

Product formulas for the relativistic and nonrelativistic conical functions

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*Dedicated to Masatoshi Noumi on the occasion of
his 60th birthday*

Abstract.

The conical function and its relativistic generalization can be viewed as eigenfunctions of the reduced 2-particle Hamiltonians of the hyperbolic Calogero-Moser system and its relativistic generalization. We prove new product formulas for these functions. As a consequence, we arrive at explicit diagonalizations of integral operators that commute with the 2-particle Hamiltonians and reduced versions thereof. The kernels of the integral operators are expressed as integrals over products of the eigenfunctions and explicit weight functions. The nonrelativistic limits are controlled by invoking novel uniform limit estimates for the hyperbolic gamma function.

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§1. Introduction

In this paper we obtain product formulas for the conical function specialization of the Gauss hypergeometric function ${}_2F_1$ and its ‘relativistic’ generalization $\mathcal{R}(a_+, a_-, b; x, y)$ from [R11]. The latter can be viewed as an eigenfunction of the Hamiltonian associated to the Calogero-Moser system of relativistic hyperbolic A_1 type. This generalizes the well-known fact that in suitable variables the conical function specialization of ${}_2F_1$ is an eigenfunction of the Hamiltonian of the Calogero-Moser system of nonrelativistic hyperbolic A_1 type.

Somewhat surprisingly, our product formulas for the conical function (obtained by taking limits of their relativistic generalizations in Theorems 4 and 5 below) seem to be new. In the context of harmonic analysis, a product formula for the more general Jacobi function is known since a long time, cf. Koornwinder’s survey [Koo84]. This formula arises from a group translate and encodes a convolution structure. By contrast, our product formulas (in Theorems 11 and 12) cannot be interpreted in terms of a generalized translate. Rather, they give rise to 1-parameter families of commuting integral operators on $L^2((0, \infty))$.

The \mathcal{R} -function was defined and studied in [R07] (see also [R11]), as a 5-variable specialization of the more general 8-variable ‘relativistic’ generalization R of ${}_2F_1$, introduced in [R94]. The definition of the R -function in *loc. cit.* is in terms of a contour integral that generalizes the Barnes representation for ${}_2F_1$. New representations of the R -function were later obtained by van de Bult [vdB06] and by van de Bult, Rains and Stokman [BRS07]. By suitable specializations the above results lead to three different representations for the \mathcal{R} -function.

More recently, it has been shown that the \mathcal{R} -function admits five further integral representations that, in contrast to previous representations, involve only four hyperbolic gamma functions [R11]. We only need one of these, which we proceed to detail.

First, throughout the paper we choose a_+ and a_- positive, and use further parameters

$$(1) \quad \alpha \equiv 2\pi/a_+a_-, \quad a \equiv (a_+ + a_-)/2,$$

$$(2) \quad a_s \equiv \min(a_+, a_-), \quad a_l \equiv \max(a_+, a_-).$$

The representation (3.51) in [R11] for the \mathcal{R} -function amounts to

$$(3) \quad \mathcal{R}(b; x, y) = (a_+a_-)^{-1/2}G(2ib - ia)J(b; x, y) \prod_{\delta=+,-} G(\delta y + ia - ib),$$

with $J(b; x, y)$ given by

$$(4) \quad J(b; x, y) = \int_{\mathbb{R}} dz \frac{G(z + x/2 - ib/2)G(z - x/2 - ib/2)}{G(z + x/2 + ib/2)G(z - x/2 + ib/2)} e^{iaz y}.$$

Here we take at first $(b, x, y) \in (0, 2a) \times \mathbb{R}^2$, and $G(z) \equiv G(a_+, a_-; z)$ denotes the hyperbolic gamma function, whose salient features are reviewed in Appendix A. (Just as we have done above, we shall suppress the dependence on a_+ , a_- , whenever this is not likely to cause ambiguities.) In particular, it is clear from the reflection equation (189) that J is even in x and y , while the conjugacy relation (192) entails real-valuedness for real arguments.

In this paper we mostly deal with the J -function (4), as opposed to the \mathcal{R} -function and further avatars introduced shortly. It naturally arises in the step from $N = 1$ to $N = 2$ in our recent recursive construction of the arbitrary- N joint eigenfunctions of the hyperbolic relativistic Calogero-Moser system [HR14], and also equals the function $B(b; x, y)$ given by Eq. (3.24) in [R11].

With an eye on making this paper somewhat more self-contained, we proceed to summarise some key properties of $J(b; x, y)$ and several related functions we have occasion to use. The analyticity properties of the R -function are known in great detail from Theorem 2.2 in [R99]. Combining this theorem with (3) and the definition of \mathcal{R} as a specialization of R , it is readily seen that $G(ib - ia)J(b; x, y)$ extends to a function that is meromorphic in b , x and y , with poles that can only be located on the affine hyperplanes

$$(5) \quad \pm x = 2ia - ib + i(ka_+ + la_-), \quad k, l \in \mathbb{N} \equiv \{0, 1, 2, \dots\},$$

$$(6) \quad \pm y = ib + i(ka_+ + la_-), \quad k, l \in \mathbb{N}.$$

Moreover, the pole order is bounded by the corresponding zero order of the product function

$$(7) \quad \prod_{\delta=+,-} E(\delta x + ib - ia)E(\delta y - ib + ia),$$

where the E -function is an entire function related to the hyperbolic gamma function by $G(z) = E(z)/E(-z)$ (see Appendix A for further details).

A pivotal role in obtaining the product formulas for the J -function is played by the explicit evaluation

$$(8) \quad J(b; ib, v) = \sqrt{a_+ a_-} G(ia - 2ib) \prod_{\delta=+,-} G(\delta v - ia + ib),$$

which follows from (3) and Eqs. (2.13) and (2.20) in [R11].

Another crucial ingredient is the asymptotic behavior of the J -function for $\operatorname{Re} x \rightarrow \infty$. This involves a specialization of Theorem 1.2 in [R03II], which deals with the 4-coupling BC_1 case, to the 1-coupling A_1 case at hand. To state the relevant result, we introduce the c -function

$$(9) \quad c(b; z) \equiv \frac{G(z + ia - ib)}{G(z + ia)},$$

the phase function

$$(10) \quad \phi(b) \equiv \exp(iab(b - 2a)/4),$$

and the scattering function

$$(11) \quad u(b; z) \equiv -c(b; z)/c(b; -z).$$

For later purposes we mention the involution symmetry

$$(12) \quad \phi(2a - b) = \phi(b), \quad u(2a - b; z) = u(b; z).$$

By contrast, the c -function and weight function,

$$(13) \quad w(b; z) \equiv 1/c(b; z)c(b; -z),$$

are not invariant under this involution, but there is a simple relation between the two distinct weight functions:

$$(14) \quad w(b; z)w(2a - b; z) = G(z + ia)^2 G(-z + ia)^2.$$

Next, we rewrite the J -function in terms of an E-function defined by

$$(15) \quad \begin{aligned} E(b; x, y) &\equiv (a_+ a_-)^{-1/2} \phi(b) G(ib - ia) \frac{J(b; x, y)}{c(b; x)c(2a - b; y)} \\ &= \phi(b) G(ib - ia) G(ia - 2ib) \frac{\mathcal{R}(b; x, y)}{c(b; x)c(b; y)}, \end{aligned}$$

where we used (3). This E-function is the A_1 specialization of the BC_1 \mathcal{E} -function dealt with in Theorem 1.2 of [R03II], cf. also (2.36)–(2.38) in [R11]. Setting

$$(16) \quad E_{\text{as}}(b; x, y) \equiv \exp(i\alpha xy/2) - u(b; -y) \exp(-i\alpha xy/2),$$

this theorem yields a bound

$$(17) \quad |(E - E_{\text{as}})(b; x, y)| < C(b, \delta, y, \operatorname{Im} x) \exp(-\rho \operatorname{Re} x),$$

where $\rho > 0, \operatorname{Re} x > \delta > 0$, and C is a positive continuous function on $(0, 2a) \times (0, \infty)^2 \times \mathbb{R}$. Moreover, specializing to $\operatorname{Im} x = 0$, it is known that the decay rate ρ can be chosen equal to any positive number r satisfying (199).

Now from the asymptotics (197) of the hyperbolic gamma function it is straightforward to infer a bound

$$(18) \quad c(b; z)/\phi(b) = \exp(-\alpha bz/2)(1 + O(\exp(-r\operatorname{Re} z))), \quad \operatorname{Re} z \rightarrow \infty,$$

uniformly on $\operatorname{Im} z$ -compacts. Thus, assuming $b \in (0, 2a)$ and $y \in (0, \infty)$, the leading asymptotic behaviour of $J(b; x, y)$ for $\operatorname{Re} x \rightarrow \infty$ is given by the function

$$(19) \quad J_{\text{as}}(b; x, y) \equiv \sqrt{a_+ a_-} G(ia - ib) \exp(-\alpha bx/2) \times \sum_{\tau=+,-} c(2a - b; \tau y) \exp(\tau i \alpha xy/2).$$

More specifically, we deduce from the above

$$(20) \quad |(J - J_{\text{as}})(b; x, y)| < C(b, \delta, y, \operatorname{Im} x) \exp(-(\alpha b/2 + \rho)\operatorname{Re} x),$$

where $\rho > 0, \operatorname{Re} x > \delta > 0$, and C is a positive continuous function on $(0, 2a) \times (0, \infty)^2 \times \mathbb{R}$ and where for $\operatorname{Im} x = 0$ the decay rate ρ can be chosen equal to any positive number r satisfying (199). Clearly, by evenness of $J(x, y)$ in x , the asymptotics for $\operatorname{Re} x \rightarrow -\infty$ is given by $J_{\text{as}}(-x, y)$.

We need to invoke some more features of the E- and J-functions that follow by specialization from results in [R03II] and [R03III], cf. also Subsection 2.2 in [R11]. First, the E-function satisfies the self-duality relation

$$(21) \quad \text{E}(b; x, y) = \text{E}(b; y, x),$$

and has the symmetry property

$$(22) \quad \text{E}(b; x, y) = \text{E}(2a - b; x, y).$$

In view of (15), this entails

$$(23) \quad J(b; x, y) = G(ia - ib)^2 J(2a - b; y, x).$$

Secondly, the J -function is a joint eigenfunction of four independent analytic difference operators (henceforth $\mathbf{A}\Delta\mathbf{O}$ s), two acting on x and two on y . The corresponding analytic difference equations (henceforth $\mathbf{A}\Delta\mathbf{E}$ s) read

$$(24) \quad A_\delta(b; x)J(b; x, y) = 2c_\delta(y)J(b; x, y), \quad \delta = +, -,$$

$$(25) \quad A_\delta(2a - b; y)J(b; x, y) = 2c_\delta(x)J(b; x, y), \quad \delta = +, -,$$

with the AΔOs given by

$$(26) \quad A_\delta(b; z) \equiv \frac{s_\delta(z - ib)}{s_\delta(z)} T_{ia-\delta}^z + \frac{s_\delta(z + ib)}{s_\delta(z)} T_{-ia-\delta}^z.$$

Here, the translation operators are defined on analytic functions by

$$(27) \quad (T_c^z f)(z) \equiv f(z - c), \quad c \in \mathbb{C}^*,$$

and we are using the notation

$$(28) \quad s_\delta(z) \equiv \sinh(\pi z/a_\delta), \quad c_\delta(z) \equiv \cosh(\pi z/a_\delta), \quad e_\delta(z) \equiv e^{\pi z/a_\delta},$$

with $\delta = +, -$.

Thirdly, at the end of Section 2 we shall use that the generalized Fourier transform

$$(29) \quad \mathcal{F}(b) : \mathcal{C} \equiv C_0^\infty((0, \infty)) \subset L^2((0, \infty)) \rightarrow L^2((0, \infty)),$$

where $b \in (0, 2a)$, defined by

$$(30) \quad (\mathcal{F}(b)\psi)(x) \equiv \left(\frac{1}{2a_+a_-} \right)^{1/2} \int_0^\infty F(b; x, y)\psi(y)dy, \quad \psi \in \mathcal{C},$$

extends to a unitary operator. The F-function occurring here is the last cousin of the \mathcal{R} -function we need (besides J and E). It can be defined by

$$(31) \quad F(b; x, y) \equiv G(ia - 2ib)G(ib - ia)w(b; x)^{1/2}\mathcal{R}(b; x, y)w(b; y)^{1/2},$$

where $b \in (0, 2a)$, $x, y > 0$, and positive square roots are understood. Alternatively, it can be rewritten as

$$(32) \quad F(b; x, y) = \phi(b)^{-1}c(b; x)w(b; x)^{1/2}E(b; x, y)c(b; y)w(b; y)^{1/2},$$

cf. (15). Moreover, using the identity (14), we see that it is related to J by

$$(33) \quad F(b; x, y) = \frac{G(ib - ia)}{\sqrt{a_+a_-}}w(b; x)^{1/2}J(b; x, y)w(2a - b; y)^{1/2}.$$

Just as the E-function this function has the symmetry properties

$$(34) \quad F(b; x, y) = F(b; y, x) = F(2a - b; x, y),$$

but by contrast to the E-function it is real-valued. (Note that the quotient of F and E is a phase factor involving a square root of the u -function (11).) Thus the transform $\mathcal{F}(b)$ is not only unitary, but also self-adjoint (hence involutory); clearly, it also satisfies

$$(35) \quad \mathcal{F}(b) = \mathcal{F}(2a - b), \quad b \in (0, 2a).$$

(The transforms $\mathcal{F}(a_-) = \mathcal{F}(a_+)$ amount to the sine transform, while the limits $\mathcal{F}(0) = \mathcal{F}(2a)$ exist and amount to the cosine transform.)

The transform diagonalizes the Hamiltonians

$$(36) \quad H_\delta(b; z) \equiv \left(\frac{s_\delta(z - ib)}{s_\delta(z)} \right)^{1/2} T_{ia-\delta}^z \left(\frac{s_\delta(z + ib)}{s_\delta(z)} \right)^{1/2} + (i \rightarrow -i),$$

where $\delta = +, -$, in the sense that on the dense subspace \mathcal{C} one has

$$(37) \quad H_\pm(b)\mathcal{F}(b) = 2\mathcal{F}(b)c_\pm(\cdot).$$

Note that \mathcal{C} is a domain of essential self-adjointness for the multiplication operators $2c_\pm(\cdot)$. Thus the unitary transform $\mathcal{F}(b)$ makes it possible to associate commuting self-adjoint operators on $L^2((0, \infty), dz)$ to the $\Lambda\Delta$ O's $H_\pm(b; z)$.

We are now in a position to sketch the results and organization of this paper. In Section 2 we first clarify the special character of the kernel function

$$(38) \quad \mathcal{K}(b; x, y, z) \equiv \prod_{\delta_1, \delta_2, \delta_3 = +, -} G((\delta_1 x + \delta_2 y + \delta_3 z - ib)/2),$$

by proving the kernel identities it satisfies, cf. Proposition 1. Then we focus on the function

$$(39) \quad F(b, v; x, y) \equiv \int_{\mathbb{R}} dz w(b; z) J(b; z, v) \mathcal{K}(b; x, y, z), \quad x, y \in \mathbb{R},$$

and show it is an eigenfunction of the $\Lambda\Delta$ O's $A_\pm(b; x)$ with eigenvalue $2c_\pm(v)$, cf. Lemma 2. By invariance under $x \leftrightarrow y$, this is also true for $A_\pm(b; y)$, and then we can appeal to previous uniqueness and continuity results to obtain

$$(40) \quad F(b, v; x, y) = \lambda(b, v) J(b; x, v) J(b; y, v),$$

where $b \in (0, 2a)$ and $x, y \in \mathbb{R}$.

It is now far from obvious, but true that the proportionality factor $\lambda(b, v)$ equals 4, hence yielding our key result

$$(41) \quad J(b; x, v) J(b; y, v) = \frac{1}{2} \int_0^\infty w(b; z) J(b; z, v) \mathcal{K}(b; x, y, z) dz,$$

where $b \in (0, 2a)$ and $x, y, v \in \mathbb{R}$, cf. Theorem 4. The proof of this identity hinges on the explicit evaluation (8) and a rather arduous asymptotic analysis carried out in Lemma 3.

Using the J -symmetry (23), we obtain a second product formula in Theorem 5. We stress that we do not invoke the unitary involution $\mathcal{F}(b)$ to arrive at these product formulas. However, we can derive some remarkable consequences when we employ $\mathcal{F}(b)$. They emerge after using (33) to switch from J to the kernel $F(b; x, y)/\sqrt{2a_+a_-}$ of $\mathcal{F}(b)$, cf. (30).

More specifically, we can completely elucidate the Hilbert space features of the family of integral operators $\mathcal{I}_z(b)$, $z \geq 0$, given by (87). Theorem 6 reveals that this is a commuting family of bounded self-adjoint operators on $L^2((0, \infty))$ diagonalized by $\mathcal{F}(b)$ as the multiplication operators $2J(b; z, \cdot)$. Moreover, the use of $\mathcal{F}(b)$ allows us to obtain the striking identity (92).

The functions \mathcal{R} , J , E and F occurring in Section 2 are related by similarity transformations that give rise to different sets of four analytic difference operators whose joint eigenfunctions they are. (Recall we detailed these $\Lambda\Delta$ O's for the J -function in (24) and (25), and for the F -function in (36).) These objects are the center-of-mass reductions of 2-particle counterparts studied in Section 3. More precisely, we only employ the 2-particle versions of J and F given by

$$(42) \quad \Psi_2(b; x, y) \equiv \exp(i\alpha(x_1 + x_2)(y_1 + y_2)/2)\Psi(b; x_1 - x_2, y_1 - y_2),$$

with $\Psi = J, F$ (cf. (99) and (109)), and do not consider the 2-particle $\Lambda\Delta$ O's whose joint eigenfunctions they are. The 2-particle counterpart of the kernel function \mathcal{K} is the function $\mathcal{S}_2(b; x, y)$ given by (94). The first result of Section 3 is that the 2-particle eigenfunction $J_2(b; z, y)$ is also an eigenfunction of the integral operator whose kernel is the product of $\mathcal{S}_2(b; x, z)$ and the weight function $w(b; z_1 - z_2)$; moreover, the eigenvalue is an explicit product of G -functions, cf. Theorem 7.

The proof of this theorem only involves the product formula (41) and the Fourier transform (202). That is, it does not involve the unitary $\mathcal{F}(b)$. When we employ the 2-particle version $\mathcal{F}_2(b)$ of $\mathcal{F}(b)$ (given by (114)), we can reformulate Theorem 7 as an explicit diagonalization of the integral operator $\mathcal{I}_2(b)$ defined by (115), cf. Theorem 9. As a consequence, the kernel of the 2-particle integral operator can be represented by a formula (namely (118)) that enables us to recover the formula (92) for the kernel of the reduced ('center-of-mass') integral operator $\mathcal{I}_z(b)$. The crucial step here is to invoke the definition (4) of the J -function, so that we come full circle.

In Section 4 we derive nonrelativistic counterparts of the results in Sections 2 and 3. The self-duality of the joint eigenfunctions is not preserved by the nonrelativistic limit. Indeed, the conical function and its 2-particle generalization are joint eigenfunctions of a hyperbolic (non-relativistic) differential operator and a rational (relativistic) difference operator, cf. [R11]. Accordingly, they admit distinct representations involving either hyperbolic functions or the rational (Euler) gamma function.

The same is true for the kernel functions, product formulas and associated integral operators. The first, hyperbolic type of limit can be handled for the integrands by using the previously known limit formula (203). The second type, which leads to the Γ -function, involves the $G \rightarrow \Gamma$ limit encoded in the equations (248)–(253).

To control the limits of the pertinent integrals, however, we need uniform bounds for the two types of limits that are strong enough to invoke the dominated convergence theorem. We obtain such bounds in Appendix B and Appendix C. These estimates are new and of independent interest.

§2. Product formulas for $J(b; x, y)$

The main purpose of this section is to obtain the product formula (41) for the J -function, given by (4). We begin by focusing on the kernel function. It is clear from its definition (38) that $\mathcal{K}(b; x, y, z)$ is a meromorphic function of b, x, y, z , with poles located only on the affine hyperplanes (cf. (196))

$$(43) \quad \delta_1 x + \delta_2 y + \delta_3 z = ib - 2ia - 2i(ka_+ + la_-), \\ \delta_1, \delta_2, \delta_3 = +, -, \quad k, l \in \mathbb{N}.$$

As demonstrated by the following proposition, it satisfies three independent kernel identities.

Proposition 1. *Letting $b \in \mathbb{C}$, we have*

$$(44) \quad A_\delta(b; x)\mathcal{K}(b; x, y, z) = A_\delta(b; y)\mathcal{K}(b; x, y, z) = A_\delta(b; z)\mathcal{K}(b; x, y, z),$$

where $\delta = +, -$.

Proof. Since $\mathcal{K}(x, y, z)$ is manifestly invariant under any permutation of the variables (x, y, z) , it suffices to establish the first equality in (44). After dividing both the left-hand side and the right-hand side by

$\mathcal{K}(b; x - ia_{-\delta}, y, z)$, we use the AΔEs (185) to rewrite the result as

$$\begin{aligned}
 (45) \quad & \frac{s_\delta(x - ib)}{s_\delta(x)} + \frac{s_\delta(x + ib)}{s_\delta(x)} \prod_{\delta_2, \delta_3 = +, -} \frac{c_\delta((x + \delta_2 y + \delta_3 z - ib)/2)}{c_\delta((x + \delta_2 y + \delta_3 z + ib)/2)} \\
 &= \frac{s_\delta(y - ib)}{s_\delta(y)} \prod_{\delta_3 = +, -} \frac{c_\delta((x - y + \delta_3 z - ib)/2)}{c_\delta((x - y + \delta_3 z + ib)/2)} \\
 & \quad + \frac{s_\delta(y + ib)}{s_\delta(y)} \prod_{\delta_3 = +, -} \frac{c_\delta((x + y + \delta_3 z - ib)/2)}{c_\delta((x + y + \delta_3 z + ib)/2)}.
 \end{aligned}$$

It is readily seen that both sides are $2ia_\delta$ -periodic functions of x with equal limits

$$(46) \quad e_\delta(\mp ib) + e_\delta(\mp 3ib), \quad \text{Re } x \rightarrow \pm\infty,$$

and that the residues at the (generically simple) poles $x = 0$ and $x = ia_\delta$ in the period strip cancel. By Liouville's theorem, it remains to verify that the residues at the poles $x = \delta_2 y + \delta_3 z - ib + ia_\delta$ cancel as well, and this amounts to a routine calculation. (In fact, one need only check the case $\delta_2 = \delta_3 = +$, since the left-hand side and right-hand side of (45) are even functions of both y and z .) Q.E.D.

Assuming $(b, v) \in (0, 2a) \times (0, \infty)$, we show next that the product $J(b; x, v)J(b; y, v)$ can be expressed as an integral over the auxiliary variable z in the integrand

$$(47) \quad I(b, v; x, y, z) \equiv w(b; z)J(b; z, v)\mathcal{K}(b; x, y, z),$$

occurring on the right-hand side of (39). (Below, we will often suppress the dependence on b and v , whenever this is not likely to cause ambiguities.)

Combining (13) with (185) and (193), we deduce

$$(48) \quad w(z) = \prod_{\delta = +, -} 2s_\delta(z) \frac{E(\delta z + ib - ia)}{E(\delta z + ia - ib)}.$$

Since the product function $J(z, v) \prod_{\delta = +, -} E(\delta z + ib - ia)$ is holomorphic for all $z \in \mathbb{C}$, it follows that poles of $I(x, y, z)$ are due either to zeros of the E -functions $E(\delta z + ia - ib)$, $\delta = +, -$, which are located at (cf. (194))

$$(49) \quad \delta z = ib + i(ka_+ + la_-), \quad \delta = +, -, \quad k, l \in \mathbb{N},$$

or poles of the kernel function $\mathcal{K}(x, y, z)$. Letting first $x, y \in \mathbb{R}$, it is clear that the integration contour \mathbb{R} stays away from these pole sequences.

Making use of the asymptotics (197) of the hyperbolic gamma function, we readily infer from (13) and (38) that

$$(50) \quad w(b; z) = \exp(\alpha b|\operatorname{Re} z|)(1 + O(\exp(-r|\operatorname{Re} z|))),$$

$$(51) \quad \mathcal{K}(b; x, y, z) = \exp(-\alpha b|\operatorname{Re} z|)(1 + O(\exp(-r|\operatorname{Re} z|))),$$

for $|\operatorname{Re} z| \rightarrow \infty$, where the decay rate is any positive number r satisfying (199) and the implied constants are uniform for $(b, \operatorname{Im} z)$ and $(b, x, y, \operatorname{Im} z)$ varying over compact subsets of $(0, 2a) \times \mathbb{R}$ and $(0, 2a) \times \mathbb{C}^2 \times \mathbb{R}$, respectively. Hence (19)–(20) entail that the integrand I decays exponentially as $|z| \rightarrow \infty$, so that the function (39) is well defined.

To motivate the next step we consider the special b -values

$$(52) \quad b_{mn} \equiv ma_+ + na_-, \quad m, n \in \mathbb{Z},$$

inasmuch as they satisfy $b_{mn} \in (0, 2a)$, and recall from Section 2.3 in [R11] that the vector space of meromorphic joint solutions $f(x)$ to the AΔEs

$$(53) \quad A_{\pm}(b_{mn}; z)f(z) = 2c_{\pm}(v)f(z), \quad v > 0, \quad a_+/a_- \notin \mathbb{Q},$$

is two-dimensional. More specifically, the subspace consisting of even meromorphic solutions is one-dimensional and is thus spanned by the function $J(b_{mn}; z, v)$. With this result in mind, we aim to prove that $F(x, y)$ (as initially defined by (39)) has a meromorphic continuation to all of \mathbb{C} as a function of x which satisfies the AΔEs (53). For this, we need not restrict attention to the special b -values (52).

Clearly, the fixed contour representation (39) and the pole locations (43) for $\mathcal{K}(x, y, z)$ entail that $F(x, y)$ is holomorphic in the domain

$$(54) \quad D \equiv \{(x, y) \in \mathbb{C}^2 \mid |\operatorname{Im} x| + |\operatorname{Im} y| < 2a - b\}.$$

For sufficiently small values of b , this already suffices to show that F is a joint eigenfunction of the AΔOs $A_{\delta}(x)$, $\delta = +, -$, with the expected eigenvalues. More precisely, we fix attention on the subset

$$(55) \quad D_s \equiv \{(x, y) \in D \mid (x + i\eta, y) \in D, \forall \eta \in [-a_l, a_l]\}, \quad b \in (0, a_s),$$

for which the shifted arguments remain within the holomorphy domain D . Note that the restriction on b is necessary in order for D_s to be non-empty.

Lemma 2. *Fixing $b \in (0, a_s)$ and $v \in (0, \infty)$, and letting $(x, y) \in D_s$, we have the eigenvalue equations*

$$(56) \quad A_{\delta}(x)F(v; x, y) = 2c_{\delta}(v)F(v; x, y), \quad \delta = +, -.$$

Proof. Thanks to the analyticity of F in D and the restriction to D_s , the shifts of x by $\pm ia_{-\delta}$ are well defined. Hence we are allowed to act with the $A\Delta O$ s under the integral sign. Using the kernel identities (44) to trade the action of $A_\delta(x)$ on $\mathcal{K}(x, y, z)$ for that of $A_\delta(z)$, we can thus rewrite the left-hand side of (56) as

$$(57) \quad \int_{\mathbb{R}} dz w(z) J(z, v) \left(\frac{s_\delta(z - ib)}{s_\delta(z)} T_{ia_{-\delta}}^z + (i \rightarrow -i) \right) \mathcal{K}(x, y, z).$$

Note that the pole at $z = 0$ due to the denominator $s_\delta(z)$ is matched by a double zero of $w(z)$, cf. (13) and the locations (194) of the G -zeros.

Changing integration variable, we obtain

$$(58) \quad \int_{\mathbb{R} - ia_{-\delta}} dz \frac{s_\delta(z + ia_{-\delta} - ib)}{s_\delta(z + ia_{-\delta})} w(z + ia_{-\delta}) J(z + ia_{-\delta}, v) \mathcal{K}(x, y, z) \\ + \int_{\mathbb{R} + ia_{-\delta}} dz \frac{s_\delta(z - ia_{-\delta} + ib)}{s_\delta(z - ia_{-\delta})} w(z - ia_{-\delta}) J(z - ia_{-\delta}, v) \mathcal{K}(x, y, z).$$

From the difference equations (185) satisfied by $G(z)$ it follows that we have

$$(59) \quad \frac{w(z \pm ia_{-\delta})}{w(z)} = \frac{s_\delta(z \pm ia_{-\delta})}{s_\delta(z \pm ia_{-\delta} \mp ib)} \frac{s_\delta(z \pm ib)}{s_\delta(z)},$$

so that the above sum of integrals equals

$$(60) \quad \int_{\mathbb{R} - ia_{-\delta}} dz w(z) \frac{s_\delta(z + ib)}{s_\delta(z)} J(z + ia_{-\delta}, v) \mathcal{K}(x, y, z) \\ + \int_{\mathbb{R} + ia_{-\delta}} dz w(z) \frac{s_\delta(z - ib)}{s_\delta(z)} J(z - ia_{-\delta}, v) \mathcal{K}(x, y, z).$$

We claim that when we shift the contour $\mathbb{R} - ia_{-\delta}$ in the former integral up by $a_{-\delta}$ and the contour $\mathbb{R} + ia_{-\delta}$ in the latter integral down by $a_{-\delta}$, then no poles are met. (The asymptotic behaviour of J , as given by (19)–(20), and the bounds (50)–(51) ensure that the shift causes no problems at the tail ends.) Taking this claim for granted, the contours of the two integrals are now both equal to \mathbb{R} , so that we are entitled to invoke the eigenvalue equation (24), which implies (56).

To complete the proof, it remains to verify the claim. Clearly, we remain within the holomorphy domain of both J and \mathcal{K} while performing the relevant contour shifts. Letting first $a_{-\delta} = a_s$, we meet only the simple poles of the factors $1/G(\pm z + ia - ib)$ at $\pm z = ib$, but they are matched by zeros of $s_\delta(z \mp ib)$. Likewise, zeros of $G(\pm z + ia)$ match

the simple poles of $1/s_\delta(z)$ at $z = 0$ and at $z = \pm ia_s$, with the latter being met when $a_s = a_l$. Thus, the first/second integrand is regular for $z \in \mathbb{R} - i\eta / z \in \mathbb{R} + i\eta$ and $\eta \in [0, a_s]$.

Finally, consider the case $a_{-\delta} = a_l$. Then we encounter simple poles of $1/G(\pm z + ia - ib)$ located at points $\pm z = ib + ika_s$ with $ka_s \leq a_l - b$, but they are again matched by zeros of $s_\delta(z \mp ib)$. Moreover, the simple poles of $1/s_\delta(z)$ at $\pm z = ka_s$ with $ka_s \leq a_l$ are matched by zeros of $G(\pm z + ia)$, so that the first/second integrand is regular for $z \in \mathbb{R} - i\eta / z \in \mathbb{R} + i\eta$ and $\eta \in [0, a_l]$. Hence, our claim is proved. Q.E.D.

Keeping the assumptions in Lemma 2, we now fix $y \in \mathbb{R}$. Then it follows that $F(x, y)$ is holomorphic in x for $|\operatorname{Im} x| < 2a - b$, and that the AΔEs (56), which take the explicit form

$$(61) \quad \frac{s_\delta(x - ib)}{s_\delta(x)} F(x - ia_{-\delta}, y) + (i \rightarrow -i) = 2c_\delta(v) F(x, y),$$

hold true for $|\operatorname{Im} x| < a_s - b$. Moreover, by the assumption $b \in (0, a_s)$, we have $2a - b > a_l$ and $a_s - b > 0$. A moment's thought reveals that this state of affairs implies that $F(x, y)$ has a meromorphic continuation to all of \mathbb{C} as a function of x . Indeed, multiplying (61) by $s_\delta(x)/s_\delta(x - ib)$, we can continue $F(x, y)$ in steps of size $a_{-\delta}$ to the lower half plane. Likewise, upon multiplication by $s_\delta(x)/s_\delta(x + ib)$, we can continue $F(x, y)$ to the upper half plane.

Appealing to the above uniqueness result, we deduce

$$(62) \quad F(b_{mn}, v; x, y) = \kappa(b_{mn}, v, y) J(b_{mn}; x, v), \quad x, y \in \mathbb{R},$$

under the assumption that the parameters satisfy the conditions

$$(63) \quad v > 0, \quad b_{mn} \in (0, a_s), \quad a_+/a_- \notin \mathbb{Q}.$$

Now due to the manifest invariance of $F(x, y)$ under the interchange $x \leftrightarrow y$, it follows that $F(x, y)$ is also meromorphic in y and satisfies the y -version of the AΔEs (61). Therefore we also have

$$(64) \quad F(b_{mn}, v; x, y) = \kappa(b_{mn}, v, x) J(b_{mn}; y, v), \quad x, y \in \mathbb{R},$$

with (63) in force. Inspecting the quotient of (62) and (64), we conclude that we have

$$(65) \quad F(b_{mn}, v; x, y) = \lambda(b_{mn}, v) J(b_{mn}; x, v) J(b_{mn}; y, v), \quad x, y \in \mathbb{R},$$

with the function $\lambda(b_{mn}, v)$ to be determined. Since the b -values b_{mn} are dense in $(0, a_s)$ when a_+/a_- is irrational, (real) analyticity in b now

entails (40). Moreover, by continuity in a_{\pm} we can allow any positive a_{\pm} -values in (40).

Until further notice, we now assume $b \in (0, a)$. Then we are allowed to set $y = ib$ in (39) with x real, since this F -representation can be analytically continued to $|\operatorname{Im} y| < 2a - b$. Hence, using the reflection equation (189) and evenness of the integrand in z , we arrive at

$$(66) \quad F(x, ib) = 2 \int_0^{\infty} dz \mathcal{I}(x, z), \quad x \in \mathbb{R}, \quad b \in (0, a),$$

with the integrand given by

$$(67) \quad \mathcal{I}(b, v; x, z) = w(b; z) J(b; z, v) \prod_{\delta_1, \delta_2 = +, -} G((\delta_1 x + \delta_2 z)/2 - ib).$$

On the other hand, combining (8) and (40) we obtain the special value

$$(68) \quad \begin{aligned} F(b, v; x, ib) &= \sqrt{a_+ a_-} G(ia - 2ib) \lambda(b, v) \\ &\times J(b; x, v) \prod_{\delta = +, -} G(\delta v - ia + ib), \end{aligned}$$

and by (19)–(20) we know the leading asymptotic behaviour of the right-hand side of (68) as $x \rightarrow \infty$. Therefore, we can determine $\lambda(b, v)$ by computing the $x \rightarrow \infty$ asymptotics of the right-hand side of (66) and comparing it with that of (68).

Substituting the right-hand side of (197) for $G(z)$, we find that

$$(69) \quad \prod_{\delta = +, -} G(\delta t/2 - ib) = \exp(-abt/2) (1 + O(\exp(-rt))), \quad t \rightarrow \infty,$$

where r is any positive number satisfying (199). We now use the bounds (50), (20) and (69) with $t = x + z$ in a telescoping argument to deduce that we have

$$(70) \quad \begin{aligned} w(z) J(z, v) \prod_{\delta = +, -} G(\delta(x+z)/2 - ib) \\ = \exp(-abx/2) \left(\exp(abz/2) J_{\text{as}}(z, v) + O(\exp(-rz)) \right), \end{aligned}$$

for $z \rightarrow \infty$. (Recall that in (20) we can choose the decay rate ρ equal to r in case $\operatorname{Im} x = 0$.) It follows, in particular, that the left-hand side of this equality is bounded for $z \in (0, \infty)$.

Choosing instead $t = x - z$ in (69), we find that the product of the remaining G -factors in the integrand \mathcal{I} decays exponentially as $x \rightarrow$

∞ and z varies over a compact subset of $(0, \infty)$. This state of affairs suggests that when we substitute (70) in \mathcal{I} , then the first term should yield the leading asymptotic behaviour of the integral in (66). We proceed to make this suggestion precise.

Lemma 3. *Assuming $(b, v) \in (0, a) \times (0, \infty)$, we have*

$$(71) \quad F(b, v; x, ib) = 2 \exp(-\alpha bx/2) \times \left[\int_{\mathbb{R}} dz \exp(\alpha bz/2) J_{\text{as}}(b; z, v) \times \prod_{\delta=+,-} G(\delta(x-z)/2 - ib) + O(\exp(-r_m x/2)) \right],$$

for $x \rightarrow \infty$, with decay rate

$$(72) \quad r_m = \min(\alpha b/2, r).$$

Proof. Taking $x \in (0, \infty)$ from now on, we multiply (70) by the product $\prod_{\delta=+,-} G(\delta(x-z)/2 - ib)$ and integrate z over $(x/2, \infty)$. Since this multiplier function is bounded, we deduce that we have

$$(73) \quad \int_{x/2}^{\infty} dz \mathcal{I}(x, z) = \exp(-\alpha bx/2) \left[\int_{x/2}^{\infty} dz \exp(\alpha bz/2) \times J_{\text{as}}(z, v) \prod_{\delta=+,-} G(\delta(x-z)/2 - ib) + O(\exp(-rx/2)) \right],$$

for $x \rightarrow \infty$.

The point is now that we want to arrive at the integral over all of \mathbb{R} in (71), since it can be evaluated explicitly (as we shall presently show, cf. (78)–(79)). In view of (66), we first need to add the integral of $\mathcal{I}(x, z)$ over $(0, x/2)$ to the left-hand side of (73), so as to obtain $F(x, ib)/2$. To estimate this addition, we use (70) as before. But now we conclude from boundedness of the function $\exp(\alpha bz/2) J_{\text{as}}(z, v)$ for $z > 0$ (cf. (19)) that there exists a positive constant C such that

$$(74) \quad \left| \int_0^{x/2} dz \mathcal{I}(x, z) \right| < C e^{-\alpha bx/2} \int_0^{x/2} dz \prod_{\delta=+,-} G(\delta(x-z)/2 - ib),$$

where $x \in (0, \infty)$. (Note that the G -product in (69) is positive for $t \in \mathbb{R}$, cf. the conjugacy relation (192) and the locations (194) of the G -zeros.)

Next, we invoke (69) with $t = x - z$ to deduce

$$(75) \quad \int_{\xi}^{x/2} dz \prod_{\delta=+,-} G(\delta(x-z)/2 - ib) = O(\exp(-\alpha bx/4)), \quad x \rightarrow \infty,$$

where the implied constant is uniform for ξ varying over any fixed interval of the form $(-\infty, c)$, $c \in \mathbb{R}$. In particular, for $\xi = 0$ we arrive at

$$(76) \quad \exp(\alpha bx/2) \int_0^{x/2} dz \mathcal{I}(x, z) = O(\exp(-\alpha bx/4)), \quad x \rightarrow \infty,$$

but the bound (75) also entails

$$(77) \quad \int_{-\infty}^{x/2} dz \exp(\alpha bz/2) J_{\text{as}}(z, v) \prod_{\delta=+,-} G(\delta(x-z)/2 - ib) = O(\exp(-\alpha bx/4)), \quad x \rightarrow \infty.$$

From this the lemma readily follows.

Q.E.D.

Taking $z \rightarrow x + 2z$ and keeping (19) in mind, a straightforward computation reveals that the integral in (71) equals

$$(78) \quad 2 \exp(\alpha bx/2) J_{\text{as}}(x, v) \int_{\mathbb{R}} dz \exp(i\alpha zv) \prod_{\delta=+,-} G(\delta z - ib).$$

An explicit evaluation of the latter integral is readily obtained from a special case of a Fourier transform formula in [R11], as reviewed in Appendix A. More specifically, setting $\nu = -\mu = ib$ in (202), we obtain

$$(79) \quad \int_{\mathbb{R}} dz e^{i\alpha zv} \prod_{\delta=+,-} G(\delta z - ib) = \frac{\sqrt{a_+ a_-}}{G(2ib - ia)} \prod_{\delta=+,-} G(\delta v - ia + ib).$$

Substituting these results into (71), we deduce

$$(80) \quad F(b, v; x, ib) \sim \frac{4\sqrt{a_+ a_-}}{G(2ib - ia)} J_{\text{as}}(b; x, v) \prod_{\delta=+,-} G(\delta v - ia + ib),$$

for $x \rightarrow \infty$. Comparing this expression for the leading asymptotic behaviour of $F(b, v; x, ib)$ as $x \rightarrow \infty$ with that given by (68) upon substituting J_{as} for J , we find that

$$(81) \quad \lambda(b, v) = 4, \quad b \in (0, a), \quad v > 0.$$

By (real) analyticity this equality immediately extends to all $b \in (0, 2a)$. Since the integrand (47) in (39) is an even function of z , we have thus established the following result.

Theorem 4. Letting $b \in (0, 2a)$ and $x, y, v \in \mathbb{R}$, we have

$$(82) \quad J(b; x, v)J(b; y, v) = \frac{1}{2} \int_0^\infty dz w(b; z)J(b; z, v) \\ \times \prod_{\delta_1, \delta_2, \delta_3 = +, -} G((\delta_1 x + \delta_2 y + \delta_3 z - ib)/2).$$

We proceed to obtain a second product formula that looks quite different at face value.

Theorem 5. Letting $b \in (0, 2a)$ and $x, t, u \in \mathbb{R}$, we have

$$(83) \quad J(b; x, t)J(b; x, u) = \frac{1}{2} G(ia - ib)^2 \int_0^\infty dv w(2a - b; v) \\ \times J(b; x, v) \prod_{\delta_1, \delta_2, \delta_3 = +, -} G((\delta_1 t + \delta_2 u + \delta_3 v + ib)/2 - ia).$$

Proof. We take $b \rightarrow 2a - b$ in (82) and then use the symmetry relation (23). Relabeling variables, we see that (83) results. Q.E.D.

It is of interest to specialize these formulas to the free cases $b = a_\pm$. From Eq. (3.4) in [HR14] we readily obtain

$$(84) \quad J(a_\delta; x, \hat{x}) = \frac{a_{-\delta} \sin(\alpha x \hat{x}/2)}{2s_{-\delta}(x)s_\delta(\hat{x})}, \quad \delta = +, -.$$

Also, using (185) we get from (13) and (9) the free weight functions

$$(85) \quad w(a_\delta; z) = 4s_{-\delta}(z)^2, \quad \delta = +, -.$$

Taking $b = a_+$ in (82), we deduce

$$(86) \quad \int_0^\infty dz \frac{s_-(z) \sin(\alpha zv/2)}{\prod_{\delta, \delta' = +, -} c_-(z + \delta x + \delta' y)/2} \\ = \frac{4a_-}{s_+(v)} \cdot \frac{\sin(\alpha xv/2)}{s_-(x)} \cdot \frac{\sin(\alpha yv/2)}{s_-(y)}.$$

Furthermore, when we take $b = a_+$ in (83) and use $G(i(a_- - a_+)/2)^2 = a_-/a_+$, then we obtain (86) with a_+ and a_- swapped.

It seems that even this elementary hyperbolic product formula is a new result. With hindsight, however, it can be obtained from a limit of the elliptic product formula in Theorem 2.2 of [R13]. (To verify this, the limit formulas Eqs. (2.92), (3.129) and (3.131) in [R97] can be used.)

We proceed to detail another perspective on the above product formulas. This arises when we rewrite them in terms of the unitary transform kernel $F(b; x, y)/(2a_+a_-)^{1/2}$, cf. (29)–(34). Let us define a family of integral operators $\mathcal{I}_z(b)$ on $L^2((0, \infty))$ by

$$(87) \quad (\mathcal{I}_z(b)f)(x) \equiv \int_0^\infty dy w(b; x)^{1/2} \mathcal{K}(b; x, y, z) w(b; y)^{1/2} f(y),$$

where $x > 0, z \geq 0, b \in (0, 2a)$, and positive square roots are taken. It follows from the G -asymptotics (197) that for fixed $(b, x, z) \in (0, 2a) \times [0, \infty)^2$ the function $\mathcal{K}(b; x, y, z)w(b; y)^{1/2}$ has exponential decay for $y \rightarrow \infty$, so the integral is absolutely convergent for any $f \in L^2((0, \infty))$. At face value, however, it is not clear that the right-hand side of (87) yields a function in $L^2((0, \infty), dx)$. Furthermore, when we fix b , there appears to be no reason for all of the operators $\mathcal{I}_z(b), z \geq 0$, to commute.

Even so, more is true: They are bounded self-adjoint operators satisfying

$$(88) \quad [\mathcal{I}_u(b), \mathcal{I}_v(b)] = 0, \quad [\mathcal{I}_u(b), \mathcal{I}_v(2a - b)] = 0, \quad u, v \geq 0.$$

This is immediate from the following result, which shows that the two families are simultaneously diagonalized by the unitary involution $\mathcal{F}(b)$, yielding bounded real-valued multiplication operators.

Theorem 6. *Let $b \in (0, 2a)$ and $z \geq 0$. Then the operator $\mathcal{I}_z(b)$ is bounded, and the operator $\mathcal{F}(b)\mathcal{I}_z(b)\mathcal{F}(b)$ on $L^2((0, \infty), dv)$ acts as multiplication by the function $2J(b; z, v)$. Moreover, the operator $\mathcal{F}(b)\mathcal{I}_z(2a - b)\mathcal{F}(b)$ on $L^2((0, \infty), dv)$ acts as multiplication by $2J(2a - b; z, v)$.*

Proof. When we swap z and y in (82) and use (33), we obtain

$$(89) \quad 2J(b; z, v) \frac{F(b; x, v)}{\sqrt{2a_+a_-}} = \int_0^\infty dy w(b; x)^{1/2} \mathcal{K}(b; x, y, z) w(b; y)^{1/2} \frac{F(b; y, v)}{\sqrt{2a_+a_-}}.$$

Integrating this with $f(v) \in C_0^\infty((0, \infty))$ we obtain

$$(90) \quad 2\mathcal{F}(b)J(b; z, \cdot)f = \mathcal{I}_z(b)\mathcal{F}(b)f.$$

From this we read off the first assertion. Using (35), we obtain the second one. Q.E.D.

Yet another illuminating version of the product formula (82) is obtained upon multiplying (89) by $F(b; t, v)/\sqrt{2a_+a_-}$ and integrating

over v . This results in

$$(91) \quad \frac{1}{a_+a_-} \int_0^\infty dv J(b; z, v) F(b; x, v) F(b; t, v) \\ = w(b; x)^{1/2} \mathcal{K}(b; x, t, z) w(b; t)^{1/2}.$$

Using (33), this can be rewritten as the identity

$$(92) \quad w(b; x)^{1/2} w(b; y)^{1/2} w(b; z)^{1/2} \mathcal{K}(b; x, y, z) \\ = \frac{G(ia - ib)}{\sqrt{a_+a_-}} \int_0^\infty \frac{dv}{w(2a - b; v)^{1/2}} F(b; x, v) F(b; y, v) F(b; z, v).$$

(Alternatively, this identity follows when we replace f in (90) by $\mathcal{F}(b)f$.) From the $b \rightarrow 2a - b$ invariance (34) of the F-function, we also obtain

$$(93) \quad w(2a - b; x)^{1/2} w(2a - b; y)^{1/2} w(2a - b; z)^{1/2} \mathcal{K}(2a - b; x, y, z) \\ = \frac{G(ib - ia)}{\sqrt{a_+a_-}} \int_0^\infty \frac{dv}{w(b; v)^{1/2}} F(b; x, v) F(b; y, v) F(b; z, v).$$

§3. An application to the hyperbolic relativistic Calogero-Moser 2-particle system

In the recent paper [HR14] we developed a recursive scheme to construct joint eigenfunctions for the commuting $A\Delta O$ s associated with the integrable N -particle systems of hyperbolic relativistic Calogero-Moser type. In this section we establish a remarkable application of the product formula (82) from Theorem 4 to the $N = 2$ case of this recursive scheme. More specifically, we show that the joint eigenfunction of the $A\Delta O$ s is also an eigenfunction of an explicit integral operator, with the eigenvalues being explicit as well.

The kernel of the pertinent integral operator is the product of the weight function $w(b; y_1 - y_2)$ and the special function

$$(94) \quad \mathcal{S}_2(b; x, y) \equiv \prod_{j,k=1}^2 \frac{G(x_j - y_k - ib/2)}{G(x_j - y_k + ib/2)},$$

which connects the 2-particle $A\Delta O$ s $A_{k,\delta}^{(2)}(x)$ to themselves via kernel identities, see (2.2) in [HR14]. For the first step $N = 1 \rightarrow N = 2$ of the recursive scheme, however, the main protagonist is a kernel function connecting these $N = 2$ $A\Delta O$ s to the elementary $N = 1$ $A\Delta O$ s

$A_{1,\delta}^{(1)}(-y_1) \equiv \exp(ia_{-\delta}\partial_{y_1})$. The latter arises from \mathcal{S}_2 by first multiplying by a suitable plane wave and then letting y_2 go to infinity, yielding

$$(95) \quad \mathcal{S}_2^\sharp(b; x, y_1) \equiv \prod_{j=1}^2 \frac{G(x_j - y_1 - ib/2)}{G(x_j - y_1 + ib/2)}.$$

Starting the recursion with the plane wave

$$(96) \quad J_1(x_1, y_1) \equiv \exp(i\alpha x_1 y_1),$$

the first step $N = 1 \rightarrow N = 2$ of the recursive scheme yields the function

$$(97) \quad J_2(b; x, y) = \exp(i\alpha y_2(x_1 + x_2)) \int_{\mathbb{R}} dz I_2(b; x, y, z),$$

where $(b, x, y) \in (0, 2a) \times \mathbb{R}^2 \times \mathbb{R}^2$, with integrand

$$(98) \quad I_2(b; x, y, z) \equiv \mathcal{S}_2^\sharp(b; x, z) J_1(z, y_1 - y_2).$$

We now recall from Section 4 in [HR14] the relation of J_2 and J . It arises by taking $z \rightarrow z + (x_1 + x_2)/2$ in the integral in (97) and comparing the result with the formula (4) for J :

$$(99) \quad \begin{aligned} J_2(b; (x_1, x_2), (y_1, y_2)) &= \exp(i\alpha(x_1 + x_2)(y_1 + y_2)/2) \\ &\quad \times J(b; x_1 - x_2, y_1 - y_2). \end{aligned}$$

We are now ready to formulate and prove the integral equation for J_2 .

Theorem 7. *Letting $(b, x, y) \in (0, 2a) \times \mathbb{R}^2 \times \mathbb{R}^2$, we have the integral equation*

$$(100) \quad \int_{\mathbb{R}^2} dz w(b; z_1 - z_2) \mathcal{S}_2(b; x, z) J_2(b; z, y) = 2\mu(b; y) J_2(b; x, y),$$

where

$$(101) \quad \mu(b; y) \equiv a_+ a_- G(ia - ib)^2 \prod_{j=1}^2 \prod_{\delta=+,-} G(\delta y_j + ib/2 - ia).$$

Proof. First, we aim to rewrite the left-hand side of (100) in such a way that the product formula (82) from Theorem 4 may be invoked. Substituting the right-hand side of (99) for J_2 and changing integration variables to

$$(102) \quad z \equiv z_1 - z_2, \quad s \equiv z_1 + z_2,$$

we use the reflection equation (189) to rewrite the left-hand side of (100) as

$$(103) \quad \frac{1}{2} \int_{\mathbb{R}} ds \exp(i\alpha s(y_1 + y_2)/2) \int_{\mathbb{R}} dz w(z) J(z, y_1 - y_2) \\ \times \prod_{\delta_1, \delta_2, \delta_3 = +, -} G((\delta_1(x_1 - x_2) + \delta_2(s - (x_1 + x_2)) + \delta_3 z - ib)/2).$$

Taking $s \rightarrow s + x_1 + x_2$ and using evenness of the latter integrand in z , we deduce that this equals

$$(104) \quad \exp(i\alpha(x_1 + x_2)(y_1 + y_2)/2) \int_{\mathbb{R}} ds \exp(i\alpha s(y_1 + y_2)/2) \\ \times \int_0^\infty dz w(z) J(z, y_1 - y_2) \mathcal{K}(x_1 - x_2, s, z),$$

where, just as in Section 2, the kernel function \mathcal{K} is given by (38). Invoking the product formula (82) as well as (99), we obtain

$$(105) \quad 2J_2(x, y) \int_{\mathbb{R}} ds \exp(i\alpha s(y_1 + y_2)/2) J(s, y_1 - y_2).$$

Next, we show that the representation (4) and the Fourier transform formula (202) allow us to compute the remaining integral explicitly. Changing integration variable according to $z \rightarrow z/2$ in (4), and once more making use of the reflection equation (189), we find that the integral is given by

$$(106) \quad \frac{1}{2} \int_{\mathbb{R}} ds \int_{\mathbb{R}} dz \exp(i\alpha(s(y_1 + y_2) + z(y_1 - y_2))/2) \\ \times \prod_{\delta_1, \delta_2 = +, -} G((\delta_1 z + \delta_2 s - ib)/2).$$

Reversing the change of variables (102), we arrive at the product of one-variable integrals

$$(107) \quad \prod_{j=1}^2 \int_{\mathbb{R}} dz_j \exp(i\alpha z_j y_j) \prod_{\delta = +, -} G(\delta z_j - ib/2).$$

Finally, setting $\nu = -\mu = ib/2$ in (202), we obtain an evaluation formula for these integrals, which yields the right-hand side of (100). Q.E.D.

Our next theorem is a simple corollary of Theorem 7.

Theorem 8. *Letting $(b, x, y) \in (0, 2a) \times \mathbb{R}^2 \times \mathbb{R}^2$, we have*

$$(108) \quad \int_{\mathbb{R}^2} dz w(2a - b; z_1 - z_2) \mathcal{S}_2(2a - b; x, z) J_2(b; y, z) \\ = 2\mu(2a - b; y) J_2(b; y, x).$$

Proof. This follows from (100) by taking $b \rightarrow 2a - b$ and using (23).
Q.E.D.

Next, we define

$$(109) \quad F_2(b; x, y) \equiv \exp(i\alpha(x_1 + x_2)(y_1 + y_2)/2) F(b; x_1 - x_2, y_1 - y_2),$$

where $x, y \in G_2$ and we have introduced

$$(110) \quad G_2 \equiv \{x \in \mathbb{R}^2 \mid x_2 < x_1\}.$$

The integrands on the left-hand sides of (100) and (108) are invariant under swapping z_1 and z_2 , so we can use (33) to rewrite these formulas in terms of F_2 . This yields

$$(111) \quad \int_{G_2} dz w(b; x_1 - x_2)^{1/2} \mathcal{S}_2(b; x, z) w(b; z_1 - z_2)^{1/2} F_2(b; z, y) \\ = \mu(b; y) F_2(b; x, y), \quad x, y \in G_2,$$

and

$$(112) \quad \int_{G_2} dz w(2a - b; x_1 - x_2)^{1/2} \mathcal{S}_2(2a - b; x, z) w(2a - b; z_1 - z_2)^{1/2} \\ \times F_2(b; z, y) = \mu(2a - b; y) F_2(b; x, y), \quad x, y \in G_2.$$

Now the generalized Fourier transform

$$(113) \quad \mathcal{F}_2(b) : \mathcal{C}_2 \equiv C_0^\infty(G_2) \subset L^2(G_2) \rightarrow L^2(G_2), \quad b \in (0, 2a),$$

defined by

$$(114) \quad (\mathcal{F}_2(b)\psi)(x) \equiv \frac{1}{a_+ a_-} \int_{G_2} F_2(b; x, y) \psi(y) dy, \quad \psi \in \mathcal{C}_2, \quad x \in G_2,$$

extends to a unitary operator. (This readily follows from the tensor product structure exhibited by the kernel F_2 .) Just as in Section 2, we proceed to consider the integral operator given by

$$(115) \quad (\mathcal{I}_2(b)f)(x) \equiv \int_{G_2} dy w(b; x_1 - x_2)^{1/2} \mathcal{S}_2(b; x, y) \\ \times w(b; y_1 - y_2)^{1/2} f(y), \quad f \in L^2(G_2), \quad b \in (0, 2a).$$

In view of (189) and (192), its kernel is positive and invariant under swapping x and y . As in Section 2, however, it is not obvious that $\mathcal{I}_2(b)$ is bounded, let alone that it satisfies

$$(116) \quad [\mathcal{I}_2(b), \mathcal{I}_2(2a - b)] = 0.$$

These properties are a direct consequence of our next result.

Theorem 9. *Letting $b \in (0, 2a)$, the operator $\mathcal{I}_2(b)$ is bounded, and the operator $\mathcal{F}_2(b)^* \mathcal{I}_2(b) \mathcal{F}_2(b)$ on $L^2(G_2, dv)$ acts as multiplication by the positive function $\mu(b; v)$ given by (101). Moreover, the operator $\mathcal{F}_2(b)^* \mathcal{I}_2(2a - b) \mathcal{F}_2(b)$ on $L^2(G_2, dv)$ acts as multiplication by $\mu(2a - b; v)$.*

Proof. This follows in the same way as before from (111) and (112).
Q.E.D.

Taking $y \rightarrow v$ in (111) and then integrating with $\overline{\mathbb{F}_2}(b; y, v)$, we obtain the identity

$$(117) \quad w(b; x_1 - x_2)^{1/2} \mathcal{S}_2(b; x, y) w(b; y_1 - y_2)^{1/2} \\ = \frac{1}{(a_+ a_-)^2} \int_{G_2} dv \mu(b; v) \mathbb{F}_2(b; x, v) \overline{\mathbb{F}_2}(b; y, v).$$

Since \mathbb{F}_2 is invariant under $b \rightarrow 2a - b$, this implies a second identity that we shall not spell out.

We conclude this section by clarifying the relation of these identities to (92) and (93). When we transform (117) to sum and difference variables, we obtain

$$(118) \quad w(b; x)^{1/2} \mathcal{K}(b; x, y, s - t) w(b; y)^{1/2} \\ = \frac{1}{2(a_+ a_-)^2} \int_{\mathbb{R}} dr \int_0^\infty dv \mu(b; (r + v)/2, (r - v)/2) \\ \times \exp(i\alpha(r(s - t)/2) \mathbb{F}(b; x, v) \mathbb{F}(b; y, v).$$

Setting $z := s - t$, we can now combine the definitions (101) and (4) of μ and J to get

$$(119) \quad \int_{\mathbb{R}} dr \mu(b; (r + v)/2, (r - v)/2) \exp(i\alpha r z / 2) \\ = 2a_+ a_- G(i\alpha - ib)^2 J(2a - b; v, z).$$

Using next the J -symmetry (23), the right-hand side of this equality becomes $2a_+ a_- J(b; z, v)$. Finally, trading J for \mathbb{F} by using (33), we arrive at (92).

§4. Nonrelativistic limit formulas

We recall that Subsection 4.2 of [R11] deals with the nonrelativistic limit of the \mathcal{R} -function. In terms of $J(b; x, y)$ (4), the starting point is the reparametrized J -function

$$(120) \quad J(\pi, \beta, \beta g; r, \beta k) = \frac{1}{2} \int_{\mathbb{R}} dt \prod_{\delta=+,-} \frac{G(\pi, \beta; (t + \delta r - i\beta g)/2)}{G(\pi, \beta; (t + \delta r + i\beta g)/2)} e^{itk},$$

and the limit amounts to taking β to 0. When we formally interchange it with the integration, we can use (203). This yields

$$(121) \quad \lim_{\beta \rightarrow 0} J(\pi, \beta, \beta g; r, \beta k) = \frac{1}{2} \int_{\mathbb{R}} dt \frac{\exp(itk)}{\prod_{\delta=+,-} [2 \cosh((t + \delta r)/2)]^g}.$$

Taking r and k real, the latter integral obviously converges for $\operatorname{Re} g > 0$. It equals the function $F(g; r, 2k)$ defined by Eq. (65) of our joint paper [HR15]. Its relation to the conical (or Mehler) function $P_{ik-1/2}^{1/2-g}(\cosh r)$ is given by

$$(122) \quad F(g; r, 2k) = \left(\frac{\pi}{4}\right)^{1/2} \frac{\Gamma(g + ik)\Gamma(g - ik)}{\Gamma(g)(2 \sinh r)^{g-1/2}} P_{ik-1/2}^{1/2-g}(\cosh r),$$

cf. Eq. 14.12.4 in [Dig10]. In addition, one can produce a number of expressions for F in terms of the hypergeometric function ${}_2F_1$. For example, the conical function specialization of Eq. 14.3.15 in [Dig10] and the duplication formula for the gamma function entail

$$(123) \quad F(g; r, 2k) = \frac{\Gamma(g + ik)\Gamma(g - ik)}{2\Gamma(2g)} \times {}_2F_1(g + ik, g - ik; g + 1/2; -\sinh^2(r/2)).$$

Now in [R11] the above interchange of limits was left uncontrolled. With Proposition 18 at our disposal, we can not only remedy this, but also obtain bounds on the exponential decay in r and k of $J(\pi, \beta, \beta g; r, \beta k)$ that are uniform for β small enough. This is detailed in the following proposition.

Proposition 10. *Let $(g, r, k) \in \mathbb{C}^3$ be restricted by*

$$(124) \quad (g, r, k) \in \{\operatorname{Re} g > 0\} \times \{|\operatorname{Im} r| < \pi/2\} \times \{|\operatorname{Im} k| < \operatorname{Re} g\} =: \mathcal{P}.$$

Then we have

$$(125) \quad \lim_{\beta \rightarrow 0} J(\pi, \beta, \beta g; r, \beta k) = F(g; r, 2k),$$

where the limit is uniform on compacts of the parameter space \mathcal{P} . Next, suppose $(g, r, k) \in (0, \infty) \times \mathbb{R}^2$. Then we have upper bounds

$$(126) \quad |J(\pi, \beta, \beta g; r, \beta k)| \leq Cr / \sinh(gr),$$

$$(127) \quad |J(\pi, \beta, \beta g; r, \beta k)| \leq C_\varepsilon \exp(-(\pi - \varepsilon)|k|), \quad \varepsilon > 0,$$

where the constants C and C_ε are independent of $\beta \in (0, \beta_0]$, with

$$(128) \quad \beta_0 \equiv \min(\pi/4, \pi/2g).$$

Finally, the limit function satisfies

$$(129) \quad |F(g; r, 2k)| \leq Cr / \sinh(gr),$$

$$(130) \quad |F(g; r, 2k)| \leq C_\varepsilon \exp(-(\pi - \varepsilon)|k|), \quad \varepsilon > 0.$$

Proof. For $|\operatorname{Re} g/2| \leq R \in [1, \infty)$, $|\operatorname{Im} r/2| \leq \rho \in [0, \pi/2)$ and t real, we obtain from Proposition 18 the majorization

$$(131) \quad \left| \frac{G(\pi, \beta; (t + \delta r - i\beta g)/2)}{G(\pi, \beta; (t + \delta r + i\beta g)/2)} \right| \leq C(g, \rho) |\exp(-g \ln 2 \cosh((t + \delta r)/2))|,$$

with C continuous on $\mathcal{S}_R \times [0, \pi/2)$ and $\beta \in (0, (\pi - 2\rho)/4R]$. Letting next $\operatorname{Re} g \in [\epsilon, 2R]$ with $\epsilon \in (0, 1]$, and restricting $k \in \mathbb{C}$ to a strip $|\operatorname{Im} k| \leq \epsilon'$ with $\epsilon' < \epsilon$, the integrand in (120) is $O(\exp(\epsilon' - \epsilon)t)$ as $t \rightarrow \infty$, with the implied constant independent of β . Thus we can invoke the dominated convergence theorem to justify the interchange of limits, so that the first assertion readily follows.

To prove (126) we use (131), yielding

$$(132) \quad |J(\pi, \beta, \beta g; r, \beta k)| \leq C(g, 0) \int_{\mathbb{R}} dt \frac{1}{\prod_{\delta=+,-} [2 \cosh((t + \delta r)/2)]^g}.$$

From the proof of Proposition 4.3 in [HR15] we then deduce (126). Finally, to show (127), we first write

$$(133) \quad e^{\eta k} J(\pi, \beta, \beta g; r, \beta k) = \frac{1}{2} \int_{-\infty}^{\infty} dt \prod_{\delta=+,-} \frac{G(\pi, \beta; (t + \delta r - i\beta g)/2)}{G(\pi, \beta; (t + \delta r + i\beta g)/2)} \exp(i(t - i\eta)k).$$

Now for $|\eta| \leq \pi - \varepsilon$ we deduce from the bound (131) with t and r swapped that we can shift the t -contour by η , after which we easily get the estimate

$$(134) \quad |e^{\eta k} J(\pi, \beta, \beta g; r, \beta k)| \leq C(g, 0) \int_{-\infty}^{\infty} dt |4 \cosh((t + i\eta + r)/2) \cosh((t + i\eta - r)/2)|^{-g}.$$

Clearly, this entails the bound (127). Finally, since the bounds (126) and (127) do not depend on β , they extend to the limit function F , so the proof is complete. Q.E.D.

It transpires from the proof that the uniform bounds (126) and (127) can be extended to suitably restricted complex g, r and k . However, for our next aim in this section, namely to obtain two product formulas for the limit function $F(g; r, 2k)$, the bounds suffice to invoke the dominated convergence theorem, as will become clear shortly.

The first product formula follows directly from Theorem 4 when we use (126) and Appendix B.

Theorem 11. *Letting $g \in (0, \infty)$ and $r, s, k \in \mathbb{R}$, we have*

$$(135) \quad F(g; r, 2k)F(g; s, 2k) = \frac{1}{2} \int_0^{\infty} dt F(g; t, 2k) \frac{[2 \sinh t]^{2g}}{\prod_{\delta_1, \delta_2 = +, -} [2 \cosh((t + \delta_1 r + \delta_2 s)/2)]^g}.$$

Proof. When we substitute

$$(136) \quad a_+ = \pi, \quad a_- = \beta, \quad b = \beta g, \quad (x, y, z, v) = \beta(r, s, t, k),$$

in (82), then the limits (125), (203) and (206) give rise to (135), provided we interchange the $\beta \rightarrow 0$ limit and the integration. To control this interchange, we need to restrict the coupling g to $[1, \infty)$. Then we can first use Proposition 18 and Proposition 19 to conclude that the product of the w -function and kernel function is bounded for $t \in (0, \infty)$, uniformly for β small enough. Now it suffices to appeal to the uniform bound (126) to obtain the desired dominating function in $L^1((0, \infty))$. Therefore (135) follows for $g \geq 1$.

In order to lift the g -restriction, we note that (121) implies that for fixed $t, k \in \mathbb{R}$ the function $F(g; t, 2k)$ on the right-hand side of (135) is real-analytic in g for $g > 0$. From (121) it is also clear that it is bounded for t, k varying over \mathbb{R} and g in a complex neighborhood of the positive real axis. For g in such a neighborhood, the remaining

hyperbolic quotient in the integrand decays exponentially as $t \rightarrow \infty$, so for $g \in (0, 1)$ the product formula (135) follows by analytic continuation.

Q.E.D.

The second product formula results from Theorem 5 by using (127) and Appendix C.

Theorem 12. *Letting $g \in (0, \infty)$ and $r, p, q \in \mathbb{R}$, we have*

$$(137) \quad F(g; r, 2p)F(g; r, 2q) = \int_0^\infty dk F(g; r, 2k) \frac{\prod_{\delta_1, \delta_2, \delta_3 = +, -} \Gamma((g + i\delta_1 p + i\delta_2 q + i\delta_3 k)/2)}{8\pi \Gamma(g)^2 \prod_{\delta = +, -} \Gamma(i\delta k) \Gamma(g + i\delta k)}.$$

Proof. We first rewrite the integrand in (83) by using (cf. (13) and (9)):

$$(138) \quad w(2a - b; v) = \prod_{\delta = +, -} G(ia + \delta v)G(ia - ib + \delta v).$$

Then we substitute $a_+ = \pi, a_- = \beta, b = \beta g$, so that the right-hand side of (83) equals the v -integral of

$$(139) \quad \frac{1}{2} J(\beta g; x, v) G(i\pi/2 + i\beta/2 - i\beta g)^2 \times \frac{\prod_{\delta = +, -} G(i\pi/2 + i\beta/2 + \delta v) G(i\pi/2 + i\beta/2 - i\beta g + \delta v)}{\prod_{\delta_1, \delta_2, \delta_3 = +, -} G(i\pi/2 + i\beta/2 + (\delta_1 t + \delta_2 u + \delta_3 v - i\beta g)/2)},$$

with $G(z) = G(\pi, \beta; z)$.

Substituting

$$(140) \quad x = r, \quad (t, u, v) = \beta(p, q, k),$$

we can rewrite (83) as

$$(141) \quad J(\pi, \beta, \beta g; r, \beta p) J(\pi, \beta, \beta g; r, \beta q) = \int_0^\infty dk J(\pi, \beta, \beta g; r, \beta k) \frac{\mathcal{G}(-ig)^2 \prod_{\delta = +, -} \mathcal{G}(\delta k) \mathcal{G}(-ig + \delta k)}{8\pi \prod_{\delta_1, \delta_2, \delta_3 = +, -} \mathcal{G}((\delta_1 p + \delta_2 q + \delta_3 k - ig)/2)},$$

where $\mathcal{G}(z) = \mathcal{G}(\beta; z)$, cf. (287). When we now formally use the limits (125) and (288), then we obtain (137). The pertinent interchange of limits can be readily justified by using the uniform bounds (289)–(292) and (127) to obtain an $L^1((0, \infty))$ -function dominating the pointwise convergence.

Q.E.D.

In the special cases $g = 1/2$ and $g = 1$ the product formula (137) was first obtained by Mizony in a somewhat different form. More precisely, in Section 3 of [Miz76] he considers the function

$$(142) \quad \phi(r, x) \equiv {}_2F_1(1/4 + ir/2, 1/4 - ir/2; 1; -\sinh^2 x).$$

Specializing (123) to $g = 1/2$, and using the quadratic transformation in Eq. 15.8.18 in [Dig10] for ${}_2F_1$ and the reflection equation for the gamma function to rewrite the resulting expression, we obtain

$$(143) \quad \phi(r, x) = \frac{2}{\pi} \cosh(\pi r) F(1/2; x, 2r).$$

Then Mizony’s product formula

$$(144) \quad \phi(r, x)\phi(s, x) = \frac{1}{2\pi} \int_0^\infty a(r, s, t)\phi(t, x)|c(t)|^{-2} dt,$$

with $|c(t)|^{-2} = \pi t \tanh(\pi t)$ and $a(r, s, t)$ given by the proposition on page 5 of [Miz76], is readily seen to amount to (137) for $g = 1/2$ (up to a factor 16). In Section 4, he also presents the corresponding product formula for the function

$$(145) \quad \phi(r, x) \equiv \frac{\sin(rx)}{r \sinh x} = \frac{2 \sinh(\pi r)}{\pi r} F(1; x, 2r),$$

and a direct computation reveals that it is equivalent to (137) with $g = 1$. (The expression for $|c(r)|^{-2}$ on page 11 of [Miz76] should read $|c(r)|^{-2} = r^2$, as can be inferred from the c -function definition on page 3.)

We mention that we can use the bounds in Appendix C in a similar way as in the above proof to arrive at a representation of $F(g; r, 2k)$ in terms of the Γ -function. Specifically, combining (23) and (120), we first obtain the alternative J -representation

$$(146) \quad J(\pi, \beta, \beta g; r, \beta k) = \frac{\mathcal{G}(-ig)^2}{8\pi} \int_{\mathbb{R}} ds \frac{\exp(isr)}{\prod_{\delta_1, \delta_2 = +, -} \mathcal{G}((\delta_1 s + \delta_2 k - ig)/2)}.$$

Taking $\beta \rightarrow 0$, we now get

$$(147) \quad F(g; r, 2k) = \frac{1}{8\pi\Gamma(g)^2} \int_{\mathbb{R}} ds \exp(isr) \prod_{\delta_1, \delta_2 = +, -} \Gamma((g + i\delta_1 s + i\delta_2 k)/2).$$

(This second representation corresponds to (4.47) in [R11], whereas the first one given by (121) corresponds to (4.48).)

We proceed to obtain two distinct limits of Theorem 7 involving the joint two-particle eigenfunction

$$(148) \quad F_2(g; r, k) = \exp(i(r_1 + r_2)(k_1 + k_2)/2)F(g; r_1 - r_2, k_1 - k_2).$$

(This function coincides with the function $F_2(\lambda; t, u)$ that we employed in [HR15], cf. Eq. (64) in *loc. cit.*) The first one is obtained by choosing parameters

$$(149) \quad a_+ = \pi, \quad a_- = \beta, \quad b = \beta g,$$

and changing variables

$$(150) \quad x = r, \quad y = \beta k/2,$$

in (100) and then taking β to zero.

Theorem 13. *Letting $(g, r, k) \in (0, \infty) \times \mathbb{R}^2 \times \mathbb{R}^2$, we have*

$$(151) \quad \int_{\mathbb{R}^2} dz \left(\frac{4 \sinh^2(z_1 - z_2)}{\prod_{j,l=1,2} [2 \cosh(r_j - z_l)]} \right)^g F_2(g; z, k) \\ = 2\mu_0(g; k)F_2(g; r, k),$$

where

$$(152) \quad \mu_0(g; k) \equiv \frac{\prod_{j=1}^2 \prod_{\delta=+,-} \Gamma((i\delta k_j + g)/2)}{4\Gamma(g)^2}.$$

Proof. With the above substitutions, the $\beta \rightarrow 0$ limit of the left-hand side of (100) yields the left-hand side of (151), cf. the proof of Theorem 11. For the eigenvalue (101) we combine the substitutions with the reparametrization (287) to get

$$(153) \quad \mathcal{G}(-ig)^2/4 \prod_{j=1}^2 \prod_{\delta=+,-} \mathcal{G}((\delta k_j - ig)/2).$$

Invoking (288), (99), (148) and (125), we see that (151) results. Q.E.D.

For the second limit we start from (108). We choose once more parameters given by (149), but now change variables

$$(154) \quad y = r, \quad z = \beta p/2, \quad x = \beta k/2.$$

Then it is clear from (125) that the function $J_2(b; y, x)$ converges to $F_2(g; r, k)$ for $\beta \rightarrow 0$. But we need a multiplicative renormalization for the right-hand side to have a finite limit. Specifically, from (101) we see that the substitutions entail

$$(155) \quad a_-^{-1}G(ia - ib)^2\mu(2a - b; y) \rightarrow \pi \prod_{j=1}^2 \prod_{\delta=+,-} G(\pi, \beta; \delta r_j - i\beta g/2),$$

so that we can invoke (203). Thus we arrive at the right-hand side of the identity in the following theorem.

Theorem 14. *Letting $(g, r, k) \in (0, \infty) \times \mathbb{R}^2 \times \mathbb{R}^2$, we have*

$$(156) \quad \frac{1}{16\pi\Gamma(g)^2} \int_{\mathbb{R}^2} dp \frac{\prod_{j,l=1,2} \prod_{\delta=+,-} \Gamma((i\delta(k_j - p_l) + g)/2)}{\prod_{\delta=+,-} \Gamma(i\delta(p_1 - p_2)/2)\Gamma(i\delta(p_1 - p_2)/2 + g)} \\ \times F_2(g; r, p) = \frac{2\pi}{[4 \cosh(r_1) \cosh(r_2)]^g} F_2(g; r, k).$$

Proof. It remains to handle the left-hand side of (108), multiplied by the factor $a_-^{-1}G(ia - ib)^2$, cf. the renormalization (155). This can be done just as in the proof of Theorem 12, and the result is (156). Q.E.D.

We proceed to obtain the nonrelativistic counterparts of the results from Sections 2 and 3 that involve Hilbert space analysis. To start with, we define an auxiliary unitary transform

$$(157) \quad \mathcal{F}_\beta(g) : \mathcal{C} \subset L^2((0, \infty), dk) \rightarrow L^2((0, \infty), dr), \quad \beta g \in (0, \pi + \beta),$$

$$(158) \quad (\mathcal{F}_\beta(g)\psi)(r) \equiv \left(\frac{1}{2\pi}\right)^{1/2} \int_0^\infty F(\pi, \beta, \beta g; r, \beta k)\psi(k)dk, \quad \psi \in \mathcal{C},$$

cf. (29)–(30). Using (33) and (287), we readily obtain

$$(159) \quad F(\pi, \beta, \beta g; r, \beta k) = \frac{2}{\mathcal{G}(\beta; -ig)} w(\pi, \beta, \beta g; r)^{1/2} \\ \times J(\pi, \beta, \beta g; r, \beta k) \left(\prod_{\delta=+,-} \mathcal{G}(\beta; \delta k)\mathcal{G}(\beta; \delta k - ig) \right)^{1/2}.$$

From this we deduce that for all $(g, r, k) \in (0, \infty)^3$ we have

$$(160) \quad \begin{aligned} F_0(g; r, k) &\equiv \lim_{\beta \rightarrow 0} F(\pi, \beta, \beta g; r, \beta k) \\ &= 2\Gamma(g)(2 \sinh r)^g F(g; r, 2k) \left(\prod_{\delta=+,-} \Gamma(i\delta k) \Gamma(i\delta k + g) \right)^{-1/2}. \end{aligned}$$

The operator on \mathcal{C} defined by

$$(161) \quad (\mathcal{F}_0(g)\psi)(r) \equiv \left(\frac{1}{2\pi}\right)^{1/2} \int_0^\infty F_0(g; r, k)\psi(k)dk, \quad \psi \in \mathcal{C},$$

gives rise to a unitary transform

$$(162) \quad \mathcal{F}_0(g) : L^2((0, \infty), dk) \rightarrow L^2((0, \infty), dr), \quad g > 0.$$

(It equals the sine transform for $g = 1$ and its $g \rightarrow 0$ limit is the cosine transform.) Although this assertion will cause no surprise, a complete proof is not immediate from our results. However, the unitarity of $\mathcal{F}_0(g)$ follows by specialization from known unitarity properties of the Jacobi function transform [Koo84].

Next, we define a family of integral operators on $L^2((0, \infty))$ by setting

$$(163) \quad (\mathcal{J}_t(g)\psi)(r) \equiv \int_0^\infty ds \frac{w_0(g; r)^{1/2} w_0(g; s)^{1/2} \psi(s)}{\prod_{\delta_1, \delta_2=+,-} [2 \cosh((t + \delta_1 s + \delta_2 r)/2)]^g},$$

$$(164) \quad w_0(g; r) \equiv (2 \sinh r)^{2g}, \quad r > 0,$$

where $g > 0, t \geq 0$ and the implied logarithm is chosen real. Hence the kernel of the integral operator is positive for all $(g, r, s, t) \in (0, \infty)^3 \times [0, \infty)$. Since it has exponential decay for $s \rightarrow \infty$, the integral is absolutely convergent, but it is not clear that the image function is square-integrable. In fact, however, the family consists of bounded self-adjoint operators satisfying

$$(165) \quad [\mathcal{J}_{t_1}(g), \mathcal{J}_{t_2}(g)] = 0, \quad t_1, t_2 \geq 0.$$

This is an obvious consequence of the following theorem.

Theorem 15. *Let $g > 0$ and $t \geq 0$. Then the operator $\mathcal{J}_t(g)$ is bounded, and the operator $\mathcal{F}_0(g)^* \mathcal{J}_t(g) \mathcal{F}_0(g)$ on $L^2((0, \infty), dk)$ acts as multiplication by $2F(g; t, 2k)$.*

Proof. When we swap t and s in (135) and use (160), the resulting identity can be written

$$(166) \quad 2F(g; t, 2k)F_0(g; r, k) = \int_0^\infty ds \frac{w_0(g; r)^{1/2}w_0(g; s)^{1/2}}{\prod_{\delta_1, \delta_2 = +, -} [2 \cosh((t + \delta_1 s + \delta_2 r)/2)]^g} F_0(g; s, k).$$

Integrating this with $\psi(k)$, $\psi \in L^2((0, \infty))$, we deduce

$$(167) \quad 2\mathcal{F}_0(g)F(g; t, 2\cdot)\psi = \mathcal{J}_t(g)\mathcal{F}_0(g)\psi.$$

From this the assertion is plain.

Q.E.D.

In order to continue, we define a dual family of integral operators on $L^2((0, \infty))$ by

$$(168) \quad (\hat{\mathcal{J}}_q(g)\phi)(k) \equiv \int_0^\infty dp \hat{w}_0(g; k)^{1/2}\hat{w}_0(g; p)^{1/2}\phi(p) \times \prod_{\delta_1, \delta_2, \delta_3 = +, -} \Gamma((g + i\delta_1 p + i\delta_2 q + i\delta_3 k)/2),$$

$$(169) \quad \hat{w}_0(g; k) \equiv 1/4\pi\Gamma(g)^2 \prod_{\delta = +, -} \Gamma(i\delta k)\Gamma(i\delta k + g),$$

where $g > 0$, $q \geq 0$ and the positive square root is taken. When we combine (271) and (293), we deduce exponential decay of the kernel of the integral operator for $p \rightarrow \infty$, so the integral is absolutely convergent. As in previous cases, it is not obvious that the image function is square-integrable. Once more, however, the family actually consists of bounded self-adjoint operators satisfying

$$(170) \quad [\hat{\mathcal{J}}_{q_1}(g), \hat{\mathcal{J}}_{q_2}(g)] = 0, \quad q_1, q_2 \geq 0,$$

as is immediate from our next theorem.

Theorem 16. *Let $g > 0$ and $q \geq 0$. Then the operator $\hat{\mathcal{J}}_q(g)$ is bounded, and the operator $\mathcal{F}_0(g)\hat{\mathcal{J}}_q(g)\mathcal{F}_0(g)^*$ on $L^2((0, \infty), dr)$ acts as multiplication by $2F(g; r, 2q)$.*

Proof. Swapping k and p in (137) and using (160), we deduce

$$(171) \quad 2F(g; r, 2q)F_0(g; r, k) = \int_0^\infty dp \hat{w}_0(g; k)^{1/2}\hat{w}_0(g; p)^{1/2} \times F_0(g; r, p) \prod_{\delta_1, \delta_2, \delta_3 = +, -} \Gamma((g + i\delta_1 p + i\delta_2 q + i\delta_3 k)/2).$$

Integrating this with $\phi(r)$, $\phi \in L^2((0, \infty))$, we obtain

$$(172) \quad 2\mathcal{F}_0(g)^*F(g; \cdot, 2q)\phi = \hat{\mathcal{J}}_q(g)\mathcal{F}_0(g)^*\phi,$$

and so the theorem follows.

Q.E.D.

Passing to the 2-particle case, we define a unitary operator by (continuous extension of)

$$(173) \quad \mathcal{F}_{0,2}(g) : \mathcal{C}_2 = C_0^\infty(G_2) \subset L^2(G_2) \rightarrow L^2(G_2), \quad g > 0,$$

where

$$(174) \quad (\mathcal{F}_{0,2}(g)\psi)(r) \equiv \frac{1}{2\pi} \int_{G_2} F_{0,2}(g; r, k)\psi(k)dk, \quad \psi \in \mathcal{C}_2, \quad r \in G_2,$$

with

$$(175) \quad F_{0,2}(g; r, k) \equiv \exp(i(r_1 + r_2)(k_1 + k_2)/2)F_0(g; r_1 - r_2, (k_1 - k_2)/2).$$

Now we define two integral operators on $L^2(G_2)$:

$$(176) \quad (\mathcal{J}_2(g)\phi)(r) \equiv \int_{G_2} dz \frac{w_0(g; r_1 - r_2)^{1/2}w_0(g; z_1 - z_2)^{1/2}}{\prod_{j,l=1,2}[2 \cosh(r_j - z_l)]^g} \phi(z),$$

$$(177) \quad (\hat{\mathcal{J}}_2(g)\psi)(k) \equiv \int_{G_2} dp \hat{w}_0(g; k_1 - k_2)^{1/2}\hat{w}_0(g; p_1 - p_2)^{1/2}\psi(p) \\ \times \prod_{j,l=1,2} \prod_{\delta=+,-} \Gamma((i\delta(k_j - p_l) + g)/2),$$

where $\phi, \psi \in L^2(G_2)$ and $g > 0$. We are now prepared for the last theorem of this section.

Theorem 17. *Let $g > 0$. The operator $\mathcal{F}_{0,2}(g)^*\mathcal{J}_2(g)\mathcal{F}_{0,2}(g)$ on $L^2(G_2, dk)$ acts as multiplication by the positive function $\mu_0(g; k)$ given by (152). Also, the operator $\mathcal{F}_{0,2}(g)\hat{\mathcal{J}}_2(g)\mathcal{F}_{0,2}(g)^*$ on $L^2(G_2, dr)$ acts as multiplication by $4\pi/[4 \cosh(r_1) \cosh(r_2)]^g$.*

Proof. Recalling (148) and (160), this can be read off from (151) and (156). Q.E.D.

We conclude this section by deriving nonrelativistic counterparts of the identities (92) and (117). First, we obtain alternative versions of the product formulas in Theorems 11 and 12, using the unitarity of

the transform $\mathcal{F}_0(g)$. To this end we multiply (166) by $F_0(g; v, k)$ and integrate over k . Now we use (160), rewritten as

$$(178) \quad F(g; r, 2k) = F_0(g; r, k)/4\sqrt{\pi}\Gamma(g)^2 w_0(g; r)^{1/2} \hat{w}_0(g; k)^{1/2}.$$

As a result, we obtain the identity

$$(179) \quad \frac{w_0(g; r)^{1/2} w_0(g; s)^{1/2} w_0(g; t)^{1/2}}{\prod_{\delta_1, \delta_2 = +, -} [2 \cosh((r + \delta_1 s + \delta_2 t)/2)]^g} \\ = \frac{1}{4\pi^{3/2} \Gamma(g)^2} \int_0^\infty dk \hat{w}_0(g; k)^{-1/2} F_0(g; r, k) F_0(g; s, k) F_0(g; t, k).$$

Likewise, we multiply (171) by $F_0(g; r, v)$ and integrate over r . Using again (178), we arrive at

$$(180) \quad \hat{w}_0(g; k)^{1/2} \hat{w}_0(g; p)^{1/2} \hat{w}_0(g; q)^{1/2} \\ \times \prod_{\delta_1, \delta_2, \delta_3 = +, -} \Gamma((g + i\delta_1 k + i\delta_2 p + i\delta_3 q)/2) \\ = \frac{1}{4\pi^{3/2} \Gamma(g)^2} \int_0^\infty dr w_0(g; r)^{-1/2} F_0(g; r, k) F_0(g; r, p) F_0(g; r, q).$$

Turning to the 2-particle case, we first use (178) to rewrite (151) as

$$(181) \quad \int_{G^2} dz \frac{w_0(g; r_1 - r_2)^{1/2} w_0(g; z_1 - z_2)^{1/2}}{\prod_{j, l = 1, 2} [2 \cosh(r_j - z_l)]^g} F_{0,2}(g; z, k) \\ = \mu_0(g; k) F_{0,2}(g; r, k).$$

Now we multiply this by $\overline{F_{0,2}}(g; s, k)/4\pi^2$ and integrate k over G_2 . This yields the identity

$$(182) \quad \frac{w_0(g; r_1 - r_2)^{1/2} w_0(g; s_1 - s_2)^{1/2}}{\prod_{j, l = 1, 2} [2 \cosh(r_j - s_l)]^g} \\ = \frac{1}{4\pi^2} \int_{G_2} dk \mu_0(g; k) F_{0,2}(g; r, k) \overline{F_{0,2}}(g; s, k).$$

Similarly, using (178) to rewrite (156) as

$$(183) \quad \int_{G^2} dp \hat{w}_0(g; k_1 - k_2)^{1/2} \hat{w}_0(g; p_1 - p_2)^{1/2} F_{0,2}(g; r, p) \\ \times \prod_{j, l = 1, 2} \prod_{\delta = +, -} \Gamma((i\delta(k_j - p_l) + g)/2) \\ = \frac{4\pi}{[4 \cosh(r_1) \cosh(r_2)]^g} F_{0,2}(g; r, k),$$

multiplying this by $\overline{F_{0,2}}(g; r, q)/4\pi^2$ and then integrating r over G_2 , we get

$$(184) \quad \hat{w}_0(k_1 - k_2)^{1/2} \hat{w}_0(q_1 - q_2)^{1/2} \prod_{j,l=1,2} \prod_{\delta=+,-} \Gamma((i\delta(k_j - q_l) + g)/2) \\ = \frac{1}{\pi} \int_{G_2} dr \frac{F_{0,2}(g; r, k) \overline{F_{0,2}}(g; r, q)}{[4 \cosh(r_1) \cosh(r_2)]^g}.$$

Finally, just as the 2-particle identity (117) leads to the reduced identity (92) upon using sum and difference variables, we can rederive the identities (179)/(180) from the identities (182)/(184) by invoking the F -representations (147)/(121) and then using (178), respectively. We note that this yields a nontrivial check on the various constants involved.

§Appendix A. The hyperbolic gamma function

In this appendix we review previously known properties of the hyperbolic gamma function $G(a_+, a_-; z)$ we have occasion to use in Sections 2 and 3. (More details can be found in [R97] and Appendix A of [R99].) Throughout the paper we choose the parameters a_+ and a_- positive. As a rule, the dependence on these parameters shall be suppressed when they are supposed to be fixed.

The hyperbolic gamma function was introduced in Section III A of [R97] as the unique minimal solution of one of the two AΔEs

$$(185) \quad \frac{G(z + ia_\delta/2)}{G(z - ia_\delta/2)} = 2c_{-\delta}(z), \quad \delta = +, -,$$

that has modulus 1 for real z and satisfies $G(0) = 1$. It is not obvious, but true, that the other one is satisfied as well. Furthermore, $G(z)$ is meromorphic in z , and it has neither poles nor zeros for z in the strip

$$(186) \quad S \equiv \{z \in \mathbb{C} \mid |\operatorname{Im}(z)| < a\}, \quad a = (a_+ + a_-)/2.$$

Hence we have

$$(187) \quad G(z) = \exp(ig(z)), \quad z \in S,$$

with $g(z)$ holomorphic in S . Explicitly, $g(z)$ has the integral representation

$$(188) \quad g(a_+, a_-; z) = \int_0^\infty \frac{dy}{y} \left(\frac{\sin 2yz}{2 \sinh(a_+y) \sinh(a_-y)} - \frac{z}{a_+a_-y} \right),$$

with $z \in S$. This clearly implies that the hyperbolic gamma function has the following properties:

$$(189) \quad G(-z) = 1/G(z), \quad (\text{reflection equation}),$$

$$(190) \quad G(a_-, a_+; z) = G(a_+, a_-; z), \quad (\text{modular invariance}),$$

$$(191) \quad G(\lambda a_+, \lambda a_-; \lambda z) = G(a_+, a_-; z), \quad \lambda > 0, \quad (\text{scale invariance}),$$

$$(192) \quad \overline{G(a_+, a_-; z)} = G(a_+, a_-; -\bar{z}).$$

From Appendix A in [R99] we recall that $G(z)$ can be written as

$$(193) \quad G(z) = E(z)/E(-z),$$

with $E(z)$ an entire function with zeros located only at the points

$$(194) \quad z = ia + ip_{kl}, \quad a = (a_+ + a_-)/2, \quad k, l \in \mathbb{N} \equiv \{0, 1, 2, \dots\},$$

where

$$(195) \quad p_{kl} \equiv ka_+ + la_-.$$

Moreover, the order of these zeros (denoted by $\mathcal{O}(kl)$), equals the number of distinct pairs $(m, n) \in \mathbb{N}^2$ such that $p_{mn} = p_{kl}$. In particular, for $a_+/a_- \notin \mathbb{Q}$ all zeros are simple. Clearly, (193) entails that G has the same zero set as E and poles of order $\mathcal{O}(kl)$ located solely at

$$(196) \quad z = -ia - ip_{kl}, \quad k, l \in \mathbb{N}.$$

We also recall that the asymptotic behaviour of $G(z)$ for $\text{Re } z \rightarrow \pm\infty$ is given by

$$(197) \quad G(z) = \exp(\mp i(\chi + \alpha z^2/4))(1 + O(\exp(-r|\text{Re } z|))),$$

where

$$(198) \quad \chi \equiv \frac{\pi}{24} \left(\frac{a_+}{a_-} + \frac{a_-}{a_+} \right),$$

the decay rate can be any positive number satisfying

$$(199) \quad r < \alpha \min(a_+, a_-),$$

and the implied constant is uniform for $\text{Im } z$ varying over compact subsets of \mathbb{R} .

Finally, in Section 2 we have occasion to make use of the Fourier transform formula from Proposition C.1 in [R11]. More specifically, let $\mu, \nu \in \mathbb{C}$ be such that

$$(200) \quad -a < \operatorname{Im} \mu < \operatorname{Im} \nu < a,$$

and assume that $y \in \mathbb{C}$ satisfies

$$(201) \quad |\operatorname{Im} y| < \operatorname{Im}(\nu - \mu)/2.$$

Then the pertinent formula is given by

$$(202) \quad \int_{\mathbb{R}} dz \exp(i\alpha zy) \frac{G(z - \nu)}{G(z - \mu)} = \sqrt{a_+ a_-} \exp(i\alpha y(\mu + \nu)/2) \\ \times G(ia + \mu - \nu) \prod_{\delta=+,-} G(\delta y - ia + (\nu - \mu)/2).$$

§Appendix B. Uniform bounds on G -function ratios

In this appendix and the next one, we reconsider and improve limits involving the hyperbolic gamma function. We shall use the results to control the nonrelativistic limit in Section 4.

We begin by recalling from Subsection III A in [R97] the limit

$$(203) \quad \lim_{\beta \downarrow 0} \frac{G(\pi, \beta; z + i\beta u)}{G(\pi, \beta; z + i\beta d)} = \exp((u - d) \ln(2 \cosh z)).$$

Here, we have $u, d \in \mathbb{R}$, the logarithm is real-valued for z real, and (203) holds true uniformly for z varying over compact subsets of the cut plane

$$(204) \quad \mathbb{C}(\pi) \equiv \mathbb{C} \setminus \{\pm i[\pi/2, \infty)\},$$

cf. *loc. cit.* (3.91). Now the definition (13) of the w -function entails

$$(205) \quad w(\pi, \beta, \beta g; z) = \frac{G(\pi, \beta; z - i\pi/2 + i\beta(g - 1/2))}{G(\pi, \beta; z - i\pi/2 + i\beta(-1/2))} \\ \times \frac{G(\pi, \beta; z + i\pi/2 + i\beta(1/2))}{G(\pi, \beta; z + i\pi/2 + i\beta(1/2 - g))},$$

so from (203) we deduce

$$(206) \quad \lim_{\beta \downarrow 0} w(\pi, \beta, \beta g; z) = \exp(2g \ln(2 \sinh z)),$$

where g is real, the limit is uniform on compacts of the open right half plane and the logarithm is real-valued for $z > 0$.

In this appendix we obtain bounds on the G -ratios occurring in (203) and (206) that are uniform for β sufficiently small and for $\operatorname{Re} z \in \mathbb{R}/z \in (0, \infty)$, so that we can appeal to the dominated convergence theorem for $(\operatorname{Re} z)$ -integrals that involve ratios of the above type. More specifically, we aim for uniform bounds that involve the limit function. To see what this entails for (203), it is helpful to inspect a simple special case. Specifically, let us take $u = v - 1/2$ and $d = v + 1/2$ in the pertinent G -ratio. Then by (185) it equals $1/2 \cosh(z + i\beta v)$. Thus for a bound

$$(207) \quad \frac{1}{|\cosh(z + i\beta v)|} < \frac{C}{|\cosh z|},$$

to hold with $C > 0$ independent of $\operatorname{Re} z \in \mathbb{R}$ and β in a sufficiently small interval $(0, \beta_0]$, we need to require $\operatorname{Im} z \in (-\pi/2, \pi/2)$ and then choose $\beta_0 > 0$ with $|\operatorname{Im} z| + \beta_0 |\operatorname{Im} v| < \pi/2$.

This example goes to show that the following proposition cannot be much improved.

Proposition 18. *Let $u, d \in \mathcal{S}_R$, where*

$$(208) \quad \mathcal{S}_R \equiv \{v \in \mathbb{C} \mid |\operatorname{Re} v| \leq R\}, \quad R \geq 1,$$

and let $z \in \mathbb{C}$ satisfy $|\operatorname{Im} z| \leq \rho$, $\rho \in [0, \pi/2)$. Choosing $\beta_0 > 0$ such that

$$(209) \quad \beta_0 R \leq (\pi - 2\rho)/4,$$

we have for all $\beta \in (0, \beta_0]$ a bound

$$(210) \quad \left| \frac{G(\pi, \beta; z + i\beta u)}{G(\pi, \beta; z + i\beta d)} \right| \leq C(u, d, \rho) |\exp((u - d) \ln(2 \cosh z))|,$$

where C is a positive continuous function on $\mathcal{S}_R^2 \times [0, \pi/2)$ and the logarithm is real for z real.

Proof. We begin by pointing out that we have

$$(211) \quad \operatorname{Im}(z + i\beta u), \operatorname{Im}(z + i\beta d) \in (-\pi/2, \pi/2),$$

by virtue of the β -restriction. Next, we show that we can use (185) to reduce the case where $|\operatorname{Re} d|$ and/or $|\operatorname{Re} u|$ is larger than $1/2$ to the case $u, d \in \mathcal{S}_{1/2}$.

Indeed, consider first the assumption $\operatorname{Re} d \in (N - 1/2, N + 1/2]$ for an integer $N \geq 1$. Then we can write

$$(212) \quad G(\pi, \beta; z + i\beta d) = G(\pi, \beta; z + i\beta(d - N)) 2^N \prod_{k=0}^{N-1} \cosh(z + i\beta(d - k - 1/2)),$$

so that $d - N \in \mathcal{S}_{1/2}$. Now in view of the restrictions (211) and $|\operatorname{Im} z| \leq \rho$, we can use the bound

$$(213) \quad 1 / \left| \prod_{k=0}^{N-1} \cosh(z + i\beta(d - k - 1/2)) \right| \leq C_+ |\cosh(z)|^{-N},$$

with $C_+ > 0$ depending on N , $\operatorname{Im} d$ and ρ , but not on β and $\operatorname{Re} z$, so as to reduce consideration to $\mathcal{S}_{1/2}$.

Likewise, assuming $\operatorname{Re} d \in [-N - 1/2, -N + 1/2]$, we can use

$$(214) \quad \left| \prod_{k=0}^{N-1} \cosh(z + i\beta(d + k + 1/2)) \right| \leq C_- |\cosh(z)|^N,$$

with C_- independent of β and $\operatorname{Re} z$ to replace the denominator $G(\pi, \beta; z + i\beta d)$ by $G(\pi, \beta; z + i\beta(d + N))$, with $d + N \in \mathcal{S}_{1/2}$. Of course, the numerator $G(\pi, \beta; z + i\beta u)$ can be treated analogously in case $|\operatorname{Re} u| > 1/2$.

Accordingly, we assume from now on

$$(215) \quad u, v \in \mathcal{S}_{1/2}.$$

We first use the integral representation resulting from (187) and (188) to write

$$(216) \quad \frac{G(\pi, \beta; z + i\beta u)}{G(\pi, \beta; z + i\beta d)} = \exp \left(\int_0^\infty \frac{dy}{y} \left(\frac{\sinh(\beta(d - u)y) \cos(2yz + i\beta(d + u)y)}{\sinh(\pi y) \sinh(\beta y)} + \frac{u - d}{\pi y} \right) \right).$$

Using the identity (cf. (3.21) in [R97])

$$(217) \quad \ln(2 \cosh z) = \int_0^\infty \frac{dy}{y} \left(\frac{1}{\pi y} - \frac{\cos 2yz}{\sinh \pi y} \right), \quad |\operatorname{Im} z| < \pi/2,$$

we now deduce

$$(218) \quad \frac{G(\pi, \beta; z + i\beta u)}{G(\pi, \beta; z + i\beta d)} \exp((d - u) \ln(2 \cosh z)) = \exp \int_0^\infty \frac{dy}{y \sinh \pi y} I(y),$$

where we have set

$$(219) \quad I(y) \equiv \frac{\sinh \beta(d - u)y}{\sinh \beta y} \cos(2yz + i\beta(d + u)y) + (u - d) \cos 2yz.$$

As a consequence, the proposition follows when we can prove a bound

$$(220) \quad \left| \int_0^\infty \frac{dy}{y \sinh \pi y} I(y) \right| \leq c(u, d, \rho),$$

with c continuous on $\mathcal{S}_{1/2}^2 \times [0, \pi/2)$.

To this end, we write the integral as a sum of integrals

$$(221) \quad I_1 \equiv \int_1^\infty \frac{dy}{y \sinh \pi y} I(y), \quad I_2 \equiv \int_0^1 \frac{dy}{y \sinh \pi y} I(y).$$

In the first integral we write

$$(222) \quad I(y) = \left(\frac{\sinh \beta(d-u)y}{\sinh \beta y} - (d-u) \right) \\ \times \cos(2yz + i\beta(d+u)y) + (d-u)(\cos(2yz + i\beta(d+u)y) - \cos 2yz).$$

Now from our assumption (215) we deduce a bound

$$(223) \quad \left| \frac{\sinh(d-u)x}{\sinh x} - (d-u) \right| < c(d-u), \quad \forall x > 0,$$

with $c(v)$ continuous on \mathcal{S}_1 ; also, using (209) we obtain

$$(224) \quad |\cos(2yz + i\beta(d+u)y)| \leq \exp(y(2|\operatorname{Im} z| + \beta|\operatorname{Re}(d+u)|)) \\ \leq \exp(y(\pi/2 + \rho)).$$

Hence we get

$$(225) \quad |I_1| \leq \int_1^\infty \frac{dy}{y \sinh \pi y} ([c(d-u) + |d-u|] \exp(y(\pi/2 + \rho)) \\ + |d-u| \exp(2y|\operatorname{Im} z|)) \leq c_1(u, d, \rho),$$

where c_1 is continuous on $\mathcal{S}_{1/2}^2 \times [0, \pi/2)$.

In the second integral I_2 we use

$$(226) \quad \frac{\sinh \beta(d-u)y}{\sinh \beta y} = (d-u)(1 + O(\beta^2 y^2)),$$

and

$$(227) \quad \cos(2yz + i\beta(d+u)y) \\ = (\cos 2yz - i\beta(d+u)y \sin 2yz)(1 + O(\beta^2 y^2)),$$

so that we wind up with

$$(228) \quad I(y) = -i\beta(d^2 - u^2)y \sin 2yz + O(\beta^2 y^2), \quad \beta y \rightarrow 0.$$

Here, the implied constant can be chosen uniformly for u and d varying over \mathbb{C} -compacts and $\text{Im } z$ varying over bounded intervals. From this we readily deduce

$$(229) \quad |I_2| \leq c_2(u, d, \text{Im } z),$$

with c_2 continuous on $\mathbb{C}^2 \times \mathbb{R}$.

Combining the estimates (225) and (229) yields (220), so the proposition follows. Q.E.D.

Consider next the weight function. Again, it is illuminating to consider a special case: Letting $N \in \mathbb{N}$ with $N > 1$, it follows from (185) that we have an explicit evaluation

$$(230) \quad w(\pi, \beta, N\beta; z) = 2^{2N} \sinh(z)^2 \prod_{k=1}^{N-1} \sinh(z + ik\beta) \sinh(z - ik\beta).$$

Hence we should not aim for a bound $C \sinh(z)^{2N}$ with C uniform for $z > 0$ and β sufficiently small, since such a bound is not valid near $z = 0$. On the other hand, this snag is not present for a bound of the form $C \sinh(Nz)^2$, and the $\beta \rightarrow 0$ limit is majorized by the latter function for $C = 2^{2N}$. Indeed, a more general inequality holds true:

$$(231) \quad \sinh(z)^a \leq \sinh(az), \quad z > 0, \quad a \geq 1.$$

(Its proof is straightforward, cf. (101)–(102) in [HR15].)

For $a < 1$ it is manifestly false that the function $\sinh(z)^a$ is majorized by $C \sinh(az)$ for all $z > 0$. Therefore the g -restriction in the following proposition cannot be relaxed.

Proposition 19. *Letting $\beta \in (0, 1]$, we have an inequality*

$$(232) \quad w(\pi, \beta, g\beta; z) < C(g) \sinh(gz)^2,$$

where $g \geq 1$, $z > 0$ and C is a positive continuous function on $[1, \infty)$.

Proof. On account of (230) and (231), the bound (232) holds true for $g = N$. Our proof for the general case involves a lot more effort. We begin by using a couple of formulas from Section V A in [R97], where the above weight function was studied in considerable detail.

The first formula is *loc. cit.* (5.15), which implies

$$(233) \quad w(\pi, \beta, (g + 1)\beta; z) = 4 \sinh(z + ig\beta) \sinh(z - ig\beta) w(\pi, \beta, g\beta; z).$$

(To be sure, (233) can be derived directly from (185).) From this it is easy to see that we need only show (232) for $g \in [1, 2]$.

Next, we introduce a new weight function w_+ by setting

$$(234) \quad w(\pi, \beta, g\beta; z) = 4 \sinh(z)^2 w_+(\pi, \beta, g\beta; z),$$

cf. *loc. cit.* (5.14). Then it suffices to prove that for all $(\beta, g, z) \in (0, 1] \times [1, 2] \times (0, \infty)$ we have a bound

$$(235) \quad w_+(\pi, \beta, g\beta; z) < C \exp(2(g-1)z).$$

To this end we invoke the integral representation

$$(236) \quad w_+(\pi, \beta, g\beta; z) = \exp\left(-2 \int_0^\infty \frac{dy}{y} \left(Q(\beta, g; y) \cos 2yz + \frac{1-g}{\pi y}\right)\right),$$

where

$$(237) \quad Q(\beta, g; y) \equiv \frac{\sinh(g-1)\beta y}{\sinh \beta y} \cdot \frac{\cosh(\pi - g\beta)y}{\sinh \pi y},$$

cf. *loc. cit.* (5.35) and (5.36). (This representation can be derived by using (187) and (188).) In order to exploit this explicit formula, we need the auxiliary integral

$$(238) \quad f(z) \equiv \frac{1}{\pi} \int_0^\infty \frac{dy}{y^2} (1 - \cos 2yz) = \frac{2z}{\pi} \int_0^\infty \frac{du}{u^2} (1 - \cos u).$$

Integrating by parts in the integral over (ϵ, R) and then taking $\epsilon \rightarrow 0$ and $R \rightarrow \infty$, we arrive at $\int_0^\infty du \sin u/u = \pi/2$, so that $f(z) = z$. We can therefore rewrite (236) as

$$(239) \quad w_+(\pi, \beta, g\beta; z) = \exp(2(g-1)z) \exp\left(-2 \int_0^\infty \frac{dy}{y} \left(Q(\beta, g; y) + \frac{1-g}{\pi y}\right) \cos 2yz\right).$$

Comparing (239) to (235), we conclude that it remains to prove

$$(240) \quad \int_0^\infty \frac{dy}{y} \left(Q(\beta, g; y) + \frac{1-g}{\pi y}\right) \cos 2yz > c,$$

with $c \in \mathbb{R}$ independent of $(\beta, g, z) \in (0, 1] \times [1, 2] \times (0, \infty)$. We begin by showing this holds for the integral over $(0, 1)$. Then we can use

$$(241) \quad \left(Q + \frac{1-g}{\pi y}\right) = (g-1) \left(\coth(\pi y) - \frac{1}{\pi y}\right) + O(\beta y), \quad \beta y \rightarrow 0,$$

where the implied constant can be chosen uniformly for $(\beta, g) \in (0, 1] \times [1, 2]$. Clearly, this entails

$$(242) \quad \left| \int_0^1 \frac{dy}{y} \left(Q + \frac{1-g}{\pi y} \right) \cos 2yz \right| \leq c_1,$$

with c_1 independent of $(\beta, g, z) \in (0, 1] \times [1, 2] \times (0, \infty)$.

Obviously uniform boundedness is also true for the integral

$$(243) \quad \int_1^\infty \frac{dy}{y^2} \cos 2yz,$$

contributing to the integration over $(1, \infty)$ in (240). Hence we are done when we can show

$$(244) \quad \int_1^\infty \frac{dy}{y} Q(\beta, g; y) \cos 2yz > c_2,$$

with c_2 independent of $(\beta, g, z) \in (0, 1] \times [1, 2] \times (0, \infty)$. This last step of our proof is the most arduous one. The difficulty is that the integral does not converge absolutely for $\beta = 0$, since Q then reduces to $\coth(\pi y)$, cf. (237).

Our method to get around this last obstacle hinges on Q being a monotonically decreasing function of y . This feature is not obvious at face value, but can be gleaned from the logarithmic derivative

$$(245) \quad Q'/Q = (g-1)\beta \coth(g-1)\beta y - \beta \coth \beta y \\ + (\pi - g\beta) \tanh(\pi - g\beta)y - \pi \coth \pi y.$$

Indeed, for all $(\beta, g, y) \in (0, 1] \times [1, 2] \times (0, \infty)$, the difference on the second line is clearly negative, whereas a moment's thought shows the first difference is nonpositive. (This follows in particular from $x \coth x$ being increasing on $(0, \infty)$.)

For a fixed $z > 0$, consider now the smallest zero $y_0(z) \in (1, \infty)$ of $\cos 2yz$ for which $\sin(2y_0(z)z) = -1$. We assert that the integral of $y^{-1}Q(\beta, g; y) \cos 2yz$ over $(y_0(z), \infty)$ is positive. To see this, note that by monotonicity of Q/y the integral over an interval between two successive zeros of the function $\cos 2yz$ on which it is positive is larger than that over the next such interval, on which it is negative. From this our assertion follows.

It remains to show that the integral over the interval $(1, y_0(z))$ is uniformly bounded below for $(\beta, g, z) \in (0, 1] \times [1, 2] \times (0, \infty)$. First we put

$$(246) \quad \Lambda \equiv Q(\beta, g; 1),$$

and note that Λ is uniformly bounded above. Next we fix attention on the z -values

$$(247) \quad z \in (0, 3\pi/4) \Rightarrow y_0(z) = 3\pi/4z.$$

Then the interval length is bounded above by $3\pi/4z$ and on this interval the integrand is bounded below by $-\Lambda$. The integral is therefore bounded below by $-3\pi\Lambda/4z$, but with this crude bound we cannot exclude a negative divergence as z goes to 0.

However, as soon as $z < \pi/4$, the function $\cos 2yz$ is positive for $y \in [1, \pi/4z)$ and negative for $y \in (\pi/4z, 3\pi/4z)$. Now the latter interval has length $\pi/2z$, but on it the integrand is actually bounded below by $-\Lambda \cdot 4z/\pi$. Hence the integral over this interval is bounded below by -2Λ .

The upshot is that the pertinent integral is uniformly bounded below for $z \in (0, 3\pi/4)$. Choosing next $z \geq 3\pi/4$, we need only use once more that the interval length on which $\cos 2yz$ is negative equals $\pi/2z$, together with $Q(\beta, g; y)/y < \Lambda$ for $y > 1$, to deduce that the contribution from the leftmost negative interval is bounded below by $-\Lambda \cdot \pi/2z \geq -2\Lambda/3$. This completes the proof that the integral is uniformly bounded below, so the proposition follows. Q.E.D.

§Appendix C. Uniform bounds on the $G \rightarrow \Gamma$ limit

In Subsection III A of [R97] it is shown that the function

$$(248) \quad H(s; z) \equiv G(1, s; sz + i/2) \exp[iz \ln(2\pi s) - 2^{-1} \ln(2\pi)], \quad s > 0,$$

converges to $1/\Gamma(iz + 1/2)$ as $s \rightarrow 0$, uniformly for z in \mathbb{C} -compacts, cf. (3.72) in *loc. cit.* This involves the product function

$$(249) \quad P(s; z) \equiv H(s; z)\Gamma(iz + 1/2),$$

and its integral representation

$$(250) \quad P(s; z) = \exp I(s; z), \quad |\operatorname{Im} z| < 1/2 + 1/s,$$

where

$$(251) \quad I(s; z) \equiv i \int_0^\infty f_s(t) [\sin(2zt) - 2z \sinh(t)] dt,$$

$$(252) \quad f_s(t) \equiv \frac{\exp(-t/s)}{2t \sinh(t) \sinh(t/s)},$$

cf. the proof of Proposition III.6 in *loc. cit.* Indeed, from this representation it is easy to check

$$(253) \quad \lim_{s \rightarrow 0} P(s; z) = 1,$$

uniformly on \mathbb{C} -compacts.

Just as in Appendix B, we now supplement this limit with bounds that are uniform for $\operatorname{Re} z \in \mathbb{R}$ and s small enough, with a view to invoke dominated convergence for G -integrals occurring in the main text, cf. Section 4. We begin by estimating $|P(s; z)|$.

Proposition 20. *For all $z \in \mathbb{C}$ with $|y| = |\operatorname{Im} z| \leq R \in [1, \infty)$ and $s \in (0, 1/R]$, we have*

$$(254) \quad |P(s; z)| = \exp(\pi s y |x|) \exp(m_P(s; z)), \quad z = x + iy,$$

where

$$(255) \quad |m_P(s; z)| < c(R) \ln(1 + |x|) + d(R),$$

with c and d positive continuous functions on $[1, \infty)$.

Proof. Letting $|y| \leq R$ and $s \in (0, 1/R]$, we have

$$(256) \quad |P(s; z)| = \exp K(s; z),$$

where

$$(257) \quad K(s; z) \equiv \int_0^\infty f_s(t) D(z, t) dt, \\ D(z, t) \equiv 2y \sinh(t) - \sinh(2yt) \cos(2xt),$$

as is easily verified from (250)–(252). Now we write

$$(258) \quad m_P(s; z) \equiv K(s; z) - \pi s y |x| \\ = K(s; z) + s y \int_0^\infty \frac{dt}{t^2} (\cos 2xt - 1),$$

and telescope the right-hand side as

$$(259) \quad \sum_{j=1}^2 \int_0^1 I_j(s; z, t) dt + \int_1^\infty f_s(t) D(z, t) dt \\ + s y \int_1^\infty \frac{dt}{t^2} (\cos 2xt - 1),$$

with

$$(260) \quad I_1 \equiv f_s(t)L_1(y, t), \quad L_1 \equiv 2y \sinh t - \sinh 2yt,$$

$$(261) \quad I_2 \equiv L_2(s; y, t)(\cos 2xt - 1)/t, \quad L_2 \equiv sy/t - tf_s(t) \sinh 2yt.$$

We proceed to bound the four summands of $m_P(s; z)$. First, we have

$$(262) \quad L_1(y, t) = \frac{t^3}{3}(y - 4y^3) + O(t^5), \quad t \rightarrow 0,$$

whence we deduce

$$(263) \quad \left| \int_0^1 I_1 dt \right| < c_1(R), \quad s \in (0, 1/R], \quad |y| \leq R,$$

with c_1 continuous on $[1, \infty)$. To estimate the second summand, we use the mean value theorem to rewrite L_2 as

$$(264) \quad L_2 = \frac{sy}{t} \left(1 - e^{-t/s} \frac{q(t)q(t/s)}{q(2yt)} \right) \\ = -sy \partial_t \left(e^{-t/s} \frac{q(t)q(t/s)}{q(2yt)} \right)_{t=t'}, \quad q(v) \equiv \frac{v}{\sinh v},$$

with $t' \in (0, t)$. From this we deduce

$$(265) \quad |L_2(s; y, t)| < c_2(R), \quad s \in (0, 1/R], \quad |y| \leq R, \quad t \in (0, 1),$$

with c_2 continuous on $[1, \infty)$. Hence we obtain

$$(266) \quad \left| \int_0^1 I_2 dt \right| < 2c_2(R) \int_0^{|x|} \frac{du}{u} \sin^2 u \\ < 2c_2(R)(c + \ln(1 + |x|)), \quad s \in (0, 1/R], \quad |y| \leq R,$$

with $c = \int_0^{\pi/2} du \sin^2 u/u$, say.

For the third summand we use

$$(267) \quad |D(z, t)| < 2R \sinh t + \sinh 2Rt, \quad |y| \leq R, \quad t > 1,$$

to infer

$$(268) \quad \left| \int_1^\infty f_s(t)D(z, t)dt \right| < c_3(R), \quad s \in (0, 1/R], \quad |y| \leq R,$$

with c_3 continuous on $[1, \infty)$. Finally, we clearly have

$$(269) \quad \left| sy \int_1^\infty \frac{dt}{t^2} (\cos 2xt - 1) \right| < 2, \quad s \in (0, 1/R], \quad |y| \leq R.$$

Combining this with the bounds (263), (266) and (268), the proposition readily follows. Q.E.D.

Next, we bound $|\Gamma(iz + 1/2)|$, $z = x + iy$, for $y < 1/2$, by using the representation

$$(270) \quad \Gamma(iz + 1/2) = (2\pi)^{1/2} \exp \left(\int_0^\infty \frac{dt}{t} \left(\frac{e^{-2izt}}{2 \sinh t} - \frac{1}{2t} + ize^{-2t} \right) \right),$$

cf. (A37) in [R97].

Proposition 21. *For all $z \in \mathbb{C}$ with $y = \text{Im } z < 1/2$, we have*

$$(271) \quad |\Gamma(iz + 1/2)| = \exp(-\pi|x|/2 - y \ln(1 + |x|) + m_\Gamma(z)), \quad z = x + iy,$$

where

$$(272) \quad |m_\Gamma(z)| < d(y),$$

with $d(y)$ a continuous function on $(-\infty, 1/2)$.

Proof. We follow the proof of the previous proposition, both in spirit and in notation. Letting $y < 1/2$, we have

$$(273) \quad |\Gamma(iz + 1/2)| = (2\pi)^{1/2} \exp K(z),$$

where

$$(274) \quad K(z) \equiv \int_0^\infty \frac{dt}{t} D(z, t), \quad D(z, t) \equiv \frac{e^{2yt}}{2 \sinh t} \cos 2xt - \frac{1}{2t} - ye^{-2t},$$

as is clear from (270). Now we write

$$(275) \quad \begin{aligned} K(z) + \pi|x|/2 + y \ln(1 + |x|) \\ &= K(z) - \frac{1}{2} \int_0^\infty \frac{dt}{t^2} (\cos 2xt - 1) + y \ln(1 + |x|) \\ &= \sum_{j=1}^2 \int_0^1 I_j(z, t) dt + \int_1^\infty \frac{dt}{t} D(z, t) - \frac{1}{2} \int_1^\infty \frac{dt}{t^2} (\cos 2xt - 1), \end{aligned}$$

where

$$(276) \quad I_1 \equiv t^{-1} L_1(y, t), \quad L_1 \equiv \frac{e^{2yt}}{2 \sinh t} - \frac{1}{2t} - ye^{-2t},$$

$$(277) \quad I_2 \equiv y \ln(1 + |x|) + L_2(y, t)(\cos 2xt - 1)/t, \quad L_2 \equiv \frac{e^{2yt}}{2 \sinh t} - \frac{1}{2t}.$$

We proceed to bound the four terms. First we observe

$$(278) \quad L_1(y, t) = t(y^2 + 2y - 1/12) + O(t^2), \quad t \rightarrow 0.$$

From this we obtain

$$(279) \quad \left| \int_0^1 I_1 dt \right| < d_1(y),$$

with d_1 continuous on \mathbb{R} . To handle the second term, we write L_2 as

$$(280) \quad L_2(y, t) = y + tr(y, t),$$

and note that we then have

$$(281) \quad |r(y, t)| < c_2(y), \quad t \in (0, 1),$$

with c_2 continuous on \mathbb{R} . Using the bound

$$(282) \quad \left| \ln(1 + a) + \int_0^a \frac{du}{u} (\cos 2u - 1) \right| < c, \quad a \geq 0,$$

whose proof is straightforward, we now obtain

$$(283) \quad \left| \int_0^1 I_2 dt \right| < d_2(y),$$

with d_2 continuous on \mathbb{R} .

In order to bound the third term, we use

$$(284) \quad |D(z, t)| < \frac{e^{2yt}}{2 \sinh t} + \frac{1}{2t} + |y|e^{-2t}, \quad y < 1/2, \quad t > 1,$$

to get

$$(285) \quad \left| \int_1^\infty \frac{dt}{t} D(z, t) \right| < d_3(y),$$

with d_3 continuous on $(-\infty, 1/2)$. Finally, we clearly have

$$(286) \quad \frac{1}{2} \left| \int_1^\infty \frac{dt}{t^2} (\cos 2xt - 1) \right| < 1,$$

so together with (279), (283) and (285), this yields the proposition.
Q.E.D.

For the applications we have in mind, it is expedient to switch from $H(s; z)$ (248) to

$$(287) \quad \mathcal{G}(\beta; z) \equiv H(\beta/\pi; z + i/2) \\ = G(\pi, \beta; i\pi/2 + i\beta/2 + \beta z) \exp(iz \ln(2\beta) - 2^{-1} \ln(4\pi\beta)).$$

(Here we used the scaling relation (191).) The last result of this paper is now readily obtained by combining the two previous propositions.

Proposition 22. *We have*

$$(288) \quad \lim_{\beta \rightarrow 0} \mathcal{G}(\beta; z) = 1/\Gamma(iz),$$

uniformly for z varying over \mathbb{C} -compacts. For all $z \in \mathbb{C}$ with $\text{Im } z \in [-R, 0)$, $R \in [1, \infty)$, and $\beta \in (0, \pi/R]$, we have

$$(289) \quad |\mathcal{G}(\beta; z)| = \exp[(\beta(\text{Im } z + 1/2) + \pi/2)|\text{Re } z|] \exp(m_{\mathcal{G}}(\beta; z)),$$

where

$$(290) \quad |m_{\mathcal{G}}(\beta; z)| < \gamma(\text{Im } z) \ln(1 + |\text{Re } z|) + \delta(\text{Im } z),$$

with γ and δ positive continuous functions on $(-\infty, 0)$. Finally, for all $k \in \mathbb{R}$ and $\beta \in (0, 1/2\pi R]$, $R \in [1, \infty)$, we have

$$(291) \quad |\mathcal{G}(\beta; k)| = \exp[(\beta/2 + \pi/2)|k|] \exp(m_{\mathcal{G}}(\beta; k)),$$

where

$$(292) \quad |m_{\mathcal{G}}(\beta; k)| < C_1 \ln(1 + |k|) + C_2.$$

Proof. Clearly, (288) follows from (253). Likewise, (289)–(290) are clear from combining the previous two propositions. To obtain (291)–(292), however, we cannot use the last one, since now we have $y = 1/2$.

On the other hand, from the reflection equation we infer

$$(293) \quad |1/\Gamma(ik)| = (k \sinh(\pi k)/\pi)^{1/2}, \quad k \in \mathbb{R},$$

and when we combine this with (254)–(255) we deduce (291)–(292).
Q.E.D.

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