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Rigged Configurations and Catalan, stretched parabolic Kostka numbers and polynomials: Polynomiality, unimodality and log-concavity

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Dedicated to Professor Masatoshi NOUMI on the occasion of his 60th Birthday

Abstract.

We will look at the Catalan numbers from the *Rigged Configurations* point of view originated [10] from an combinatorial analysis of the Bethe Ansatz Equations associated with the higher spin anisotropic Heisenberg models. Our strategy is to take a combinatorial interpretation of the Catalan number C_n as the number of standard Young tableaux of rectangular shape (n^2), or equivalently, as the Kostka number $K_{(n^2),1^{2n}}$, as the starting point of our research. We observe that the rectangular (or multidimensional) Catalan numbers $C(m,n)$, introduced and studied by P. MacMahon [23], [34], see also [35], can be identified with the corresponding Kostka numbers $K_{(n^m),1^{mn}}$, and therefore can be treated by the Rigged Configurations technique. Based on this technique we study the stretched Kostka numbers and polynomials, and give a proof of a *strong rationality* of the stretched Kostka polynomials. This result implies a polynomiality property of the stretched Kostka and stretched Littlewood–Richardson coefficients [8], [28], [17]. Finally, we give a brief introduction to a rigged configuration version of the Robinson–Schensted–Knuth correspondence.

Another application of the Rigged Configuration technique presented, is a new family of counterexamples to Okounkov's log-concavity conjecture [27].

Finally, we apply Rigged Configurations technique to give a combinatorial proof of the unimodality of the principal specialization of

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the internal product of Schur functions. In fact we prove a combinatorial (fermionic) formula for generalized q-Gaussian polynomials which is a far generalization of the so-called KOH-identity [26], as well as it manifests the unimodality property of the q-Gaussian polynomials.

§1. Introduction

The literature devoted to the study of Catalan [1] and Narayana numbers [2], their different combinatorial interpretations (more than 200 in fact, [33]), numerous generalizations, applications to Combinatorics, Algebraic Geometry, Probability Theory and so on and so forth, are enormous, see [33] and the literature quoted therein. There exists a wide variety of different generalizations of Catalan numbers, such as the Fuss–Catalan numbers [3] and the Schröder numbers [4], higher *genus* multivariable Catalan numbers [24], higher dimensional Catalan [5] and Narayana numbers [23], [34], and many and varied other generalizations. Each a such generalization comes from a generalization of a certain combinatorial interpretation of Catalan numbers, taken as a starting point for investigation. One a such interpretation of Catalan numbers has been taken as the starting point of the present paper, is the well-known fact that the Catalan number C_n is equal to the number of *standard* Young tableaux of the shape (n^2).

Now let us look at the Catalan numbers from *Rigged Configurations* side. Since $C_n = K_{(n^2),1^{2n}}$ we can apply a *fermionic formula* for Kostka polynomials [9], and come to the following combinatorial expressions for Catalan and Narayana numbers

$$C_n = \sum_{\nu \vdash n} \prod_{j \geq 1} \binom{2n - 2(\sum_{a \leq j} \nu_a) + \nu_j - \nu_{j+1}}{\nu_j - \nu_{j+1}},$$

[1] $en.wikipedia.org/wiki/Catalan\_number$

[2] $en.wikipedia.org/wiki/Narayana\_number$

[3] $en.wikipedia.org/wiki/Fuss-Catalan\_number$

[4] wolfram.com/SchröederNumber.html

[5] We denote the multidimensional Catalan numbers (as well as the set thereof) by $C(m,n)$. It might be well to point out that the set $C(m,n)$ is different from the set of *Fuss-Catalan* paths (or numbers) denoted commonly by $C_n^{(m)}$.

where the sum runs over all partitions ν of size n;

$$N(n,k) = \sum_{\substack{\nu \vdash n \\ \nu_1 = k}} \prod_{j \geq 1} \binom{2n - 2(\sum_{a \leq j} \nu_a) + \nu_j - \nu_{j+1}}{\nu_j - \nu_{j+1}},$$

where the sum runs over all partitions ν of size n, $\nu_1 = k$.
A q-versions of these formulas one can find, for example, in [13].
Let us illustrate our combinatorial formulas for $n = 6$. There are 11 partitions of size 6. We display below the distribution of contributions to the combinatorial formulae for the Catalan and Narayana numbers presented above, which come from partitions ν of size 6 and $k = 1, 2, \ldots, 6$.
$N(6,1) = 1$,
$N(6,2) = \binom{9}{1} + \binom{5}{4} + 1 = 9 + 5 + 1 = 15$,
$N(6,3) = \binom{8}{6} + \binom{3}{1}\binom{7}{1} + 1 = 28 + 21 + 1 = 50$,
$N(6,4) = \binom{7}{4} + \binom{6}{4} = 35 + 15 = 50$,
$N(6,5) = \binom{6}{4} = 15$, $N(6,6) = 1$.

A few comments in order.

• In [11] we gave a combinatorial interpretation of the shape of first (admissible) configuration $\nu^{(1)}$ corresponding to a given semistandard tableau T in terms of the set of *secondary descent sets* associated with the Young tableau T in question. In the case of *standard* Young tableaux of rectangular shape (n,n) there exist only one admissible configuration $\nu, |\nu| = n$, and a combinatorial rule how to describe partition ν stated in [11], can be restated as follows:
By the use of classical bijection between the set of standard Young tableaux of shape (n,n) and that of rooted plane trees with n nodes, see e.g. [33], one can associate to a given tableau $T \in STY((n,n))$ a rooted plane tree $\mathcal{T}$ on n nodes (out of the root). The number of external nodes of a tree $\mathcal{T}$ is equal to $p := p(T) = \#(DES(T))$, where $DES(T)$ denotes the descent set of the tableau in question. Now for any external node b of the tree $\mathcal{T}$ mentioned, denote by π_b a unique path in the tree $\mathcal{T}$ from the node b to the root. Let $\kappa_b(\mathcal{T})$ stands for the number of *edges* in the path π_b.

Lemma 1.1. *Let $T \in STY((n,n)$ be a standard Young tableau of shape (n,n), and $\nu \vdash n$ be a configuration corresponding to T under the Rigged Configuration bijection. Then*

$$\nu_1 = \kappa^{(1)}(\mathcal{T}) := \max(\kappa_1(\mathcal{T}), \ldots, \kappa_p(\mathcal{T})).$$

Now we proceed by induction. Namely, consider the most left node b in the tree $\mathcal{T}$ such that $\kappa_b = \nu_1$. Let $\mathcal{T}_1$ denotes a *forest* of rooted trees associated with the complement $\mathcal{T} \setminus \pi_b$. Let $\mathcal{T}_1 = \mathcal{T}_1^{(1)} \bigcup \mathcal{T}_1^{(2)} \ldots \bigcup \mathcal{T}_1^{(s)}$ be the union of distinct rooted trees making up the forest $\mathcal{T}_1$. Let now b be a node which belongs, say, to a (unique) subtree $\mathcal{T}_1^{(a)}$ of the forest $\mathcal{T}$, denote as before, by $\pi_b^{(1)}$ and $\kappa_b^{(2)}$ a unique path from the node b to the root of the tree $\mathcal{T}_1^{(a)}$, and its number of edges. Then

$$\nu_2 = \kappa^{(2)}(\mathcal{T}) := max(\kappa_b^{(2)}),$$

where maximum is taken over the all nodes of the forest $\mathcal{T}_1$. Now consider forest $\mathcal{T}_2 = \mathcal{T}_1 \setminus \pi_b^{(1)}$ and repeat the above procedure. As a result we obtain a sequence of numbers $\kappa = (\kappa^{(1)}, \ldots, \kappa^{(p)})$ such that

$$\nu^{(1)} = \kappa.$$

It is easy to see that for a given partition $\lambda \vdash n$, $\lambda = (m_1^{a_1}, \ldots, m_k^{a_k})$, $m_1 > m_2 > \ldots > m_k > 0$, $a_i \geq 1, \forall i$, the rooted plane tree corresponding to the *maximal* rigged configuration of type λ [12], looks as follows. It is a rooted plane tree $\mathcal{T}_{max}$ with a unique branching point at the root and external nodes $b_1, \ldots, b_k$ such that $\kappa_{b_1}(\mathcal{T}_{max}) = \cdots = \kappa_{b_{a_1}}(\mathcal{T}_{max}) = m_1$, $\kappa_{b_{a_1}+1}(\mathcal{T}_{max}) = \cdots = \kappa_{b_{a_1+a_2}}(\mathcal{T}_{max}) = m_2$, and so on. The rooted plane tree corresponding to the *minimal* rigged configuration, i.e. that with all zero riggings, corresponds to the *mirror image* of the tree $\mathcal{T}_{max}$.

The Rigged Configuration Bijection allows to attach a non-negative integer to each node of the corresponding rooted plane tree, It is an interesting **Problem** to read off these numbers from the associated tree directly.

- q-versions of formulas for Catalan and Narayana numbers displayed above coincide with the Carlitz–Riordan q-analog of Catalan numbers [32] and q-analog of Narayana numbers correspondingly.
- It is well-known that partitions of n with respect to the dominance ordering, form a lattice denoted by L_n. One (A.K) can define an ordering [6] on the set of admissible configurations of type (λ, μ) as well. In the case $\lambda = (n^2)$, $\mu = (1^{2n})$ the poset of admissible configurations of type (λ, μ) is essentially the same as the lattice of partitions L_n. Therefore, to each vertex ν of the lattice L_n one can attach the space of rigged configurations $RC_{\lambda,\mu}(\nu)$ associated with partition ν. Under a certain evolution a configuration (ν, J) evolves and touches the boundary of the set $RC_{\lambda,\mu}(\nu)$. When such is the case, "state" (ν, J) suffers "a phase transition", executes the wall-crossing, and end up as a newborn state of some space $RC_{\lambda,\nu}(\nu^{'})$. A precise description of this process is the essence of the Rigged Configuration Bijection [15], [16]. It seems an interesting task to write out in full the evolution process going on in

the space of triangulations of a convex $(n+2)$-gon under the Rigged Configuration Bijection (*work in progress*).

• It is an open **Problems** to count the number of admissible configurations associated with the multidimensional Catalan numbers $C(m,n)$ for general n and $m \geq 3$, and describe a structure of the corresponding poset on the set of admissible configurations, as well as to trace out a dynamics of riggings in the poset associated, for example, with the set $SYT((n,n))$. If $m=3$, the set of of admissible configurations consists of pairs of partitions $(\nu^{(1)}, \nu^{(2)})$ such that $\nu^{(2)} \vdash n$ and $\nu^{(1)} \geq \nu^{(2)} \vee \nu^{(2)}$ [7]. One can check that the number of admissible configurations of type $(n^3, 1^{3n})$ is equal to $1,3,6,16,33,78$, for $n=1,2,3,4,5,6$.

• It is well-known that the q-Narayana numbers [8] obey the symmetry property, namely, $N(k,n) = N(n-k+1,n)$. Therefore it implies some non trivial relations among the products of q-binomial coefficients, combinatorial proofs of whose are desirable.

• It is well-known that the Narayana number $N(k,n)$ counts the number of Dyck paths of the semilength n with exactly k peaks, see e.g. [29], $A001263$. Therefore, the set of rigged configurations $\{\nu\}$ which associated with the Catalan number C_n and have fixed $\nu_1 = k$, is in one-to-one correspondence with the set of the semilength n Dyck paths with exactly k peaks, as well as the number of rooted plane trees with n edges and k ends.

Thus it looks natural to find and study combinatorial properties of the number of standard Young tableaux of an arbitrary rectangular shape (n^m), that is the Kostka number $K_{(n^m),1^{mn}}$, which are inherent in the classical Catalan and Narayana numbers. For example, one can expect that a multidimensional Catalan number is the sum of multidimensional Narayana ones (this is so !), or expect that a multidimensional Narayana polynomial is the δ-vector of a certain convex lattice polytope, see e.g. [31] for the case of classical Catalan and Narayana numbers [9].

• Combinatorial analysis of the Bethe Ansatz Equations [10], gives rise to a natural interpretation of the Catalan and rectangular Catalan and Narayana numbers in terms of rigged configurations, and pose

[7] Recall that for any partitions λ and μ, $\lambda \vee \mu$ denotes partition corresponding to composition $(\lambda_1, \mu_1, \lambda_2, \mu_2, \ldots)$.

[8] Recall that the q- Narayana number $N(k,n \mid q) = \frac{1-q}{1-q^n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k+1 \end{bmatrix}_q$.

[9] The multidimensional Catalan and Narayana numbers, as well as the first *expectation*, had been introduced and proved by P.MacMahon [23]. The second *expectation* will be treated in the present paper, Section 3.

Problem to elaborate combinatorial structures induced by rigged configurations on any chosen combinatorial interpretation of Catalan numbers. For example, how to describe all triangulations of a convex $(n+2)$-gon which are in a "natural" bijection with the set of all rigged configurations (μ, J) corresponding to a given configuration ν of type $((n^2), 1^{2n})$? One can ask similar questions concerning Dyck paths and its multidimensional generalizations [35], and so on.

In Section 5.1 we present an example to illustrate some basic properties of the Rigged Configuration Bijection.

■

In the present paper we are interested in to investigate combinatorics related with the higher dimensional Catalan numbers, had been introduced and studied in depth by P. MacMahon [23]. It is highly possible that the starting point to introduce the higher dimensional Catalan numbers in [23] was an interpretation of classical Catalan numbers as the number of rectangular shape (n^2) standard Young tableaux mentioned above.

Our main objective in the present paper is to look on the multidimensional Catalan numbers $C(m,n) := C(m,n|1)$, defined as the value of the Kostka– Foulkes polynomials $K_{(n^m),(1^{mn})}(q)$ at $q=1$, from the point of view of Rigged Configurations Theory. In other words, we want to study the multidimensional Catalan and Narayana numbers introduced in [23], [34], by means of a fermionic formula for parabolic Kostka polynomials due to the author, e.g. [14], [17]. In particular, we apply the fermionic formula for parabolic Kostka polynomials cited above, to the study a stretched (parabolic) Kostka polynomials $K_{N\lambda, N\{\mathcal{R}\}}(q)$. At this way we obtain the following results.

Theorem 1.2. (*Strong polynomiality*)
Let λ be partition and $\{\mathcal{R}\}$ be a dominant sequence of rectangular shape partitions. Then

$$\sum_{N\ge 0} K_{N\lambda, N\mathcal{R}}(q)\ t^N = \frac{P_{\lambda,\mathcal{R}}(q,t)}{Q_{\lambda,\mathcal{R}}(q,t)},$$

were a polynomial $P_{\lambda,\mathcal{R}}(q,t)$ is such that $P_{\lambda,\mathcal{R}}(0,0)=1$;
a polynomial $Q_{\lambda,\mathcal{R}}(q,t) = \prod_{s\in S}(1-q^s t)$ for a some set of non-negative integers $S := S(\lambda,\mathcal{R})$, depending on data λ and $\mathcal{R}$.

Corollary 1.3. ([8], [28], [17])
Let λ be partition and $\{\mathcal{R}\}$ be a dominant sequence of rectangular shape partitions. Then

• *$\mathcal{K}_{\lambda,\mathcal{R}}(N) := K_{N\lambda, N\mathcal{R}}(1)$ is a polynomial of N with rational coefficients.*

• (*Littlewood–Richardson polynomials*, [22], [28], [17])
Let λ, μ and ν be partitions such that $|\lambda|+|\mu|=|\nu|$.
The Littlewood–Richardson number $c^{\nu}_{\lambda,\mu}(N) := c^{N\nu}_{N\lambda,N\mu}$ is a polynomial of N with rational coefficients.

Problem 1.4. *Compute* [10] *the degree of polynomial $\mathcal{K}_{\lambda,,\mathcal{R}}(N)$.*

Our next objective is to define a lattice convex polytope $\mathcal{P}(n,m)$ which has the δ-vector [11] equals to the sequence of multidimensional Narayana numbers $\{N(m,n;k|1),\ 1 \le k \le (m-1)(n-1)\}$, see [17], pp. 100–103.

As a preliminary step we recall the definition of a Gelfand -Tsetlin polytope.
Let $\lambda=(\lambda_1,\ldots,\lambda_n)$ be partition and $\mu=(\mu_1,\ldots,\mu_n)$ be composition, $|\lambda|=|\mu|$. The Gelfand– Tsetlin polytope of type (λ,μ), denoted by $GT(\lambda,\mu)$, is the convex hull of all points $(x_{ij})_{1\le i\le j\le n} \in \mathbb{R}_{+}^{\binom{n+1}{2}}$ which satisfy the following set of inequalities and equalities

$$x_{i,j+1} \ge x_{ij} \ge x_{i+1,j+1} \ge 0,\ x_{1j}=\lambda_j,\ 1 \le j \le n,\ \sum_{a=1}^{j} x_{aj} = \sum_{a=1}^{j} \mu_a.$$

It is well-known that the number of integer points in the Gelfand–Tsetlin polytope $GT(\lambda,\mu)$, i.e. points $(x_{ij}) \in GT(\lambda,\mu)$ such that $x_{ij} \in \mathbb{Z}_{\ge 0},\ \forall 1 \le i \le j \le n$, is equal to the Kostka number $K_{\lambda,\mu}(1)$. Therefore the stretched Kostka number $K_{N\lambda,N\mu}(1)$ counts the number of integer points in the polytope $GT(N\lambda,N\mu) = N \cdot GT(\lambda,\mu)$. As far as is we know, there is no general criterion to decide where or not the Gelfand–Tsetlin polytope $GT(\lambda,\mu)$ has only integral vertices, but see [4], [8], [1] for particular cases treated.

In the present paper we are interested in the h-vectors of Gelfand–Tsetlin polytopes $GT(n,1^d)$ and that $GT((n^k,1^{kd}),(1^k)^{n+d})$. We expect (cf [1]) that the polytope $GT(n,1^d)$ is an integral one, but we don't know

[10]It seems that the formulas for the degree of the stretched Kostka polynomials stated in [8], [28]. [17] are valid only for a special choice of λ, μ or $\mathcal{R}$.

[11]By definition the δ-vector of an integral convex polytope $\mathcal{P} \subset \mathbb{R}^N$ of dimension d is equal to

$$\delta(\mathcal{P}) = \sum_{J=0}^{d} \delta_j t^j = (1-t)^{d+1} \sum_{m=0}^{\infty} \iota(\mathcal{P},m)\ t^m,$$

where $\iota(\mathcal{P},m) := \#(m\mathcal{P} \cap \mathbb{Z}^N)$ denotes the number of integer points in the stretched polytope $m\mathcal{P} := \{mx \mid x \in \mathcal{P}\}$, $m \ge 1$, and we set $\iota(\mathcal{P},0)=1$.

how to describe the set of parameters (n,k,d) such that the polytope $GT((n^k,(1^k)^{n+d})$ is an integral one [12].

Theorem 1.5.
(1) *Let* $\lambda := \lambda_{n,d} = (n,1^d)$ *and* $\mu = \mu_{n,d} := (1^{n+d})$. *Then*

$$\sum_{N \ge 0} K_{N\lambda,N\mu}(q)\ t^N = \frac{C_{d,n-1}(q^{\binom{n}{2}}\ t,q)}{(q^{\binom{n}{2}}\ t;q))_{d(n-1)+1}},$$

where $C_{d,m}(t,q) = \sum_{k=1}^{(d-1)(m-1)} N(d,m,k \mid q)\ t^{k-1}$ *stands for a* (q,t)*–analogue of the rectangular* (d,n)*-Catalan number.*
In particular, the normalized volume of the Gelfand–Tsetlin polytope $GT((n,1^d),1^{n+d})$ *is equal to the d–dimensional Catalan number*

$$C_{d,n}(1,1) := (dn)!\ \prod_{j=0}^{d-1} \frac{j!}{(n+j)!} = f^{(n^d)} = f^{(d^n)},$$

(2) *Let* $\lambda := \lambda_{n,1,2} = (n^2,1^2)$ *and* $\mu = ((1,1)^{n+1})$, $n \ge 2$. *Then*

$$\sum_{N \ge 0} K_{N(n,n,1,1),N(1,1)^{n+1}}(1)\ t^N = \frac{P_{2,n}(t)}{(1-t)^{4n-6}},$$

and $P_{2,n}(1) = C_{n-3}\ C_{n-2}$, *i.e. equal to the product of two Catalan numbers.*
(3) *Let* $\lambda := \lambda_{n,k,d} = (n^k,1^{kd})$ *and* $\mu = ((1^k)^{n+d})$, $d \ge 1$. *Then*

$$\sum_{N \ge 0} K_{N(n^k,1^{kd}).N(1^k)^{n+d}}(1)\ t^N = \frac{P_{k,d,n}(t)}{Q_{k,d,n}(t)}.$$

Moreover, $P_{k,d,n}(0) = 1$,

$$Q_{k,d,n}(t) = (1-t)^{k^2(d(n-1)-1)+2+(k-1)\delta_{n,2}\ \delta_{d,1})},$$

and the polynomial $P_{k,d,n}(t)$ *is symmetric with respect to variable* t;

$$deg_t(P_{k,k,n}(t)) = (k-1)(k(n-2)+2(\delta_{n,2}-1)).$$

[12] Here we have used and will use throughout this paper, a standard notation

$$(1^k)^n = \underbrace{((1^k),\ldots,(1^k))}_{n},\ (1^k) = \underbrace{(1,\ldots,1)}_{k}.$$

One can see from Theorem 1.5, (1), that [13] the degree [14] of the stretched Kostka polynomial $\mathcal{K}_{(n,1),1^{n+1}}(N) := K_{N(n,1),N(1^{n+1})}(1)$ is equal to $n-1$, whereas it follows from Theorem 1.5, (2) that

$$deg_N(\mathcal{K}_{2(n,1),2(1)^{n+1}}(N)) = 4n-7 > 3\ deg_N(\mathcal{K}_{(n,1),1^{n+1}}(N)),\ if\ N > 4.$$

Therefore one comes to an infinite family of counterexamples to Okounkov log-concavity conjecture for the Littlewood–Richardson coefficients [26].

Corollary 1.6.

• *Let $n \geq 3$. There exists an integer $N_0(n)$ such that*

(1.1)
$$K_{2N(n,1),2N(1)^{n+1}}(1)\ >\ \Big(K_{N(n,1),N(1)^{n+1}}(1)\Big)^2\ for\ all\ N \geq N_0(n);$$

• *Let $n \geq 5$ be an integer, choose ϵ, $0 \leq \epsilon < \frac{n-4}{n-1}$. There is an integer $N_0(n;\epsilon)$ such that*

$$K_{2N(n,1),2N(1)^{n+1}}(1)\ >\ \Big(K_{N(n,1),N(1)^{n+1}}(1)\Big)^{3+\epsilon}\ for\ all\ N \geq N_0(n;\epsilon).$$

(3) *Let $n > 1 + \frac{k^2+2}{k^2 d}$. There exist an integer $N_0(n,k,d)$ such that*

$$K_{2N(n^k,1^{kd}),2N(1^k)^{n+d}}(1)\ >\ \Big(K_{N(n^k,1^{kd}),N(1^k)^{n+d}}(1)\Big)^3 \quad for\ all\ N \geq N_0(n,k,d).$$

[13] Let us say a few words about the case $d=1$ of Theorem 1.5. In this case, as easily seen from definition, $C_{1,n}(q,t) = 1$. Based on *rigged configurations* theory, see e.g. [17] and the literature quoted therein, one can prove that

$$K_{N(n,1),(N)^{n+1}}(q) = q^{N\binom{n}{2}} \begin{bmatrix} n+N-1 \\ n-1 \end{bmatrix}_q.$$

Therefore the the degree of the stretched Kostka polynomial $K_{N(n,1),(N)^{n+1}}(1)$, as a polynomial of N, is equal to $n-1$. Moreover, identity stated in Theorem 1.5, (1) in the case $d=1$ is a consequence of the well-known formula in the theory of hypergeometric functions, namely

$$\frac{1}{(t;q)_n} = \sum_{N \geq 0} \begin{bmatrix} n+N-1 \\ n-1 \end{bmatrix}_q t^N.$$

[14] Clearly that the degree of stretched Kostka polynomial $K_{N\lambda,N\mu}(1)$ as a polynomial of N, is equal to $\kappa - 1$, where $\kappa := \kappa(\lambda,\mu)$ is the order of the pole at $t=1$ of the series $\sum_{N\geq 0} K_{N\lambda,N\mu}(1)\ t^N$.

This Corollary is an easy consequence of the results stated in Theorem 1.5, namely that the degrees of stretched Kostka polynomials involved, are $4n-7$ and $n-1$ correspondingly.

For example,

- $K_{2N(5,1),2N(1,1)^6}(1) > (K_{N(5,1),N(1^6)}(1))^3$

if and only if $N \geq 49916$.

Let us recall the well-known fact that any parabolic Kostka number $K_{\lambda,\mathcal{R}}(1)$ can be realized as the Littlewood–Richardson coefficient $c_{\lambda,M}^{\Lambda}$ for uniquely defined partitions Λ and M, see Section 3.2 for details

It should be stressed that for $n=3$ the example (1.1) has been discovered in [3], and independently by the author (unpublished notes [19]). In this case the minimal value of $N_0(3)$ is equal to 23; one can show (A.K.) that $N_0(4)=8$.

Our next objective of the present paper is to prove the unimodality of the principal specialization $s_\alpha * s_\beta(q,\ldots,q^{N-1})$ of Schur functions [13], [14]. Proofs given in *loc. cit.* is based on an identification of the principal specialization of internal product of Schur functions with a certain parabolic Kostka polynomial.

Theorem 1.7. (**Principal specialization of the internal product of Schur functions and parabolic Kostka polynomials**, [17], *Theorem 6.6*)

Let α,β be partitions such that $|\alpha|=|\beta|$, $\alpha_1 \leq r$ and $\beta_1 \leq k$. For given integer N such that $\alpha_1+\beta_1 \leq Nr$, consider partition

$$\lambda_N := (rN-\beta_k^{'}, rN-\beta_{k-1}^{'}, \ldots, rN-\beta_1^{'}, \alpha^{'})$$

and a sequence of rectangular shape partitions

$$R_N := (\underbrace{(r^k),\ldots,(r^k)}_{N}).$$

Then [15]

$$K_{\lambda_N,R_N}(q) \doteq s_\alpha * s_\beta(q,\ldots,q^{N-1}). \tag{1.2}$$

[15] Hereinafter we shall use the notation $A(q) \doteq B(q)$ to mean that the ratio $A(q)/B(q)$ is a certain power of q.

Now we state a fermionic formula for polynomials

$$V_{\alpha,\beta}^{N}(q) := s_{\alpha} * s_{\beta}(q,\ldots,q^{N-1}),$$

which is our main tool to give a combinatorial proof of the unimodality of the principal specialization of the Schur functions, and that of the generalized q-Gaussian polynomials $\begin{bmatrix} N \\ \lambda \end{bmatrix}_q$ associated with a partition λ, as a special case.

Theorem 1.8. ([17], *Corollary 6.7*)
Let α and β be two partitions of the same size, and $r := \ell(\alpha)$ be the length of α. Then

$$s_{\alpha} * s_{\beta}(q,\ldots,q^{N-1}) =$$

$$\sum_{\{\nu\}} q^{c(\{\nu\})} \prod_{k,j\geq 1} \begin{bmatrix} P_j^{(k)}(\nu) + m_j(\nu^{(k)}) + N(k-1)\delta_{j,\beta_1}\theta(r-k) \\ P_j^{(k)}(\nu) \end{bmatrix}_q, \tag{1.3}$$

where the sum runs over the set of admissible configurations $\{\nu\}$ of type $([\alpha,\beta]_N,(\beta_1)^N)$. Here for any partition λ, λ_j denotes its j-th component.

See Section 4, Theorem 4.2 for details concerning notation. An important property which is specific to admissible configurations of type $[\alpha,\beta]_N,\beta_1^n)$, is the following relations

$$2c(\nu) + \sum_{k,j\geq 1} P_j^{(k)}(\nu)\Big[m_j(\nu^{(k)}) + N(k-1)\delta_{j,\beta_1}\theta(r-k)\Big] = N|\alpha|,$$

which imply the unimodality of polynomials $V_{\alpha,\beta}^{N}(q)$, and $\begin{bmatrix} N \\ \alpha \end{bmatrix}_q =$ $V_{\alpha,(|\alpha|)}^{N}(q)$. Let us stress that the sum in the $RHS(1.3)$ runs over the set of admissible configurations of type $([\alpha,\beta]_N,(\beta_1)^N)$. Remark, that the $RHS(1.3)$ has a natural generalization to the case $|\alpha| \equiv |\beta|$ $(mod\ N)$, but in this case a representation-theoretical meaning of the $LHS(1.3)$ is unclear to the author.

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§2. Higher dimensional Catalan and Narayana numbers, [23], [17], [34]

2.1. Rectangular Catalan and Narayana polynomials, and MacMahon polytope, [17]

2.1.1. *Rectangular Catalan and Narayana numbers and polynomials* Define **rectangular Catalan polynomial**

$$C(n,m|q)=\frac{(q;q)_{nm}}{\prod_{i=1}^{n}\prod_{j=1}^{m}(1-q^{i+j-1})}=[n\ m]_q!\prod_{j=0}^{d-1}\frac{[j]_q!}{[n+j]_q!},\tag{2.4}$$

where $[n]_q:=\frac{1-q^n}{1-q}$ stands for the q-analogue of an integer n, and by definition $[n]_q!:=\prod_{j=1}^{n}[j]_q$.

The next statement is apparent from the q-hook formula for the Kostka polynomials of a form $K_{\lambda,(1^{|\lambda|})}$, see e.g. [21], and (2.4).

Proposition 2.1. (*Cf* [17], (2.12))

$$q^{m\binom{n}{2}}C(n,m|q)=K_{(n^m),(1^{nm})}(q).\tag{2.5}$$

Thus, $C(n,m|q)$ is a polynomial of degree $nm(n-1)(m-1)/2$ in the variable q with non–negative integer coefficients. Moreover,

$$C(n,2|q)=C(2,n|q)=c_n(q)=\frac{1-q}{1-q^{n+1}}\begin{bmatrix}2n\\ n\end{bmatrix}_q$$

coincides with "the most obvious" q–analog of the Catalan numbers, see e.g. [5], p.255, or [32], and [23],

$$C(n,3|q)=\frac{[2]_q\ [3\ n]_q\ !}{[n]_q\ !\ [n+1]_q\ !\ [n+2]_q\ !}.$$

It follows from (2.5) that the rectangular Catalan number $C(n,m|1)$ counts the number of *lattice* words

$$w=a_1a_2\cdots a_{nm}$$

of weight (m^n), i.e. lattice words in which each i between 1 and m occurs exactly n times. Let us recall that a word $a_1 \cdots a_p$ in the symbols $1, \ldots, m$ is said to be a *lattice* word, if for $1 \le r \le p$ and $1 \le j \le m-1$, the number of occurrences of the symbol j in $a_1 \cdots a_r$ is not less than the number of occurrences of $j+1$:

$$\#\{i|1 \le i \le r \text{ and } a_i = j\} \ge \#\{i|1 \le i \le r \text{ and } a_i = j+1\}. \tag{2.6}$$

For any word $w = a_1 \cdots a_k$, in which each a_i is a positive integer, define the major index

$$\mathrm{maj}(w) = \sum_{i=1}^{k-1} i\chi(a_i > a_{i+1}),$$

and the number of descents

$$\mathrm{des}(w) = \sum_{i=1}^{k-1} \chi(a_i > a_{i+1}).$$

Finally, for any integer k between 0 and $(n-1)(m-1)$, define **rectangular q–Narayana** number

$$N(n, m; k \mid q) = \sum_{w} q^{\mathrm{maj}(w)},$$

where w ranges over all lattice words of weight (m^n) such that $\mathrm{des}(w) = k$.

Equivalently, $N(n, m; k)$ is equal to the number of rectangular standard Young tableaux with n rows and m columns having k descents, i.e. k occurrences of an integer j appearing in a lower row that that $j+1$.

Example 2.2. Take $n = 4$, $m = 3$, then

$$\sum_{k=0}^{6} N(3, 4; k \mid 1)t^k = 1 + 22t + 113t^2 + 190t^3 + 113t^4 + 22t^5 + t^6.$$

We summarize the basic known properties of the rectangular Catalan and Narayana numbers in Proposition 2.3 below.

Proposition 2.3. ([23], [34], [14])

(**A**) (**Lattice words and rectangular Catalan numbers**)

$C(n, m|q) = \sum_{w} q^{\mathrm{maj}(w)}$, *where w ranges over all lattice words of weight (m^n);*

(B) (Bosonic formula for multidimensional Narayana numbers)
(2.7)
$$N(n,m;k \mid q) = \sum_{a=0}^{k} (-1)^{k-a}\, q^{\binom{k-a}{2}} \left[\begin{array}{c} n\ m+1 \\ k-a \end{array} \right]_q \prod_{b=0}^{n-1} \frac{[b]!\ [m+a+b]!}{[m+b]!\ [a+b]!},$$

(C) (Summation formula) *Let r be a positive integer, then*

$$\sum_{k=0}^{r} \left[\begin{array}{c} n\ m+r-k \\ r-k \end{array} \right]_q N(n,m;k \mid q) = \prod_{a=0}^{m-1} \frac{[a]!\ [n+r+a]!}{[n+a]!\ [r+a]!}$$
$$= \prod_{a=0}^{n-1} \frac{[a]!\ [m+r+a]!}{[m+a]!\ [r+a]!} = \prod_{a=0}^{r-1} \frac{[a]!\ [n+m+a]!}{[n+a]!\ [m+a]!}.$$

(D) (Symmetry)

$$N(n,m;k \mid q) = q^{nm((n-1)(m-1)/2-k)} N(n,m;(n-1)(m-1)-k \mid q^{-1})$$
$$= N(m,n;k \mid q),$$

for any integer k, $0 \le k \le (n-1)(m-1)/2$;

(E) (q-Narayana numbers)

$$N(2,n;k \mid q) = q^{k(k+1)} \frac{1-q}{1-q^n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k+1 \end{bmatrix}_q \doteq \dim_q V_{(k,k)}^{\mathfrak{gl}(n-k+1)},$$
$$0 \le k \le n-1,$$

where $V_{(k,k)}^{\mathfrak{gl}(n-k+1)}$ stands for the irreducible representation of the Lie algebra $\mathfrak{gl}(n-k+1)$ corresponding to the two row partition (k,k); recall that for any finite dimensional $\mathfrak{gl}(N)$–module V the symbol $\dim_q V$ denotes its q–dimension, i.e. the principal specialization of the character of the module V:

$$\dim_q V = (\mathrm{ch} V)(1,q,\ldots,q^{N-1});$$

(F) $N(n,m;1 \mid 1) = \sum_{j \ge 2} \binom{n}{j}\binom{m}{j} = \binom{n+m}{n} - nm - 1;$

(G) (Fermionic formula for q–Narayana numbers, [17])

(2.8) $$q^{m\binom{n}{2}} N(n,m;l \mid q) = \sum_{\{\nu\}} q^{c(\nu)} \prod_{k,j \ge 1} \left[\begin{array}{c} P_j^{(k)}(\nu) + m_j(\nu^{(k)}) \\ m_j(\nu^{(k)}) \end{array} \right]_q,$$

summed over all sequences of partitions $\{\nu\} = \{\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(m-1)}\}$ *such that*

- $|\nu^{(k)}| = (m-k)n$, $1 \le k \le m-1$;
- $(\nu^{(1)})'_1 = (m-1)n - l$, *i.e. the length of the first column of the diagram* $\nu^{(1)}$ *is equal to* $(m-1)n - l$, $l = 0, \ldots, (m-1)(n-1)$;
- $P_j^{(k)}(\nu) := Q_j(\nu^{(k-1)}) - 2Q_j(\nu^{(k)}) + Q_j(\nu^{(k+1)}) \ge 0$, *for all* $k, j \ge 1$,

where by definition we put $\nu^{(0)} = (1^{nm})$; *for any diagram* λ *the number* $Q_j(\lambda) = \lambda'_1 + \cdots \lambda'_j$ *is equal to the number of cells in the first* j *columns of the diagram* λ, *and* $m_j(\lambda)$ *is equal to the number of parts of* λ *of size* j;

- $c(\nu) = \sum_{k,j \ge 1} \binom{(\nu^{(k-1)})'_j - (\nu^{(k)})'_j}{2}$.

Example 2.4. *Consider the case* $m = 3$, $n = 4$. *In this case* $C(3, 4 \mid 1) = 462$, *and the sequences of Narayana numbers is* $(1, 22, 113, 190, 113, 22, 1)$. *Let us display below the distribution of Narayana numbers which is coming from the counting the number of admissible rigged configurations of type* $((4^3), (1^{12}))$ *according to the number* $(m-1)n - \ell(\nu^{(1)})$, *where* $\ell(\nu^{(1)})$ *denotes the length of the first configuration* $\nu^{(1)}$:
$N(3, 4; 0 \mid 1) = 1$, $N(3, 4; 1 \mid 1) = 1 + 21$, $N(3, 4; 2 \mid 1) = 15 + 35 + 63$,
$N(3, 4; 3 \mid 1) = 140 + 15 + 35$, $N(3, 4; 4 \mid 1) = 21 + 28 + 63$,
$N(3, 4; 5 \mid 1) = 6 + 16$, $N(3, 4; 6 \mid 1) = 1$.

Conjecture 2.5. *If* $1 \le k \le (n-1)(m-1)/2$, *then*

$$N(n, m; k-1 \mid 1) \le N(n, m; k \mid 1),$$

i.e. the sequence of rectangular Narayana numbers
$\{N(n, m; k \mid 1)\}_{k=0}^{(n-1)(m-1)}$ *is symmetric and* **unimodal**.

For definition of unimodal polynomials/sequences see e.g. [31], where one may find a big variety of examples of unimodal sequences which frequently appear in Algebra, Combinatorics and Geometry.

2.1.2. *Volume of the MacMahon polytope and rectangular Catalan and Narayana numbers* Let $\mathfrak{M}_{mn}$ be the convex polytope in $\mathbb{R}^{nm}$ of all points $\mathbf{x} = (x_{ij})_{1 \le i \le n, 1 \le j \le m}$ satisfying the following conditions

$$0 \le x_{ij} \le 1, \; x_{ij} \ge x_{i-1,j}, \; x_{ij} \ge x_{i,j-1}, \tag{2.9}$$

for all pairs of integers (i, j) such that $1 \le i \le n$, $1 \le j \le m$, and where by definition we set $x_{i0} = 0 = x_{0j}$.

We will call the polytope $\mathfrak{M}_{nm}$ by *MacMahon polytope.* The MacMahon polytope is an integral polytope of dimension nm with $\binom{m+n}{n}$ vertices which correspond to the set of (0,1)–matrices satisfying (2.9).

If k is a positive integer, define $i(\mathfrak{M}_{nm};k)$ to be the number of points $\mathbf{x} \in \mathfrak{M}_{nm}$ such that $k\mathbf{x} \in \mathbb{Z}^{nm}$. Thus, $i(\mathfrak{M}_{nm};k)$ is equal to the number of plane partitions of rectangular shape (n^m) with all parts do not exceed k. By a theorem of MacMahon (see e.g. [21], Chapter I, §5, Example 13)

$$i(\mathfrak{M}_{nm};k) = \prod_{i=1}^{n}\prod_{j=1}^{m} \frac{k+i+j-1}{i+j-1}. \tag{2.10}$$

It follows from (2.10) that the Ehrhart polynomial $\mathcal{E}(\mathfrak{M}_{nm};t)$ of the MacMahon polytope $\mathfrak{M}_{nm}$ is completely resolved into linear factors:

$$\mathcal{E}(\mathfrak{M}_{nm};t) = \prod_{i=1}^{n}\prod_{j=1}^{m} \frac{t+i+j-1}{i+j-1}.$$

Hence, the normalized volume

$$\widetilde{\mathrm{vol}}(\mathfrak{M}_{nm}) = (nm)!\mathrm{vol}(\mathfrak{M}_{nm})$$

of the MacMahon polytope $\mathfrak{M}_{nm}$ is equal to the rectangular Catalan number $C(n,m|1)$, i.e. the number of standard Young tableaux of rectangular shape (n^m). We refer the reader to [32], Section 4.6, and [7], Chapter IX, for definition and basic properties of the Ehrhart polynomial $\mathcal{E}(\mathfrak{P};t)$ of a convex integral polytope $\mathfrak{P}$.

Proposition 2.6. (Cf [17], (2.17))

$$\sum_{k\geq 0} i(\mathfrak{M}_{nm};k)z^k = \left(\sum_{j=0}^{(n-1)(m-1)} N(n,m;j)z^j\right) / (1-z)^{nm+1}, \tag{2.11}$$

where

$$N(n,m;j) := N(n,m;j|1)$$

denotes the rectangular Narayana number.

Thus, the sequence of Narayana numbers

$$(1 = N(n,m;0), N(n,m;1), \ldots, N(n,m;(n-1)(m-1)) = 1)$$

is the δ-*vector* (see e.g. [32], p. 235) of the MacMahon polytope. In the case $n=2$ (or $m=2$) all these results may be found in [32], Chapter 6, Exercise 6.31.

Question. (*Higher associahedron*) Does there exist an $(m-1)(n-1)$-dimensional integral convex (simplicial?) polytope $Q_{n,m}$ which has δ-vector

$$\delta = (\delta_0(Q_{n,m}), \delta_1(Q_{n,m}), \ldots, \delta_{(n-1)(m-1)}(Q_{n,m}))$$

given by the *rectangular Narayana numbers* $N(n,m;k)$:

$$\sum_{i=0}^{(n-1)(m-1)} \delta_i(Q_{n,m}) t^i = C(n,m|t) \ ?$$

We refer the reader to [7], Chapter I, §6 and Chapter III, for definitions and basic properties of the h-vector and δ-vector of a simplicial polytope; see also, R. Stanley (J. Pure and Appl. Algebra **71** (1991), 319-331).

An answer on this question is known if either n or m is equal to 2, see e.g. R. Simion (Adv. in Appl. Math. **18** (1997), 149-180, Example **4** (the Associahedron)).

Definition 2.7. ([23], [34]) *Define* **rectangular** *Schröder polynomial*

$$S(n,m|t) := C(n,m|1+t),$$

and put

$$S(n,m|t) = \sum_{k\geq 0}^{(n-1)(m-1)} S(n,m||k) t^k.$$

A combinatorial interpretations of the numbers $S(n,m||k)$ and $S(n,m|1)$ have been done by R. Sulanke [34].

2.1.3. *Rectangular Narayana and Catalan numbers, and d dimensional lattice paths*, [34] Let $\mathcal{C}(d,n)$ denote the set of d–dimensional lattice paths using the steps

$$X_1 = (1,0,\cdots,0), \ X_2 = (0,1,\cdots,0), \cdots, X_d = (0,0,\cdots,1),$$

running from $(0,0,\cdots,0)$ to $(n,n,\cdots,n)$, and lying in the region

$$\{(x_1,x_2,\cdots,x_d) \in \mathbb{R}^d_{\geq 0} \mid x_1 \leq x_2 \leq \cdots \leq x_d\}.$$

For each path $P := p_1 p_2 \cdots p_{nd} \in \mathcal{C}(d,n)$ define the statistics

$$asc(P) := \#\{j \mid p_j p_{j+1} = X_k \ X_l, \ k < l\}.$$

Definition 2.8. *The n–th d–dimensional MacMahon–Narayana number of level k, $MN(d,n,k)$ counts the paths $P \in \mathcal{C}(d,n)$ with $asc(P) = k$.*

Proposition 2.9. (*Cf* [34]) *For any $d \ge 2$ and for $0 \le k \le (d-1)(n-1)$,*

$$MN(d,n,k) = \sum_{j=0}^{k} (-1)^{k-j} \binom{dn+1}{k-j} \prod_{a=0}^{j-1} \frac{a!\ (d+n+a)!}{(d+a)!\ (n+a)!}.$$

Note that the product $\prod_{a=0}^{j-1} \frac{a!\ (d+n+a)!}{(d+a)!\ (n+a)!}$ is equal to the number of plane partitions of the rectangular shape (n^d), all the parts do not exceed j.

Definition 2.10. *For $d \ge 3$ and $n \ge 1$ the n-th d-Narayana polynomial defined to be*

$$N_{d,n}(t) = \sum_{k=0}^{(d-1)(n-1)} MN(d,n,k)\ t^k.$$

Corollary 2.11. (*Recurrence relations*, [34]) *For any integer $m \ge 0$ one has*

$$\sum_{k=0}^{m} \binom{dn+m-k}{m-k} MN(d,n,k) = \prod_{a=0}^{d-1} \frac{a!\ (n+m+a)!}{(n+a)!\ (m+a)!}.$$

Corollary 2.12. *The MacMahon–Narayana number $MN(d,n,k)$ is equal to the rectangular Narayana number $N(d,n;k)$.*

This Corollary follows from Proposition 2.3, (**B**) with $q = 1$, and Proposition 2.9.

2.1.4. *Gelfand–Tsetlin polytope $GT((n,1^d),(1)^{n+d})$ and rectangular Narayana numbers*

Theorem 2.13. *Let $\lambda := \lambda_{n,d} = (n,1^d)$ and $\mu = \mu_{n,d} := (1^{n+d})$. Then*

$$\sum_{N \ge 0} K_{N\lambda,N\mu}(q)\ t^N = \frac{C_{d,n-1}(q^{\binom{n}{2}}\ t, q)}{(q^{\binom{n}{2}}\ t; q))_{d(n-1)+1}},$$

where $C_{d,m}(t,q) = \sum_{k=0}^{(d-1)(m-1)} N(d,m,k \mid q)\ t^k$ stands for a (q,t)–analog of the rectangular (d,n)-Catalan number.
In particular, the normalized volume of the Gelfand–Tsetlin polytope $GT((n,1^d),1^{n+d})$ is equal to the d–dimensional Catalan number

$$C_{d,n}(1,1) := (dn)! \prod_{j=0}^{d-1} \frac{j!}{(n+j)!} = f^{(n^d)} = f^{(d^n)},$$

where for any partition λ, f^{λ} *denotes the number of standard Young tableaux of shape* λ.

The proof of Theorem 2.13 (as well as Theorem 3.1, (2)) is rather long and technical, and is based essentially on the properties of **Rigged Configuration Bijection**, cf [17], [18], and will appear in a separate publication.

§3. Rigged configurations, stretched Kostka numbers, log-concavity and unimodality

3.1. Stretched Kostka numbers $K_{N(n^k.1^{kd}),N(1^k)^{n+d}}(1)$

Theorem 3.1. (1)

$$\sum_{N\geq 0} K_{N(n,n,1,1)),N((1,1)^{n+1})}(1)\ t^N = \frac{P_{2,n}(t)}{(1-t)^{4n-6}},$$

and $P_{2,n}(1)=C_{n-3}\ C_{n-2}$.

(2) *Let* $d\geq 1$, *then*

$$\sum_{N\geq 0} K_{N(n^k,1^{kd}).N(1^k)^{n+d}}(1)\ t^N = \frac{P_{k,d,n}(t)}{Q_{k,d,n}(t)}.$$

Moreover, $P_{k,d,n}(0)=1$,

$$Q_{k,d,n}(t)=(1-t)^{k^2(d(n-1)-1)+2+(k-1)\delta_{n,2}\ \delta_{d,1})},$$

and the polynomial $P_{k,d,n}(t)$ *is symmetric with respect to variable* t;

$$deg_t(P_{k,k,n}(t)) = (k-1)(k(n-2)+2(\delta_{n,2}-1)).$$

■

For example, assume that $d=1$ and set $P_{k,n}(t):=P_{k,1,n}(t)$. Then
$P_{2,3}(t)=1$, $P_{2,4}(t)=(1,0,1)$, $P_{2,5}(t)=(1,1,6,1,1)$,
$P_{2,6}(t)=(1,3,21,20,21,3,1)$, $P_{2,7}(t)=(1,6,56,126,210,126,56,6,1)$,
$P_{2,8}(t)=(1,10,125,500,1310,1652,1310,500,125,10,1)$,
$P_{3,3}(t)=(1,-1,1)$, $P_{3,4}(t)=(1,0,20,20,55,20,20,0,1)$,
$P_{3,5}(t)=(1,6,141,931,4816,13916,27531,33391,27531,13916,4816,931,141,6,1)$,
$P_{4,1,3}(t)=(1,-3,9,-8,9,-3,1)=P_{4,2,2}(t)$.

It follows from the duality theorem for parabolic Kostka polynomials [17] that

$$K_{(Nn,N^d)',((N)^{n+d})')}(1) = K_{((d+1)^N,1^{N(n-1)}),(1^N)^{n+d})}(1),$$

and

$$K_{((2d+2)^N,2^{N(n-1)}),((2)^N)^{n+d}}(1) = K_{(Nn,Nn,N^{2d}),((N,N)^{n+d})}(1).$$

Now consider the case $d = 1$, that is $\lambda = (n, 1)$, $\mu = (1^{n+1})$. Then

$$K_{N\lambda,N\mu}(1) = K_{(Nn,N),(N^{n+1})}(1) = \binom{N+n-1}{n-1}.$$

The second equality follows from a more general result [13], [17],

Proposition 3.2. *Let λ be a partition and N be a positive integer. Consider partitions $\lambda_N := (N|\lambda|, \lambda)$ and $\mu_N := (|\lambda|^{N+1}) = (\underbrace{|\lambda|, \ldots, |\lambda|}_{N+1})$. Then*

$$K_{\lambda_N,\mu_N}(q) \stackrel{\bullet}{=} \begin{bmatrix} N \\ \lambda \end{bmatrix} = dim_q V_\lambda^{\mathfrak{gl}(N)},$$

where the symbol $P(q) \stackrel{\bullet}{=} R(q)$ means that the ratio $P(q)/R(q)$ is a power of q; the symbol $\left[\begin{smallmatrix} N \\ \lambda \end{smallmatrix}\right]$ stands for the generalized Gaussian coefficient corresponding to a partition λ, see [21] *for example.*

3.2. Counterexamples to Okounkov's log-concavity conjecture

On the other hand,

$$K_{2\lambda_N,2\mu_N}(1) = K_{N(n,n,1,1),N(1,1)^{n+1}}(1) = Coeff_{t^N}\left(\frac{P_{2,n}(t)}{(1-t)^{4n-6}}\right).$$

Therefore the number $K_{2\lambda_N,2\mu_N}(1)$ is a **polynomial** of the degree $4\,n-7$ with respect to parameter N. Recall that the number $K_{N\lambda,N\mu}(1) = \binom{N+n-1}{n-1}$ is a polynomial of degree $n-1$ with respect to parameter N. Therefore we come to the following infinite set of examples which violate the log-concavity Conjecture stated by A. Okounkov [27].

Corollary 3.3. *For any integer $n > 4$, there exists a constant $N_0(n)$ such that*

$$K_{2\lambda_N,2\mu_N}(1) > \Big(K_{N\lambda,N\mu}(1)\Big)^3,$$

for all $N > N_0(n)$.

Recall that $\lambda = (n, 1)$, $\mu = (1^{n+1})$.

<u>Now take $n = 3$.</u> One has [3]

$$K_{N(3,1),N(1^4)}(1) = \binom{N+2}{2}, \; K_{N(3,3,1,1),N(1,1)^4}(1) = \binom{N+5}{5}.$$

One can check [3] that

$$K_{N(3,3,1,1),N(1,1)^4}(1) > \Big(K_{N(3,1),N(1^3)}(1)\Big)^2.$$

if (and only if) $N \geq 21$.

Indeed,

$$K_{N(3,3,1,1),N(1,1)^4}(1) - \Big(K_{N(3,1),N(1^3)}(1)\Big)^2 = \frac{N^2 - 18N - 43}{20} \binom{n+2}{3}.$$

<u>Now take $n = 4$.</u> One has

$$K_{N(4,1),N(1^5)}(1) = \binom{N+3}{3},$$
$$K_{N(4,4,1,1),N(1,1)^5}(1) = \binom{N+9}{9} + \binom{N+7}{9}.$$

One can check that

$$K_{N(4,4,1,1),N(1,1)^5}(1) > \Big(K_{N(4,1),N(1^5)}(1)\Big)^2,$$

if (and only if) $N \geq 8$.

<u>Now take $n = 5$.</u>

Proposition 3.4. *Let $\nu_N := N(5,1)$ and $\eta_N := N(1)^6$. Then*

- $K_{2\nu_N,2\eta_N}(1) > (K_{\nu_N,\eta_N}(1))^2$

if and only if $N \geq 6$,

- $K_{2\nu_N,2\eta_N}(1) > (K_{\nu_N,\eta_N}(1))^3$

if and only if $N \geq 49916$.

Indeed,

$$K_{N(5,1),N(1^6)}(1) = \binom{N+4}{4},$$

$$K_{N(5,5,1,1),N(1,1)^6}(1) =$$
$$\binom{N+13}{13} + \binom{N+12}{13} + 6\binom{N+11}{13} + \binom{N+10}{13} + \binom{N+9}{13},$$

and $51891840\times \left[K_{N(5,5,1,1),N(1,1)^6}(1)-\left(K_{N(5,1),N(1^6)}(1)\right)^3\right]=\binom{N+4}{5}\times$ $(-78631416-172503780\ N-174033932\ N^2-101206400\ N^3$ $-35852065\ N^4-7638110\ N^5-899548\ N^6-44990\ N^7+N^8)$.

Note, see e.g. [21], that for any set of partitions $\lambda,\mu^{(1)},\ldots,\mu^{(p)}$ the parabolic Kostka number $K_{\lambda,\mu^{(1)},\ldots,\mu^{(p)}}(1)$ is equal to the Littlewood–Richardson number $c^{\Lambda}_{\lambda,M}$, where partitions $\Lambda\supset M$ are such that $\Lambda\backslash M=\coprod_i \mu^{(i)}$ is a disjoint union of partitions $\mu^{(i)},\ i=1,\ldots,p$.

§4. Internal product of Schur functions

The irreducible characters χ^{λ} of the symmetric group S_n are indexed in a natural way by partitions λ of n. If $w\in S_n$, then define $\rho(w)$ to be the partition of n whose parts are the cycle lengths of w. For any partition λ of m of length l, define the power–sum symmetric function

$$p_\lambda=p_{\lambda_1}\cdots p_{\lambda_l},$$

where $p_n(x)=\sum x_i^n$. For brevity write $p_w:=p_{\rho(w)}$. The Schur functions s_λ and power–sums p_μ are related by a famous result of Frobenius

$$s_\lambda=\frac{1}{n!}\sum_{w\in S_n}\chi^{\lambda}(w)p_w. \tag{4.1}$$

For a pair of partitions α and β, $|\alpha|=|\beta|=n$, let us define the internal product $s_\alpha * s_\beta$ of Schur functions s_α and s_β:

$$s_\alpha * s_\beta=\frac{1}{n!}\sum_{w\in S_n}\chi^{\alpha}(w)\chi^{\beta}(w)p_w. \tag{4.2}$$

It is well–known that

$$s_\alpha * s_{(n)}=s_\alpha,\ \ s_\alpha * s_{(1^n)}=s_{\alpha'},$$

where α' denotes the conjugate partition to α.

Let α,β,γ be partitions of a natural number $n\geq 1$, consider the following numbers

$$g_{\alpha\beta\gamma}=\frac{1}{n!}\sum_{w\in S_n}\chi^{\alpha}(w)\chi^{\beta}(w)\chi^{\gamma}(w). \tag{4.3}$$

The numbers $g_{\alpha\beta\gamma}$ coincide with the structural constants for multiplication of the characters χ^α of the symmetric group S_n:

$$\chi^\alpha\chi^\beta=\sum_\gamma g_{\alpha\beta\gamma}\chi^\gamma. \tag{4.4}$$

Hence, $g_{\alpha\beta\gamma}$ are non–negative integers. It is clear that

$$s_\alpha * s_\beta = \sum_\gamma g_{\alpha\beta\gamma} s_\gamma. \tag{4.5}$$

4.1. Internal product of Schur functions, principal specialization, fermionic formulas and unimodality

Let $N \geq 2$, consider the principal specialization $x_i = q^i$, $1 \leq i \leq N-1$, and $x_i = 0$, if $i \geq N$, of the internal product of Schur functions s_α and s_β:

(4.6)

$$s_\alpha * s_\beta(q, q^2, \ldots, q^{N-1}) = \frac{1}{n!} \sum_{w \in S_n} \chi^\alpha(w) \chi^\beta(w) \prod_{k \geq 1} \left(\frac{q^k - q^{kN}}{1-q^k} \right)^{\rho_k(w)},$$

where $\rho_k(w)$ denotes the number of the length k cycles of w.

By a result of R.-K. Brylinski [2], Corollary 5.3, the polynomials

$$s_\alpha * s_\beta(q, \ldots, q^{N-1})$$

admit the following interpretation. Let $P_{n,N}$ denote the variety of n by n complex matrices z such that $z^N = 0$. Denote by

$$R_{n,N} := \mathbb{C}[P_{n,N}]$$

the coordinate ring of polynomial functions on $P_{n,N}$ with values in the field of complex numbers $\mathbb{C}$. This is a graded ring:

$$R_{n,N} = \oplus_{k \geq 0} R_{n,N}^{(k)},$$

where $R_{n,N}^{(k)}$ is a finite dimensional $\mathfrak{gl}(n)$–module with respect to the adjoint action. Let α and β be partitions of common size. Then [2]

$$s_\alpha * s_\beta(q, \ldots, q^{N-1}) = \sum_{k \geq 0} \langle V_{[\alpha,\beta]_n}, P_{n,N}^{(k)} \rangle q^k,$$

as long as $n \geq \max(Nl(\alpha), Nl(\beta), l(\alpha)+l(\beta))$. Here the symbol $\langle \bullet\ , \bullet\ \rangle$ denotes the scalar product on the ring of symmetric functions such that $\langle s_\lambda\ , s_\mu\ \rangle = \delta_{\lambda,\mu}$. In other words, if V and W be two $GL(N)$-modules, then $\langle V, W \rangle = \dim Hom_{GL(N)}(V, W)$.

Let us remind below one of the main result obtained in [17], namely, Theorem 6.6, which connects the principal specialization of the internal product of Schur functions with certain parabolic Kostka polynomials, and gives, via Corollary 6.7, [17], an effective method for computing the polynomials $s_\alpha * s_\beta(q,\ldots,q^{N-1})$ which, turns out to be for the first time, does not use the character table of the symmetric group S_n, $n=|\alpha|$.

Let α and β be partitions, $\ell(\alpha)=r$, $\ell(\beta)=s$ and $|\alpha|=|\beta|$. Let N be an integer such that $r+s<N$. Consider partition

$$[\alpha,\beta]_N := (\alpha_1+\beta_1,\alpha_2+\beta_1,\ldots,\alpha_r+\beta_1,\underbrace{\beta_1,\ldots,\beta_1}_{N-r-s},\beta_1-\beta_s,\ldots,\beta_1-\beta_2].$$

Clearly, $|[\alpha,\beta]_N|=\beta_1 N$, $\ell([\alpha,\beta]_N=N-1$.

Theorem 4.1. i) *Let α,β be partitions, $|\alpha|=|\beta|$, $l(\alpha)\le r$, and $l(\alpha)+l(\beta)\le Nr$. Consider the sequence of rectangular shape partitions*

$$R_N=\{\underbrace{(\beta_1^r),\ldots,(\beta_1^r)}_{N}\}.$$

Then

$$s_\alpha * s_\beta(q,\ldots,q^{N-1}) \doteq K_{[\alpha,\beta]_{\mathbf{Nr}},R_N}(q). \tag{4.7}$$

ii) (*Dual form*) *Let α,β be partitions such that $|\alpha|=|\beta|$, $\alpha_1\le r$ and $\beta_1\le k$. For given integer N such that $\alpha_1+\beta_1\le Nr$, consider partition*

$$\lambda_N := (rN-\beta_k^{'},rN-\beta_{k-1}^{'},\ldots,rN-\beta_1^{'},\alpha^{'})$$

and a sequence of rectangular shape partitions

$$R_N := (\underbrace{(r^k),\ldots,(r^k)}_{N}).$$

Then

$$K_{\lambda_N R_N}(q) \doteq s_\alpha * s_\beta(q,\ldots,q^{N-1}) \tag{4.8}$$

Theorem 4.2. (**Fermionic formula for the principal specialization of the internal product of Schur functions**).

Let α and β be two partitions of the same size, and $r:=\ell(\alpha)$ be the length of α. Then

$$\begin{aligned} & s_\alpha * s_\beta(q,\ldots,q^{N-1}) = \\ & \sum_{\{\nu\}} q^{c(\{\nu\})} \prod_{k,j\ge 1} \begin{bmatrix} P_j^{(k)}(\nu)+m_j(\nu^{(k)})+N(k-1)\delta_{j,\beta_1}\theta(r-k) \\ P_j^{(k)}(\nu) \end{bmatrix}_q, \end{aligned} \tag{4.9}$$

where the sum runs over the set of admissible configurations $\{\nu\}$ of type $([\alpha,\beta]_N,(\beta_1)^N)$. Here for any partition λ, λ_j denotes its j-th component.

Let us explain notations have used in Theorem 4.2.

• A configuration $\{\nu\}$ of type $[\alpha,\beta]_N$ consists of a collection of partitions $\{\nu^{(1)},\ldots,\nu^{(N-1)}\}$ such that $|\nu^k| = \sum_{j>k}\Big([\alpha,\beta]_N\Big)_j$; by definition we set $\nu^{(0)} := (\beta_1)^N$;

• $P_j^{(k)}(\nu) := N\min(j,\beta_1)\ \delta_{k,1} + Q_j(\nu^{(k-1)}) - 2Q_j(\nu^{(k)}) + Q_j(\nu^{k+1})$; here for any partition λ we set $Q_n(\lambda) := \sum_{j\le n}\ \min(n,\lambda_j)$;

• For any partition λ, $m_j(\lambda)$ denotes the number of parts of λ are equal to j;

• A configuration $\{\nu\}$ of type $[\alpha,\beta]_N$ is called *admissible configuration of type* $([\alpha,\beta]_N,(\beta_1)^N)$, if $P_j^{(k)}(\nu)\ge 0,\ \forall j,k\ge 1$;

• Here $\delta_{n,m}$ denotes Kronecker's delta function, and we define $\theta(x) = 1,\ if\ x\ge 0$, and $\theta(x)=0,\ if\ x<0$;

• $c(\{\nu\}) = \sum_{n,k\ge 1}\binom{\lambda_n^{(k-1)}-\lambda_n^{(k)}}{2}$ denotes the **charge** of a configuration $\{\nu\}$; by definition, $\binom{x}{2} := x(x-1)/2,\ \forall x\in\mathbb{R}$.

Let us draw attention to the fact that the summation in (4.9) runs over the set of all admissible configurations of type $([\alpha,\beta]_N,(\beta_1)^N)$, other than that of type $([\alpha,\beta]_{Nr},(\beta_1^r)^N)$.

Corollary 4.3. ([13], [18])

For any partitions of the same size α and β, the polynomial $s_\alpha * s_\beta(q,\ldots,q^{N-1})$ is symmetric and **unimodal**. *In particular, the generalized Gaussian polynomial $\begin{bmatrix} N\\ \alpha\end{bmatrix}_q$ is symmetric and* **unimodal** *for any partition α.*

Indeed, in the case $\beta=(n)$, $n:=|\beta|$, one has $s_\alpha * s_{(n)} = s_\alpha$.

Our proof of Corollary 4.3 is proceeded by induction on size λ and the following identity

$$2c(\nu) + \sum_{k,j\ge 1} P_j^{(k)}(\nu)\Big[m_j(\nu^{(k)}) + N(k-1)\delta_{j,\beta_1}\theta(r-k)\Big] = N|\alpha|,$$

which can be checked directly by the use of properties of admissible configurations, see e.g., either [14] or [17] for details. This identity shows that the all polynomials associated with a given admissible configuration involved, are symmetric and have the same "symmetry center" $N|\alpha|/2$, and therefore the resulting polynomial $s_\alpha * s_\beta(q,\ldots,q^{N-1})$ is symmetric and <u>unimodal</u>. The latter statement is a consequence of the induction assumption, since the all q-binomial coefficient which appear in the $RHS(4.9)$ correspond to partitions of the form (m), $m<|\lambda|$, and plus the well-known fact, see e.g., [31], that the product of symmetric and unimodal polynomials is also symmetric and unimodal.

• (Combinatorial Hard Lefschetz Theorem for internal product of Schur functions)

We want to stress that in fact the Rigged Configuration Bijection defines an *embedding* of the sets $PL_k(\alpha,\beta) \subset PL_{k+1k}(\alpha,\beta)$, if $k < N|\alpha|/2$, where the set $PL_k(\alpha,\beta)$ denotes the set of all rigged configurations $(\{\nu\},\mathbf{J})$ with **charge** equal to k, see e.g.,[17] or Appendix to the preset paper. Note that in the special case $\alpha=(Nm,m)$ and $\beta=(N(m+1))$ the Rigged Configuration Bijection (essentially) *coincides* with the bijection constructed by K.O'Hara in [26], see [14] for details.

Corollary 4.4. *Let α and β be partitions of the same size, and $K_{\beta,\alpha}(q,t)$ denotes the Kostka–Macdonald polynomial associated with partitions α and β,* [21]*. One has*

$$K_{\beta,\alpha}(q,q) = H_\alpha(q)\left(\sum_{\{\nu\}} q^{c(\{\nu\})} \prod_{k=2}^{r} \frac{1}{\left[m_{\beta_1}(\nu^{(k)})\right]_q!} \prod_{\substack{k\geq 1\\ j\geq 1,\ j\neq\beta_1}} \begin{bmatrix} P_j^{(k)}(\nu)+m_j(\nu^{(k)}) \\ m_j(\nu^{(k)}) \end{bmatrix}_q\right),$$

where the sum runs over the same set of admissible configurations as in Theorem 4.2, and $[m]_q! := \prod_{j=1}^{m}(1-q^j)$ stands for the q-factorial of an positive integer m, and by definition $[0]_q! = 1$; $H_\alpha(q) := \prod_{x\in\alpha}(1-q^{h(x)})$ denotes the hook polynomial associated with partition α, see e.g. [21].

Indeed, one can show [30] that

$$s_\alpha * s_\beta(1,q,q^2,\ldots) = \frac{K_{\beta,\alpha}(q,q)}{H_\alpha(q)},$$

and therefore,

$$\frac{K_{\beta,\alpha}(q,q)}{H_\alpha(q)} = \lim_{N\to\infty} s_\alpha * s_\beta(1,q,q^2,\ldots,q^N) = \lim_{N\to\infty} K_{[\alpha,\beta]_N,(\beta_1)^N}(q).$$

Now by using the fermionic formula from Theorem 4.2, one can prove, see [14], [13], the formula stated in Corollary 4.4.

A fermionic formula for the principal specialization of the internal product of Schur functions, and therefore that for the generalized Gaussian polynomials, is a far generalization of the so-called KOH-identity [26] which is equivalent to the fermionic formula for the Kostka number $K_{(Nk,k),(k)^{N+1}}(1)$. The rigged configuration bijection gives rise to a combinatorial proof of Theorem 4.2, and therefore to a combinatorial proof of unimodality of the generalized Gaussian polynomials [13], as well as to give an interpretation of the statistics introduced in [26] in terms of rigged configurations data, see [14], Section 10.2.

Example 4.5. Let $\alpha = (4,2),\ \beta = \alpha^{'} = (2,2,1,1)$. We want to compute the principal specialization of the internal product of Schur functions $s_\alpha * s_\beta(q,\ldots,q^{N-1})$ by means of a fermionic formula (4.9). First of all, there are 8 admissible configurations of type $(\lambda = [\alpha = (4,2), \beta = (2,2,1,1)]_N, \mu = (6)^N)$. In fact, it is a general fact that for given partitions λ and μ, the number of admissible configurations of type $(N\lambda, N\mu)$ **doesn't depend** on N, if $N > N_0$ for a certain number $N_0 := N_0(\lambda,\mu)$ depending on λ and μ only. This fact is a direct consequence of constraints are imposed by the set of inequalities $\{P_j^{(k)}(\nu) \geq 0,\ \forall j,k \geq 1\}$.
Now let us list the **conjugate** of the first configurations $\nu^{(1)} \in \{\nu\}$ for all admissible configurations $\{\nu\}$ of type $(\lambda = [(4,2),(2,2,1,1)_N], \mu = 6^N)$, together with all non-zero numbers $P_j^{(k)}(\nu),\ j,k \geq 1$.

$$(N-3,N-3),\ P_2^{(1)} = 2, P_2^{(2)} = 2,\ c = 9,$$

$$(N-2,N-4),\ P_2^{(1)} = 2, P_1^{(2)} = 1, P_2^{(2)} = 2,\ 9,$$

$$(N-3,N-4,1),\ P_1^{(1)} = 2, P_1^{(2)} = 4, P_1^{(3)} = 2, P_2^{(2)} = 1,\ c = 11,$$

$$(N-2,N-5,1),\ P_1^{(2)} = 1, P_2^{(1)} = 4, P_3^{(1)} = 2, P_2^{(2)} = 1,\ c = 13,$$

$$(N-3,N-5,1,1),\ P_1^{(1)} = 2, P_2^{(1)} = 6, (P_2^{(2)} = 0), P_3^{(1)} = 2,\ c = 15,$$

$$(N-3,N-5,2),\ P_1^{(1)} = 2, P_2^{(1)} = 6, (P_2^{(2)} = 0), P_3^{(1)} = 2,\ c = 17,$$

$$(N-2,N-6,1,1),\ P_1^{(2)} = 1, P_2^{(1)} = 6, (P_2^{(2)} = 0), P_4^{(1)} = 2,\ c = 19,$$

$$(N-2,N-6,2),\ P_1^{(2)} = 1, P_2^{(1)} = 6, (P_2^{(2)} = 0), P_3^{(1)} = 2,\ c = 21.$$

These are data related with the first configurations $\nu^{(1)}$ in the set of all admissible configurations of type $([(42),(2211)]_N,(6)^N)$, and the set of all nonzero numbers $P_j^{(k)}(\nu)$. All other diagrams $\nu^{(k)},\ k > 1$ from the set of admissible configurations in question, are the same, and are displayed below

$$\nu^{(k)} = (N-2-k, max(N-4-k,0)),\ 2 \leq k \leq N-3,$$

so that $m_1^{(2)} = 2,\ m_2^{(2)} = N-6.$

Therefore,

$$(\clubsuit)\ K_{[(4,2),(2,2,1,1)]_{2N},(2,2)^N}(q) \doteq q^9 \begin{bmatrix} N-1 \\ 2 \end{bmatrix} \begin{bmatrix} 2N-4 \\ 2 \end{bmatrix} +$$

$$q^9 \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} N-2 \\ 2 \end{bmatrix} \begin{bmatrix} 2N-4 \\ 2 \end{bmatrix} + q^{11} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} N-1 \\ 4 \end{bmatrix} \begin{bmatrix} 2N-5 \\ 1 \end{bmatrix} +$$

$$q^{13} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} N-2 \\ 4 \end{bmatrix} \begin{bmatrix} 2N-5 \\ 1 \end{bmatrix} + q^{15} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \begin{bmatrix} N \\ 6 \end{bmatrix} +$$

$$q^{17} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \begin{bmatrix} N-1 \\ 6 \end{bmatrix} + q^{19} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} N-1 \\ 6 \end{bmatrix} + q^{21} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \begin{bmatrix} N-2 \\ 6 \end{bmatrix}.$$

On the other hand,

$$s_{42} * s_{2211} =$$

$$\frac{1}{720}\left(81p_1^6 - 135p_1^4p_2 + 45p_1p_2^2 - 90p_1^2p_4 + 144p_1p_5 - 135p_2^3 + 90p_2p_4\right).$$

Here $p_k := \sum_{i \geq 1} x_i^k$ stands for the power sum symmetric functions degree of k. One can check that $s_{42} * s_{2211}(q, \ldots, q^{N-1}) = RHS(\clubsuit) \doteq K_{[(4,2),(2,2,1,1)]_{2N},(2,2)^N}(q)$, as expected.
Finally one can check that $\lim_{N \to \infty}\ s_{42} * s_{2211}(q, \ldots, q^{N-1}) = q^9(2,1,2,2,1,1) = K_{2211,42}(q,q)$.

4.2. Polynomiality of stretched Kostka and Littlewood–Richardson numbers

• As it was mentioned above, for a given partitions λ and μ (resp. λ and a sequence of rectangular shape partitions $\{\mathcal{R}\}$), the number of **admissible** configurations of type $(N\lambda, N\mu)$ (resp. of type $(N\lambda, \{\mathcal{R}\})$) **doesn't** depend on $N > N_0$, where the number N_0 depends on λ and μ (resp. λ and $\{\mathcal{R}\}$) only.

• Each admissible configuration $\{\nu\}$ provides a contribution of a form

$$a_{\{\nu\}}(q) \prod_{j,k \geq 1} \begin{bmatrix} b_{j,k}(\nu)N + d_{jk}(\nu) \\ d_{jk}(\nu) \end{bmatrix}_q,$$

to the parabolic Kostka polynomial $K_{N\lambda, N\{\mathcal{R}\}}(q)$, where a polynomial $a_\nu(q)$ and a **finite set** of numbers $\{b_{jk}(\nu), d_{jk}(\nu)\}_{j,k \geq 1}$ both doesn't depend on $N > N_0$ for some $N_0 := N_0(\lambda, \{\mathcal{R}\})$.

• It is clear that the sum $\sum_{N \geq 0} \binom{aN+b}{b} t^N$ is a **rational** function of variable t. It is well-known (and easy to prove) that the *Hadamard product* [16] of *rational* functions is again a *rational* function. Therefore,

Corollary 4.6. ([17], [8], [28])
For any partition λ and a sequence of rectangular shape partitions $\{\mathcal{R}\}$, the generating function

$$\sum_{N \geq 0} K_{N\lambda, N\{\mathcal{R}\}}(1)\ t^N$$

is a rational function of variable t with a unique pole at $t = 1$.

More generally using a q-version of Hadamard's product Theorem, we can show

Theorem 4.7. ([17])
For any partition λ and a dominant sequence of rectangular shape partitions $\{\mathcal{R}\}$, the generating function of stretched parabolic Kostka polynomials

$$\sum_{N \geq 0} K_{N\lambda, N\{\mathcal{R}\}}(q)\ t^N$$

is a rational function of variables q and t of a form $P_{\lambda,\{\mathcal{R}\}}(q,t)/Q_{\lambda,\{\mathcal{R}\}}(q,t)$, where the dominator $Q_{\lambda,\{\mathcal{R}\}}(q,t)$ has the following form

$$Q_{\lambda,\{\mathcal{R}\}}(q,t) = \prod_{s \in S} (1 - q^s\ t)$$

for a certain finite set $S := S(\lambda, \{\mathcal{R}\})$ depending on λ and $\{\mathcal{R}\}$.

Clearly that Corollary 4.6 is a special case $q = 1$ of Theorem 4.7.

• (Littlewood–Richardson polynomials) Let λ be a partition and $\{\mathcal{R}\}$ be a dominant sequence of rectangular shape partitions. Write

$$K_{\lambda,\{\mathcal{R}\}}(q) = b(\lambda, \mathcal{R})\ q^{a(\lambda,\mathcal{R})} + \ \textit{higher degree terms.}$$

(1) (Generalized saturation theorem [18])

$$a(N\lambda, N\{\mathcal{R}\}) = N\ a(\lambda, \{\mathcal{R}\}).$$

Therefore,

$$\sum_{N \geq 0} b(N\lambda, N\{\mathcal{R}\})\ t^N = \left. \frac{P_{\lambda,\mathcal{R}}(q, q^{-a(\lambda,\{\mathcal{R}\})}\ t)}{Q_{\lambda,\mathcal{R}}(q, q^{-a(\lambda,\{\mathcal{R}\})}\ t)} \right|_{q=0}$$

[16] See e.g. *wikipedia.org/wiki/Hadamard_product_(matrices)* and the literature quoted therein.

is a **rational** function of t which has a unique pole at $t = 1$ with multiplicity equals to $\#|s \in S(\lambda, \mathcal{R}) \mid s = a(\lambda, \mathcal{R})|$.

(2) Now let λ, μ and ν be partitions such that $|\lambda| + |\mu| = |\nu|$. Consider an integer $N \geq max(\ell(\lambda), \mu_1)$, and define partition $\Lambda = \Lambda(N, \lambda, \mu)$ $:= ((N^N) \oplus \lambda, \mu)$ and the dominant rearrangement of the set of rectangular shape partitions $\{(N^N), \nu_1, \ldots, \nu_{\ell(\nu)}\}$, denoted by $M := M(N, \nu)$. Here we have used standard notation: if λ and μ partitions, then $\lambda \oplus \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots)$, and $(\lambda, \mu) = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}, \mu_1, \ldots, \mu_{\ell(\mu)})$.

Proposition 4.8. ([18])
One has

$$b(\Lambda, M) := c_{\lambda,\mu}^{\nu},$$

where $c_{\lambda,\mu}^{\nu}$ denotes the Littlewood–Richardson number corresponding to partitions λ, μ and ν, that is, the multiplicity of Schur function s_ν in the product of Schur functions s_λ s_μ.

Theorem 4.9. ([18], [28])
Given three partitions λ, μ and ν such that $|\lambda| + |\mu| = |\nu|$. The the generating function

$$\sum_{N \geq 0} c_{N\lambda, N\mu}^{N\nu} \ t^N$$

is a **rational** *function of variable t with a unique pole at $t = 1$. Therefore, $c_{N\lambda,N\mu}^{N\nu}$ is a* **polynomial** *in N with rational coefficients.*

It is well-known that there exists a rational convex polytope, called the Gelfand–Tsetlin polytope $GT(\lambda, \mu, \nu) \subset \mathbb{R}^{\binom{n+1}{2}}$, where $n = \ell(\lambda)$, such that $GT(N\lambda, N\mu, N\nu) \bigcap \mathbb{Z}^{\binom{n+1}{2}} = c_{N\lambda,N\mu}^{N\nu}$. We **expect** that for any partition Λ and a dominant sequence of rectangular shape partitions $\{\mathcal{R}\}$ there exists a rational convex polytope $\Gamma(\Lambda, \{\mathcal{R}\}) \subset \mathbb{R}^{\binom{\ell+1}{2}}$, $\ell = \ell(\Lambda)$, such that $\Gamma(N\Lambda, N\{\mathcal{R}\}) \bigcap \mathbb{Z}^{\binom{\ell+1}{2}} = b(N\Lambda, N\{\mathcal{R}\})$.

Example 4.10. ([14], [17]) (*MacMahon polytope and multidimensional Narayana numbers again*)

Take $\lambda = (n + k, n, n - 1, \ldots, 2)$ and $\mu = \lambda' = (n, n, n - 1, n - 2, \ldots, 2, 1^k)$. One can show [17] that if $n \geq k \geq 1$, then for any positive integer N

- $a(N\lambda, N\mu) = (2k - 1)N$;
- $b(N\lambda, N\mu) = \dim V_{((n-k+1)^{k-1})}^{\mathfrak{gl}(N+k-1)} = \displaystyle\prod_{i=1}^{k-1} \prod_{j=1}^{n-k+1} \frac{N + i + j - 1}{i + j - 1}$.

In other words, the number $b(N\lambda, N\mu)$ is equal to the number of (weak) plane partitions of rectangular shape $((n - k + 1)^{k-1})$ whose parts do not exceed N. According to Exercise 1, **c**, [17], pp. 102–103, $b(N\lambda, N\mu)$ is equal also to the number $i(\mathfrak{M}_{k-1,n-k+1}; N)$ of rational points $\mathbf{x}$ in the MacMahon polytope $\mathfrak{M}_{k-1,n-k+1}$ such that the points $N\mathbf{x}$ have integer coordinates. It follows from Proposition 2.3 ,(**G**),(2.8), that the generating function for numbers $b(n\lambda, n\mu)$ has the following form

$$\sum_{n \geq 0} b(n\lambda, n\mu)t^n =$$
$$\left(\sum_{j=0}^{(k-2)(n-k)} N(k-1, n-k+1; j)t^j \right) / (1-t)^{(k-1)(n-k+1)+1},$$

where $N(k, n; j)$, $0 \leq j \leq (k-1)(n-1)$, denote rectangular Narayana's numbers, see e.g. [23], [34].

One can show (A.K.) that

- if $r := k - \binom{n+2}{2} \geq 0$, then $b(\lambda, \mu) = 1$, and

$$a(\lambda, \mu) = 2\binom{n+3}{3} + (n+1)(2r-1) + \binom{r}{2};$$

- if $1 \leq k < \binom{n+2}{2}$, then there exists a unique p, $1 \leq p \leq n$, such that

$$(p-1)(2n-p+4)/2 < k \leq p(2n-p+3)/2.$$

In this case

$$a(\lambda, \mu) = p(2k-(p-1)n-p) + 2\binom{p}{3},$$

and one can take $\Gamma(\lambda, \mu)$ to be equal to the MacMahon polytope $\mathfrak{M}_{r(k),s(k)}$ with

$r(k) := k - 1 - (p-1)(2n-p+4)/2$, and $s(k) := p(2n-p+3)/2 - k$.

This Example gives some flavor how intricate the piecewise linear function $a(\lambda, \mu)$ may be.

Conjecture 4.11. *Let λ and μ be partitions of the same size. Then*

- ([21]) $a(\lambda, \mu) = a(\mu', \lambda')$,
- ([14]) $b(\lambda, \mu) = b(\mu', \lambda')$.

■

Definition 4.12. ([17]) *Let α and β be partitions of the same size. Define Liskova polynomials $L^{\mu}_{\alpha,\beta}(q)$ through the decomposition of the internal product of Schur functions in terms of Hall-Littlewood polynomials*

$$s_\alpha * s_\beta(X) = \sum_{\mu} L^{\mu}_{\alpha,\beta} \; P_\mu(X; q).$$

Clearly, $L^{\mu}_{\alpha,\beta} \in \mathbb{N}[q]$, *and* $L^{\mu}_{\alpha,(|\alpha|)}(q) = K_{\alpha,\mu}(q)$, *so that the Liskova polynomials are natural generalization of Kostka–Foulkes polynomials.*

Problem 4.13. *Find for Liskova polynomials an analogue of a fermionic formula for Kostka–Foulkes polynomials stated, for example, in* [10], [14].

4.3. Rigged Configurations and RSK

The classical Robinson-Schensted-Knuth correspondence (RSK for short) associatesto a matrix with nonnegative integer coefficients a pair of semistandard Young tableaux of the same shape. More precisely, let α and β be two compositions of the same size N, the RSK correspondence establishes a bijection

$$M_{n\times n}(\alpha,\beta) :\cong \coprod_{\lambda\vdash N} STY(\lambda,\alpha)\times STY(\lambda,\beta),$$

where $M_{n\times n}(\alpha,\beta) = \{(m_{ij}) \in Mat_{n\times n}(\mathbb{Z}_{\geq 0}) \mid \sum_j m_{ij} = \alpha_i,\ \sum_i m_{ij} = \beta_j\}$, and for a partition λ, $STY(\lambda,\alpha)$ stands for the set of semistandard Young tableaus of shape λ and content/weight α.

The literature concerning the RSK, its construction, the study of algebraic, combinatorial, geometric, probabilistic, *etc*, properties of RSK with a vide variety of applications in different areas of Mathematics, is enormous and includes thousands of items. For our purposes we will use the construction of RSK due to D. Knuth, [20]. Let us briefly recall this bijection.

Let $M := (m_{ij}) \in Mat_{n\times n}(\alpha,\beta)$ be a transportation matrix. One can define a multipermutation $\pi(M)$ as follows: $\pi(M) =$

$$\left(\begin{array}{cccccc} \overbrace{1,\ldots,1,}^{\alpha_1} & & \overbrace{2,\ldots,2,}^{\alpha_2}\overbrace{3,\ldots,3,}^{\alpha_3} & \ldots & ,\overbrace{n,\ldots,n}^{\alpha_n} \\ \underbrace{1,\ldots,1}_{m_{1,1}},\underbrace{2,\ldots,2}_{m_{1,2}},\ldots,\underbrace{n,\ldots,n}_{m_{1,n}}, & & \ldots & \ldots, & \underbrace{1,\ldots,1}_{m_{1,n}},\ldots,\underbrace{n,\ldots,n}_{m_{n,n}} \end{array}\right).$$

We will write

$$\pi_{(2)}(M) :=$$
$$(1^{m_{1,1}}2^{m_{1,2}}\ldots n^{m_{1,n}}1^{m_{2,1}}2^{m_{2,2}}\cdots n^{m_{2,n}}\cdots 1^{m_{n,1}}2^{m_{2,n}}\cdots n^{m_{n,n}})$$

for the second row of a multipermutation $\pi(M)$. We denote by $I(M)$ the length of any maximal *increasing* subsequence in $\pi_{(2)}(M)$.

Now one can apply the classical *row insertion algorithm*, see., e.g. [20], to a multipermutation $\pi(M)$. As output of this algorithm one obtains a pair of semistandard Young tableaux $(P \in STY(\lambda,\alpha), Q \in$

$STY(\lambda, \beta))$ for some shape/partition $\lambda = (\lambda_1, \ldots)$. According to Schensted's theorem, see e.g., [6], $\lambda_1 = I(M)$. Our nearest goal is to replace the pair of semistandard Young tableaux (P, Q) obtained by the use of RSK algorithm, by a semistandard Young tableau of the rectangular shape $\Lambda := (\underbrace{I(M), \ldots, I(M)}_{n})$ and weight $\Psi(M) := ((I(M)^n) - \alpha, \beta)$. For this purpose we consider the Gelfand–Tsetlin patterns $GT(P)$ and $GT(Q)$ which correspond in a natural and unique way to the semistandard Young tableaux P and Q correspondingly. Since the tableaux P and Q have the same shape, it is clear that the Gelfand–Tsetlin patterns $GT(P)$ and $GT(Q)$ have the same first row. So one can "glue" together the GT-pattern $GT(P)$ and that $GT(Q)$ by identifying their first rows. As a result one obtains a *plane partition* which is displayed as a diamond. Clearly, there is a unique way to include this diamond to the Gelfand–Tsetlin pattern $GT(P, Q)$ of the highest weight $\Lambda = (\underbrace{I(M), \ldots, I(M)}_{n})$ and weight $\Psi(M)$. Finally, we replace the Gelfand–Tsetlin pattern $GT(P, Q)$ by the corresponding semistandard Young tableau. Using the fact that RSK is a bijection, we come to a bijection

$$\{M \in Mat_{n\times n}(\alpha, \beta) \mid I(M) = L\} \;\cong\; STY((L^n), ((L^n) - \alpha, \beta)).$$

Corollary 4.14. (**Algebraic version of the Robinson–Schensted–Knuth correspondence**)
Let λ and μ be partitions of the same size, n and N be integers such that $N \geq max(\lambda_1 + \mu_1, |\lambda|)$, $n \geq max(\ell(\lambda), \ell(\mu))$. Define partitions $\Lambda := (N^n)$ and $\nu := ((N)^n - \overleftarrow{\lambda}, \mu)$. Then,

$$K_{\Lambda,\nu}(q) \doteq \sum_{\eta} K_{\eta,\lambda}(1)\ K_{\eta,\mu}(q),$$

where $\overleftarrow{\lambda} := (\lambda_{\ell(\lambda)}, \lambda_{\ell(\lambda)-1}, \ldots, \lambda_2, \lambda_1)$.

In other words, under the above assumptions, RSK correspondence gives rise to a bijection between the set of semistandard Young tableaux of rectangular shape Λ and content ν defined above, and the set of transportation matrices $M_{n\times n}(\overleftarrow{\lambda}, \mu)$, and such that it is compatible with the Lascoux–Schützenberger statistics *charge*.

Finally we apply the Rigged Configuration Bijection to the set $STY((L^n), ((L^n) - \alpha, \beta))$. As a result we associate with a multipermutation $\pi(M)$ a rigged configuration $(\{\nu^{(k)}\}_{1\leq k\leq n-1}, \mathbf{J})$. where $\nu^{(k)}$ is a partition of size $(n-k)I(M)$, $k = 1, \ldots, n-1$. In particular, $|\nu^{(n-1)}| = I(M)$ that is $|\nu^{(n-1)}|$ is equal to the length of the maximal increasing subsequence in the multipermutation $\pi_{(2)}(M)$. In a separate publication we are planning to present a direct construction of a RC-type bijection

$$\{M \in M_{n\times n}(\alpha,\beta) \mid I(M)=L\} \ \cong\ RC((L^n),((L^n)-\alpha,\beta))$$

and describe some combinatorial properties of the latter.

In the present paper we illustrate our construction by the following example. Let us take transportation matrix
$M := \begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 2 \\ 3 & 3 & 2 \end{pmatrix} \in M_{3\times 3}((678),(786))$. The corresponding multipermutation is

$$\pi(M) := \begin{pmatrix} 111111222222233333333 \\ 122233111223311122233 \end{pmatrix}.$$

One can check that the output of the RSK algorithm is the following pairs of the Gelfand–Tsetlin patterns

$$(GT(P),GT(Q)) = \left(\begin{pmatrix} 12 & 7 & 2 \\ 10 & 5 & \\ 7 & & \end{pmatrix}, \begin{pmatrix} 12 & 7 & 2 \\ 8 & 5 & \\ 6 & & \end{pmatrix} \right)$$

and the corresponding semistandard Young tableau is

$$T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & 5 & 5 \\ 3 & 3 & 4 & 4 & 5 & 5 & 6 & 6 & 6 & 6 & 6 & 6 \end{bmatrix}$$
$$\in STY((12,12,12),(7,8,6,4,5,6)).$$

Now we apply the Rigged Configuration Bijection to this tableau and come to the following rigged configuration

$$\nu^{(1)}=(12,7,5),\ \nu^{(2)}=(12),\ J_5^{(1)}=3,\ J_7^{(1)}=2,\ J_{12}^{(1)}=J_{12}^{(2)}=0.$$

One can check that
$|M_{3\times 3}((678),(786))| = |STY((12,12,12),(7,8,6,4,5,6))| = 180$. Moreover, there are 24 admissible configurations of type $((12^3),(7,8,6,4,5,6))$, namely, there are :
90 rigged configurations corresponding to seven admissible configurations with $\nu^{(2)}=(12)$;
52 rigged configurations associated with six admissible configurations with $\nu^{(2)}=(11,1)$;

26 rigged configurations associated with 5 admissible configurations with $\nu^{(2)} = (10, 2)$;
10 rigged configurations corresponding to 4 admissible configurations with $\nu^{(2)} = (9, 3)$;
two rigged configurations associated with two admissible configurations with $\nu^{(2)} = (8, 4)$, namely, $((8, 8, 4, 4), (8, 4); \mathbf{J} = \mathbf{0})$ and $((8, 7, 5, 4), (8, 4)); \mathbf{J} = \mathbf{0})$.

Problem 4.15.

(1) *Describe matrices in the set* $M_{n\times n}(\alpha, \beta)$ *which corresponds to rigged configurations* $((I(M)^n), ((I(M)^n) - \alpha, \beta); \mathbf{J} = \mathbf{0})$ *that is transportation matrices which have only* <u>*zero*</u> *riggings* (= *"quantum numbers"*).

(2) *Give a combinatorial interpretation of riggings* $\{\mathbf{J} := \mathbf{J}(\mathbf{M})\}$ *which correspond to a given transportation matrix via the RC-bijection.*

(3) *Give an interpretation of the C. Greene invariants of a multi-permutation* [6] *in terms of the corresponding rigged configurations data, cf* [11].

(4) *Study asymptotic and probabilistic properties of the RC bijection.*

§5. Appendix. Rigged Configurations: a brief review

Let λ be a partition and $R = ((\mu_a^{\eta_a}))_{a=1}^p$ be a sequence of rectangular shape partitions such that

$$|\lambda| = \sum_a |R_a| = \sum_a \mu_a \eta_a.$$

Definition 5.1.

The configuration of type (λ, R) *is a sequence of* **partitions** $\{\nu\} = (\nu^{(1)}, \nu^{(2)}, \ldots)$ *such that*

$$|\nu^{(k)}| = \sum_{j>k} \lambda_j - \sum_{a\geq 1} \mu_a \max(\eta_a - k, 0) = -\sum_{j\leq k} \lambda_j + \sum_{a\geq 1} \mu_a \min(k, \eta_a)$$

for each $k \geq 1$.

Note that if $k \geq l(\lambda)$ and $k \geq \eta_a$ for all a, then $\nu^{(k)}$ is empty.

In the sequel we make the convention that $\nu^{(0)}$ is the empty partition[17].

[17]However, in some cases it is more convenient to set $\nu^{(0)} = (\mu_{i_1}, \ldots, \mu_{i_s})$, where we assume that $\eta_{i_a} = 1,\ a = 1, \ldots, s$. We will give an indication of such choice if it is necessary.

For a partition μ and an integer $j \geq 1$ define the number

$$Q_j(\mu) = \mu'_1 + \cdots + \mu'_j,$$

which is equal to the number of cells in the first j columns of μ.

The *vacancy* numbers $P_j^{(k)}(\nu) := P_j^{(k)}(\nu; R)$ of the configuration $\{\nu\}$ of type (λ, R) are defined by

$$P_j^{(k)}(\nu) = Q_j(\nu^{(k-1)}) - 2Q_j(\nu^{(k)}) + Q_j(\nu^{(k+1)}) + \sum_{a \geq 1} \min(\mu_a, j)\delta_{\eta_a,k}$$

for $k, j \geq 1$, where $\delta_{a,b}$ is the Kronecker delta.

Definition 5.2. *The configuration $\{\nu\}$ of type (λ, R) is called* **admissible**, *if*

$$P_j^{(k)}(\nu; R) \geq 0 \text{ for all } k, j \geq 1.$$

We denote by $C(\lambda; R)$ the set of all admissible configurations of type (λ, R), and call the vacancy number $P_j^{(k)}(\nu, R)$ *essential*, if $m_j(\nu^{(k)}) > 0$.

Finally, for configuration $\{\nu\}$ of type (λ, R) let us define its **charge**

$$c(\nu) = \sum_{k,j \geq 1} \begin{pmatrix} \alpha_j^{(k-1)} - \alpha_j^{(k)} + \sum_a \theta(\eta_a - k)\theta(\mu_a - j) \\ 2 \end{pmatrix},$$

and cocharge

$$\overline{c}(\nu) = \sum_{k,j \geq 1} \begin{pmatrix} \alpha_j^{(k-1)} - \alpha_j^{(k)} \\ 2 \end{pmatrix},$$

where $\alpha_j^{(k)} = (\nu^{(k)})'_j$ denotes the size of the j–th column of the k–th partition $\nu^{(k)}$ of the configuration $\{\nu\}$; for any real number $x \in \mathbb{R}$ we put $\theta(x) = 1$, if $x \geq 0$, and $\theta(x) = 0$, if $x < 0$.

Theorem 5.3. (**Fermionic formula for parabolic Kostka polynomials** [14])

Let λ be a partition and R be a dominant sequence of rectangular shape partitions. Then

$$K_{\lambda R}(q) = \sum_{\nu} q^{c(\nu)} \prod_{k,j \geq 1} \begin{bmatrix} P_j^{(k)}(\nu; R) + m_j(\nu^{(k)}) \\ m_j(\nu^{(k)}) \end{bmatrix}_q, \tag{5.10}$$

summed over all admissible configurations ν of type $(\lambda; R)$; $m_j(\lambda)$ denotes the number of parts of the partition λ of size j.

Corollary 5.4. (**Fermionic formula for Kostka–Foulkes polynomials** [10])

Let λ and μ be partitions of the same size. Then

$$K_{\lambda\mu}(q)=\sum_{\nu}q^{c(\nu)}\prod_{k,j\geq 1}\left[\begin{array}{c}P_j^{(k)}(\nu,\mu)+m_j(\nu^{(k)})\\ m_j(\nu^{(k)})\end{array}\right]_q, \tag{5.11}$$

summed over all sequences of partitions $\nu=\{\nu^{(1)},\nu^{(2)},\ldots\}$ such that

- $|\nu^{(k)}|=\sum_{j>k}\lambda_j,\ k=1,2,\ldots;$
- $P_j^{(k)}(\nu,\mu):=Q_j(\nu^{(k-1)})-2Q_j(\nu^{(k)})+Q_j(\nu^{(k+1)})\geq 0$ *for all* $k,j\geq 1$, *where by definition we* put $\nu^{(0)}=\mu$;

$$\bullet\ c(\nu)=\sum_{k,j\geq 1}\binom{(\nu^{(k-1)})'_j-(\nu^{(k)})'_j}{2}. \tag{5.12}$$

It is frequently convenient to represent an admissible configuration $\{\nu\}$ by a matrix $m(\nu)=(m_{ij}),\ m_{ij}\in\mathbb{Z},\forall i,j\geq 1$, which must meets certain conditions. Namely, starting from the collection of partitions $\{\nu\}=(\nu^{(1)},\nu^{(2)},\ldots,...)$ corresponding to configuration $\{\nu\}$, define matrix

$$m(\nu):=(m_{ij}),\ m_{ij}=(\nu^{(i-1)})'_j-(\nu^{(i)})'_j+\sum_{a\geq 1}\theta(\eta_a-i)\theta(\mu_a-j),\ \nu^{(0)}:=\emptyset,$$

where we set by definition $\theta(x)=1,\ if\ x\in\mathbb{R}_{\geq 0}$ and $\theta(x)=0,\ x\in\mathbb{R}_{<0}$. One can check that a configuration $\{\nu\}$ of type (λ,R) is **admissible** if and only if the matrix $m(\nu)$ meets the following conditions

(0) $m_{ij}\in\mathbb{Z}$,
(1) $\sum_{i\geq 1}m_{ij}=\sum_{a\geq 1}\eta_a\theta(\mu_a-j)$,
(2) $\sum_{j\geq 1}m_{ij}=\lambda_i$,
(3) $\sum_{j\leq k}(m_{ij}-m_{i+1,j})\geq 0$, for all i,j,k
(4) $\sum_{a\geq 1}\min(\eta_a,k)\delta_{\mu_a,j}\geq\sum_{i\leq k}(m_{ij}-m_{i,j+1})$, for all i,j,k.

One can check that if matrix (m_{ij}) satisfies the conditions $(0)-(4)$, then the set of partitions $\{\nu\}=(\nu^{(1)},\nu_{(2)},\ldots,...)$

$$(\nu^{(k)})'_j:=\sum_{i>k}m_{ij}-\sum_{a}\max(\eta_a-k,0)\theta(\mu_a-j)$$

defines an admissible configuration of type $(\lambda,R=\{(\mu_a)^{\eta_a}\})$.

Example 5.5. *Take* $\lambda=(44332),\ R=\{(2^3),(2^2),(2^2),(1),(1)\}$, *so that*

$$\{\mu_a\}=(2,2,2,1,1)\ \textit{and}\ \{\eta_a\}=(3,2,2,1,1),\ a=1,\ldots,5.$$

Therefore $|\nu^{(1)}|=4$, $|\nu^{(2)}|=6$, $|\nu^{(3)}|=5$, *and* $|\nu^{(4)}|=2$. *It is not hard to check that there exist* 6 *admissible configurations. They are*:
(1) $\{\nu^{(1)}=(3,1),\ \nu^{(2)}=(3,3),\ \nu^{(3)}=(3,2),\ \nu_{(4)}=(2)\}$,
(2) $\{\nu^{(1)}=(3,1),\ \nu^{(2)}=(3,2,1),\ \nu^{(3)}=(3,2),\ \nu^{(4)}=(2)\}$,
(3) $\{\nu^{(1)}=(2,2),\ \nu^{(2)}=(2,2,2),\ \nu^{(3)}=(3,2),\ \nu^{(4)}=(2)\}$,
(4) $\{\nu^{(1)}=(4),\ \nu^{(2)}=(3,3),\ \nu^{(3)}=(3,2),\ \nu^{(4)}=(2)\}$,
(5) $\{\nu^{(1)}=(3,1),\ \nu^{(2)}=(2,2,1,1),\ \nu^{(3)}=(2,2,1),\ \nu^{(4)}=(2)\}$,
(6) $\{\nu^{(1)}=(3,1),\ \nu^{(2)}=(2,2,1,1),\ \nu^{(3)}=(3,1,1),\ \nu^{(4)}=(2)\}$,
Let us compute the matrix (m_{ij}) *corresponding to the configuration* (2). *Clearly*,
$(m_{ij})=((\nu^{(i-1)})_j'-(\nu^{(i)})_j')+(\sum_{a\geq 1}\theta(\eta_a-i)\theta(\mu_a-j)):=U+W$. *One can check that*

$$U=\begin{pmatrix} -3 & -1 & 0 & 0 & 0\\ 0 & -1 & -1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 2 & 1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0\end{pmatrix},\ W=\begin{pmatrix} 5 & 3 & 0\\ 3 & 3 & 0\\ 1 & 1 & 0\end{pmatrix}.$$

Therefore,

$$m(\{\nu\})=\begin{pmatrix} 2 & 2 & 0 & 0 & 0\\ 3 & 2 & -1 & 0 & 0\\ 1 & 1 & 1 & 0 & 0\\ 2 & 1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0\end{pmatrix}.$$

One can read off directly from the matrix m_{ij} the all additional quantities need to compute the parabolic Kostka polynomial corresponding to λ and a (dominant) sequence of rectangular shape partitions R. Namely,

$$\begin{aligned} P_j^{(k)} &= \sum_{i\geq j}(m_{ki}-m_{k+1,i}),\ m_j(\nu^{(k)})\\ &= \sum_{a\geq 1}\min(\eta_a,k)\delta_{\mu_a,j}-\sum_{i\leq k}(m_{ij}-m_{i,j+1}), c(\nu)=\sum_{i,j\geq 1}\binom{m_{ij}}{2}. \end{aligned}$$

For example, in our example, we have $c(\nu)=8$, $P_1^{(1)}=1, P_2^{(2)}=1$, $P_3^{(2)}=1$, $P_2^{(3)}=1$ are all non-zero vacancy numbers, and the contribution of the configuration in question to the parabolic Kostka polynomial is equal to $q^8\begin{bmatrix}2\\1\end{bmatrix}^4$. Treating in a similar fashion other configurations, we come to a fermionic formula

$$K_{44332,\{(2^3),(2^2),(2^2),(1),(1)\}}(q) =$$
$$q^{10}\begin{bmatrix}3\\1\end{bmatrix} + q^8\begin{bmatrix}2\\1\end{bmatrix}^4 + q^8\begin{bmatrix}3\\2\end{bmatrix} + q^{12} + q^6\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} + q^8.$$

■

If $\eta_a = 1,\ \forall a$, then $\sum_{a\geq 1}\eta_a\theta(\mu_a - j) = \mu_j^{'}$, and we set $\nu^{(0)} = \mu^{'}$. In this case one can rewrite the conditions (1) – (4) as follows

$(1^{'})$ $\sum_{i\geq 1} m_{ij} = \mu_j^{'}$,

$(2^{'})$ $\sum_{j\geq 1} m_{ij} = \lambda_i$,

$(3^{'})$ $\sum_{j\leq k}(m_{ij} - m_{i+1,j}) \geq 0$, for all i, j, k

$(4^{'})$ $\sum_{i>k}(m_{ij} - m_{i,j+1}) \geq 0$, for all $i, j.k$

Let us remark that if $m_{ij} \in \mathbb{Z}_{\geq 0}$ then the matrix (m_{ij}) defines a lattice plane partition of shape λ. For example, take $\lambda = (6,4,2,2,1,1)$, $\mu = (2^8)$ and admissible configuration $\{\nu\} = \{(5,5),(4,2),(3,1),(2),(1)\}$. The corresponding matrix and lattice plane partition of shape λ are

$$(m_{ij}) = \begin{pmatrix} 3 & 3 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ \textit{and plane partition} \ \begin{matrix} 3 & 3 \\ 1 & 3 \\ 1 & 1 \\ 1 & 1 \\ 1 & \\ 1 & \end{matrix}.$$

The corresponding lattice word is 111222.1222.12.12.1.1.

In the case $\eta_a = 1,\ \forall a$, there exists a unique admissible configuration of type (λ,μ) denoted by $\Delta(\lambda,\mu)$ such that $\max(c((\Delta(\lambda,\mu),J))) = n(\mu) - n(\lambda)$, where the maximum is taken over all rigged configurations associated with configuration $\Delta(\lambda,\mu)$. Recall that for any partition λ,

$$n(\lambda) = \sum_{j\geq 1}\binom{\lambda_a^{'}}{2},$$

and if $\lambda \geq \mu$ with respect to the *dominance order*, then the degree of Kostka polynomial $K_{\lambda,\mu}(q)$ is equal to $n(\mu) - n(\lambda)$, see, e.g. [21], Chapter 1, for details. Namely, the configuration $\Delta(\lambda,\mu)$ corresponds to the following matrix:

$$m_{1j} = \mu_j^{'} - \max(\lambda_j^{'} - 1, 0),\ j \geq 1,$$
$$m_{ij} = 1,\ if\ (i,j) \in \lambda,\ i \geq 2,$$
$$m_{ij} = 0,\ if\ (i,j) \notin \lambda.$$

In other words, the configuration $\Delta(\lambda,\mu)$ consists of the following partitions $(\lambda[1],\lambda[2],\ldots)$, where $\lambda[k]=(\lambda_{k+1},\lambda_{k+2},\ldots)$. It is not difficult to see that the contribution to the Kostka polynomial $K_{\lambda,\mu}(q)$ coming from the maximal configuration, is equal to

$$K_q(\Delta(\lambda,\mu)) := q^{c(\Delta(\lambda,\mu))} \prod_{j=1}^{\lambda_2} \begin{bmatrix} Q_j(\mu)-Q_j(\lambda)+\lambda_j^{'}-\lambda_{j+1}^{'} \\ \lambda_j^{'}-\lambda_{j+1}^{'} \end{bmatrix}_q,$$

where $c(\Delta(\lambda,\mu)) = n(\lambda)+n(\mu)-\sum_{j\geq 1} \mu_J^{'}(\lambda_j^{'}-1)$. Therefore,

$$K_{\lambda,\mu}(q) \geq K_q(\Delta(\lambda,\mu)). \tag{5.13}$$

It is clearly seen that if $\lambda \geq \mu$, then $Q_j(\mu)\geq Q_j(\lambda)$, $\forall\ j\geq 1$, and thus, $K_{q=1}(\Delta(\lambda,\mu)) \geq 1$, and the inequality (5.13) can be considered as a "quantitative" generalization of the Gale– Ryser theorem, see, e.g. [21], Chapter I, Section 7, or [12] for details.

Now let us **stress** that for a fixed k, the all partitions $\nu^{(k)}$ which contribute to the set of admissible configurations of type (λ,μ) have the same size equals to $\sum_{j\geq k+1}\lambda_j$, and thus the size of each $\nu^{(k)}$ doesn't depend on μ. However the Rigged Configuration bijection

$$RC_{\lambda,\mu} : STY(\lambda,\mu) \longrightarrow RC(\lambda,\mu)$$

happens to be essentially depends on μ. One can check that the map $RC_{\lambda,\mu}$ is compatible with the familiar *Bender–Knuth* transformations on the set of semistandard Young tableaux of a fixed shape.

As it was mentioned above, for a fixed k the all (admissible) configurations have the same size. Therefore, the set of admissible configurations admits a partial ordering denoted by "$\succcurlyeq$". Namely, if $\{\nu\}$ and $\{\xi\}$ are two admissible configurations of the same type (λ,μ), we will write $\{\nu\} \succcurlyeq \{\xi\}$, if either $\{\nu\}=\{\xi\}$ or there exists an integer ℓ such that $\nu^{(a)}=\xi^{(a)}$ if $1\leq a\leq \ell$, and $\nu^{(\ell+1)} > \xi^{(\ell+1)}$ with respect of the dominance order on the set of the same size partitions. It seems an interesting **Problem** to study poset structures on the set of admissible configurations of type (λ,μ), especially to investigate the posets of admissible configurations associated with the multidimensional Catalan numbers, (work in progress).

■

Theorem 5.6. (**Duality theorem for parabolic Kostka polynomials** [14])

Let λ be partition and $R=\{(\mu_a^{\eta_a})\}$ be a dominant sequence of rectangular shape partitions. Denote by $\lambda^{'}$ the conjugate of λ, and by $R^{'}$

a dominant rearrangement of a sequence of rectangular shape partitions $\{(\eta_a^{\mu_a})\}$. *Then*

$$K_{\lambda,R}(q)=q^{n(R)}K_{\lambda',R'}(q^{-1}),$$

where

$$n(R)=\sum_{a<b}\min(\mu_a,\mu_b)\min(\eta_a,\eta_b).$$

A technical proof is based on checking of the statement that the map

$$\iota : m_{ij} \longrightarrow \hat{m}_{ij}=-m_{ji}+\theta(\lambda_j-i)+\sum_{a\geq 1}\theta(\mu_a-j)\theta(\eta_a-i)$$

establishes bijection between the sets of admissible configurations of types (λ,R) and $(\lambda^{'},R^{'})$, and $\iota(c(m_{ij}))=c((\hat{m}_{ij}))$.

5.1. Example

Let $n=6$, consider for example, a standard Young tableau

$$T=\begin{matrix}1&2&3&6&8&9\\4&5&7&10&11&12\end{matrix},\ c(T)=48.$$

The corresponding rigged configuration (ν,J) is

$$\nu=(321),\ J=(J_3=0,\ J_2=2,J_1=6),$$
$$(m_{ij})(\nu)=\begin{pmatrix}9&-2&-1\\3&2&1\end{pmatrix},\ c(\nu)=44.$$

Recall that $c(T)$ and $c(\nu)$ denote the charge of tableau T and configuration ν correspondingly.

• One can see that $c(T)=c(\nu)+J_3+J_2+J_1$, as it should be in general.

• Now, the descent set and descent number of tableau T are $Des(T)=\{3,6,9\}$, $des(T)=3$. One can see that $des(T)=3=\nu_1^{'}$, as it should be in general [18].

• One can check that our tableau T is invariant under the action of the Schützenberger involution [19] on the set of standard Young tableaux of a shape λ. It is clearly seen from the set of riggings J [20] that the

[18] In fact the shape of the first configuration $\nu^{(1)}$ of type (λ,μ) can be read off from the set of "secondary" descent sets $\{Des^{(1)}(T)=Des(T)$, $Des^{(2)}(T),\ldots,...)$, cf [11].

[19] *http://en.wikipedia.org/wiki/Jeu_de_taquin*

[20] In our example $J=(0,1,3)$.

rigged configuration (ν, J) corresponding to tableau T, is invariant under the Flip involution [21] on the set of rigged configurations of type $(\lambda, 1^{|\lambda|})$, as it should be in general, see [16] for a complete proof of the statement that the action of the Schützenberger transformation on a Littlewood– Richardson tableau $T \in LR(\lambda, R)$, under the Rigged Configuration Bijection transforms tableau T to a Littlewood–Richardson tableau corresponding to the rigged configuration $\nu\kappa(J))$, where (νJ) is the rigged configuration corresponding to tableau T we are started with.

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[21] Recall that a rigging of an admissible configuration ν is a collection of integers

$$J = (\{J_{s,r}^{(k)}\}\ ,\ 1 \le s \le m_r(\nu^{(k)})$$

such that for a given k, r one has

$$0 \le J_{1,r}^{(k)} \le J_{2,r}^{(k)} \le \ldots \le J_{m_r(\nu^{(k)}),r}^{(k)} \le P_r^{(k)}(\nu).$$

The Flip involution κ is defined as follows:

$$\kappa(\nu, \{J_{s,r}^{(k)}\}) = (\nu, \{J_{m_r(\nu^{(k)})-s+1,r}^{(k)}\}).$$

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