Advanced Studies in Pure Mathematics 76, 2018
Representation Theory, Special Functions and Painlevé Equations — RIMS 2015
pp. 247–274

A unified approach to transformations for multiple basic hypergeometric series of type A

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Dedicated to Professor Masatoshi Noumi for his 60th birthday

Abstract.

We propose a unified approach to transformation and summation formulas for multiple basic hypergeometric series of type A on the basis of balanced duality transformations. We study two classes of transformations, one between an n-ple sum and an m-ple sum, and the other between two n-ple sums. Though the latter are not simple special cases of the former, we can still derive them from the former in a systematic way. Our derivation utilizes the fact that some multiple basic hypergeometric series have hidden symmetry which originates from the relation to basic hypergeometric series in one variable. We also give remarks on the related summation formulas.

§1. Introduction

In this paper we propose a unified approach to transformation and summation formulas for multiple basic hypergeometric series of type A on the basis of duality transformations that relate n-ple sums and m-ple sums. This work can be considered as an extension of our previous works [10] and [15].

In a series of papers [18, 20, 22, 24], S.C. Milne and his collaborators developed the theory of $SU(n)$ (or A_{n-1}) basic hypergeometric series, and proposed various transformation and summation formulas for them,

Received October 22, 2015.
Revised July 7, 2016.
2010 *Mathematics Subject Classification.* 33D67; 33D15, 33D20.
Key words and phrases. basic hypergeometric series, multivariate basic hypergeometric series.

using a rational function identity which is called as Milne's fundamental lemma (see final section of [20]) and multiple generalizations of the Bailey lemma (see the exposition [23] and the references therein).

On the other hand, we derived in [10] a multiple generalization of the Euler transformation for the basic hypergeometric series ${}_2\phi_1$, from a kernel identity of Cauchy type for Macdonald's q-difference operators. Interpreting the multiple Euler transformation as a generating series, we further obtained several types of multiple hypergeometric summations and transformations [11, 12], as well as some multiple elliptic hypergeometric transformations [15].

The *balanced duality transformation formula* (2.9), generalizing the ${}_{10}W_9$ transformation (2.10), is one of the most important results of [10], from which various multiple basic hypergeometric transformations between n-ple sums and m-ple sums are obtained by limiting procedures (degenerations). In this paper we propose an approach to understanding a variety of multiple basic hypergeometric transformations and summations of A type, on the basis of the balanced duality transformation (2.9).

We remark that there exist a number of multiple basic hypergeometric identities between two n-ple sums that cannot be obtained simply by putting $m = n$ in the balanced duality transformations and their degenerations. There is, however, a method to obtain such identities by the balanced duality transformations: first transform the n-ple sum to a single sum by a balanced duality transformation, utilize the symmetry in the single sum, and then go back to the n-ple sum [9, 15]. The main purpose of this paper is to give a systematic derivation of multiple basic hypergeometric identities of type A, taking full advantage of this method.

The contents of this paper are as follows. In the next section, after recalling some definitions and notations, we present the balanced duality transformation (2.9) for multiple hypergeometric series of type A.

In Section 3, by certain limiting procedures we derive from (2.9) various useful transformations for multiple basic hypergeometric series that relate n-ple sums and m-ple sums. They include most of the results in [10] and two new transformations generalizing ${}_8W_7$ transformations. An interesting feature of the results in this section is the equivalence of four transformations (2.9), (3.2), (3.6) and (3.14). This equivalence is explained in Subsection 3.3.

In Section 4, we employ the method mentioned above to obtain several multiple basic hypergeometric transformations of type A between n-ple sums which generalize Watson, Sears and nonterminating ${}_8W_7$

transformations. Some of them are new; others are known ([24], [9]) but the derivation here seems to be simpler.

In Section 5, we give remarks on summation formulas which are obtained as special cases of the transformations studied in Section 3. We also give a new proof of the basic analogue of the Minton–Karlsson summation due to Gasper [6].

§2. Preliminaries: Φ series and W series

In this section, we introduce two types of A_{n-1} multiple basic hypergeometric series, $\Phi^n_{m,r}$ *series* and $W^{n,m}$ *series*. We also recall from [10] a transformation between $W^{n,m+2}$ and $W^{m,n+2}$ series, called the *balanced duality transformation* (2.9) (Proposition 2.1). This identity can be regarded as the master identity for various transformations we are going to investigate in this paper. Throughout this paper, we fix once for all a complex number q such that $0 < |q| < 1$. We denote by $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of all nonnegative integers.

2.1. $\Phi^n_{m,r}$ series and $W^{n,m}$ series

Recall that the basic hypergeometric series ${}_{r+1}\phi_r$ and the *very well-poised* basic hypergeometric series ${}_{r+3}W_{r+2}$ are defined by

$$(2.1)\qquad {}_{r+1}\phi_r\left[\begin{matrix} a_0, \{a_i\}_r \\ \{c_i\}_r \end{matrix}; q, u\right]$$

$$:= {}_{r+1}\phi_r\left[\begin{matrix} a_0, a_1, \ldots, a_r \\ c_1, \ldots, c_r \end{matrix}; q, u\right] = \sum_{k\in\mathbb{N}} \frac{(a_0, a_1, \ldots, a_r)_k}{(q, c_1, \ldots, c_r)_k} u^k$$

and

$$(2.2)\qquad {}_{r+3}W_{r+2}\big[s; \{a_i\}_r; q, u\big]$$

$$:= {}_{r+3}\phi_{r+2}\left[\begin{matrix} s, q\sqrt{s}, -q\sqrt{s}, & \{a_i\}_r \\ \sqrt{s}, \ -\sqrt{s}, & \{sq/a_i\}_r \end{matrix}; q, u\right]$$

$$= \sum_{k\in\mathbb{N}} \frac{1 - sq^{2k}}{1 - s} \frac{(s, a_1, \ldots, a_r)_k}{(q, sq/a_1, \ldots, sq/a_r)_k} u^k,$$

where

$$(a)_\infty := (a;q)_\infty = \prod_{n\in\mathbb{N}}(1-aq^n),$$
$$(a)_k := (a;q)_k = \frac{(a)_\infty}{(aq^k)_\infty} \quad \text{for } k\in\mathbb{C},$$
$$(a_1,a_2,\ldots,a_n)_k := (a_1)_k(a_2)_k\cdots(a_n)_k.$$

In these formulas, the symbol $\{a_i\}_r$ stands for the sequence $a_1,\ldots,a_r$ with i regarded as the running index. (For the standard notations of basic hypergeometric series, we refer the reader to Gasper–Rahman [7].) We generalize these basic hypergeometric series ${}_{r+1}\phi_r$ and ${}_{r+3}W_2$ to A_{n-1} multiple basic hypergeometric series as follows. We define the $\Phi^n_{m,n}$ *series* for $n,m,r\in\mathbb{N}$ by

$$\Phi^n_{m,r}\left(\begin{matrix}\{a_i\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\,\begin{matrix}\{b_k\}_m\\ \{d_k\}_m\end{matrix}\,\middle|\,\begin{matrix}\{c_s\}_r\\ \{e_s\}_r\end{matrix}\,\middle|\, q,u\right) \tag{2.3}$$
$$:= \sum_{\gamma\in\mathbb{N}^n} u^{|\gamma|}\,\frac{\Delta(xq^\gamma)}{\Delta(x)}\prod_{1\le i,j\le n}\frac{(a_jx_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}}$$
$$\times\left(\prod_{1\le k\le m}\prod_{1\le i\le n}\frac{(b_kx_i)_{\gamma_i}}{(d_kx_i)_{\gamma_i}}\right)\left(\prod_{1\le s\le r}\frac{(c_s)_{|\gamma|}}{(e_s)_{|\gamma|}}\right),$$

and the $W^{n,m}$ *series* for $n,m\in\mathbb{N}$ by

$$W^{n,m}\left(\begin{matrix}\{a_i\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\, s;\{u_k\}_m;\{v_k\}_m;q,z\right) \tag{2.4}$$
$$:= \sum_{\gamma\in\mathbb{N}^n} z^{|\gamma|}\prod_{1\le i<j\le n}\frac{\Delta(xq^\gamma)}{\Delta(x)}\prod_{1\le i\le n}\frac{1-sq^{|\gamma|+\gamma_i}x_i}{1-sx_i}$$
$$\times\prod_{1\le j\le n}\left[\frac{(sx_j)_{|\gamma|}}{((sq/a_j)x_j)_{|\gamma|}}\prod_{1\le i\le n}\frac{(a_jx_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}}\right]$$
$$\times\prod_{1\le k\le m}\left[\frac{(v_k)_{|\gamma|}}{(sq/u_k)_{|\gamma|}}\prod_{1\le i\le n}\frac{(u_kx_i)_{\gamma_i}}{((sq/v_k)x_i)_{\gamma_i}}\right],$$

where $\gamma=(\gamma_1,\cdots,\gamma_n)\in\mathbb{N}^n$, $|\gamma|=\gamma_1+\cdots+\gamma_n$, and

$$\Delta(x)=\prod_{1\le i<j\le n}(x_i-x_j) \quad\text{and}\quad \Delta(xq^\gamma)=\prod_{1\le i<j\le n}(x_iq^{\gamma_i}-x_jq^{\gamma_j}).$$

We remark that, when $n = 1$ and $x_1 = 1$, the $\Phi^1_{m,r}$ series reduces to ${}_{m+r+1}\phi_{m+r}$ as

$$\Phi^1_{m,r}\left(\begin{matrix} a \\ 1 \end{matrix}\,\middle|\,\begin{matrix} \{b_k\}_m \\ \{d_k\}_m \end{matrix}\,\middle|\,\begin{matrix} \{c_s\}_r \\ \{e_s\}_r \end{matrix}\,\middle|\, q, u\right) = {}_{m+r+1}\phi_{m+r}\left[\begin{matrix} a, \{b_k\}_m, \{c_s\}_r \\ \{d_k\}_m, \{e_s\}_r \end{matrix}; q, u\right], \tag{2.5}$$

and the $W^{1,m}$ series to ${}_{2m+4}W_{2m+3}$ as

$$W^{1,m}\left(\begin{matrix} a \\ 1 \end{matrix}\,\middle|\, s; \{u_k\}_m; \{v_k\}_m; q, z\right) = {}_{2m+4}W_{2m+3}\big[s; a, \{u_k\}_r, \{v_k\}_r; q, z\big], \tag{2.6}$$

respectively. When $n = 0$, both $\Phi^0_{m,r}$ and $W^{0,m}$ are understood as the constant 1.

Remark 2.1. *The very well-poised multiple series $W^{n,m}$ was introduced in our previous work* [15]*. It can also be regarded as a special case of the $[H]^{(n)}$ series introduced by S.C. Milne* [20, Definition 1.10]*.*

The following variants $W^{n,m}_l$ ($l \in \mathbb{Z}$) of $W^{n,m} = W^{n,m}_0$ will also play auxiliary roles in this paper:

$$\begin{aligned} W^{n,m}_l&\left(\begin{matrix} \{a_i\}_n \\ \{x_i\}_n \end{matrix}\,\middle|\, s; \{u_k\}_{m-l}; \{v_k\}_{m+l}; q, z\right) \\ &:= \sum_{\gamma\in\mathbb{N}^n} x_1^{l\gamma_1}\cdots x_n^{l\gamma_n}\; q^{-le_2(\gamma)}\;\; z^{|\gamma|}\, \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1\le i\le n} \frac{1 - sq^{|\gamma|+\gamma_i}x_i}{1 - sx_i} \\ &\times \prod_{1\le j\le n}\left[\frac{(sx_j)_{|\gamma|}}{((sq/a_j)x_j)_{|\gamma|}} \prod_{1\le i\le n} \frac{(a_jx_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}}\right] \\ &\times \prod_{1\le k\le m-l} \frac{\prod\limits_{1\le i\le n} (u_kx_i)_{\gamma_i}}{(sq/u_k)_{|\gamma|}} \prod_{1\le k\le m+l} \frac{(v_k)_{|\gamma|}}{\prod\limits_{1\le i\le n} ((sq/v_k)x_i)_{\gamma_i}}, \end{aligned} \tag{2.7}$$

where $e_2(\gamma) = \sum\limits_{1\le i<j\le n} \gamma_i\gamma_j$ denotes the elementary symmetric function of degree 2. We remark that these $W^{n,m}_l$ series are obtained formally from the $W^{n,m+|l|}$ series through a limiting procedure.

2.2. Balanced duality transformation for W series

The following proposition provides the key identity of this paper, from which all the transformations in the next section will be derived.

Proposition 2.1. ([10], Corollary 6.3) *Under the balancing condition*

$$a^{m+2}q^{m+1+N} = BCdef^m g, \quad N \in \mathbb{N}, \tag{2.8}$$

where $B = b_1 \cdots b_n$ *and* $C = c_1 \cdots c_m$, *the following identity holds*:

$$\begin{aligned}
&W^{n,m+2}\left(\begin{matrix} \{b_i\}_n \\ \{x_i\}_n \end{matrix} \,\middle|\, a; \{c_k y_k\}_m, d, e; \{f y_k^{-1}\}_m, g, q^{-N}; q, q \right) \\
&= \frac{(\mu df/a, \mu ef/a)_N}{(aq/d, aq/e)_N} \prod_{1\le k\le m} \frac{((\mu c_k f/a)y_k, f y_k^{-1})_N}{(\mu q y_k, (aq/c_k)y_k^{-1})_N} \\
&\quad \times \prod_{1\le i\le n} \frac{(aqx_i, (\mu b_i f/a)x_i^{-1})_N}{((aq/b_i)x_i, (\mu f/a)x_i^{-1})_N} \\
&\quad \times W^{m,n+2}\left(\begin{matrix} \{aq/c_k f\}_m \\ \{y_k\}_m \end{matrix} \,\middle|\, \mu; \{(aq/b_i f)x_i\}_n, aq/df, aq/ef; \right. \\
&\qquad\qquad \left. \{(\mu f/a)x_i^{-1}\}_n, g, q^{-N}; q, q \right),
\end{aligned} \tag{2.9}$$

where $\mu = a^{m+2}q^{m+1}/BCdef^{m+1} = q^{-N}g/f$.

When $m = n = 1$ and $x_1 = y_1 = 1$, the balanced duality transformation (2.9) reduces to the following terminating balanced ${}_{10}W_9$ transformation:

$$\begin{aligned}
&{}_{10}W_9\left[a; b, c, d, e, f, \mu f q^N, q^{-N}; q, q\right] \\
&= \frac{(\mu bf/a, \mu cf/a, \mu df/a, \mu ef/a, aq, f)_N}{(aq/b, aq/c, aq/d, aq/e, \mu q, \mu f/a)_N} \\
&\times {}_{10}W_9\left[\mu; aq/bf, aq/cf, aq/df, aq/ef, \mu f/a, \mu f q^N, q^{-N}; q, q\right],
\end{aligned} \tag{2.10}$$

where $\mu = a^3q^2/bcdef^2$ ([7, Exercise 2.19]). We also remark that, in [15], a direct proof of Proposition 2.1 is given on the basis of the Cauchy determinant.

We remark that, when $m=1$ and $y_1=1$, (2.9) implies the transformation

$$W^{n,3}\left(\begin{array}{c}\{b_i\}_n\\ \{x_i\}_n\end{array}\middle| a;c,d,e;f,\mu f q^N,q^{-N};q,q\right) \tag{2.11}$$
$$=\frac{(\mu cf/a,\mu df/a,\mu ef/a,f)_N}{(\mu q,aq/c,aq/d,aq/e)_N}\prod_{1\le i\le n}\frac{(aqx_i,(\mu b_i f/a)x_i^{-1})_N}{((aq/b_i)x_i,(\mu f/a)x_i^{-1})_N}$$
$$\times\ {}_{2n+8}W_{2n+7}\big[\mu;\ aq/cf,aq/df,aq/ef,g,q^{-N},$$
$$\{(aq/b_i f)x_i\}_n,\{(\mu f/a)x_i^{-1}\}_n;\ q,q\big],$$

between a multiple sum and a single sum, where $\mu=a^3q^2/Bcdef^2$. Also, when $m=0$, it gives the summation formula

$$W^{n,2}\left(\begin{array}{c}\{b_i\}_n\\ \{x_i\}_n\end{array}\middle| a;d,e;\lambda q^N,q^{-N};q,q\right) \tag{2.12}$$
$$=\frac{(\lambda d/a,\lambda e/a)_N}{(aq/d,aq/e)_N}\prod_{1\le i\le n}\frac{(aqx_i,(\lambda b_i/a)x_i^{-1})_N}{((aq/b_i)x_i,(\lambda/a)x_i^{-1})_N},$$

where $\lambda=a^2q/Bde$.

Remark 2.2. *Equality* (2.9) *is equivalent to the transformation*

$$\sum_{\substack{\gamma\in\mathbb{N}^n\\ |\gamma|=N}}\frac{\Delta(xq^\gamma)}{\Delta(x)}\prod_{1\le i,j\le n}\frac{(a_jx_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}}\prod_{\substack{1\le i\le n\\ 1\le k\le m}}\frac{(b_kx_iy_k)_{\gamma_i}}{(cx_iy_k)_{\gamma_i}} \tag{2.13}$$
$$=\sum_{\substack{\delta\in\mathbb{N}^m\\ |\delta|=N}}\frac{\Delta(yq^\delta)}{\Delta(y)}\prod_{\substack{1\le k,\\ l\le m}}\frac{((c/b_l)y_k/y_l)_{\delta_k}}{(qy_k/y_l)_{\delta_k}}\prod_{\substack{1\le i\le n\\ 1\le k\le m}}\frac{((c/a_i)x_iy_k)_{\delta_k}}{(cx_iy_k)_{\delta_k}},$$

under the condition $a_1\cdots a_n b_1\cdots b_m=c^m$ ([10, Proposition 6.2]). *In fact, Proposition 2.1 is obtained by rewriting the both sides of* (2.13) *in terms of W series.*

§3. Transformations between n-ple sums and m-ple sums

As the preparation for the next section, we derive below several types of balanced duality transformations among Φ series and W series from the balanced duality transformation (2.9) through limiting procedures. They include two new transformations that generalize ${}_8W_7$ transformations (Propositions 3.4 and 3.5).

3.1. Transformation between W series and Φ series

First, we present a transformation between $W^{n,m+1}$ series and $\Phi^{m}_{n+1,1}$ series. It generalizes the following transformation between terminating ${}_8W_7$ series and terminating balanced ${}_4\phi_3$ series:

$$(3.1)\qquad {}_8W_7\left[a;b,c,d,e,q^{-N};q,\frac{a^2q^{N+2}}{bcde}\right] = \frac{(a^2q^2/bcde,e,aq)_N}{(aq/b,aq/c,aq/d)_N}\,{}_4\phi_3\left[\begin{matrix}q^{-N},aq/be,aq/ce,aq/de\\ q^{1-N}/e,a^2q^2/bcde,aq/e\end{matrix};q,q\right].$$

Proposition 3.1. ([10] **Proposition 6.1)** *We have*

(3.2)
$$W^{n,m+1}\left(\begin{matrix}\{b_i\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\, a;c,\{d_ky_k\}_m;q^{-N},\{ey_k^{-1}\}_m;q,\frac{a^{m+1}q^{N+m+1}}{BcDe^m}\right)$$
$$= \frac{(a^{m+1}q^{m+1}/BcDe^m)_N}{(aq/c)_N}\prod_{1\le i\le n}\frac{(aqx_i)_N}{((aq/b_i)x_i)_N}\prod_{1\le k\le m}\frac{(ey_k^{-1})_N}{((aq/d_i)y_k^{-1})_N}$$
$$\times\ \Phi^{m}_{n+1,1}\left(\begin{matrix}\{aq/d_ke\}_m\\ \{y_k\}_m\end{matrix}\,\middle|\,\begin{matrix}\{(aq/b_ie)x_i\}_n,aq/ce\\ \{(aq/e)x_i\}_n,q^{1-N}/e\end{matrix}\,\middle|\,\begin{matrix}q^{-N}\\ a^{m+1}BcDe^m\end{matrix}\,\middle|\, q,q\right),$$

where $B=b_1\cdots b_n$ and $D=d_1\cdots d_m$.

Proof. In (2.9), setting $g=\mu f q^N$, take the limit $e\to\infty$ and change the parameter as $f\to e$. Q.E.D.

We remark that, when $m=1$ and $y_1=1$, (3.2) reduces to

$$(3.3)\qquad W^{n,2}\left(\begin{matrix}\{b_i\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\, a;c,d;q^{-N},e;q,\frac{a^2q^{N+2}}{Bcde}\right) = \frac{(a^2q^2/Bcde,e)_N}{(aq/c,aq/d)_N}\prod_{1\le i\le n}\frac{(aqx_i)_N}{((aq/b_i)x_i)_N}$$
$$\times\ {}_{n+3}\phi_{n+2}\left[\begin{matrix}q^{-N},\{(aq/b_ie)x_i\}_n,aq/ce,aq/de\\ q^{1-N}/e,\{(aq/e)x_i\}_n,a^2q^2/Bcde,\end{matrix};q,q\right].$$

Also, when $n = 1$ and $x_1 = 1$, (3.2) reduces to

$$(3.4)\quad {}_{2m+6}W_{2m+5}\left[a; b, \{c_k y_k\}_m, d, \{e y_k^{-1}\}_m, q^{-N}; q, \frac{a^{m+1}q^{N+m+1}}{bCde^m}\right]$$
$$= \frac{(a^{m+1}q^{m+1}/bCde^m, aq)_N}{(aq/b, aq/d)_N} \prod_{1\le k\le m} \frac{(e y_k^{-1})_N}{((aq/c_k)y_k^{-1})_N}$$
$$\times\ \Phi^m_{2,1}\left(\begin{matrix}\{aq/d_k e\}_m\\ \{y_k\}_m\end{matrix}\ \middle|\ \begin{matrix}aq/be, aq/ce\\ aq/e, q^{1-N}/e\end{matrix}\ \middle|\ \begin{matrix}q^{-N}\\ a^{m+1}bcDe^m\end{matrix}\ \middle|\ q, q\right).$$

When $m = n = 1$ and $x_1 = y_1 = 1$, (3.2) reduces to (3.1). Special transformations like (3.3) and (3.4), where one side is a single sum, will be crucial in the argument of the next section.

Remark 3.1. *In* [10], (3.2) *was proved by taking the coefficient of* u^N *in the both side of the multiple Euler transformation* (3.14) *below. In* [27] *Rosengren gave another proof using his reduction formula of Karlsson-Minton type. As is mentioned in* [27], *the balanced duality transformation* (2.9) *itself can also be considered as a special case of* (3.2).

3.2. Two transformations among Φ series

We now propose a transformation between terminating balanced $\Phi^n_{m,2}$ series and terminating balanced $\Phi^m_{n,2}$ series. It generalizes the Sears transformation for terminating balanced ${}_4\phi_3$ series:

$$(3.5)\quad {}_4\phi_3\left[\begin{matrix}q^{-N}, a, b, c\\ d, e, abcq^{1-N}/de\end{matrix}; q, q\right]$$
$$= \frac{(e/a, de/bc)_N}{(e, de/abc)_N}\,{}_4\phi_3\left[\begin{matrix}q^{-N}, a, d/b, d/c\\ d, aq^{1-N}/e, de/bc\end{matrix}; q, q\right].$$

Proposition 3.2. ([10, Proposition 7.1]) *We have*

$$(3.6)\quad \Phi^n_{m,2}\left(\begin{matrix}\{b_i\}_n\\ \{x_i\}_n\end{matrix}\ \middle|\ \begin{matrix}\{c_k y_k\}_m\\ \{(d y_k\}_m\end{matrix}\ \middle|\ \begin{matrix}a,\ q^{-N}\\ e,\ aBCq^{1-N}/d^m e\end{matrix}\ \middle|\ q, q\right)$$
$$= \frac{(e/a, d^m e/BC)_N}{(e, d^m e/aBC)_N}$$
$$\times\ \Phi^m_{n,2}\left(\begin{matrix}\{d/c_k\}_m\\ \{y_k\}_m\end{matrix}\ \middle|\ \begin{matrix}\{(d/b_i)x_i\}_n\\ \{d x_i\}_n\end{matrix}\ \middle|\ \begin{matrix}a,\ q^{-N}\\ d^m e/BC,\ q^{1-N}a/e\end{matrix}\ \middle|\ q, q\right).$$

Proof. In (2.9), replace the parameters d, e and f by $aq/d, aq/e$ and aq/f respectively, and take the limit $a \to 0$. Then change the parameters again as $f \to d$ and $d \to aBCq^{1-N}/def^m$. Q.E.D.

When $m = 1$ and $y_1 = 1$, (3.6) reduces to

$$\tag{3.7} \begin{aligned} &\Phi^n_{1,2}\left(\begin{matrix}\{b_i\}_n\\ \{x_i\}_n\end{matrix}\middle|\begin{matrix}c\\ d\end{matrix}\middle|\begin{matrix}a, q^{-N}\\ e, aBcq^{1-N}/de\end{matrix}\middle|\, q, q\right)\\ &= \frac{(e/a, de/Bc)_N}{(e, de/aBc)_N}\ {}_{n+3}\phi_{n+2}\left[\begin{matrix}q^{-N}, a, \{(d/b_i)x_i\}_n, d/c\\ q^{1-N}a/e, \{dx_i\}_n, de/Bc\end{matrix}; q, q\right]. \end{aligned}$$

When $m = n = 1$ and $x_1 = y_1 = 1$, (3.6) reduces to (3.5). (Further information on the multiple Sears transformation (3.6) can be found in [9].)

We next derive a transformation between terminating balanced $\Phi^n_{1,m+1}$ series and terminating balanced $\Phi^m_{1,n+1}$ series. It generalizes the following transformation formula for terminating balanced ${}_4\phi_3$ series:

$$\tag{3.8} \begin{aligned} &{}_4\phi_3\left[\begin{matrix}q^{-N}, a, b, c\\ d, e, f\end{matrix}; q, q\right]\\ &= \frac{(ef/ab, ef/ac, a)_N}{(e, f, ef/abc)_N}\,{}_4\phi_3\left[\begin{matrix}q^{-N}, ef/abc, e/a, f/a\\ def/a^2bc, ef/ab, ef/ac\end{matrix}; q, q\right]\\ &\qquad (abcq^{1-N} = def). \end{aligned}$$

We remark that (3.8) is obtained by reversing the order of summation in the Sears transformation (3.5), or alternatively by iterating the Sears transformation twice.

Proposition 3.3. **(**[10, Proposition 7.2]**)** *Under the balancing condition $a^mBcq^{1-N} = dEf$, we have*

$$\tag{3.9} \begin{aligned} &\Phi^n_{1,m+1}\left(\begin{matrix}\{b_i\}_n\\ \{x_i\}_n\end{matrix}\middle|\begin{matrix}c\\ d\end{matrix}\middle|\begin{matrix}\{ay_k\}_m, q^{-N}\\ \{e_ky_k\}_m, f\end{matrix}\middle|\, q, q\right)\\ &= \frac{(Ef/a^mB)_N}{(f)_N}\prod_{1\le k\le m}\frac{(ay_k)_N}{(e_ky_k)_N}\prod_{1\le i\le n}\frac{((Ef/a^mc)z_i)_N}{((Ef/a^mBc)z_i)_N}\\ &\times\ \Phi^m_{1,n+1}\left(\begin{matrix}\{e_l/a\}_m\\ \{w_k\}_m\end{matrix}\middle|\begin{matrix}f/a\\ dEf/a^mBc\end{matrix}\middle|\begin{matrix}\{(Ef/a^mb_ic)z_i\}_n,\ q^{-N}\\ \{(Ef/a^mc)z_i\}_n,\ Ef/a^mB\end{matrix}\middle|\, q, q\right), \end{aligned}$$

where $z_i = b_i/Bx_i$ $(1 \le i \le n)$ and $w_k = y_k^{-1}$ $(1 \le k \le m)$ respectively.

Proof. In (2.9), change the parameters as $c_k \to aq/c_k$ $(1 \le k \le m)$ and $e \to aq/e$, and put $a = 0$. Next change the parameters again as $f \to a,\ d \to c,\ c_k \to e_k \quad (1 \le k \le m),\ e \to f$. Q.E.D.

When $m = 1$ and $y_1 = 1$, (3.9) reduces to

$$\Phi^{n}_{1,2}\left(\begin{matrix}\{b_i\}_n\\ \{x_i\}_n\end{matrix}\ \middle|\ \begin{matrix}c\\ d\end{matrix}\ \middle|\ \begin{matrix}a,\ q^{-N}\\ e,\ f\end{matrix}\ \middle|\ q,q\right) \tag{3.10}$$

$$= \frac{(ef/aB, a)_N}{(e,f)_N}\prod_{1\le i\le n}\frac{((ef/ac)z_i)_N}{((ef/aBc)z_i)_N}$$

$$\times\ {}_{n+3}\phi_{n+2}\left[\begin{matrix}q^{-N}, e/a, f/a, \{(ef/ab_ic)z_i\}_n\\ def/a^2Bc, ef/aB, \{(ef/ac)z_i\}_n\end{matrix}; q, q\right]$$

under the condition $aBcq^{1-N} = def$, where $z_i = b_i/Bx_i$ $(1 \le i \le n)$. When $m = n = 1$ and $x_1 = y_1 = 1$, (3.9) reduces to (3.8).

3.3. Passage to the multiple basic Euler transformation

Let N tend to infinity in the multiple Sears transformation (3.6). Then we obtain a transformation between $\Phi^{n}_{m,1}$ series and $\Phi^{m}_{n,1}$ series which generalizes the transformation

$$ {}_3\phi_2\left[\begin{matrix}a,b,c\\ d,e\end{matrix}; q, \frac{de}{abc}\right] = \frac{(e/a, de/bc)_\infty}{(e, de/abc)_\infty}\ {}_3\phi_2\left[\begin{matrix}a, d/b, d/c\\ d, de/bc\end{matrix}; q, \frac{e}{a}\right] \tag{3.11}$$

for nonterminating ${}_3\phi_2$ series.

Lemma 3.1 ([9]). *We have*

$$\Phi^{n}_{m,1}\left(\begin{matrix}\{b_i\}_n\\ \{x_i\}_n\end{matrix}\ \middle|\ \begin{matrix}\{c_ky_k\}_m\\ \{dy_k\}_m\end{matrix}\ \middle|\ \begin{matrix}a\\ e\end{matrix}\ \middle|\ q, \frac{d^m e}{aBC}\right) \tag{3.12}$$

$$= \frac{(e/a, d^m e/BC)_\infty}{(e, d^m e/aBC)_\infty}$$

$$\times\ \Phi^{m}_{n,1}\left(\begin{matrix}\{d/c_k\}_m\\ \{y_k\}_m\end{matrix}\ \middle|\ \begin{matrix}\{(d/b_i)x_i\}_n\\ \{dx_i\}_n\end{matrix}\ \middle|\ \begin{matrix}a\\ d^m e/BC\end{matrix}\ \middle|\ q, \frac{e}{a}\right)$$

under the convergence condition $\max(|d^m e/aBC|, |e/a|) < 1$.

For the convergence of multiple series, see Milne–Newcomb [25, 26] and Milne [22]. When $m=1$ and $y_1=1$, (3.12) reduces to

$$\Phi^{n}_{1,1}\left(\begin{array}{c|c|c|c} \{b_i\}_n & c & a & \\ \{x_i\}_n & d & e & q,\dfrac{d^m e}{aBC} \end{array}\right) = \frac{(e/a, de/Bc)_\infty}{(e, de/aBc)_\infty}\, {}_{n+2}\phi_{n+1}\left[\begin{matrix} d/c, a, \{(d/b_i)x_i\}_n \\ de/Bc, \{dx_i\}_n \end{matrix}; q, \frac{e}{a}\right] \tag{3.13}$$

under the convergence condition $\max(|d^m e/aBC|, |e/a|) < 1$. When $m=n=1$ and $x_1=y_1=1$, (3.12) reduces to (3.11).

Changing the parameter $e \to aBCu/d^m$ and taking limit as $a \to \infty$ in (3.12), we obtain the following multiple basic Euler transformation.

Theorem 3.1. ([10, Theorem 1.1]) *Under the convergence condition* $\max(|u|, |ABu/c^m|) < 1$, *we have*

$$\Phi^{n}_{m,0}\left(\begin{array}{c|c|c|c} \{a_i\}_n & \{b_k y_k\}_m & \cdot & \\ \{x_i\}_n & \{cy_k\}_m & \cdot & q,u \end{array}\right) = \frac{(ABu/c^m)_\infty}{(u)_\infty}\,\Phi^{m}_{n,0}\left(\begin{array}{c|c|c|c} \{c/b_k\}_m & \{(c/a_i)x_i\}_n & \cdot & \\ \{y_k\}_m & \{cx_i\}_n & \cdot & q,\dfrac{ABu}{c^m} \end{array}\right), \tag{3.14}$$

where $A=a_1\cdots a_n$ *and* $B=b_1\cdots b_m$.

Note that, when $n=m=1$ and $x_1=y_1=1$, (3.14) reduces to the basic Euler transformation formula for ${}_2\phi_1$-series: :

$${}_2\phi_1\left[\begin{matrix} a,b \\ c \end{matrix}; q,u\right] = \frac{(abu/c)_\infty}{(u)_\infty}\,{}_2\phi_1\left[\begin{matrix} c/b, c/a \\ c \end{matrix}; q, abu/c\right]. \tag{3.15}$$

Remark 3.2. *We have investigated so far the following four typical transformations for multiple basic hypergeometric series of type A.*

a) *Balanced duality transformation* (2.9) (*terminating* ${}_{10}W_9$)
b) *Watson type transformation* (3.2) (*terminating* ${}_8W_7$ *and* ${}_4\phi_3$)
c) *Multiple Sears transformation* (3.6) (*terminanting* ${}_4\phi_3$)
d) *Multiple basic Euler transformation* (3.14) (*nonterminating* ${}_2\phi_1$)

We have shown in fact the implications a) $\Rightarrow$ b) *and* a) $\Rightarrow$ c) $\Rightarrow$ d), *regarding* a) *as the master identity. We also have gaven a remark on* b) $\Rightarrow$ a) *in Remark 3.1. Note that the multiple basic Euler transformation* (3.14) *itself is the main theorem of our paper* [10, Theorem 1.1], *from which we have deduced the other three transformations. This means that these four transformations are equivalent to each other, and that any* *of them can play the part of a master identity for multiple basic hypergeometric transformations of type A. This fact could be considered as one of the most remarkable features of multiple basic hypergeometric series and their transformations.*

Remark 3.3. *In our previous work* [11], *we derived some multiple generalizations, as well as their applications, of other Heine transformations and a basic Pfaff transformation from* (3.6) *and* (3.9) *in a similar way as in this paper.*

3.4. Transformations for W series

Here we present two types of new transformations for W series which are derived from the balanced duality transformation (2.9)

We first give a transformation between nonterminating $W_{-1}^{n,m+1}$ series and nonterminating $W_{-1}^{m,n+1}$ series. It generalizes the following transformation formula for nonterminating ${}_8W_7$ series:

$$\begin{aligned}(3.16)\qquad & {}_8W_7\left[a;b,c,d,e,f;q,a^2q^2/bcdef\right]\\ &= \frac{(\mu bf/a,\mu cf/a,\mu df/a,\mu ef/a,aq,f)_\infty}{(aq/b,aq/c,aq/d,aq/e,\mu q,\mu f/a)_\infty}\\ &\quad\times {}_8W_7\left[\mu;aq/bf,aq/cf,aq/df,aq/ef,\mu f/a;q,f\right],\end{aligned}$$

where $\mu=a^3q^2/bcdef^2$, under the condition $\max(|a^2q^2/bcdef|,|f|)<1$.

Proposition 3.4. *Assume that* $\left|a^{m+1}q^{m+1}/BCdef^m x_i\right|<1$ *for all* $i=1,\cdots,n$ *and* $|fy_k^{-1}|<1$ *for all* $k=1,\cdots,m$. *Then we have*

(3.17)

$$\begin{aligned}&W_{-1}^{n,m+1}\left(\begin{matrix}\{b_i\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\,a;\{c_ky_k\}_m,d,e;\{fy_k^{-1}\}_m;q,\frac{a^{m+1}q^{m+1}}{BCdef^m}\right)\\ &=\frac{(\mu df/a,\mu ef/a)_\infty}{(aq/d,aq/e)_\infty}\prod_{1\le k\le m}\frac{((\mu c_kf/a)y_k,fy_k^{-1})_\infty}{(\mu qy_k,(aq/c_k)y_k^{-1})_\infty}\\ &\quad\times\prod_{1\le i\le n}\frac{(aqx_i,(\mu b_if/a)x_i^{-1})_\infty}{((aq/b_i)x_i,(\mu f/a)x_i^{-1})_\infty}\\ &\quad\times W_{-1}^{m,n+1}\left(\begin{matrix}\{aq/c_kf\}_m\\ \{y_k\}_m\end{matrix}\,\middle|\,\mu;\left\{\frac{aq}{b_if}x_i\right\}_n,\frac{aq}{df},\frac{aq}{ef};\left\{\frac{\mu f}{a}x_i^{-1}\right\}_n;q,f\right),\end{aligned}$$

where $\mu=a^{m+2}q^{m+1}/BCdef^{m+1}$.

Proof. Setting $g=\mu fq^N$ in (2.9), let N tend to infinity. Q.E.D.

When $m=1$ and $y_1=1$, (3.17) can be stated as

$$(3.18)\ W^{n,2}_{-1}\left(\begin{array}{c}\{b_i\}_n\\ \{x_i\}_n\end{array}\,\middle|\, a;c,d,e;f;q,\frac{a^2q^2}{Bcdef}\right)$$
$$=\frac{(\mu cf/a,\mu df/a,\mu ef/a,f)_\infty}{(aq/c,aq/d,aq/e,\mu q)_\infty}\prod_{1\le i\le n}\frac{(aqx_i,(\mu b_if/a)x_i^{-1})_\infty}{((aq/b_i)x_i,(\mu f/a)x_i^{-1})_\infty}$$
$$\times\ {}_{2n+4}W_{2n+3}\left[\mu;\{(aq/b_if)x_i\}_n,aq/cf,aq/df,\right.$$
$$\left.aq/ef,\{(\mu f/a)x_i^{-1}\}_n;q,f\right],$$

where $\mu=a^3q^2/Bcdef^2$. When $m=n=1$ and $x_1=y_1=1$, (3.17) reduces to (3.16).

We next give a transformation between terminating $W^{n,m+1}_{+1}$ series and terminating $W^{m,n+1}_{+1}$ series. It generalizes the following transformation formula for terminating ${}_8W_7$ series:

$$(3.19)\qquad {}_8W_7\left[a;b,c,d,e,q^{-N};q,\frac{a^2q^{N+2}}{bcde}\right]$$
$$=\frac{(aq/be,aq/ce,aq,d)_N}{(aq/b,aq/c,aq/e,d/e)_N}$$
$$\times\ {}_8W_7\left[q^{-N}e/d;aq/bd,q^{-N}e/a,aq/cd,e,q^{-N};q,\frac{bc}{a}\right].$$

Proposition 3.5. *The following identity holds*:

(3.20)

$$W^{n,m+1}_{+1}\left(\begin{array}{c}\{b_i\}_n\\ \{x_i\}_n\end{array}\,\middle|\, a;\{c_ky_k\}_m;\{dy_k^{-1}\}_m,e,q^{-N};q,\frac{a^{m+1}q^{N+m+1}}{BCd^me}\right)$$
$$=\prod_{1\le k\le m}\frac{((aq/c_ke)y_k^{-1},dy_k^{-1})_N}{((aq/c_k)y_k^{-1},(d/e)y_k^{-1})_N}\prod_{1\le i\le n}\frac{((aq/b_ie)x_i,aqx_i)_N}{((aq/b_i)x_i,(aq/e)x_i)_N}$$
$$\times W^{m,n+1}_{+1}\left(\begin{array}{c}\{aq/c_kd\}_m\\ \{y_k\}_m\end{array}\,\middle|\,\frac{q^{-N}e}{d};\left\{\frac{aq}{b_id}x_i\right\}_n;\left\{\frac{q^{-N}e}{a}x_i^{-1}\right\}_n,e,q^{-N};q,\frac{Bcd^{m-1}}{a^m}\right).$$

Proof. In (2.9), write $e=a^{m+2}q^{m+N+1}/BCdf^mg$ and let d tend to infinity. Then change the parameters as $f\to d$ and $g\to e$ to get (3.20). Q.E.D.

When $m = 1$ and $y_1 = 1$, (3.20) can be stated as:

$$
\begin{aligned}
(3.21)\quad & W^{n,2}_{+1}\left(\begin{matrix} \{b_i\}_n \\ \{x_i\}_n \end{matrix} \,\middle|\, a; c; d, e, q^{-N}; q, \frac{a^2 q^{N+2}}{Bcde} \right) \\
&= \frac{(aq/ce, d)_N}{(aq/c, d/e)_N} \prod_{1\le i\le n} \frac{((aq/b_i e)x_i, aqx_i)_N}{((aq/b_i)x_i, (aq/e)x_i)_N} \\
&\times {}_{2n+6}W_{2n+5}\left[q^{-N}e/d; \{(aq/b_i d)x_i\}_n, \{(q^{-N}e/a)x_i^{-1}\}_n,\right. \\
&\qquad\qquad \left. aq/cd, e, q^{-N}; q, Bc/a\right].
\end{aligned}
$$

When $m = n = 1$ and $x_1 = y_1 = 1$, (3.20) reduces to (3.19).

§4. Transformations of n-ple sums

We have shown in Section 3 that the balanced duality transformation (2.9) gives rise to various transformations between n-ple sums and m-ple sums. In the case of a transformation that relate a multiple sum to a single sum, it is a common feature that the single-sum side has bigger symmetry which is *not* apparent in the multiple-sum side. In this section, we use this symmetry of single sums for constructing various transformations between multiple series within an equal number of running indices. They include a number of transformations previously known by Milne–Lilly [24] and our paper [9], as well as new transformations.

We remark that this method of symmetry, based on transformations from multiple sums to single sums, was first used in our previous work [9], and was developed by [15] in the study of multiple elliptic hypergeometric transformations.

4.1. Watson transformations between $W^{n,2}$ and $\Phi^{n}_{1,2}$ series

In this subsection and the next, we derive two A_{n-1} generalizations of the Watson transformation between terminating ${}_8W_7$ series and terminating balanced ${}_4\phi_3$ series ([7, (2.5.1)]):

$$
\begin{aligned}
(4.1)\qquad & {}_8W_7\left[a; b, c, d, e, q^{-N}; q, \frac{a^2 q^{2+N}}{bcde}\right] \\
&= \frac{(aq, aq/de)_N}{(aq/d, aq/e)_N}\, {}_4\phi_3\left[\begin{matrix} q^{-N}, d, e, aq/bc \\ aq/b, aq/c, deq^{-N}/a \end{matrix}; q, q\right].
\end{aligned}
$$

Proposition 4.1. *We have*

$$W^{n,2}\left(\begin{matrix}\{b_i\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\, a;c,d;e,q^{-N};q,\frac{a^2q^{N+2}}{Bcde}\right) \tag{4.2}$$
$$= \frac{(aq/Bd)_N}{(aq/d)_N}\prod_{1\le i\le n}\frac{(ax_i)_N}{((aq/b_i)x_i)_N}$$
$$\times\ \Phi^n_{1,2}\left(\begin{matrix}\{b_i\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\,\begin{matrix}d\\ aq/e\end{matrix}\,\middle|\,\begin{matrix}aq/ce,\ q^{-N}\\ aq/c, Bdq^{1-N}/a\end{matrix}\,\middle|\, q,q\right).$$

Proof. In (3.6), set $n=1$ and change m to n to get

$$ {}_{n+3}\phi_{n+2}\left[\begin{matrix}q^{-N},a,c,\{u_i\}_n\\ e,\{v_i\}_n, acUq^{1-N}/eV\end{matrix};q,q\right] = \frac{(e,eV/acU)_N}{(e/a,eV/cU)_N} \tag{4.3}$$
$$\times\ \Phi^n_{1,2}\left(\begin{matrix}\{v_i/u_i\}_n\\ \{v_i\}_n\end{matrix}\,\middle|\,\begin{matrix}1/c\\ 1\end{matrix}\,\middle|\,\begin{matrix}a,\ q^{-N}\\ eV/cU,\ aq^{1-N}/e\end{matrix}\,\middle|\, q,q\right).$$

Equation (4.2) is obtained by identifying the terminating balanced ${}_{n+3}\phi_{n+2}$ series appearing in (4.3) and (3.3). In view of the set of variables

$$(\{(aq/b_ie)x_i\}_n, aq/ce, aq/de, a^2q^2/Bcde, \{(aq/e)x_i\}_n) \tag{4.4}$$

in the ${}_{n+3}\phi_{n+2}$ series in (3.3), apply the change of variables

$$\begin{aligned} &a\to aq/ce, && c\to aq/de, && e\to a^2q^2/Bcde,\\ &u_i\to (aq/b_ie)x_i, && v_i\to (aq/e)x_i && (i=1,\cdots,n).\end{aligned} \tag{4.5}$$

to (4.3). Then the left-hand side of the resulting formula coincides with the ${}_{n+3}\phi_{n+2}$-series in (3.3). Q.E.D.

Note that both the W series and the Φ series in (4.2) terminate with respect to the length of multi-indices (by confining the running multi-indices to those with length $\le N$); we call such terminating multiple series *triangular*. As compared with those triangular cases, we call a finite multiple series *rectangular* if it terminates with respect to a fixed multi-index $M=(m_1,\dots,m_n)\in\mathbb{N}^n$ as a sum over $\gamma=(\gamma_1,\dots,\gamma_n)\in\mathbb{N}^n$ such that $\gamma_i\le m_i$ $(i=1,\dots,n)$. Using a general recipe, one can translate identities for triangular multiple series into those for rectangular multiple series, and *vice versa*.

Corollary 4.1. (S.C. Milne and G.M. Lilly [24, Theorem 6.1]**)** *The following identity holds*:

$$
(4.6)\qquad W^{n,2}\left(\begin{matrix}\{q^{-m_i}\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\, a;c,d;b,e;q,\frac{a^2q^{|M|+2}}{bcde}\right)
$$

$$
= \frac{(aq/bd)_{|M|}}{(aq/d)_{|M|}}\prod_{1\le i\le n}\frac{(aqx_i)_{m_i}}{((aq/b)x_i)_{m_i}}
$$

$$
\times\ \Phi^n_{1,2}\left(\begin{matrix}\{q^{-m_i}\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\,\begin{matrix}d\\ aq/e\end{matrix}\,\middle|\,\begin{matrix}aq/ce,\ b\\ aq/c, bdq^{1-|M|}/a\end{matrix}\,\middle|\, q,q\right).
$$

Proof. We first write the prefactor in the right-hand side of (4.2) as a quotient of infinite products. Set $b_i = q^{-m_i}$ in (4.2), and notice that (4.6) is true for $b = q^{-N}$ for all nonnegative integer N. Clear the denominators in (4.6). Then we find that it is a polynomial identity in b^{-1} with an infinite number of roots. Thus, (4.6) is true for arbitrary b. Q.E.D.

We remark that all the corollaries of rectangular type in this section can be proved from the triangular versions by similar polynomial arguments. For this reason we omit proofs for the corollaires below.

The next proposition is a variant of (4.2) which is obtained by combining (3.3) and (3.9) with $n = 1$.

Proposition 4.2. *We have*

$$
(4.7)\qquad W^{n,2}\left(\begin{matrix}\{b_i\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\, a;c,d;e,q^{-N};q,\frac{a^2q^{N+2}}{Bcde}\right)
$$

$$
= \prod_{1\le i\le n}\frac{(ax_i,(aq/b_ie)x_i)_N}{((aq/b_i)x_i,(aq/e)x_i)_N}
$$

$$
\times\ \Phi^n_{1,2}\left(\begin{matrix}\{b_i\}_n\\ \{z_i\}_n\end{matrix}\,\middle|\,\begin{matrix}aq/cd\\ Beq^{-N}/a\end{matrix}\,\middle|\,\begin{matrix}e,\ q^{-N}\\ aq/c,\ aq/d\end{matrix}\,\middle|\, q,q\right),
$$

where $z_i = b_i/Bx_i$ *for* $1 \le i \le n$.

Proof. In (3.9), set $n=1$ and change m to n:

$$(4.8)\qquad {}_{n+3}\phi_{n+2}\left[\begin{matrix} q^{-N}, a, c, \{u_i\}_n \\ e, \{v_i\}_n, acUq^{1-N}/eV \end{matrix}; q, q\right]$$

$$= \frac{(eV/aU, eV/cU)_N}{(eV/acU, e)_N} \prod_{1\le i\le n} \frac{(u_i)_N}{(v_i)_N}$$

$$\times\ \Phi^n_{1,2}\left(\begin{matrix} \{v_i/u_i\}_n \\ \{u_i^{-1}\}_n \end{matrix} \middle| \begin{matrix} e \\ q^{1-N} \end{matrix} \middle| \begin{matrix} eV/acU, q^{-N} \\ eV/aU, eV/cU \end{matrix} \middle| q, q\right).$$

Equation (4.7) is obtained by identifying the terminating balanced ${}_{n+3}\phi_{n+2}$ series appearing in (4.8) and (3.3) through the change of variables (4.5).

Q.E.D.

Corollary 4.2.

$$(4.9)\qquad W^{n,2}\left(\begin{matrix} \{q^{-m_i}\}_n \\ \{x_i\}_n \end{matrix} \middle| a; c, d; b, e; q, \frac{a^2 q^{|M|+2}}{bcde}\right)$$

$$= \prod_{1\le i\le n} \frac{(aqx_i, (aq/be)x_i)_{m_i}}{((aq/b)x_i, (aq/e)x_i)_{m_i}}$$

$$\times\ \Phi^n_{1,2}\left(\begin{matrix} \{b_i\}_n \\ \{z_i\}_n \end{matrix} \middle| \begin{matrix} aq/cd \\ Beq^{-N}/a \end{matrix} \middle| \begin{matrix} e,\ q^{-N} \\ aq/c,\ aq/d \end{matrix} \middle| q, q\right),$$

where $z_i = q^{-m_i+|M|} x_i^{-1}$ *for* $i = 1, \cdots, n$.

Remark 4.1. *The transformation* (4.7) *can also be proved by a limiting procedure from the multiple Bailey transformation for* $W^{n,3}$ *series* ([15, (4.36)]):

$$(4.10)\qquad W^{n,3}\left(\begin{matrix} \{e_i\}_n \\ \{x_i\}_n \end{matrix} \middle| a; b, c, d; q^{-N}, f, \frac{a\lambda q^{1+N}}{Ef}; q, q\right)$$

$$= \prod_{1\le i\le n} \frac{(aqx_i, (aq/e_i f)x_i, (\lambda q/e_i)z_i, (\lambda q/f)z_i)_N}{((aq/e_i)x_i, (aq/f)x_i, \lambda q z_i, (\lambda q/e_i f)z_i)_N}$$

$$\times\ W^{n,3}\left(\begin{matrix} \{e_i\}_n \\ \{z_i\}_n \end{matrix} \middle| \lambda; \frac{aq}{cd}, \frac{aq}{bd}, \frac{aq}{bc}; q^{-N}, f, \frac{a\lambda q^{1+N}}{Ef}; q, q\right),$$

where $\lambda = a^2q/bcd$ *and* $z_i = e_i/Ex_i$ *for* $1 \le i \le n$.

4.2. Watson transformation between $W^{n,2}_{+1}$ and $\Phi^n_{2,1}$ series

We next present Watson transformations between terminating $W^{n,2}_{+1}$ series and terminating balanced $\Phi^n_{2,1}$ series. They are proved by combining (3.21) and (3.4) through terminating ${}_{2n+6}W_{2n+5}$ series.

Proposition 4.3. *We have*

$$
\begin{aligned}
&W^{n,2}_{+1}\left(\begin{matrix}\{b_i\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\, a;c;d,e,q^{-N};q,\frac{a^2q^{N+2}}{Bcde}\right)\\
&=\frac{(aq/Bc)_N}{(aq/c)_N}\prod_{1\le i\le n}\frac{(aqx_i)_N}{((aq/b_i)x_i)_N}\\
&\quad\times\Phi^n_{2,1}\left(\begin{matrix}\{b_i\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\,\begin{matrix}c,\ aq/de\\ aq/d,\ aq/e\end{matrix}\,\middle|\,\begin{matrix}q^{-N}\\ q^{-N}Bc/a\end{matrix}\,\middle|\, q,q\right).
\end{aligned}
\tag{4.11}
$$

Proof. We combine (3.21) and (3.4). Changing the parameterization, (3.4) can be rewritten as

$$
\begin{aligned}
&{}_{2n+6}W_{2n+5}\left[a;b,\{u_i\}_n,d,\{v_i\}_n,q^{-N};q,\frac{a^{n+1}q^{N+n+1}}{bdUV}\right]\\
&=\frac{(a^{n+1}q^{n+1}/bdUV,aq)_N}{(aq/b,aq/d)_N}\prod_{1\le i\le n}\frac{(v_i)_N}{(aq/u_i)_N}\\
&\quad\times\Phi^n_{2,1}\left(\begin{matrix}\{aqv_i/u_i\}_n\\ \{v_i^{-1}\}_n\end{matrix}\,\middle|\,\begin{matrix}aq/b,\ aq/d\\ aq,\ q^{1-N}\end{matrix}\,\middle|\,\begin{matrix}q^{-N}\\ a^{n+1}q^{n+1}/bdUV\end{matrix}\,\middle|\, q,q\right).
\end{aligned}
\tag{4.12}
$$

The very well-poised balanced ${}_{2n+6}W_{2n+5}$ series in (3.21) and (4.12) are identified through the change of variables

$$
\begin{aligned}
&a\to q^{-N}e/d, && b\to aq/cd, && d\to e,\\
&u_i\to (aq/b_id)x_i, && v_i\to (q^{-N}e/a)x_i^{-1} && (i=1,\cdots,n)
\end{aligned}
\tag{4.13}
$$

in (4.12). Q.E.D.

Corollary 4.3. *The following identity holds:*

$$
\begin{aligned}
&W^{n,2}_{+1}\left(\begin{matrix}\{q^{-m_i}\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\, a;c;b,d,e;q,\frac{a^2q^{|M|+2}}{bcde}\right)\\
&=\frac{(aq/bc)_{|M|}}{(aq/c)_{|M|}}\prod_{1\le i\le n}\frac{(aqx_i)_{m_i}}{((aq/b)x_i)_{m_i}}\\
&\quad\times\Phi^n_{2,1}\left(\begin{matrix}\{q^{-m_i}\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\,\begin{matrix}c,\ aq/de\\ aq/d,\ aq/e\end{matrix}\,\middle|\,\begin{matrix}b\\ q^{-|M|}bc/a\end{matrix}\,\middle|\, q,q\right).
\end{aligned}
\tag{4.14}
$$

4.3. Sears transformations for $\Phi^n_{1,2}$ series

In this subsection and the next, we present two $\Phi^n_{1,2}$ generalizations of the Sears transformation formula for terminating balanced ${}_4\phi_3$ series (3.5). They are derived by using special cases of (3.6) and (3.9) through terminating balanced ${}_{n+3}\phi_{n+2}$ series.

Proposition 4.4. *We have*

$$\Phi^n_{1,2}\left(\begin{matrix}\{b_i\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\,\begin{matrix}c\\ d\end{matrix}\,\middle|\,\begin{matrix}a,\ q^{-N}\\ e,\ aBcq^{1-N}/de\end{matrix}\,\middle|\, q,q\right) \tag{4.15}$$
$$= \frac{(e/B,de/ac)_N}{(e,de/aBc)_N}\,\Phi^n_{1,2}\left(\begin{matrix}\{b_i\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\,\begin{matrix}d/a\\ d\end{matrix}\,\middle|\,\begin{matrix}d/c,\ q^{-N}\\ de/ac,\ Bq^{1-N}/e\end{matrix}\,\middle|\, q,q\right).$$

Proof. We combine (3.10) and (4.8). The terminating balanced ${}_{n+3}\phi_{n+2}$ series in these formulas are identified through the change of variables

$$\begin{aligned} &a\to e/a, \qquad c\to f/a, \qquad e\to def/a^2Bc,\\ &u_i\to \frac{ef}{ab_ic}z_i, \quad v_i\to\frac{ef}{ac}z_i \quad (1\le i\le n).\end{aligned} \tag{4.16}$$

Q.E.D.

Corollary 4.4. *The following identity holds*:

$$\Phi^n_{1,2}\left(\begin{matrix}\{q^{-m_i}\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\,\begin{matrix}c\\ d\end{matrix}\,\middle|\,\begin{matrix}a,\ b\\ e,\ abcq^{1-|M|}/de\end{matrix}\,\middle|\, q,q\right) \tag{4.17}$$
$$= \frac{(e/b,de/ac)_{|M|}}{(e,de/abc)_{|M|}}\,\Phi^n_{1,2}\left(\begin{matrix}\{q^{-m_i}\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\,\begin{matrix}d/a\\ d\end{matrix}\,\middle|\,\begin{matrix}d/c, b\\ de/ac,\ bq^{1-|M|}/e\end{matrix}\,\middle|\, q,q\right).$$

Proposition 4.5. *We have*

$$\begin{aligned}&\Phi^n_{1,2}\left(\begin{matrix}\{b_i\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\,\begin{matrix}c\\ d\end{matrix}\,\middle|\,\begin{matrix}a, q^{-N}\\ e, aBcq^{1-N}/de\end{matrix}\,\middle|\, q,q\right)\\ &= \frac{(de/ac)_N}{(de/aBc)_N}\prod_{1\le i\le n}\frac{((d/b_i)x_i)_N}{(dx_i)_N}\\ &\quad\times \Phi^n_{1,2}\left(\begin{matrix}\{b_i\}_n\\ \{z_i\}_n\end{matrix}\,\middle|\,\begin{matrix}e/c\\ q^{-N}B/d\end{matrix}\,\middle|\,\begin{matrix}e/a,\ q^{-N}\\ de/ac,\ e\end{matrix}\,\middle|\, q,q\right),\end{aligned} \tag{4.18}$$

where $z_i = b_i/Bx_i$ for $1\le i\le n$.

Proof. Combine (3.7) and (4.8) through the change of variables

$$
\begin{aligned}
&a \to a, \qquad c \to d/c, \quad e \to de/Bc, \\
&u_i \to \frac{d}{b_i} x_i, \quad v_i \to dx_i \quad (1 \le i \le n).
\end{aligned} \tag{4.19}
$$

Q.E.D.

Corollary 4.5. (S.C. Milne and G.M. Lilly [24, Theorem 6.5]**)** *The following identity holds*:

$$
\begin{aligned}
&\Phi^n_{1,2}\left(\begin{matrix}\{q^{-m_i}\}_n \\ \{x_i\}_n\end{matrix}\;\middle|\;\begin{matrix} c \\ d\end{matrix}\;\middle|\;\begin{matrix} a, b \\ e, abcq^{1-|M|}/de\end{matrix}\;\middle|\; q, q\right) \\
&= \frac{(de/ac)_{|M|}}{(de/abc)_{|M|}} \prod_{1\le i\le n} \frac{((d/b)x_i)_{m_i}}{(dx_i)_{m_i}} \\
&\quad \times \Phi^n_{1,2}\left(\begin{matrix}\{q^{-m_i}\}_n \\ \{z_i\}_n\end{matrix}\;\middle|\;\begin{matrix} e/c \\ q^{-|M|}b/d\end{matrix}\;\middle|\;\begin{matrix} e/a,\ b \\ de/ac,\ e\end{matrix}\;\middle|\; q, q\right),
\end{aligned} \tag{4.20}
$$

where $z_i = q^{-m_i+|M|}x_i^{-1}$ *for* $1 \le i \le n$.

Remark 4.2. *Equations* (4.15) *and* (4.17) *can be proved by the iteration of* (4.18) *and* (4.20), *respectively. Including this fact, in our previous work* [12] *we gave a description of the transformation properties of* $\Phi^n_{1,2}$ *series in the limit* $q \to 1$.

Remark 4.3. *The two Watson transformations* (4.2) *and* (4.7) *transforms to each other by the Sears transformation* (4.18). *Similarly,* (4.6) *and* (4.9) *transform to each other by* (4.20).

4.4. Sears transformation for $\Phi^n_{2,1}$ series

Here we propose the Sears transformation for $\Phi^n_{2,1}$ series. It is proved by using two special cases of (3.2) through terminating ${}_{2n+6}W_{2n+5}$ series.

Proposition 4.6. *We have*

$$
\begin{aligned}
&\Phi^n_{2,1}\left(\begin{matrix}\{a_i\}_n \\ \{x_i\}_n\end{matrix}\;\middle|\;\begin{matrix} b,\ c \\ e,\ Abcq^{1-N}/de\end{matrix}\;\middle|\;\begin{matrix} q^{-N} \\ d\end{matrix}\;\middle|\; q, q\right) \\
&= \prod_{1\le i\le n} \frac{((de/bc)z_i, (e/a_i)x_i)_N}{((de/a_ibc)z_i, ex_i)_N} \\
&\quad \times \Phi^n_{2,1}\left(\begin{matrix}\{a_i\}_n \\ \{z_i\}_n\end{matrix}\;\middle|\;\begin{matrix} d/b,\ d/c \\ de/bc,\ Aq^{1-N}/e\end{matrix}\;\middle|\;\begin{matrix} q^{-N} \\ d\end{matrix}\;\middle|\; q, q\right),
\end{aligned} \tag{4.21}
$$

where $z_i = a_i/Ax_i$ *for* $i = 1, 2, \cdots, n$.

Proof. Rewriting (3.4) as

$$\begin{aligned}
(4.22)\quad & \Phi^n_{2,1}\left(\begin{matrix}\{a_i\}_n \\ \{x_i\}_n\end{matrix}\,\middle|\,\begin{matrix}b,\ c \\ e,\ Abcq^{1-N}/de\end{matrix}\,\middle|\,\begin{matrix}q^{-N} \\ d\end{matrix}\,\middle|\, q,q\right) \\
&= \frac{(de/Ac, de/Ab)_N}{(de^2/Abc, d)_N}\prod_{1\le i\le n}\frac{((dea_i/Abc)x_i^{-1})_N}{((de/Abc)x_i^{-1})_N} \\
&\quad \times {}_{2n+6}W_{2n+5}\left[de^2q^{-1}/Abc; \{(e/a_i)x_i\}_n,\right. \\
&\qquad\qquad \left.\{(deq^{-1}/Abc)x_i^{-1}\}_n, e/b, e/c, q^{-N}; q, dq^N\right],
\end{aligned}$$

combine (4.22) and (4.12). The very well-poised balanced ${}_{2n+6}W_{2n+5}$ series in these formulas are identified through change of variables:

$$(4.23)\qquad \begin{aligned}
&a \to \frac{de^2q^{-1}}{Abc}, && b \to e/b, && d \to e/c, \\
&u_i \to (deq^{-1}/Abc)x_i^{-1}, && v_i \to (e/a_i)x_i && (i = 1, \cdots, n).
\end{aligned}$$

Q.E.D.

Corollary 4.6. (S.C. Milne and G.M. Lilly [24, Theorem 6.8]**)** *The following identity holds*:

$$\begin{aligned}
(4.24)\qquad & \Phi^n_{2,1}\left(\begin{matrix}\{q^{-m_i}\}_n \\ \{x_i\}_n\end{matrix}\,\middle|\,\begin{matrix}b,\ c \\ e,\ abcq^{1-|M|}/de\end{matrix}\,\middle|\,\begin{matrix}a \\ d\end{matrix}\,\middle|\, q,q\right) \\
&= \prod_{1\le i\le n}\frac{((de/bc)z_i, (e/a)x_i)_{m_i}}{((de/abc)z_i, ex_i)_{m_i}} \\
&\quad \times \Phi^n_{2,1}\left(\begin{matrix}\{a_i\}_n \\ \{z_i\}_n\end{matrix}\,\middle|\,\begin{matrix}d/b,\ d/c \\ de/bc,\ aq^{1-|M|}/e\end{matrix}\,\middle|\,\begin{matrix}a \\ d\end{matrix}\,\middle|\, q,q\right),
\end{aligned}$$

where $z_i = q^{m_i-|M|}x_i^{-1}$ *for* $i = 1, 2, \cdots, n$.

Remark 4.4. *In the original work of Milne–Lilly* [24], (4.24) *is referred to as the* C_r *Sears transformation. The equation* (4.21) *can be obtained from the multiple Bailey transformation* (4.10) *in the following way*: *First replace the parameters as* $d \to aq/d$ *and* $f \to aq/f$ *in* (4.10), *and then let* a *tends to infinity in the resulting formula.*

4.5. Transformation for nonterminating $W^{n,2}_{-1}$ series

Here we derive a transformation for nonterminating $W^{n,2}_{-1}$ series which generalizes the nonterminating ${}_8W_7$ transformation

$$(4.25)\quad {}_8W_7\left[a;b,c,d,e,f;q,\frac{a^2q^2}{bcdef}\right]$$
$$= \frac{(aq,aq/ef,\lambda q/e,\lambda q/f)_\infty}{(aq/e,aq/f,\lambda q,\lambda q/ef)_\infty}{}_8W_7\left[\lambda;\lambda b/a,\lambda c/a,\lambda d/a,e,f;q,\frac{aq}{ef}\right],$$

for where $\lambda = a^2q/bcd$. It is proved by using (3.18) through nonterminating ${}_{2n+6}W_{2n+5}$ series.

Proposition 4.7. *We have*

$$(4.26)\qquad W^{n,2}_{-1}\left(\begin{array}{c}\{e_i\}_n\\ \{x_i\}_n\end{array}\middle| a;b,c,d;f;q,\frac{a^2q^2}{bcdEf}\right)$$
$$= \prod_{1\le i\le n}\frac{(aqx_i,(aq/e_if)x_i,(\lambda q/e_i)z_i,(\lambda q/f)z_i)_\infty}{((aq/e_i)x_i,(aq/f)x_i,(\lambda q/e_if)z_i,\lambda qz_i)_\infty}$$
$$\times W^{n,2}_{-1}\left(\begin{array}{c}\{e_i\}_n\\ \{z_i\}_n\end{array}\middle| \lambda;aq/cd,aq/bd,aq/bc;f;q,\frac{aq}{Ef}\right),$$

where $\lambda = a^2q/bcd$ *and* $z_i = \dfrac{e_i}{E}x_i^{-1}$.

Proof. In the ${}_{2n+6}W_{2n+5}$ series appearing in (3.18), interchange the parameters $(aq/b_if)x_i$ and $(\mu f/a)x_i^{-1}$ $(i=1,\cdots,n)$ simultaneously. Q.E.D.

Remark 4.5. *Equation* (4.26) *can also be obtained by taking the limit* $N\to\infty$ *in the multiple Bailey transformation* (4.10).

§5. Summation formulas

In this final section, we give some remarks on summation formulas which are obtained as special cases of multiple basic hypergeometric transformations in Section 3. We also give a new proof of the basic analogue of the Minton–Karlsson summation due to Gasper [6].

5.1. Summation formulas for W series

The transformation (3.2) with $m = 0$ implies Milne's generalization of Rogers' summation formula [7, (2.4.2)]

$$(5.1)\qquad {}_6W_5\left[a;b,c,q^{-N};q,\frac{aq^{N+1}}{bc}\right] = \frac{(aq,aq/bc)_N}{(aq/b,aq/c)_N}$$

to terminating $W^{n,1}$ series.

Corollary 5.1. (S.C. Milne [22, Theorem 2.1]**)** *We have*

$$W^{n,1}\left(\begin{matrix}\{b_i\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\, a;c;q^{-N};q,\frac{aq^{N+1}}{Bc}\right) = \frac{(aq/Bc)_N}{(aq/c)_N}\prod_{1\le i\le n}\frac{(aqx_i)_N}{((aq/b_i)x_i)_N}. \tag{5.2}$$

Remark 5.1. *The summation* (5.2) *is equivalent to the A_{n-1} multiple q-binomial theorem*

$$\sum_{\gamma\in\mathbb{N}^n,|\gamma|=N}\frac{\Delta(xq^{\gamma})}{\Delta(x)}\prod_{1\le i,j\le n}\frac{(a_jx_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}} = \frac{(A)_N}{(q)_N}. \tag{5.3}$$

From this formula, it follows that the $\Phi^n_{m,r}$ series with $m=0$ essentially reduce to ${}_{r+1}\phi_r$ series as

$$\Phi^n_{0,r}\left(\begin{matrix}\{a_i\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\,\begin{matrix}\cdot\\ \cdot\end{matrix}\,\middle|\,\begin{matrix}\{c_s\}_r\\ \{e_s\}_r\end{matrix}\,\middle|\, q,u\right) = {}_{r+1}\phi_r\left[\begin{matrix}A,\{c_s\}_r\\ \{e_s\}_r\end{matrix};q;u\right]. \tag{5.4}$$

The transformation (3.20) with $m=0$ gives a generalization of Rogers' summation formula to terminating $W^{n,1}_{+1}$ series.

Corollary 5.2. *The following identity holds*:

$$W^{n,1}_{+1}\left(\begin{matrix}\{b_i\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\, a;\cdot;c,q^{-N};q,\frac{a^2q^{N+2}}{Bc}\right) = \prod_{1\le i\le n}\frac{((aq/b_ic)x_i,aqx_i)_N}{((aq/b_i)x_i,(aq/c)x_i)_N}. \tag{5.5}$$

As the special case of (3.17) with $m=0$, we obtain Milne's generalization of Jackson's q-Dougall summation [7]

$$ {}_6W_5\left[a;b,c,d;q,\frac{aq}{bcd}\right] = \frac{(aq,aq/cd,aq/bd,aq/bc)_\infty}{(aq/bcd,aq/b,aq/c,aq/d)_\infty} \tag{5.6}$$

to nonterminating $W^{n,1}_{-1}$ series.

Corollary 5.3. (S.C. Milne [21, Theorem 4.27]**)** *Assume that* $\left|\dfrac{aq}{Bcd}x_i^{-1}\right| < 1$ *for all* $i = 1, \cdots, n$*. Then we have*

$$W^{n,1}_{-1}\left(\begin{array}{c}\{b_i\}_n\\ \{x_i\}_n\end{array}\middle|\, a; c, d; \cdot; q, \frac{a^2q^2}{Bcd}\right) \tag{5.7}$$
$$= \frac{(aq/Bc, aq/Bd)_\infty}{(aq/c, aq/d)_\infty} \prod_{1\le i\le n} \frac{(aqx_i, (aqb_i/Bcd)x_i^{-1})_\infty}{((aq/b_i)x_i, (aq/Bcd)x_i^{-1})_\infty}.$$

5.2. Summation formulas for Φ series

A multiple generalization of the Pfaff-Saalschütz summation formula [7, (1.7.2)]

$$ {}_3\phi_2\left[\begin{matrix}a, b, q^{-N}\\ c, abq^{1-N}/c\end{matrix}; q, q\right] = \frac{(c/a, c/b)_N}{(c, c/ab)_N}. \tag{5.8}$$

is obtained as the special case of (3.2) with $m = 0$.

Corollary 5.4. (S.C. Milne [22, Theorem 4.15]**)** *We have*

$$\Phi^n_{1,1}\left(\begin{array}{c|c|c|c}\{a_i\}_n & b & q^{-N} & \\ \{x_i\}_n & c & Abq^{1-N}/c & q, q\end{array}\right) \tag{5.9}$$
$$= \frac{(c/b)_N}{(c/Ab)_N} \prod_{1\le i\le n} \frac{((c/a_i)x_i)_N}{(cx_i)_N}.$$

By the general recipe as in the proof of Corollary 4.1, from the multiple summation (5.9) of triangular type we obtain the following rectangular version.

Corollary 5.5. (S.C. Milne [22, Theorem 2.2]**)** *We have*

$$\Phi^n_{1,1}\left(\begin{array}{c|c|c|c}\{q^{-m_i}\}_n & b & a & \\ \{x_i\}_n & c & abq^{1-|M|}/c & q, q\end{array}\right) \tag{5.10}$$
$$= \frac{(c/b)_{|M|}}{(c/ab)_{|M|}} \prod_{1\le i\le n} \frac{((c/a)x_i)_{m_i}}{(cx_i)_{m_i}}.$$

We now apply the transformation (3.13) to rewrite the left-hand side of (5.10) into an ${}_{n+2}\phi_{n+1}$ series. Then, the resulting formula turns out to be Gasper's q-analogue of the Minton–Karlsson summation.

Corollary 5.6. (G. Gasper [6, Equation (19)]**)**

$$(5.11) \qquad {}_{n+2}\phi_{n+1}\left[\begin{matrix} a, b, \{cq^{m_i}x_i\}_n \\ bq, \{cx_i\}_n \end{matrix}; q, a^{-1}q^{1-|M|}\right] = b^{|M|}\,\frac{(q, bq/a)_\infty}{(bq, q/a)_\infty}\prod_{1\le i\le n}\frac{((c/b)x_i)_{m_i}}{(cx_i)_{m_i}}.$$

This means that Gasper's q-analogue of the Minton–Karlsson summation (5.11) for ${}_{n+2}\phi_{n+1}$ series and the multiple q-Pfaff-Saalschütz summation (5.10) transform into each other through the multiple ${}_3\phi_2$ transformation (3.12).

We have shown that various A_{n-1} multiple hypergeometric summation formulas due to Milne and his collaborators are interpreted as special cases of the transformations discussed in Section 3. (For their series of works, see the references cited in Milne [23] and Milne–Newcomb [26])

In view of the results in Section 4 and this section, we may say that our multiple basic hypergeometric transformations in Section 3 recover a large extent of previously known results on A_{n-1} multiple summations and transformations as special cases. Also, we have clarified in this paper that a variety of identities that characterize A_{n-1} multiple basic hypergeometric series arise from the symmetry hidden behind the transformations between multiple sums and single sums.

Acknowledgments

I would like to express my sincere thanks to Professor Etsuro Date and especially to my former adviser Professor Masatoshi Noumi for their encouragements. I also thank Professor S.C. Milne for valuable comments. Finally, my thanks go to one of the editors and the referees for their suggestions and criticisms for the improvement of this paper.

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