{\bar{\epsilon}_j} + h_{\bar{\epsilon}_j}$. We have the paring: $< Q_\alpha, P_\beta >= (\alpha, \beta) =< \alpha, h_\beta >\ \ \alpha, \beta \in \bar{\mathfrak{h}}^*$, $< \Lambda_0, c >= 1$, and the others vanish. Let $\mathcal{M}_{H^*}$ be the field of meromorphic functions on H^*. We denote by $\widehat{f} = f(P+h, P)$ an element of $\mathcal{M}_{H^*}$, where $P + h = \sum_j a_j(P+h)_{\bar{\epsilon}_j}, P = \sum_j b_j P_{\bar{\epsilon}_j} \in H$. The function $\widehat{f}$ is evaluated at $\mu \in H^*$ as $\widehat{f}(\mu) = f(< \mu, P+h >, < \mu, P >)$ etc.. Hereafter we set $\mathbb{F} = \mathcal{M}_{H^*}$.

Let $\hbar$ and p be indeterminates. We set $q = e^\hbar$. Through this paper we also use $p^* = pq^{-2c}$. The following notations are often used.

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad \Theta_p(z) = (z;p)_\infty (p/z;p)_\infty (p;p)_\infty,$$

$$(x; q_1, q_2, \cdots, q_k)_\infty = \prod_{n_1, n_2, \cdots, n_k = 0}^{\infty} (1 - x q_1^{n_1} q_2^{n_2} \cdots q_k^{n_k}),$$

$$\{z\} = (z; p, q^{2N})_\infty,$$

$$(x_1, x_2, \cdots, x_l; q_1, q_2, \cdots, q_k)_\infty = \prod_{i=1}^{l} (x_i; q_1, q_2, \cdots, q_k)_\infty.$$

2.1. The elliptic dynamical R-matrices

Let $V = \oplus_{i=1}^N \mathbb{C}v_i$, $E_{ij}v_k = \delta_{j,k}v_i$. We consider the following elliptic dynamical R-matrices $R^\pm(z,s) \in \mathrm{End}(V \otimes V)$ of type $A_{N-1}^{(1)}$. For $s \in H$,

$$\begin{aligned}
R^\pm(z,s) &= \rho^\pm(z)\overline{R}(z,s), \\
\overline{R}(z,s) &= \sum_{j=1}^N E_{jj} \otimes E_{jj} \\
&\quad + \sum_{1 \le j < l \le N} \Big(b(z, s_{j,l}) E_{jj} \otimes E_{ll} + \bar{b}(z) E_{ll} \otimes E_{jj} \\
&\qquad + c(z, s_{j,l}) E_{jl} \otimes E_{lj} + \bar{c}(z, s_{j,l}) E_{lj} \otimes E_{jl} \Big),
\end{aligned} \tag{2.3}$$

where $s_{j,l} = s_{\bar{\epsilon}_j} - s_{\bar{\epsilon}_l} \quad (1 \le j < l \le N)$ and

$$\rho^+(z)=q^{-\frac{N-1}{N}}\frac{\{q^2z\}\{q^{2N-2}z\}\{p/z\}\{pq^{2N}/z\}}{\{z\}\{q^{2N}z\}\{pq^2/z\}\{pq^{2N-2}/z\}}, \tag{2.4}$$

$$\rho^-(z)=q^{\frac{N-1}{N}}\frac{\{pq^2z\}\{pq^{2N-2}z\}\{1/z\}\{q^{2N}/z\}}{\{pz\}\{pq^{2N}z\}\{q^2/z\}\{q^{2N-2}/z\}}, \tag{2.5}$$

$$b(z,s)=q\frac{\Theta_p(q^2q^{2s})\Theta_p(q^{-2}q^{2s})\Theta_p(z)}{\Theta_p(q^{2s})^2\Theta_p(q^2z)}, \tag{2.6}$$

$$\bar{b}(z)=q\frac{\Theta_p(z)}{\Theta_p(q^2z)}, \tag{2.7}$$

$$c(z,s)=\frac{\Theta_p(q^2)\Theta_p(q^{2s}z)}{\Theta_p(q^{2s})\Theta_p(q^2z)}, \qquad \bar{c}(z,s)=c(z,-s). \tag{2.8}$$

We also denote by $R^{\pm *}(z,s)$ the R matrices obtained from $R^{\pm}(z,s)$ by replacing p with p^*. Note that

$$\rho^+(zp)=q^{-2\frac{N-1}{N}}\rho^-(z). \tag{2.9}$$

In particular,

$$R^-(z,s)^{-1}=PR^+(z^{-1},s)P. \tag{2.10}$$

Furthermore if we set

$$\rho(z)=\frac{\rho^{+*}(z)}{\rho^+(z)} \tag{2.11}$$

where $\rho^{\pm *}(z)=\rho^{\pm}(z)|_{p\to p^*}$, we have

$$\rho(z)^{-1}=\rho(z^{-1}), \qquad \rho(z)=\frac{\rho^{-*}(z)}{\rho^-(z)}. \tag{2.12}$$

Proposition 2.1. [19] *The $R^+(z,s)$ satisfies the following dynamical Yang-Baxter equation.*

$$\begin{aligned}(2.13)\; & R^{+(12)}(z_1/z_2,P+\pi_V(h)^{(3)})R^{+(13)}(z_1,P)R^{+(23)}(z_2,P+\pi_V(h)^{(1)})\\ &=R^{+(23)}(z_2,P)R^{+(13)}(z_1,P+\pi_V(h)^{(2)})R^{+(12)}(z_1/z_2,P),\end{aligned}$$

where $\pi_V(h)^{(1)}=\pi_V(h)\otimes 1\otimes 1$ and $\pi_V(h)^{(1)}_{j,l}=\pi_V(h_{\bar{\epsilon}_j})^{(1)}-\pi_V(h_{\bar{\epsilon}_l})^{(1)}$ with $\pi_V(h_{\bar{\epsilon}_j})=E_{jj}-\frac{1}{N}\mathrm{I}$ etc.. Here I denotes the $N\times N$ unit matrix.

Remark 1. The elliptic dynamical R-matrix (2.3) is gauge equivalent to the $A^{(1)}_{N-1}$ type face weight obtained by Jimbo, Miwa and Okado[33]. See Appendix D.

Remark 2. The R-matrix preserves the weights

$$[R^{\pm}(z,s), \pi_V(h)\otimes\pi_V(h)] = 0 \qquad \forall h \in \bar{\mathfrak{h}}. \tag{2.14}$$

Now let us set

$$\rho_0(z) = q^{-\frac{N-1}{N}} \frac{(q^2z;q^{2N})_\infty (q^{2N-2}z;q^{2N})_\infty}{(z;q^{2N})_\infty (q^{2N}z;q^{2N})_\infty}, \tag{2.15}$$

$$\alpha(z) = \frac{\{pq^2z\}\{pq^{2N-2}z\}\{p/z\}\{pq^{2N}/z\}}{\{pz\}\{pq^{2N}z\}\{pq^2/z\}\{pq^{2N-2}/z\}} \tag{2.16}$$

$$\Xi_p(z) = (pz;p)_\infty (p/z;p)_\infty. \tag{2.17}$$

Then we have

$$\begin{aligned}\rho^{\pm}(z)b(z,s) = {} & q^{\pm1}\rho_0(z^{\pm1})^{\pm1} \frac{(1-(q^{2s}q^{-2})^{\pm1})(1-(q^{2s}q^2)^{\pm1})}{(1-q^{\pm2s})^2} \\ & \times \frac{1-z^{\pm1}}{1-(q^2z)^{\pm1}} \alpha(z) \frac{\Xi_p(q^{2s}q^{-2})\Xi_p(q^{2s}q^2)}{\Xi_p(q^{2s})^2} \frac{\Xi_p(z)}{\Xi_p(q^2z)},\end{aligned} \tag{2.18}$$

$$\rho^{\pm}(z)\bar{b}(z) = q^{\pm1}\rho_0(z^{\pm1})^{\pm1} \frac{1-z^{\pm1}}{1-(q^2z)^{\pm1}} \alpha(z) \frac{\Xi_p(z)}{\Xi_p(q^2z)}, \tag{2.19}$$

$$\begin{aligned}\rho^{\pm}(z)c(z,s) = {} & \rho_0(z^{\pm1})^{\pm1} \frac{1-q^{\pm2}}{1-q^{\pm2s}} \frac{1-(q^{2s}z)^{\pm1}}{1-(q^2z)^{\pm1}} \\ & \times \alpha(z) \frac{\Xi_p(q^2)\Xi_p(q^{2s}z)}{\Xi_p(q^{2s})\Xi_p(q^2z)}.\end{aligned} \tag{2.20}$$

In [24] a similar expression was obtained for Baxter's elliptic R matrix. In $R^{\pm}(z,s)$, we specify the factors $\frac{1-z^{\pm1}}{1-(q^2z)^{\pm1}}$ and $\frac{1-(q^{2s}z)^{\pm1}}{1-(q^2z)^{\pm1}}$ in (2.18)-(2.20) to be power series in $z^{\pm1}$. We then treat $R^{\pm}(z,s)$ as formal Laurent series in z

$$R^{\pm}(z,s) = \sum_{n\in\mathbb{Z}} R^{\pm}(s)_n z^n \tag{2.21}$$

whose coefficients are in the ring $\mathbb{F}[[p]]$ of formal power series in p. Note that $\alpha(z)$, $\frac{\Xi(z)}{\Xi(q^2z)}$ and $\frac{\Xi(q^{2s_{ij}}z)}{\Xi(q^2z)}$ are well defined formal Laurent series in z with coefficients in $\mathbb{F}[[p]]$. Then the matrices $R^{\pm}(z,s)$ satisfy

$$R^{\pm}(s)_n \equiv 0 \qquad \text{mod } p^{\max(\mp n,0)}\mathbb{F}[[p]] \qquad \forall n \in \mathbb{Z}. \tag{2.22}$$

In particular, at $p=0$ $R^{+}(z,s)$ (reps. $R^{-}(z,s)$) contains only non-negative (reps. non-positive) powers in z. Explicitly we have $R_0^{\pm}(z,s) \equiv R^{\pm}(z,s)\Big|_{p=0}$,

$$R_0^{\pm}(z,s)=\rho_0^{\pm}(z^{\pm})^{\pm 1}\overline{R}_0^{\pm}(z,s), \tag{2.23}$$

$$\begin{aligned}\overline{R}_0^{\pm}(z,s)=&\sum_{j=1}^{N}E_{jj}\otimes E_{jj}\\ &+\sum_{1\le j<l\le N}\Big(b_0^{\pm}(z,s_{j,l})E_{jj}\otimes E_{ll}+\bar{b}_0^{\pm}(z)E_{ll}\otimes E_{jj}\\ &+c_0^{\pm}(z,s_{j,l})E_{jl}\otimes E_{lj}+\bar{c}_0^{\pm}(z,s_{j,l})E_{lj}\otimes E_{jl}\Big),\end{aligned} \tag{2.24}$$

where

$$\begin{aligned}&\rho_0^{+}(z)=\rho_0(z),\qquad \rho_0^{-}(z)=\rho_0(z^{-1})^{-1},\\ &b_0^{\pm}(z,s)=\frac{(1-q^{2(s-1)})(1-q^{2(s+1)})}{(1-q^{2s})^2}q^{\pm 1}\frac{1-z^{\pm 1}}{1-(q^2z)^{\pm 1}},\\ &\bar{b}_0^{\pm}(z)=q^{\pm 1}\frac{1-z^{\pm 1}}{1-(q^2z)^{\pm 1}},\\ &c_0^{\pm}(z,s)=\frac{1-q^{2s}z}{1-q^{2s}}\frac{1-q^{\pm 2}}{1-(q^2z)^{\pm 1}}\times\begin{cases}1 & \text{for } +\\ z^{-1} & \text{for } -\end{cases},\\ &\bar{c}_0^{\pm}(z,s)=\frac{1-q^{2s}z^{-1}}{1-q^{2s}}\frac{1-q^{\pm 2}}{1-(q^2z)^{\pm 1}}\times\begin{cases}z & \text{for } +\\ 1 & \text{for } -\end{cases}.\end{aligned}$$

Hence one can regard the matrix element $R_0(z,s)_{ij}^{kl}$ as a formal power series in the (multiplicative) dynamical variables $q^{2s_{i,j}}$. The 0-th order term in $q^{2s_{i,j}}$ coincides with the corresponding component of the standard trigonometric R matrix

$$\begin{aligned}R_0(z)&=\rho_0(z)\bar{R}_0(z),\\ \bar{R}_0(z)&=\sum_{j=1}^{N}E_{jj}\otimes E_{jj}\\ &\quad+\sum_{1\le j<l\le N}\Big(q\frac{1-z}{1-q^2z}(E_{jj}\otimes E_{ll}+E_{ll}\otimes E_{jj})\\ &\quad+\frac{1-q^2}{1-q^2z}E_{jl}\otimes E_{lj}+z\frac{1-q^2}{1-q^2z}E_{lj}\otimes E_{jl}\Big),\end{aligned} \tag{2.25}$$

Note that one can parametrize the elliptic dynamical R matrices associated with the other types of affine Lie algebras, at least $\widehat{\mathfrak{g}}=B_N^{(1)},C_N^{(1)},D_N^{(1)}$ [33, 48], in a similar way to (2.18)-(2.20) so that they have the same property at $p=0$.

§3. The Elliptic Quantum Algebras $U_{q,p}(\widehat{\mathfrak{gl}}_N)$ and $E_{q,p}(\widehat{\mathfrak{gl}}_N)$

In this section we define two elliptic algebras $U_{q,p}(\widehat{\mathfrak{gl}}_N)$ and $E_{q,p}(\widehat{\mathfrak{gl}}_N)$ as topological algebras over $\mathbb{F}[[p]]$.

3.1. $U_{q,p}(\widehat{\mathfrak{gl}}_N)$

Definition 3.1. *The elliptic algebra* $U_{q,p}(\widehat{\mathfrak{gl}}_N)$ *is a topological algebra over* $\mathbb{F}[[p]]$ *generated by* $e_{j,m}, f_{j,m}, k_{l,m}$, $(1 \le j \le N-1, 1 \le l \le N, m \in \mathbb{Z})$, $\widehat{d}$ *and the central element* $q^{\pm c/2}$. *We set*

$$e_j(z) = \sum_{m \in \mathbb{Z}} e_{j,m} z^{-m}, \quad f_j(z) = \sum_{m \in \mathbb{Z}} f_{j,m} z^{-m}, \tag{3.1}$$

$$k_l^+(z) = \sum_{m \in \mathbb{Z}_{\ge 0}} k_{l,-m} z^m + \sum_{m \in \mathbb{Z}_{>0}} k_{l,m} p^m z^{-m}, \tag{3.2}$$

$$k_l^-(z) = q^{2h_{\bar{\epsilon}_l}} k_l^+(zp^* q^c). \tag{3.3}$$

The defining relations are as follows. For $g(P), g(P+h) \in \mathbb{F}$,

$$\begin{aligned} &g(P+h)e_j(z) = e_j(z)g(P+h), \\ &g(P)e_j(z) = e_j(z)g(P- < Q_{\alpha_j}, P >), \end{aligned} \tag{3.4}$$

$$\begin{aligned} &g(P+h)f_j(z) = f_j(z)g(P+h- < \alpha_j, P+h >), \\ &g(P)f_j(z) = f_j(z)g(P), \end{aligned} \tag{3.5}$$

$$\begin{aligned} &g(P)k_l^+(z) = k_l^+(z)g(P- < Q_{\bar{\epsilon}_l}, P >), \\ &g(P+h)k_l^+(z) = k_l^+(z)g(P+h- < Q_{\bar{\epsilon}_l}, P >), \end{aligned} \tag{3.6}$$

$$[\widehat{d}, g(P+h)] = 0 = [\widehat{d}, g(P)], \tag{3.7}$$

$$\begin{aligned} &[\widehat{d}, k_l^+(z)] = -z\frac{\partial}{\partial z}k_l^+(z), \quad [\widehat{d}, e_j(z)] = -z\frac{\partial}{\partial z}e_j(z), \\ &[\widehat{d}, f_j(z)] = -z\frac{\partial}{\partial z}f_j(z), \end{aligned} \tag{3.8}$$

$$\rho_+^+(z_2/z_1)k_l^+(z_1)k_l^+(z_2) = \rho_+^+(z_1/z_2)k_l^+(z_2)k_l^+(z_1), \tag{3.9}$$

$$\begin{aligned} &\rho_+^+(z_2/z_1)\frac{(p^* z_2/z_1;p^*)_\infty (pq^2 z_2/z_1;p)_\infty}{(p^* q^2 z_2/z_1;p^*)_\infty (pz_2/z_1;p)_\infty} k_j^+(z_1)k_l^+(z_2) \\ &\quad = \rho_+^+(z_1/z_2)\frac{(q^{-2} z_1/z_2;p^*)_\infty (z_1/z_z;p)_\infty}{(z_1/z_2;p^*)_\infty (q^{-2} z_1/z_2;p)_\infty} k_l^+(z_2)k_j^+(z_1) \\ &\qquad (1 \le j < l \le N), \end{aligned} \tag{3.10}$$

$$\frac{(p^*q^{c+2-j}z_2/z_1;p^*)_\infty}{(p^*q^{c-j}z_2/z_1;p^*)_\infty}k_j^+(z_1)e_j(z_2) = q^{-1}\frac{(q^{-c+j}z_1/z_2;p^*)_\infty}{(q^{-c-2+j}z_1/z_2;p^*)_\infty}e_j(z_2)k_j^+(z_1), \tag{3.11}$$

$$\frac{(p^*q^{c-2-j}z_2/z_1;p^*)_\infty}{(p^*q^{c-j}z_2/z_1;p^*)_\infty}k_{j+1}^+(z_1)e_j(z_2) = q\frac{(q^{-c+j}z_1/z_2;p^*)_\infty}{(q^{-c+2+j}z_1/z_2;p^*)_\infty}e_j(z_2)k_{j+1}^+(z_1), \tag{3.12}$$

$$k_l^+(z_1)e_j(z_2)k_l^+(z_1)^{-1} = e_j(z_2) \qquad (l \neq j, j+1), \tag{3.13}$$

$$\frac{(pq^{-j}z_2/z_1;p)_\infty}{(pq^{2-j}z_2/z_1;p)_\infty}k_j^+(z_1)f_j(z_2) = q\frac{(q^{-2+j}z_1/z_2;p)_\infty}{(q^j z_1/z_2;p)_\infty}f_j(z_2)k_j^+(z_1), \tag{3.14}$$

$$\frac{(pq^{-j}z_2/z_1;p)_\infty}{(pq^{-2-j}z_2/z_1;p)_\infty}k_{j+1}^+(z_1)f_j(z_2) = q^{-1}\frac{(q^{2+j}z_1/z_2;p)_\infty}{(q^j z_1/z_2;p)_\infty}f_j(z_2)k_{j+1}^+(z_1), \tag{3.15}$$

$$k_l^+(z_1)f_j(z_2)k_l^+(z_1)^{-1} = f_j(z_2) \qquad (l \neq j, j+1), \tag{3.16}$$

$$z_1\frac{(q^2z_2/z_1;p^*)_\infty}{(p^*q^{-2}z_2/z_1;p^*)_\infty}e_j(z_1)e_j(z_2) = -z_2\frac{(q^2z_1/z_2;p^*)_\infty}{(p^*q^{-2}z_1/z_2;p^*)_\infty}e_j(z_2)e_j(z_1), \tag{3.17}$$

$$z_1\frac{(q^{-1}z_2/z_1;p^*)_\infty}{(p^*qz_2/z_1;p^*)_\infty}e_j(z_1)e_{j+1}(z_2) = -z_2\frac{(q^{-1}z_1/z_2;p^*)_\infty}{(p^*qz_1/z_2;p^*)_\infty}e_{j+1}(z_2)e_j(z_1), \tag{3.18}$$

$$e_j(z_1)e_l(z_2) = e_l(z_2)e_j(z_1) \qquad (|j-l|>1) \tag{3.19}$$

$$z_1\frac{(q^{-2}z_2/z_1;p)_\infty}{(pq^2z_2/z_1;p)_\infty}f_j(z_1)f_j(z_2) = -z_2\frac{(q^{-2}z_1/z_2;p)_\infty}{(pq^2z_1/z_2;p)_\infty}f_j(z_2)f_j(z_1), \tag{3.20}$$

$$z_1\frac{(qz_2/z_1;p)_\infty}{(pq^{-1}z_2/z_1;p)_\infty}f_j(z_1)f_{j+1}(z_2) = -z_2\frac{(qz_1/z_2;p)_\infty}{(pq^{-1}z_1/z_2;p)_\infty}f_{j+1}(z_2)f_j(z_1), \tag{3.21}$$

$$f_j(z_1)f_l(z_2)=f_l(z_2)f_j(z_1) \qquad (|j-l|>1) \tag{3.22}$$

$$\begin{aligned}&[e_i(z_1),f_j(z_2)]\\ &=\frac{\delta_{i,j}\kappa}{q-q^{-1}}\left(\delta(q^{-c}z_1/z_2)k_j^-(q^{-\frac{c}{2}}z_1)k_{j+1}^-(q^{-\frac{c}{2}}z_1)^{-1}\right.\\ &\qquad\left.-\,\delta(q^{c}z_1/z_2)k_j^+(q^{-\frac{c}{2}}z_2)k_{j+1}^+(q^{-\frac{c}{2}}z_2)^{-1}\right),\end{aligned} \tag{3.23}$$

$$\begin{aligned}&\frac{(p^*q^2z_2/z_1;p^*)_\infty}{(p^*q^{-2}z_2/z_1;p^*)_\infty}\\ &\times\left\{\frac{(p^*q^{-1}z_1/w;p^*)_\infty}{(p^*qz_1/w;p^*)_\infty}\frac{(p^*q^{-1}z_2/w;p^*)_\infty}{(p^*qz_2/w;p^*)_\infty}e_j(w)e_i(z_1)e_i(z_2)\right.\\ &\quad-[2]_q\frac{(p^*q^{-1}w/z_1;p^*)_\infty}{(p^*qw/z_1;p^*)_\infty}\frac{(p^*q^{-1}z_2/w;p^*)_\infty}{(p^*qz_2/w;p^*)_\infty}e_i(z_1)e_j(w)e_i(z_2)\\ &\quad\left.+\frac{(p^*q^{-1}w/z_1;p^*)_\infty}{(p^*qw/z_1;p^*)_\infty}\frac{(p^*q^{-1}w/z_2;p^*)_\infty}{(p^*qw/z_2;p^*)_\infty}e_i(z_1)e_i(z_2)e_j(w)\right\}\\ &\qquad+(z_1\leftrightarrow z_2)=0,\end{aligned} \tag{3.24}$$

$$\begin{aligned}&\frac{(pq^{-2}z_2/z_1;p)_\infty}{(pq^{2}z_2/z_1;p)_\infty}\\ &\times\left\{\frac{(pqz_1/w;p)_\infty}{(pq^{-1}z_1/w;p)_\infty}\frac{(pqz_2/w;p)_\infty}{(pq^{-1}z_2/w;p)_\infty}f_j(w)f_i(z_1)f_i(z_2)\right.\\ &\quad-[2]_q\frac{(pqw/z_1;p)_\infty}{(pq^{-1}w/z_1;p)_\infty}\frac{(pqz_2/w;p)_\infty}{(pq^{-1}z_2/w;p)_\infty}f_i(z_1)f_j(w)f_i(z_2)\\ &\quad\left.+\frac{(pqw/z_1;p)_\infty}{(pq^{-1}w/z_1;p)_\infty}\frac{(pqw/z_2;p)_\infty}{(pq^{-1}w/z_2;p)_\infty}f_i(z_1)f_i(z_2)f_j(w)\right\}\\ &\qquad+(z_1\leftrightarrow z_2)=0\quad |i-j|=1.\end{aligned} \tag{3.25}$$

where $\delta(z)=\sum_{n\in\mathbb{Z}}z^n$, $\rho(z)$ *is given in* (2.11),

$$\rho_+^+(z)=\frac{\{q^2z\}^*\{q^{-2}q^{2N}z\}^*\{z\}\{q^{2N}z\}}{\{z\}^*\{q^{2N}z\}^*\{q^2z\}\{q^{-2}q^{2N}z\}}, \tag{3.26}$$

and κ *is given by*

$$\kappa=\frac{(p;p)_\infty(p^*q^2;p^*)_\infty}{(p^*;p^*)_\infty(pq^2;p)_\infty}. \tag{3.27}$$

We call $e_j(z),f_j(z),k_l^\pm(z)$ *the elliptic currents. We also denote by* $U'_{q,p}(\widehat{\mathfrak{gl}}_N)$ *the subalgebra obtained by removing* $\widehat{d}$.

We treat these relations as formal Laurent series in z,w and z_j's. In each term of (3.10)-(3.22) and (3.24)-(3.25), the expansion direction of the structure function given by a ratio of infinite products is chosen according to the order of the accompanied product of the elliptic currents. For example, in the l.h.s of (3.17), $\frac{(q^2z_2/z_1;p^*)_\infty}{(p^*q^{-2}z_2/z_1;p^*)_\infty}$ should be expanded in z_2/z_1, whereas in the r.h.s $\frac{(q^2z_1/z_2;p^*)_\infty}{(p^*q^{-2}z_1/z_2;p^*)_\infty}$ should be expanded in z_1/z_2. All the coefficients in z_j's are well defined in the p-adic topology.

For a practical use, we remark that in the sense of analytic continuation (3.9)-(3.15) and (3.17)-(3.21) can be rewritten as follows.

$$k_l^+(z_1)k_l^+(z_2)=\rho(z_1/z_2)k_l^+(z_2)k_l^+(z_1), \qquad (1\le l\le N), \tag{3.28}$$

$$\begin{aligned} &k_j^+(z_1)k_l^+(z_2)\\ &=\rho(z_1/z_2)\frac{\Theta_{p^*}(q^{-2}z_1/z_2)\Theta_p(z_1/z_2)}{\Theta_{p^*}(z_1/z_2)\Theta_p(q^{-2}z_1/z_2)}k_l^+(z_2)k_j^+(z_1)\\ &\qquad (1\le j<l\le N),\end{aligned} \tag{3.29}$$

$$k_j^+(z_1)e_j(z_2)k_j^+(z_1)^{-1}=q^{-1}\frac{\Theta_{p^*}(q^{-c+j}z_1/z_2)}{\Theta_{p^*}(q^{-c-2+j}z_1/z_2)}e_j(z_2), \tag{3.30}$$

$$k_{j+1}^+(z_1)e_j(z_2)k_{j+1}^+(z_1)^{-1}=q\frac{\Theta_{p^*}(q^{-c+j}z_1/z_2)}{\Theta_{p^*}(q^{-c+2+j}z_1/z_2)}e_j(z_2), \tag{3.31}$$

$$k_j^+(z_1)f_j(z_2)k_j^+(z_1)^{-1}=q\frac{\Theta_p(q^{-2+j}z_1/z_2)}{\Theta_p(q^jz_1/z_2)}f_j(z_2), \tag{3.32}$$

$$k_{j+1}^+(z_1)f_j(z_2)k_{j+1}^+(z_1)^{-1}=q^{-1}\frac{\Theta_p(q^{2+j}z_1/z_2)}{\Theta_p(q^jz_1/z_2)}f_j(z_2) \tag{3.33}$$

$$e_j(z_1)e_j(z_2)=-\frac{z_2}{z_1}\frac{\Theta_{p^*}(q^2z_1/z_2)}{\Theta_{p^*}(q^2z_2/z_1)}e_j(z_2)e_j(z_1), \tag{3.34}$$

$$e_j(z_1)e_{j+1}(z_2)=-\frac{z_2}{z_1}\frac{\Theta_{p^*}(q^{-1}z_1/z_2)}{\Theta_{p^*}(q^{-1}z_2/z_1)}e_{j+1}(z_2)e_j(z_1), \tag{3.35}$$

$$f_j(z_1)f_j(z_2)=-\frac{z_2}{z_1}\frac{\Theta_{p^*}(q^{-2}z_1/z_2)}{\Theta_p(q^{-2}z_2/z_1)}f_j(z_2)f_j(z_1), \tag{3.36}$$

$$f_j(z_1)f_{j+1}(z_2)=-\frac{z_2}{z_1}\frac{\Theta_{p^*}(qz_1/z_2)}{\Theta_p(qz_2/z_1)}f_{j+1}(z_2)f_j(z_1). \tag{3.37}$$

Proposition 3.2. *Let us set*

$$K(z)=k_1^+(z)k_2^+(q^{-2}z)\cdots k_N^+(q^{-2(N-1)}z).$$

Then $K(z)$ *belongs to the center of* $U'_{q,p}(\widehat{\mathfrak{gl}}_N)$.

Proof. Direct calculation using (3.6), (3.13), (3.16), (3.28)-(3.33) shows that $K(z)$ commutes with $\mathbb{F}$ and all of the elliptic currents of $U_{q,p}(\widehat{\mathfrak{gl}}_N)$.

In particular, $[K(z), k_l^+(w)] = 0$ $(1 \le l \le N)$ follows from the identity

$$\prod_{j=1}^{N} \rho(q^{-2(j-1)}z) = \frac{\Theta_{p^*}(z)\Theta_p(q^{-(N-1)}z)}{\Theta_{p^*}(q^{-(N-1)}z)\Theta_p(z)}.$$

Q.E.D.

Remark. In Appendix E we identify $K(z)$ with the q-determinant of the L-operator.

The elliptic algebra $U'_{q,p}(\widehat{\mathfrak{sl}}_N)$ is identified with the quotient algebra $U'_{q,p}(\widehat{\mathfrak{gl}}_N)/ < K(z) - 1 >$. More explicitly, one can realize $k_l^+(z)$ $(1 \le l \le N)$ satisfying (3.6), (3.9)-(3.10) and $K(z) = 1$ as follows. Let $A = (a_{ij})_{i,j\in I\cup\{0\}}$ be the $A_{N-1}^{(1)}$ type generalized Cartan matrix. Let $\alpha_{i,m}$ $(i \in I, m \in \mathbb{Z}_{\neq 0})$ be the Heisenberg algebra satisfying

$$[\alpha_{i,m}, \alpha_{j,n}] = \delta_{m+n,0}\frac{[a_{ij}m]_q[cm]_q}{m}\frac{1-p^m}{1-p^{*m}}q^{-cm}. \tag{3.38}$$

Let us consider the following $\mathcal{E}_m^{+l}$ $(1 \le l \le N, m \in \mathbb{Z}_{\neq 0})$, which we call the elliptic bosons of the orthonormal basis type[17]. For $1 \le l \le N-1$

$$\begin{aligned}\mathcal{E}_m^{+l} = {} & \frac{q^{lm}}{(q-q^{-1})[m]_q^2[Nm]_q} \\ & \times \left(-q^{-Nm}\sum_{k=1}^{l-1}[km]_q\alpha_{k,m} + \sum_{k=l}^{N-1}[(N-k)m]_q\alpha_{k,m}\right),\end{aligned}$$

$$\mathcal{E}_m^{+N} = -\frac{1}{(q-q^{-1})[m]_q^2[Nm]_q}\sum_{k=1}^{N}[km]_q\alpha_{k,m}.$$

They satisfy

$$[\mathcal{E}_m^{+l}, \mathcal{E}_n^{+l}] = \delta_{m+n,0}\frac{[cm]_q[(N-1)m]_q}{m(q-q^{-1})^2[m]_q^3[Nm]_q}\frac{1-p^m}{1-p^{*m}}q^{-cm}, \tag{3.39}$$

$$\begin{aligned}[\mathcal{E}_m^{+j}, \mathcal{E}_n^{+l}] = {} & -\delta_{m+n,0}q^{(\mathrm{sgn}(l-j)N-l+j)m} \\ & \times \frac{[cm]_q}{m(q-q^{-1})^2[m]_q^2[Nm]_q}\frac{1-p^m}{1-p^{*m}}q^{-cm},\end{aligned} \tag{3.40}$$

$$\begin{aligned}[\alpha_{i,m}, \mathcal{E}_n^{+l}] = {} & \delta_{m+n,0}\frac{[cm]_q}{m(q^m-q^{-m})} \\ & \times \frac{1-p^m}{1-p^{*m}}q^{-cm}(q^{-m}\delta_{i,l} - \delta_{i,l-1}).\end{aligned} \tag{3.41}$$

Let $K^+_{\bar{\epsilon}_j}$ satisfies for $g(P), g(P+h) \in \mathbb{F}$

$$g(P)K^+_{\bar{\epsilon}_j} = K^+_{\bar{\epsilon}_j} g(P- < Q_{\bar{\epsilon}_j}, P >), \tag{3.42}$$

$$g(P+h)K^+_{\bar{\epsilon}_j} = K^+_{\bar{\epsilon}_j} g(P+h- < Q_{\bar{\epsilon}_j}, P >). \tag{3.43}$$

Then the following $k^+_l(z)$ satisfy the desired relations.

$$k^+_l(z) = K^+_{\bar{\epsilon}_l} : \exp\left\{ \sum_{m\neq 0} \frac{(q^m - q^{-m})^2 p^m}{1-p^m} \mathcal{E}^{+l}_m (q^l z)^{-m} \right\} : . \tag{3.44}$$

Furthermore if we further require that $\alpha_{i,m}$ and $K^+_{\bar{\epsilon}_i}$ satisfy

$$[g(P), \alpha_{i,m}] = [g(P+h), \alpha_{i,n}] = 0, \tag{3.45}$$

$$[\widehat{d}, \alpha_{j,n}] = n\alpha_{j,n}, \tag{3.46}$$

$$[\alpha_{i,m}, e_j(z)] = \frac{[a_{ij}m]_q}{m} \frac{1-p^m}{1-p^{*m}} q^{-cm} z^m e_j(z), \tag{3.47}$$

$$[\alpha_{i,m}, f_j(z)] = -\frac{[a_{ij}m]_q}{m} z^m f_j(z), \tag{3.48}$$

$$\begin{aligned} K^+_{\bar{\epsilon}_i} e_j(z) &= q^{-<\alpha_j, h_{\bar{\epsilon}_i}>} e_j(z) K^+_{\bar{\epsilon}_i}, \\ K^+_{\bar{\epsilon}_i} f_j(z) &= q^{<\alpha_j, h_{\bar{\epsilon}_i}>} f_j(z) K^+_{\bar{\epsilon}_i}, \end{aligned} \tag{3.49}$$

then $k^+_l(z)$ satisfy the remaining relations (3.8), (3.11)-(3.16).

Now let us define $\psi^\pm_j(z)$ $(1 \le j \le N-1)$ by[1]

$$\begin{aligned} \psi^+_j(q^{-c/2} q^j z) &= \kappa k^+_j(z) k^+_{j+1}(z)^{-1}, \\ \psi^-_j(q^{-c/2} q^j z) &= \kappa k^-_j(z) k^-_{j+1}(z)^{-1}. \end{aligned}$$

We have

$$\begin{aligned} \psi^+_j(q^{-\frac{c}{2}} z) = K^+_j \exp&\left(-(q-q^{-1}) \sum_{n>0} \frac{\alpha_{j,-n}}{1-p^n} z^n \right) \\ &\times \exp\left((q-q^{-1}) \sum_{n>0} \frac{p^n \alpha_{j,n}}{1-p^n} z^{-n} \right), \end{aligned} \tag{3.50}$$

and $\psi^-_j(z) = q^{2h_j} \psi^+_j(zpq^{-c})$ where we set $K^+_j = K^+_{\bar{\epsilon}_j} K^{+-1}_{\bar{\epsilon}_{j+1}}$.

---

[1] Our $\psi^\pm_j(z)$ are $\psi^\mp_j(z)$ in [36, 43].

Proposition 3.3. *The elliptic algebra* $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ *is characterized by* (3.45)-(3.48) *and the following relations. For* $g(P), g(P+h) \in \mathbb{F}$,

$$g(P+h)e_j(z) = e_j(z)g(P+h), \quad g(P)e_j(z) = e_j(z)g(P - < Q_{\alpha_j}, P >), \tag{3.51}$$

$$g(P+h)f_j(z) = f_j(z)g(P+h - < \alpha_j, P+h >), \quad g(P)f_j(z) = f_j(z)g(P), \tag{3.52}$$

$$[\widehat{d}, g(P+h,P)] = 0, \tag{3.53}$$

$$[\widehat{d}, e_j(z)] = -z\frac{\partial}{\partial z}e_j(z), \quad [\widehat{d}, f_j(z)] = -z\frac{\partial}{\partial z}f_j(z), \tag{3.54}$$

$$z_1\frac{(q^{a_{ij}}z_2/z_1;p^*)_\infty}{(p^*q^{-a_{ij}}z_2/z_1;p^*)_\infty}e_i(z_1)e_j(z_2) = -z_2\frac{(q^{a_{ij}}z_1/z_2;p^*)_\infty}{(p^*q^{-a_{ij}}z_1/z_2;p^*)_\infty}e_j(z_2)e_i(z_1), \tag{3.55}$$

$$z_1\frac{(q^{-a_{ij}}z_2/z_1;p)_\infty}{(pq^{a_{ij}}z_2/z_1;p)_\infty}f_i(z_1)f_j(z_2) = -z_2\frac{(q^{-a_{ij}}z_1/z_2;p)_\infty}{(pq^{a_{ij}}z_1/z_2;p)_\infty}f_j(z_2)f_i(z_1), \tag{3.56}$$

$$[e_i(z_1), f_j(z_2)] = \frac{\delta_{i,j}}{q-q^{-1}}\left(\delta(q^{-c}z_1/z_2)\psi_j^-(q^{\frac{c}{2}}z_2) - \delta(q^c z_1/z_2)\psi_j^+(q^{-\frac{c}{2}}z_2)\right), \tag{3.57}$$

$$\begin{aligned}\sum_{\sigma\in S_a}\prod_{1\le m<k\le a}&\frac{(p^*q^2z_{\sigma(k)}/z_{\sigma(m)};p^*)_\infty}{(p^*q^{-2}z_{\sigma(k)}/z_{\sigma(m)};p^*)_\infty}\\ &\times\sum_{s=0}^{a}(-1)^s\begin{bmatrix} a\\ s\end{bmatrix}_q\prod_{1\le m\le s}\frac{(p^*q^{a_{ij}}w/z_{\sigma(m)};p^*)_\infty}{(p^*q^{-a_{ij}}w/z_{\sigma(m)};p^*)_\infty}\\ &\times\prod_{s+1\le m\le a}\frac{(p^*q^{a_{ij}}z_{\sigma(m)}/w;p^*)_\infty}{(p^*q^{-a_{ij}}z_{\sigma(m)}/w;p^*)_\infty}\\ &\times e_i(z_{\sigma(1)})\cdots e_i(z_{\sigma(s)})e_j(w)e_i(z_{\sigma(s+1)})\cdots e_i(z_{\sigma(a)}) = 0,\end{aligned} \tag{3.58}$$

$$
\begin{aligned}
(3.59)\quad & \sum_{\sigma\in S_a}\prod_{1\le m<k\le a}\frac{(pq^{-2}z_{\sigma(k)}/z_{\sigma(m)};p)_\infty}{(pq^{2}z_{\sigma(k)}/z_{\sigma(m)};p)_\infty}\\
&\times\sum_{s=0}^{a}(-1)^s\left[\begin{array}{c}a\\ s\end{array}\right]_q\prod_{1\le m\le s}\frac{(pq^{-a_{ij}}w/z_\sigma(m);p)_\infty}{(pq^{a_{ij}}w/z_{\sigma(m)};p)_\infty}\\
&\times\prod_{s+1\le m\le a}\frac{(pq^{-a_{ij}}z_{\sigma(m)}/w;p)_\infty}{(pq^{a_{ij}}z_{\sigma(m)}/w;p)_\infty}\\
&\times f_i(z_{\sigma(1)})\cdots f_i(z_{\sigma(s)})f_j(w)f_i(z_{\sigma(s+1)})\cdots f_i(z_{\sigma(a)})=0\\
&\qquad (i\neq j, a=1-a_{ij}),
\end{aligned}
$$

Proposition 3.4. *In the sense of analytic continuation, we have*

$$(3.60)\quad \psi_i^+(z_1)\psi_j^+(z_2)=\frac{\Theta_{p^*}(q^{a_{ij}}z_1/z_2)\Theta_p(q^{-a_{ij}}z_1/z_2)}{\Theta_{p^*}(q^{-a_{ij}}z_1/z_2)\Theta_p(q^{a_{ij}}z_1/z_2)}\psi_j^+(z_2)\psi_i^+(z_1),$$

$$(3.61)\quad \psi_i^+(z_1)e_j(z_2)=q^{-a_{ij}}\frac{\Theta_{p^*}(q^{a_{ij}-c/2}z_1/z_2)}{\Theta_{p^*}(q^{-a_{ij}-c/2}z_1/z_2)}e_j(z_2)\psi_i^+(z_1),$$

$$(3.62)\quad \psi_i^+(z_1)f_j(z_2)=q^{a_{ij}}\frac{\Theta_{p}(q^{-a_{ij}+c/2}z_1/z_2)}{\Theta_{p}(q^{a_{ij}+c/2}z_1/z_2)}f_j(z_2)\psi_i^+(z_1).$$

Let $U_q(\widehat{\mathfrak{g}})$ be the quantum affine algebra over $\mathbb{C}$ associated with the untwisted affine Lie algebra $\widehat{\mathfrak{g}}$ in the Drinfeld realization[11] and $x_j^\pm(z), k_{0,l}^\pm(z)$ be the Drinfeld currents. See Appendix A for the $\widehat{\mathfrak{gl}}_N$ case. The other cases can be found, for example in [17]. Then $U_{q,p}(\widehat{\mathfrak{g}})$ is a natural face type (i.e. dynamical) elliptic deformation of $U_q(\widehat{\mathfrak{g}})$ in the following sense.

Theorem 3.5. [17]

$$U_{q,p}(\widehat{\mathfrak{g}})/pU_{q,p}(\widehat{\mathfrak{g}})\cong(\mathbb{F}\otimes_{\mathbb{C}}U_q(\widehat{\mathfrak{g}}))\sharp\mathbb{C}[\mathcal{R}_Q]$$

by the following identification at $p=0$*.*

$$e_j(z)=x_j^+(z)e^{-Q_{\alpha_j}},\quad f_j(z)=x_j^-(z),\quad k_l^\pm(z)=k_{0,l}^\pm(z)e^{-Q_{\bar{\epsilon}_l}}.$$

Here the smash product $\sharp$ *is defined as follows.*

$$
\begin{aligned}
&g(P,P+h)a\otimes e^{Q_\alpha}\cdot f(P,P+h)b\otimes e^{Q_\beta}\\
&=g(P,P+h)f(P-<Q_\alpha,P>,P+h-<Q_\alpha+\mathrm{wt}(a),P+h>)ab\\
&\qquad\otimes e^{Q_\alpha+Q_\beta}
\end{aligned}
$$

where $\mathrm{wt}(a)\in\bar{\mathfrak{h}}^*$ *s.t.* $q^h a q^{-h}=q^{<\mathrm{wt}(a),h>}a$ *for* $a,b\in U_q(\widehat{\mathfrak{g}}), f(P),g(P)\in\mathbb{F}, e^{Q_\alpha},e^{Q_\beta}\in\mathbb{C}[\mathcal{R}_Q]$.

Definition 3.6. *Let us introduce the multiplicative dynamical parameters* $x = (x_1, \cdots, x_N), x_i = q^{2P_{\bar{\epsilon}_i}}$. *We set* $U_{q,x}(\widehat{\mathfrak{g}}) = U_{q,p}(\widehat{\mathfrak{g}})/pU_{q,p}(\widehat{\mathfrak{g}})$ *and call it the dynamical quantum affine algebra in the Drinfeld realization.*

3.2. $E_{q,p}(\widehat{\mathfrak{gl}}_N)$

Let $\bar{L}_{ij,n}$ $(n \in \mathbb{Z}, 1 \le i,j \le N)$ be abstract symbols. We define $L^+(z) = \sum_{1 \le i,j \le N} E_{ij} L^+_{ij}(z)$ by

$$L^+_{ij}(z) = \sum_{n \in \mathbb{Z}} L_{ij,n} z^{-n}, \qquad L_{ij,n} = p^{\max(n,0)} \bar{L}_{ij,n}. \tag{3.63}$$

Definition 3.7. *Let* $R^+(z,s)$ *be the same R matrix as in Sec.2.1. The elliptic algebra* $E_{q,p}(\widehat{\mathfrak{gl}}_N)$ *is a topological algebra over* $\mathbb{F}[[p]]$ *generated by* $\bar{L}_{ij,n}$, $\widehat{d}$ *and the central element* $q^{\pm c/2}$ *satisfying the following relations.*

$$\begin{aligned} R^{+(12)}(z_1/z_2, P+h) L^{+(1)}(z_1) L^{+(2)}(z_2) \\ = L^{+(2)}(z_2) L^{+(1)}(z_1) R^{+*(12)}(z_1/z_2, P), \end{aligned} \tag{3.64}$$

$$g(P+h)\bar{L}_{ij,n} = \bar{L}_{ij,n}\, g(P+h- < Q_{\bar{\epsilon}_i}, P+h >), \tag{3.65}$$

$$g(P)\bar{L}_{ij,n} = \bar{L}_{ij,n}\, g(P- < Q_{\bar{\epsilon}_j}, P >), \tag{3.66}$$

$$[\widehat{d}, L^+(z)] = -z\frac{\partial}{\partial z} L^+(z), \tag{3.67}$$

where $g(P+h), g(P) \in \mathbb{F}$ *and*

$$L^{+(1)}(z) = L^+(z) \otimes \mathrm{id}, \qquad L^{+(2)}(z) = \mathrm{id} \otimes L^+(z).$$

We regard $L^+(z) \in \mathrm{End}V \otimes E_{q,p}(\widehat{\mathfrak{gl}}_N)$. We treat (3.64) as a formal Laurent series in z_1 and z_2. Then the coefficients of z_1, z_2 are well defined in the p-adic topology. See [24] for a similar formulation for the vertex type elliptic quantum algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$. Note also that due to the RLL-relation (3.64) the L-operator $L^+(z)$ is invertible. See Appendix E.

For later convenience we define $L^-(z) = \sum_{1 \le i,j \le N} E_{ij} L^-_{ij}(z)$ by[37]

$$L^-(z) = \left(\mathrm{Ad}(q^{-2\theta_V(P)}) \otimes \mathrm{id}\right) \left(q^{2T_V} L^+(zp^* q^c)\right), \tag{3.68}$$

$$\theta_V(P) = -\sum_{j=1}^{N-1} \left(\frac{1}{2}\pi_V(h_j)\pi_V(h^j) + P_j \pi_V(h^j)\right), \tag{3.69}$$

$$T_V = \sum_{j=1}^{N-1} \pi_V(h_j) \otimes h^j. \tag{3.70}$$

Here $(\mathrm{Ad}X)Y = XYX^{-1}$, $h^j = h_{\bar{\Lambda}_j}$ $(j \in I)$, $\pi_V(h_j) = E_{jj} - E_{j+1j+1}$ and $\pi_V(h^j) = \sum_{i=1}^{j} \pi_V(h_{\bar{\epsilon}_i})$ $(j \in I)$. Then one can verify the following.

Proposition 3.8. *The L operators $L^+(z)$ and $L^-(z)$ satisfy the following relations.*

$$\begin{aligned} &R^{-(12)}(z_1/z_2, P+h)L^{-(1)}(z_1)L^{-(2)}(z_2) \\ &\qquad = L^{-(2)}(z_2)L^{-(1)}(z_1)R^{-*(12)}(z_1/z_2, P), \end{aligned} \tag{3.71}$$

$$\begin{aligned} &R^{\pm(12)}(q^{\pm c}z_1/z_2, P+h)L^{\pm(1)}(z_1)L^{\mp(2)}(z_2) \\ &\qquad = L^{\mp(2)}(z_2)L^{\pm(1)}(z_1)R^{\pm*(12)}(q^{\mp c}z_1/z_2, P). \end{aligned} \tag{3.72}$$

proof) Replace z_i with $z_i p^* q^c$ $(i = 1, 2)$ in (3.64). Note that (3.65), (3.66) and (3.68) yields

$$L^+(p^*q^c z) = q^{2\frac{N-1}{N}} \sum_{i,j} q^{-2(P+h)_{\bar{\epsilon}_i}} q^{2P_{\bar{\epsilon}_j}} E_{ij} L^-_{ij}(z).$$

By a componentwise comparison we obtain

$$R^+(z_1/z_2, P+h)L^-(z_1)L^-(z_2) = L^-(z_2)L^-(z_1)R^{+*}(z_1/z_2, P).$$

Then noting (2.9) and (2.12), we obtain (3.71).

Similarly let us replace z_1 by $z_1 p^* q^c$ in (3.64). Noting $p^* q^c = pq^{-c}$, the components of R^+ are changed as

$$\begin{aligned} &\rho^+(zpq^{-c}) = q^{-2\frac{N-1}{N}} \rho^-(zq^{-c}), \\ &b(zpq^{-c}, s) = q^2 b(zq^{-c}, s), \qquad \bar{b}(zpq^{-c}) = q^2 \bar{b}(zq^{-c}), \\ &c(zpq^{-c}, \pm s) = q^{\mp 2s+2} c(zq^{-c}, \pm s) \end{aligned} \tag{3.73}$$

and similarly for R^{+*}. Then from (2.3) and (2.9), we obtain the second (lower sign) relation in (3.72). Note that a factor arising from the action of $\mathrm{Ad}(q^{-2\theta_V(P)}) \otimes \mathrm{id}$ on the L-operators cancels the extra factors in (3.73).

To obtain the first relation in (3.72), exchange z_1 and z_2 in the second relation of (3.72). Then we have

$$R^-(q^{-c}z_2/z_1, P+h)^{-1}L^+(z_1)L^-(z_2) = L^-(z_2)L^+(z_1)R^{-*}(q^c z_2/z_1, P)^{-1}.$$

Using (2.10), we obtain the desired result. Q.E.D.

Remark. We can expand (3.64) and (3.71) in both $z = z_1/z_2$ and $z^{-1} = z_2/z_1$. However (3.72) admits an expansion only in z (resp. z^{-1}) for the upper (resp. lower) sign case for the sake of the well-definedness in the p-adic topology. It is instructive to compare this with the trigonometric case [9].

In the component form, (3.64), (3.71) and (3.72) are

$$\begin{aligned}
&(3.74) & &\sum_{i',j'} R^{\pm}(z_1/z_2, P+h)_{ij}^{i'j'} L^{\pm}_{i'i''}(z_1) L^{\pm}_{j'j''}(z_2) \\
& & &\quad = \sum_{i',j'} L^{\pm}_{jj'}(z_2) L^{\pm}_{ii'}(z_1) R^{\pm *}(z_1/z_2, P)_{i'j'}^{i''j''}, \\
&(3.75) & &\sum_{i',j'} R^{\pm}(q^{\pm c} z_1/z_2, P+h)_{ij}^{i'j'} L^{\pm}_{i'i''}(z_1) L^{\mp}_{j'j''}(z_2) \\
& & &\quad = \sum_{i',j'} L^{\mp}_{jj'}(z_2) L^{\pm}_{ii'}(z_1) R^{\pm *}(q^{\mp c} z_1/z_2, P)_{i'j'}^{i''j''},
\end{aligned}$$

We call (3.74) the $(i,j),(i'',j'')$ component of (3.64), etc.

Remark. In order to obtain a 'fully' dynamical RLL-relations used in [19, 21] with a central extension one may introduce the L-operators $L^{\pm}(z,P)$ related to our $L^{\pm}(z)$ by[36, 43]

$$(3.76) \qquad L^{\pm}(z,P) = L^{\pm}(z) e^{\sum_{i=1}^{N} \pi_V(h_{\epsilon_i}) \otimes Q_{\bar{\epsilon}_i}},$$

where $\pi_V(h_{\epsilon_i}) = E_{i,i}$. In fact from (3.65) and (3.66) we have

$$(3.77) \qquad [L^{\pm}_{ij}(z,P), f(P)] = 0,$$

$$(3.78) \qquad g(h) L^{\pm}_{ij}(z,P) = L^{\pm}_{ij}(z,P) g(h - <\bar{\epsilon}_i - \bar{\epsilon}_j, h>),$$

$$(3.79) \qquad [\widehat{d}, L^{\pm}(z)] = -z \frac{\partial}{\partial z} L^{\pm}(z).$$

(3.77) indicates that $L^{+}(z,P)$ is independent of $\mathbb{C}[\mathcal{R}_Q]$. Furthermore from (2.2), (3.64) (3.71) and (3.72), $L^{\pm}(z,P)$ satisfy the following full dynamical RLL-relations

$$\begin{aligned}
(3.80)\, & R^{\pm(12)}(z_1/z_2, P+h) L^{\pm(1)}(z_1,P) L^{\pm(2)}(z_2, P+\pi_V(h)^{(1)}) \\
&\quad = L^{\pm(2)}(z_2,P) L^{\pm(1)}(z_1, P+\pi_V(h)^{(2)}) R^{\pm *(12)}(z_1/z_2, P), \\
(3.81)\, & R^{\pm(12)}(q^{\pm c} z_1/z_2, P+h) L^{\pm(1)}(z_1,P) L^{\mp(2)}(z_2, P+\pi_V(h)^{(1)}) \\
&\quad = L^{\mp(2)}(z_2,P) L^{\pm(1)}(z_1, P+\pi_V(h)^{(2)}) R^{\pm *(12)}(q^{\mp c} z_1/z_2, P).
\end{aligned}$$

Here the generators are clear. If we set $L^{\pm}(z,P) = \sum_{i,j} E_{i,j} L^{\pm}_{ij}(z,P)$ with $L^{\pm}_{ij}(z,P) = \sum_{m \in \mathbb{Z}} L^{\pm}_{ij,n}(P) z^{-n}$, then from (3.63) we have $L^{\pm}_{ij,n}(P) = L^{\pm}_{ij,n} e^{-Q_{\bar{\epsilon}_j}}$.

Remark. The dynamical RLL relations (3.80)-(3.81) coincides with those derived from the universal DYBE for $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{g}})$ in [37, 36].

3.3. Reflection equations

Following [59], let us set

$$\mathcal{L}(z) = L^+(zq^c)L^-(z)^{-1}.$$

Then using (3.64), (3.71)-(3.72), one can show the following relations.

Proposition 3.9.

$$\begin{aligned} &R^{+(12)}(z_1/z_2, P+h)\mathcal{L}^{(1)}(z_1)R^{+(21)}(q^{2c}z_2/z_1, P+h)\mathcal{L}^{(2)}(z_2) \\ &\quad = \mathcal{L}^{(2)}(z_2)R^{+(12)}(q^{2c}z_1/z_2, P+h)\mathcal{L}^{(1)}(z_1)R^{+(21)}(z_2/z_1, P+h), \\ &R^{+(12)}(z_1/z_2, P+h)L^{+(1)}(z_1q^c)\mathcal{L}^{(2)}(z_2) \\ &\quad = \mathcal{L}^{(2)}(z_2)R^{+(12)}(q^{2c}z_1/z_2, P+h)L^{+(1)}(z_1q^c). \end{aligned}$$

3.4. The trigonometric limit

Let us consider the trigonometric counterpart of $E_{q,p}(\widehat{\mathfrak{g}})$ according to an idea described in [50]. Set $L^{\pm}_{0;ij}(z) = L^{\pm}_{ij}(z)|_{p=0}$. From (3.63) and (3.68), we have

$$L^{\pm}_{0;ij}(z) = \sum_{m\in\mathbb{Z}_{\geq 0}} L^{\pm}_{0;ij,\mp m}z^{\pm m} \qquad (1 \leq i,j \leq N)$$

where for $m \in \mathbb{Z}_{\geq 0}$,

$$L^{+}_{0;ij,-m} = \bar{L}_{ij,-m}|_{p=0}, \quad L^{-}_{0;ij,m} = q^{2(P+h)_{\bar{\epsilon}_i}}\bar{L}_{ij,m}|_{p=0}q^{-2P_{\bar{\epsilon}_j}}q^{cm}.$$

Let $R^{\pm}_0(z,s)$ be the trigonometric dynamical R matrix in (2.23). From (3.64), (3.71) and (3.72), $L^{\pm}_0(z) = \sum_{1\leq i,j\leq N} E_{ij}L^{\pm}_{0;ij}(z)$ satisfy for $g(P+h), g(P) \in \mathbb{F}$

$$\begin{aligned} &R_0^{\pm(12)}(z_1/z_2, P+h)L_0^{\pm(1)}(z_1)L_0^{\pm(2)}(z_2) \\ &\qquad = L_0^{\pm(2)}(z_2)L_0^{\pm(1)}(z_1)R_0^{\pm(12)}(z_1/z_2, P), \end{aligned} \tag{3.82}$$

$$\begin{aligned} &R_0^{\pm(12)}(q^{\pm c}z_1/z_2, P+h)L_0^{\pm(1)}(z_1)L_0^{\mp(2)}(z_2) \\ &\qquad = L_0^{\mp(2)}(z_2)L_0^{\pm(1)}(z_1)R_0^{\pm(12)}(q^{\mp c}z_1/z_2, P). \end{aligned} \tag{3.83}$$

$$g(P+h)L^{\pm}_{0;ij}(z) = L^{\pm}_{0;ij}(z)g(P+h- < Q_{\bar{\epsilon}_i}, P+h >), \tag{3.84}$$

$$g(P)L^{\pm}_{0;ij}(z) = L^{\pm}_{0;ij}(z)g(P- < Q_{\bar{\epsilon}_j}, P >), \tag{3.85}$$

$$[\widehat{d}, L^{\pm}_0(z)] = -z\frac{\partial}{\partial z}L^{\pm}_0(z). \tag{3.86}$$

Definition 3.10. *Let $x = (x_1, \cdots, x_N), x_i = q^{2P_{\bar{\epsilon}_i}}$ as before. We denote by $U^R_{q,x}(\widehat{\mathfrak{gl}}_N)$ the unital associative algebra over $\mathbb{F}$ generated by*

$L^{\pm}_{0;ij,\mp m}$ $(m \in \mathbb{Z}_{\geq 0}), \widehat{d}$ *and the central element* $q^{\pm c/2}$ *subject to* (3.82)-(3.86). *We call* $U^R_{q,x}(\widehat{\mathfrak{gl}}_N)$ *the dynamical quantum affine algebra in the FRST formulation.*

Hence we have

Proposition 3.11.

$$E_{q,p}(\widehat{\mathfrak{gl}}_N)/pE_{q,p}(\widehat{\mathfrak{gl}}_N) \cong U^R_{q,x}(\widehat{\mathfrak{gl}}_N).$$

In order to clarify a relation between the dynamical $U^R_{q,x}(\widehat{\mathfrak{gl}}_N)$ and the usual quantum affine algebra $U^R_q(\widehat{\mathfrak{gl}}_N)$ in the FRST formulation[59], one needs to further remove the $\mathbb{C}[\mathcal{R}_Q]$ dependence from $U^R_{q,x}(\widehat{\mathfrak{gl}}_N)$. This can be done by considering the algebra generated by the trigonometric limit $L^{\pm}_0(z,P)$ of $L^{\pm}(z,P)$ in (3.76). Then $L^{\pm}_0(z,P)$ satisfy the same relations as (3.77)-(3.79) as well as the trigonometric limit of the dynamical RLL-relations (3.80)-(3.81), where $R^{\pm}(z,s)$ and $R^{*\pm}(z,s)$ are replaced by $R^{\pm}_0(z,s)$ and $R^{*\pm}_0(z,s)$, respectively. We set $L^{\pm}_{0;ij}(z,P) = \sum_{m\in\mathbb{Z}_{\geq 0}} L^{\pm}_{0;ij,\mp m}(P)z^{\pm m}$ and denote by $\widetilde{U}^R_{q,x}(\widehat{\mathfrak{gl}}_N)$ the unital associative algebra over $\mathbb{F}$ generated by $L^{\pm}_{0;ij,m}(P)$. Then we have

$$U^R_{q,x}(\widehat{\mathfrak{gl}}_N) \cong \widetilde{U}^R_{q,x}(\widehat{\mathfrak{gl}}_N)\sharp\mathbb{C}[\mathcal{R}_Q].$$

Recall that $R^{\pm}_0(z,P)^{kl}_{ij}$ can be expanded to a formal power series in $x_{i,j} = q^{2P_{i,j}}$ and the 0-th order term gives the trigonometric R matrix in (2.25). We assume the same property for $L^{\pm}_0(z,P)$. Let $L^{\pm}_0(z,P) = \sum_{|k|=0}^{\infty}\sum_{k\in\mathbb{N}^N} L^{\pm}_0(z;k)x^k,\ L^{\pm}_0(z;k) = \sum_{1\leq i,j\leq N} E_{ij}L^{\pm}_{0;ij}(z,k)$, where $k = (k_1,\cdots,k_N), |k| = k_1+\cdots+k_N$ and $x^k = x_1^{k_1}\cdots x_N^{k_N}$. Then $L^{\pm}_0(z;0) = \sum_{1\leq i,j\leq N} E_{ij}L^{\pm}_{0;ij}(z,0), L^{\pm}_{0;ij}(z;0) = \sum_{m\in\mathbb{Z}_{\geq 0}} L^{\pm}_{0;ij,\mp m}(0)z^{\pm m}$ satisfy the same RLL-relations as the quantum affine algebra $U^R_q(\widehat{\mathfrak{gl}}_N)$ over $\mathbb{C}$ in the FRST formulation[59]. Hence

$$\widetilde{U}^R_{q,x}(\widehat{\mathfrak{gl}}_N)/\Big(\sum_i x_i\widetilde{U}^R_{q,x}(\widehat{\mathfrak{gl}}_N)\Big) \cong U^R_q(\widehat{\mathfrak{gl}}_N).$$

§4. Hopf Algebroid Structure

In this section, we introduce an H-Hopf algebroid structure[15, 40, 50] into the elliptic algebras $E_{q,p}(\widehat{\mathfrak{gl}}_N)$ and $U_{q,p}(\widehat{\mathfrak{gl}}_N)$, and formulate them as elliptic quantum groups.

4.1. $U_{q,p}(\widehat{\mathfrak{gl}}_N)$ and $E_{q,p}(\widehat{\mathfrak{gl}}_N)$ as H-Algebras

Let $\mathcal{A}$ be an associative algebra, $\mathcal{H}$ be a commutative subalgebra of $\mathcal{A}$, and $\mathcal{M}_{\mathcal{H}^*}$ be the field of meromorphic functions on $\mathcal{H}^*$ the dual space of $\mathcal{H}$.

Definition 4.1. *An associative algebra $\mathcal{A}$ with 1 is said to be an $\mathcal{H}$-algebra, if it is bigraded over $\mathcal{H}^*$, $\mathcal{A} = \bigoplus_{\alpha,\beta\in\mathcal{H}^*} \mathcal{A}_{\alpha\beta}$, and equipped with two algebra embeddings $\mu_l, \mu_r : \mathcal{M}_{\mathcal{H}^*} \to \mathcal{A}_{00}$ (the left and right moment maps), such that*

$$\mu_l(\widehat{f})a = a\mu_l(T_\alpha\widehat{f}), \quad \mu_r(\widehat{f})a = a\mu_r(T_\beta\widehat{f}), \qquad a \in \mathcal{A}_{\alpha\beta},\ \widehat{f} \in \mathcal{M}_{\mathcal{H}^*},$$

where T_α denotes the automorphism $(T_\alpha\widehat{f})(\lambda) = \widehat{f}(\lambda+\alpha)$ of $\mathcal{M}_{\mathcal{H}^*}$.

Definition 4.2. *An $\mathcal{H}$-algebra homomorphism is an algebra homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ between two $\mathcal{H}$-algebras $\mathcal{A}$ and $\mathcal{B}$ preserving the bigrading and the moment maps, i.e. $\pi(\mathcal{A}_{\alpha\beta}) \subseteq \mathcal{B}_{\alpha\beta}$ for all $\alpha, \beta \in \mathcal{H}^*$ and $\pi(\mu_l^{\mathcal{A}}(\widehat{f})) = \mu_l^{\mathcal{B}}(\widehat{f}), \pi(\mu_r^{\mathcal{A}}(\widehat{f})) = \mu_r^{\mathcal{B}}(\widehat{f})$.*

Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{H}$-algebras. The tensor product $\mathcal{A}\widetilde{\otimes}\mathcal{B}$ is the $\mathcal{H}^*$-bigraded vector space with

$$(\mathcal{A}\widetilde{\otimes}\mathcal{B})_{\alpha\beta} = \bigoplus_{\gamma\in\mathcal{H}^*} (\mathcal{A}_{\alpha\gamma} \otimes_{\mathcal{M}_{\mathcal{H}^*}} \mathcal{B}_{\gamma\beta}),$$

where $\otimes_{\mathcal{M}_{\mathcal{H}^*}}$ denotes the usual tensor product modulo the following relation.

$$\mu_r^{\mathcal{A}}(\widehat{f})a \otimes b = a \otimes \mu_l^{\mathcal{B}}(\widehat{f})b, \qquad a \in \mathcal{A}, b \in \mathcal{B}, \widehat{f} \in \mathcal{M}_{\mathcal{H}^*}. \tag{4.1}$$

The tensor product $\mathcal{A}\widetilde{\otimes}\mathcal{B}$ is again an $\mathcal{H}$-algebra with the multiplication $(a \otimes b)(c \otimes d) = ac \otimes bd$ and the moment maps

$$\mu_l^{\mathcal{A}\widetilde{\otimes}\mathcal{B}} = \mu_l^{\mathcal{A}} \otimes 1, \qquad \mu_r^{\mathcal{A}\widetilde{\otimes}\mathcal{B}} = 1 \otimes \mu_r^{\mathcal{B}}.$$

Let $\mathcal{D}$ be the algebra of automorphisms $\mathcal{M}_{\mathcal{H}^*} \to \mathcal{M}_{\mathcal{H}^*}$

$$\mathcal{D} \;=\; \Big\{ \sum_i \widehat{f}_i T_{\beta_i} \mid \widehat{f}_i \in \mathcal{M}_{\mathcal{H}^*},\ \beta_i \in \mathcal{H}^* \Big\}.$$

Equipped with the bigrading $\mathcal{D}_{\alpha\alpha} = \{\ \widehat{f}T_{-\alpha} \mid \widehat{f} \in \mathcal{M}_{\mathcal{H}^*},\ \alpha \in \mathcal{H}^*\ \}$, $\mathcal{D}_{\alpha\beta} = 0$ $(\alpha \neq \beta)$ and the moment maps $\mu_l^{\mathcal{D}}, \mu_r^{\mathcal{D}} : \mathcal{M}_{\mathcal{H}^*} \to \mathcal{D}_{00}$ defined by $\mu_l^{\mathcal{D}}(\widehat{f}) = \mu_r^{\mathcal{D}}(\widehat{f}) = \widehat{f}T_0$, $\mathcal{D}$ is an $\mathcal{H}$-algebra. For any $\mathcal{H}$-algebra $\mathcal{A}$, we have the canonical isomorphism as an $\mathcal{H}$-algebra

$$\mathcal{A} \cong \mathcal{A}\widetilde{\otimes}\mathcal{D} \cong \mathcal{D}\widetilde{\otimes}\mathcal{A} \tag{4.2}$$

by $a \cong a\widetilde{\otimes}T_{-\beta} \cong T_{-\alpha}\widetilde{\otimes}a$ for all $a \in \mathcal{A}_{\alpha\beta}$.

Now let H be the same as defined in Sec.2 and take $\mathcal{H} = H$.

Proposition 4.3. *The* $\mathcal{U} = U_{q,p}(\widehat{\mathfrak{gl}}_N)$ *is an* H*-algebra by*

$$(4.3)\qquad \mathcal{U} = \bigoplus_{\alpha,\beta\in H^*} \mathcal{U}_{\alpha,\beta}$$

$$\mathcal{U}_{\alpha\beta} = \left\{ a \in \mathcal{U} \;\middle|\; \begin{array}{c} q^{P+h}aq^{-(P+h)} = q^{<\alpha,P+h>}a,\ q^{P}aq^{-P} = q^{<\beta,P>}a, \\ \forall P+h, P \in H \end{array} \right\}$$

and $\mu_l, \mu_r : \mathbb{F} \to \mathcal{U}_{0,0}$ *defined by* [50]

$$\mu_l(\widehat{f}) = f(P+h,p) \in \mathbb{F}[[p]], \qquad \mu_r(\widehat{f}) = f(P,p^*) \in \mathbb{F}[[p]].$$

Proposition 4.4. *The* $\mathcal{E} = E_{q,p}(\widehat{\mathfrak{gl}}_N)$ *is an* H*-algebra by*

$$(4.4)\quad \mathcal{E} = \bigoplus\nolimits_{\alpha,\beta\in H^*} \mathcal{E}_{\alpha,\beta},$$

$$\mathcal{E}_{\alpha,\beta} = \left\{ \ a \in \mathcal{E} \ \middle|\; \begin{array}{c} q^{P+h}aq^{-(P+h)} = q^{<\alpha,P+h>}a,\ q^{P}aq^{-P} = q^{<\beta,P>}a, \\ \forall P+h, P \in H \end{array} \right\}$$

and $\mu_l, \mu_r : \mathbb{F} \to \mathcal{E}_{0,0}$ *defined by the same* μ_l, μ_r *as in* $\mathcal{U}$. *Note that* $\bar{L}_{ij,n} \in \mathcal{E}_{-Q_{\bar{\epsilon}_i},-Q_{\bar{\epsilon}_j}}$.

We regard $T_\alpha = e^{-Q_\alpha} \in \mathbb{C}[\mathcal{R}_Q]$ as the shift operator $\mathbb{F}[[p]] \to \mathbb{F}[[p]]$

$$(T_\alpha\mu_r(\widehat{f})) = e^{-Q_\alpha}f(P,p^*)e^{Q_\alpha} = f(P+ < Q_\alpha, P >, p^*),$$
$$(T_\alpha\mu_l(\widehat{f})) = e^{-Q_\alpha}f(P+h,p)e^{Q_\alpha} = f(P+h+ < Q_\alpha, P+h >, p).$$

Then $\mathcal{D} = \mathbb{F}\otimes_{\mathbb{C}}\mathbb{C}[\mathcal{R}_Q]$ becomes the H-algebra having the property (4.2) for $\mathcal{A} = \mathcal{U}, \mathcal{E}$.

Hereafter we abbreviate $f(P+h,p)$ and $f(P,p^*)$ as $f(P+h)$ and $f^*(P)$, respectively.

4.2. H-Hopf algebroids $E_{q,p}(\widehat{\mathfrak{gl}}_N)$ and $U_{q,p}(\widehat{\mathfrak{gl}}_N)$

Let us first recall the $\mathcal{H}$-Hopf algebroid following [15, 40].

Definition 4.5. *An* $\mathcal{H}$*-bialgebroid is an* $\mathcal{H}$*-algebra* $\mathcal{A}$ *equipped with two* $\mathcal{H}$*-algebra homomorphisms* $\Delta : \mathcal{A} \to \mathcal{A}\widetilde{\otimes}\mathcal{A}$ (*the comultiplication*) *and* $\varepsilon : \mathcal{A} \to \mathcal{D}$ (*the counit*) *such that*

$$(\Delta\widetilde{\otimes}\mathrm{id}) \circ \Delta = (\mathrm{id}\widetilde{\otimes}\Delta) \circ \Delta,$$
$$(\varepsilon\widetilde{\otimes}\mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id}\widetilde{\otimes}\varepsilon) \circ \Delta,$$

under the identification (4.2).

Definition 4.6. *An $\mathcal{H}$-Hopf algebroid is an $\mathcal{H}$-bialgebroid $\mathcal{A}$ equipped with a $\mathbb{C}$-linear map $S:\mathcal{A}\to\mathcal{A}$ (the antipode), such that*

$$
\begin{aligned}
&S(\mu_r(\widehat{f})a)=S(a)\mu_l(\widehat{f}),\quad S(a\mu_l(\widehat{f}))=\mu_r(\widehat{f})S(a),\quad \forall a\in\mathcal{A},\widehat{f}\in\mathcal{M}_{\mathcal{H}^*},\\
&m\circ(\mathrm{id}\widetilde{\otimes}S)\circ\Delta(a)=\mu_l(\varepsilon(a)1),\quad \forall a\in\mathcal{A},\\
&m\circ(S\widetilde{\otimes}\mathrm{id})\circ\Delta(a)=\mu_r(T_\alpha(\varepsilon(a)1)),\quad \forall a\in\mathcal{A}_{\alpha\beta},
\end{aligned}
$$

where $m:\mathcal{A}\widetilde{\otimes}\mathcal{A}\to\mathcal{A}$ denotes the multiplication and $\varepsilon(a)1$ is the result of applying the difference operator $\varepsilon(a)$ to the constant function $1\in\mathcal{M}_{\mathcal{H}^*}$.

Remark. [40] Definition 4.6 yields that the antipode of an $\mathcal{H}$-Hopf algebroid uniquely exists and gives the algebra antihomomorphism.

The $\mathcal{H}$-algebra $\mathcal{D}$ is an $\mathcal{H}$-Hopf algebroid with $\Delta_{\mathcal{D}}:\mathcal{D}\to\mathcal{D}\widetilde{\otimes}\mathcal{D}$, $\varepsilon_{\mathcal{D}}:\mathcal{D}\to\mathcal{D}$, $S_{\mathcal{D}}:\mathcal{D}\to\mathcal{D}$ defined by

$$
\begin{aligned}
&\Delta_{\mathcal{D}}(\widehat{f}T_{-\alpha})=\widehat{f}T_{-\alpha}\widetilde{\otimes}T_{-\alpha},\\
&\varepsilon_{\mathcal{D}}=\mathrm{id},\qquad S_{\mathcal{D}}(\widehat{f}T_{-\alpha})=T_\alpha\widehat{f}=(T_\alpha\widehat{f})T_\alpha.
\end{aligned}
$$

Now let us consider the H-algebras $\mathcal{E}$ and $\mathcal{U}$. Let us first consider the H-Hopf algebroid structure on $\mathcal{E}$. We define two H-algebra homomorphisms, the co-unit $\varepsilon:\mathcal{E}\to\mathcal{D}$ and the co-multiplication $\Delta:\mathcal{E}\to\mathcal{E}\widetilde{\otimes}\mathcal{E}$ by

$$\varepsilon(L_{ij,n})=\delta_{i,j}\delta_{n,0}T_{\bar{\epsilon}_i}\quad(n\in\mathbb{Z}),\qquad \varepsilon(e^Q)=e^Q, \tag{4.5}$$

$$\varepsilon(\mu_l(\widehat{f}))=\varepsilon(\mu_r(\widehat{f}))=\widehat{f}T_0, \tag{4.6}$$

$$\Delta(L^+_{ij}(z))=\sum_k L^+_{ik}(z)\widetilde{\otimes}L^+_{kj}(z), \tag{4.7}$$

$$\Delta(e^Q)=e^Q\widetilde{\otimes}e^Q,\qquad \Delta(\widehat{d})=\widehat{d}\,\widetilde{\otimes}1+1\widetilde{\otimes}\widehat{d}, \tag{4.8}$$

$$\Delta(\mu_l(\widehat{f}))=\mu_l(\widehat{f})\widetilde{\otimes}1,\quad \Delta(\mu_r(\widehat{f}))=1\widetilde{\otimes}\mu_r(\widehat{f}). \tag{4.9}$$

One can check that Δ preserves the relation in Definition 3.7.

Lemma 4.7. *The maps ε and Δ satisfy*

$$(\Delta\widetilde{\otimes}\mathrm{id})\circ\Delta=(\mathrm{id}\widetilde{\otimes}\Delta)\circ\Delta, \tag{4.10}$$

$$(\varepsilon\widetilde{\otimes}\mathrm{id})\circ\Delta=\mathrm{id}=(\mathrm{id}\widetilde{\otimes}\varepsilon)\circ\Delta. \tag{4.11}$$

Proof. Straightforward. Q.E.D.

We also have the following formulae.

Proposition 4.8.

$$\Delta\left(\frac{f(P,p^*)}{f(P+h,p)}\right)=\frac{f(P,p^*)}{f(P+h,p)}\widetilde{\otimes}\frac{f(P,p^*)}{f(P+h,p)}. \tag{4.12}$$

Hence $(\mathcal{E},\Delta,\mathcal{M}_{H^*},\mu_l,\mu_r,\varepsilon)$ is a H-bialgebroid.

We define an algebra antihomomorphism (the antipode) $S:\mathcal{E}\to\mathcal{E}$ by

$$S(L^+_{ij}(z))=(L^+(z)^{-1})_{ij},\tag{4.13}$$

$$S(e^Q)=e^{-Q},\quad S(\mu_r(\widehat{f}))=\mu_l(\widehat{f}),\quad S(\mu_l(\widehat{f}))=\mu_r(\widehat{f}).\tag{4.14}$$

The explicit formula for (4.13) in terms of the components of the L-operator is given in Appendix E. Then S preserves the RLL relation (3.64) and satisfies the antipode axioms. We hence obtain

Theorem 4.9. *The H-algebra $\mathcal{E}$ equipped with (Δ,ε,S) is an H-Hopf algebroid.*

Definition 4.10. *We call the H-Hopf algebroid $(\mathcal{E},H,\mathcal{M}_{H^*},\mu_l,\mu_r,\Delta,\varepsilon,S)$ the elliptic quantum group $E_{q,p}(\widehat{\mathfrak{gl}}_N)$.*

Remark. The coproduct for $L^+(z,P)$ used in [19, 21] is essentially obtained from (4.7) via (3.76):

$$\Delta(L^+(z,P))=L^+(z,P)\otimes L^+(z,P+h^{(1)}).$$

By making use of the isomorphism between $\mathcal{U}$ and $\mathcal{E}$ given in Sec.6, we can define the L-operators of $\mathcal{U}$ by identifying them with those of $\mathcal{E}$ in (6.1). Then the H-Hopf algebroid structure of $\mathcal{U}$ is defined by using the same Δ,ε,S as $\mathcal{E}$. See [50] for the $\widehat{\mathfrak{sl}}_2$ case.

Definition 4.11. *We call the H-Hopf algebroid $(\mathcal{U},H,\mathcal{M}_{H^*},\mu_l,\mu_r,\Delta,\varepsilon,S)$ the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{gl}}_N)$.*

Hence the isomorphism obtained in Sec.6 can be extended to as an H-Hopf algebroid.

Remark. $U_{q,p}(\widehat{\mathfrak{g}})$ admits another co-algebra structure through another coproduct called the Drinfeld coproduct[36, 51].

§5. Dynamical Representations

5.1. Definition

We summarize some basic facts on the dynamical representation of $U_{q,p}(\widehat{\mathfrak{gl}}_N)$. Most of them can be extended to the arbitrary untwisted affine Lie algebra $\widehat{\mathfrak{g}}$ case[17].

Let us consider a vector space $\widehat{V}$ over $\mathbb{F}[[p]]$, which is H-diagonalizable, i.e.

$$\widehat{V}=\bigoplus_{\lambda,\mu\in H^*}\widehat{V}_{\lambda,\mu},$$
$$\widehat{V}_{\lambda,\mu}=\left\{v\in\widehat{V}\;\middle|\;\begin{array}{c}q^{P+h}\cdot v=q^{<\lambda,P+h>}v,\ q^{P}\cdot v=q^{<\mu,P>}v\\ \forall P+h,P\in H\end{array}\right\}.$$

Let us define the H-algebra $\mathcal{D}_{H,\widehat{V}}$ of the $\mathbb{C}$-linear operators on $\widehat{V}$ by

$$\mathcal{D}_{H,\widehat{V}}=\bigoplus_{\alpha,\beta\in H^*}(\mathcal{D}_{H,\widehat{V}})_{\alpha\beta},$$
$$(\mathcal{D}_{H,\widehat{V}})_{\alpha\beta}$$
$$=\left\{\;X\in\mathrm{End}_{\mathbb{C}}\widehat{V}\;\middle|\;\begin{array}{c}f(P+h)X=Xf(P+h+<\alpha,P+h>),\\ f(P)X=Xf(P+<\beta,P>),\\ X\cdot\widehat{V}_{\lambda,\mu}\subseteq\widehat{V}_{\lambda+\alpha,\mu+\beta},\quad\forall f(P),f(P+h)\in\mathbb{F}[[p]]\end{array}\right\},$$
$$\mu_l^{\mathcal{D}_{H,\widehat{V}}}(\widehat{f})v=f(<\lambda,P+h>,p)v,\quad \mu_r^{\mathcal{D}_{H,\widehat{V}}}(\widehat{f})v=f(<\mu,P>,p^*)v,$$
$$\widehat{f}\in\mathbb{F}[[p]],\ v\in\widehat{V}_{\lambda,\mu}.$$

Definition 5.1. *We define a dynamical representation of* $U_{q,p}(\widehat{\mathfrak{gl}}_N)$ *on* $\widehat{V}$ *to be an* H*-algebra homomorphism* $\pi:U_{q,p}(\widehat{\mathfrak{gl}}_N)\to\mathcal{D}_{H,\widehat{V}}$. *By the action* π *of* $U_{q,p}(\widehat{\mathfrak{gl}}_N)$ *we regard* $\widehat{V}$ *as a* $U_{q,p}(\widehat{\mathfrak{gl}}_N)$*-module.*

Definition 5.2. *For* $k\in\mathbb{C}$, *we say that a* $U_{q,p}(\widehat{\mathfrak{gl}}_N)$*-module has level* k *if* $q^{\pm c/2}$ *acts as the scalar* $q^{\pm k/2}$ *on it.*

Definition 5.3. *Let* $\mathfrak{H},\mathfrak{N}_+,\mathfrak{N}_-$ *be the subalgebras of* $U_{q,p}(\widehat{\mathfrak{gl}}_N)$ *generated by* $q^{\pm c/2},d,k_{i,0}$ $(1\le i\le N)$, *by* $k_{i,n}$ $(1\le i\le N,n\in\mathbb{Z}_{>0})$, $e_{j,n}$ $(j\in I,n\in\mathbb{Z}_{\ge0})$ $f_{j,n}$ $(j\in I,n\in\mathbb{Z}_{>0})$ *and by* $k_{i,-n}$ $(1\le i\le N,n\in\mathbb{Z}_{>0})$, $e_{j,-n}$ $(j\in I,n\in\mathbb{Z}_{>0})$, $f_{j,-n}$ $(i\in I,n\in\mathbb{Z}_{\ge0})$, *respectively.*

Definition 5.4. *For* $k\in\mathbb{C}$, $\lambda,\mu\in H^*$, *a dynamical* $U_{q,p}(\widehat{\mathfrak{g}})$*-module* $\widehat{V}(\lambda,\mu)$ *is called the level-k highest weight module with the highest weight* (λ,μ), *if there exists a vector* $v\in\widehat{V}(\lambda,\mu)$ *such that*

$$\widehat{V}(\lambda,\mu)=U_{q,p}(\widehat{\mathfrak{g}})\cdot v,\qquad \mathfrak{N}_+\cdot v=0,\qquad q^{\pm c/2}\cdot v=q^{\pm k/2}v,$$
$$k_{i,0}\cdot v=q^{-<\lambda-\mu,h_{\bar{\epsilon}_i}>}v,\quad f(P)\cdot v=f(<\mu,P>)v,$$
$$f(P+h)\cdot v=f(<\lambda,P+h>)v.$$

5.2. The evaluation H-algebra homomorphism

Let $k^{\pm}_{0,i}(z), x^{\pm}_j(z)$ be the Drinfeld currents of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$. See Appendix A. Let us introduce the currents $u^+_{\varepsilon_i}(z,p) \in (U_q(\widehat{\mathfrak{gl}}_N)[[p]])[[z]], u^-_{\varepsilon_i}(z,p) \in (U_q(\widehat{\mathfrak{gl}}_N)[[p]])[[z^{-1}]]$ $(1 \le i \le N)$ by

$$u^+_{\varepsilon_i}(z,p) = \prod_{n=1}^{\infty} \left(k^-_{i,0} \cdot k^+_{0,i}(p^{*n} q^{c-i} z)\right), \tag{5.1}$$

$$u^-_{\varepsilon_i}(z,p) = \prod_{n=1}^{\infty} \left(k^+_{i,0} \cdot k^-_{0,i}(p^{-n} q^{c-i} z)\right). \tag{5.2}$$

We also set

$$u^{\pm}_j(z,p) = u^{\pm}_{\varepsilon_j}(z,p) u^{\pm}_{\varepsilon_{j+1}}(qz,p)^{-1} \qquad (1 \le j \le N-1). \tag{5.3}$$

These are well defined elements in $(U_q(\widehat{\mathfrak{gl}}_N)[[p]])[[z,z^{-1}]]$ in the p-adic topology.

Now let us define the 'dressed' currents $x^{\pm}_j(z,p)$ $(1 \le j \le N-1)$, $k^{\pm}_i(z,p)$ $(1 \le i \le N)$ by

$$x^+_j(z,p) = u^+_j(z,p) x^+_j(z) e^{-Q_{\alpha_j}}, \tag{5.4}$$

$$x^-_j(z,p) = x^-_j(z) u^-_j(z,p), \tag{5.5}$$

$$k^+_i(z,p) = u^+_{\varepsilon_i}(q^{-c+j} z,p) k^+_{0,i}(z) u^-_{\varepsilon_i}(q^j z,p) e^{-Q_{\bar{\varepsilon}_i}}, \tag{5.6}$$

$$k^-_i(z,p) = u^+_{\varepsilon_i}(q^j z,p) k^-_{0,i}(z) u^-_{\varepsilon_i}(q^{-c+j} z,p) e^{-Q_{\bar{\varepsilon}_i}}. \tag{5.7}$$

Theorem 5.5. *The map*
$\phi_p : U_{q,p}(\widehat{\mathfrak{gl}}_N)[[z,z^{-1}]] \to (\mathbb{F}[[p]] \otimes_{\mathbb{C}} U_q(\widehat{\mathfrak{gl}}_N))[[z,z^{-1}]] \sharp \mathbb{C}[\mathcal{R}_Q]$ *defined by*

$$e_i(z) \mapsto x^+_i(z,p), \qquad f_i(z) \mapsto x^-_i(z,p), \qquad k^{\pm}_i(z) \mapsto k^{\pm}_i(z,p)$$

is an H-algebra homomorphism. We call ϕ_p the evaluation H-algebra homomorphism.

Proof. Direct calculations using Lemma B.1. Q.E.D.

Let (φ_V, V) be a representation of $U_q(\widehat{\mathfrak{gl}}_N)$. We assume V is an $\mathfrak{h}$-diagonalizable vector space over $\mathbb{C}$. We set $V_{\mathbb{F}[[p]]} = \mathbb{F}[[p]] \otimes_{\mathbb{C}} V$. Let V_Q be a vector space over $\mathbb{C}$, on which an action of e^Q is defined appropriately. Two important examples of V_Q are $V_Q = \mathbb{C}1$ and $V_Q = \mathbb{C}[\mathcal{R}_Q]$, where 1 denotes the vacuum state satisfying $e^Q.1 = 1$. Let us consider the vector space $\widehat{V}_{\mathbb{F}[[p]]} = V_{\mathbb{F}[[p]]} \otimes_{\mathbb{C}} V_Q$, on which the actions of $f(P,h,p) \in \mathbb{F}[[p]]$ and e^Q are defined as follows. For $v \otimes \xi \in V \otimes V_Q$,

$$f(P,h,p).(v \otimes \xi) = f(P, \mathrm{wt}(v), p) v \otimes \xi,$$
$$e^Q.(f(P,h,p) v \otimes \xi) = f(P - <Q,P>, h, p) v \otimes e^Q \xi,$$

where $h.v = \mathrm{wt}(v)v$. We extend $\varphi_V : U_q(\widehat{\mathfrak{gl}}_N) \to \mathrm{End}_{\mathbb{C}}V$ to a dynamical representation $\varphi_V : (\mathbb{F}[[p]] \otimes_C U_q(\widehat{\mathfrak{gl}}_N))\sharp\mathbb{C}[\mathcal{R}_Q] \to \mathcal{D}_{H,\widehat{V}_{\mathbb{F}[[p]]}}$ by

$$\varphi_V(f(P)) = f(P), \qquad \varphi_V(e^{Q_\alpha}) = e^{Q_\alpha} \qquad \forall e^{Q_\alpha} \in \mathbb{C}[\mathcal{R}_Q].$$

Note that if we specialize p as 0, $\widehat{V}_{\mathbb{F}[[p]]}$ becomes $V_{\mathbb{F}} \otimes V_Q$, where $V_{\mathbb{F}} = \mathbb{F} \otimes_{\mathbb{C}} V$.

Then from Theorem 5.5 we obtain the following.

Corollary 5.6. *A map* $\varphi_V^p = \varphi_V \circ \phi_p : U_{q,p}(\widehat{\mathfrak{gl}}_N) \to \mathcal{D}_{H,\widehat{V}_{\mathbb{F}[[p]]}}$ *gives a dynamical representation of* $U_{q,p}(\widehat{\mathfrak{gl}}_N)$ *on* $\widehat{V}_{\mathbb{F}[[p]]}$. *We call* $(\varphi_V^p, \widehat{V}_{\mathbb{F}[[p]]})$ *the evaluation dynamical representation.*

Due to this corollary, any representation of $U_q(\widehat{\mathfrak{gl}}_N)$ admits an 'elliptic and dynamical deformation' for generic p. One can easily extend this to any untwisted affine Lie algebra case by using the evaluation homomorphism given in Appendix A of [36] . See [50] for $\widehat{\mathfrak{sl}}_2$ case.

§6. Isomorphism Between $U_{q,p}(\widehat{\mathfrak{gl}}_N)$ and $E_{q,p}(\widehat{\mathfrak{gl}}_N)$

We introduce the Gauss components of the L operator of $\mathcal{E} = E_{q,p}(\widehat{\mathfrak{gl}}_N)$ and the half currents of $\mathcal{U} = U_{q,p}(\widehat{\mathfrak{gl}}_N)$. Then we show the isomorphism between $\mathcal{U}$ and $\mathcal{E}$.

6.1. The L-operators of $\mathcal{E}$

Let us set

$$\mathcal{E}^{\pm} = \left\{A(z) \in \mathcal{E}[[p]][[z, z^{-1}]] \;\middle|\; A(z) \in \mathcal{E}[[z^{\pm 1}]] \mod p\mathcal{E}[[p]][[z, z^{-1}]]\right\}.$$

Then it is easy to show

Lemma 6.1. *For* $A(z), B(z) \in \mathcal{E}^{\pm}$, *the product* $A(z)B(z)$ *is a well-defined element in* $\mathcal{E}^{\pm}$ *in the* p*-adic topology, respectively. Conversely, if* $A(z), B(z) \in \mathcal{E}[[p]][[z, z^{-1}]]$ *satisfy* $A(z)B(z) \in \mathcal{E}^{\pm}$, *then* $A(z), B(z) \in \mathcal{E}^{\pm}$, *respectively.*

Definition 6.1. *We define the Gauss components* $E^{\pm}_{l,j}(z), F^{\pm}_{j,l}(z)$, $K^{\pm}_m(z)$ $(1 \le j < l \le N, 1 \le m \le N)$ *of the* L*-operator* $L^{\pm}(z)$ *of* $\mathcal{E}$ *as follows.*

(6.1)

$$L^{\pm}(z) = \begin{pmatrix} 1 & F^{\pm}_{1,2}(z) & F^{\pm}_{1,3}(z) & \cdots & F^{\pm}_{1,N}(z) \\ 0 & 1 & F^{\pm}_{2,3}(z) & \cdots & F^{\pm}_{2,N}(z) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & F^{\pm}_{N-1,N}(z) \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} K^{\pm}_1(z) & 0 & \cdots & 0 \\ 0 & K^{\pm}_2(z) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & K^{\pm}_N(z) \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ E^{\pm}_{2,1}(z) & 1 & \ddots & & \vdots \\ E^{\pm}_{3,1}(z) & E^{\pm}_{3,2}(z) & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ E^{\pm}_{N,1}(z) & E^{\pm}_{N,2}(z) & \cdots & E^{\pm}_{N,N-1}(z) & 1 \end{pmatrix}.$$

In particular we call $E^{\pm}_{j+1,j}(z), F^{\pm}_{j,j+1}(z), K^{\pm}_m(z)$ the basic Gauss components.

Remark. By definition the matrix elements $L^{\pm}_{i,j}(z)$ are the elements in $\mathcal{E}^{\pm}$, respectively. Then from Lemma 6.1, the matrix elements $E^{\pm}_{l,j}(z)$, $F^{\pm}_{j,l}(z), K^{\pm}_m(z)$ of the right hand side of (6.1) are elements in $\mathcal{E}^{\pm}$, respectively and their products are well defined formal Laurent series in z in the p-adic topology. In addition, since $L^{\pm}(z)$ are invertible, $K^{\pm}_m(z)$ $(1 \leq m \leq N)$ are invertible. Therefore all the components $E^{\pm}_{l,j}(z), F^{\pm}_{j,l}(z)$ and $K^{\pm}_m(z)$ are determined uniquely by $L^{\pm}_{ij}(z)$, respectively.

Hence we define the coefficients of the Gauss components $E^{+}_{l,j}(z)$, $F^{+}_{j,l}(z), K^{+}_m(z)$ as follows.

Definition 6.2.

$$E^{+}_{l,j}(z) = \sum_{n\in\mathbb{Z}_{\geq 0}} E^{+}_{l,j,-n} z^n + \sum_{n\in\mathbb{Z}_{>0}} E^{+}_{l,j,n} p^n z^{-n}, \tag{6.2}$$

$$F^{+}_{l,j}(z) = \sum_{n\in\mathbb{Z}_{\geq 0}} F^{+}_{l,j,-n} z^n + \sum_{n\in\mathbb{Z}_{>0}} F^{+}_{l,j,n} p^n z^{-n}, \tag{6.3}$$

$$K^{+}_{j}(z) = \sum_{n\in\mathbb{Z}_{\geq 0}} K^{+}_{j,-n} z^n + \sum_{n\in\mathbb{Z}_{>0}} K^{+}_{j,n} p^n z^{-n}. \tag{6.4}$$

In addition, from the definition of $L^-(z)$ (3.68), we have

$$E^-_{j,i}(z) = q^{2P_{\bar{\epsilon}_j}} E^+_{j,i}(zpq^{-c}) q^{-2P_{\bar{\epsilon}_i}}, \tag{6.5}$$

$$F^-_{i,j}(z) = q^{2(P+h)_{\bar{\epsilon}_i}} F^+_{i,j}(zpq^{-c}) q^{-2(P+h)_{\bar{\epsilon}_j}}, \tag{6.6}$$

$$K^-_{i}(z) = q^{\frac{2(N-1)}{N}} q^{2(P+h)_{\bar{\epsilon}_i}} K^+_{i}(zpq^{-c}) q^{-2P_{\bar{\epsilon}_i}}. \tag{6.7}$$

Hence we define

Definition 6.3.

$$E^-_{j,i,n} = q^{2P_{\bar{\epsilon}_j}} E^+_{j,i,n} q^{-2P_{\bar{\epsilon}_i}}, \quad F^-_{i,j,n} = q^{2(P+h)_{\bar{\epsilon}_i}} F^+_{i,j,n} q^{-2(P+h)_{\bar{\epsilon}_j}}, \tag{6.8}$$
$$K^-_{i,n} = q^{\frac{2(N-1)}{N}} q^{2(P+h)_{\bar{\epsilon}_i}} K^+_{i,n} q^{-2P_{\bar{\epsilon}_i}},$$

for $n \in \mathbb{Z}$.

Then we have

$$E^-_{j,i}(z) = \sum_{n\in\mathbb{Z}_{>0}} E^-_{l,j,-n} p^n (q^{-c}z)^n + \sum_{n\in\mathbb{Z}_{\geq 0}} E^-_{l,j,n} (q^{-c}z)^{-n}, \tag{6.9}$$

$$F^-_{i,j}(z) = \sum_{n\in\mathbb{Z}_{>0}} F^-_{l,j,-n} p^n (q^{-c}z)^n + \sum_{n\in\mathbb{Z}_{\geq 0}} F^-_{l,j,n} (q^{-c}z)^{-n}, \tag{6.10}$$

$$K^-_{i}(z) = \sum_{n\in\mathbb{Z}_{>0}} K^-_{i,-n} p^n (q^{-c}z)^n + \sum_{n\in\mathbb{Z}_{\geq 0}} K^-_{i,n} (q^{-c}z)^{-n}. \tag{6.11}$$

6.2. Subalgebras

For $1 \leq l < N$, let us define the reduced R-matrix and L-operators by

$$R^\pm_l(z,s) = (R^\pm(z,s)^{i'j'}_{ij})_{l\leq i,j,i',j'\leq N}, \tag{6.12}$$

$$L^\pm_l(z) = (L^\pm_{ij}(z))_{l\leq i,j\leq N}. \tag{6.13}$$

Note that up to overall factors $R^\pm_l(z,s)$ are the elliptic dynamical R matrix of type $A^{(1)}_{N-l}$. Note also that if $R^\pm(z,s)^{i'j'}_{ij} \neq 0$ for $1 \leq i,j \leq l$ (resp. $l \leq i,j \leq N$), then $1 \leq i',j' \leq l$ (resp. $l \leq i',j' \leq N$). Hence we obtain

Proposition 6.4. *The reduced L-operators* $L^\pm_l(z)$ *satisfy*

$$\begin{aligned} R^{\pm(12)}_l(z_1/z_2, P+h) L^{\pm(1)}_l(z_1) L^{\pm(2)}_l(z_2) \\ = L^{\pm(2)}_l(z_2) L^{\pm(1)}_l(z_1) R^{-*(12)}_l(z_1/z_2, P), \end{aligned} \tag{6.14}$$

$$\begin{aligned} R^{\pm(12)}_l(q^{\pm c}z_1/z_2, P+h) L^{\pm(1)}_l(z_1) L^{\mp(2)}_l(z_2) \\ = L^{\mp(2)}_l(z_2) L^{\pm(1)}_l(z_1) R^{\pm*(12)}_l(q^{\mp c}z_1/z_2, P). \end{aligned} \tag{6.15}$$

Therefore $L_l^+(z), \widehat{d}, q^{\pm c/2}$ generate a subalgebra of $E_{q,p}(\widehat{\mathfrak{gl}}_N)$, which is isomorphic to $E_{q,p}(\widehat{\mathfrak{gl}}_{N-l+1})$.

Note that from (6.1) we have

(6.16)
$$
\begin{aligned}
&L_l^{\pm}(z)\\
&= \begin{pmatrix}
1 & F^{\pm}_{l,l+1}(z) & F^{\pm}_{l,l+2}(z) & \cdots & F^{\pm}_{l,N}(z) \\
0 & 1 & F^{\pm}_{l+1,l+2}(z) & \cdots & F^{\pm}_{l+1,N}(z) \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & 1 & F^{\pm}_{N-1,N}(z) \\
0 & \cdots & \cdots & 0 & 1
\end{pmatrix}
\begin{pmatrix}
K^{\pm}_l(z) & 0 & \cdots & 0 \\
0 & K^{\pm}_{l+1}(z) & & \vdots \\
\vdots & & \ddots & 0 \\
0 & \cdots & 0 & K^{\pm}_N(z)
\end{pmatrix}\\
&\qquad \times \begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 \\
E^{\pm}_{l+1,l}(z) & 1 & \ddots & & \vdots \\
E^{\pm}_{l+2,l}(z) & E^{\pm}_{l+2,l+1}(z) & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & 1 & 0 \\
E^{\pm}_{N,l}(z) & E^{\pm}_{N,l+1}(z) & \cdots & E^{\pm}_{N,N-1}(z) & 1
\end{pmatrix}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&L_l^{\pm}(z)^{-1}\\
&= \begin{pmatrix}
K_l^{\pm-1} & -K_l^{\pm-1}F^{\pm}_{l,l+1} & K_l^{\pm-1}x_l & \cdots \\
-E^{\pm}_{l+1,l}K_l^{\pm-1} & E^{\pm}_{l+1,l}K_l^{\pm-1}F^{\pm}_{l,l+1} + K^{\pm-1}_{l+1} & -E^{\pm}_{l+1,l}K_l^{\pm-1}x_l - K^{\pm-1}_{l+1}F^{\pm}_{l+1,l+2} & \cdots \\
y_l K_l^{\pm-1} & -y_l K_l^{\pm-1}F^{\pm}_{l,l+1} - E^{\pm}_{l+2,l+1}K^{\pm-1}_{l+1} & \ddots & \vdots \\
\vdots & \vdots & \cdots & \ddots
\end{pmatrix},
\end{aligned}
$$

where we omitted the argument z and set

$$x_l(z) = F^{\pm}_{l,l+1}(z)F^{\pm}_{l+1,l+2}(z) - F^{\pm}_{l,l+2}(z), \tag{6.17}$$

$$y_l(z) = E^{\pm}_{l+2,l+1}(z)E^{\pm}_{l+1,l}(z) - E^{\pm}_{l+2,l}(z). \tag{6.18}$$

Furthermore, for $l < m \le N$ let us define

$$R_l^{m\pm}(z,s) := (R^{\pm}(z,s)_{ij}^{i'j'})_{l\le i,j,i',j'\le m}, \tag{6.19}$$

$$(L_l^{\pm}(z)^{-1})^m := ((L_l^{\pm}(z)^{-1})_{ij})_{l\le i,j\le m}. \tag{6.20}$$

Proposition 6.5. *If we set*

$$L_l^{m\pm}(z) = \begin{pmatrix} 1 & F^{\pm}_{l,l+1}(z) & F^{\pm}_{l,l+2}(z) & \cdots & F^{\pm}_{l,m}(z) \\ 0 & 1 & F^{\pm}_{l+1,l+2}(z) & \cdots & F^{\pm}_{l+1,m}(z) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & F^{\pm}_{m-1,m}(z) \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} K^{\pm}_l(z) & 0 & \cdots & 0 \\ 0 & K^{\pm}_{l+1}(z) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & K^{\pm}_m(z) \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ E^{\pm}_{l+1,l}(z) & 1 & \ddots & & \vdots \\ E^{\pm}_{l+2,l}(z) & E^{\pm}_{l+2,l+1}(z) & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ E^{\pm}_{N,l}(z) & E^{\pm}_{m,l+1}(z) & \cdots & E^{\pm}_{m,m-1}(z) & 1 \end{pmatrix}.$$

Then we have

$$(L_l^{+}(z)^{-1})^m = L_l^{m\pm}(z)^{-1}.$$

Note that

$$L_l^{m\pm}(z) \neq (L^{\pm}_{ij}(z))_{l\leq i,j\leq m}.$$

Hence we have

Lemma 6.6. *The restriction of the relations*

$$\begin{aligned} L_l^{\pm(1)}(z_1)^{-1}L_l^{\pm(2)}(z_2)^{-1}R_l^{\pm(12)}(z,P+h) \\ = R_l^{-*(12)}(z,P)L_l^{\pm(2)}(z_2)^{-1}L_l^{\pm(1)}(z_1)^{-1}, \end{aligned} \tag{6.21}$$

$$\begin{aligned} L_l^{\pm(1)}(z_1)^{-1}L_l^{\mp(2)}(z_2)^{-1}R_l^{\pm(12)}(zq^{\pm c},P+h) \\ = R_l^{\pm*(12)}(zq^{\mp c},P)L_l^{\mp(2)}(z_2)^{-1}L_l^{\pm(1)}(z_1)^{-1} \end{aligned} \tag{6.22}$$

to the $(i,j),(i'',j'')$ *components with* $l\leq i,j,i'',j''\leq m$ *are equivalent to*

$$\begin{aligned} R_l^{m\pm(12)}(z,P+h)L_l^{m\pm(1)}(z_1)L_l^{m\pm(2)}(z_2) \\ = L_l^{m\pm(2)}(z_2)L_l^{m\pm(1)}(z_1)R_l^{m\pm*(12)}(z,P+h), \end{aligned} \tag{6.23}$$

$$\begin{aligned} R_l^{m\pm(12)}(zq^{\pm c},P+h)L_l^{m\pm(1)}(z_1)L_l^{m\mp(2)}(z_2) \\ = L_l^{m\mp(2)}(z_2)L_l^{m\pm(1)}(z_1)R_l^{m\pm*(12)}(zq^{\mp c},P), \end{aligned} \tag{6.24}$$

where $z=z_1/z_2$.

Therefore we obtain the following statement.

Theorem 6.7. $L_l^{m+}(z), \widehat{d}, q^{\pm c/2}$ *generate the subalgebra of* $E_{q,p}(\widehat{\mathfrak{gl}}_N)$*, which is isomorphic to* $E_{q,p}(\widehat{\mathfrak{gl}}_{m-l+1})$.

From (6.23) and (6.24), we have

Proposition 6.8.

$$\begin{aligned}(6.25)\quad & L_l^{m\pm(2)}(z_2)^{-1}R_l^{m\pm(12)}(z,P+h)L_l^{m\pm(1)}(z_1)\\ &= L_l^{m\pm(1)}(z_1)R_l^{m\pm*(12)}(z,P)L_l^{m\pm(2)}(z_2)^{-1},\\ (6.26)\quad & L_l^{m\pm(1)}(z_1)^{-1}L_l^{m\pm(2)}(z_2)^{-1}R_l^{m\pm(12)}(z,P+h)\\ &= R_l^{m\pm*(12)}(z,P)L_l^{m\pm(2)}(z_2)^{-1}L_l^{m\pm(1)}(z_1)^{-1},\\ (6.27)\quad & L_l^{m\mp(2)}(z_2)^{-1}R_l^{m\pm(12)}(zq^{\pm c},P+h)L_l^{m\pm(1)}(z_1)\\ &= L_l^{m\pm(1)}(z_1)R_l^{m\pm*(12)}(zq^{\mp c},P)L_l^{m\mp(2)}(z_2)^{-1},\\ (6.28)\quad & L_l^{m\pm(1)}(z_1)^{-1}L_l^{m\mp(2)}(z_2)^{-1}R_l^{m\pm(12)}(zq^{\pm c},P+h)\\ &= R_l^{m\pm*(12)}(zq^{\mp c},P)L_l^{m\mp(2)}(z_2)^{-1}L_l^{m\pm(1)}(z_1)^{-1}.\end{aligned}$$

Lemma 6.9. *For* $2 \le l+1 < m \le N$*, we have*

$$\begin{aligned}(6.29)\quad & \rho^{\pm}(z)\bar{b}(z)K_l^{\pm}(z_2)^{-1}K_m^{\pm}(z_1) = \rho^{*\pm}(z)\bar{b}(z)^*K_m^{\pm}(z_1)K_l^{\pm}(z_2)^{-1},\\ (6.30)\quad & \rho^{\pm}(q^{\pm c}z)\bar{b}(q^{\pm c}z)K_l^{\mp}(z_2)^{-1}K_m^{\pm}(z_1)\\ &= \rho^{*\pm}(q^{\mp c}z)\bar{b}(q^{\mp c}z)^*K_m^{\pm}(z_1)K_l^{\mp}(z_2)^{-1},\\ (6.31)\quad & [E_{l+1,l}^{\pm}(z_1),K_m^{\pm}(z_2)] = 0 = [E_{m,l+1}^{\pm}(z_1),K_l^{\pm}(z_2)],\\ (6.32)\quad & [E_{l+1,l}^{\mp}(z_2),K_m^{\pm}(z_1)] = 0 = [E_{m,l+1}^{\pm}(z_1),K_l^{\mp}(z_2)]\end{aligned}$$

where $z = z_1/z_2$.

Proof. (6.29) and (6.30) follows from the $(m,l),(m,l)$ component of (6.25) and (6.27), respectvely. Similarly, (6.31) (resp. (6.32)) follows from (6.29) and the $(m,l+1),(m,l)$ component of (6.25)(resp. (6.30) and the same component of (6.27)). Q.E.D.

Other relations among the basic Gauss components are given in Appendix C.

The following lemma indicates that the whole Gauss components of $L^{\pm}(z)$ can be determined recursively by the basic ones.

Lemma 6.10. *Let* $I_{a,b} = \{\ (j,k) \mid a \le j \le b-1,\ j+1 \le k \le b\} \setminus \{(a,b)\}$. *For* $2 \le l+1 < m \le N$, $E_{m,l}^{+}(z)$ *(resp.* $F_{l,m}^{+}(z)$*) is determined by* $\{E_{k,j}^{+}(z)\ (j,k) \in I_{l,m}, K_j^{+}(z)\ l \le j \le m\}$ *(resp.* $\{F_{j,k}^{+}(z)\ (j,k) \in I_{l,m}, K_j^{+}(z)\ l \le j \le m\}$*).*

Proof. Let us consider $E^{+}_{m,l}(z)$. The $F^{+}_{l,m}(z)$ case is similar. From (6.29) and the $(m, l+1), (l+1, l)$ component of (6.25) we have

$$\begin{aligned}(6.33)\quad & -\bar{b}^*(z)E^{\pm}_{l+1,l}(z_2^{\pm})E^{\pm}_{m,l+1}(z_1^{\pm}) \\ & = E^{\pm}_{m,l}(z_1^{\pm})c^*(z, P_{l,l+1}) - E^{\pm}_{m,l+1}(z_1^{\pm})E^{\pm}_{l+1,l}(z_2^{\pm}) \\ & \quad + \bar{c}^*(z, P_{l+1,m})(L_l^{m\pm}(z_2^{\pm})^{-1})_{ml}K_l^{\pm}(z_2^{\pm}) \\ & \quad + \sum_{l+2\le k\le m-1} E^{\pm}_{m,k}(z_1^{\pm})\bar{c}^*(z, P_{l+1,k})(L_l^{m\pm}(z_2^{\pm})^{-1})_{kl}K_l^{\pm}(z_2^{\pm}).\end{aligned}$$

Similarly, from (6.30) and the $(m, l+1), (l+1, l)$ component of (6.27), we have

$$\begin{aligned}(6.34)\quad & -\bar{b}^*(z)E^{\mp}_{l+1,l}(z_2^{\mp})E^{\pm}_{m,l+1}(z_1^{\pm}) \\ & = E^{\pm}_{m,l}(z_1^{\pm})c^*(z, P_{l,l+1}) - E^{\pm}_{m,l+1}(z_1^{\pm})E^{\mp}_{l+1,l}(z_2^{\mp}) \\ & \quad + \bar{c}^*(z, P_{l+1,m})(L_l^{m\mp}(z_2^{\mp})^{-1})_{ml}K_l^{\mp}(z_2^{\mp}) \\ & \quad + \sum_{l+2\le k\le m-1} E^{\pm}_{m,k}(z_1^{\pm})\bar{c}^*(z, P_{l+1,k})(L_l^{m\mp}(z_2^{\mp})^{-1})_{kl}K_l^{\mp}(z_2^{\pm}).\end{aligned}$$

Subtracting (6.34) with the upper and lower signs reversed from (6.33), we have

(6.35)

$$\begin{aligned}& E^{\pm}_{m,l}(z_1^{\pm}) - E^{\mp}_{m,l}(z_1^{\mp}) \\ & = \frac{1}{c^*(z, P_{l,l+1})}\Bigg(\Big(E^{\pm}_{m,l+1}(z_1^{\pm}) - E^{\mp}_{m,l+1}(z_1^{\mp})\Big)\, E^{\pm}_{l+1,l}(z_2^{\pm}) \\ & \qquad -\bar{b}^*(z)E^{\pm}_{l+1,l}(z_2^{\pm})\Big(E^{\pm}_{m,l+1}(z_1^{\pm}) - E^{\mp}_{m,l+1}(z_1^{\mp})\Big) \\ & - \sum_{l+2\le k\le m-1}(E^{\pm}_{m,k}(z_1^{\pm}) - E^{\mp}_{m,k}(z_1^{\mp}))\bar{c}^*(z, P_{l+1,k})(L_l^{m\pm}(z_2^{\pm})^{-1})_{kl}K_l^{\pm}(z_2^{\pm})\Bigg).\end{aligned}$$

Then due to (6.2) and (6.9), in each sign case the right hand side of (6.35) determines both $E^{+}_{m,l}(z)$ and $E^{-}_{m,l}(z)$ uniquely as formal Laurent series in z. Q.E.D.

Remark. The upper and the lower sign cases of (6.35) give two expressions for $E^{+}_{m,l}(z_1^{+}) - E^{-}_{m,l}(z_1^{-})$. It is instructive to derive their consistency condition. Equating them, we obtain

$$\begin{aligned}
&\left(E^+_{m,l+1}(z_1^+)-E^-_{m,l+1}(z_1^-)\right)E^+_{l+1,l}(z_2^+)\\
&\qquad\quad -\bar{b}^*(z)E^+_{l+1,l}(z_2^+)\left(E^+_{m,l+1}(z_1^+)-E^+_{m,l+1}(z_1^-)\right)\\
&-\sum_{l+2\le k\le m-1}(E^+_{m,k}(z_1^+)-E^-_{m,k}(z_1^-))\bar{c}^*(z,P_{l+1,k})(L^{m+}_l(z_2^+)^{-1})_{kl}K^+_l(z_2^+)\\
&=\left(E^+_{m,l+1}(z_1^+)-E^-_{m,l+1}(z_1^-)\right)E^-_{l+1,l}(z_2^-)\\
&\qquad\quad -\bar{b}^*(z)E^-_{l+1,l}(z_2^-)\left(E^+_{m,l+1}(z_1^+)-E^-_{m,l+1}(z_1^-)\right)\\
&-\sum_{l+2\le k\le m-1}(E^+_{m,k}(z_1^+)-E^-_{m,k}(z_1^-))\bar{c}^*(z,P_{l+1,k})(L^{m-}_l(z_2^-)^{-1})_{kl}K^-_l(z_2^-).
\end{aligned}$$

Hence the consistency condition is

$$\begin{aligned}
&\left(E^+_{m,l+1}(z_1^+)-E^-_{m,l+1}(z_1^-)\right)\left(E^+_{l+1,l}(z_2^+)-E^-_{l+1,l}(z_2^-)\right)\\
&-\bar{b}^*(z)\left(E^+_{l+1,l}(z_2^+)-E^-_{l+1,l}(z_2^-)\right)\left(E^+_{m,l+1}(z_1^+)-E^+_{m,l+1}(z_1^-)\right)\\
&=\sum_{l+2\le k\le m-1}(E^+_{m,k}(z_1^+)-E^-_{m,k}(z_1^-))\bar{c}^*(z,P_{l+1,k})\\
&\qquad\times\left((L^{m+}_l(z_2^+)^{-1})_{kl}K^+_l(z_2^+)-(L^{m-}_l(z_2^-)^{-1})_{kl}K^-_l(z_2^-)\right).
\end{aligned}$$

In particular, for $m=l+2$ we have

$$\begin{aligned}
(6.36)\quad &\left(E^+_{l+2,l+1}(z_1^+)-E^-_{l+2,l+1}(z_1^-)\right)\left(E^+_{l+1,l}(z_2^+)-E^-_{l+1,l}(z_2^-)\right)\\
&=\bar{b}^*(z)\left(E^+_{l+1,l}(z_2^+)-E^-_{l+1,l}(z_2^-)\right)\left(E^+_{l+2,l+1}(z_1^+)-E^+_{l+2,l+1}(z_1^-)\right)
\end{aligned}$$

for $1\le l\le N-2$. In Appendix C we identify these relations with the commutation relations of the total elliptic currents of $U_{q,p}(\widehat{\mathfrak{gl}}_N)$. This provides an example suggesting the injectivity of the H-algebra homomorphism $\Phi:\mathcal{U}\to\mathcal{E}$ given in the next subsection.

6.3. The half currents of $\mathcal{U}$

Let us define the basic half currents of $\mathcal{U}$ as follows. For $1 \le j \le N-1$

$$
\begin{aligned}
e^+_{j+1,j}(z) &= \frac{a^*_{j+1,j}\Theta_{p^*}(q^2)}{(p^*;p^*)^3_\infty}\left(\sum_{m\ge 0} e_{j,-m}\frac{1}{1-q^{-2(P_{\alpha_j}-1)}p^{*m}}(zq^{j-c})^m\right.\\
&\qquad\left.-\sum_{m>0} e_{j,m}\frac{q^{2(P_{\alpha_j}-1)}p^{*m}}{1-q^{2(P_{\alpha_j}-1)}p^{*m}}(zq^{j-c})^{-m}\right),\\
f^+_{j,j+1}(z) &= \frac{a_{j,j+1}\Theta_p(q^2)}{(p;p)^3_\infty}\left(\sum_{m\ge 0} f_{j,-m}\frac{1}{1-q^{2((P+h)_{\alpha_j}-1)}p^m}(zq^j)^m\right.\\
&\qquad\left.-\sum_{m>0} f_{j,m}\frac{q^{-2((P+h)_{\alpha_j}-1)}p^m}{1-q^{-2((P+h)_{\alpha_j}-1)}p^m}(zq^j)^{-m}\right),\\
e^-_{j+1,j}(z) &= q^{2P_{\bar\epsilon_{j+1}}}\, e^+_{j+1,j}(zp^*q^c)q^{-2P_{\bar\epsilon_j}}\\
&= \frac{a^*_{j+1,j}\Theta_{p^*}(q^2)}{(p^*;p^*)^3_\infty}\left(\sum_{m\ge 0} e_{j,-m}\frac{q^{-2(P_{\alpha_j}-1)}p^{*m}}{1-q^{-2(P_{\alpha_j}-1)}p^{*m}}(zq^j)^m\right.\\
&\qquad\left.-\sum_{m>0} e_{j,m}\frac{1}{1-q^{2(P_{\alpha_j}-1)}p^{*m}}(zq^j)^{-m}\right),\\
f^-_{j,j+1}(z) &= q^{2(P+h)_{\bar\epsilon_j}}\, f^+_{j,j+1}(zpq^{-c})q^{-2(P+h)_{\bar\epsilon_{j+1}}}\\
&= \frac{a_{j,j+1}\Theta_p(q^2)}{(p;p)^3_\infty}\left(\sum_{m\ge 0} f_{j,-m}\frac{q^{2((P+h)_{\alpha_j}-1)}p^m}{1-q^{2((P+h)_{\alpha_j}-1)}p^m}(zq^{j-c})^m\right.\\
&\qquad\left.-\sum_{m>0} f_{j,m}\frac{1}{1-q^{-2((P+h)_{\alpha_j}-1)}p^m}(zq^{j-c})^{-m}\right),
\end{aligned}
$$

where $a^*_{j+1,j}$ and $a_{j,j+1}$ are constants given by

$$
a^*_{j+1,j} = q^{-1}\frac{(p^*;p^*)_\infty}{(p^*q^2;p^*)_\infty}, \qquad a_{j,j+1} = q^{-1}\frac{(p;p)_\infty}{(pq^{-2};p)_\infty}. \tag{6.37}
$$

We then obtain

$$
\begin{aligned}
e^+_{j+1,j}(z^+) - e^-_{j+1,j}(z^-) &= \frac{a^*_{j+1,j}\Theta_{p^*}(q^2)}{(p^*;p^*)^3_\infty}e_j(zq^{j-c/2}),\\
f^+_{j,j+1}(z^-) - f^-_{j,j+1}(z^+) &= \frac{a_{j,j+1}\Theta_p(q^2)}{(p;p)^3_\infty}f_j(zq^{j-c/2}).
\end{aligned}
$$

Here we set $z^{\pm} = zq^{\pm c/2}$.

Note that at $p = 0$

$$\begin{aligned}
e^+_{j+1,j}(z) &= a^*_{j+1,j}(1-q^2)\left(e_{j,0}\frac{1}{1-q^{-2(P_{\alpha_j}-1)}} + \sum_{m>0} e_{j,-m}(zq^{j-c})^m\right),\\
e^-_{j+1,j}(z) &= a^*_{j+1,j}(1-q^2)\left(e_{j,0}\frac{q^{-2(P_{\alpha_j}-1)}}{1-q^{-2(P_{\alpha_j}-1)}} - \sum_{m\geq 0} e_{j,m}(zq^{j})^{-m}\right),\\
f^+_{j,j+1}(z) &= a_{j,j+1}(1-q^2)\left(f_{j,0}\frac{1}{1-q^{2((P+h)_{\alpha_j}-1)}} + \sum_{m\geq 0} f_{j,-m}(zq^{j})^m\right),\\
f^-_{j,j+1}(z) &= a_{j,j+1}(1-q^2)\left(f_{j,0}\frac{q^{2((P+h)_{\alpha_j}-1)}}{1-q^{2((P+h)_{\alpha_j}-1)}} - \sum_{m\geq 0} f_{j,m}(zq^{j-c})^{-m}\right).
\end{aligned}$$

Noting (3.1) and the expansion formula

$$\begin{aligned}
\frac{\Theta_p(q^{2s}z)(p;p)^3_\infty}{\Theta_p(q^{2s})\Theta_p(z)} &= \sum_{n\in\mathbb{Z}} \frac{1}{1-q^{2s}p^n} z^n\\
&= \sum_{l\in\mathbb{Z}_{\geq 0}} \left(\frac{q^{2sl}}{1-p^l z} - \frac{q^{-2s(l+1)}p^{l+1}/z}{1-p^{l+1}/z}\right),
\end{aligned} \tag{6.38}$$

for $|p| < |z| < 1$, one can express the basic half currents as follows.

Proposition 6.11.

$$\begin{aligned}
e^+_{j+1,j}(z) &= a^*_{j+1,j}\oint_{C^*}\frac{dz'}{2\pi i z'} e_j(z')\frac{\Theta_{p^*}(zq^{j-c}q^{2(1-P_{\alpha_j})}/z')\Theta_{p^*}(q^2)}{\Theta_{p^*}(zq^{j-c}/z')\Theta_{p^*}(q^{2(P_{\alpha_j}-1)})},\\
f^+_{j,j+1}(z) &= a_{j,j+1}\oint_{C}\frac{dz'}{2\pi i z'} f_j(z')\frac{\Theta_{p}(zq^{j}q^{2((P+h)_{\alpha_j}-1)}/z')\Theta_{p}(q^2)}{\Theta_{p}(zq^{j}/z')\Theta_{p}(q^{2((P+h)_{\alpha_j}-1)})},
\end{aligned}$$

where $C^* : |q^{j-c}z| < |z'| < |p^{*-1}q^{j-c}z|$, $C : |q^j z| < |z'| < |p^{-1}q^j z|$.

Proposition 6.12. [36, 43] *The basic half currents* $e^+_{j+1,j}(z)$, $f^+_{j,j+1}(z)$ $(j \in I)$ *and* $k^+_l(z)$ $(1 \le l \le N)$ *satisfy the following relations.*

$$
\begin{aligned}
(6.39)\quad & k^+_{j+1}(z_1)^{-1} e^+_{j+1,j}(z_2) k^+_{j+1}(z_1) \\
&\qquad = e^+_{j+1,j}(z_2)\frac{1}{\bar{b}^*(z)} - e^+_{j+1,j}(z_1)\frac{c^*(z,P_{j,j+1})}{\bar{b}^*(z)},
\end{aligned}
$$

$$
\begin{aligned}
(6.40)\quad & k^+_{j+1}(z_1) f^+_{j,j+1}(z_2) k^+_{j+1}(z_1)^{-1} \\
&\qquad = \frac{1}{\bar{b}(z)} f^+_{j,j+1}(z_2) - \frac{\bar{c}(z,(P+h)_{j,j+1})}{\bar{b}(z)} f^+_{j,j+1}(z_1),
\end{aligned}
$$

$$
\begin{aligned}
(6.41)\quad & \frac{1}{\bar{b}^*(1/z)} e^+_{j+1,j}(z_1) e^+_{j+1,j}(z_2) - e^+_{j+1,j}(z_2)^2 \frac{c^*(1/z, P_{j,j+1}-2)}{\bar{b}^*(1/z)} \\
&\quad = \frac{1}{\bar{b}^*(z)} e^+_{j+1,j}(z_2) e^+_{j+1,j}(z_1) - e^+_{j+1,j}(z_1)^2 \frac{c^*(z, P_{j,j+1}-2)}{\bar{b}^*(z)},
\end{aligned}
$$

$$
\begin{aligned}
(6.42)\quad & \frac{1}{\bar{b}(z)} f^+_{j,j+1}(z_1) f^+_{j,j+1}(z_2) - f^+_{j,j+1}(z_1)^2 \frac{\bar{c}(z,(P+h)_{j,j+1}-2)}{\bar{b}(z)} \\
& = \frac{1}{\bar{b}(1/z)} f^+_{j,j+1}(z_2) f^+_{j,j+1}(z_1) - f^+_{j,j+1}(z_2)^2 \frac{\bar{c}(1/z,(P+h)_{j,j+1}-2)}{\bar{b}(1/z)},
\end{aligned}
$$

$$
\begin{aligned}
(6.43)\quad & [e^+_{j+1,j}(z_1), f^+_{j,j+1}(z_2)] \\
&\quad = k^+_j(z_2) k^+_{j+1}(z_2)^{-1} \frac{\bar{c}^*(z, P_{j,j+1}-1)}{\bar{b}^*(z)} \\
&\qquad - k^+_{j+1}(z_1)^{-1} k^+_j(z_1) \frac{\bar{c}(z,(P+h)_{j,j+1}-1)}{\bar{b}(z)},
\end{aligned}
$$

where $z = z_1/z_2$.

Proof. Direct calculation using Proposition 6.11, the relations in Definition 3.1 and (3.28)-(3.37). Q.E.D.

For $1 \le j \le N-1$, let us consider the subalgebra $U^{(j)}_{q,p}(\widehat{\mathfrak{gl}}_2)$ of $\mathcal{U}$ generated by $e_j(z), f_j(z), k^+_j(z)$, $k^+_{j+1}(z), q^{\pm c/2}, \widehat{d}$. Let us define the L-operator by the associated basic half currents by

$$
\begin{aligned}
\mathcal{L}^\pm_j(z) &= \begin{pmatrix} 1 & f^\pm_{j,j+1}(z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k^\pm_j(z) & 0 \\ 0 & k^\pm_{j+1}(z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^\pm_{j+1,j}(z) & 1 \end{pmatrix} \\
&= \begin{pmatrix} k^\pm_j(z) + f^\pm_{j,j+1}(z) k^\pm_{j+1}(z) e^\pm_{j+1,j}(z) & f^\pm_{j,j+1}(z) k^\pm_{j+1}(z) \\ k^\pm_{j+1}(z) e^\pm_{j+1,j}(z) & k^\pm_{j+1}(z) \end{pmatrix}.
\end{aligned}
$$

Then comparing the relations in Proposition 6.12 and those of the basic Gauss components of $L^\pm(z)$ in $E_{q,p}(\widehat{\mathfrak{gl}}_2)$ in Sec.C.1, we obtain the following.

Theorem 6.13. [36, 43] *For each j, the L-operators $\mathcal{L}^{\pm}_j(z)$ satisfy the same RLL-relations* (6.23)-(6.24) *at $l = j, m = j+1$ replacing $L^{m\pm}_l(z)$ with $\mathcal{L}^{\pm}_j(z)$. Hence the following map gives a surjective H-algebra homomorphism.*

$$\begin{aligned}
&\Phi^{(j)} \ : \ U^{(j)}_{q,p}(\widehat{\mathfrak{gl}}_2) \to E_{q,p}(\widehat{\mathfrak{gl}}_2),\\
&e^{+}_{j+1,j}(z) \mapsto E^{+}_{j+1,j}(z), \quad f^{+}_{j,j+1}(z) \mapsto F^{+}_{j,j+1}(z),\\
&k^{+}_{j}(z) \mapsto K^{+}_{j}(z) \quad k^{+}_{j+1}(z) \mapsto K^{+}_{j+1}(z).
\end{aligned}$$

Now let us consider the canonical extension of the map $\Phi^{(j)}$ to $\Phi : \mathcal{U} \to \mathcal{E}$ by

$$\begin{aligned}
&e^{+}_{j+1,j}(z) \mapsto E^{+}_{j+1,j}(z), \quad f^{+}_{j,j+1}(z) \mapsto F^{+}_{j,j+1}(z),\\
&k^{+}_{l}(z) \mapsto K^{+}_{l}(z) \qquad (j \in I, \ 1 \le l \le N).
\end{aligned}$$

Theorem 6.14. Φ *gives an isomorphism as a topological H-algebra over $\mathbb{F}[[p]]$.*

Proof. 1) Surjectivity: From Theorem 6.7 with $l = j, m = j+1$ the basic Gauss components $E^{+}_{j+1,j}(z)$, $F^{+}_{j,j+1}(z)$, $K^{+}_{j}(z)$ and $\widehat{d}$ generate the subalgebra $E_{q,p}(\widehat{\mathfrak{gl}}_2)$. From Lemma 6.10 the *RLL* relations allows us to construct the other Gauss components $E^{\pm}_{k,j}(z)$, $F^{\pm}_{j,k}(z)$ $(3 \le j+2 \le k \le N)$ recursively from the basic ones $E^{\pm}_{j+1,j}(z), F^{\pm}_{j,j+1}(z)$, $K^{\pm}_{j}(z), K^{\pm}_{j+1}(z)$. Then the surjectivity follows from Theorem 6.13.
2) Injectivity: Let $(\varphi^{q}_{\lambda,k}, V)$ be a highest weight representation of $\mathcal{U}_q = U_q(\widehat{\mathfrak{gl}}_N)$ with the highest weight λ and the level k. We extend $\varphi^{q}_{\lambda,k}$ to the dynamical representation $(\varphi^{q,p}_{\lambda,k} = \varphi^{q}_{\lambda,k} \circ \phi_p, \widehat{V}_{\mathbb{F}[[p]]})$ of $\mathcal{U}$ as in Corollary 5.6. Then we define $\widetilde{\varphi}^{q,p}_{\lambda,k} : \mathcal{E} \to \mathrm{End}_{\mathbb{C}}\widehat{V}_{\mathbb{F}[[p]]}$ by $\widetilde{\varphi}^{q,p}_{\lambda,k}(A) = \varphi^{q,p}_{\lambda,k}(a)$ for $A = \Phi(a), a \in \mathcal{U}$. Let $\pi_p : \mathcal{U} \to \mathcal{U}/p\mathcal{U}$ be the canonical projection. From the remark above Corollary 5.6, we also have the corresponding canonical projection $\pi_p : \mathrm{End}_{\mathbb{C}}\widehat{V}_{\mathbb{F}[[p]]} \to \mathrm{End}_{\mathbb{C}}(V_{\mathbb{F}} \otimes V_Q)$. We then consider the following diagram.

$$\begin{tikzcd}
\mathcal{U} \arrow[r, "\Phi"] \arrow[d, "\pi_p"'] \arrow[dr, "\phi_p"] & \mathcal{E} \arrow[r, "\widetilde{\varphi}^{q,p}_{\lambda,k}"] & \mathcal{D}_{H,\widehat{V}_{\mathbb{F}[[p]]}} \arrow[dd, "\pi_p"] \\
\mathcal{U}/p\mathcal{U} \arrow[d, phantom, "\wr|"] & (\mathbb{F}[[p]] \otimes_{\mathbb{C}} \mathcal{U}_q)\sharp\mathbb{C}[\mathcal{R}_Q] \arrow[ur, "\varphi^{q}_{\lambda,k}"] \arrow[dl, "\pi_p"] & \\
(\mathbb{F} \otimes_{\mathbb{C}} \mathcal{U}_q)\sharp\mathbb{C}[\mathcal{R}_{\mathcal{Q}}] \arrow[rr, "\varphi^{q}_{\lambda,k}"] & & \mathcal{D}_{H,\widehat{V}_{\mathbb{F}}}
\end{tikzcd}$$

Lemma 6.15.

(i) $\pi_p \circ \phi_p = \pi_p$

(ii) $\pi_p \circ \varphi^q_{\lambda,k} = \varphi^q_{\lambda,k} \circ \pi_p$ *on* $(\mathbb{F}[[p]] \otimes_{\mathbb{C}} \mathcal{U}_q)\sharp\mathbb{C}[\mathcal{R}_Q]$

Proof. (i) follows from $u^{\pm}_{\varepsilon_i}(z,p) = 1$ at $p = 0$.

(ii) follows from (i) and $\widehat{V}_{\mathbb{F}[[p]]} = (\mathbb{F}[[p]] \otimes_{\mathbb{C}} V) \otimes V_Q$. Q.E.D.

Lemma 6.16. Ker $\Phi \subset p\mathcal{U}$.

Proof. Assume Ker $\Phi \not\subset p\mathcal{U}$. Then there exists a non zero element $a \in$ Ker Φ such that $\pi_p(a) \neq 0$. Then from Lemma 6.15 for any level-k highest weight representation $\varphi^q_{\lambda,k}$ of $\mathcal{U}_q$, we have

$$(6.44)\quad 0 \;=\; \pi_p \circ \widetilde{\varphi}^{q,p}_{\lambda,k} \circ \Phi(a) = \pi_p \circ \varphi^q_{\lambda,k} \circ \phi_p(a) = \varphi^q_{\lambda,k} \circ \pi_p(a).$$

This contradicts the fact $\bigcap_{\lambda,k}$ Ker $\varphi^q_{\lambda,k} = 0$ given in [9]. Q.E.D.

Proof of the injectivity. Let us assume Ker $\Phi \neq 0$. Let $a \neq 0 \in$ Ker Φ. Then from Lemma 6.16 there exists $\widetilde{a} \in \mathcal{U}$ such that $a = p^n \widetilde{a}$ for some positive integer n and $\pi_p(\widetilde{a}) \neq 0 \in \mathcal{U}_q$. Then the same argument as (6.44) yields for any $\varphi^q_{\lambda,k}$

$$0 \;=\; \pi_p \circ \varphi^{q,p}_{\lambda,k} \circ \Phi(a) = p^n \varphi^q_{\lambda,k} \circ \pi_p(\widetilde{a}).$$

This again contradicts $\bigcap_{\lambda,k}$ Ker $\varphi^q_{\lambda,k} = 0$.

Q.E.D.

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§Appendix A. Quantum Affine Algebra $U_q(\widehat{\mathfrak{gl}}_N)$

Definition A.1. *The quantum affine algebra* $U_q(\widehat{\mathfrak{gl}}_N)$ *is a topological algebra over* $\mathbb{C}$ *generated by* $k^{\pm}_{i,m},\ x^{\pm}_{j,n}, d$ $(1 \leq i \leq N,\ 1 \leq j \leq N-1,\ m \in \mathbb{Z}_{\geq 0},\ n \in \mathbb{Z})$ *and the central element* $q^{\pm c/2}$. *The defining relations are conveniently written in terms of the generating functions called the Drinfeld currents* :

$$k^{\pm}_{0,i}(z) = \sum_{m \in \mathbb{Z}_{\geq 0}} k^{\pm}_{i,m} z^{\pm m},$$

$$x^{\pm}_j(z) = \sum_{n \in \mathbb{Z}} x^{\pm}_{j,n} z^{-n}.$$

The relations are given by

$$[d,k_i^{\pm}(z)]=\pm z\frac{\partial}{\partial z}k_i^{\pm}(z),\qquad [d,x_j^{\pm}(z)]=-z\frac{\partial}{\partial z}x_j^{\pm}(z),$$

$$k_{i,0}^{+}k_{i,0}^{-}=1=k_{i,0}^{-}k_{i,0}^{+},$$

$$k_{0,j}^{\pm}(z_1)k_{0,l}^{\pm}(z_2)=k_{0.l}^{\pm}(z_2)k_{0,j}^{\pm}(z_1),$$

$$k_{0,j}^{-}(z_1)k_{0,j}^{+}(z_2)$$
$$=\frac{(q^{c+2}z_2/z_1,q^{2N}q^{c-2}z_2/z_1,q^{2N}q^{-c}z_2/z_1,q^{-c}z_2/z_1;q^{2N})_\infty}{(q^{-c+2}z_2/z_1,q^{2N}q^{-c-2}z_2/z_1,q^{2N}q^{c}z_2/z_1,q^{c}z_2/z_1;q^{2N})_\infty}k_{0,j}^{+}(z_2)k_{0,j}^{-}(z_1),$$

$$k_{0,j}^{-}(z_1)k_{0,l}^{+}(z_2)$$
$$=\frac{(q^{2N}q^{c+2}z_2/z_1,q^{2N}q^{c-2}z_2/z_1,q^{2N}q^{-c}z_2/z_1,q^{2N}q^{-c}z_2/z_1;q^{2N})_\infty}{(q^{2N}q^{-c+2}z_2/z_1,q^{2N}q^{-c-2}z_2/z_1,q^{2N}q^{c}z_2/z_1,q^{2N}q^{c}z_2/z_1;q^{2N})_\infty}$$
$$\times k_{0,l}^{+}(z_2)k_{0,j}^{-}(z_1)\quad (j<l),$$

$$k_{0,j}^{-}(z_1)k_{0,l}^{+}(z_2)$$
$$=\frac{(q^{c+2}z_2/z_1,q^{c-2}z_2/z_1,q^{-c}z_2/z_1,q^{-c}z_2/z_1;q^{2N})_\infty}{(q^{-c+2}z_2/z_1,q^{-c-2}z_2/z_1,q^{c}z_2/z_1,q^{c}z_2/z_1;q^{2N})_\infty}k_{0,l}^{+}(z_2)k_{0,j}^{-}(z_1)\quad (j>l),$$

$$k_{0,j}^{+}(z_1)x_j^{+}(z_2)k_{0,j}^{+}(z_1)^{-1}=q^{-1}\frac{1-q^{-c+j}z_1/z_2}{1-q^{-c-2+j}z_1/z_2}x_j^{+}(z_2),$$

$$k_{0,j+1}^{+}(z_1)x_j^{+}(z_2)k_{0,j+1}^{+}(z_1)^{-1}=q\frac{1-q^{-c+j}z_1/z_2}{1-q^{-c+2+j}z_1/z_2}x_j^{+}(z_2),$$

$$k_{0,j}^{+}(z_1)x_j^{-}(z_2)k_{0,j}^{+}(z_1)^{-1}=q\frac{1-q^{-2+j}z_1/z_2}{1-q^{j}z_1/z_2}x_j^{-}(z_2),$$

$$k_{0,j+1}^{+}(z_1)x_j^{-}(z_2)k_{0,j+1}^{+}(z_1)^{-1}=q^{-1}\frac{1-q^{2+j}z_1/z_2}{1-q^{j}z_1/z_2}x_j^{-}(z_2),$$

$$k_{0,l}^{+}(z_1)x_j^{\pm}(z_2)k_{0,l}^{+}(z_1)^{-1}=x_j^{\pm}(z_2)\qquad (l\neq j,j+1),$$

$$k_{0,j}^{-}(z_1)^{-1}x_j^{+}(z_2)k_{0,j}^{-}(z_1)=q^{-1}\frac{1-q^{2-j}z_2/z_1}{1-q^{-j}z_2/z_1}x_j^{+}(z_2),$$

$$k_{0,j+1}^{-}(z_1)^{-1}x_j^{+}(z_2)k_{0,j+1}^{-}(z_1)=q\frac{1-q^{-2-j}z_2/z_1}{1-q^{-j}z_2/z_1}x_j^{+}(z_2),$$

$$k_{0,j}^{-}(z_1)^{-1}x_j^{-}(z_2)k_{0,j}^{-}(z_1)=q\frac{1-q^{c-j}z_2/z_1}{1-q^{c+2-j}z_2/z_1}x_j^{-}(z_2),$$

$$k_{0,j+1}^{-}(z_1)^{-1}x_j^{-}(z_2)k_{0,j+1}^{-}(z_1)=q^{-1}\frac{1-q^{c-j}z_2/z_1}{1-q^{c-2-j}z_2/z_1}x_j^{-}(z_2),$$

$$k_{0,l}^{-}(z_1)^{-1}x_j^{\pm}(z_2)k_{0,l}^{-}(z_1)=x_j^{\pm}(z_2)\qquad (l\neq j,j+1),$$

$$
\begin{aligned}
&z_1(1-q^{\pm 2}z_2/z_1)x^{\pm}_j(z_1)x^{\pm}_j(z_2)=-z_2(1-q^{\pm 2}z_1/z_2)x^{\pm}_j(z_2)x^{\pm}_j(z_1),\\
&z_1(1-q^{\mp 1}z_2/z_1)x^{\pm}_j(z_1)x^{\pm}_{j+1}(z_2)=-z_2(1-q^{\mp 1}z_1/z_2)x^{\pm}_{j+1}(z_2)x^{\pm}_j(z_1),\\
&x^{\pm}_j(z_1)x^{\pm}_l(z_2)=x^{\pm}_l(z_2)x^{\pm}_j(z_1) \qquad (l\neq j,j+1),\\
&[x^{+}_i(z_1),x^{-}_j(z_2)]\\
&=\frac{\delta_{i,j}}{q-q^{-1}}\Big(\delta\big(q^{-c}z_1/z_2\big)k^{-}_{0,i}(q^{-c/2}z_1)k^{-}_{0,i+1}(q^{-c/2}z_1)^{-1}\\
&\qquad\qquad -\delta\big(q^{c}z_1/z_2\big)k^{+}_{0,i}(q^{-c/2}z_2)k^{+}_{0,i+1}(q^{-c/2}z_2)^{-1}\Big),
\end{aligned}
$$

$$
\begin{aligned}
&\big\{x^{\pm}_i(z_1)x^{\pm}_i(z_2)x^{\pm}_j(w)-(q+q^{-1})x^{\pm}_i(z_1)x^{\pm}_j(w)x^{\pm}_i(z_2)+x^{\pm}_j(w)x^{\pm}_i(z_1)x^{\pm}_i(z_2)\big\}\\
&\qquad\qquad +(z_1\leftrightarrow z_2)=0, \qquad |i-j|=1.
\end{aligned}
$$

§Appendix B. Relations Among $u^{\pm}_{\varepsilon_i}(z,p)$ and $u^{\pm}_j(z,p)$

From the relations in Definition A.1 the following commutation relations hold.

Lemma B.1. *Let us set* $z=z_1/z_2$.

$$
\begin{aligned}
&k^{+}_{0,j}(z_1)u^{-}_{\varepsilon_j}(q^jz_2,p)\\
&\quad=\frac{(p^*q^2z,p^*q^{2N}q^{-2}z,pq^{2N}z,pz;p,q^{2N})_\infty}{(pq^2z,pq^{2N}q^{-2}z,p^*q^{2N}z,p^*z;p,q^{2N})_\infty}u^{-}_{\varepsilon_j}(q^jz_2,p)k^{+}_{0,j}(z_1),\\
&k^{+}_{0,j}(z_1)u^{-}_{\varepsilon_l}(q^lz_2,p)\\
&\quad=\frac{(p^*q^2z,p^*q^{-2}z,pz,pz;p,q^{2N})_\infty}{(pq^2z,pq^{-2}z,p^*z,p^*z;p,q^{2N})_\infty}u^{-}_{\varepsilon_l}(q^lz_2,p)k^{+}_{0,j}(z_1)\ (j<l),\\
&k^{+}_{0,j}(z_1)u^{-}_{\varepsilon_l}(q^lz_2,p)=\frac{(p^*q^{2N}q^2z,p^*q^{2N}q^{-2}z,pq^{2N}z,pq^{2N}z;p,q^{2N})_\infty}{(pq^{2N}q^2z,pq^{2N}q^{-2}z,p^*q^{2N}z,p^*q^{2N}z;p,q^{2N})_\infty}\\
&\qquad\qquad\qquad\qquad\times u^{-}_{\varepsilon_l}(q^lz_2,p)k^{+}_{0,j}(z_1) \qquad (j>l),\\
&k^{+}_{0,j}(z_1)u^{-}_j(q^jz_2,p)=\frac{(pq^{-2}z;p)_\infty(p^*z;p)_\infty}{(p^*q^{-2}z;p)_\infty(pz;p)_\infty}u^{-}_j(q^jz_2,p)k^{+}_{0,j}(z_1),\\
&k^{+}_{0,j+1}(z_1)u^{-}_j(q^jz_2,p)=\frac{(pq^{2}z;p)_\infty(p^*z;p)_\infty}{(p^*q^{2}z;p)_\infty(pz;p)_\infty}u^{-}_j(q^jz_2,p)k^{+}_{0,j+1}(z_1),\\
&k^{+}_{0,l}(z_1)u^{-}_j(q^jz_2,p)=u^{-}_j(q^jz_2,p)k^{+}_{0,l}(z_1) \qquad (l\neq j,j+1),\\
&u^{+}_{\varepsilon_j}(q^jz_1,p)k^{-}_{0,j}(z_2)\\
&\quad=\frac{(p^*q^2z,p^*q^{2N}q^{-2}z,pq^{2N}z,pz;p^*,q^{2N})_\infty}{(pq^2z,pq^{2N}q^{-2}z,p^*q^{2N}z,p^*z;p^*,q^{2N})_\infty}k^{-}_{0,j}(z_2)u^{+}_{\varepsilon_j}(q^jz_1,p),
\end{aligned}
$$

$$\begin{aligned} u^+_{\varepsilon_l}(q^lz_1,p)k^-_{0,j}(z_2) &= \frac{(p^*q^{2N}q^2z,p^*q^{2N}q^{-2}z,pq^{2N}z,pq^{2N}z;p^*,q^{2N})_\infty}{(pq^{2N}q^2z,pq^{2N}q^{-2}z,p^*q^{2N}z,p^*q^{2N}z;p^*,q^{2N})_\infty} \\ &\qquad \times k^-_{0,j}(z_2)u^+_{\varepsilon_l}(q^lz_1,p) \qquad (j<l), \end{aligned}$$

$$\begin{aligned} &u^+_{\varepsilon_l}(q^lz_1,p)k^-_{0,j}(z_2) \\ &\quad = \frac{(p^*q^2z,p^*q^{-2}z,pz,pz;p^*,q^{2N})_\infty}{(pq^2z,pq^{-2}z,p^*z,p^*z;p^*,q^{2N})_\infty}k^-_{0,j}(z_2)u^+_{\varepsilon_l}(q^lz_1,p)\ (j>l), \end{aligned}$$

$$u^+_j(q^jz_1,p)k^-_{0,j}(z_2) = \frac{(p^*q^2z;p^*)_\infty(pz;p^*)_\infty}{(pq^2z;p^*)_\infty(p^*z;p^*)_\infty}k^-_{0,j}(z_2)u^+_j(q^jz_1,p),$$

$$u^+_j(q^jz_1,p)k^-_{0,j+1}(z_2) = \frac{(p^*q^{-2}z;p^*)_\infty(pz;p^*)_\infty}{(pq^{-2}z;p^*)_\infty(p^*z;p^*)_\infty}k^-_{0,j+1}(z_2)u^+_j(q^jz_1,p),$$

$$u^+_j(q^jz_1,p)k^-_{0,l}(z_2) = k^-_{0,l}(z_2)u^+_j(q^jz_1,p) \qquad (l\neq j,j+1),$$

$$u^\pm_{\varepsilon_j}(z_1,p)u^\pm_{\varepsilon_l}(z_2,p) = u^\pm_{\varepsilon_l}(z_2,p)u^\pm_{\varepsilon_j}(z_1,p) \quad (\forall j,l),$$

$$\begin{aligned} &u^+_{\varepsilon_j}(q^jz_1,p)u^-_{\varepsilon_j}(q^jz_2,p) \\ &\quad = \frac{(p^{*2}q^{c+2}z,p^{*2}q^{2N}q^{c-2}z,pp^*q^{2N}q^cz,pp^*q^cz;p,p^*,q^{2N})_\infty}{(pp^*q^{c+2}z,pp^*q^{2N}q^{c-2}z,p^{*2}q^{2N}q^cz,p^{*2}q^cz;p,p^*,q^{2N})_\infty} \\ &\qquad \times u^-_{\varepsilon_j}(q^jz_2,p)(z_2)u^+_{\varepsilon_j}(q^jz_1,p), \end{aligned}$$

$$\begin{aligned} &u^+_{\varepsilon_l}(q^lz_1,p)u^-_{\varepsilon_j}(q^jz_2,p) \\ &\quad = \frac{(p^{*2}q^{2N}q^{c+2}z,p^{*2}q^{2N}q^{c-2}z,pp^*q^{2N}q^cz,pp^*q^{2N}q^cz;p,p^*,q^{2N})_\infty}{(pp^*q^{2N}q^{c+2}z,pp^*q^{2N}q^{c-2}z,p^{*2}q^{2N}q^cz,p^{*2}q^{2N}q^cz;p,p^*,q^{2N})_\infty} \\ &\qquad \times u^-_{\varepsilon_j}(q^jz_2,p)u^+_{\varepsilon_l}(q^lz_1,p) \qquad (j<l), \end{aligned}$$

$$\begin{aligned} &u^+_{\varepsilon_l}(q^lz_1,p)u^-_{\varepsilon_j}(q^jz_2,p) \\ &\quad = \frac{(p^{*2}q^{c+2}z,p^{*2}q^{c-2}z,pp^*q^cz,pp^*q^cz;p,p^*,q^{2N})_\infty}{(pp^*q^{c+2}z,pp^*q^{c-2}z,p^{*2}q^cz,p^{*2}q^cz;p,p^*,q^{2N})_\infty} \\ &\qquad \times u^-_{\varepsilon_j}(q^jz_2,p)u^+_{\varepsilon_l}(q^lz_1,p) \qquad (j>l), \end{aligned}$$

$$\begin{aligned} &u^+_j(q^jz_1,p)u^-_{\varepsilon_j}(q^jz_2,p) \\ &\quad = \frac{(p^*q^{c+2}z;p^*)_\infty(p^*q^cz;p)_\infty}{(p^*q^{c+2}z;p)_\infty(p^*q^cz;p^*)_\infty}u^-_{\varepsilon_j}(q^jz_2,p)u^+_j(q^jz_1,p), \end{aligned}$$

$$\begin{aligned} &u^+_j(q^jz_1,p)u^-_{\varepsilon_{j+1}}(q^{j+1}z_2,p) \\ &\quad = \frac{(p^*q^{c-2}z;p^*)_\infty(p^*q^cz;p)_\infty}{(p^*q^{c-2}z;p)_\infty(p^*q^cz;p^*)_\infty}u^-_{\varepsilon_{j+1}}(q^{j+1}z_2,p)u^+_j(q^jz_1,p), \end{aligned}$$

$$u^+_j(q^jz_1,p)u^-_{\varepsilon_l}(q^lz_2,p) = u^-_{\varepsilon_l}(q^lz_2,p)u^+_j(q^jz_1,p) \qquad (l\neq j,j+1),$$

$$\begin{aligned} &u^+_{\varepsilon_j}(q^jz_1,p)u^-_j(q^jz_2,p) \\ &\quad = \frac{(p^*q^{c-2}z;p)_\infty(p^*q^cz;p^*)_\infty}{(p^*q^{c-2}z;p^*)_\infty(p^*q^cz;p)_\infty}u^-_j(q^jz_2,p)u^+_{\varepsilon_j}(q^jz_1,p), \end{aligned}$$

$$
\begin{aligned}
&u^{+}_{\varepsilon_{j+1}}(q^{j+1}z_1,p)u^{-}_{j}(q^{j}z_2,p) = \\
&\qquad \frac{(p^*q^{c+2}z;p)_\infty(p^*q^{c}z;p^*)_\infty}{(p^*q^{c+2}z;p^*)_\infty(p^*q^{c}z;p)_\infty}u^{-}_{j}(q^{j}z_2,p)u^{+}_{\varepsilon_{j+1}}(q^{j+1}z_1,p),\\
&u^{+}_{\varepsilon_l}(q^{l}z_1,p)u^{-}_{j}(q^{j}z_2,p) = u^{-}_{j}(q^{j}z_2,p)u^{+}_{\varepsilon_l}(q^{l}z_1,p) \qquad (l\neq j,j+1),\\
&u^{+}_{j}(z_1,p)u^{-}_{j}(z_2,p) = \frac{(pq^{-c-2}z;p)_\infty}{(pq^{-c+2}z;p)_\infty}\frac{(p^*q^{c+2}z;p^*)_\infty}{(p^*q^{c-2}z;p^*)_\infty}u^{-}_{j}(z_2,p)u^{+}_{j}(z_1,p),\\
&u^{+}_{j}(z_1,p)u^{-}_{j+1}(z_2,p) = \frac{(pq^{-c+1}z;p)_\infty}{(pq^{-c-1}z;p)_\infty}\frac{(p^*q^{c-1}z;p^*)_\infty}{(p^*q^{c+1}z;p^*)_\infty}u^{-}_{j+1}(z_2,p)u^{+}_{j}(z_1,p),\\
&u^{+}_{j}(z_1,p)u^{-}_{l}(z_2,p) = u^{-}_{l}(z_2,p)u^{+}_{j}(z_1,p) \qquad (l\neq j,j+1),\\
&u^{+}_{\varepsilon_j}(q^{-c+j}z_1,p)x^{+}_{j}(z_2) = \frac{(p^*q^{-c+j}z;p^*)_\infty}{(p^*q^{-c-2+j}z;p^*)_\infty}x^{+}_{j}(z_2)u^{+}_{\varepsilon_j}(q^{-c+j}z_1,p),\\
&u^{-}_{\varepsilon_j}(q^{j}z_1,p)x^{+}_{j}(z_2) = \frac{(pq^{-c-j}/z;p)_\infty}{(pq^{-c+2-j}/z;p)_\infty}x^{+}_{j}(z_2)u^{-}_{\varepsilon_j}(q^{j}z_1,p),\\
&u^{+}_{\varepsilon_{j+1}}(q^{-c+j+1}z_1,p)x^{+}_{j}(z_2) = \frac{(p^*q^{-c+j}z;p^*)_\infty}{(p^*q^{-c+2+j}z;p^*)_\infty}x^{+}_{j}(z_2)u^{+}_{\varepsilon_{j+1}}(q^{-c+j+1}z_1,p),\\
&u^{-}_{\varepsilon_{j+1}}(q^{j+1}z_1,p)x^{+}_{j}(z_2) = \frac{(pq^{-c-j}/z;p)_\infty}{(pq^{-c-2-j}/z;p)_\infty}x^{+}_{j}(z_2)u^{-}_{\varepsilon_{j+1}}(q^{j+1}z_1,p),\\
&u^{\pm}_{\varepsilon_l}(z_1,p)x^{+}_{j}(z_2) = x^{+}_{j}(z_2)u^{\pm}_{\varepsilon_l}(z_1,p) \qquad (l\neq j,j+1),\\
&u^{+}_{j}(z_1,p)x^{+}_{j}(z_2) = \frac{(p^*q^{2}z;p^*)_\infty}{(p^*q^{-2}z;p^*)_\infty}x^{+}_{j}(z_2)u^{+}_{j}(z_1,p),\\
&u^{+}_{j+1}(z_1,p)x^{+}_{j}(z_2) = \frac{(p^*q^{-1}z;p^*)_\infty}{(p^*qz;p^*)_\infty}x^{+}_{j}(z_2)u^{+}_{j+1}(z_1,p),\\
&u^{-}_{j}(z_1,p)x^{+}_{j}(z_2) = \frac{(pq^{-2-c}z;p)_\infty}{(pq^{2-c}z;p)_\infty}x^{+}_{j}(z_2)u^{-}_{j}(z_1,p),\\
&u^{-}_{j+1}(z_1,p)x^{+}_{j}(z_2) = \frac{(pq^{1-c}z;p)_\infty}{(pq^{-1-c}z;p)_\infty}x^{+}_{j}(z_2)u^{-}_{j+1}(z_1,p),\\
&u^{\pm}_{l}(z_1,p)x^{+}_{j}(z_2) = x^{+}_{j}(z_2)u^{\pm}_{l}(z_1,p) \qquad (l\neq j,j+1),\\
&u^{+}_{\varepsilon_j}(q^{-c+j}z_1,p)x^{-}_{j}(z_2) = \frac{(p^*q^{-2+j}z;p^*)_\infty}{(p^*q^{j}z;p^*)_\infty}x^{-}_{j}(z_2)u^{+}_{\varepsilon_j}(q^{-c+j}z_1,p),\\
&u^{-}_{\varepsilon_j}(q^{j}z_1,p)x^{-}_{j}(z_2) = \frac{(pq^{2-j}/z;p)_\infty}{(pq^{-j}/z;p)_\infty}x^{-}_{j}(z_2)u^{-}_{\varepsilon_j}(q^{j}z_1,p),\\
&u^{+}_{\varepsilon_{j+1}}(q^{-c+j+1}z_1,p)x^{-}_{j}(z_2) = \frac{(p^*q^{2+j}z;p^*)_\infty}{(p^*q^{j}z;p^*)_\infty}x^{-}_{j}(z_2)u^{+}_{\varepsilon_{j+1}}(q^{-c+j+1}z_1,p),\\
&u^{-}_{\varepsilon_{j+1}}(q^{j+1}z_1,p)x^{-}_{j}(z_2) = \frac{(pq^{-2-j}/z;p)_\infty}{(pq^{-j}/z;p)_\infty}x^{-}_{j}(z_2)u^{-}_{\varepsilon_{j+1}}(q^{j+1}z_1,p),
\end{aligned}
$$

$$\begin{aligned}
u^{\pm}_{\varepsilon_l}(z_1,p)x^-_j(z_2) &= x^-_j(z_2)u^{\pm}_{\varepsilon_l}(z_1,p) \qquad (l\neq j,j+1),\\
u^{+}_{j}(z_1,p)x^-_j(z_2) &= \frac{(p^*q^{-2+c}z;p^*)_\infty}{(p^*q^{2+c}z;p^*)_\infty}x^-_j(z_2)u^{+}_{j}(z_1,p),\\
u^{+}_{j+1}(z_1,p)x^-_j(z_2) &= \frac{(p^*q^{1+c}z;p^*)_\infty}{(p^*q^{-1+c}z;p^*)_\infty}x^-_j(z_2)u^{+}_{j+1}(z_1,p),\\
u^{-}_{j}(z_1,p)x^-_j(z_2) &= \frac{(pq^{2}z;p)_\infty}{(pq^{-2}z;p)_\infty}x^-_j(z_2)u^{-}_{j}(z_1,p),\\
u^{-}_{j+1}(z_1,p)x^-_j(z_2) &= \frac{(pq^{-1}z;p)_\infty}{(pqz;p)_\infty}x^-_j(z_2)u^{-}_{j+1}(z_1,p),\\
u^{\pm}_{l}(z_1,p)x^-_j(z_2) &= x^-_j(z_2)u^{\pm}_{l}(z_1,p) \qquad (l\neq j,j+1).
\end{aligned}$$

§Appendix C. From the Basic Gauss Components to the Elliptic Currents

C.1. Relations among the basic Gauss components

Let us consider the $m=j+1, l=j$ case of the $L^{m\pm}_j(z)$ and the subalgebra $E_{q,p}(\widehat{\mathfrak{gl}}_2)$ generated by them.

$$\begin{aligned}
L^{j+1\pm}_j(z) &= \begin{pmatrix} 1 & F^{\pm}_{j,j+1}(z)\\ 0 & 1\end{pmatrix}\begin{pmatrix} K^{\pm}_j(z) & 0\\ 0 & K^{\pm}_{j+1}(z)\end{pmatrix}\begin{pmatrix} 1 & 0\\ E^{\pm}_{j+1,j}(z) & 1\end{pmatrix}\\
&= \begin{pmatrix} K^{\pm}_j(z)+F^{\pm}_{j,j+1}(z)K^{\pm}_{j+1}(z)E^{\pm}_{j+1,j}(z) & F^{\pm}_{j,j+1}(z)K^{\pm}_{j+1}(z)\\ K^{\pm}_{j+1}(z)E^{\pm}_{j+1,j}(z) & K^{\pm}_{j+1}(z)\end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
&L^{j+1\pm}_j(z)^{-1}\\
&= \begin{pmatrix} K^{\pm}_j(z)^{-1} & -K^{\pm}_j(z)^{-1}F^{\pm}_{j,j+1}(z)\\ -E^{\pm}_{j+1,j}(z)K^{\pm}_j(z)^{-1} & E^{\pm}_{j+1,j}(z)K^{\pm}_j(z)^{-1}F^{\pm}_{j,j+1}(z)+K^{\pm}_{j+1}(z)^{-1}\end{pmatrix}.
\end{aligned}$$

From $(j,j),(j,j)$ of (6.26) and $(j+1,j+1),(j+1,j+2)$ of (6.23)

$$K^{\pm}_l(z_1)K^{\pm}_l(z_2)=\rho(z)K^{\pm}_l(z_2)K^{\pm}_l(z_1)\ (l=j,j+1), \tag{C.1}$$

where $z=z_1/z_2$. Similarly, from $(j,j),(j,j)$ of (6.28) and $(j+1,j+1)$, $(j+1,j+1)$ of (6.24)

$$K^{\pm}_l(z_1)K^{\mp}_l(z_2)=\frac{\rho^{\pm *}(zq^{\mp c})}{\rho^{\pm}(zq^{\pm c})}K^{\mp}_l(z_2)K^{\pm}_l(z_1),\ (l=j,j+1). \tag{C.2}$$

From $(j+1,j),(j+1,j)$ of (6.25)

$$K^{\pm}_j(z_2)K^{\pm}_{j+1}(z_1)=\rho(z)\frac{\bar{b}(z)}{\bar{b}^*(z)}K^{\pm}_{j+1}(z_1)K^{\pm}_j(z_2). \tag{C.3}$$

Inparticular, in the limit $z_1 \to z_2$, we have

$$K^{\pm}_{j}(z_2)K^{\pm}_{j+1}(z_2)=\frac{(p;p)^3_\infty}{(p^*;p^*)^3_\infty}\frac{\Theta_{p^*}(q^2)}{\Theta_p(q^2)}K^{\pm}_{j+1}(z_2)K^{\pm}_{j}(z_2). \tag{C.4}$$

From $(j+1,j),(j+1,j)$ of (6.27)

$$K^{\pm}_{j+1}(z_1)K^{\mp}_{j}(z_2)=\frac{\rho^{\pm *}(zq^{\mp c})}{\rho^{\pm}(zq^{\pm c})}\frac{\overline{b}^*(zq^{\mp c})}{\overline{b}(zq^{\pm c})}K^{\mp}_{j}(z_2)K^{\pm}_{j+1}(z_1). \tag{C.5}$$

From $(j+1,j+1),(j+1,j)$ of (6.23), we obtain

$$\begin{aligned} &K^{\pm}_{j+1}(z_1)^{-1}E^{\pm}_{j+1,j}(z_2)K^{\pm}_{j+1}(z_1)\\ &\quad=E^{\pm}_{j+1,j}(z_2)\frac{1}{\overline{b}^*(z)}-E^{\pm}_{j+1,j}(z_1)\frac{c^*(z,P_{j,j+1})}{\overline{b}^*(z)}. \end{aligned} \tag{C.6}$$

From $(j+1,j+1),(j+1,j)$ of (6.24), we obtain

$$\begin{aligned} &K^{\pm}_{j+1}(z_1)^{-1}E^{\mp}_{j+1,j}(z_2)K^{\pm}_{j+1}(z_1)\\ &\quad=E^{\mp}_{j+1,j}(z_2)\frac{1}{\overline{b}^*(zq^{\mp c})}-E^{\pm}_{j+1,j}(z_1)\frac{c^*(zq^{\mp c},P_{j,j+1})}{\overline{b}^*(zq^{\mp c})}. \end{aligned} \tag{C.7}$$

Similarly, from $(j+1,j),(j+1,j+1)$ of (6.23),

$$\begin{aligned} &K^{\pm}_{j+1}(z_1)F^{\pm}_{j,j+1}(z_2)K^{\pm}_{j+1}(z_1)^{-1}\\ &\quad=\frac{1}{\overline{b}(z)}F^{\pm}_{j,j+1}(z_2)-\frac{\overline{c}(z,(P+h)_{j,j+1})}{\overline{b}(z)}F^{\pm}_{j,j+1}(z_1). \end{aligned} \tag{C.8}$$

From $(j+1,j),(j+1,j+1)$ of (6.24),

$$\begin{aligned} &K^{\pm}_{j+1}(z_1)F^{\mp}_{j,j+1}(z_2)K^{\pm}_{j+1}(z_1)^{-1}\\ &\quad=\frac{1}{\overline{b}(zq^{\pm c})}F^{\mp}_{j,j+1}(z_2)-\frac{\overline{c}(zq^{\pm c},(P+h)_{j,j+1})}{\overline{b}(zq^{\pm c})}F^{\pm}_{j,j+1}(z_1). \end{aligned} \tag{C.9}$$

From $(j+1,j),(j,j)$ of (6.26)

$$\begin{aligned} &K^{\pm}_{j}(z_2)^{-1}E^{\pm}_{j+1,j}(z_1)K^{\pm}_{j}(z_2)\\ &\quad=\frac{1}{\overline{b}^*(z)}E^{\pm}_{j+1,j}(z_1)-\frac{\overline{c}^*(z,P_{j,j+1})}{\overline{b}^*(z)}E^{\pm}_{j+1,j}(z_2). \end{aligned} \tag{C.10}$$

From $(j+1,j),(j,j)$ of (6.28)

$$\begin{aligned} &K^{\mp}_{j}(z_2)^{-1}E^{\pm}_{j+1,j}(z_1)K^{\mp}_{j}(z_2)\\ &\quad=\frac{1}{\overline{b}^*(zq^{\mp c})}E^{\pm}_{j+1,j}(z_1)-\frac{\overline{c}^*(zq^{\mp c},P_{j,j+1})}{\overline{b}^*(zq^{\mp c})}E^{\mp}_{j+1,j}(z_2). \end{aligned} \tag{C.11}$$

From $(j,j),(j+1,j)$ of (6.26)

$$\begin{aligned} &K_j^{\pm}(z_2)F^{\pm}_{j,j+1}(z_1)K_j^{\pm}(z_2)^{-1} \\ &\quad = F^{\pm}_{j,j+1}(z_1)\frac{1}{\bar{b}(z)} - F^{\pm}_{j,j+1}(z_2)\frac{\bar{c}(z,(P+h)_{j,j+1})}{\bar{b}(z)}. \end{aligned} \tag{C.12}$$

From $(j+1,j+1),(j,j)$ of (6.23), we obtain

$$\begin{aligned} &K^{\pm}_{j+1}(z_2)^{-1}E^{\pm}_{j+1,j}(z_1)K^{\pm}_{j+1}(z_2)E^{\pm}_{j+1,j}(z_2) \\ &\quad = K^{\pm}_{j+1}(z_1)^{-1}E^{\pm}_{j+1,j}(z_2)K^{\pm}_{j+1}(z_1)E^{\pm}_{j+1,j}(z_1). \end{aligned} \tag{C.13}$$

From (C.6) and (C.13), we obtain

$$\begin{aligned} &E^{\pm}_{j+1,j}(z_1)E^{\pm}_{j+1,j}(z_2)\frac{1}{\bar{b}^*(1/z)} - E^{\pm}_{j+1,j}(z_2)^2\frac{c^*(1/z,P_{j,j+1}-2)}{\bar{b}^*(1/z)} \\ &\quad = E^{\pm}_{j+1,j}(z_2)E^{\pm}_{j+1,j}(z_1)\frac{1}{\bar{b}^*(z)} - E^{\pm}_{j+1,j}(z_1)^2\frac{c^*(z,P_{j,j+1}-2)}{\bar{b}^*(z)}. \end{aligned} \tag{C.14}$$

From $(j+1,j+1),(j,j)$ of (6.24), we obtain

$$\begin{aligned} &K^{\mp}_{j+1}(z_2)^{-1}E^{\pm}_{j+1,j}(z_1)K^{\mp}_{j+1}(z_2)E^{\mp}_{j+1,j}(z_2) \\ &\quad = K^{\pm}_{j+1}(z_1)^{-1}E^{\mp}_{j+1,j}(z_2)K^{\pm}_{j+1}(z_1)E^{\pm}_{j+1,j}(z_1). \end{aligned} \tag{C.15}$$

From (C.7) and (C.15), we obtain

$$\begin{aligned} &E^{\pm}_{j+1,j}(z_1)E^{\mp}_{j+1,j}(z_2)\frac{1}{\bar{b}^*(1/(zq^{\mp c}))} - E^{\mp}_{j+1,j}(z_2)^2\frac{c^*(1/(zq^{\mp c}),P_{j,j+1}-2)}{\bar{b}^*(1/(zq^{\mp c}))} \\ &\quad = E^{\mp}_{j+1,j}(z_2)E^{\pm}_{j+1,j}(z_1)\frac{1}{\bar{b}^*(zq^{\mp c})} - E^{\pm}_{j+1,j}(z_1)^2\frac{c^*(zq^{\mp c},P_{j,j+1}-2)}{\bar{b}^*(zq^{\mp c})}. \end{aligned} \tag{C.16}$$

In a similar way, we obtain

$$\begin{aligned} &F^{\pm}_{j,j+1}(z_1)F^{\pm}_{j,j+1}(z_2)\frac{1}{\bar{b}(z)} - F^{\pm}_{j,j+1}(z_1)^2\frac{\bar{c}(z,(P+h)_{j,j+1}-2)}{\bar{b}(z)} \\ &\quad = F^{\pm}_{j,j+1}(z_2)F^{\pm}_{j,j+1}(z_1)\frac{1}{\bar{b}(1/z)} - F^{\pm}_{j,j+1}(z_2)^2\frac{\bar{c}(1/z,(P+h)_{j,j+1}-2)}{\bar{b}(1/z)}, \end{aligned} \tag{C.17}$$

(C.18)

$$\begin{aligned}
&F^{\pm}_{j,j+1}(z_1)F^{\mp}_{j,j+1}(z_2)\frac{1}{\bar{b}(zq^{\pm c})}+F^{\pm}_{j,j+1}(z_1)^2\frac{\bar{c}(zq^{\pm c},(P+h)_{j,j+1}-2)}{\bar{b}(zq^{\pm c})}\\
&\qquad=F^{\mp}_{j,j+1}(z_2)F^{\pm}_{j,j+1}(z_1)\frac{1}{\bar{b}(1/(zq^{\pm c}))}\\
&\qquad\quad-F^{\pm}_{j,j+1}(z_1)^2\frac{\bar{c}(1/(zq^{\pm c}),(P+h)_{j,j+1}-2)}{\bar{b}(1/(zq^{\pm c}))}.
\end{aligned}$$

From $(j+1,j),(j,j+1)$ of (6.23),

$$\begin{aligned}
&\rho^{\pm}(z)\left\{\bar{b}(z)K^{\pm}_{j+1}(z_1)E^{\pm}_{j+1,j}(z_1)F^{\pm}_{j,j+1}(z_2)K^{\pm}_{j+1}(z_2)\right.\\
&\left.+\bar{c}(z,(P+h)_{j,j+1})\left(K^{\pm}_{j}(z_1)+F^{\pm}_{j,j+1}(z_1)K^{\pm}_{j+1}(z_1)E^{\pm}_{j+1,j}(z_1)\right)K^{\pm}_{j+1}(z_2)\right\}\\
&=\rho^{*\pm}(z)\left\{\left(K^{\pm}_{j}(z_2)+F^{\pm}_{j,j+1}(z_2)K^{\pm}_{j+1}(z_2)E^{\pm}_{j+1,j}(z_2)\right)K^{\pm}_{j+1}(z_1)\bar{c}^{*}(z,P_{j,j+1})\right.\\
&\left.+F^{\pm}_{j,j+1}(z_2)K^{\pm}_{j+1}(z_2)K^{\pm}_{j+1}(z_1)E^{\pm}_{j+1,j}(z_1)b^{*}(z,P_{j,j+1})\right\}.
\end{aligned}$$

Using $(j+1,j),(j+1,j+1)$ and $(j+1,j+1),(j,j+1)$ of (6.23), we get

(C.19)

$$\begin{aligned}
&[E^{\pm}_{j+1,j}(z_1),F^{\pm}_{j,j+1}(z_2)]\\
&=K^{\pm}_{j}(z_2)K^{\pm}_{j+1}(z_2)^{-1}\frac{\bar{c}^{*}(z,P_{j,j+1}-1)}{\bar{b}^{*}(z)}\\
&\qquad-K^{\pm}_{j+1}(z_1)^{-1}K^{\pm}_{j}(z_1)\frac{\bar{c}(z,(P+h)_{j,j+1}-1)}{\bar{b}(z)}.
\end{aligned}$$

Similarly, from $(j+1,j),(j,j+1)$ of (6.24),

$$\begin{aligned}
&\rho^{\pm}(zq^{\pm c})\left(\bar{b}(zq^{\pm c})K^{\pm}_{j+1}(z_1)E^{\pm}_{j+1,j}(z_1)F^{\mp}_{j,j+1}(z_2)K^{\mp}_{j+1}(z_2)\right.\\
&\left.+\bar{c}(zq^{\pm c},(P+h)_{j,j+1})(K^{\pm}_{j}(z_1)+F^{\pm}_{j,j+1}(z_1)K^{\pm}_{j+1}(z_1)E^{\pm}_{j+1,j}(z_1))K^{\mp}_{j+1}(z_2)\right)\\
&=\rho^{*\pm}(zq^{\mp c})\left((K^{\mp}_{j}(z_2)+F^{\mp}_{j,j+1}(z_2)K^{\mp}_{j+1}(z_2)E^{\mp}_{j+1,j}(z_2))\right.\\
&\qquad\qquad K^{\pm}_{j+1}(z_1)\bar{c}^{*}(zq^{\mp c},P_{j,j+1})\\
&\left.+F^{\mp}_{j,j+1}(z_2)K^{\mp}_{j+1}(z_2)K^{\pm}_{j+1}(z_1)E^{\pm}_{j+1,j}(z_1)b^{*}(zq^{\mp c},P_{j,j+1})\right).
\end{aligned}$$

Using $(j+1,j),(j+1,j+1)$ and $(j+1,j+1),(j,j+1)$ of (6.24), we get

(C.20)

$$\begin{aligned}
&[E^{\pm}_{j+1,j}(z_1),F^{\mp}_{j,j+1}(z_2)]\\
&=K^{\mp}_{j}(z_2)K^{\mp}_{j+1}(z_2)^{-1}\frac{\bar{c}^{*}(zq^{\mp c},P_{j,j+1}-1)}{\bar{b}^{*}(zq^{\mp c})}\\
&\qquad-K^{\pm}_{j+1}(z_1)^{-1}K^{\pm}_{j}(z_1)\frac{\bar{c}(zq^{\pm c},(P+h)_{j,j+1}-1)}{\bar{b}(zq^{\pm c})}.
\end{aligned}$$

C.2. Identification with the elliptic currents

In this section we demonstrate an identification of a combination of the basic Gauss components with the elliptic currents of $U_{q,p}(\widehat{\mathfrak{gl}}_N)$.

Let us define the total current by $E_j(zq^{j-c/2}) = \mu^*(E^+_{j+1,j}(z^+) - E^-_{j+1,j}(z^-))$ with $\mu^* = \frac{(p^*;p^*)^3_\infty}{a^*_{j+1,j}\Theta_{p^*}(q^2)}$. Then we obtain from (C.6) and (C.7)

$$
\begin{aligned}
\text{(C.21)}\quad & K^+_{j+1}(z_1)^{-1}E_j(z_2q^{j-c/2})K^+_{j+1}(z_1)\\
&= K^+_{j+1}(z_1)^{-1}\mu^*\left(E^+_{j+1,j}(z_2^+) - E^-_{j+1,j}(z_2^-)\right)K^+_{j+1}(z_1)\\
&= \mu^*\Bigg(E^+_{j+1,j}(z_2^+)\frac{1}{\bar{b}^*(zq^{-c/2})} - E^+_{j+1,j}(z_1)\frac{c^*(zq^{-c/2},P_{j,j+1})}{\bar{b}^*(zq^{-c/2})}\\
&\qquad -E^-_{j+1,j}(z_2^-)\frac{1}{\bar{b}^*(zq^{-c/2})} + E^+_{j+1,j}(z_1)\frac{c^*(zq^{-c/2},P_{j,j+1})}{\bar{b}^*(zq^{-c/2})}\Bigg)\\
&= E_j(z_2q^{j-c/2})\frac{1}{\bar{b}^*(zq^{-c/2})}.
\end{aligned}
$$

Comparing this with (3.31), we identify $E_j(z), K^+_j(z)$ with $e_j(z), k^+_j(z)$, respectively.

Next, inserting (C.14) and (C.16) into $(E^+_{j+1,j}(z_1^+) - E_{j+1,j}(z_1^-))$ $\times\, (E^+_{j+1,j}(z_2^+) - E_{j+1,j}(z_2^-))$, we obtain

$$
E_j(z_1)E_j(z_2) = -\frac{1}{z}\frac{\Theta_{p^*}(zq^2)}{\Theta_{p^*}(q^2/z)}E_j(z_2)E_j(z_1).
$$

This is consistent to (3.34).

To obtain (3.35) we use (6.36), which is derived from (6.27) with $m = l+2$.

Similarly, if we define $F_j(zq^{j-c/2}) = \mu(F^+_{j,j+1}(z_-) - F^-_{j,j+1}(z_+))$ with $\mu = \frac{(p;p)^3_\infty}{a_{j,j+1}\Theta_p(q^2)}$, we recover the relations (3.36)-(3.37) from (C.8)-(C.9), (C.17)-(C.18) and the F^+ counterpart of (6.36). Hence we identify $F_j(z)$ with $f_j(z)$.

Finally let us check the relation (3.23). From (C.19) and (C.20), we have

$$
\begin{aligned}
\text{(C.22)}\quad & (\mu\mu^*)^{-1}[E(z_1),F(z_2)]\\
&= [E^+_{j+1,j}(z_1^+) - E^-_{j+1,j}(z_1^-), F^+_{j,j+1}(z_2^-) - F^-_{j,j+1}(z_2^+)]\\
&= [E^+_{j+1,j}(z_1^+), F^+_{j,j+1}(z_2^-)] + [E^-_{j+1,j}(z_1^-), F^-_{j,j+1}(z_2^+)]\\
&\qquad -[E^+_{j+1,j}(z_1^+), F^-_{j,j+1}(z_2^+)] - [E^-_{j+1,j}(z_1^-), F^+_{j,j+1}(z_2^-)].
\end{aligned}
$$

Let us substitute (C.19) and (C.20) into this. Noting the remark below Proposition 3.8, one finds that the terms containing $K^-_j(z_2^+)K^-_{j+1}(z_2^+)^{-1}$ from the 2nd and 3rd terms in (C.22) cancel out each other and the same is true for the terms containing $K^+_{j+1}(z_1^+)^{-1}K^+_j(z_1^+)$ from the 1st and the 3rd terms in (C.22). We obtain

$$\begin{aligned}
&(\mu\mu^*)^{-1}[E_j(z_1),F_j(z_2)]\\
&= K^+_j(z_2^-)K^+_{j+1}(z_2^-)^{-1}q^{-1}\Theta_{p^*}(q^2)\\
&\times\left(\left.\frac{\Theta_{p^*}(q^{-2(P_{j,j+1}-1)}q^cz)}{\Theta_{p^*}(q^{-2(P_{j,j+1}-1)})\Theta_{p^*}(zq^c)}\right|_+ - \left.\frac{\Theta_{p^*}(q^{-2(P_{j,j+1}-1)}q^cz)}{\Theta_{p^*}(q^{-2(P_{j,j+1}-1)})\Theta_{p^*}(zq^c)}\right|_-\right)\\
&-K^-_{j+1}(z_1^-)^{-1}K^-_j(z_1^-)q^{-1}\Theta_p(q^2)\\
&\times\left(\left.\frac{\Theta_p(q^{-2((P+h)_{j,j+1}-1)}q^{-c}z)}{\Theta_p(q^{-2((P+h)_{j,j+1}-1)})\Theta_p(zq^{-c})}\right|_+ - \left.\frac{\Theta_p(q^{-2((P+h)_{j,j+1}-1)}q^{-c}z)}{\Theta_p(q^{-2((P+h)_{j,j+1}-1)})\Theta_p(zq^{-c})}\right|_-\right).
\end{aligned}$$

Here

$$\text{(C.23)}\quad \left.\frac{\Theta_p(q^{2s}z)(p;p)^3_\infty}{\Theta_p(q^{2s})\Theta_p(z)}\right|_+ \begin{aligned}[t]
&= \frac{\Theta_p(q^{2s}z)(p;p)^3_\infty}{\Theta_p(q^{2s})\Theta_p(z)}\\
&= \sum_{n\in\mathbb{Z}}\frac{1}{1-q^{2s}p^n}z^n\\
&= \sum_{l\in\mathbb{Z}_{\geq 0}}\left(\frac{q^{2sl}}{1-p^lz}-\frac{q^{-2s(l+1)}p^{l+1}/z}{1-p^{l+1}/z}\right)
\end{aligned}$$

for $|p|<|z|<1$ and

$$\text{(C.24)}\quad \left.\frac{\Theta_p(q^{2s}z)(p;p)^3_\infty}{\Theta_p(q^{2s})\Theta_p(z)}\right|_- \begin{aligned}[t]
&= -\frac{\Theta_p(q^{-2s}/z)(p;p)^3_\infty}{\Theta_p(q^{-2s})\Theta_p(1/z)}\\
&= -\sum_{n\in\mathbb{Z}}\frac{1}{1-q^{-2s}p^n}z^{-n}\\
&= -\sum_{l\in\mathbb{Z}_{\geq 0}}\left(\frac{q^{-2sl}p^l/z}{1-p^l/z}-\frac{q^{2s(l+1)}}{1-p^{l+1}z}\right)
\end{aligned}$$

for $1<|z|<|p^{-1}|$. Then the difference between these two expansions turns out to be the formal delta function $\delta(z)=\sum_{n\in\mathbb{Z}}z^n$. We thus obtain

$$\begin{aligned}
&[E_j(z_1), F_j(z_2)]\\
&= \mu\mu^* q^{-1} \left\{ K_j^+(z_2^-)K_{j+1}^+(z_2^-)^{-1} \frac{\Theta_{p^*}(q^2)}{(p^*;p^*)_\infty^3} \delta\left(\frac{z_1}{z_2}q^c\right) \right.\\
&\qquad\qquad \left. - K_{j+1}^-(z_1^-)^{-1} K_j^-(z_1^-) \frac{\Theta_p(q^2)}{(p;p)_\infty} \delta\left(\frac{z_1}{z_2}q^{-c}\right) \right\}\\
&= -\frac{\kappa}{q-q^{-1}} \left\{ K_j^+(z_2^-)K_{j+1}^+(z_2^-)^{-1} \delta\left(\frac{z_1}{z_2}q^c\right) \right.\\
&\qquad\qquad \left. - K_j^-(z_1^-) K_{j+1}^-(z_1^-)^{-1} \delta\left(\frac{z_1}{z_2}q^{-c}\right) \right\}.
\end{aligned}$$

In the last line we used (C.4) and

$$q^{-1}\mu\mu^* \frac{\Theta_{p^*}(q^2)}{(p^*;p^*)_\infty^3} = -\frac{\kappa}{q-q^{-1}}.$$

This is consistent to (3.23).

§Appendix D. Relation to Jimbo-Miwa-Okado's Face Weight

In this Appendix we give a relationship between our R-matrix (2.3) and Jimbo-Miwa-Okado's $A_{n-1}^{(1)}$ type face weight in [33].

D.1. Fractional powers in z

So far we have defined the elliptic algebras $U_{q,p}(\widehat{\mathfrak{gl}}_N)$ and $E_{q,p}(\widehat{\mathfrak{gl}}_N)$ by using the generating functions $e_j(z), f_j(z), k_j(z)$ and $L_{ij}(z)$, respectively. The coefficients of their relations are given in terms of the theta function $\Theta_p(z)$, which enables us to expand every things to power series in p.

However for a practical use it is convenient to introduce operators such as $z^{\pm\frac{P+h-1}{r}}$ and $z^{\pm\frac{P-1}{r^*}}$ into the algebras. Here r and r^* are introduced by $p = q^{2r}$ and $p^* = pq^{-2c} = q^{2r^*}$ with $r^* = r - c$. The main reason for this can be seen in the following example. Let us consider Jacobi's theta function

$$\vartheta_1(u,\tau) = i\sum_{n=-\infty}^{\infty} (-1)^n e^{\pi i\tau(n-1/2)^2} e^{2\pi i u(n-1/2)}.$$

Identifying $z = q^{2u}$, $p = e^{-\frac{2\pi i}{\tau}}$ we have

$$\vartheta_1\left(\frac{u}{r},\tau\right) = e^{\frac{\pi i}{4}}\tau^{-\frac{1}{2}} p^{\frac{1}{8}} q^{\frac{u^2}{r}-u}\Theta_p(z).$$

Then we have

$$z^{\frac{s-1}{r}}c(z,s)=z^{\frac{s-1}{r}}\frac{\Theta_p(q^2)\Theta_p(q^{2s}z)}{\Theta_p(q^{2s})\Theta_p(q^2z)}=\frac{\vartheta_1(\frac{1}{r},\tau)\vartheta_1(\frac{s+u}{r},\tau)}{\vartheta_1(\frac{s}{r},\tau)\vartheta_1(\frac{1+u}{r},\tau)}. \tag{D.1}$$

This is invariant under the shift $z\mapsto zp$ i.e. $u\mapsto u+r$. Compare this with (3.73).

Motivated by this, let us consider the transformation

$$\widehat{R}^{\pm}(z,s)=\left(\mathrm{Ad}z^{-\frac{1}{r}\theta_V(s)}\otimes\mathrm{id}\right)\left(z^{\frac{1}{r}T_{V,V}}R^{\pm}(z,s)\right), \tag{D.2}$$

where $\theta_V(s)$ is given in (3.69) and

$$T_{V,V}=\sum_{j=1}^{N-1}\pi_V(h_j)\otimes\pi_V(h^j)$$

Then one finds

$$\begin{aligned}\widehat{R}^{\pm}(u,s)=&\ \widehat{\rho}^{\pm}(u)\widehat{\overline{R}}(u,s),\\ \widehat{\overline{R}}(u,s)=&\sum_{j=1}^{N}E_{jj}\otimes E_{jj}+\sum_{1\le j<l\le N}\left(\widehat{b}(z,s_{j,l})E_{jj}\otimes E_{ll}+\widehat{\overline{b}}(z)E_{ll}\otimes E_{jj}\right)\\ &+\sum_{1\le j<l\le N}\left(\widehat{c}(z,s_{j,l})E_{jl}\otimes E_{lj}+\widehat{c}(z,-s_{j,l})E_{lj}\otimes E_{jl}\right),\end{aligned} \tag{D.3}$$

with

$$\widehat{\rho^{+}}(u)=q^{-\frac{N-1}{N}}z^{\frac{N-1}{rN}}\frac{\{q^2z\}\{q^{2N-2}z\}\{p/z\}\{pq^{2N}/z\}}{\{z\}\{q^{2N}z\}\{pq^2/z\}\{pq^{2N-2}/z\}}, \tag{D.4}$$

$$\widehat{\rho^{-}}(u)=q^{\frac{N-1}{N}}z^{\frac{N-1}{rN}}\frac{\{pq^2z\}\{pq^{2N-2}z\}\{1/z\}\{q^{2N}/z\}}{\{pz\}\{pq^{2N}z\}\{q^2/z\}\{q^{2N-2}/z\}}, \tag{D.5}$$

$$\widehat{b}(u,s)=\frac{[s+1][s-1][u]}{[s]^2[u+1]},\ \widehat{\overline{b}}(u)=\frac{[u]}{[u+1]}, \tag{D.6}$$

$$\widehat{c}(u,\pm s)=\frac{[1][s+u]}{[s][u+1]}, \tag{D.7}$$

where we set

$$[u]=\vartheta_1\left(\frac{u}{r},\tau\right).$$

We also need for $p^*=e^{-\frac{2\pi i}{\tau^*}}$

$$[u]^*=\vartheta_1\left(\frac{u}{r^*},\tau^*\right).$$

We have

$$\text{(D.8)}\quad \widehat{\rho}^+(u+r)=\widehat{\rho}^-(u),\qquad \frac{\widehat{\rho}^+(u)}{\widehat{\rho}^{+*}(u)}=\frac{\widehat{\rho}^-(u)}{\widehat{\rho}^{-*}(u)},\qquad \widehat{\rho}^+(u)\widehat{\rho}^-(-u)=1,$$

$$\text{(D.9)}\quad \widehat{R}^-(u,s)^{-1}=P\widehat{R}^+(-u,s)P$$

for $\widehat{\rho}^{\pm*}(u)=\widehat{\rho}^{\pm}(u)|_{r\to r^*,p\to p^*}$. Hence we obtain

Proposition D.1.

$$\text{(D.10)}\qquad \widehat{R}^+(u+r,s)=\widehat{R}^-(u,s).$$

Let us further define $\widehat{L}^{\pm}(u)$ from $L^+(z)$ in Sec.3.2 by

$$\text{(D.11)}\qquad \begin{aligned}\widehat{L}^+(u)&=\left(z^{-\frac{1}{r}\theta_V(P)}\otimes\mathrm{id}\right)z^{\frac{1}{r}T_V}L^+(z)\left(z^{\frac{1}{r^*}\theta_V(P)}\otimes\mathrm{id}\right),\\ \widehat{L}^-(u)&=\widehat{L}^+(u+r^*+c/2),\end{aligned}$$

where $z=q^{2u}$, and T_V is given in (3.70). Note that we do not need the extra Ad $q^{-2\theta_V(P)}$ action to define $\widehat{L}^-(u)$ unlike (3.68).

Proposition D.2. $\widehat{L}^{\pm}(u)$ *satisfy the same relations as* (3.64), (3.71) *and* (3.72) *with replacing* $R^{\pm}(z,s)$ *by* $\widehat{R}^{\pm}(u,s)$. *Namely,*

$$\text{(D.12)}\qquad \widehat{R}^{\pm}(u,P+h)\widehat{L}^{\pm}(u_1)\widehat{L}^{\pm}(u_2)=\widehat{L}^{\pm}(u_2)\widehat{L}^{\pm}(u_1)\widehat{R}^{\pm*}(u,P),$$

$$\text{(D.13)}\qquad \begin{aligned}&\widehat{R}^{\pm}(u\pm c/2,P+h)\widehat{L}^{\pm}(u_1)\widehat{L}^{\mp}(u_2)\\ &\qquad=\widehat{L}^{\mp}(u_2)\widehat{L}^{\pm}(u_1)\widehat{R}^{\pm*}(u\mp c/2,P),\end{aligned}$$

where $u=u_1-u_2$.

Remark. The R-matrices and L-operators in the previous works [47, 36, 43, 50] are $\widehat{R}^{\pm}(u,s)$ and $\widehat{L}^{\pm}(u)$ given in this section except for the prefactor: $\rho^{\pm}(u)$ in the previous works are $\widehat{\rho}^{\mp}(u)$ in (D.5) and (D.4).

D.2. Gauge transformation from Jimbo-Miwa-Okado's W

For $1\le j\le N$ let us define $F(P,P+\widehat{j})$ by

$$F(s,s+\widehat{j})\quad=\quad\left(\prod_{m=j+1}^{N}\frac{[s_{j,m}+1]}{[s_{j,m}]}\right)^{\frac{1}{2}},$$

where $s_{j,m} = s_{\bar{\epsilon}_j} - s_{\bar{\epsilon}_m}$ as in Sec.2.

Our R-matrix is related to Jimbo-Miwa-Okado's W denoted by W_{JMO} as follows. Let us set

$$\widehat{R}^{+}(u,s)^{ij}_{kl} = W\left(\begin{matrix} s & s+\widehat{i} \\ s+\widehat{l} & a+\widehat{i}+\widehat{j}\end{matrix}\middle|\, u\right) \qquad (i+j=k+l).$$

Then we have

$$W\left(\begin{matrix} s & s+\widehat{i} \\ s+\widehat{l} & s+\widehat{i}+\widehat{j}\end{matrix}\middle|\, u\right) = \widehat{\rho}^{+}(u)\frac{[1]}{[u+1]}\frac{F(s,s+\widehat{i})F(s+\widehat{i},s+\widehat{i}+\widehat{j})}{F(,s+\widehat{l})F(s+\widehat{l},s+\widehat{l}+\widehat{k})}$$
$$\times W_{JMO}\left(\begin{matrix} s & s+\widehat{i} \\ s+\widehat{l} & s+\widehat{i}+\widehat{j}\end{matrix}\middle|\, u\right).$$

Here $(s+\widehat{i})_{\bar{\epsilon}_j} = s_{\bar{\epsilon}_i} + \delta_{i,j}$ etc. and

$$W_{JMO}\left(\begin{matrix} s & s+\widehat{i} \\ s+\widehat{i} & s+2\widehat{i}\end{matrix}\middle|\, u\right) = \frac{[1+u]}{[u]},$$
$$W_{JMO}\left(\begin{matrix} s & s+\widehat{i} \\ s+\widehat{i} & s+\widehat{i}+\widehat{j}\end{matrix}\middle|\, u\right) = \frac{[s_{i,j}-u]}{[s_{i,j}]},$$
$$W_{JMO}\left(\begin{matrix} s & s+\widehat{j} \\ s+\widehat{i} & s+\widehat{i}+\widehat{j}\end{matrix}\middle|\, u\right) = \frac{[u]}{[1]}\left(\frac{[s_{i,j}+1][s_{i,j}-1]}{[s_{i,j}]^2}\right)^{1/2}.$$

§Appendix E. Elliptic Quantum Determinant

E.1. Jimbo-Kuniba-Miwa-Okado's projection

Quantum determinant of the L-operator depends on a choice of the gauge for the R-matrix. Let $\mathcal{V} = \mathbb{F}\otimes_{\mathbb{C}} V,\ V = \oplus_{a=1}^{N}\mathbb{C}v_a$ and $E_{i,j}v_a = \delta_{j,a}v_i$. Let us consider the following R matrix $R'(z,s) \in \mathrm{End}(\mathcal{V}\otimes\mathcal{V})$ given by

(E.1)

$$R'(u,s)$$
$$=\rho^{+'}(u)\left[\sum_{j=1}^{N}\alpha(u)E_{jj}\otimes E_{jj}+\sum_{j\neq l}\Big(\beta(u,s_{l,j})E_{jj}\otimes E_{ll}+\gamma(u,s_{l,j})E_{jl}\otimes E_{lj}\Big)\right],$$

where

$$\rho^{+'}(u) = \widehat{\rho}^{+}(u)\frac{[1]}{[u+1]},$$
$$\alpha(u) = \frac{[u+1]}{[1]},\quad \beta(u,s) = \frac{[u][s+1]}{[1][s]},\quad \gamma(u,s) = \frac{[s-u]}{[s]}.$$

Note that $R'(u,s)^{ij}_{kl} = W\left(\begin{array}{cc} s & s+\widehat{i} \\ s+\widehat{l} & s+\widehat{i}+\widehat{j} \end{array}\middle| u\right)$ is the face weight in [34], which is gauge equivalent to $\widehat{R}^+(u,s)$ in (D.3).

Instead of giving the gauge transformation of the R matrix we give the gauge transformation of the L-operator $\widehat{L}^+(u)$ in (D.11) to the one satisfying the RLL relation with the new R matrix (E.1). Namely we define $L(u) = \sum_{1\le i,j\le N} E_{i,j} L_{ij}(u)$ by

$$L_{ij}(u) = \prod_{m=i+1}^{N} \frac{[(P+h)_{im}+1]}{[1]}\, \widehat{L}^+_{ij}(u) \prod_{n=1}^{j-1} \frac{[1]^*}{[P_{nj}+1]^*}. \tag{E.2}$$

Then from (D.12) we obtain

Proposition E.1.

$$\begin{aligned} &R'^{(12)}(u_1-u_2, P+h) L^{(1)}(u_1) L^{(2)}(u_2) \\ &\quad = L^{(2)}(u_2) L^{(1)}(u_1) R'^{*(12)}(u_1-u_2, P). \end{aligned} \tag{E.3}$$

One has

$$R'(-1,s) = \rho_0 \sum_{j\neq l} \frac{[s_{l,j}+1]}{[s_{l,j}]} \Big(E_{jj}\otimes E_{ll} - E_{jl}\otimes E_{lj} \Big),$$

where

$$\rho_0 = -\lim_{u\to -1\ (z\to q^{-2})} \rho^{+'}(u).$$

Hence

$$R'(-1,s) v_a\otimes v_b = \rho_0 \left(\tfrac{[s_{b,a}+1]}{[s_{b,a}]} v_a\otimes v_b - \tfrac{[s_{a,b}+1]}{[s_{a,b}]} v_b\otimes v_a \right) \in \mathcal{V}\wedge\mathcal{V}.$$

In order to generalize this it is convenient to consider the 'transposition' of $R'(u,s)$:

$$R(u,s) = {}^{t_1 t_2} R'^{(21)}(u,s), \qquad R^*(u,s) = R(u,s)|_{r\to r^*, p\to p^*}. \tag{E.4}$$

In fact this yields

$$R(-1,s) = \rho_0 \sum_{j\neq l} \frac{[s_{l,j}+1]}{[s_{l,j}]} \Big(E_{ll}\otimes E_{jj} - E_{jl}\otimes E_{lj} \Big).$$

Hence

$$R(-1,s) v_a\otimes v_b = \rho_0 \frac{[s_{a,b}+1]}{[s_{a,b}]} (v_a\otimes v_b - v_b\otimes v_a) \in \mathcal{V}\wedge\mathcal{V}.$$

Accordingly taking the transpositions ^{t_1} and ^{t_2} of (E.3), flipping the two tensor components and exchanging u_1 and u_2, we obtain

$$\begin{aligned} R^{*(12)}(u_2-u_1,P)\ {}^tL^{(1)}(u_1){}^tL^{(2)}(u_2) \\ = {}^tL^{(2)}(u_2){}^tL^{(1)}(u_1)R^{(12)}(u_2-u_1,P+h). \end{aligned} \tag{E.5}$$

Let us generalize these formulas as follows[34, 29, 23]. For $2\le k\le N$, define $R(u_1,\cdots,u_{k-1};u_k,s)^{1\cdots k-1;k}$ and $\Pi_k(u_1,\cdots,u_{k-1};u_k,s)\in \mathrm{End}_{\mathbb{F}}(\mathcal{V}^{\otimes k})$ by

(E.6)

$$\begin{aligned} &R(u_1,\cdots,u_{k-1};u_k,s)^{1\cdots k-1;k} \\ &= R^{(k-1k)}(u_k-u_{k-1},s)R^{(k-2k)}(u_k-u_{k-2},s+h^{(k-1)}) \\ &\qquad\qquad \cdots R^{(1k)}\left(u_k-u_1,s+\sum_{j=2}^{k-1}h^{(j)}\right), \end{aligned}$$

(E.7)

$$\begin{aligned} &\Pi_k(u_1,\cdots,u_{k-1};u_k,s) \\ &= \frac{1}{k!}R(u_1,\cdots,u_{k-1};u_k,s)^{1\cdots k-1;k}R(u_1,\cdots,u_{k-2};u_{k-1},s)^{1\cdots k-2;k-1} \\ &\qquad\qquad \cdots R(u_1,u_2;u_3,s)^{12;3}R(u_1;u_2,s)^{1;2}. \end{aligned}$$

We also need $\Pi^*_k(s)$ defined by the same formula as (E.7) with replacing $R(u,s)$ by $R^*(u,s)$. By using the DYBE (2.13) repeatedly, one obtains another expression of $\Pi_k(u_1,\cdots,u_{k-1};u_k,s)$

Proposition E.2.

$$\begin{aligned} &\Pi_k(u_1,\cdots,u_{k-1};u_k,s) \\ &=\frac{1}{k!}R(u_1;u_2,\cdots,u_k,s)^{1;2\cdots k}R(u_2;u_3,\cdots,u_k,s)^{2;3\cdots k}\cdots R(u_{k-1};u_k,s)^{k-1k}. \end{aligned}$$

where for $j<k$

$$\begin{aligned} &R(u_j;u_{j+1},\cdots,u_k,s)^{j;j+1\cdots k} \\ &= R^{(jj+1)}\left(u_{j+1}-u_j,s+\sum_{\substack{i=1\\ \neq j,j+1}}^{k}h^{(i)}\right)R^{(jj+2)}\left(u_{j+2}-u_j,s+\sum_{\substack{i=1\\ \neq j,j+1,j+2}}^{k}h^{(i)}\right) \\ &\qquad\qquad \cdots R^{(jk)}(u_k-u_j,s). \end{aligned}$$

Proposition E.3. *Let* $L(z)$ *be the* L *operator defined by* (E.2). *Then we have*

(E.8)

$$\Pi^*_k(u_1,\cdots,u_{k-1};u_k,P)\ {}^tL^{(1)}(u_1){}^tL^{(2)}(u_2)\cdots{}^tL^{(k)}(u_k)$$

$$={}^tL^{(k)}(u_k){}^tL^{(k-1)}(u_{k-1})\cdots{}^tL^{(1)}(u_1)\Pi_k\left(u_1,\cdots,u_{k-1};u_k,P+h-\sum_{j=1}^{k}h^{(j)}\right).$$

Note that

$$R^{(ij)}(u,s+h^{(i)}+h^{(j)})=R^{(ij)}(u,s).$$

Now let us consider the operators $\Pi_k(u_1,\cdots,u_{k-1};u_k,P+h)$ and $\Pi^*_k(u_1,\cdots,u_{k-1};u_k,P)$ with the specialization of the spectrum parameters $(u_1,u_2,\cdots,u_k)=(u,u-1,\cdots,u-(k-1))$. We denote the resultant operators by $\Pi_k(P+h)$ and $\Pi^*_k(P)$, respectively. Let us set $[1,N]=\{1,2,\cdots,N\}$, $I=\{i_1,i_2,\cdots,i_k\}\subseteq[1,N]$ with $i_1<i_2<\cdots<i_k$ and define

$$v_I=\mathcal{C}_I\ v_{i_1}\wedge v_{i_2}\wedge\cdots\wedge v_{i_k},$$

$$v_{i_1}\wedge v_{i_2}\wedge\cdots\wedge v_{i_k}=\frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}\mathrm{sgn}\sigma\ v_{i_{\sigma(1)}}\widetilde{\otimes}v_{i_{\sigma(2)}}\widetilde{\otimes}\cdots\widetilde{\otimes}v_{i_{\sigma(k)}},$$

$$\mathcal{C}_I=\prod_{1\le a<b\le k}\sqrt{\frac{\rho^*_0[a]^*}{[1]^*}\frac{\rho_0[a]}{[1]}}\frac{[(P+h)_{i_a,i_b}+1]}{[P_{i_a,i_b}]^*}.$$

Proposition E.4. *For* $2\le k\le N$ *and* $s=P,P+h\in H$*,*

$$\mathrm{Im}\ \Pi_k(s)=\wedge^k\mathcal{V}. \tag{E.9}$$

In particular, $\mathrm{Im}\,\Pi_N(s)$ *is the one dimensional subspace of* $\mathcal{V}^{\widetilde{\otimes}N}$ *spanned by*

$$v_{[1,N]}=\mathcal{C}_{[1,N]}\ v_1\wedge v_2\wedge\cdots\wedge v_N.$$

Proof. By induction one has

$$\Pi^*_k(P)v_{i_1}\widetilde{\otimes}v_{i_2}\widetilde{\otimes}\cdots\widetilde{\otimes}v_{i_k}$$

$$=\prod_{1\le a<b\le k}\rho^*_0\frac{[a]^*[P_{i_a,i_b}+1]^*}{[1]^*[P_{i_a,i_b}]^*}v_{i_1}\wedge v_{i_2}\wedge\cdots\wedge v_{i_k}\quad\in\wedge^k\mathcal{V}.$$

Then noting the identity

$$\prod_{1\le a<b\le k}\rho_0^*\frac{[a]^*[P_{i_a,i_b}+1]^*}{[1]^*[P_{i_a,i_b}]^*}=\mathcal{N}_I\mathcal{C}_I \tag{E.10}$$

where

$$\mathcal{N}_I=\prod_{1\le a<b\le k}\sqrt{\frac{\rho_0^*[a]^*[1]}{\rho_0[a][1]^*}\frac{[P_{i_a,i_b}+1]^*}{[(P+h)_{i_a,i_b}+1]}}, \tag{E.11}$$

one obtains

$$\Pi_k^*(P)v_{i_1}\widetilde{\otimes}v_{i_2}\widetilde{\otimes}\cdots\widetilde{\otimes}v_{i_k}=\mathcal{N}_I\ v_I.$$

Similarly using the identity

$$\prod_{1\le a<b\le k}\rho_0\frac{[a][(P+h)_{i_a,i_b}+1]}{[1][(P+h)_{i_a,i_b}]}=\mathcal{N}_I'\mathcal{C}_I, \tag{E.12}$$

with

$$\mathcal{N}_I'=\prod_{1\le a<b\le k}\sqrt{\frac{\rho_0[a][1]^*}{\rho_0^*[a]^*[1]}\frac{[P_{i_a,i_b}]^*}{[(P+h)_{i_a,i_b}]}}, \tag{E.13}$$

one obtains

$$\Pi_k(P+h)v_{i_1}\widetilde{\otimes}v_{i_2}\widetilde{\otimes}\cdots\widetilde{\otimes}v_{i_k}=\mathcal{N}_I'\ v_I.$$

Q.E.D.

Consider the projection operator $A_k:\mathcal{V}^{\otimes k}\to\wedge^k\mathcal{V}$

$$A_k=\frac{1}{k!}\sum_{\substack{1\le j_1,\cdots,j_k\le N\\ j_a\ne j_b(a\ne b)}}\sum_{\sigma\in\mathfrak{S}_k}\mathrm{sgn}\sigma E_{j_{\sigma(1)},j_1}\widetilde{\otimes}\cdots\widetilde{\otimes}E_{j_{\sigma(k)},j_k}.$$

Note that $A_k\Pi_k^*(P)=\Pi_k^*(P)$.

Definition E.5. *Let* $I=\{i_1,i_2,\cdots,i_k\},J=\{j_1,j_2,\cdots,j_k\}\subset[1,N]$ *with* $i_a<i_b,j_a<j_b$ *for* $1\le a<b\le k$. *We define the quantum minor determinant* $l(z)_I^J$ *of* $L(z)$ *by*

$$\begin{aligned}&\Pi_k^*(P)\ {}^tL^{(1)}(u){}^tL^{(2)}(u-1)\cdots{}^tL^{(k)}(u-(k-1))v_{i_1}\widetilde{\otimes}\cdots\widetilde{\otimes}v_{i_k}\\&=A_k\ {}^tL^{(k)}(u-(k-1)){}^tL^{(k-1)}(u-(k-2))\cdots{}^tL^{(1)}(u)\Pi_k(P+h)v_{i_1}\widetilde{\otimes}\cdots\widetilde{\otimes}v_{i_k}\\&=\sum_{1\le j_1<\cdots<j_k\le N}l(z)_I^J\ v_J.\end{aligned}$$

For $\tau \in \mathfrak{S}_k$ we set $\tau(I) = \{i_{\tau(1)}, \cdots, i_{\tau(k)}\}$ and define[28]

$$\mathrm{sgn}_I(\tau, P+h) = \prod_{\substack{1\le a<b\le k\\ \tau(a)>\tau(b)}} \frac{[(P+h)_{i_{\tau(a)},i_{\tau(b)}}+1]}{[(P+h)_{i_{\tau(b)},i_{\tau(a)}}+1]},$$

$$\mathrm{sgn}^*_I(\tau, P) = \prod_{\substack{1\le a<b\le k\\ \tau(a)>\tau(b)}} \frac{[P_{i_{\tau(a)},i_{\tau(b)}}+1]^*}{[P_{i_{\tau(b)},i_{\tau(a)}}+1]^*}.$$

Then we have

Proposition E.6.

$$\begin{aligned} &l(u)_I^J \\ &= \mathcal{N}_J \sum_{\sigma\in\mathfrak{S}_k} \mathrm{sgn}^*_J(\sigma, P) L_{i_1 j_{\sigma(1)}}(u) L_{i_2 j_{\sigma(2)}}(u-1)\cdots L_{i_k j_{\sigma(k)}}(u-(k-1)) \\ &= \mathcal{N}'_J \frac{F_I(P+h)}{F_J(P+h)} \sum_{\sigma\in\mathfrak{S}_k} \mathrm{sgn}\sigma\ L_{i_{\sigma(k)} j_k}(u-(k-1)) L_{i_{\sigma(k-1)} j_{k-1}}(u-(k-2)) \\ &\qquad\qquad \cdots L_{i_{\sigma(1)} j_1}(u), \end{aligned}$$

where

$$F_I(P+h) = \prod_{1\le a<b\le k} \frac{[(P+h)_{i_a,i_b}+1]}{[(P+h)_{i_a,i_b}]}.$$

In particular, in the case $I = J = [1, N]$ we obtain the quantum determinant of $L(z)$:

$$\begin{aligned} &q\text{-}\mathrm{det} L(u) \\ &= \mathcal{N}_{[1,N]} \sum_{\sigma\in\mathfrak{S}_N} \mathrm{sgn}^*_{[1,N]}(\sigma, P) L_{1\sigma(1)}(u) L_{2\sigma(2)}(u-1)\cdots L_{N\sigma(N)}(u-(N-1)) \\ &= \mathcal{N}'_{[1,N]} \sum_{\sigma\in\mathfrak{S}_N} \mathrm{sgn}\sigma\ L_{\sigma(N)N}(u-(N-1)) L_{\sigma(N-1)N-1}(u-(N-2)) \\ &\qquad\qquad \cdots L_{\sigma(1)1}(u). \end{aligned}$$

Proof. The statement follows from a standard calculation given for example in [55] and the formulas

$$\mathrm{sgn}^*_{[1,N]}(\sigma, P) = \prod_{\substack{1\le a<b\le N\\ \sigma(a)>\sigma(b)}} \frac{[P_{i_{\sigma(a)},i_{\sigma(b)}}+1]^*}{[P_{i_{\sigma(b)},i_{\sigma(a)}}+1]^*} = \prod_{1\le a<b\le N} \frac{[P_{i_{\sigma(a)},i_{\sigma(b)}}+1]^*}{[P_{i_a,i_b}+1]^*},$$

$$\mathrm{sgn}\sigma = \prod_{1\le a<b\le N} \frac{[P_{i_a,i_b}]^*}{[P_{i_{\sigma(a)},i_{\sigma(b)}}]^*} = \prod_{1\le a<b\le N} \frac{[(P+h)_{i_a,i_b}]}{[(P+h)_{i_{\sigma(a)},i_{\sigma(b)}}]}.$$

Q.E.D.

Note that the formulas in Proposition E.6 are consistent to the ones obtained by Hartwig using the co-module algebras[28].

Now using the identity $\mathrm{sgn}\tau\ \mathcal{C}_{\tau(I)}/\mathcal{C}_I = \mathrm{sgn}_I(\tau, P+h)$ we have

Proposition E.7.

$$v_{\tau(I)} = \mathrm{sgn}_I(\tau, P+h) v_I. \tag{E.14}$$

Then we obtain

Proposition E.8. *For* $\tau \in \mathfrak{S}_k$,

$$l(u)^J_{\tau(I)} = \mathrm{sgn}_I(\tau, P+h)\ l(u)^J_I, \tag{E.15}$$

$$l(u)^{\tau(J)}_I = \frac{1}{\mathrm{sgn}_J(\tau, P+h)}\ l(u)^J_I. \tag{E.16}$$

Then noting (4.9) and Proposition 4.8 we obtain

Proposition E.9.

$$\Delta(l(u)^J_I) = \sum_{1\le l_1<\cdots<l_k\le N} \frac{\mathcal{N}_J}{\mathcal{N}_L}\ l(u)^L_I \ \widetilde{\otimes}\ l(u)^J_L. \tag{E.17}$$

In particular,

$$\Delta(q\text{-}\mathrm{det}L(u)) = q\text{-}\mathrm{det}L(u)\widetilde{\otimes}q\text{-}\mathrm{det}L(u). \tag{E.18}$$

Next for $\le l \le k$ let us set $\widehat{i_l} = I\backslash\{i_l\}$ and define

$$\mathcal{N}^{(l)}_I = \frac{\mathcal{N}_I}{\mathcal{N}_{\widehat{i_l}}}, \qquad F^{(l)}_I(P+h) = \frac{F_I(P+h)}{F_{\widehat{i_l}}(P+h)}$$

etc.. Note that

$$F^{(l)}_I(P+h) = \prod_{1\le a<l} \frac{[(P+h)_{i_a,i_l}+1]}{[(P+h)_{i_a,i_l}]} \prod_{l<a\le k} \frac{[(P+h)_{i_l,i_a}+1]}{[(P+h)_{i_l,i_a}]}.$$

Then by a direct calculation using the expressions of $l(u)^J_I$ in Proposition E.6 we obtain

Proposition E.10.

$$\begin{aligned}
l(u)^J_I &= \sum_{l=1}^{k} \mathcal{N}^{(l)}_J \prod_{l<a\le k} \frac{[P_{j_a,j_l}+1]^*}{[P_{j_l,j_a}+1]^*}\ l(u)^{\widehat{j_l}}_{\widehat{i_k}}\ L_{i_k j_l}(u-(k-1)),\\
&= \sum_{l=1}^{k} L_{i_1 j_l}(u)\ l(u-1)^{\widehat{j_l}}_{\widehat{i_1}}\ \mathcal{N}^{(l)}_J \prod_{1\le a<l} \frac{[P_{j_l,j_a}+1]^*}{[P_{j_a,j_l}+1]^*},\\
&= \sum_{l=1}^{k} L_{i_l j_k}(u-(k-1))\ l(u)^{\widehat{j_k}}_{\widehat{i_l}}\ \mathcal{N}^{(k)}_J \frac{F^{(l)}_I(P+h)}{F^{(k)}_J(P+h)}(-1)^{k-l},\\
&= \sum_{l=1}^{k} \mathcal{N}^{(1)}_J \frac{F^{(l)}_I(P+h)}{F^{(1)}_J(P+h)}(-1)^{k-l}\ l(u-1)^{\widehat{j_1}}_{\widehat{i_l}}\ L_{i_l j_1}(u)
\end{aligned}$$

Proposition E.11. *For* $1\le i\le N$,

$$\begin{aligned}
&q\text{-}\mathrm{det}L(u)\\
&= \sum_{l=1}^{N} \mathcal{N}^{(l)}_{[1,N]} \prod_{i<a\le N} \frac{[(P{+}h)_{i,a}{+}1]}{[(P{+}h)_{a,i}{+}1]} \prod_{l<a\le N} \frac{[P_{a,l}{+}1]^*}{[P_{l,a}{+}1]^*}\ l(u)^{\widehat{l}}_{\widehat{i}}\ L_{il}(u{-}(N{-}1)),\\
&= \sum_{l=1}^{N} L_{il}(u)\ l(u-1)^{\widehat{l}}_{\widehat{i}}\ \mathcal{N}^{(l)}_{[1,N]} \prod_{1\le a<i} \frac{[(P+h)_{a,i}+1]}{[(P+h)_{i,a}+1]} \prod_{1\le a<l} \frac{[P_{l,a}+1]^*}{[P_{a,l}+1]^*},\\
&= \sum_{l=1}^{N} L_{li}(u-(N-1))\ l(u)^{\widehat{i}}_{\widehat{l}}\ (-1)^{N-l}\mathcal{N}^{(i)}_{[1,N]} \frac{F^{(l)}_{[1,N]}(P+h)}{F^{(i)}_{[1,N]}(P+h)}\\
&\qquad\qquad \prod_{i<a\le N} \frac{[(P+h)_{i,a}+1]}{[(P+h)_{a,i}+1]},\\
&= \sum_{l=1}^{N} (-1)^{N-l}\mathcal{N}^{(i)}_{[1,N]} \frac{F^{(l)}_{[1,N]}(P{+}h)}{F^{(i)}_{[1,N]}(P{+}h)} \prod_{1\le a<j} \frac{[(P{+}h)_{a,i}+1]}{[(P{+}h)_{i,a}+1]}\ l(u-1)^{\widehat{j}}_{\widehat{l}}\ L_{li}(u)
\end{aligned}$$

Proof. Consider the case $I=J=[1,N]$ in Proposition E.10 and use Proposition E.8 for the cyclic permutations $\tau=(i,i+1,\cdots,N)$ in the 1st and the 4th lines, whereas for $\tau=(i,i-1,\cdots,2,1)$ in the 2nd and the 3rd lines.

E.2. Gauge transformation

Inserting (E.2) into the expressions of $l(u)_I^J$ in Proposition E.6 we define the quantum minor determinant $\widehat{l}^+(u)_I^J$ of $\widehat{L}^+(u)$ by

$$l(u)_I^J = \left(\prod_{1\le a<b\le k}\frac{[1]^*}{[1]}\frac{[(P+h)_{i_a,i_b}+1]}{[(P+h)_{j_a,j_b}+1]}\right)\widehat{l}^+(u)_I^J. \tag{E.19}$$

For $\sigma\in\mathfrak{S}$ we set

$$\widehat{\mathrm{sgn}}_I(\sigma,P+h)=\prod_{\substack{1\le a<b\le k\\ \sigma(a)>\sigma(b)}}\frac{[(P+h)_{i_{\sigma(a)},i_{\sigma(b)}}+1]}{[(P+h)_{i_{\sigma(b)},i_{\sigma(a)}}]},$$

$$\widehat{\mathrm{sgn}}^*_I(\sigma,P)=\prod_{\substack{1\le a<b\le k\\ \sigma(a)>\sigma(b)}}\frac{[P_{i_{\sigma(a)},i_{\sigma(b)}}+1]^*}{[P_{i_{\sigma(b)},i_{\sigma(a)}}]^*}.$$

Proposition E.12.

$$\begin{aligned}&\widehat{l}^+(u)_I^J\\ &=\mathcal{N}_k\sum_{\sigma\in\mathfrak{S}_k}\widehat{\mathrm{sgn}}^*_J(\sigma,P)\widehat{L}^+_{i_1j_{\sigma(1)}}(u)\widehat{L}^+_{i_2j_{\sigma(2)}}(u-1)\cdots\widehat{L}^+_{i_kj_{\sigma(k)}}(u-(k-1)),\\ &=\mathcal{N}_k^{-1}\sum_{\sigma\in\mathfrak{S}_k}\widehat{\mathrm{sgn}}_I(\sigma,P+h)\widehat{L}^+_{i_{\sigma(k)}j_k}(u-(k-1))\widehat{L}^+_{i_{\sigma(k-1)}j_{k-1}}(u-(k-2))\\ &\qquad\cdots\widehat{L}^+_{i_{\sigma(1)}j_1}(u),\end{aligned}$$

where

$$\mathcal{N}_k=\prod_{1\le a<b\le k}\sqrt{\frac{\rho_0^*[a]^*[1]}{\rho_0[a][1]^*}}.$$

Proof. Inserting (E.2) and using (3.65) and (3.66), we have

$$\begin{aligned}
&L_{i_1 j_{\sigma(1)}}(u)L_{i_2 j_{\sigma(2)}}(u-1)\cdots L_{i_k j_{\sigma(k)}}(u-(k-1))\\
&=\prod_{1\le a<b\le k}\frac{[(P+h)_{i_a,i_b}+1]}{[1]}\ \widehat{L}^+_{i_1 j_{\sigma(1)}}(u)\widehat{L}^+_{i_2 j_{\sigma(2)}}(u-1)\cdots\widehat{L}^+_{i_k j_{\sigma(k)}}(u-(k-1))\\
&\qquad\times\prod_{1\le a<b\le k}\frac{[1]^*}{[P_{j_a,j_b}+1]^*}\prod_{\substack{1\le a<b\le k\\ \sigma(a)>\sigma(b)}}\frac{[P_{j_{\sigma(b)},j_{\sigma(a)}}+1]^*}{[P_{j_{\sigma(b)},j_{\sigma(a)}}]^*}
\end{aligned}$$

$$\begin{aligned}
&L_{i_{\sigma(k)} j_k}(u-(k-1))L_{i_{\sigma(k-1)} j_{k-1}}(u-(k-2))\cdots L_{i_{\sigma(1)} j_1}(u)\\
&=\prod_{1\le a<b\le k}\frac{[(P+h)_{i_a,i_b}]}{[1]}\prod_{\substack{1\le a<b\le k\\ \sigma(a)>\sigma(b)}}\frac{[(P+h)_{i_{\sigma(a)},i_{\sigma(b)}}+1]}{[(P+h)_{i_{\sigma(a)},i_{\sigma(b)}}]}\\
&\times\widehat{L}^+_{i_{\sigma(k)} j_k}(u-(k-1))\widehat{L}^+_{i_{\sigma(k-1)} j_{k-1}}(u-(k-2))\cdots\widehat{L}^+_{i_{\sigma(1)} j_1}(u)\prod_{1\le a<b\le k}\frac{[1]^*}{[P_{j_a,j_b}]^*}.
\end{aligned}$$

Q.E.D.

Corollary E.13.

$$\begin{aligned}
&q\text{-}\mathrm{det}\widehat{L}^+(u)\\
&=\mathcal{N}_N\sum_{\sigma\in\mathfrak{S}_N}\widehat{\mathrm{sgn}}^*_{[1,N]}(\sigma,P)\widehat{L}^+_{1\sigma(1)}(u)\widehat{L}^+_{2\sigma(2)}(u-1)\cdots\widehat{L}^+_{N\sigma(N)}(u-(N-1)),\\
&=\mathcal{N}_N^{-1}\sum_{\sigma\in\mathfrak{S}_N}\widehat{\mathrm{sgn}}_{[1,N]}(\sigma,P+h)\widehat{L}^+_{\sigma(N)N}(u-(N-1))\\
&\qquad\qquad\widehat{L}^+_{\sigma(N-1)N-1}(u-(N-2))\cdots\widehat{L}^+_{\sigma(1)1}(u).
\end{aligned}$$

Proposition E.14.

$$\widehat{l}^+(u)^J_{\tau(I)}=\widehat{\mathrm{sgn}}_I(\tau,P+h)\ \widehat{l}^+(u)^J_I, \tag{E.20}$$

$$\widehat{l}^+(u)^{\tau(J)}_I=\widehat{\mathrm{sgn}}^*_J(\tau,P)\ \widehat{l}^+(u)^J_I. \tag{E.21}$$

Proposition E.15.

$$\Delta(\widehat{l}^+(u)^J_I)=\sum_{1\le l_1<\cdots<l_k\le k}\widehat{l}^+(u)^L_I\ \widetilde{\otimes}\ \widehat{l}^+(u)^J_L. \tag{E.22}$$

Proof. Note $\Delta(\mathcal{N}_k)=\mathcal{N}_k\widetilde{\otimes}\mathcal{N}_k$ and

$$\Delta(\widehat{\mathrm{sgn}}^*_I(\sigma,P))=1\widetilde{\otimes}\ \widehat{\mathrm{sgn}}^*_I(\sigma,P),\quad \Delta(\widehat{\mathrm{sgn}}_I(\sigma,P+h))=\widehat{\mathrm{sgn}}_I(\sigma,P+h)\widetilde{\otimes}1.$$

Q.E.D.

Proposition E.16.

$$\begin{aligned}
\widehat{l}^+(u)^J_I &= \sum_{l=1}^{k} {\mathcal{N}'}_k \prod_{l<a\leq k} \frac{[P_{j_a,j_l}+1]^*}{[P_{j_l,j_a}]^*}\ \widehat{l}^+(u)^{\widehat{j_l}}_{\widehat{i_k}}\ \widehat{L}^+_{i_k j_l}(u-(k-1)),\\
&= \sum_{l=1}^{k} \widehat{L}^+_{i_1 j_l}(u)\ \widehat{l}^+(u-1)^{\widehat{j_l}}_{\widehat{i_1}}\ {\mathcal{N}'}_k \prod_{1\leq a<l} \frac{[P_{j_l,j_a}+1]^*}{[P_{j_a,j_l}]^*},\\
&= \sum_{l=1}^{k} \widehat{L}^+_{i_l j_k}(u-(k-1))\ \widehat{l}^+(u)^{\widehat{j_k}}_{\widehat{i_l}}\ {{\mathcal{N}'}_k}^{-1} \prod_{l<a\leq k} \frac{[(P+h)_{i_a,i_l}+1]}{[(P+h)_{i_l,i_a}]},\\
&= \sum_{l=1}^{k} {{\mathcal{N}'}_k}^{-1} \prod_{1\leq a<l} \frac{[(P+h)_{i_l,i_a}+1]}{[(P+h)_{i_a,i_l}]}\ \widehat{l}^+(u-1)^{\widehat{j_1}}_{\widehat{i_l}}\ \widehat{L}^+_{i_l j_1}(u),
\end{aligned}$$

where

$$\mathcal{N}'_k = \frac{\mathcal{N}_k}{\mathcal{N}_{k-1}} = \prod_{1\leq a\leq k-1} \sqrt{\frac{\rho_0^*[a]^*[1]}{\rho_0[a][1]^*}}.$$

Proposition E.17. *For* $1 \leq i \leq N$,

$$\begin{aligned}
&q\text{-det}\widehat{L}^+(u)\\
&= \sum_{l=1}^{N} {\mathcal{N}'}_N \prod_{i<a\leq N} \frac{[(P+h)_{i,a}]}{[(P+h)_{a,i}+1]} \prod_{l<a\leq N} \frac{[P_{a,l}+1]^*}{[P_{l,a}]^*}\ \widehat{l}^+(u)^{\widehat{l}}_{\widehat{i}}\ \widehat{L}^+_{il}(u-(N-1)),\\
&= \sum_{l=1}^{N} \widehat{L}^+_{il}(u)\ \widehat{l}^+(u-1)^{\widehat{l}}_{\widehat{i}}\ {\mathcal{N}'}_N \prod_{1\leq a<i} \frac{[(P+h)_{a,i}]}{[(P+h)_{i,a}+1]} \prod_{1\leq a<l} \frac{[P_{l,a}+1]^*}{[P_{a,l}]^*},\\
&= \sum_{l=1}^{N} \widehat{L}^+_{li}(u-(N-1))\ \widehat{l}^+(u)^{\widehat{i}}_{\widehat{l}}\ {{\mathcal{N}'}_N}^{-1} \prod_{l<a\leq N} \frac{[(P+h)_{a,l}+1]}{[(P+h)_{l,a}]} \prod_{i<a\leq N} \frac{[P_{a,i}+1]^*}{[P_{i,a}]^*},\\
&= \sum_{l=1}^{N} {{\mathcal{N}'}_N}^{-1} \prod_{1\leq a<l} \frac{[(P+h)_{l,a}+1]}{[(P+h)_{a,l}]} \prod_{1\leq a<i} \frac{[P_{a,i}]^*}{[P_{i,a}+1]^*}\ \widehat{l}^+(u-1)^{\widehat{i}}_{\widehat{l}}\ \widehat{L}^+_{li}(u)
\end{aligned}$$

Comparing this and the axiom for the antipode S, we determine the action of S on $\widehat{L}^+_{il}(u)$ and $\widehat{l}^+(u)^{\widehat{i}}_{\widehat{l}}$. For each there are four different expressions. For example,

Proposition E.18.

$$\begin{aligned}
S(\widehat{L}^+_{il}(u)) &= \widehat{l}^+(u-1)^{\widehat{i}}_{\widehat{l}}\,\mathcal{N}'_N \prod_{1\le a<l} \frac{[(P+h)_{a,l}]}{[(P+h)_{l,a}+1]} \prod_{1\le a<i} \frac{[P_{i,a}+1]^*}{[P_{a,i}]^*} \\
&\qquad\times(q\text{-}\mathrm{det}\widehat{L}^+(u))^{-1}, \\
S(\widehat{l}^+(u)^{\widehat{i}}_{\widehat{l}}) &= \mathcal{N}'_N \prod_{i<a\le N} \frac{[(P+h)_{i,a}]}{[(P+h)_{a,i}+1]} \prod_{l<a\le N} \frac{[P_{a,l}+1]^*}{[P_{l,a}]^*}(q\text{-}\mathrm{det}\widehat{L}^+(u))^{-1}.
\end{aligned}$$

Combining these we obtain

Proposition E.19.

$$S^2(\widehat{L}^+_{il}(u)) = \prod_{a\in\widehat{i}} \frac{[(P+h)_{i,a}+1]}{[(P+h)_{i,a}]}\ \widehat{L}^+_{il}(u-N) \prod_{a\in\widehat{l}} \frac{[P_{a,l}+1]^*}{[P_{a,l}]^*}.$$

Proposition E.16 also yields

Proposition E.20.

$$(\widehat{L}^+(u)^{-1})_{ij} = S(\widehat{L}^+_{ij}(u)).$$

E.3. Formulas for the half currents

In this section we follow the idea in [31].

For $1\le a,b\le N$ let us define $\widehat{L}^+(u)_{a,a} = (\widehat{L}^+_{i,j}(u))_{a\le i,j\le N}$ and

$$\begin{aligned}
\widehat{L}^+(u)_{a,b} &= \begin{pmatrix} \widehat{L}^+_{ab}(u) & \widehat{L}^+_{aa+1}(u) & \cdots & \widehat{L}^+_{aN}(u) \\ \widehat{L}^+_{a+1b}(u) & \widehat{L}^+_{a+1a+1}(u) & \cdots & \widehat{L}^+_{a+1N}(u) \\ \vdots & \vdots & & \vdots \\ \widehat{L}^+_{Nb}(u) & \widehat{L}^+_{Na+1}(u) & \cdots & \widehat{L}^+_{NN}(u) \end{pmatrix} \quad \text{for } a>b \\
&= \begin{pmatrix} \widehat{L}^+_{ab}(u) & \widehat{L}^+_{ab+1}(u) & \cdots & \widehat{L}^+_{aN}(u) \\ \widehat{L}^+_{b+1b}(u) & \widehat{L}^+_{b+1b+1}(u) & \cdots & \widehat{L}^+_{b+1N}(u) \\ \vdots & \vdots & & \vdots \\ \widehat{L}^+_{Nb}(u) & \widehat{L}^+_{Nb+1}(u) & \cdots & \widehat{L}^+_{NN}(u) \end{pmatrix} \quad \text{for } a<b.
\end{aligned}$$

Then we have the following Gauss decompositions.

Lemma E.21. *For $a > b$*

$$\begin{aligned}
&\widehat{L}^+(u)_{a,b}\\
&=\begin{pmatrix}
1 & F^+_{a,a+1}(z) & F^+_{a,a+2}(z) & \cdots & F^+_{a,N}(z)\\
0 & 1 & F^+_{a+1,a+2}(z) & \cdots & F^+_{a+1,N}(z)\\
\vdots & \ddots & \ddots & \ddots & \vdots\\
\vdots & & \ddots & 1 & F^+_{N-1,N}(z)\\
0 & \cdots & \cdots & 0 & 1
\end{pmatrix}
\begin{pmatrix}
K^+_a(z)E^+_{a,b}(u) & 0 & \cdots & 0\\
0 & K^+_{a+1}(z) & & \vdots\\
\vdots & & \ddots & 0\\
0 & \cdots & 0 & K^+_N(z)
\end{pmatrix}\\
&\qquad\times\begin{pmatrix}
1 & 0 & \cdots & \cdots & 0\\
E^+_{a+1,b}(z) & 1 & \ddots & & \vdots\\
E^+_{a+2,b}(z) & E^+_{a+2,a+1}(z) & \ddots & \ddots & \vdots\\
\vdots & \vdots & \ddots & 1 & 0\\
E^+_{N,b}(z) & E^+_{N,a+1}(z) & \cdots & E^+_{N,N-1}(z) & 1
\end{pmatrix}.
\end{aligned}$$

For $a < b$

$$\begin{aligned}
&\widehat{L}^+(u)_{a,b}\\
&=\begin{pmatrix}
1 & F^+_{a,b+1}(z) & F^+_{a,b+2}(z) & \cdots & F^+_{a,N}(z)\\
0 & 1 & F^+_{b+1,b+2}(z) & \cdots & F^+_{b+1,N}(z)\\
\vdots & \ddots & \ddots & \ddots & \vdots\\
\vdots & & \ddots & 1 & F^+_{N-1,N}(z)\\
0 & \cdots & \cdots & 0 & 1
\end{pmatrix}
\begin{pmatrix}
F^+_{a,b}(u)K^+_b(z) & 0 & \cdots & 0\\
0 & K^+_{b+1}(z) & & \vdots\\
\vdots & & \ddots & 0\\
0 & \cdots & 0 & K^+_N(z)
\end{pmatrix}\\
&\qquad\times\begin{pmatrix}
1 & 0 & \cdots & \cdots & 0\\
E^+_{b+1,b}(z) & 1 & \ddots & & \vdots\\
E^+_{b+2,b}(z) & E^+_{b+2,b+1}(z) & \ddots & \ddots & \vdots\\
\vdots & \vdots & \ddots & 1 & 0\\
E^+_{N,b}(z) & E^+_{N,b+1}(z) & \cdots & E^+_{N,N-1}(z) & 1
\end{pmatrix}.
\end{aligned}$$

The formula for $\widehat{L}^+(u)_{a,a}$ is the same as (6.16) *with $l = a$.*

Let us write the Gauss decomposition of $\widehat{L}^+(u)_{a,b}$ in the above Lemma as

$$\widehat{L}^+(u)_{a,b} = \mathcal{F}_{a,b}(u)\mathcal{K}_{a,b}(u)\mathcal{E}_{a,b}(u).$$

Then we have

$$\mathcal{F}_{a,b}(u)^{-1} = \mathcal{K}_{a,b}(u)\mathcal{E}_{a,b}(u)\widehat{L}^+(u)^{-1}_{a,b}.$$

Comparing the $(1,1)$ component of the both sides we obtain the following.

Lemma E.22.

$$K^+_a(u) = \frac{1}{(\widehat{L}^+(u)^{-1}_{a,a})_{11}} \qquad \textit{for } a = b, \tag{E.23}$$

$$E^+_{a,b}(u) = (\widehat{L}^+(u)^{-1}_{a,a})_{11}\frac{1}{(\widehat{L}^+(u)^{-1}_{a,b})_{11}} \qquad \textit{for } a > b, \tag{E.24}$$

$$F^+_{a,b}(u) = \frac{1}{(\widehat{L}^+(u)^{-1}_{a,b})_{11}}(\widehat{L}^+_b(u)^{-1})_{11} \qquad \textit{for } a < b. \tag{E.25}$$

Noting

$$(\widehat{L}^+(u)^{-1}_{a,b})_{11} = (\widehat{l}^+(u-1)_{a,b})^{\hat{1}}_{\hat{1}}(q\text{-det}\widehat{L}^+(u)_{a,b})^{-1}\mathcal{N}'_{N-a+1},$$
$$(\widehat{l}^+(u-1)_{a,b})^{\hat{1}}_{\hat{1}} = q\text{-det}\widehat{L}^+(u-1)_{a+1,a+1},$$

we have

Theorem E.23.

$$K^+_a(u) = \frac{q\text{-det}\widehat{L}^+(u)_{a,a}}{\mathcal{N}'_{N-a+1}}\frac{1}{q\text{-det}\widehat{L}^+(u-1)_{a+1,a+1}}, \tag{E.26}$$

$$E^+_{a,b}(u) = \frac{1}{q\text{-det}\widehat{L}^+(u)_{a,a}}q\text{-det}\widehat{L}^+(u)_{a,b} \qquad \textit{for } a > b, \tag{E.27}$$

$$F^+_{a,b}(u) = q\text{-det}\widehat{L}^+(u)_{a,b}\frac{1}{q\text{-det}\widehat{L}^+(u)_{b,b}} \qquad \textit{for } a < b. \tag{E.28}$$

Corollary E.24. *Let us define*

$$K(u) = K^+_1(u)K^+_2(u-1)\cdots K^+_N(u-(N-1)). \tag{E.29}$$

Then

$$q\text{-det}\widehat{L}^+(u) = \mathcal{N}_N\ K(u). \tag{E.30}$$

Moreover the q*-determinant* $q\text{-det}\widehat{L}^+(u)$ *belongs to the center of* $E_{q,p}(\widehat{\mathfrak{gl}}_N)$.

Proof. Since $K^+_l(v), E_i(v) = const.(E^+_{i+1,i}(v+c/4) - E^-_{i+1,i}(v-c/4))$, $F_i(v) = const'.(F^+_{i,i+1}(v-c/4) - F^-_{i,i+1}(v+c/4))$ $(1 \leq l \leq N, 1 \leq i \leq N-1)$ satisfy the same commutation relations as the elliptic currents of $U_{q,p}(\widehat{\mathfrak{gl}}_N)$, $K(u)$ commutes with $K^+_l(v), E_i(v), F_i(v), \mathbb{F}$ due to Proposition 3.2. Hence by Definition 6.2 $K(u)$ commutes with $K^\pm_l(v), E^\pm_{i+1,i}(v)$, $F^\pm_{i,i+1}(v)$ so that $K(u)$ commutes with $\widehat{L}^+(v)$. Q.E.D.

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