Remarks on τ-functions for the difference Painlevé equations of type E_8

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Abstract.

   We investigate the structure of τ-functions for the elliptic difference Painlevé equation of type E_8. Introducing the notion of ORG τ-functions for the E_8 lattice, we construct some particular solutions which are expressed in terms of elliptic hypergeometric integrals. Also, we discuss how this construction is related to the framework of lattice τ-functions associated with the configuration of generic nine points in the projective plane.

§ Introduction

   There are several approaches to difference (or discrete) Painlevé equations associated with the root system of type E_8. In the geometric approach of Sakai [15], there are three discrete Painlevé equations with affine Weyl group symmetry of type $E_8^{(1)}$, which may be called rational, trigonometric and elliptic. They are formulated in the language of certain rational surfaces obtained from $\mathbb{P}^2$ by blowing-up at generic nine points, and regarded as master families of second order discrete Painlevé equations. On the other hand, Ohta–Ramani–Grammaticos [11] introduced the elliptic (difference) Painlevé equation of type E_8 and its τ-functions as a discrete system on the root lattice of type E_8. Equivalence of these two approaches has been clarified by Kajiwara et al. [4, 5] in the framework of the birational affine Weyl group action on the configuration space of generic nine points in $\mathbb{P}^2$ and the associated lattice τ-functions. Besides these approaches, the elliptic Painlevé equation is interpreted as the compatibility condition of certain linear difference equations in two

ways by Rains [14] and by Noumi–Tsujimoto–Yamada [10]. Also, it is known that the elliptic difference Painlevé equation has particular solutions which are expressible in terms of elliptic hypergeometric functions as in Kajiwara et al. [4], Rains [12, 14] and Noumi–Tsujimoto–Yamada [10].

In this paper we introduce the notion of *ORG τ-functions*, which is a reformulation of τ-functions associated with the E_8 lattice proposed by Ohta–Ramani–Grammaticos [11]. We fix a realization of the root lattice $P = Q(E_8)$ of type E_8 in the 8-dimensional complex vector space $V = \mathbb{C}^8$ endowed with the canonical symmetric bilinear form $(\cdot|\cdot) : V \times V \to \mathbb{C}$ (Section 1), and take a subset $D \subseteq V$ such that $D + P\delta = D$, where $\delta \in \mathbb{C}^*$ is a nonzero constant. A function $\tau(x)$ defined on D is called an *ORG τ-function* if it satisfies a system of non-autonomous Hirota equations

$$(0.1) \quad [(b \pm c|x)]\tau(x \pm a\delta) + [(c \pm a|x)]\tau(x \pm b\delta) + [(a \pm b|x)]\tau(x \pm c\delta) = 0$$

for all C_3-frames $\{\pm a, \pm b, \pm c\}$ relative to P (Section 2). Here $[z]$ denotes any nonzero entire function in $z \in \mathbb{C}$ which satisfies the three term relation (2.1), and each double sign in (0.1) indicates the product of two functions with different signs. When the domain D is a disjoint union

$$(0.2) \qquad D_c = \bigsqcup_{n \in \mathbb{Z}} H_{c+n\delta}, \quad H_{c+n\delta} = \{x \in V \mid (\phi|x) = c + n\delta\},$$

of parallel hyperplanes perpendicular to $\phi = (\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$, each ORG τ-function $\tau = \tau(x)$ on D_c (of type E_8) is regarded as an infinite chain of ORG τ-functions $\tau^{(n)} = \tau|_{H_{c+n\delta}}$ on $H_{c+n\delta}$ of type E_7 ($n \in \mathbb{Z}$) (Section 3). We say that a meromorphic ORG τ-function $\tau(x)$ on D_c is a *hypergeometric τ-function* if $\tau^{(n)}(x) = 0$ ($n < 0$) and $\tau^{(0)}(x) \not\equiv 0$. Supposing that $\omega \in \Omega$ is a period of $[z]$, consider the case where $D = D_\omega$. In such a case, on the basis of a recursion theorem (Theorem 3.3) one can show that, if a given pair of functions $\tau^{(0)}(x)$ ($x \in H_\omega$) and $\tau^{(1)}(x)$ ($x \in H_{\omega+\delta}$) satisfies certain initial conditions, then there exists a unique hypergeometric τ-function $\tau(x)$ ($x \in D_\omega$) having those $\tau^{(0)}(x)$, $\tau^{(1)}(x)$ as the first two components (Theorem 4.2). Furthermore, if one can specify a gauge factor for $\tau^{(1)}$ with respect to a C_3-frame of type II_1, then for each $n = 0, 1, 2, \dots$ the nth component $\tau^{(n)}(x)$ ($x \in H_{\omega+n\delta}$) of the hypergeometric τ-function is expressed in terms of a *2-directional Casorati determinant* with respect to the C_3-frame of type II_1 (Theorem 4.3). A proof of the recursion theorem (Thoerem 3.3) will be given in Appendix A.

In the latter half of this paper, we apply our arguments to the elliptic case for constructing hypergeometric ORG τ-functions which are expressible in terms of elliptic hypergeometric integrals of Spiridonov [16, 17] and Rains [12, 13]. The fundamental elliptic hypergeometric integral is the meromorphic function $I(u; p, q)$ in eight variables $u = (u_0, u_1, \ldots, u_7) \in (\mathbb{C}^*)^8$ defined by

$$(0.3) \qquad I(u; p, q) = \frac{(p; p)_\infty (q; q)_\infty}{4\pi\sqrt{-1}} \int_C \frac{\prod_{k=0}^7 \Gamma(u_k z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{z}.$$

After recalling basic facts concerning elliptic hypergeometric integrals in Section 5, we present in Section 6 two types of explicit representations for the $W(E_7)$-invariant hypergeometric ORG τ-functions, one by determinants (Theorem 6.1) and the other by multiple integrals (Theorem 6.2). Theorems 6.1 and 6.2 will be proved in Section 7. In particular we show there how the 2-directional Casorati determinant gives rise to the multiple elliptic hypergeometric integral of Rains [13]:

$$
\begin{aligned}
(0.4) \quad & I_n(u; p, q) = \frac{(p; p)_\infty^n (q; q)_\infty^n}{2^n n! (2\pi\sqrt{-1})^n} \\
& \cdot \int_{C^n} \prod_{i=1}^n \frac{\prod_{k=0}^7 \Gamma(u_k z_i^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 2}; p, q)} \prod_{1 \le i < j \le n} \theta(z_i^{\pm 1} z_j^{\pm 1}; p) \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n}.
\end{aligned}
$$

We also give some remarks in Section 8 on variations of hypergeometric ORG τ-functions obtained by transformations in Theorem 2.3.

In the final section, we discuss how the notion of ORG τ-functions is related to that of lattice τ-functions as discussed by Kajiwara et al. [5] in the context of the configuration space of generic nine points in $\mathbb{P}^2$. Some remarks are also given on the similar picture in the case of the configuration space of generic eight points in $\mathbb{P}^1 \times \mathbb{P}^1$ as in Kajiwara–Noumi–Yamada [6].

On this occasion I would like to give some personal comments on the position of this paper. The contents of this paper are not completely new, and many things presented here may be found in the literature. In fact, in the group of coauthors of [5], basic structures of the ORG τ-functions were already known around the end of 2004, including the 2-directional Casorati determinant representation of the hypergeometric τ-functions. (Some part of our discussion is reflected in the work on Masuda [8].) It was almost at the same time that Rains [12] clarified that his multiple elliptic hypergeometric integrals (0.4) satisfy certain quadratic relations of Hirota type that should be understood in the context of the elliptic Painlevé equation. It took a couple of years,

however, for us to be able to confirm that the 2-directional Casorati determinants for a particular choice of C_3-frame of type II_1 certainly give rise to the multiple elliptic hypergeometric integrals of Rains. As to further delay in presenting the detail of such an argument, I apologize just adding that it requires much more time and effort than might be imagined to accomplish a satisfactory paper in the language of a cultural sphere to which its author does not belong.

Contents

§1. E_8 lattice and C_l-frames

We begin by recalling some basic facts concerning the root lattice of type E_8.

Let $V = \mathbb{C}^8 = \mathbb{C}v_0 \oplus \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_7$ be the 8-dimensional complex vector space with canonical basis $\{v_0, v_1, \ldots, v_7\}$, and $(\cdot|\cdot) : V \times V \to \mathbb{C}$ the scalar product (symmetric bilinear form) such that $(v_i|v_j) = \delta_{ij}$ $(i, j \in \{0, 1, \ldots, 7\})$. Setting

$$(1.1) \qquad \phi = (\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}) = \tfrac{1}{2}(v_0 + v_1 + \cdots + v_7) \in V,$$

we realize the root lattice $Q(E_8)$ and the root system $\Delta(E_8)$ of type E_8 as

$$(1.2)$$
$$P = \big\{ a \in \mathbb{Z}^8 \cup (\phi + \mathbb{Z}^8) \,\big|\, (\phi|a) \in \mathbb{Z} \big\} \subset V, \quad \Delta(E_8) = \big\{ \alpha \in P \,\big|\, (\alpha|\alpha) = 2 \big\},$$

respectively (see [1] for instance). This set P forms a free $\mathbb{Z}$-module of rank 8 and the scalar product takes integer values on P. The theta

series of the lattice $P = Q(E_8)$ is given by

$$(1.3) \quad \sum_{a \in P} q^{(a|a)} = 1 + 240q^2 + 2160q^4 + 6720q^6 + 17520q^8 + 30240q^{10} + \cdots ,$$
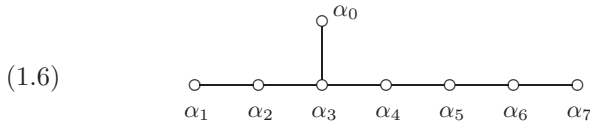
and $\Delta(E_8)$ consists of the following 240 vectors in P:
(1.4)
(1) $\pm v_i \pm v_j$ $\qquad\qquad (0 \le i < j \le 7)$ $\qquad\qquad \cdots \binom{8}{2} \cdot 4 = 112,$
(2) $\frac{1}{2}(\pm v_0 \pm v_1 \pm \cdots \pm v_7)$ (even number of $-$ signs) $\cdots \quad 2^7 = 128.$

In this root system, we take the *simple roots*
(1.5)
$$\alpha_0 = \phi - v_0 - v_1 - v_2 - v_3, \quad \alpha_j = v_j - v_{j+1} \ (j = 1, \ldots, 6), \quad \alpha_7 = v_7 + v_0$$

corresponding to the Dynkin diagram

(1.6)

so that $P = Q(E_8) = \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_7$. (For mutually distinct $i, j \in \{0, 1, \ldots, 7\}$, $(\alpha_i|\alpha_j) = -1$ if the two nodes named α_i and α_j are connected by an edge, and $(\alpha_i|\alpha_j) = 0$ otherwise.) The vector

$$(1.7) \qquad \phi = 3\alpha_0 + 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7$$

is called the *highest root* with respect to the simple roots $\alpha_0, \alpha_1, \ldots, \alpha_7$. Note also that the weight lattice $P(E_8)$ coincides with the root lattice $Q(E_8)$ in this E_8 case.

For each $\alpha \in V$ with $(\alpha|\alpha) \ne 0$, we define the *reflection* $r_\alpha : V \to V$ with respect to α by

$$(1.8) \qquad\qquad r_\alpha(v) = v - (\alpha^\vee|v)\alpha \qquad (v \in V),$$

where $\alpha^\vee = 2\alpha/(\alpha|\alpha)$. The Weyl group $W(E_8) = \langle r_\alpha \ (\alpha \in \Delta(E_8)) \rangle$ of type E_8 acts on V as a group of isometries; it stabilizes the root lattice $P = Q(E_8)$ and the root system $\Delta(E_8)$. We denote by $s_j = r_{\alpha_j}$ $(j = 0, 1, \ldots, 7)$ the *simple reflections*. Then, $W(E_8) = \langle s_0, s_1, \ldots, s_7 \rangle$ is the Coxeter group associated with the Dynkin diagram (1.6). This group is generated by $s_0, s_1, \ldots, s_7$ with fundamental relations $s_j^2 = 1$ $(j = 0, 1, \ldots, 7)$, $s_i s_j = s_j s_i$ for distinct $i, j \in \{0, 1, \ldots, 7\}$ with $(\alpha_i|\alpha_j) = 0$ and $s_i s_j s_i = s_j s_i s_j$ for distinct $i, j \in \{0, 1, \ldots, 7\}$ with $(\alpha_i|\alpha_j) = -1$.

In the E_8 lattice $P = Q(E_8)$, the root lattice $Q(E_7)$ and the root system $\Delta(E_7)$ of type E_7 are realized as

(1.9)
$$Q(E_7) = \{a \in P \mid (\phi|a) = 0\} = \mathbb{Z}\,\alpha_0 \oplus \mathbb{Z}\,\alpha_1 \oplus \cdots \oplus \mathbb{Z}\,\alpha_6,$$
$$\Delta(E_7) = \{\alpha \in \Delta(E_8) \mid (\phi|\alpha) = 0\},$$

respectively. The root system $\Delta(E_7)$ consists of the following 126 vectors:

(1.10)

(1) $\pm(v_i - v_j)$ $\qquad\qquad\qquad (0 \le i < j \le 7)$ $\quad \cdots \quad$ $\binom{8}{2} \cdot 2 = 56,$

(2) $\frac{1}{2}(\pm v_0 \pm v_1 \pm \cdots \pm v_7)$ (four − signs) $\quad \cdots \quad$ $\binom{8}{4} = 70.$

The highest root of $\Delta(E_7)$ with respect to the simple roots $\alpha_0, \alpha_1, \ldots, \alpha_6$ is given by

(1.11) $\qquad v_1 - v_0 = 2\alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6.$

We denote by $W(E_7) = \langle r_\alpha \ (\alpha \in \Delta(E_7)) \rangle = \langle s_0, s_1, \ldots, s_6 \rangle$ the Weyl group of type E_7. Note that $W(E_7)$ contains the symmetric group $\mathfrak{S}_8 = \langle r_{v_0 - v_1}, s_1, \ldots, s_6 \rangle$ acting on V through the permutation of $v_0, v_1, \ldots, v_7$, and is generated by this $\mathfrak{S}_8$ together with the reflection s_0 with respect to $\alpha_0 = \phi - v_0 - v_1 - v_2 - v_3$.

In this paper, the following notion of C_l-frames plays a fundamental role.

Definition 1.1. For $l = 1, 2, \ldots, 8$, a set $A = \{\pm a_0, \pm a_1, \ldots, \pm a_{l-1}\}$ of $2l$ vectors in V is called a C_l-frame (relative to P), if the following two conditions are satisfied:

(1) $(a_i|a_j) = \delta_{ij}$ $\quad (0 \le i, j < l),$

(2) $a_i \pm a_j \in P$ $\ (0 \le i < j < l),$ $\quad 2a_i \in P$ $\ (0 \le i < l).$

This condition for $A = \{\pm a_0, \ldots, \pm a_{l-1}\}$ means that the set of $2l^2$ vectors

(1.12) $\Delta_A(C_l) = \{\pm a_i \pm a_j \mid 0 \le i < j < l)\} \cup \{\pm 2a_i \mid 0 \le i < l\} \subset P$

form a root system of type C_l. For $l = 1, 2, \ldots, 8$, we denote by $\mathcal{C}_l$ the set of all C_l-frames.

Example 1.2 (Typical examples of C_8-frames).

(0) $A_0 = \{\pm v_0, \pm v_1, \ldots, \pm v_7\}$.

(1) $A_1 = \{\pm a_0, \pm a_1, \ldots, \pm a_7\}$,

$$a_0 = \tfrac{1}{2}(v_0 + v_1 + v_2 + v_3), \quad a_4 = \tfrac{1}{2}(v_4 - v_5 - v_6 + v_7),$$
$$a_1 = \tfrac{1}{2}(v_0 + v_1 - v_2 - v_3), \quad a_5 = \tfrac{1}{2}(-v_4 + v_5 - v_6 + v_7),$$
$$a_2 = \tfrac{1}{2}(v_0 - v_1 + v_2 - v_3), \quad a_6 = \tfrac{1}{2}(-v_4 - v_5 + v_6 + v_7),$$
$$a_3 = \tfrac{1}{2}(v_0 - v_1 - v_2 + v_3), \quad a_7 = \tfrac{1}{2}(v_4 + v_5 + v_6 + v_7).$$

(2) $A_2 = \{\pm a_0, \pm a_1, \ldots, \pm a_7\}$,

$$a_0 = \tfrac{1}{2}(\phi + v_0 - v_7), \quad a_7 = \tfrac{1}{2}(\phi - v_0 + v_7),$$
$$a_j = v_j + \tfrac{1}{2}(v_0 + v_7 - \phi) \quad (j = 1, \ldots, 6).$$

In the following, we denote by $N(v) = (v|v)$ the square norm of $v \in V$, and by $\varphi(v) = (\phi|v)$ the scalar product of v with ϕ. Also, for a subset $S \subseteq V$ given, we use the notations

(1.13)
$$S_{N=k} = \{v \in S \,|\, (v|v) = k\}, \quad S_{\varphi=k} = \{v \in S \,|\, (\phi|v) = k\} \qquad (k \in \mathbb{C})$$

to refer to the level sets of N and φ respectively.

Note that each C_l-frame $(l = 1, 2, \ldots, 8)$ is formed by vectors in $(\tfrac{1}{2}P)_{N=1} = \tfrac{1}{2}(P_{N=4})$. The set $P_{N=4}$ of all vectors in P with square norm 4 consists of the following 2160 vectors that are classified into three groups under the action of the symmetric group $\mathfrak{S}_8$:

(1.14)

(0) $\pm 2v_0$ $\cdots$ $8 \cdot 2 = 16$,

(1) $\pm v_0 \pm v_1 \pm v_2 \pm v_3$ $\cdots$ $\binom{8}{4} \cdot 2^4 = 1120$,

(2) $\tfrac{1}{2}(\pm 3v_0 \pm v_1 \pm \cdots \pm v_7)$ (odd number of $-$ signs) $\cdots$ $8 \cdot 2^7 = 1024$.

The Weyl group $W(E_8)$ acts on $P_{N=4}$ transitively. In fact we have $P_{N=4} \cap P^+ = \{\phi - v_0 + v_1\}$, where $P^+ = \{v \in P \,|\, (\alpha_j|v) \geq 0(0 \leq j \leq 7)\}$ stands for the cone of dominant integral weights. It turns out that the stabilizer of $\phi - v_0 + v_1$ is $W(D_7)$ and that

(1.15) $P_{N=4} \xleftarrow{\sim} W(E_8)/W(D_7), \quad |P_{N=4}| = |W(E_8)/W(D_7)| = 2160$.

The following two propositions can be verified directly on the basis of this transitive action of $W(E_8)$ on $P_{N=4}$.

Proposition 1.3.

(1) *For each $a \in (\tfrac{1}{2}P)_{N=1}$, there exists a unique C_8-frame containing a.*

(2) *The set $(\frac{1}{2}P)_{N=1}$ is the disjoint union of all C_8-frames: $(\frac{1}{2}P)_{N=1} = \bigsqcup_{A \in \mathcal{C}_8} A$.*

(3) *The number of C_8-frames is given by $|\mathcal{C}_8| = 2160/16 = 135$.*   □

Proposition 1.4. *Fix a positive integer $l \in \{1, \ldots, 8\}$.*

(1) *The Weyl group $W(E_8)$ acts transitively on the set $\mathcal{C}_l$ of all C_l-frames.*

(2) *Each C_l-frame is contained in a unique C_8-frame.*

(3) *The number of C_l-frames is given by $|\mathcal{C}_l| = 135 \cdot \binom{8}{l}$.*   □

§2.   ORG τ-functions

In this section we introduce the notion of *ORG τ-functions*, which is a reformulation of τ-functions associated with the E_8 lattice proposed by Ohta–Ramani–Grammaticos [11].

We fix once for all a nonzero entire function $[z]$ in $z \in \mathbb{C}$ satisfying the three-term relation

$$(2.1) \quad [\beta \pm \gamma][z \pm \alpha] + [\gamma \pm \alpha][z \pm \beta] + [\alpha \pm \beta][z \pm \gamma] = 0 \qquad (z, \alpha, \beta, \gamma \in \mathbb{C}).$$

Throughout this paper, we use the abbreviation $[\alpha \pm \beta] = [\alpha + \beta][\alpha - \beta]$ with a double sign indicating the product of two factors. From (2.1) it follows that that $[z]$ is an odd function $([-z] = -[z], [0] = 0)$. We remark that the three-term relation (2.1) can be written alternatively as

$$(2.2) \quad [z \pm \alpha][w \pm \beta] - [z \pm \beta][w \pm \alpha] = [z \pm w][\alpha \pm \beta] \qquad (z, w, \alpha, \beta \in \mathbb{C}),$$

or

$$(2.3) \qquad \frac{[z \pm \alpha]}{[z \pm \beta]} - \frac{[w \pm \alpha]}{[w \pm \beta]} = \frac{[z \pm w][\alpha \pm \beta]}{[z \pm \beta][w \pm \beta]} \qquad (z, w, \alpha, \beta \in \mathbb{C}).$$

It is known that the functional equation (2.1) for $[z]$ implies that the set of zeros $\Omega = \{\omega \in \mathbb{C} \mid [\omega] = 0\}$ form a closed discrete subgroup of the additive group $\mathbb{C}$. Furthermore, such a function $[z]$ belongs to one of the following three classes, *rational*, *trigonometric* or *elliptic*, according to the rank of Ω ([19]):

(2.4)

(0)	rational	: $[z] = e(c_0 z^2 + c_1) z$	$(\Omega = 0)$,
(1)	trigonometric	: $[z] = e(c_0 z^2 + c_1) \sin(\pi z/\omega_1)$	$(\Omega = \mathbb{Z}\omega_1)$,
(2)	elliptic	: $[z] = e(c_0 z^2 + c_1) \sigma(z|\Omega)$	$(\Omega = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$,

where $e(z) = e^{2\pi\sqrt{-1}z}$, and $c_0, c_1 \in \mathbb{C}$. In the elliptic case, Ω is generated by complex numbers ω_1, ω_2 which are linearly independent over $\mathbb{R}$, and

$\sigma(z|\Omega)$ stands for the Weierstrass sigma function associated with the period lattice $\Omega = \mathbb{Z}\,\omega_1 \oplus \mathbb{Z}\,\omega_2$. In the trigonometric and elliptic cases, $[z]$ is quasi-periodic with respect to Ω in the following sense:

$$(2.5) \qquad [z + \omega] = \epsilon_\omega e(\eta_\omega(z + \tfrac{\omega}{2}))[z] \quad (\omega \in \Omega),$$

where $\eta_\omega \in \mathbb{C}$ ($\omega \in \Omega$) are constants such that $\eta_{\omega+\omega'} = \eta_\omega + \eta_{\omega'}$ ($\omega, \omega' \in \Omega$), and $\epsilon_\omega = +1$ or -1 according as $\omega \in 2\Omega$ or $\omega \notin 2\Omega$.

In what follows we fix a nonzero constant $\delta \in \mathbb{C}$ such that $\mathbb{Z}\delta \cap \Omega = \{0\}$. Let D be a subset of $V = \mathbb{C}^8$ stable under the translation by $P\delta$, namely $D + P\delta = D$.

Definition 2.1. A function $\tau(x)$ defined over D is called an *ORG* τ-*function* if it satisfies the non-autonomous Hirota equations
$$(2.6)$$
$$\mathrm{H}(a, b, c): \quad [(b{\pm}c|x)]\tau(x{\pm}a\delta)+[(c{\pm}a|x)]\tau(x{\pm}b\delta)+[(a{\pm}b|x)]\tau(x{\pm}c\delta) = 0$$

for all C_3-frames $\{\pm a, \pm b, \pm c\}$ relative to P.

A C_3-frame $\{\pm a, \pm b, \pm c\}$ defines an octahedron in V of which the twelve edges and the three diagonals are vectors in the E_8 lattice P. Hence, if one of the six vertices $\{x \pm a\delta, x \pm b\delta, x \pm c\delta\}$ belongs to D, the other five belong to D as well by the property of a C_3-frame. Also, by Proposition 1.4 the number of C_3-frames is $|\mathcal{C}_3| = 135 \cdot \binom{8}{3} = 7560$. Hence, the equation to be satisfied by an ORG τ-function is a system of 7560 non-autonomous Hirota equations, which we call the *ORG system* of type E_8. (A bilinear equation of the form (2.6) is also called a *Hirota-Miwa equation*.)

In Definition 2.1, as the independent variables of $\tau(x)$ one can take both discrete and continuous variables. The two extreme cases of the domain D are:
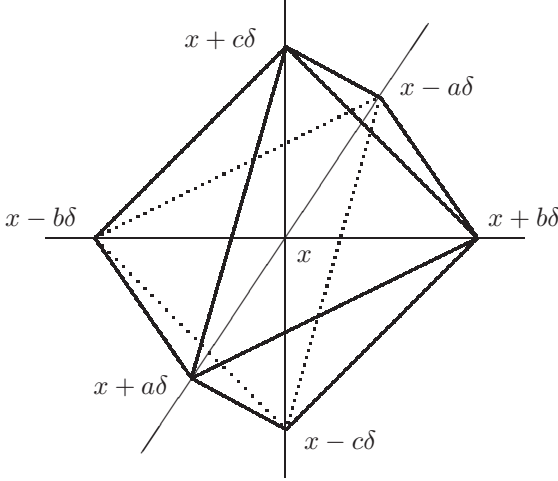
$$(2.7) \quad (1) \ \ D = v + P\delta \ \text{(fully discrete)}, \quad (2) \ \ D = V \ \text{(fully continuous)}.$$

There are intermediate cases where D is a disjoint union of a countable family of affine subspaces. In such cases, we assume that $\tau(x)$ is a holomorphic (or meromorphic) function on D.

Proposition 2.2. *For any constant $c \in \mathbb{C}$, the entire function*

$$(2.8) \qquad \tau(x) = [\tfrac{1}{2\delta}(x|x) + c] \qquad (x \in V)$$

is an ORG τ-function on V.

$$[(b \pm c|x)]\tau(x \pm a\delta) + [(c \pm a|x)]\tau(x \pm b\delta) + [(a \pm b|x)]\tau(x \pm c\delta) = 0$$

Fig. 1. Non-autonomous Hirota equation

Proof. Noting that $\tau(x \pm a\delta) = [\frac{1}{2\delta}(x|x) + \frac{\delta}{2} + c \pm (a|x)]$, set

$$(2.9) \qquad z = \frac{1}{2\delta}(x|x) + \frac{\delta}{2} + c, \quad \alpha = (a|x), \quad \beta = (b|x), \quad \gamma = (c|x).$$

Then the Hirota equation $H(a, b, c)$ reduces to the functional equation (2.1). Q.E.D.

The ORG τ-function (2.8) can be regarded as the *canonical solution* of the ORG system. We give below some remarks on transformations of an ORG τ-function.

Theorem 2.3. *Let D be a subset of V with $D + P\delta = D$, and $\tau(x)$ an ORG τ-function on D.*
(1) (*Multiplication by an exponential function*) *For any constants $k, c \in \mathbb{C}$, any vector $v \in V$ and $\epsilon = \pm 1$, the function*

$$(2.10) \qquad \widetilde{\tau}(x) = e\big(k(x|x) + (v|x) + c\big)\tau(\epsilon x) \qquad (x \in \epsilon D)$$

is an ORG τ-function on ϵD.
(2) (*Transformation by $W(E_8)$*) *For any $w \in W(E_8)$, the function $w.\tau$*

defined by

$$(2.11) \qquad (w.\tau)(x) = \tau(w^{-1}.x) \qquad (x \in w.D)$$

is an ORG τ-function on $w.D$.

(3) (*Translation by a period*) *For any period $\omega \in \Omega$ and any $v \in P$, the function $\widetilde{\tau}$ defined by*

$$(2.12) \qquad \begin{aligned} \widetilde{\tau}(x) &= e(S(x; v, \omega))\tau(x - v\omega) \qquad (x \in D + v\omega), \\ S(x; v, \omega) &= \tfrac{\eta_\omega}{2\delta^2}(v|x)(x|x - v\omega), \end{aligned}$$

is an ORG τ-function on $D + v\omega$.

    Proof. Since Statements (1) and (2) are straightforward, we give a proof of (3) only. Set $y = x - v\omega \in D$ so that

$$(2.13) \quad [(b \pm c|x)]\widetilde{\tau}(x \pm a\delta) = [(b \pm c|y + v\omega)]e(S(x \pm a\delta; v, \omega))\tau(y \pm a\delta).$$

Since

$$(2.14)$$
$$\begin{aligned} [(b + c|y + v\omega)] &= [(b + c|y) + (b + c|v)\omega] \\ &= \epsilon_{(b+c|v)\omega}e(\eta_\omega(b + c|v)(b + c|y + v\tfrac{\omega}{2}))[(b + c|y)], \end{aligned}$$

we have

$$(2.15)$$
$$\begin{aligned} &[(b \pm c|y + v\omega)] \\ &= \epsilon_{(b+c|v)\omega}\epsilon_{(b-c|v)\omega}[(b \pm c|y)]e\left(2\eta_\omega((b|v)b + (c|v)c|y + v\tfrac{\omega}{2})\right) \\ &= \epsilon_{(b+c|v)\omega}\epsilon_{(b-c|v)\omega}[(b \pm c|y)]e\left(2\eta_\omega((b|v)b + (c|v)c|x - v\tfrac{\omega}{2})\right). \end{aligned}$$

On the other hand,

$$(2.16) \qquad \begin{aligned} &S(x + a\delta; v, w) + S(x - a\delta; v, w) \\ &= \tfrac{\eta_\omega}{\delta^2}(v|x)\left((v|x - v\omega) + \delta^2\right) + 2\eta_\omega(a|v)(a|x - v\tfrac{\omega}{2}). \end{aligned}$$

This implies

$$(2.17)$$
$$\begin{aligned} &[(b \pm c|x)]\widetilde{\tau}(x \pm a\delta) \\ &= \epsilon_{(b+c|v)\omega}\epsilon_{(b-c|v)\omega}[(b \pm c|y)]\tau(y \pm a\delta)e\left(\tfrac{\eta_\omega}{\delta^2}(v|x)((v|x - v\omega) + \delta^2)\right) \\ &\quad \cdot e\left(2\eta_\omega((a|v)a + (b|v)b + (c|v)c|x - v\tfrac{\omega}{2})\right). \end{aligned}$$

Hence, validity of the Hirota equation for $\widetilde{\tau}(x)$ reduces to proving

$$(2.18) \qquad \epsilon_{(b+c|v)\omega}\epsilon_{(b-c|v)\omega} = \epsilon_{(c+a|v)\omega}\epsilon_{(c-a|v)\omega} = \epsilon_{(a+b|v)\omega}\epsilon_{(a-b|v)\omega}.$$

Since this holds trivially for $\omega \in 2\Omega$, we assume $\omega \notin 2\Omega$. In view of the transitive action of $W(E_8)$ on $\mathcal{C}_3$, we may assume $\{\pm a, \pm b, \pm c\} = \{\pm v_0, \pm v_1, \pm v_2\}$. Then, for distinct $i, j \in \{0, 1, 2\}$, we have

$$(2.19) \qquad \begin{aligned} v \in \mathbb{Z}^8 &\implies (v_i + v_j | v) \equiv (v_i - v_j | v) \mod 2, \\ v \in \phi + \mathbb{Z}^8 &\implies (v_i + v_j | v) \not\equiv (v_i - v_j | v) \mod 2. \end{aligned}$$

Since

$$(2.20) \qquad \epsilon_{k\omega}\epsilon_{l\omega} = \begin{cases} +1 & (k \equiv l \mod 2), \\ -1 & (k \not\equiv l \mod 2) \end{cases}$$

for $\omega \notin 2\Omega$, $\epsilon_{(v_i+v_j|v)\omega}\epsilon_{(v_i-v_j|v)\omega}$ takes the value $+1$ or -1 according as $v \in P$ belongs to $\mathbb{Z}^8$ or $\phi + \mathbb{Z}^8$, regardless of the choice of the pair i, j. This completes the proof of (3).   Q.E.D.

We remark that the composition of two translations of (3) by $a\omega$ and by $b\omega$ for $a, b \in P$ and $\omega \in \Omega$ results essentially in the same transformation as the translation by $(a + b)\omega$. In fact we have

$$(2.21) \qquad \begin{aligned} &e(S(x; b\omega)e(S(x - b\omega; a, \omega))\tau(x - (a + b)\omega) \\ &= e(k(x|x) + (v|x) + c)S(x; a + b, \omega)\tau(x - (a + b)\omega) \end{aligned}$$

for some $k, c \in \mathbb{C}$ and $v \in V$.

§3. E_8 τ-function as an infinite chain of E_7 τ-functions

Recall that the root lattice $Q(E_7)$ of type E_7 is the orthogonal complement of ϕ in $P = Q(E_8)$. In what follows, we denote by

$$(3.1) \qquad H_\kappa = \{x \in V \mid (\phi|x) = \kappa\} \quad (\kappa \in \mathbb{C})$$

the hyperplanes in V defined as the level sets of $\varphi = (\phi|\cdot)$. Fixing a constant $c \in \mathbb{C}$, we now consider the case where the domain D of an ORG τ-function is a disjoint union of parallel hyperplanes

$$(3.2) \qquad D_c = \bigsqcup_{n \in \mathbb{Z}} H_{c+n\delta}.$$

Note that each component $H_{c+n\delta}$ $(n \in \mathbb{Z})$ is invariant under the action of $W(E_7)$ and the translation by $Q(E_7)\delta$, and that the whole set D is invariant under the translation by $Q(E_8)\delta$. In this situation, we regard a function $\tau(x)$ on D_c as an infinite family of functions $\tau^{(n)}(x)$ on $H_{c+n\delta}$ $(n \in \mathbb{Z})$ defined by restriction as $\tau^{(n)} = \tau\big|_{H_{c+n\delta}}$ for $n \in \mathbb{Z}$. In order

to investigate the system of Hirota equations for $\tau^{(n)}(x)$ $(n \in \mathbb{Z})$, we classify them under the action of the Weyl group $W(E_7)$.
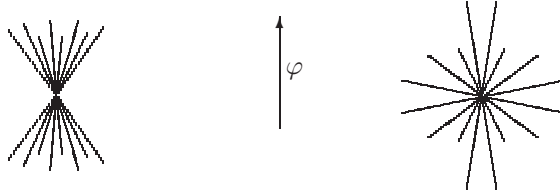
For a C_l-frame $A = \{\pm a_0, \ldots, \pm a_{l-1}\}$ given, we consider the *multiset*

$$(3.3) \qquad \varphi(A) = \{\pm\varphi(a_0), \ldots, \pm\varphi(a_{l-1})\},$$

where $\varphi(v) = (\phi|v)$ $(v \in V)$. As to the three C_8-frames of Example 1.2, we have

$$(3.4) \qquad \varphi(A_0) = \{(\pm\tfrac{1}{2})^8\}, \quad \varphi(A_1) = \varphi(A_2) = \{(\pm 1)^2, 0^{12}\}.$$

Here the symbol c^n (resp. $(\pm c)^n$) indicates that c appears (resp. both $+c$ and $-c$ appear) with multiplicity n in the multiset. We say that a C_8-frame is *of type* I if $\varphi(A) = \{(\pm\tfrac{1}{2})^8\}$, and *of type* II if $\varphi(A) = \{(\pm 1)^2, 0^{12}\}$.

$$(3.5) \qquad C_8\text{-frame of type I} \qquad\qquad C_8\text{-frame of type II}$$

Proposition 3.1. *Any C_8-frame is either of type I or of type II. Furthermore, these two types give the decomposition of the set $\mathcal{C}_8$ of all C_8-frames into $W(E_7)$-orbits:*

$$(3.6) \quad \begin{aligned} &\mathcal{C}_8 = \mathcal{C}_{8,\mathrm{I}} \sqcup \mathcal{C}_{8,\mathrm{II}}, \qquad |\mathcal{C}_{8,\mathrm{I}}| = 72, \quad |\mathcal{C}_{8,\mathrm{II}}| = 63; \\ &\mathcal{C}_{8,\mathrm{I}} = W(E_7)A_0, \quad \mathcal{C}_{8,\mathrm{II}} = W(E_7)A_1 = W(E_7)A_2. \end{aligned}$$

In order to analyze the $W(E_7)$-orbits in $\mathcal{C}_8$, we first decompose $P_{N=4}$ into $W(E_7)$-orbits. As a result, $P_{N=4}$ decomposes into the form
$$(3.7)$$
$$P_{N=4} = P_{N=4,\varphi=2} \sqcup P_{N=4,\varphi=1} \sqcup P_{N=4,\varphi=0} \sqcup P_{N=4,\varphi=-1} \sqcup P_{N=4,\varphi=-2},$$

and each level set of φ forms a single $W(E_7)$-orbit. The five $W(E_7)$-orbits are described as follows.
$$(3.8)$$

φ	2	1	0	-1	-2
representative	$\phi-v_0+v_1$	$\phi-2v_0$	$\phi-2v_0-v_6-v_7$	$-2v_0$	$-\phi-v_0+v_1$
stabilizer	$W(D_6)$	$W(A_6)$	$W(D_5 \times A_1)$	$W(A_6)$	$W(D_6)$
cardinality	126	576	756	576	126

The "representative" indicates a unique vector v in the orbit such that $(\alpha_j|v) \geq 0$ $(j = 0, 1, \ldots, 6)$. According to this decomposition of $P_{N=4}$, $(\frac{1}{2}P)_{N=1} = \frac{1}{2}(P_{N=4})$ decomposes into the five $W(E_7)$-orbits with $\varphi = 1, \frac{1}{2}, 0, -\frac{1}{2}, -1$. Proposition 3.1 follows from the fact that $\frac{1}{2}\phi - v_0$ and v_0 belong to C_8-frames of type I, and $\frac{1}{2}(\phi - v_0 + v_1)$, $\frac{1}{2}(\phi - v_6 - v_7) - v_0$, and $\frac{1}{2}(-\phi - v_0 + v_1)$ to those of type II. Note that, among the 63 C_8-frames of type II, 35 are obtained from A_1, and 28 from A_2 of Example 1.2 by the action of the symmetric group $\mathfrak{S}_8 \subset W(E_7)$.

We remark that in any C_8-frame $\{\pm a_0, \pm a_1, \ldots, \pm a_7\}$ with

$$(3.9) \qquad \text{(I)}: \quad (\phi|a_j) = \tfrac{1}{2} \quad (j = 0, 1, \ldots, 7),$$

we have $\frac{1}{2}(a_0 + a_1 + \cdots + a_7) = \phi$. Also, in any C_8-frame $\{\pm a_0, \pm a_1, \ldots, \pm a_7\}$ with

$$(3.10) \qquad \text{(II)}: \quad (\phi|a_0) = (\phi|a_7) = 1, \quad (\phi|a_j) = 0 \quad (j = 1, \ldots, 6),$$

we have $a_0 + a_7 = \phi$. Since these statements are $W(E_7)$-invariant, by Proposition 3.1 we have only to check the cases of A_0 and A_1 of Example 1.2, respectively.

By Proposition 1.4, each C_3-frame is contained in a unique C_8-frame. Hence, by Proposition 3.1 we obtain the following classification of C_3-frames.

Proposition 3.2. *The set $\mathcal{C}_3$ of all C_3-frames decomposes into four $W(E_7)$-orbits:*

$$(3.11) \qquad \mathcal{C}_3 = \mathcal{C}_{3,\mathrm{I}} \sqcup \mathcal{C}_{3,\mathrm{II}_0} \sqcup \mathcal{C}_{3,\mathrm{II}_1} \sqcup \mathcal{C}_{3,\mathrm{II}_2}.$$

These four $W(E_7)$-orbits are characterized as follows.

(3.12)

type	I	II$_0$	II$_1$	II$_2$
φ	$(\pm\frac{1}{2})^3$	0^6	$(\pm 1)\, 0^4$	$(\pm 1)^2\, 0^2$
cardinality	$56 \cdot 72$	$20 \cdot 63$	$30 \cdot 63$	$6 \cdot 63$

According to the four types of C_3-frames, the Hirota equations for $\tau^{(n)}(x)$ are classified as follows. For each C_3-frame $\{\pm a_0, \pm a_1, \pm a_2\}$ of type I with

$$(3.13) \qquad \text{(I)}: \quad (\phi|a_0) = (\phi|a_1) = (\phi|a_2) = \tfrac{1}{2},$$

the Hirota equation $H(a_0, a_1, a_2)$ takes the form

$$(\mathrm{I})_{n+1/2}: \quad [(a_1 \pm a_2|x)]\tau^{(n)}(x - a_0\delta)\tau^{(n+1)}(x + a_0\delta)$$
$$(3.14) \qquad\qquad + [(a_2 \pm a_0|x)]\tau^{(n)}(x - a_1\delta)\tau^{(n+1)}(x + a_1\delta)$$
$$\qquad\qquad + [(a_0 \pm a_1|x)]\tau^{(n)}(x - a_2\delta)\tau^{(n+1)}(x + a_2\delta) = 0$$
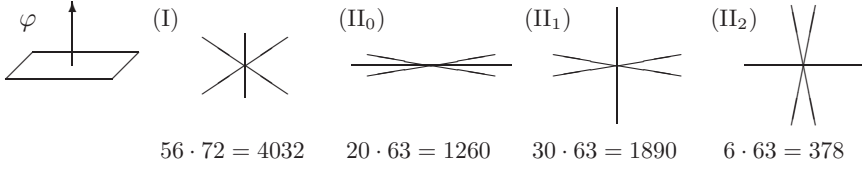
Fig. 2. Four types of 7560 C_3-frames

for $x \in H_{c+(n+1/2)\delta}$. This bilinear equation describes the relationship (*Bäcklund transformation*) between the two τ-functions $\tau^{(n)}(x)$ and $\tau^{(n+1)}(x)$ on $H_{c+n\delta}$ and $H_{c+(n+1)\delta}$, respectively. When $\{\pm a_0, \pm a_1, \pm a_2\}$ is C_3-frame of type II_0, II_1, II_2, we choose a_0, a_1, a_2 so that

$$
\begin{aligned}
&(II_0): \quad (\phi|a_0) = (\phi|a_1) = (\phi|a_2) = 0, \\
(3.15) \quad &(II_1): \quad (\phi|a_0) = 1, \quad (\phi|a_1) = (\phi|a_2) = 0, \\
&(II_2): \quad (\phi|a_0) = (\phi|a_1) = 1, \quad (\phi|a_2) = 0.
\end{aligned}
$$

Then the corresponding Hirota equations are given by

(3.16)
$$
\begin{aligned}
(II_0)_n: \quad &[(a_1 \pm a_2|x)]\tau^{(n)}(x \pm a_0\delta) + [(a_2 \pm a_0|x)]\tau^{(n)}(x \pm a_1\delta) \\
&+ [(a_0 \pm a_1|x)]\tau^{(n)}(x \pm a_2\delta) = 0, \\
(II_1)_n: \quad &[(a_1 \pm a_2|x)]\tau^{(n-1)}(x - a_0\delta)\tau^{(n+1)}(x + a_0\delta) \\
&= [(a_0 \pm a_2|x)]\tau^{(n)}(x \pm a_1\delta) - [(a_0 \pm a_1|x)]\tau^{(n)}(x \pm a_2\delta), \\
(II_2)_n: \quad &[(a_1 \pm a_2|x)]\tau^{(n-1)}(x - a_0\delta)\tau^{(n+1)}(x + a_0\delta) \\
&- [(a_0 \pm a_2|x)]\tau^{(n-1)}(x - a_1\delta)\tau^{(n+1)}(x + a_1\delta) \\
&= [(a_1 \pm a_0|x)]\tau^{(n)}(x \pm a_2\delta)
\end{aligned}
$$

for $x \in H_{c+n\delta}$. Note that the Hirota equation of type $(II_0)_n$ is an equation for $\tau^{(n)}(x)$ on $H_{c+n\delta}$ only, while those of types $(II_1)_n$ and $(II_2)_n$ are equations among the three τ-functions $\tau^{(n-1)}(x)$, $\tau^{(n)}(x)$, $\tau^{(n+1)}(x)$. A bilinear equation of type $(II_1)_n$ can be regarded as a discrete version of the *Toda equation*.

For each $n \in \mathbb{Z}$, the system of 1260 Hirota equations $(II_0)_n$ for $\tau^{(n)}(x)$ on $H_{c+n\delta}$ can be regarded as an ORG system of type E_7. In this way, the whole ORG system of type E_8 for $\tau(x)$ on D_c can be regarded

as an infinite chain of ORG systems of type E_7 for $\tau^{(n)}(x)$ on $H_{c+n\delta}$ $(n \in \mathbb{Z})$.

Theorem 3.3. *For an integer $n \in \mathbb{Z}$, let $\tau^{(n-1)}(x)$ and $\tau^{(n)}(x)$ be meromorphic functions on $H_{c+(n-1)\delta}$ and $H_{c+n\delta}$, respectively. Suppose that $\tau^{(n-1)}(x) \not\equiv 0$ and that the following two conditions are satisfied:*

(A1): $\tau^{(n-1)}(x)$ *and* $\tau^{(n)}(x)$ *satisfy all the bilinear equations of type* $(\mathrm{I})_{n-1/2}$.

(A2): $\tau^{(n)}(x)$ *satisfies all the bilinear equations of type* $(\mathrm{II}_0)_n$.

Then there exists a unique meromorphic function $\tau^{(n+1)}(x)$ on $H_{c+(n+1)\delta}$ such that

(B): $\tau^{(n-1)}(x)$, $\tau^{(n)}(x)$, $\tau^{(n+1)}(x)$ *satisfy all the bilinear equations of type* $(\mathrm{II}_1)_n$.

Furthermore, this $\tau^{(n+1)}(x)$ satisfies the following conditions:

(C1): $\tau^{(n-1)}(x)$, $\tau^{(n)}(x)$, $\tau^{(n+1)}(x)$ *satisfy all the bilinear equations of type* $(\mathrm{II}_2)_n$.

(C2): $\tau^{(n)}(x)$, $\tau^{(n+1)}(x)$ *satisfy all the bilinear equations of type* $(\mathrm{I})_{n+1/2}$.

(C3): $\tau^{(n+1)}(x)$ *satisfies all the bilinear equations of type* $(\mathrm{II}_0)_{n+1}$.

This theorem can be proved essentially by the same argument as that of Masuda [8, Section 3]. For completeness, we include a proof of Theorem 3.3 in Appendix A.

§4.   Hypergeometric ORG τ-functions

Keeping the notations in the previous section, we consider an ORG τ-function $\tau = \tau(x)$ on

$$(4.1) \quad D_c = \bigsqcup_{n \in \mathbb{Z}} H_{c+n\delta}, \quad H_{c+n\delta} = \{x \in V \mid (\phi|x) = c + n\delta\} \quad (n \in \mathbb{Z}),$$

where $c \in \mathbb{C}$. For each $n \in \mathbb{Z}$ we denote by $\tau^{(n)} = \tau\big|_{H_{c+n\delta}}$ the restriction of τ to $H_{c+n\delta}$.

Definition 4.1. A meromorphic ORG τ-function $\tau(x)$ on D_c is called a *hypergeometric τ-function* if $\tau^{(n)}(x) = 0$ for $n < 0$, and $\tau^{(0)}(x) \not\equiv 0$.

We now apply Theorem 3.3 for constructing hypergeometric τ-functions. Since $\tau^{(-1)}(x) = 0$ $(x \in H_{c-\delta})$, for any C_8-frame $\{\pm a_0, \ldots, \pm a_7\}$ of type II with

$$(4.2) \qquad (\phi|a_0) = (\phi|a_7) = 1, \quad (\phi|a_i) = 0 \quad (i = 1, \ldots, 6),$$

$\tau^{(0)}(x)$ $(x \in H_c)$ must satisfy the following three types of equations:

$$(\text{II}_2)_0: \quad [(a_0 \pm a_7|x)]\tau^{(0)}(x \pm a_i\delta) = 0,$$

$$(4.3) \quad (\text{II}_1)_0: \quad [(a_r \pm a_j|x)]\tau^{(0)}(x \pm a_i\delta) = [(a_r \pm a_i|x)]\tau^{(0)}(x \pm a_j\delta),$$

$$(\text{II}_0)_0: \quad [(a_j \pm a_k|x)]\tau^{(0)}(x \pm a_i\delta) + [(a_k \pm a_i|x)]\tau^{(0)}(x \pm a_j\delta)$$
$$+ [(a_i \pm a_j|x)]\tau^{(0)}(x \pm a_k\delta) = 0,$$

where $r = 0, 7$ and $i, j, k \in \{1, \ldots, 6\}$. Noting that $a_0 + a_7 = \phi$, in order to fulfill $(\text{II}_2)_0$ for any C_8-frame of type II, we consider the case where $c = \omega \in \Omega$ is a period of $[z]$, so that $[(a_0 + a_7|x)] = [(\phi|x)] = [\omega] = 0$. Equations of type $(\text{II}_0)_0$ follow from those of type $(\text{II}_0)_1$. In fact, since

$$(4.4) \quad \tau^{(0)}(x \pm a_j\delta) = \frac{[(a_0 \pm a_j|x)]}{[(a_0 \pm a_k|x)]}\tau^{(0)}(x \pm a_k\delta)$$

for any distinct $j, k \in \{1, \ldots, 6\}$, equations $(\text{II}_0)_0$ reduce to the functional equation (2.1) of $[z]$.

Theorem 4.2. *Let $\omega \in \Omega$ be a period of the function $[z]$. Let $\tau^{(0)}(x)$ and $\tau^{(1)}(x)$ be nonzero meromorphic functions on H_ω and $H_{\omega+\delta}$, respectively. Suppose that*

$$(4.5) \quad \frac{\tau^{(0)}(x \pm a_1\delta)}{\tau^{(0)}(x \pm a_2\delta)} = \frac{[(a_0 \pm a_1|x)]}{[(a_0 \pm a_2|x)]} \quad (x \in H_\omega)$$

for any C_3-frame $\{\pm a_0, \pm a_1, \pm a_2\}$ of type II_1 with $(\phi|a_0) = 1$, and

$$(4.6) \quad \begin{aligned} &[(a_1 \pm a_2|x)]\tau^{(0)}(x - a_0\delta)\tau^{(1)}(x + a_0\delta) \\ &+ [(a_2 \pm a_0|x)]\tau^{(0)}(x - a_1\delta)\tau^{(1)}(x + a_1\delta) \\ &+ [(a_0 \pm a_1|x)]\tau^{(0)}(x - a_2\delta)\tau^{(1)}(x + a_2\delta) = 0 \quad (x \in H_{\omega+\delta/2}) \end{aligned}$$

for any C_3-frame $\{\pm a_0, \pm a_1, \pm a_2\}$ with $(\phi|a_0) = (\phi|a_1) = (\phi|a_2) = \frac{1}{2}$. Then there exists a unique hypergeometric τ-function $\tau = \tau(x)$ on D_ω such that $\tau^{(n)}(x) = 0$ for $n < 0$ and

$$(4.7) \quad \tau^{(0)}(x) = \tau(x) \quad (x \in H_\omega), \quad \tau^{(1)}(x) = \tau(x) \quad (x \in H_{\omega+\delta}).$$

Proof. We apply Theorem 3.3 to nonzero meromorphic functions $\tau^{(n-1)}(x)$, $\tau^{(n)}(x)$ on $H_{\omega+(n-1)\delta}$, $H_{c+n\delta}$, for constructing $\tau^{(n+1)}(x)$ on $H_{\omega+(n+1)\delta}$ recursively for $n = 1, 2, \ldots$. At each step, we need to show that the meromorphic function $\tau^{(n+1)}(x)$ determined by Theorem 3.3 is not identically zero. If $\tau^{(n+1)}(x) \equiv 0$, the bilinear equations $(\text{II}_2)_n$ imply

$$(4.8) \quad [(a_0 \pm a_1|x)]\tau^{(n)}(x \pm a_2\delta) = 0 \quad (x \in H_{\omega+n\delta})$$
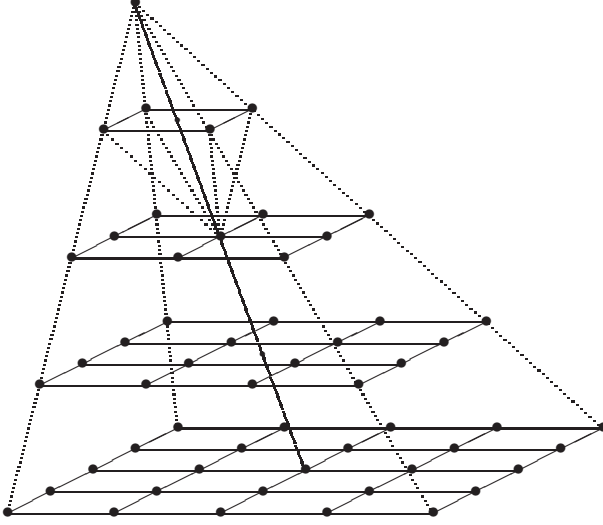
Fig. 3. 2-Directional Casorati determinants

for any C_3-frame $\{\pm a_0, \pm a_1, \pm a_2\}$ of type II_2 with $(\phi|a_0) = (\phi|a_1) = 1$, $(\phi|a_2) = 0$. Since $a_0 + a_1 = \phi$, $[(a_0 + a_1|x)] = [\omega + n\delta] \neq 0$. Also, since $[(a_0 - a_1|x)] \neq 0$, we have $\tau^{(n)}(x \pm a_2\delta) = 0$ and hence $\tau^{(n)}(x) \equiv 0$ on $H_{\omega+n\delta}$, contrarily to the hypothesis.   Q.E.D.

We now fix a C_3-frame $\{\pm a_0, \pm a_1, \pm a_2\}$ of type II_1 with $(\phi|a_0) = 1$, $(\phi|a_1) = (\phi|a_2) = 0$. Then the τ-functions $\tau^{(n)}$ on $H_{\omega+n\delta}$ for $n = 2, 3, \ldots$ are uniquely determined by the bilinear equations

(4.9)

$$(II_1)_n : \quad [(a_1 \pm a_2|x)]\tau^{(n-1)}(x - a_0\delta)\tau^{(n+1)}(x + a_0\delta)$$
$$= [(a_0 \pm a_2|x)]\tau^{(n)}(x \pm a_1\delta) - [(a_0 \pm a_1|x)]\tau^{(n)}(x \pm a_2\delta)$$

of Toda type. From this recursive structure, it follows that the τ-functions $\tau^{(n)}(x)$ are expressed in terms of *2-directional Casorati determinants*.

Theorem 4.3. *Under the assumption of Theorem 4.2, suppose that* $\tau^{(1)}(x)$ *on* $H_{\omega+\delta}$ *is expressed in the form* $\tau^{(1)}(x) = g^{(1)}(x)\psi(x)$ *with a*

nonzero meromorphic function $g^{(1)}(x)$ such that

$$(4.10) \qquad \frac{g^{(1)}(x \pm a_1\delta)}{g^{(1)}(x \pm a_2\delta)} = \frac{[(a_0 \pm a_1|x)]}{[(a_0 \pm a_2|x)]} \qquad (x \in H_{\omega+\delta}).$$

for a C_3-frame $\{\pm a_0, \pm a_1, \pm a_2\}$ of type II_1 with $(\phi|a_0) = 1$, $(\phi|a_1) = (\phi|a_2) = 0$. Then the components $\tau^{(n)}(x)$ of the hypergeometric τ-function $\tau(x)$ are expressed as follows in terms of 2-directional Casorati determinants:

$$(4.11) \qquad \begin{aligned} \tau^{(n)}(x) &= g^{(n)}(x)K^{(n)}(x), \\ K^{(n)}(x) &= \det\left(\psi_{ij}^{(n)}(x)\right)_{i,j=1}^n \quad (x \in H_{\omega+n\delta}) \end{aligned}$$

for $n = 0, 1, 2, \ldots$, where

$$(4.12) \quad \psi_{ij}^{(n)}(x) = \psi(x - (n-1)a_0\delta + (n+1-i-j)a_1\delta + (j-i)a_2\delta)$$

for $i, j = 1, \ldots, n$. The gauge factors $g^{(n)}(x)$ are determined inductively from $g^{(0)}(x) = \tau^{(0)}(x)$ and $g^{(1)}(x)$ by

$$(4.13) \quad \frac{g^{(n-1)}(x - a_0\delta)g^{(n+1)}(x + a_0\delta)}{g^{(n)}(x \pm a_1\delta)} = \frac{[(a_0 \pm a_2|x)]}{[(a_1 \pm a_2|x)]} \quad (x \in H_{\omega+n\delta})$$

for $n = 1, 2, \ldots$.

    Lemma 4.4. *The gauge factors $g^{(n)}(x)$ $(x \in H_{\omega+n\delta})$ defined by* (4.13) *satisfy*

$$(4.14) \qquad \frac{g^{(n)}(x \pm a_1\delta)}{g^{(n)}(x \pm a_2\delta)} = \frac{[(a_0 \pm a_1|x)]}{[(a_0 \pm a_2|x)]} \qquad (n = 0, 1, 2, \ldots).$$

    Proof. Formulas (4.14) for $n = 0, 1$ are included in the assumption for $g^{(0)}(x) = \tau^{(0)}(x)$ and $g^{(1)}(x)$. For $n = 1, 2, \ldots$, we show inductively that $g^{(n+1)}(x)$ defined by (4.13) satisfies this condition. From (4.14) for $g^{(n)}(x)$, we have

$$(4.15) \qquad \begin{aligned} g^{(n+1)}(x + a_0\delta) &= \frac{[(a_0 \pm a_2|x)]}{[(a_1 \pm a_2|x)]} \frac{g^{(n)}(x \pm a_1\delta)}{g^{(n-1)}(x - a_0\delta)} \\ &= \frac{[(a_0 \pm a_1|x)]}{[(a_1 \pm a_2|x)]} \frac{g^{(n)}(x \pm a_2\delta)}{g^{(n-1)}(x - a_0\delta)} \end{aligned}$$

and hence

$$(4.16) \qquad \begin{aligned} g^{(n+1)}(x) &= \frac{[(a_0 \pm a_2|x) - \delta]}{[(a_1 \pm a_2|x)]} \frac{g^{(n)}(x - a_0\delta \pm a_1\delta)}{g^{(n-1)}(x - 2a_0\delta)} \\ &= \frac{[(a_0 \pm a_1|x) - \delta]}{[(a_1 \pm a_2|x)]} \frac{g^{(n)}(x - a_0\delta \pm a_2\delta)}{g^{(n-1)}(x - 2a_0\delta)}. \end{aligned}$$

From these two expressions of $g^{(n+1)}(x)$ we obtain

(4.17)

$$g^{(n+1)}(x \pm a_2\delta) = \frac{[(a_0 \pm a_2|x)][(a_0 \pm a_2|x) - 2\delta]}{[(a_1 \pm a_2|x) \pm \delta]} \frac{g^{(n)}(x - a_0\delta \pm a_1\delta \pm a_2\delta)}{g^{(n-1)}(x - 2a_0\delta \pm a_2\delta)}$$

$$g^{(n+1)}(x \pm a_1\delta) = \frac{[(a_0 \pm a_1|x)][(a_0 \pm a_1|x) - 2\delta]}{[(a_1 \pm a_2|x) \pm \delta]} \frac{g^{(n)}(x - a_0\delta \pm a_1\delta \pm a_2\delta)}{g^{(n-1)}(x - 2a_0\delta \pm a_1\delta)}.$$

Then by (4.14) for $g^{(n-1)}(x)$ we obtain

(4.18)

$$\frac{g^{(n+1)}(x \pm a_1\delta)}{g^{(n+1)}(x \pm a_2\delta)} = \frac{[(a_0 \pm a_1|x)][(a_0 \pm a_1|x) - 2\delta]}{[(a_0 \pm a_2|x)][(a_0 \pm a_2|x) - 2\delta]} \frac{g^{(n-1)}(x - 2a_0\delta \pm a_2\delta)}{g^{(n-1)}(x - 2a_0\delta \pm a_1\delta)}$$

$$= \frac{[(a_0 \pm a_1|x)]}{[(a_0 \pm a_2|x)]}$$

as desired.   Q.E.D.

Proof of Theorem 4.3. Using the gauge factors $g^{(n)}(x)$ defined as above, we set

(4.19) $\tau^{(0)}(x) = g^{(0)}(x), \quad \tau^{(1)}(x) = g^{(1)}(x)\psi(x),$

and define $K^{(n)}(x)$ by

(4.20) $\tau^{(n)}(x) = g^{(n)}(x)K^{(n)}(x) \quad (n = 0, 1, 2, \ldots).$

Then the bilinear equation (4.9) is written as

(4.21)

$$[(a_1 \pm a_2|x)]g^{(n-1)}(x - a_0\delta)g^{(n+1)}(x + a_0\delta)$$
$$\cdot K^{(n-1)}(x - a_0\delta)K^{(n+1)}(x + a_0\delta)$$
$$= [(a_0 \pm a_2|x)]g^{(n)}(x \pm a_1\delta)K^{(n)}(x \pm a_1\delta)$$
$$- [(a_0 \pm a_1|x)]g^{(n)}(x \pm a_2\delta)K^{(n)}(x \pm a_2\delta).$$

By Lemma 4.4, we have

(4.22)
$$[(a_1 \pm a_2|x)]g^{(n-1)}(x - a_0\delta)g^{(n+1)}(x + a_0\delta)$$
$$= [(a_0 \pm a_2|x)]g^{(n)}(x \pm a_1\delta) = [(a_0 \pm a_1|x)]g^{(n)}(x \pm a_2\delta).$$

Therefore, the main factors $K^{(n)}(x)$ are determined by

(4.23) $K^{(n-1)}(x - a_0\delta)K^{(n+1)}(x + a_0\delta) = K^{(n)}(x \pm a_1\delta) - K^{(n)}(x \pm a_2\delta)$

for $n = 1, 2, \ldots$ starting from $K^{(0)}(x) = 1$, $K^{(1)}(x) = \psi(x)$. For example, we have

(4.24)

$$K^{(2)}(x + a_0\delta) = \psi(x \pm a_1\delta) - \psi(x \pm a_2\delta) = \det \begin{bmatrix} \psi(x + a_1\delta) & \psi(x + a_2\delta) \\ \psi(x - a_2\delta) & \psi(x - a_1\delta) \end{bmatrix},$$
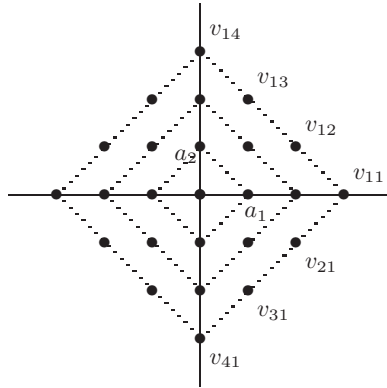
$$K^{(3)}(x + 2a_0\delta)$$

$$= \det \begin{bmatrix} \psi(x + 2a_1\delta) & \psi(x + a_1\delta + a_2\delta) & \psi(x + 2a_2\delta) \\ \psi(x + a_1\delta - a_2\delta) & \psi(x) & \psi(x - a_1\delta + a_2\delta) \\ \psi(x - 2a_2\delta) & \psi(x - a_1\delta - a_2\delta) & \psi(x - 2a_1\delta) \end{bmatrix}.$$

In general, this recurrence (4.23) for $K^{(n)}(x)$ is solved by the Lewis Carroll formula for the Casorati determinants with respect to the two directions $a_1 + a_2$ and $a_1 - a_2$. Namely, for $n = 1, 2, \ldots$, we have
(4.25)
$$K^{(n)}(x + (n - 1)a_0\delta) = \det \left(\psi(x + (n + 1 - i - j)a_1\delta + (j - i)a_2\delta) \right)_{i,j=1}^{n}$$

with the vectors $v_{ij} = (n + 1 - i - j)a_1 + (j - i)a_2$ $(i, j = 1, \ldots, n)$ arranged as follows.

(4.26)

This implies the expression (4.11) for $K^{(n)}(x)$.   Q.E.D.

§5.   Elliptic hypergeometric integrals

In this section, we recall fundamental facts concerning the elliptic hypergeometric integrals of Spiridonov [16, 17] and Rains [12, 13].

Fixing two bases $p, q \in \mathbb{C}^* = \mathbb{C} \backslash \{0\}$ with $|p| < 1$, $|q| < 1$, we use the multiplicative notations

$$(5.1) \qquad \theta(z; p) = (z; p)_\infty (p/z; p)_\infty, \qquad (z; p)_\infty = \prod_{i=0}^{\infty} (1 - p^i z),$$

for the *Jacobi theta function*, and

$$(5.2) \qquad \Gamma(z; p, q) = \frac{(pq/z; p, q)_\infty}{(z; p, q)_\infty}, \qquad (z; p, q)_\infty = \prod_{i,j=0}^{\infty} (1 - p^i q^j z)$$

for the *Ruijsenaars elliptic gamma function*. These functions satisfy the functional equations

$$(5.3) \qquad \begin{aligned} &\theta(pz; p) = -z^{-1}\theta(z; p), \quad \theta(p/z; p) = \theta(z; p), \\ &\Gamma(qz; p, q) = \theta(z; p)\Gamma(z; p, q), \qquad \Gamma(pq/u; p, q) = \Gamma(u; p, q)^{-1}. \end{aligned}$$

The multiplicative theta function $\theta(z; p)$ satisfies the three-term relation
$$(5.4)$$
$$c\,\theta(bc^{\pm 1}; p)\theta(az^{\pm 1}; p) + a\,\theta(ca^{\pm 1}; p)\theta(bz^{\pm 1}; p) + b\,\theta(ab^{\pm 1}; p)\theta(cz^{\pm 1}; p) = 0,$$

corresponding to (2.1). Here we have used the abbreviation $\theta(ab^{\pm 1}; p) = \theta(ab; p)\theta(ab^{-1}; p)$ to refer to the product of two factors with different signs. Note also that
$$(5.5)$$
$$\frac{1}{\Gamma(z^{\pm 1}; p, q)} = (1 - z^{\pm 1})(pz^{\pm 1}; p)_\infty(qz^{\pm 1}; q)_\infty = -z^{-1}\theta(z; p)\theta(z; q).$$

Following Spiridonov [16, 17], we consider the elliptic hypergeometric integral $I(u; p, q)$ in eight variables $u = (u_0, u_1, \ldots, u_7)$ defined by

$$(5.6) \qquad I(u; p, q) = \frac{(p; p)_\infty (q; q)_\infty}{4\pi\sqrt{-1}} \int_C \frac{\prod_{k=0}^{7} \Gamma(u_k z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{z}.$$

Here we assume that $u = (u_0, u_1, \ldots, u_7)$ is generic in the sense that $u_k u_l \notin p^{-\mathbb{N}} q^{-\mathbb{N}}$ for any $k, l \in \{0, 1, \ldots, 7\}$ ($\mathbb{N} = \{0, 1, 2, \ldots\}$). This condition is equivalent to saying that the two sets

$$(5.7) \qquad \begin{aligned} S_0 &= \{p^i q^j u_k \,|\, i, j \in \mathbb{N}, \ k \in \{0, 1, \ldots, 7\}\}, \\ S_\infty &= \{p^{-i} q^{-j} u_k^{-1} \,|\, i, j \in \mathbb{N}, \ k \in \{0, 1, \ldots, 7\}\} \end{aligned}$$

of possible poles of the integrand are disjoint. For the contour C we take a homology cycle in $\mathbb{C}^* \backslash (S_0 \cup S_\infty)$ such that $n(C, a) = 1$ for all $a \in S_0$ and $n(C, a) = 0$ for all $a \in S_\infty$, where $n(C; a)$ stands for the winding

number of C around $z = a$. Note also that, if $|u_k| < 1$ $(k = 0, 1, \ldots, 7)$, one can take the unit circle $|z| = 1$ oriented positively as the cycle C.

The following transformation formulas are due to Spiridonov [16] and Rains [13].

Theorem 5.1. *Suppose that the parameters $u = (u_0, u_1, \ldots, u_7)$ satisfy the balancing condition $u_0 u_1 \cdots u_7 = p^2 q^2$. Then the following transformation formulas hold:*

(5.8)

(1) $\displaystyle I(u; p, q) = I(\widetilde{u}; p, q) \prod_{0 \le i < j \le 3} \Gamma(u_i u_j; p, q) \prod_{4 \le i < j \le 7} \Gamma(u_i u_j; p, q),$

$$\widetilde{u} = (\widetilde{u}_0, \widetilde{u}_1, \ldots, \widetilde{u}_7), \quad \widetilde{u}_i = \begin{cases} u_i \sqrt{pq/u_0 u_1 u_2 u_3} & (i = 0, 1, 2, 3), \\ u_i \sqrt{pq/u_4 u_5 u_6 u_7} & (i = 4, 5, 6, 7), \end{cases}$$

and

(5.9)

(2) $\displaystyle I(u; p, q) = I(\widehat{u}; p, q) \prod_{0 \le i < j \le 7} \Gamma(u_i u_j; p, q),$

$$\widehat{u} = (\widehat{u}_0, \ldots, \widehat{u}_7), \quad \widehat{u}_i = \sqrt{pq}/u_i \quad (i = 0, 1, \ldots, 7).$$

Note that, if the parameters $u = (u_0, u_1, \ldots, u_7)$ satisfy $u_0 u_1 \cdots u_7 = p^2 q^2$ and $|u_k| = |pq|^{\frac{1}{4}}$ $(k = 0, 1, \ldots, 7)$, then both $\widetilde{u}$ and $\widehat{u}$ satisfy the two conditions as well.

Taking another base $r \in \mathbb{C}^*$ with $|r| < 1$, we set

(5.10) $\displaystyle \Psi(u; p, q, r) = I(u; p, q) \prod_{0 \le i < j \le 7} \Gamma(u_i u_j; p, q, r),$

where

$$\Gamma(z; p, q, r) = (z; p, q, r)_\infty (pqr/z; p, q, r)_\infty,$$

(5.11)
$$(z; p, q, r)_\infty = \prod_{i,j,k=0}^{\infty} (1 - p^i q^j r^k z).$$

Note that
(5.12)
$$\Gamma(rz; p, q, r) = \Gamma(z; p, q)\Gamma(z; p, q, r), \quad \Gamma(pqr/z; p, q, r) = \Gamma(z; p, q, r).$$

Proposition 5.2. *Under the condition $u_0 u_1 \cdots u_7 = p^2 q^2$, the function $\Psi(u; p, q, r)$ defined by (5.10) is invariant with respect to the transformations $u \to \widetilde{u}$ and $u \to \widehat{u}$.*

Proof. When $\{i, j, k, l\} = \{0, 1, 2, 3\}$ or $\{4, 5, 6, 7\}$, one has

(5.13)
$$\Gamma(\widetilde{u}_i\widetilde{u}_j; p, q, r) = \Gamma(pq/u_ku_l; p, q, r) = \Gamma(ru_ku_l; p, q, r)$$
$$= \Gamma(u_k, u_l; p, q)\Gamma(u_ku_l; p, q, r).$$

Also, for distinct $i, j \in \{0, 1, \ldots, 7\}$,

(5.14)
$$\Gamma(\widehat{u}_i\widehat{u}_j; p, q, r) = \Gamma(pq/u_iu_j; p, q, r) = \Gamma(ru_iu_j; p, q, r)$$
$$= \Gamma(ru_iu_j; p, q)\Gamma(u_iu_j; p, q, r).$$

Using these formulas, Theorem 5.1 can be reformulated as

(5.15)
$$\Psi(u; p, q, r) = \Psi(\widetilde{u}; p, q, r) = \Psi(\widehat{u}; p, q, r)$$

under the condition $u_0u_1 \cdots u_7 = p^2q^2$. Q.E.D.

Returning to the elliptic hypergeometric integral (5.6), we notice that the integrand

(5.16)
$$H(z, u; p, q) = \frac{\prod_{k=0}^{7}\Gamma(u_kz^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)}$$

satisfies

(5.17)
$$T_{q,u_k}H(z, u; p, q) = \theta(u_kz^{\pm 1}; p)H(z, u; p, q)$$

with respect to the q-shift operator T_{q,u_k} in u_k ($k \in \{0, 1, \ldots, 7\}$):

(5.18)
$$T_{q,u_k}f(u_0, u_1, \ldots, u_7) = f(u_0, \ldots, qu_k, \ldots, u_7).$$

Hence, by the functional equation (5.4) we have

(5.19)
$$\left(u_k\theta(u_ju_k^{\pm 1}; p)T_{q,u_i} + u_i\theta(u_ku_i^{\pm 1}; p)T_{q,u_j} + u_j\theta(u_iu_j^{\pm 1}p)T_{q,u_k}\right)H(z, u; p, q)$$
$$= 0$$

for any triple $i, j, k \in \{0, 1, \ldots, 7\}$. Passing to the integral, we obtain the following contiguity relations for the elliptic hypergeometric integral.

Proposition 5.3. *The elliptic hypergeometric integral* (5.6) *satisfies the three-term relation*

(5.20)
$$\left(u_k\theta(u_ju_k^{\pm 1}; p)T_{q,u_i} + u_i\theta(u_ku_i^{\pm 1}; p)T_{q,u_j} + u_j\theta(u_iu_j^{\pm 1}; p)T_{q,u_k}\right)I(u; p, q)$$
$$= 0$$

for any triple $i, j, k \in \{0, 1, \ldots, 7\}$.

It is known that the elliptic hypergeometric integral $I(u; p, q)$ of (5.6) gives rise to terminating elliptic hypergeometric series in the special cases where $pq/u_k u_l = p^{-M} q^{-N}$ for distinct $k, l \in \{0, 1, \ldots, 7\}$ and $M, N \in \mathbb{N}$ (see for example Komori [7]). Here we give a remark on the case where $pq/u_0 u_7 = q^{-N}$ for simplicity. We use the notation of very well-poised elliptic hypergeometric series

(5.21)

$$_{12}V_{11}(a_0; a_1, \ldots, a_7; q, p) = \sum_{k=0}^{\infty} \frac{\theta(q^{2k} a_0; p)}{\theta(a_0; p)} \left(\prod_{i=0}^{7} \frac{\theta(a_i; p; q)_k}{\theta(q a_0/a_i; p; q)_k} \right) q^k,$$

$$\theta(z; p; q)_k = \frac{\Gamma(q^k z; p, q)}{\Gamma(z; p, q)} = \theta(z; p)\theta(qz; p) \cdots \theta(q^{k-1} z; p) \quad (k = 0, 1, 2, \ldots),$$

assuming that $a_i \in p^{\mathbb{Z}} q^{-N}$ for some $i \in \{0, 1, \ldots, 7\}$ and $N = 0, 1, 2, \ldots$.

Proposition 5.4. *Under the balancing condition* $u_0 u_1 \cdots u_7 = q^2$, *we assume either* $q/u_0 u_i = q^{-N}$ *for some* $i \in \{1, \ldots, 6\}$ *or* $q/u_0 u_7 = pq^{-N}$, *where* $N = 0, 1, 2, \ldots$. *Then we have*

$$I(pu_0, u_1, \ldots, u_6, pu_7; p, q)$$

(5.22)
$$= \prod_{1 \leq k < l \leq 6} \Gamma(u_k u_l; p, q) \frac{\Gamma(q^2/u_0^2; p, q)\Gamma(u_0/u_7; p, q)}{\prod_{k=1}^{6} \Gamma(q u_i/u_0; p, q)\Gamma(q/u_i u_7; p, q)}$$
$$\cdot\, {}_{12}V_{11}(q/u_0^2; q/u_0 u_1, \ldots, q/u_0 u_6, q/u_0 u_7; q, p).$$

Sketch of proof.   Under the balancing condition $u_0 u_1 \cdots u_7 = p^2 q^2$, we set $t = (t_0, t_1, \ldots, t_7)$, $t_i = \sqrt{pq}/u_i$ $(i = 0, 1, \ldots, 7)$, so that
(5.23)
$$I(u; p, q) = I(t; p, q) \prod_{0 \leq k < l \leq 7} \Gamma(u_k u_l; p, q) = \frac{I(t; p, q)}{\prod_{0 \leq k < l \leq 7} \Gamma(t_k t_l; p, q)}.$$

We investigate the behavior of the both sides in the limit as $pq/u_0 u_7 \to q^{-N}$, namely, $t_0 t_7 \to q^{-N}$. In this limit $u_7 \to pq^{1+N}/u_0$, $t_7 \to q^{-N}/t_0$, the integral $I(u; p, q)$ on the left-hand side has a finite limit, while $I(t; p, q)$ gives rise to singularities due to pinching of the contour at

(5.24) $q^k t_7 \to q^{-N+k}/t_0, \quad q^{-k} t_7^{-1} \to q^{N-k} t_0 \quad (k = 0, 1, \ldots, N).$

By the residue calculus around these points, we can compute the limit

(5.25)
$$\lim_{t_7 \to q^{-N} t_0} \frac{I(t; p, q)}{\prod_{0 \le k < l \le 7} \Gamma(t_k t_l; p, q)}$$

$$= \frac{1}{\prod_{0 \le i < j \le 6} \Gamma(t_i t_j; p, q)} \prod_{\nu=0}^{N-1} \frac{\prod_{i=1}^{6} \theta(q^{-N+\nu} t_i / t_0; p)}{\theta(q^{-N+\nu}/t_0^2; p)}$$

$$\cdot {}_{12}V_{11}(p t_0^2; t_0 t_1, \dots, t_0 t_6, pq^{-N}; q, p).$$

Hence, under the conditions $u_0 u_1 \cdots u_7 = p^2 q^2$ and $pq/u_0 u_7 = q^{-N}$, we obtain

(5.26)

$$I(u; p, q) = \Gamma(p^2 q^2 / u_0^2; p, q) \prod_{i=1}^{7} \Gamma(u_0/u_i; p, q) \prod_{1 \le i < j \le 7} \Gamma(u_i u_j; p, q)$$

$$\cdot {}_{12}V_{11}(p^2 q / u_0^2; pq/u_0 u_1, \dots, pq/u_0 u_6, p^2 q/u_0 u_7; q, p).$$

Noting that ${}_{12}V_{11}(a_0; a_1, \dots, a_7; q, p)$ is invariant under the p-shift operator $T_{p,a_i} T_{p,a_j}^{-1}$ for distinct $i, j \in \{1, \dots, 7\}$, we see that the same formula (5.26) holds if we replace the condition "$pq/u_0 u_7 = q^{-N}$" by "$pq/u_0 u_i = q^{-N}$ for some $i \in \{1, \dots, 6\}$". Then, replacing u_0, u_7 by pu_0, pu_7 respectively, we obtain (5.22). Q.E.D.

§6. $W(E_7)$-invariant hypergeometric τ-function

In the following, we present an explicit hypergeometric τ-function for the ORG system of type E_8 in terms of elliptic hypergeometric integrals.

We denote by $x = (x_0, x_1, \dots, x_7)$ the canonical coordinates of $V = \mathbb{C}^8$ so that

(6.1)
$$x = (x_0, x_1, \dots, x_7) = x_0 v_0 + \cdots + x_7 v_7; \quad x_i = (v_i | x) \quad (i = 0, 1, \dots, 7).$$

Note that the highest root $\phi = \frac{1}{2}(v_0 + v_1 + \cdots + v_7)$ of $\Delta(E_8)$ corresponds to the linear function

(6.2)
$$(\phi | x) = \frac{1}{2}(x_0 + x_1 + \cdots + x_7).$$

We relate the additive coordinates $x = (x_0, x_1, \dots, x_7)$ and the multiplicative coordinates $u = (u_0, u_1, \dots, u_7)$ through $u_i = e(x_i) = e^{2\pi \sqrt{-1} x_i}$ $(i = 0, 1, \dots, 7)$. We also use the notation of exponential functions

(6.3)
$$u^\lambda = e((\lambda | x)) \quad (\lambda \in P),$$

so that $u^{v_i} = u_i$ $(i = 0, 1, \ldots, 7)$ and $u^\phi = (u_0 u_1 \cdots u_7)^{\frac{1}{2}}$.

We now consider the case where the fundamental function $[\zeta]$ $(\zeta \in \mathbb{C})$ is quasi-periodic with respect to the $\Omega = \mathbb{Z}1 \oplus \mathbb{Z}\varpi$, $\text{Im}(\varpi) > 0$, and is expressed as

$$(6.4) \qquad\qquad [\zeta] = z^{-\frac{1}{2}} \theta(z; p), \quad z = e(\zeta)$$

with base $p = e(\varpi)$, $|p| < 1$. This function has the quasi-periodicity

$$(6.5) \qquad [\zeta + 1] = -[\zeta], \quad [\zeta + \varpi] = -e(-\zeta - \tfrac{\varpi}{2})[\zeta]$$

and hence $\eta_1 = 0$, $\eta_\varpi = -1$. Note also that

$$(6.6) \qquad [\alpha \pm \beta] = a^{-1}\theta(ab^{\pm 1}; p) \quad (a = e(\alpha), b = e(\beta)),$$

and that the three-term relation (5.4) for $\theta(z; p)$ corresponds to (2.1) for $[\zeta]$. As to the constant $\delta \in \mathbb{C}$, we assume $\text{Im}(\delta) > 0$, and set $q = e(\delta)$ so that $|q| < 1$.

As in Section 1, we take the simple roots

$$(6.7) \qquad \alpha_0 = \phi - v_0 - v_1 - v_2 - v_3, \quad \alpha_j = v_j - v_{j+1} \quad (j = 1, \ldots, 6)$$

for the root system $\Delta(E_7)$. Since $v_1 - v_0$ is the highest root, we see that the Weyl group $W(E_7)$ is generated by $\mathfrak{S}_8 = \langle r_{v_j - v_{j+1}}(j = 0, 1, \ldots, 6)\rangle$ and the reflection $s_0 = r_{\alpha_0}$ by $\alpha_0 = r_{\phi - v_0 - v_1 - v_2 - v_3}$. The symmetric group $\mathfrak{S}_8$ acts on the coordinates $x = (x_0, x_1, \ldots, x_7)$ and $u = (u_0, u_1, \ldots, u_7)$ through the permutation of indices, while s_0 acts on the additive coordinates as

$$(6.8) \quad s_0(x_i) = \begin{cases} x_i + \frac{1}{2}((\phi|x) - x_0 - x_1 - x_2 - x_3) & (i = 0, 1, 2, 3), \\ x_i + \frac{1}{2}((\phi|x) - x_4 - x_5 - x_6 - x_7) & (i = 4, 5, 6, 7), \end{cases}$$

and on the multiplicative coordinates $u_0, u_1, \ldots, u_7$ as

$$(6.9) \qquad s_0(u_i) = \begin{cases} u_i(u^\phi/u_0 u_1 u_2 u_3)^{\frac{1}{2}} & (i = 0, 1, 2, 3), \\ u_i(u^\phi/u_4 u_5 u_6 u_7)^{\frac{1}{2}} & (i = 4, 5, 6, 7). \end{cases}$$

We now restrict the coordinates x_i and u_i to the level set

$$(6.10) \quad H_\kappa = \left\{ x \in V \,\middle|\, (\phi|x) = \tfrac{1}{2}(x_0 + x_1 + \cdots + x_7) = \kappa \right\} \quad (\kappa \in \mathbb{C})$$

so that $u^\phi = (u_0 u_1 \cdots u_7)^{\frac{1}{2}} = e(\kappa)$. Then the action of s_0 is given by

$$(6.11) \quad s_0(x_i) = \begin{cases} x_i + \frac{1}{2}(\kappa - x_0 - x_1 - x_2 - x_3) & (i = 0, 1, 2, 3), \\ x_i + \frac{1}{2}(\kappa - x_4 - x_5 - x_6 - x_7) & (i = 4, 5, 6, 7). \end{cases}$$

and by

$$(6.12) \qquad s_0(u_i) = \begin{cases} u_i\sqrt{e(\kappa)/u_0 u_1 u_2 u_3} & (i = 0, 1, 2, 3), \\ u_i\sqrt{e(\kappa)/u_4 u_5 u_6 u_7} & (i = 4, 5, 6, 7). \end{cases}$$

respectively. Suppose $\kappa = \varpi + \delta$ so that $e(\kappa) = e(\varpi + \delta) = pq$. In this case, we have $u_0 u_1 \cdots u_7 = p^2 q^2$ on $H_{\varpi+\delta}$, and the action of s_0 coincides with the transformation $u_i \to \widetilde{u}_i$ in (5.8). Proposition 5.2 thus implies that the function

$$(6.13) \qquad \Psi(u; p, q, r) = I(u; p, q) \prod_{0 \le k < l \le 7} \Gamma(u_k u_l; p, q, r)$$

regarded as a function on $H_{\varpi+\delta}$, $u^\phi = pq$, is invariant under the action of s_0. Since $\Psi(u; p, q, r)$ is manifestly symmetric with respect to $u = (u_0, u_1, \ldots, u_7)$, we see that $\Psi(u; p, q, r)$ is a $W(E_7)$-invariant meromorphic function on $H_{\varpi+\delta}$. We remark that the transformation $u_i \to \widehat{u}_i$ in (5.9) coincides with the action of

$$(6.14) \qquad w = r_{07} r_{12} r_{34} r_{56} r_{0127} r_{0347} r_{0567} \in W(E_7),$$

where $r_{ij} = r_{v_i - v_j}$ and $r_{ijkl} = r_{\phi - v_i - v_j - v_k - v_l}$. This means that the transformation formula (2) of Theorem 5.1 follows from (1).

We rewrite the contiguity relations of (5.20) as

$$(6.15)$$
$$u_j^{-1}\theta(u_j u_k^{\pm 1}; p) u_i^{-1} T_{q,u_i} I(u; p, q) + u_k^{-1}\theta(u_k u_i^{\pm 1}; p) u_j^{-1} T_{q,u_j} I(u; p, q)$$
$$+ u_i^{-1}\theta(u_i u_j^{\pm 1}; p) u_k^{-1} T_{q,u_k} I(u; p, q) = 0.$$

Since $u_i^{-1}\theta(u_i u_j^{\pm 1}; p) = [x_i \pm x_j]$, this means that

$$(6.16) \qquad \begin{aligned} [x_j \pm x_k] u_i^{-1} T_{q,u_i} I(u; p, q) + [x_k \pm x_i] u_j^{-1} T_{q,u_j} I(u; p, q) \\ + [x_i \pm x_j] u_k^{-1} T_{q,u_k} I(u; p, q) = 0. \end{aligned}$$

In view of this formula, we set

$$(6.17) \qquad J(x) = e(-Q(x)) I(u; p, q), \quad Q(x) = \tfrac{1}{2\delta}(x|x).$$

Note that

$$(6.18) \qquad Q(x + a\delta) = Q(x) + (a|x) + \tfrac{1}{2}(a|a)\delta$$

for any $a \in P$. Since

$$(6.19) \qquad Q(x + v_i\delta) = Q(x) + (v_i|x) + \tfrac{1}{2}\delta = Q(x) + x_i + \tfrac{1}{2}\delta$$

we have
(6.20)
$$J(x + v_i\delta) = e(-Q(x))\, q^{-\frac{1}{2}} u_i^{-1} T_{q,u_i} I(u; p, q) \quad (i \in \{0, 1, \ldots, 7\}).$$

Hence by (6.16) we obtain the three-term relations

(6.21) $[x_j \pm x_k]J(x + v_i\delta) + [x_k \pm x_i]J(x + v_j\delta) + [x_i \pm x_j]J(x + v_k\delta) = 0,$

namely
(6.22)
$$[(v_j \pm v_k|x)]J(x + v_i\delta) + [(v_k \pm v_i|x)]J(x + v_j\delta) + [(v_i \pm v_j|x)]J(x + v_k\delta) = 0$$

for any triple $i, j, k \in \{0, 1, \ldots, 7\}$.

On the basis of these observations, we construct a hypergeometric τ-function on

(6.23)
$$D_\varpi = \bigsqcup_{n \in \mathbb{Z}} H_{\varpi + n\delta} \subset V$$

with the initial level $c = \varpi$. For this purpose we introduce the holomorphic function

(6.24)
$$\mathcal{F}(x) = \prod_{0 \le i < j \le 7} \Gamma(u_i u_j; p, q, q) \qquad (x \in V).$$

Theorem 6.1. *There exists a unique hypergeometric τ-function* $\tau(x)$ *on* D_ϖ *such that* $\tau^{(n)}(x) = 0$ *($n < 0$) and*

(6.25)
$$\tau^{(0)}(x) = \mathcal{F}(x + \phi\delta) = \prod_{0 \le i < j \le 7} \Gamma(qu_i u_j; p, q, q) \quad (x \in H_\varpi),$$

$$\tau^{(1)}(x) = \mathcal{F}(x)J(x)$$
$$= e(-Q(x))I(u; p, q) \prod_{0 \le i < j \le 7} \Gamma(u_i u_j; p, q, q) \quad (x \in H_{\varpi + \delta}).$$

Furthermore $\tau(x)$ *is a* $W(E_7)$-*invariant meromorphic function on* D_ϖ.

Proof. We need to show that $\tau^{(0)}(x)$ and $\tau^{(1)}(x)$ satisfy the two conditions of Theorem 4.2. We first show that $\tau^{(0)}(x)$ ($x \in H_\varpi$) is $W(E_7)$-invariant. Since the $\mathfrak{S}_8$-invariance is manifest, we have only to show that it is invariant under the action of s_0. Noting that

(6.26) $\qquad s_0(u_i u_j) = \begin{cases} p/u_k u_l & (\{i, j, k, l\} = \{0, 1, 2, 3\}), \\ u_i u_j & (i \in \{0, 1, 2, 3\}, j \in \{4, 5, 6, 7\}), \\ p/u_k u_l & (\{i, j, k, l\} = \{4, 5, 6, 7\}) \end{cases}$

for $x \in H_\varpi$, we have

(6.27) $s_0\big(\Gamma(qu_iu_j;p,q,q)\big) = \Gamma(pq/u_ku_l;p,q,q) = \Gamma(qu_ku_l;p,q,q)$

for $\{i,j,k,l\} = \{0,1,2,3\}$ or $\{4,5,6,7\}$. Since $\Gamma(u_iu_j;p,q,q)$ is s_0-invariant for $i \in \{0,1,2,3\}$, $j \in \{4,5,6,7\}$, we have $s_0(\tau^{(0)}(x)) = \tau^{(0)}(x)$. We now verify that our $\tau^{(0)}(x)$ satisfy the condition (4.5) for any C_3-frame $\{\pm a_0,\pm a_1,\pm a_2\}$ of type II_1 as in (3.15). Since the C_3-frames of type II_1 form a single $W(E_7)$-orbit, by the $W(E_7)$-invariance of $\tau^{(0)}(x)$ we may take

(6.28)
$$a_0 = \tfrac{1}{2}(v_0 + v_1 + v_2 + v_3),$$
$$a_1 = \tfrac{1}{2}(v_0 + v_1 - v_2 - v_3), \quad a_2 = \tfrac{1}{2}(v_0 - v_1 + v_2 - v_3)$$

so that

(6.29)
$$\{a_0 \pm a_1\} = \{v_0 + v_1, v_2 + v_3\}, \quad \{a_0 \pm a_2\} = \{v_0 + v_2, v_1 + v_3\}.$$

In this case one can directly check

(6.30)
$$\frac{\tau^{(0)}(x \pm a_1\delta)}{\tau^{(0)}(x \pm a_2\delta)} = \frac{\theta(u_0u_1;p)\theta(u_2u_3;p)}{\theta(u_0u_2;p)\theta(u_1u_3;p)} = \frac{[x_0 + x_1][x_2 + x_3]}{[x_0 + x_2][x_1 + x_3]}$$
$$= \frac{[(a_0 \pm a_1|x)]}{[(a_0 \pm a_2|x)]}.$$

We next verify that $\tau^{(0)}(x)$ and $\tau^{(1)}(x)$ satisfy (4.6) for any C_3-frame $\{\pm a_0,\pm a_1,\pm a_2\}$ with $(\phi|a_i) = \tfrac{1}{2}$ $(i = 0,1,2)$. Since $\tau^{(1)}(x) = e(-Q(x))\Psi(u;p,q,q)$ is $W(E_7)$-invariant, we have only to check (4.6) for $\{\pm a_0,\pm a_1,\pm a_2\} = \{\pm v_0,\pm v_1,\pm v_2\}$, namely,

(6.31)
$$[x_1 \pm x_2]\tau^{(0)}(x - v_0\delta)\tau^{(1)}(x + v_0\delta) + [x_2 \pm x_0]\tau^{(0)}(x - v_1\delta)\tau^{(1)}(x + v_1\delta)$$
$$+ [x_0 \pm x_1]\tau^{(0)}(x - v_2\delta)\tau^{(1)}(x + v_2\delta) = 0.$$

Since

(6.32) $\mathcal{F}(x + \phi\delta - v_k\delta)\mathcal{F}(x + v_k\delta) = \mathcal{F}(x + \phi\delta)\mathcal{F}(x),$

we have

(6.33)
$$\tau^{(0)}(x - v_k\delta)\tau^{(1)}(x + v_k\delta) = \mathcal{F}(x + \phi\delta - v_k\delta)\mathcal{F}(x + v_k\delta)J(x + v_k\delta)$$
$$= \mathcal{F}(x + \phi\delta)\mathcal{F}(x)J(x + v_k\delta)$$

for each $k = 0, 1, \ldots, 7$. Hence the three-term relations (6.21) for $J(x)$ imply

(6.34)
$$[x_j \pm x_k]\tau^{(0)}(x - v_i\delta)\tau^{(1)}(x + v_i\delta) + [x_k \pm x_i]\tau^{(0)}(x - v_j\delta)\tau^{(1)}(x + v_j\delta)$$
$$+ [x_i \pm x_j]\tau^{(0)}(x - v_k\delta)\tau^{(1)}(x + v_j\delta) = 0$$

for any tripe $i, j, k \in \{0, 1, \ldots, 7\}$. The $W(E_7)$-invariance of $\tau(x)$ on D_ϖ follows from the uniqueness of $\tau(x)$ and the $W(E_7)$-invariance of $\tau^{(0)}(x)$ and $\tau^{(1)}(x)$.   Q.E.D.

We next investigate the determinant formula of Theorem 4.3 for the hypergeometric τ-function of Theorem 6.1 with initial condition

(6.35)
$$\tau^{(0)}(x) = \prod_{0 \le i < j \le 7} \Gamma(qu_iu_j; p, q, q) \quad (x \in H_\varpi),$$

$$\tau^{(1)}(x) = e(-Q(x))\, I(u; p, q) \prod_{0 \le i < j \le 7} \Gamma(u_iu_j; p, q, q) \quad (x \in H_{\varpi+\delta}).$$

For the recursive construction of $\tau^{(n)}(x)$ $(x \in H_{\varpi+n\delta})$ for $n = 2, 3, \ldots$, we use the C_8-frame $A_1 = \{\pm a_0, \pm a_1, \ldots, \pm a_7\}$ of type II of Example 1.2, where

(6.36)
$$\begin{aligned}
a_0 &= \tfrac{1}{2}(v_0 + v_1 + v_2 + v_3), & a_4 &= \tfrac{1}{2}(v_4 - v_5 - v_6 + v_7), \\
a_1 &= \tfrac{1}{2}(v_0 + v_1 - v_2 - v_3), & a_5 &= \tfrac{1}{2}(-v_4 + v_5 - v_6 + v_7). \\
a_2 &= \tfrac{1}{2}(v_0 - v_1 + v_2 - v_3), & a_6 &= \tfrac{1}{2}(-v_4 - v_5 + v_6 + v_7), \\
a_3 &= \tfrac{1}{2}(v_0 - v_1 - v_2 + v_3), & a_7 &= \tfrac{1}{2}(v_4 + v_5 + v_6 + v_7).
\end{aligned}$$

Note here that $(\phi|a_0) = (\phi|a_7) = 1$, $(\phi|a_i) = 0$ $(i = 1, \ldots, 6)$ and $a_0 + a_7 = \phi$. This C_8-frame A_1 contains the following 30 C_3-frame of type II$_1$:

(6.37) $\{\pm a_0, \pm a_i, \pm a_j\}$, $\{\pm a_7, \pm a_i, \pm a_j\}$ $(1 \le i < j \le 6)$.

Since

(6.38) $\alpha_0 = \phi - v_0 - v_1 - v_2 - v_3 = a_7 - a_0,$

we have

(6.39) $s_0(a_0) = a_7$, $s_0(a_7) = a_0$, $s_0(a_i) = a_i$ $(i = 1, \ldots, 6).$

This means that the C_3-frames $\{\pm a_0, \pm a_i, \pm a_j\}$ and $\{\pm a_7, \pm a_i, \pm a_j\}$ are transformed to each other by s_0. To fix the idea, we consider below the cases of C_3-frames $\{\pm a_0, \pm a_1, \pm a_2\}$ and $\{\pm a_7, \pm a_1, \pm a_2\}$.

Theorem 6.2 (Determinant formula). *The $W(E_7)$-invariant hypergeometric τ-function of Theorem 6.1 is expressed in terms of 2-directional Casorati determinants as*

$$(6.40) \qquad \tau^{(n)}(x) = g^{(n)}(x) \det \left(\psi_{ij}^{(n)}(x) \right)_{i,j=1}^n \quad (x \in H_{\varpi + n\delta})$$

for $n = 0, 1, 2, \ldots$, where the gauge factors $g^{(n)}(x)$ and the matrix elements $\psi_{ij}^{(n)}(x)$ are given as follows according to the C_3-frame of type II_1 chosen for the recurrence.
(1) *Case of the C_3-frame $\{\pm a_0, \pm a_1, \pm a_2\}$:*

(6.41)

$$g^{(n)}(x) = \frac{p^{\binom{n}{2}} e(-nQ(x))}{d^{(n)}(x)}$$

$$\prod_{\substack{0 \le i < j \le 3 \\ \text{or } 4 \le i < j \le 7}} \Gamma(q u_i u_j; p, q, q) \prod_{\substack{0 \le i \le 3 \\ 4 \le j \le 7}} \Gamma(q^{1-n} u_i u_j; p, q, q),$$

$$d^{(n)}(x) = q^{2\binom{3}{2}} (pq/u_0 u_1)^{\binom{n}{2}} \prod_{k=1}^n \theta(q^{1-n} u_0 u_3; p; q)_{k-1} \theta(q^{k-n} u_0/u_3; p; q)_{k-1},$$

$$\cdot \prod_{k=1}^n \theta(q^{1-n} u_1 u_2; p; q)_{k-1} \theta(q^{k-n} u_1/u_2; p; q)_{k-1},$$

$$\psi_{ij}^{(n)}(x) = I(q^{n-i} t_0, q^{n-j} t_1, q^{j-1} t_2, q^{i-1} t_3, t_4, t_5, t_6, t_7; p, q),$$

$$t_k = \begin{cases} u_k \sqrt{pq/u_0 u_1 u_2 u_3} & (k = 0, 1, 2, 3), \\ u_k \sqrt{pq/u_4 u_5 u_6 u_7} & (k = 4, 5, 6, 7). \end{cases}$$

(2) *Case of the C_3-frame $\{\pm a_7, \pm a_1, \pm a_2\}$:*

(6.42)

$$g^{(n)}(x) = \frac{p^{\binom{n}{2}} e(-nQ(x))}{d^{(n)}(x)} \prod_{0 \le i < j \le 7} \Gamma(q^{1-n} u_i u_j; p, q, q),$$

$$d^{(n)}(x) = q^{-\binom{n+1}{3}} (u_2 u_3)^{\binom{n}{2}} \prod_{k=1}^n \theta(q^{1-k} u_0 u_3; p; q)_{k-1} \theta(q^{k-n} u_0/u_3; p; q)_{k-1}$$

$$\cdot \prod_{k=1}^n \theta(q^{1-k} u_1 u_2; p; q)_{k-1} \theta(q^{k-n} u_1/u_2; p; q)_{k-1},$$

$$\psi_{ij}^{(n)}(x) = I(q^{n-i} t_0, q^{n-j} t_1, q^{1-j} t_2, q^{1-i} t_3, t_4, t_5, t_6, t_7; p, q),$$

$$t_k = q^{\frac{1}{2}(1-n)} u_k.$$

Rewriting the 2-directional Casorati determinants above, we obtain expressions of the $W(E_7)$-invariant hypergeometric τ-function in terms of multiple elliptic hypergeometric integrals. For the variables $t = (t_0, t_1, \ldots, t_7)$, we consider the multiple integrals as in Rains [12, 13]:

$$
\begin{aligned}
(6.43) \quad I_n(t; p, q) &= I_n(t_0, t_1, \ldots, t_7; p, q) \\
&= \frac{(p;p)_\infty^n (q;q)_\infty^n}{2^n n! (2\pi\sqrt{-1})^n} \int_{C^n} \prod_{i=1}^n \frac{\prod_{k=0}^7 \Gamma(t_k z_i^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 2}; p, q)} \\
&\qquad \cdot \prod_{1 \le i < j \le n} \theta(z_i^{\pm 1} z_j^{\pm 1}; p) \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n}.
\end{aligned}
$$

We remark that $I_n(t; p, q)$ is a special case (with $s = q$) of the BC_n elliptic hypergeometric integral

$$
\begin{aligned}
(6.44) \quad & \frac{(p;p)_\infty^n (q;q)_\infty^n}{2^n n! (2\pi\sqrt{-1})^n} \int_{C^n} \prod_{i=1}^n \frac{\prod_{k=0}^7 \Gamma(t_k z_i^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 2}; p, q)} \\
&\qquad \cdot \prod_{1 \le i < j \le n} \frac{\Gamma(s z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n}
\end{aligned}
$$

of type II.

Theorem 6.3 (Multiple integral representation). *The $W(E_7)$-invariant hypergeometric τ-function of Theorem 6.1 is expressed as follows in terms of multiple elliptic hypergeometric integrals:*

$$
(6.45)
$$
$$
\tau^{(n)}(x) = p^{\binom{n}{2}} e(-nQ(x)) \, I_n(q^{\frac{1}{2}(1-n)} u; p, q) \prod_{0 \le i < j \le 7} \Gamma(q^{1-n} u_i u_j; p, q, q)
$$

$$
= p^{\binom{n}{2}} e(-nQ(x)) \, I_n(\widetilde{u}; p, q)
$$
$$
\prod_{\substack{0 \le i < j \le 3 \\ \text{or } 4 \le i < j \le 7}} \Gamma(q u_i u_j; p, q, q) \prod_{\substack{0 \le i \le 3 \\ 4 \le j \le 7}} \Gamma(q^{1-n} u_i u_j; p, q, q)
$$

$(x \in H_{\varpi + n\delta}, \ n = 0, 1, 2, \ldots),$

where

$$
(6.46) \qquad \widetilde{u}_k = \begin{cases} u_k \sqrt{pq/u_0 u_1 u_2 u_3} & (k = 0, 1, 2, 3), \\ u_k \sqrt{pq/u_4 u_5 u_6 u_7} & (k = 4, 5, 6, 7). \end{cases}
$$

Theorems 6.2 and 6.3 will be proved in the next section.

We remark here that the equality of two expressions in (6.45) implies the transformation formula

(6.47)
$$I_n(t;p,q) = I_n(\widetilde{t};p,q) \prod_{0 \le i < j \le 3} \frac{\Gamma(q^n t_i t_j; p, q, q)}{\Gamma(t_i t_j; p, q, q)} \prod_{4 \le i < j \le 7} \frac{\Gamma(q^n t_i t_j; p, q, q)}{\Gamma(t_i t_j; p, q, q)}$$

for the hypergeometric integral $I_n(t;p,q)$ on $H_{\varpi + (2-n)\delta}$ $(t_0 t_1 \cdots t_7 = p^2 q^{4-2n})$, where

(6.48)
$$\widetilde{t} = (\widetilde{t}_0, \widetilde{t}_1, \ldots, \widetilde{t}_7), \quad \widetilde{t}_k = s_0(t_k) = \begin{cases} t_k \sqrt{pq^{2-n}/t_0 t_1 t_2 t_3} & (k = 0, 1, 2, 3), \\ t_k \sqrt{pq^{2-n}/t_4 t_5 t_6 t_7} & (k = 4, 5, 6, 7). \end{cases}$$

This is a special case of a transformation formula of Rains [13]. Note also, that the meromorphic function

(6.49)
$$\Psi_n(t;p,q) = I_n(t;p,q) \prod_{0 \le i < j \le 7} \Gamma(t_i t_j; p, q, q)$$

on $H_{\varpi + (2-n)\delta}$ is $W(E_7)$-invariant. The invariance of $\Psi_n(t;p,q)$ with respect to w of (6.14) gives rise to the transformation formula

(6.50)
$$I_n(t;p,q) = I_n(\widehat{t};p,q) \prod_{0 \le i < j \le 7} \frac{\Gamma(q^n t_i t_j; p, q, q)}{\Gamma(t_i t_j; p, q, q)}$$

$$\widehat{t} = (\widehat{t}_0, \widehat{t}_1, \ldots, \widehat{t}_7), \quad \widehat{t}_k = w(t_k) = \sqrt{pq^{2-n}}/t_k \quad (k = 0, 1, \ldots, 7)$$

under the condition $t_0 t_1 \cdots t_7 = p^2 q^{4-2n}$.

Applying this transformation formula, we obtain another expression of $\tau(x)$ of Theorem 6.3:

(6.51)
$$\tau^{(n)}(x) = p^{\binom{n}{2}} e(-nQ(x)) I_n(\sqrt{pq}u^{-1};p,q) \prod_{0 \le i < j \le 7} \Gamma(qu_i u_j; p, q, q)$$

$$= p^{\binom{n}{2}} e(-nQ(x)) I_n(\sqrt{pq}u^{-1};p,q) \prod_{0 \le i < j \le 7} \Gamma(pq/u_i u_j; p, q, q).$$

In the notation of (6.49), the $W(E_7)$-invariant hypergeometric τ-function is expressed as

(6.52)
$$\tau^{(n)}(x) = p^{\binom{n}{2}} e(-nQ(x)) \Psi_n(q^{\frac{1}{2}(1-n)}u;p,q)$$

$$= p^{\binom{n}{2}} e(-nQ(x)) \Psi_n(p^{\frac{1}{2}} q^{\frac{1}{2}} u^{-1};p,q)$$

for $x \in H_{\varpi + n\delta}$ $(n = 0, 1, 2, \ldots)$.

§7.   Proof of Theorems 6.2 and 6.3

In this section we prove Theorems 6.2 and 6.3 simultaneously.

We take the C_3-frame $\{\pm a_0, \pm a_1, \pm a_2\}$ of type II_1 as in (6.36) for constructing $\tau^{(n)}(x)$ $n = 2, 3, \ldots$. In this case, we have

(7.1)
$$\{a_0 \pm a_1\} = \{v_0 + v_1, v_2 + v_3\}, \quad \{a_0 \pm a_2\} = \{v_0 + v_2, v_1 + v_3\}$$
$$\{a_1 \pm a_2\} = \{v_0 - v_3, v_1 - v_2\}.$$

Hence, in order to apply Theorem 4.3, we need to decompose $\tau^{(1)}(x)$ into the product

(7.2)
$$\tau^{(1)}(x) = g^{(1)}(x)\psi(x) \quad (x \in H_{\varpi + \delta})$$

with a gauge factor satisfying
(7.3)
$$\frac{g^{(1)}(x \pm a_1\delta)}{g^{(1)}(x \pm a_2\delta)} = \frac{[(a_0 \pm a_1|x)]}{[(a_0 \pm a_2|x)]} = \frac{\theta(u_0 u_1; p)\theta(u_2 u_3; p)}{\theta(u_0 u_2; p)\theta(u_1 u_3; p)} \quad (x \in H_{\varpi + \delta}).$$

Then the gauge factors $g^{(n)}(x)$ $n = 2, 3, \ldots$ are determined by the recurrence formula

(7.4)
$$\frac{g^{(n-1)}(x - a_0\delta)g^{(n+1)}(x + a_0\delta)}{g^{(n)}(x \pm a_1\delta)} = \frac{[(a_0 \pm a_2|x)]}{[(a_1 \pm a_2|x)]}$$
$$= \frac{\theta(u_0 u_2; p)\theta(u_1 u_3; p)}{u_2 u_3 \theta(u_0/u_3; p)\theta(u_1/u_2; p)}$$

for $n = 1, 2, \ldots$ starting from $g^{(0)}(x) = \tau^{(0)}(x)$ and $g^{(1)}(x)$.

For this purpose, using the transformation formula (1) of Theorem 5.1, we rewrite $\tau^{(1)}(x)$ as

(7.5)
$$\tau^{(1)}(x) = e(-Q(x))\,I(\widetilde{u}; p, q) \prod_{0 \le i < j \le 7} \Gamma(u_i u_j; p, q, q) \prod_{\substack{0 \le i < j \le 3 \\ \text{or } 4 \le i < j \le 7}} \Gamma(u_i u_j; p, q, q)$$

$$= e(-Q(x))\,I(\widetilde{u}; p, q) \prod_{\substack{0 \le i < j \le 3 \\ \text{or } 4 \le i < j \le 7}} \Gamma(q u_i u_j; p, q, q) \prod_{\substack{0 \le i \le 3 \\ 4 \le j \le 7}} \Gamma(u_i u_j; p, q, q),$$

where
(7.6)
$$\widetilde{u} = (\widetilde{u}_0, \widetilde{u}_1, \ldots, \widetilde{u}_7), \quad \widetilde{u}_i = \begin{cases} u_i \sqrt{pq/u_0 u_1 u_2 u_3} & (i = 0, 1, 2, 3), \\ u_i \sqrt{pq/u_4 u_5 u_6 u_7} & (i = 4, 5, 6, 7), \end{cases}$$

and set

(7.7)
$$g^{(1)}(x) = e(-Q(x)) \prod_{\substack{0 \le i < j \le 3 \\ \text{or } 4 \le i < j \le 7}} \Gamma(qu_i u_j; p, q, q) \prod_{0 \le i \le 3, 4 \le j \le 7} \Gamma(u_i u_j; p, q),$$

$$\psi(x) = I(\widetilde{u}; p, q) \quad (x \in H_{\varpi + \delta}).$$

Then one can directly verify that $g^{(1)}(x)$ satisfies the condition (7.3). For the moment we set $t = (t_0, t_1, \ldots, t_7)$, $t_i = \widetilde{u}_i$ $(i = 0, 1, \ldots, 7)$, so that $\psi(x) = I(t; p, q)$.

We now compute the determinant

$$K^{(n)}(x) = \det\left(\psi_{ij}^{(n)}(x)\right)_{i,j=1}^n, \quad \psi_{ij}^{(n)}(x) = \psi(x + v_{ij}^{(n)}\delta),$$

(7.8)
$$v_{ij}^{(n)} = (1 - n)a_0 + (n + 1 - i - j)a_1 + (j - i)a_2$$
$$= (1 - i, 1 - j, j - n, i - n, 0, 0, 0, 0).$$

Noting that the multiplicative coordinates of $x + v_{ij}^{(n)}\delta$ are given by

(7.9)
$$(q^{1-i}u_0, q^{1-j}u_1, q^{j-n}u_2, q^{i-n}u_3, u_4, u_5, u_6),$$

we obtain

(7.10)
$$\psi_{ij}^{(n)}(x) = I(q^{n-i}t_0, q^{n-j}t_1, q^{j-1}t_2, q^{i-1}t_3, t_4, t_5, t_6, t_7; p, q).$$

Hence $\psi_{ij}^{(n)}(x)$ is expressed as

$$\psi_{ij}^{(n)}(x) = \kappa \int_C h(z) f_i(z) g_j(z) \frac{dz}{z}, \quad \kappa = \frac{(p; p)_\infty (q; q)_\infty}{4\pi\sqrt{-1}},$$

(7.11)
$$h(z) = \frac{\prod_{k=0}^7 \Gamma(t_k z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)},$$

$$f_i(z) = \theta(t_0 z^{\pm 1}; p; q)_{n-i} \, \theta(t_3 z^{\pm 1}; p; q)_{i-1},$$

$$g_j(z) = \theta(t_1 z^{\pm 1}; p; q)_{n-j} \, \theta(t_2 z^{\pm 1}; p; q)_{j-1},$$

for $i, j = 1, 2, \ldots, n$, where $\theta(z; p; q)_k = \theta(z; p)\theta(qz; p)\cdots\theta(q^{k-1}z; p)$ $(k = 0, 1, 2, \ldots)$. We rewrite the determinant $K^{(n)}(x) = \det\left(\psi_{ij}^{(n)}(x)\right)_{i,j=1}^n$

as

(7.12)

$$K^{(n)}(x) = \frac{1}{n!} \sum_{\sigma,\tau \in \mathfrak{S}_n} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) \prod_{k=1}^{n} \psi_{\sigma(k),\tau(k)}^{(n)}(x)$$

$$= \frac{\kappa^n}{n!} \sum_{\sigma,\tau \in \mathfrak{S}_n} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) \int_{C^n} \prod_{k=1}^{n} h(z_k) f_{\sigma(k)}(z_k) g_{\sigma(k)}(z_k) \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n}$$

$$= \frac{\kappa^n}{n!} \int_{C^n} h(z_1) \cdots h(z_n) \det(f_i(z_j))_{i,j=1}^{n} \det(g_i(z_j))_{i,j=1}^{n} \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n}.$$

Then the determinants $\det(f_j(z_i))_{i,j=1}^{n}$, $\det(g_j(z_i))_{i,j=1}^{n}$ can be evaluated by means of Warnaar's elliptic extension of the Krattenthaler determinant [18] (see also [9]).

Lemma 7.1 (Warnaar [18]). *For a set of complex variables* $(z_1, \ldots, z_n)$ *and two parameters* a, b, *one has*

$$\det\left(\theta(az_i^{\pm 1}; p; q)_{j-1}\theta(bz_i^{\pm 1}; p; q)_{n-j}\right)_{i,j=1}^{n}$$

(7.13)

$$= q^{\binom{n}{3}} a^{\binom{n}{2}} \prod_{k=1}^{n} \theta(b(q^{k-1}a)^{\pm 1}; p; q)_{n-k} \prod_{1 \le i < j \le n} z_i^{-1}\theta(z_i z_j^{\mp}; p).$$

Sketch of proof. The left-hand side is invariant under the inversion $z_i \to z_i^{-1}$ for each $i = 1, \ldots, n$, and alternating with respect to $(z_1, \ldots, z_n)$. Hence it is divisible by $\prod_{1 \le i < j \le n} z_i^{-1}\theta(z_i z_j^{\mp}; p)$. Also, the ratio of these two functions is elliptic with respect to the additive variable ζ_i with $z_i = e(\zeta_i)$ for each $i = 1, \ldots, n$, and hence is constant. The constant on the right-hand side is determined by the substitution $z_i = q^{i-1}a$ $(i = 1, \ldots, n)$ which makes the matrix on the left-hand side lower triangular.   Q.E.D.

We thus obtain

(7.14) $$K^{(n)}(x) = \det\left(\psi_{ij}^{(n)}(x)\right)_{i,j=1}^{n} = d^{(n)}(x)\, I_n(t; p, q)$$

where
(7.15)

$$I_n(t; p, q) = \frac{(p; p)_{\infty}^{n}(q; q)_{\infty}^{n}}{2^n n!(2\pi\sqrt{-1})^n} \int_{C^n} \prod_{i=1}^{n} h(z_i) \prod_{1 \le i < j \le n} \theta(z_i^{\pm 1} z_j^{\pm 1}; p) \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n},$$

and

(7.16)

$$d^{(n)}(x) = q^{2\binom{n}{3}}(t_2 t_3)^{\binom{n}{2}} \prod_{k=1}^{n} \theta(t_0(q^{k-1}t_3)^{\pm 1}; p; q)_{n-k}\theta(t_1(q^{k-1}t_2)^{\pm 1}; p; q)_{n-k}$$

$$= q^{2\binom{n}{3}}(pq/u_0 u_1)^{\binom{n}{2}}$$

$$\prod_{(i,j)=(0,3),(1,2)} \prod_{k=1}^{n} \theta(q^{1-n}u_i u_j; p; q)_{k-1}\theta(q^{k-n}u_i/u_k; p; q)_{k-1}.$$

Finally we determine the gauge factors $g^{(n)}(x)$ $n = 2, 3, \ldots$. We set

$$(7.17) \quad \mathcal{G}^{(n)}(x) = \prod_{\substack{0 \le i < j \le 3 \\ \text{or } 4 \le i < j \le 7}} \Gamma(q u_i u_j; p, q, q) \prod_{\substack{0 \le i \le 3 \\ 4 \le j \le 7}} \Gamma(q^{1-n}u_i u_j; p, q, q),$$

so that $g^{(0)}(x) = \mathcal{G}^{(0)}(x)$, $g^{(1)}(x) = e(-Q(x))\mathcal{G}^{(1)}(x)$. By a direct computation, we see that $\mathcal{G}^{(n)}(x)$ satisfy the recurrence formula

(7.18)

$$\frac{\mathcal{G}^{(n-1)}(x-a_0\delta)\mathcal{G}^{(n+1)}(x+a_0\delta)}{\mathcal{G}^{(n)}(x \pm a_1\delta)} = \theta(u_0 u_2; p)\theta(u_0 u_3; p)\theta(u_1 u_2; p)\theta(u_1 u_3; p).$$

If we set $g^{(n)}(x) = \mathcal{G}^{(n)}(x)c^{(n)}(x)$, the recurrence formula to be satisfied by $c^{(n)}(x)$ is given by

$$(7.19) \quad \frac{c^{(n-1)}(x - a_0\delta)c^{(n+1)}(x + a_0\delta)}{c^{(n)}(x \pm a_1\delta)} = \frac{1}{u_2 u_3 \theta(u_0 u_3^{\pm 1}; p)\theta(u_1 u_2^{\pm 1}; p)}$$

with the initial conditions $c^{(0)}(x) = 1$, $c^{(1)}(x) = e(-Q(x))$. On the other hand, one can verify that the functions $d^{(n)}(x)$, which appeared in the evaluation of the determinant $K^{(n)}(x)$, satisfy

(7.20)

$$\frac{d^{(n-1)}(x - a_0\delta)d^{(n+1)}(x + a_0\delta)}{d^{(n)}(x \pm a_1\delta)^2} = (p/u_0 u_1)\theta(u_0 u_3^{\pm 1}; p)\theta(u_1 u_2^{\pm 1}; p).$$

If we set $c^{(n)}(x) = e^{(n)}(x)/d^{(n)}(x)$, the recurrence formula for $e^{(n)}(x)$ is determined as

$$(7.21) \quad \frac{e^{(n-1)}(x - a_0\delta)e^{(n+1)}(x + a_0\delta)}{e^{(n)}(x \pm a_1\delta)} = p/u_0 u_1 u_2 u_3 = pe(-2(a_0|x)).$$

With the initial conditions $e^{(0)}(x) = 1$, $e^{(1)}(x) = e(-Q(x))$, this recurrence is solved as

$$(7.22) \qquad e^{(n)}(x) = p^{\binom{n}{2}}e(-nQ(x)) \quad (n = 0, 1, 2, \ldots).$$

Hence the gauge factors $g^{(n)}(x)$ are determined as

$$(7.23) \qquad g^{(n)}(x) = \frac{p^{\binom{n}{2}}e(-nQ(x))}{d^{(n)}(x)}\mathcal{G}^{(n)}(x) \quad (n = 0, 1, 2, \ldots).$$

Since $K^{(n)}(x) = d^{(n)}(x)I_n(t; p, q)$ with $t = \widetilde{u}$ as in (7.6), we also obtain

$$(7.24) \qquad \begin{aligned} \tau^{(n)}(x) &= \frac{p^{\binom{n}{2}}e(-nQ(x))}{d^{(n)}(x)}\mathcal{G}^{(n)}(x)\det\left(\psi_{ij}^{(n)}(x)\right)_{i,j=1}^{n} \\ &= p^{\binom{n}{2}}e(-nQ(x))\mathcal{G}^{(n)}(x)I_n(\widetilde{u}; p, q). \end{aligned}$$

This proves Theorem 6.2, (1) and the second equality of Theorem 6.3.

We already know by Theorem 6.1 that $\tau^{(n)}(x)$ $(x \in H_{\varpi+n\delta})$ are $W(E_7)$-invariant for all $n = 0, 1, 2, \ldots$. Namely $w(\tau^{(n)}(x)) = \tau^{(n)}(x)$ for any $w \in W(E_7)$. This means that, for each $w \in W(E_7)$, $\tau^{(n)}(x)$ $(x \in H_{\varpi+n\delta})$ has a determinant formula

$$(7.25) \qquad \tau^{(n)}(x) = \frac{p^{\binom{n}{2}}e(-nQ(x))}{\gamma^{(n)}(x)}\mathcal{F}^{(n)}(x)\det\left(\varphi_{ij}^{(n)}(x)\right)_{i,j=1}^{n}$$

and a multiple integral representation

$$(7.26) \qquad \tau^{(n)}(x) = p^{\binom{n}{2}}e(-nQ(x))\mathcal{F}^{(n)}(x)I_n(t; p, q),$$

where $\gamma^{(n)}$, $\mathcal{F}^{(n)}$, $\varphi_{ij}^{(n)}$ $(i, j = 1, \ldots, n)$ and t_k $(k = 0, 1, \ldots, 7)$ are specified by applying w to the functions $d^{(n)}$, $\mathcal{G}^{(n)}$, $\psi_{ij}^{(n)}$ and $\widetilde{u}_k$ on $H_{\varpi+n\delta}$, respectively. When $w = s_0$, by the transformation

$$(7.27) \qquad s_0(u_i) = \begin{cases} u_i\sqrt{pq^n/u_0u_1u_2u_3} & (i = 0, 1, 2, 3), \\ u_i\sqrt{pq^n/u_4u_5u_6u_7} & (i = 4, 5, 6, 7), \end{cases}$$

we obtain

$$(7.28)$$
$$\begin{aligned} \gamma^{(n)}(x) &= s_0(d^{(n)}(x)) \\ &= q^{2\binom{n}{3}}(q^{1-n}u_2u_3)^{\binom{n}{2}} \\ &\qquad \prod_{(i,j)=(0,3),(1,2)}\prod_{k=1}^{n}\theta(pq/u_iu_j; p; q)_{k-1}\theta(q^{k-n}u_i/u_j; p; q)_{k-1} \\ &= q^{-\binom{n+1}{3}}(u_2u_3)^{\binom{n}{2}} \\ &\qquad \prod_{(i,j)=(0,3),(1,2)}\prod_{k=1}^{n}\theta(q^{1-k}u_iu_j; p; q)_{k-1}\theta(q^{k-n}u_i/u_j; p; q)_{k-1} \end{aligned}$$

and

(7.29)
$$\mathcal{F}^{(n)}(x) = s_0(\mathcal{G}^{(n)}(x)) = \prod_{0 \le i < j \le n} \Gamma(q^{1-n} u_i u_j; p, q),$$

$$\varphi_{ij}^{(n)}(x) = s_0(\psi_{ij}^{(n)}(x)) = I(q^{n-i} t_0, q^{n-j} t_1, q^{j-1} t_2, q^{i-1} t_3, t_4, t_5, t_6, t_7; p, q),$$

$$t_k = q^{\frac{1}{2}(1-n)} u_k.$$

Formulas (7.25) and (7.26) for this case $w = s_0$ give Theorem 6.2, (2) and the first equality of Theorem 6.3.

§8.   Transformation of hypergeometric τ-functions

From the $W(E_7)$-invariant hypergeometric τ-function on D_{ϖ} discussed above, one can construct a class of τ-functions of hypergeometric type by the transformations in Theorem 2.3. In this section we give some remarks on this class of ORG τ-functions.

In what follows, for a root $\alpha \in \Delta(E_8)$ and a constant $\kappa \in \mathbb{C}$, we denote by

(8.1)
$$H_{\alpha, \kappa} = \{ x \in V \mid (\alpha|x) = \kappa \}$$

the hyperplane of level κ with respect to α. We consider a meromorphic function $\tau(x)$ on the disjoint union of parallel hyperplanes

(8.2)
$$D_{\alpha, \kappa} = \bigsqcup_{n \in \mathbb{Z}} H_{\alpha, \kappa + n\delta} \subset V.$$

Denoting by $\tau^{(n)}(x)$ the restriction of $\tau(x)$ to the nth hyperplane $H_{\kappa + n\delta}$, we say that an ORG τ-function $\tau(x)$ on $D_{\alpha, \kappa}$ is a *hypergeometric τ-function* of direction α with initial level κ if $\tau^{(n)}(x) = 0$ for $n < 0$ and $\tau^{(0)}(x) \not\equiv 0$.

In what follows, we use the notation

(8.3)
$$\Psi_n(t; p, q) = I_n(t; p, q) \prod_{0 \le i < j \le 7} \Gamma(t_i t_j; p, q, q).$$

As we have seen in (6.50), this function satisfies

(8.4)
$$\Psi_n(t; p, q) = \Psi_n(p^{\frac{1}{2}} q^{\frac{1}{2}(2-n)} t^{-1}; p, q)$$

under the balancing condition $t_0 t_1 \cdots t_7 = p^2 q^{4-2n}$.

Theorem 8.1. *With respect to the directions $\pm\phi$ and the initial levels $\pm\varpi$ ($e(\varpi) = p$), the following four functions are $W(E_7)$-invariant hypergeometric τ-functions.*

(0) $\tau_{++}(x)$ *on* $D_{\phi,\varpi} = \bigsqcup_{n\in\mathbb{Z}} H_{\phi,\varpi+n\delta}$:

$$
\begin{aligned}
\tau_{++}^{(n)}(x) &= p^{\binom{n}{2}} e(-nQ(x)) \Psi_n(q^{\frac{1}{2}(1-n)} u; p, q) \\
&= p^{\binom{n}{2}} e(-nQ(x)) \Psi_n(p^{\frac{1}{2}} q^{\frac{1}{2}} u^{-1}; p, q).
\end{aligned}
$$
(8.5)

(1) $\tau_{+-}(x)$ *on* $D_{\phi,-\varpi} = \bigsqcup_{n\in\mathbb{Z}} H_{\phi,-\varpi+n\delta}$:

(8.6) $\qquad \tau_{+-}^{(n)}(x) = \Psi_n(p^{\frac{1}{2}} q^{\frac{1}{2}(1-n)} u; p, q) = \Psi_n(q^{\frac{1}{2}} u^{-1}; p, q).$

(2) $\tau_{-+}(x)$ *on* $D_{-\phi,\varpi} = \bigsqcup_{n\in\mathbb{Z}} H_{-\phi,\varpi+n\delta} = \bigsqcup_{n\in\mathbb{Z}} H_{\phi,-\varpi-n\delta}$:

$$
\begin{aligned}
\tau_{-+}^{(n)}(x) &= p^{\binom{n}{2}} e(-nQ(x)) \Psi_n(p^{\frac{1}{2}} q^{\frac{1}{2}} u; p, q) \\
&= p^{\binom{n}{2}} e(-nQ(x)) \Psi_n(q^{\frac{1}{2}(1-n)} u^{-1}; p, q).
\end{aligned}
$$
(8.7)

(3) $\tau_{--}(x)$ *on* $D_{-\phi,-\varpi} = \bigsqcup_{n\in\mathbb{Z}} H_{-\phi,-\varpi+n\delta} = \bigsqcup_{n\in\mathbb{Z}} H_{\phi,\varpi-n\delta}$:

(8.8) $\qquad \tau_{--}^{(n)}(x) = \Psi_n(q^{\frac{1}{2}} u; p, q) = \Psi_n(p^{\frac{1}{2}} q^{\frac{1}{2}(1-n)} u^{-1}; p, q).$

Proof. The function $\tau_{++}(x)$ is the $W(E_8)$-invariant hypergeometric τ-function of direction ϕ with initial level ϖ discussed in previous sections. We apply the translation by $-\phi\varpi$ to $\tau(x) = \tau_{++}(x)$ as in Theorem 2.3, (3) to construct a τ-function

(8.9) $\qquad\qquad \widetilde{\tau}(x) = e(S(x; -\phi, \varpi))\tau(x + \phi\varpi)$

on $D_{\phi,\varpi} - \phi\varpi = D_{\phi,-\varpi}$. We look at the prefactor of $\widetilde{\tau}^{(n)}(x)$ ($x \in H_{\phi,-\varpi+n\delta}$):
(8.10)
$$
e(S(x; -\phi, \varpi)) p^{\binom{n}{2}} e(-nQ(x)) = e(S(x; -\phi, \varpi)) e(-nQ(x) + \binom{n}{2}\varpi).
$$

Noting that $\eta_\varpi = -1$ in this case, we compute

$$
\begin{aligned}
S(x; -\phi, \varpi) &= \tfrac{1}{2\delta^2}(\phi|x)(x|x + \phi\varpi) = \tfrac{1}{2\delta^2}(\phi|x)(x|x) + \tfrac{\varpi}{2\delta^2}(\phi|x)^2 \\
&= \tfrac{n}{2\delta}(x|x) + \tfrac{n^2}{2}\varpi - \tfrac{\varpi}{2\delta^2}(x|x) - \tfrac{n}{2}\tfrac{\varpi^2}{\delta} + \tfrac{\varpi^3}{2\delta^2}.
\end{aligned}
$$
(8.11)

for $(\phi|x) = -\varpi + n\delta$. Combining this with

(8.12) $\qquad -nQ(x + \phi\varpi) = -\tfrac{n}{2\delta}(x + \phi\varpi|x + \phi\varpi) = -\tfrac{n}{2\delta}(x|x) - n^2\varpi,$

we obtain

$$(8.13) \quad \begin{aligned} S(x; -\phi, \varpi) - nQ(x) + \binom{n}{2}\varpi &= -\tfrac{\varpi}{2\delta^2}(x|x) - n\tfrac{\varpi(\varpi+\delta)}{2\delta} + \tfrac{\varpi^3}{2\delta^2} \\ &= -\tfrac{\varpi}{2\delta^2}(x|x) - \tfrac{\varpi(\varpi+\delta)}{2\delta^2}(\phi|x) - \tfrac{\varpi^2}{2\delta} \end{aligned}$$

by $n = ((\phi|x) + \varpi)/\delta$. Hence the prefactor (8.10) for $\widetilde{\tau}^{(n)}(x)$ is determined as

$$(8.14) \quad e\left(-\tfrac{\varpi}{2\delta^2}(x|x) - \tfrac{\omega(\omega+\delta)}{2\delta^2}(\phi|x) - \tfrac{\omega^2}{2\delta}\right).$$

This factor does not effect on the Hirota equations, and can be eliminated by Theorem 2.3, (1). We thus obtain the τ-function $\tau_{+-}(x)$ of (8.6) by replacing $e(x) = u$ in the last two factors of $\tau_{++}(x)$ with $e(x + \phi\omega) = p^\phi u = p^{\frac{1}{2}}u$. The other two functions $\tau_{-+}(x)$ and $\tau_{--}(x)$ are obtained by replacing x in $\tau_{++}(x)$ and $\tau_{+-}(x)$ with $-x$, respectively. Q.E.D.

We now apply Theorem 2.3, (3) for constructing hypergeometric τ-functions with initial level 0. Let $a \in P$ be a vector with $(\phi|a) = 1$, so that

$$(8.15) \quad D_{\phi,-\varpi} + a\varpi = D_{\phi,0} = \bigsqcup_{n \in \mathbb{Z}} H_{\phi,n\delta}.$$

Then, from

$$(8.16) \quad \tau_{+-}^{(n)}(x) = \Psi_n(p^{\frac{1}{2}}q^{\frac{1}{2}(1-n)}u; p, q) = \Psi_n(q^{\frac{1}{2}}u^{-1}; p, q)$$

we obtain a hypergeometric τ-function

$$(8.17) \quad \begin{aligned} \tau_a(x) &= e(S(x; a, \varpi))\tau_{+-}(x - a\varpi) \\ &= e(S(x; a, \varpi))\Psi_n(p^{-a}p^{\frac{1}{2}}q^{\frac{1}{2}(1-n)}u; p, q) \\ &= e(S(x; a, \varpi))\Psi_n(p^a q^{\frac{1}{2}}u^{-1}; p, q) \end{aligned}$$

on $D_{\phi,0}$, where

$$(8.18) \quad S(x; a, \varpi) = -\tfrac{1}{2\delta^2}(a|x)(x|x - a\varpi).$$

When $a \in \Delta(E_8)$, namely $(a|a) = 2$, there are 56 choices of a with $(\phi|a) = 1$:

$$(8.19) \quad a = v_k + v_l, \quad \phi - v_k - v_l \quad (0 \le k < l \le 7).$$

Those τ-functions $\tau_a(x)$ on $D_{\phi,0}$ correspond to the 56 hypergeometric τ-functions in the trigonometric case studied by Masuda [8].

In general, let $\omega = k + l\varpi \in \Omega$ $(k, l \in \mathbb{Z})$ a period, and take two vectors $a, b \in P$ with $(\phi|a) = k$, $(\phi|b) = l + 1$. Then

$$(8.20) \qquad e(S(x; b, \varpi))\tau_{+-}(x - a - b\varpi)$$

is a hypergeometric τ-function on $D_{\phi,\omega}$. Furthermore, let $\alpha \in \Delta(E_8)$ be an arbitrary root of the root system of type E_8, and choose a $w \in W(E_8)$ such that $\alpha = w.\phi$. Then
(8.21)
$$w(e(S(x; b, \varpi))\tau_{+-}(x-a-b\varpi)) = e(S(w^{-1}.x; b, \varpi))\tau_{+-}(w^{-1}.x-a-b\varpi)$$

provides a hypergeometric τ-function on $D_{\alpha,\omega}$ in the direction α with initial level ω.

§9.   Relation to the framework of point configurations

In this section, we give some remarks on how ORG τ-functions are related to the notion of lattice τ-functions associated with the configuration of generic nine points in $\mathbb{P}^2$.

9.1.   Realization of the affine root system $E_8^{(1)}$

We first recall from [5] the realization of the affine root system of type $E_8^{(1)}$ in the context of the configuration of generic nine points in $\mathbb{P}^2$ (see also Dolgachev-Ortland [2]). We consider the 10-dimensional complex vector space

$$(9.1) \qquad \mathfrak{h} = \mathfrak{h}_{3,9} = \mathbb{C}e_0 \oplus \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_9$$

with basis $\{e_0, e_1, \ldots, e_9\}$, and define a scalar product (nondegenerate symmetric $\mathbb{C}$-bilinear form) $(\cdot|\cdot) : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$ by

$$(9.2) \qquad \begin{aligned} (e_0|e_0) &= -1, \quad (e_j|e_j) = 1 \quad (j \in \{1, \ldots, 9\}), \\ (e_i|e_j) &= 0 \quad (i, j \in \{0, 1, \ldots, 9\}; \ i \neq j). \end{aligned}$$

This vector space is regarded as the complexification of the lattice

$$(9.3) \qquad L = L_{3,9} = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \cdots \oplus \mathbb{Z}e_9 \subset \mathfrak{h} = \mathfrak{h}_{3,9}$$

endowed with the symmetric $\mathbb{Z}$-bilinear form $(\cdot|\cdot) : L \times L \to \mathbb{Z}$. In geometric terms, $L = L_{3,9}$ is the *Picard lattice* associated with the blowup of $\mathbb{P}^2$ at generic nine points $p_1, \ldots, p_9$. The vectors e_0 and $e_1, \ldots, e_9$ denote the class of lines in $\mathbb{P}^2$ and those of exceptional divisors corresponding to $p_1, \ldots, p_9$ respectively, and the scalar product $(\cdot|\cdot)$ on L represents the intersection form of divisor classes multiplied by -1.

The root lattice of type $E_8^{(1)}$ is realized as

$$(9.4) \qquad Q(E_8^{(1)}) = \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_8 \subset L,$$

where the *simple roots* $\alpha_0, \alpha_1, \ldots, \alpha_8 \in \mathfrak{h}$ are defined by

$$(9.5) \qquad \alpha_0 = e_0 - e_1 - e_2 - e_3, \quad \alpha_j = e_j - e_{j+1} \quad (j = 1, \ldots, 8)$$

with Dynkin diagram

$$(9.6)$$

(These α_j are called the *coroots* h_j in [5]). Note also that

$$(9.7)$$
$$c = 3e_0 - e_1 - \cdots - e_9$$
$$= 3\alpha_0 + 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8 \in Q(E_8^{(1)})$$

satisfies $(c|\alpha_j) = 0$ for $j = 0, 1, \ldots, 8$. Denoting by $\mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ the dual space of $\mathfrak{h}$, we take the linear functions $\varepsilon_j = (e_j | \cdot) \in \mathfrak{h}^*$ $(j = 0, 1, \ldots, 9)$ so that $\mathfrak{h}^* = \mathbb{C}\varepsilon_0 \oplus \mathbb{C}\varepsilon_1 \oplus \cdots \oplus \mathbb{C}\varepsilon_9$, and regard $\varepsilon = (\varepsilon_0; \varepsilon_1, \ldots, \varepsilon_9)$ as the canonical coordinates for $\mathfrak{h}$. We often identify $\mathfrak{h}^*$ with $\mathfrak{h}$ through the isomorphism $\nu : \mathfrak{h} \overset{\sim}{\to} \mathfrak{h}^*$ defined by $\nu(h) = (h | \cdot)$ $(h \in \mathfrak{h})$, and denote the induced scalar product by the same symbol $(\cdot | \cdot)$. When we regard the simple roots as $\mathbb{C}$-linear functions on $\mathfrak{h}$, they are expressed as $\nu(\alpha_0) = \varepsilon_0 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3$ and $\nu(\alpha_j) = \varepsilon_j - \varepsilon_{j+1}$ $(j = 1, \ldots, 8)$. Setting $\delta = (c | \cdot) = 3\varepsilon_0 - \varepsilon_1 - \cdots - \varepsilon_9 \in \mathfrak{h}^*$, we regard below this null root $\delta \in \mathfrak{h}^*$ as the scaling unit for difference equations.

The root lattice of type E_8 is specified as $Q(E_8) = \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_7 \subset Q(E_8^{(1)})$. The vector space $\mathfrak{h}$ is decomposed accordingly as

$$(9.8) \quad \mathfrak{h} = \overset{\circ}{\mathfrak{h}} \oplus \mathbb{C}c \oplus \mathbb{C}d, \quad \overset{\circ}{\mathfrak{h}} = \mathbb{C}\alpha_0 \oplus \mathbb{C}\alpha_1 \oplus \cdots \oplus \mathbb{C}\alpha_7, \quad d = -e_9 - \tfrac{1}{2}c,$$

where

$$(9.9) \quad (c|h) = 0, \quad (d|h) = 0 \quad (h \in \overset{\circ}{\mathfrak{h}}); \quad (c|c) = 0, \quad (c|d) = 1, \quad (d|d) = 0.$$

The 8-dimensional subspace $\overset{\circ}{\mathfrak{h}} \subset \mathfrak{h}$ can be identified with the vector space $V = \mathbb{C}^8$ that we have used throughout this paper for the realization of the root lattice $P = Q(E_8)$. Noting that $\overset{\circ}{\mathfrak{h}} = \{h \in \mathfrak{h} \,|\, (c|h) = 0, \ (d|h) = 0\}$

and $Q(E_8) = L \cap \overset{\circ}{\mathfrak{h}}$, we define the orthonormal basis $\{v_0, v_1, \ldots, v_7\}$ for $\overset{\circ}{\mathfrak{h}}$ by

(9.10) $\qquad v_j = e_j - \frac{1}{2}(e_0 - e_9) + \frac{1}{2}c \quad (j = 1, \ldots, 8), \quad v_0 = -v_8.$

Then the highest root and the simple roots of type E_8 are expressed as

(9.11)
$$\phi = \frac{1}{2}(v_0 + v_1 + \cdots + v_7) = c - \alpha_8,$$
$$\alpha_0 = \phi - v_0 - v_1 - v_2 - v_3, \quad \alpha_j = v_j - v_{j+1} \quad (j = 1, \ldots, 7)$$

respectively, which recovers the situation of Section 1. In what follows, we identify $\overset{\circ}{\mathfrak{h}}$ with V through this orthonormal basis $\{v_0, v_1, \ldots, v_7\}$. The corresponding $\mathbb{C}$-linear functions $x_j = (v_j | \cdot) \in \mathfrak{h}^*$ are realized as

(9.12) $\qquad x_j = \varepsilon_j - \frac{1}{2}(\varepsilon_0 - \varepsilon_9) + \frac{1}{2}\delta \quad (j = 1, \ldots, 8), \quad x_0 = -x_8.$

For each $\alpha \in \mathfrak{h}$ with $(\alpha|\alpha) \neq 0$, the *reflection* $r_\alpha : \mathfrak{h} \to \mathfrak{h}$ with respect to α is defined in the standard way by

(9.13) $\qquad r_\alpha(h) = h - (\alpha^\vee | h)\alpha \quad (h \in \mathfrak{h}), \quad \alpha^\vee = 2\alpha/(\alpha|\alpha).$

The affine Weyl group $W(E_8^{(1)}) = \langle s_0, s_1, \ldots, s_8 \rangle$ of type $E_8^{(1)}$ (Coxeter group associated with diagram (9.6)) then acts faithfully on $\mathfrak{h}$ through the *simple reflections* $s_j = r_{\alpha_j}$ $(j = 0, 1, \ldots 8)$. We remark that this group contains the symmetric group $\mathfrak{S}_9 = \langle s_1, \ldots, s_8 \rangle$ as a subgroup which permutes $e_1, \ldots, e_9$. The affine Weyl group $W(E_8^{(1)})$ also acts on the dual space $\mathfrak{h}^*$ through $s_j = r_{\alpha_j} : \mathfrak{h}^* \to \mathfrak{h}^*$ defined in the same way as (9.13). These actions of $W(E_8^{(1)})$ on $\mathfrak{h}$ and $\mathfrak{h}^*$ leave the two scalar products invariant. Note also that $c \in \mathfrak{h}$ and $\delta \in \mathfrak{h}^*$ are invariant under the action of $W(E_8^{(1)})$.

Setting $\mathfrak{h}_0 = \{\alpha \in \mathfrak{h} \mid (c|\alpha) = 0\}$, for each $\alpha \in \mathfrak{h}_0$ we define the *Kac translation* ([3]) $T_\alpha : \mathfrak{h} \to \mathfrak{h}$ with respect to α by

(9.14) $\qquad T_\alpha(h) = h + (c|h)\alpha - \left(\frac{1}{2}(\alpha|\alpha)(c|h) + (\alpha|h)\right)c \quad (h \in \mathfrak{h}).$

It is directly verified that these $\mathbb{C}$-linear transformations $T_\alpha \in \mathrm{GL}(\mathfrak{h})$ $(\alpha \in \mathfrak{h}_0)$ satisfy
(9.15)
    (1) $(T_\alpha(h)|T_\alpha(h')) = (h|h') \quad (\alpha \in \mathfrak{h}_0; \ h, h' \in \mathfrak{h})$,
    (2) $T_\alpha T_\beta = T_\beta T_\alpha = T_{\alpha+\beta} \quad (\alpha, \beta \in \mathfrak{h}_0), \quad T_{kc} = \mathrm{id}_\mathfrak{h} \quad (k \in \mathbb{C})$,
    (3) $w T_\alpha w^{-1} = T_{w.\alpha} \quad (\alpha \in \mathfrak{h}_0, w \in W(E_8^{(1)}))$.

Note also that

(9.16) $T_\alpha(h) = h - (\alpha|h)c \quad (h \in \mathfrak{h}, (c|h) = 0), \quad T_\alpha(c) = c$

for any $\alpha \in \mathfrak{h}_0$. For a $\mathbb{Z}$-submodule Q of $\mathfrak{h}_0$ given, we denote by $T(Q) \subset$ $\mathrm{GL}(\mathfrak{h})$ the abelian subgroup of Kac translations T_α $(\alpha \in Q)$. We remark that, if $\alpha \in \mathfrak{h}_0$ and $(\alpha|\alpha) \neq 0$, then the Kac translation T_α is expressed in the form $T_\alpha = r_{c-\alpha^\vee} r_{\alpha^\vee}$ as a product of two reflections. This implies in particular that the Kac translations T_{α_j} $(j = 0, 1, \ldots, 8)$ by simple roots belong to the affine Weyl group $W(E_8^{(1)}) = \langle s_0, s_1, \ldots, s_8 \rangle \subset \mathrm{GL}(\mathfrak{h})$. It is known ([3]) in fact that $W(E_8^{(1)})$ splits into the semidirect product

(9.17) $W(E_8^{(1)}) = T(Q(E_8)) \rtimes W(E_8), \quad W(E_8) = \langle s_0, s_1, \ldots, s_7 \rangle.$

We remark here that the linear action of $W(E_8^{(1)})$ on $\mathfrak{h}$ extends to a bigger group $T(\mathfrak{h}_0)W(E_8^{(1)}) = T(V) \rtimes W(E_8)$ including the abelian group $T(V)$ of Kac translations with respect to $V = \mathbb{C} \otimes_{\mathbb{Z}} Q(E_8)$. Note that $T(V) \rtimes W(E_8)$ acts also on $\mathfrak{h}^*$ so that $\nu : \mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$ intertwines its linear actions on $\mathfrak{h}$ and $\mathfrak{h}^*$.

Before proceeding further, we clarify how the linear actions of Kac translations on $\mathfrak{h}$ are related to the affine-linear actions of parallel translations on V. Note first that, for each $\kappa \in \mathbb{C}$, the hyperplane

(9.18) $\mathfrak{h}_\kappa = \{ h \in \mathfrak{h} \mid (c|h) = \kappa \} \subset \mathfrak{h}$

is stable by $W(E_8^{(1)})$ and by $T(\mathfrak{h}_0)$. On this hyperplane $\mathfrak{h}_\kappa$ of level κ, the null root $\delta = (c|\cdot) \in \mathfrak{h}^*$ is identified with the constant function $\delta = \kappa$. For each $(\mu, \kappa) \in \mathbb{C} \times \mathbb{C}^*$, we define a quadratic mapping $\gamma_{(\mu,\kappa)} : V \to \mathfrak{h}_\kappa$ by

(9.19) $\gamma_{(\mu,\kappa)}(x) = T_{\kappa^{-1}x}(\kappa d) - \mu c = x - (\frac{1}{2\kappa}(x|x) + \mu)c + \kappa d \quad (x \in V).$

These mappings $\gamma_{(\mu,\kappa)}$ induce the parametrization $\gamma_\kappa : V \times \mathbb{C} \xrightarrow{\sim} \mathfrak{h}_\kappa :$ $(x; \mu) \mapsto \gamma_{(\mu,\kappa)}(x)$ of $\mathfrak{h}_\kappa$ for each $\kappa \in \mathbb{C}^*$, as well as the isomorphism $\gamma : V \times \mathbb{C} \times \mathbb{C}^* \xrightarrow{\sim} \mathfrak{h} \backslash \mathfrak{h}_0$ of affine varieties such that

(9.20) $\gamma(x; \mu, \kappa) = x - (\frac{1}{2\kappa}(x|x) + \mu)c + \kappa d \quad (x \in V, \mu \in \mathbb{C}, \kappa \in \mathbb{C}^*).$

Furthermore, this isomorphism is equivariant with respect to the action of the group $T(V) \rtimes W(E_8)$ on $V \times \mathbb{C} \times \mathbb{C}^*$ specified by

(9.21) $T_v w.(x; \mu, \kappa) = (w.x + \kappa v; \mu, \kappa) \quad (v \in V, w \in W(E_8)).$

Through this isomorphism, the coordinates $(x; \mu, \kappa) = (x_0, x_1, \ldots, x_7; \mu, \kappa)$ for $V \times \mathbb{C} \times \mathbb{C}^*$ and $\varepsilon = (\varepsilon_0; \varepsilon_1, \ldots, \varepsilon_9)$ for $\mathfrak{h} \backslash \mathfrak{h}_0$ are transformed into each other through

$$(9.22) \quad \begin{aligned} x_j &= \varepsilon_j - \tfrac{1}{2}(\varepsilon_0 - \varepsilon_9) + \tfrac{1}{2}\delta \quad (j = 1, \ldots, 8), \quad x_0 = -x_8, \\ \mu &= -\tfrac{1}{2\delta}(\varepsilon|\varepsilon), \quad \kappa = \delta, \end{aligned}$$

where $(\varepsilon|\varepsilon) = -\varepsilon_0^2 + \varepsilon_1^2 + \cdots + \varepsilon_9^2$, and by

$$(9.23) \quad \begin{aligned} \varepsilon_0 &= 2x_0 - 2(\phi|x) + 3(\tfrac{1}{2\kappa}(x|x) + \mu + \tfrac{1}{2}\kappa), \\ \varepsilon_j &= x_j + x_0 - (\phi|x) + \tfrac{1}{2\kappa}(x|x) + \mu + \tfrac{1}{2}\kappa \quad (j = 1, \ldots, 7), \\ \varepsilon_8 &= -(\phi|x) + \tfrac{1}{2\kappa}(x|x) + \mu + \tfrac{1}{2}\kappa, \quad \varepsilon_9 = \tfrac{1}{2\kappa}(x|x) + \mu - \tfrac{1}{2}\kappa, \end{aligned}$$

where $(\phi|x) = \tfrac{1}{2}(x_0 + x_1 + \cdots + x_7)$, $(x|x) = x_0^2 + x_1^2 + \cdots + x_7^2$.

9.2. Lattice τ-functions vs. ORG τ-functions

We now consider the $W(E_8^{(1)})$-orbit

$$(9.24) \quad M = M_{3,9} = W(E_8^{(1)})\{e_1, \ldots, e_9\} = W(E_8^{(1)})e_9 \subset L = L_{3,9}$$

in the Picard lattice. Noting that $e_9 \in M$ is $W(E_8)$-invariant, we see that the natural mapping $W(E_8^{(1)}) \to M : w \mapsto w.e_9$ induces the bijection $Q(E_8) \xrightarrow{\sim} M : \alpha \mapsto T_\alpha.e_9$. The orbit $M = W(E_8^{(1)})e_9$ is intrinsically characterized as

$$(9.25) \quad M = \{\Lambda \in L \mid (\Lambda|\Lambda) = 1, \ (c|\Lambda) = -1\}.$$

To see this, suppose that $\Lambda \in L$ satisfies $(\Lambda|\Lambda) = 1$ and $(c|\Lambda) = -1$. Then, the difference $\beta = \Lambda - e_9$ satisfies $(\beta|\beta) + 2(e_9|\beta) = 0$ and $(c|\beta) = 0$. This implies $\beta = \Lambda - e_9 \in Q(E_8^{(1)})$ and $T_\beta.e_9 = e_9 + \beta = \Lambda$. Taking $\alpha = \beta + (e_9|\beta)c \in Q(E_8)$, we see that Λ is uniquely expressed in the form

$$(9.26) \quad \Lambda = e_9 + \alpha + \tfrac{1}{2}(\alpha|\alpha)c, \quad \alpha \in Q(E_8),$$

and hence $\Lambda = T_{\Lambda - e_9}^{-1}.e_9 = T_\alpha^{-1}.e_9$. We remark that α is the unique element in $Q(E_8)$ such that $\Lambda - e_9 \equiv \alpha \pmod{\mathbb{C}c}$, which we call the *classical part* of $\Lambda - e_9 \in \mathfrak{h}_0$.

In [5], a system of *lattice τ-functions* associated with the configurations of generic nine points in $\mathbb{P}^2$ is defined as a family of dependent variables τ_Λ indexed by $\Lambda \in M$ which admit an action of the affine Weyl group $W(E_8^{(1)})$ such that

$$(9.27) \quad w.\tau_\Lambda = \tau_{w.\Lambda} \quad (\Lambda \in M, \ w \in W(E_8^{(1)}))$$

and satisfy the quadratic relations
(9.28)
$$[\varepsilon_{jk}][\varepsilon_{jkl}]\tau_{e_i}\tau_{e_0-e_i-e_l} + [\varepsilon_{ki}][\varepsilon_{kil}]\tau_{e_j}\tau_{e_0-e_j-e_l} + [\varepsilon_{ij}][\varepsilon_{ijl}]\tau_{e_k}\tau_{e_0-e_k-e_l} = 0$$

for all quadruples of mutually distinct $i,j,k,l \in \{1,\ldots,9\}$, where $\varepsilon_{ij} = \varepsilon_i - \varepsilon_j$ and $\varepsilon_{ijk} = \varepsilon_0 - \varepsilon_i - \varepsilon_j - \varepsilon_k$. (Here we use the notation τ_Λ instead of $\tau(\Lambda)$ as in [5] to make clear that Λ is *not* an independent variable, but an index.) Under the condition (9.27), τ_{e_9} is $W(E_8)$-invariant, and all the functions τ_Λ ($\Lambda \in M$) are expressed as $\tau_\Lambda = T^{-1}_{\Lambda-e_9}.\tau$ in terms of a single $W(E_8)$-invariant function $\tau = \tau_{e_9}$. We also remark that, if equation (9.28) holds for some quadruple of distinct $i,j,k,l \in \{1,\ldots,9\}$, then it holds for all quadruples as a result of the action of $\mathfrak{S}_9 \subset W(E_8^{(1)})$. In the following, we use the notation $\sigma_h = [\nu(h)] = [(h|\cdot)]$ for $h \in \mathfrak{h}$, so that $\sigma_{e_{ij}} = [\varepsilon_{ij}]$ and $\sigma_{e_{ijk}} = [\varepsilon_{ijk}]$ where $e_{ij} = e_i - e_j$ and $e_{ijk} = e_0 - e_i - e_j - e_k$. In this notation, equation (9.28) is rewritten as
(9.29)
$$\sigma_{e_{jk}}\sigma_{e_{jkl}}\tau_{e_i}\tau_{e_0-e_i-e_l} + \sigma_{e_{ki}}\sigma_{e_{kil}}\tau_{e_j}\tau_{e_0-e_j-e_l} + \sigma_{e_{ij}}\sigma_{e_{ijl}}\tau_{e_k}\tau_{e_0-e_k-e_l} = 0.$$

As we will see below, equations (9.28) can be rewritten in a $W(E_8^{(1)})$-invariant form.

To clarify the situation, let X be a left $W(E_8^{(1)})$-set. Noting that functions of the form $\sigma_\alpha = [(\alpha|\cdot)]$ ($\alpha \in Q(E_8^{(1)})$) are defined over $\overline{\mathfrak{h}} = \mathfrak{h}/\mathbb{C}c$, we suppose that a $W(E_8^{(1)})$-equivariant mapping $\gamma : X \to \overline{\mathfrak{h}}$ is given. Regarding those σ_α as functions defined on X through $\gamma : X \to \overline{\mathfrak{h}}$, we can consider systems of lattice τ-functions τ_Λ ($\Lambda \in M$) defined on X. In order to compare this notion with that of ORG τ-functions on X, we assume that the extension $T(\frac{1}{2}Q(E_8)) \rtimes W(E_8)$ of $W(E_8^{(1)})$ acts on X so that γ is an equivariant mapping. For a function φ defined on a subset $U \subseteq X$, we define the action of $w \in T(\frac{1}{2}Q(E_8)) \rtimes W(E_8)$ on φ to be the function $w.\varphi$ on $w.U$ such that $(w.\varphi)(x) = \varphi(w^{-1}.x)$ ($x \in w.U$).

Proposition 9.1. *Let τ be a $W(E_8)$-invariant function on X and set $\tau_\Lambda = T^{-1}_{\Lambda-e_9}.\tau$ for each $\Lambda \in M$. Then the following three conditions are equivalent*:
(a) *The equation*
(9.30)
$$\sigma_{e_{jk}}\sigma_{e_{jkl}}\tau_{e_i}\tau_{e_0-e_i-e_l} + \sigma_{e_{ki}}\sigma_{e_{kil}}\tau_{e_j}\tau_{e_0-e_j-e_l} + \sigma_{e_{ij}}\sigma_{e_{ijl}}\tau_{e_k}\tau_{e_0-e_k-e_l} = 0$$

holds for each quadruple of mutually distinct $i,j,k,l \in \{1,\ldots,9\}$.
(b) *The equation*
(9.31)
$$\sigma_{u_1-u_2}\sigma_{u_1-u'_2}\tau_{u_0}\tau_{u'_0} + \sigma_{u_2-u_0}\sigma_{u_2-u'_0}\tau_{u_1}\tau_{u'_1} + \sigma_{u_0-u_1}\sigma_{u_0-u'_1}\tau_{u_2}\tau_{u'_2} = 0$$

holds for each sextuple of points $u_i, u_i' \in M$ $(i = 0, 1, 2)$ such that

(9.32)
$$u_0 + u_0' = u_1 + u_1' = u_2 + u_2', \quad (u_i - u_i'|u_j - u_j') = 4\delta_{ij} \quad (i, j = 0, 1, 2).$$

(c) *The equation*

(9.33) $\quad \sigma_{a_1 \pm a_2} T_{a_0}.\tau \, T_{a_0}^{-1}.\tau + \sigma_{a_2 \pm a_0} T_{a_1}.\tau \, T_{a_1}^{-1}.\tau + \sigma_{a_0 \pm a_1} T_{a_2}.\tau \, T_{a_2}^{-1}.\tau = 0$

holds for each triple of vectors $a_0, a_1, a_2 \in \frac{1}{2}Q(E_8)$ such that

(9.34) $\qquad (a_i|a_j) = \delta_{i,j}, \quad \pm a_i \pm a_j \in Q(E_8) \quad (i, j = 0, 1, 2).$

$\quad$ *Proof.* Note first that equation (9.30) is a special case of (9.31) where $u_0 = e_i, u_1 = e_j, u_2 = e_k$ and $v = e_0 - e_l$. Hence condition (b) implies (a). We consider equation (9.31) for a sextuple $u_i, u_i' \in M$ $(i = 0, 1, 2)$ as in (b). Introducing

(9.35)
$$g = \tfrac{1}{2}(u_i + u_i'), \quad b_i = \tfrac{1}{2}(u_i - u_i') \quad (i = 0, 1, 2),$$
$$u_i = g + b_i, \quad u_i' = g - b_i \quad (i = 0, 1, 2),$$

we rewrite equation (9.31) into the equation

(9.36) $\quad \sigma_{b_1 \pm b_2} \tau_{g+b_0} \tau_{g-b_0} + \sigma_{b_2 \pm b_0} \tau_{g+b_1} \tau_{g-b_1} + \sigma_{b_0 \pm b_1} \tau_{g+b_2} \tau_{g-b_2} = 0$

for a sextuple of points $g \pm b_i \in M$ $(i = 0, 1, 2)$ such that

(9.37) $\qquad (b_i|b_j) = \delta_{i,j}, \quad \pm b_i \pm b_j \in Q(E_8^{(1)}) \quad (i, j = 0, 1, 2).$

In this setting, b_0, b_1, b_2 and g are characterized by the conditions

(9.38)
$$g \in \tfrac{1}{2}L, \quad (g|g) = 0, \quad (c|g) = -1$$
$$b_i \in \tfrac{1}{2}L, \quad (c|b_i) = 0, \quad (g|b_i) = 0, \quad (b_i|b_j) = \delta_{ij}, \quad \pm b_i \pm b_j \in Q(E_8^{(1)}).$$

Since $b_i \in \frac{1}{2}Q(E_8^{(1)})$ $(i = 0, 1, 2)$, they are expressed as

(9.39) $\qquad b_i = a_i + k_i c, \quad k_i = (g|a_i) \in \tfrac{1}{2}\mathbb{Z} \quad (i = 0, 1, 2),$

where

(9.40) $\quad a_i \in \tfrac{1}{2}Q(E_8), \quad (g|a_i) \in \tfrac{1}{2}\mathbb{Z}, \quad (a_i|a_j) = \delta_{ij}, \quad \pm a_i \pm a_j \in Q(E_8).$

Note that $(c|g - e_9) = 0$ and hence $g - e_9 \in \frac{1}{2}Q(E_8^{(1)})$. In this situation we have

(9.41) $\qquad T_{g-e_9}^{-1}.a_i = a_i + (g - e_9|a_i)c = a_i + k_i c = b_i \quad (i = 0, 1, 2),$

and hence we see that equations (9.36) and (9.33) are transformed into each other by the actions of T_{g-e_9} and $T_{g-e_9}^{-1}$. These arguments show that condition (c) implies (b). Note also that equation (9.33) is a special case of (9.36) where $g = e_9 + \frac{1}{2}c$ and $b_i = a_i$ ($i = 0, 1, 2$). We finally show that (a) implies (c). Suppose that equation (9.30) holds for some quadruple of distinct $i, j, k, l \in \{1, \ldots, 9\}$. Since it is a special case of (9.31), we see that equation (9.33) holds for some triple $a_0, a_1, a_2 \in \frac{1}{2}Q(E_8)$ satisfying (9.34), namely for some C_3-frame $\{\pm a_0, \pm a_1, \pm a_2\}$ in the terminology of Section 1. Since the Weyl group $W(E_8)$ acts transitively on the set of all C_3-frames, we see that equation (9.33) holds for all C_3-frames, which implies (c).   Q.E.D.

By abuse of terminology, we say that a function τ is an ORG τ-function if it satisfies the non-autonomous Hirota equations (9.33) for all C_3-frames relative to $Q(E_8)$. Proposition 9.1 means that a family of functions τ_Λ ($\Lambda \in M$) on X is a system of lattice τ-functions if and only if $\tau = \tau_{e_9}$ is a $W(E_8)$-invariant ORG τ-function on X. General ORG τ-functions which are not necessarily $W(E_8)$-invariant can be interpreted as the system of lattice τ-functions over a covering space of X.

Setting

$$(9.42) \qquad \widetilde{X} = W(E_8) \times X = \big\{(w, x) \,\big|\, w \in W(E_8), \ x \in X\big\},$$

we define an action of $T(\frac{1}{2}Q(E_8)) \rtimes W(E_8)$ on $\widetilde{X}$ by

$$(9.43) \quad T_\alpha w.(w', x) = (ww', T_\alpha w.x) \qquad (\alpha \in \tfrac{1}{2}Q(E_8), \ w \in W(E_8))$$

so that the projection $\widetilde{X} \to X$ is equivariant. Let U be a subset of X and suppose that U is stable by the group $T(Q(E_8))$ of translations. Then the subset $\widetilde{U} \subseteq \widetilde{X}$ defined as

$$(9.44) \ \widetilde{U} = \big\{(w, x) \in \widetilde{X} \,\big|\, w \in W(E_8), \ x \in w.U\big\} = \bigsqcup_{w \in W(E_8)} \{w\} \times w.U$$

is stable by the action of $W(E_8^{(1)}) = T(Q(E_8)) \rtimes W(E_8)$. To a function τ defined on U, we associate a function $\widetilde{\tau}$ on $\widetilde{U}$ by setting

$$(9.45) \quad \widetilde{\tau}(w, x) = (w.\tau)(x) = \tau(w^{-1}.x) \quad (w \in W(E_8), \ x \in w.U).$$

The function $\widetilde{\tau}$ on $\widetilde{U}$ is $W(E_8)$-invariant and τ on U is recovered from $\widetilde{\tau}$ as $\widetilde{\tau}(1, x) = \tau(x)$ ($x \in U$). Also, any $W(E_8)$-invariant function on $\widetilde{U}$ is obtained in this way from a function on U. Note also that the function τ on U satisfies equations

$$(9.46) \ \sigma_{a_1 \pm a_2} T_{a_0}.\tau \ T_{a_0}^{-1}.\tau + \sigma_{a_2 \pm a_0} T_{a_1}.\tau \ T_{a_1}^{-1}.\tau + \sigma_{a_0 \pm a_1} T_{a_2}.\tau \ T_{a_2}^{-1}.\tau = 0$$

for all C_3-frames $\{\pm a_0, \pm a_1, \pm a_2\}$ relative to $Q(E_8)$ if and only if the $W(E_8)$-invariant function $\widetilde{\tau}$ on $\widetilde{U}$ satisfies

$$(9.47) \quad \sigma_{a_1 \pm a_2} T_{a_0}.\widetilde{\tau}\, T_{a_0}^{-1}.\widetilde{\tau} + \sigma_{a_2 \pm a_0} T_{a_1}.\widetilde{\tau}\, T_{a_1}^{-1}.\widetilde{\tau} + \sigma_{a_0 \pm a_1} T_{a_2}.\widetilde{\tau}\, T_{a_2}^{-1}.\widetilde{\tau} = 0$$

for all C_3-frames $\{\pm a_0, \pm a_1, \pm a_2\}$ relative to $Q(E_8)$. Applying Proposition 9.1 to the mapping

$$(9.48) \qquad \widetilde{\gamma} : \ \widetilde{U} \to \overline{\mathfrak{h}} : \quad \widetilde{\gamma}(w, x) = \gamma(x) \quad (w \in W(E_8), x \in w.U),$$

and $\widetilde{\tau}$ on $\widetilde{U}$, we obtain the following characterization of an ORG τ-function on U.

Proposition 9.2. *For a function τ on a subset $U \subseteq X$, consider the function $\widetilde{\tau}$ defined on $\widetilde{U}$. Then τ is an ORG τ-function on U if and only if the functions $\widetilde{\tau}_\Lambda = T_{\Lambda - e_9}^{-1}.\widetilde{\tau}$ $(\Lambda \in M)$ form a system lattice τ-functions on $\widetilde{U}$.*

We apply this proposition for constructing lattice τ-functions from ORG τ-functions discussed in this paper. Fixing a nonzero constant $\kappa \in \mathbb{C}^*$, let $D \subset V$ be a subset such that $D + Q(E_8)\kappa = D$, and take an ORG τ-function $\tau = \tau(x)$ on D with $\delta = \kappa$ in the sense of Definition 2.1. We then define the action of $T_v w \in T(V) \rtimes W(E_8)$ $(v \in V, \ w \in W(E_8))$ on V by

$$(9.49) \qquad\qquad T_v w.x = w.x + \kappa v \qquad (x \in V).$$

We denote by $\pi : \mathfrak{h} \to V$ the orthogonal projection to $\overset{\circ}{\mathfrak{h}} = V$ in (9.8), and by $\pi_\kappa : \mathfrak{h}_\kappa \to V$ its restriction to $\mathfrak{h}_\kappa$. This projection $\pi_\kappa : \mathfrak{h}_\kappa \to V$ is equivariant with respect to the action of $T(V) \rtimes W(E_8)$ and compatible with the scalar product in the sense $(v|\pi_\kappa(h)) = (v|h)$, $(v \in V, h \in \mathfrak{h})$. Introducing a subset U of $\mathfrak{h}_\kappa$ by

$$(9.50) \qquad U = \pi_\kappa^{-1}(D) = \big\{ h \in \mathfrak{h}_\kappa \,\big|\, \pi_\kappa(h) \in D \big\} \subset \mathfrak{h}_\kappa,$$

we regard τ as a function on U through the projection $\pi_\kappa : U \to D$. Note that the isomorphism

$$(9.51) \qquad \gamma_\kappa : V \times \mathbb{C} \overset{\sim}{\to} \mathfrak{h}_\kappa : \ \gamma_\kappa(x, \mu) = x - (\tfrac{1}{2\kappa}(x|x) + \mu)c + \kappa d,$$

induces the parametrization $\gamma_\kappa : D \times \mathbb{C} \overset{\sim}{\to} U$ of $U = \pi_\kappa^{-1}(D)$ with an invariant parameter $\mu \in \mathbb{C}$. Then obtain a $W(E_8)$-invariant function $\widetilde{\tau}$ on

$$(9.52) \qquad\qquad \widetilde{U} = \bigsqcup_{w \in W(E_8)} \{w\} \times w.U \subset W(E_8) \times \mathfrak{h}_\kappa$$

by setting

$$(9.53) \quad \widetilde{\tau}(w, h) = \tau(\pi_\kappa(w^{-1}.h)) = \tau(w^{-1}.x) \quad (w \in W(E_8),\ h \in w.U),$$

where $x = \pi_\kappa(h) \in w.D$. By Proposition 9.2, the family of functions $\widetilde{\tau}_\Lambda = T^{-1}_{\Lambda - e_9}.\widetilde{\tau}$ gives a system of lattice τ-functions on $\widetilde{U}$. If we take the classical part $\alpha = \pi_0(\Lambda - e_9) \in Q(E_8)$ of $\Lambda - e_9$, then we have $\widetilde{\tau}_\Lambda = T^{-1}_\alpha.\widetilde{\tau}$, and hence

$$(9.54) \quad \widetilde{\tau}_\Lambda(w, h) = \widetilde{\tau}(w, T_\alpha.h) = \tau(w^{-1}.\pi_\kappa(T_\alpha.h)) = \tau(w^{-1}.(x + \kappa\alpha)),$$

for any $w \in W(E_8)$ and $h \in w.U$, where $x = \pi_\kappa(h) \in w.D$.

Theorem 9.3. *Let $\kappa \in \mathbb{C}^*$ be a generic constant. Suppose that a subset $D \subseteq V$ is stable by $W(E_8)$ and $D + Q(E_8)\kappa = D$. Let $\tau(x)$ be an ORG τ-function on D with $\delta = \kappa$. For each $\Lambda \in M$, define a function $\widetilde{\tau}_\Lambda$ on $\widetilde{U}$ of (9.52) by*

$$(9.55) \qquad \widetilde{\tau}_\Lambda(w, h) = \tau(w^{-1}.(x + \kappa\alpha)) \quad (w \in W(E_8),\ h \in w.U)$$

with $\alpha = \pi_0(\Lambda - e_9) \in Q(E_8)$ and $x = \pi_\kappa(h) \in w.D$. Then $\widetilde{\tau}_\Lambda$ $(\Lambda \in M)$ form a system of lattice τ-functions on $\widetilde{U}$.

For example, we consider the hypergeometric the ORG τ-function $\tau(x) = \tau_{+-}(x)$ of Theorem 8.1 defined on

$$(9.56) \qquad \begin{aligned} D &= D_{\phi, -\varpi} = \bigsqcup_{n \in \mathbb{Z}} H_{\phi, -\varpi + n\kappa}, \\ H_{\phi, -\varpi + n\kappa} &= \left\{ x \in V \,\middle|\, (\phi|x) = -\varpi + n\kappa \right\} \quad (n \in \mathbb{Z}), \end{aligned}$$

under the identification $\delta = \kappa$. The components $\tau^{(n)} = \tau|_{H_{\phi, -\varpi + n\kappa}}$ $(n \in \mathbb{Z})$ are then given by $\tau^{(n)}(x) = 0$ $(n < 0)$ and

$$(9.57)$$
$$\tau^{(n)}(x) = \Psi_n(p^{\frac{1}{2}} q^{\frac{1}{2}(1-n)} u; p, q) = \Psi_n(q^{\frac{1}{2}} u^{-1}; p, q) \quad (n = 0, 1, 2, \ldots)$$

in terms of the elliptic hypergeometric integral (6.49). Here, $p = e(\varpi)$, $q = e(\kappa)$ and $u_j = e(x_j)$ denote the multiplicative variables corresponding to x_j $(j = 0, 1, \ldots, 7)$. In this case, the subset $U = \pi_\kappa^{-1}(D) \subset \mathfrak{h}_\kappa$ is specified as $U = \bigsqcup_{n \in \mathbb{Z}} U_n$, where

$$(9.58) \qquad \begin{aligned} U_n &= \left\{ h \in \mathfrak{h}_\kappa \,\middle|\, (\phi|h) = -\varpi + n\kappa \right\} \\ &= \left\{ h \in \mathfrak{h}_\kappa \,\middle|\, (\alpha_8|h) = \varpi + (1 - n)\kappa \right\} \quad (n \in \mathbb{Z}). \end{aligned}$$

We regard the ORG τ-function $\tau(x)$ as a function on U through the coordinates

$$(9.59) \qquad x_j = \varepsilon_j - \tfrac{1}{2}(\varepsilon_0 - \varepsilon_9) + \tfrac{1}{2}\kappa \quad (j = 1, \ldots, 8), \quad x_0 = -x_8.$$

The corresponding lattice τ-functions $\widetilde{\tau}_\Lambda$ $(\Lambda \in M)$ on

$$(9.60) \qquad \widetilde{U} = \bigsqcup_{w \in W(E_8)} \{w\} \times w.U$$

are defined as

$$(9.61) \qquad \widetilde{\tau}_\Lambda(w, h) = \tau(w^{-1}.(x + \kappa\alpha)) \quad (w \in W(E_8),\ h \in w.U),$$

where $\alpha = \pi_0(\Lambda - e_9) \in Q(E_8)$ and $x = \pi_\kappa(h)$. Since

$$(9.62) \qquad e_j - e_9 = v_0 + v_j - \phi + c \quad (j = 1, \ldots, 8),$$

the nine fundamental τ-functions $\widetilde{\tau}_{e_j}$ $(j = 1, \ldots, 9)$ are specified as

$$(9.63) \qquad \begin{aligned} \widetilde{\tau}_{e_j}(w, h) &= \tau(w^{-1}.(x + \kappa(v_0 + v_j - \phi))) \quad (j = 1, \ldots, 7), \\ \widetilde{\tau}_{e_8}(w, h) &= \tau(w^{-1}.(x - \kappa\phi)), \qquad \widetilde{\tau}_{e_9}(w, h) = \tau(w^{-1}.x). \end{aligned}$$

for $w \in W(E_8)$ and $h \in w.U$.

9.3. Remarks on the $\mathbb{P}^1 \times \mathbb{P}^1$ picture

The difference Painlevé equation of type E_8 can also be formulated in terms of the configuration of generic eight points in $\mathbb{P}^1 \times \mathbb{P}^1$. In this case, following the formulation of [6] we use the Picard lattice

$$(9.64) \qquad L = \mathbb{Z}\mathsf{h}_1 \oplus \mathbb{Z}\mathsf{h}_2 \oplus \mathbb{Z}\mathsf{e}_1 \oplus \mathbb{Z}\mathsf{e}_2 \oplus \cdots \oplus \mathbb{Z}\mathsf{e}_8$$

with the symmetric bilinear form $(\cdot|\cdot) : L \times L \to \mathbb{Z}$ defined by

$$(9.65) \qquad \begin{aligned} &(\mathsf{h}_i|\mathsf{h}_i) = 0 \ \ (i = 1, 2), \quad (\mathsf{h}_1|\mathsf{h}_2) = (\mathsf{h}_2|\mathsf{h}_1) = -1, \\ &(\mathsf{e}_i|\mathsf{e}_j) = \delta_{ij} \quad (i, j = 1, \ldots, 8). \end{aligned}$$

If we denote by (f, g) the inhomogeneous coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$, h_1 and h_2 represent the classes of lines $f = \mathrm{const.}$ and $g = \mathrm{const.}$ respectively, and $\mathsf{e}_1, \ldots, \mathsf{e}_8$ the classes of exceptional divisors corresponding to the generic eight points. This Picard lattice and its symmetric bilinear form are identified with those we have used in the $\mathbb{P}^2$ picture through the change of bases

$$(9.66)$$
$$\mathsf{h}_1 = e_0 - e_2, \ \mathsf{h}_2 = e_0 - e_1, \ \mathsf{e}_1 = e_0 - e_1 - e_2, \ \mathsf{e}_j = e_{j+1} \ (j = 2, \ldots, 8),$$

and
(9.67)
$$e_0 = h_1 + h_2 - e_1, \ e_1 = h_1 - e_1, \ e_2 = h_2 - e_1, \ e_j = e_{j-1} \quad (j = 3, \ldots, 9).$$

The simple roots $\alpha_0, \alpha_1, \ldots, \alpha_8$ are now expressed as
(9.68)
$$\alpha_0 = e_1 - e_2, \ \alpha_1 = h_1 - h_2, \ \alpha_2 = h_2 - e_1 - e_2, \ \alpha_j = e_{j-1} - e_j \ (j = 3, \ldots, 8).$$

The complex vector space $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ is decomposed as $\mathfrak{h} = \overset{\circ}{\mathfrak{h}} \oplus \mathbb{C}c \oplus \mathbb{C}d$ where

(9.69) $$c = 2h_1 + 2h_2 - e_1 - \cdots - e_8, \quad d = -e_8 - \tfrac{1}{2}c.$$

In this realization, the orthonormal basis $\{v_0, v_1, \ldots, v_8\}$ for $V = \overset{\circ}{\mathfrak{h}}$ is given by
(9.70)
$$v_1 = h_1 - e_1 - \tfrac{1}{2}(h_1 + h_2 - e_1 - e_8) + \tfrac{1}{2}c,$$
$$v_2 = h_2 - e_1 - \tfrac{1}{2}(h_1 + h_2 - e_1 - e_8) + \tfrac{1}{2}c,$$
$$v_j = e_{j-1} - \tfrac{1}{2}(h_1 + h_2 - e_1 - e_8) + \tfrac{1}{2}c \ (j = 3, \ldots, 8), \quad v_0 = -v_8.$$

Accordingly, the coordinates $x = (x_0, x_1, \ldots, x_7)$ for V are given by
(9.71)
$$x_1 = \eta_1 - \epsilon_1 - \tfrac{1}{2}(\eta_1 + \eta_2 - \epsilon_1 - \epsilon_8) + \tfrac{1}{2}\delta,$$
$$x_2 = \eta_2 - \epsilon_1 - \tfrac{1}{2}(\eta_1 + \eta_2 - \epsilon_1 - \epsilon_8) + \tfrac{1}{2}\delta,$$
$$x_j = \epsilon_{j-1} - \tfrac{1}{2}(\eta_1 + \eta_2 - \epsilon_1 - \epsilon_8) + \tfrac{1}{2}\delta \ (j = 3, \ldots, 8), \quad x_0 = -x_8.$$

where $\eta_i = (h_i | \cdot) \ (i = 1, 2)$ and $\epsilon_j = (e_j | \cdot) \ (j = 1, \ldots, 8)$.
   In this framework, a system of lattice τ-functions is defined as a family of dependent variables τ_Λ, indexed by the same orbit

(9.72) $$M = W(E_8^{(1)})\{e_1, \ldots, e_8\} = W(E_8^{(1)})e_8 \subset L,$$

which admit an action of $W(E_8^{(1)})$ such that

(9.73) $$w.\tau_\Lambda = \tau_{w.\Lambda} \quad (w \in W(E_8^{(1)}), \ \Lambda \in M)$$

and satisfy the quadratic relations

(9.74) $$\sigma_{e_{jk}} \sigma_{e_{r;jk}} \tau_{e_i} \tau_{h_r - e_i} + \sigma_{e_{ki}} \sigma_{e_{r;ki}} \tau_{e_j} \tau_{h_r - e_j} + \sigma_{e_{ij}} \sigma_{e_{r;ij}} \tau_{e_k} \tau_{h_r - e_k} = 0$$

for $r = 1, 2$ and for mutually distinct $i, j, k \in \{1, \ldots, 8\}$, where $e_{ij} = e_i - e_j$ and $e_{r;ij} = h_r - e_i - e_j$. Through the expression $\tau_\Lambda = T_{\Lambda - e_8}^{-1} \tau_{e_8}$, this notion of lattice τ-functions is interpreted by that of ORG τ-functions in the same way as in Proposition 9.2 and Theorem 9.3.

§Appendix A.   Proof of Theorem 3.3

In this Appendix, we give a proof of Theorem 3.3. The following proof is essentially the same as the argument in Masuda [8, Section 3]. We first prove Theorem 3.3 under an additional assumption that $\tau^{(n-1)}(x)$ and $\tau^{(n)}(x)$ are $W(E_7)$-invariant. After that we explain how the general case can be reduced to the invariant case.

A.1.   Preliminary remark

We begin by a general remark on the Hirota equations associated with C_3-frames. Fixing a C_l-frame $A = \{\pm a_1, \ldots, \pm a_l\}$ $(l = 3, \ldots, 8)$, we suppose that a function $\tau(x)$ satisfies the Hirota equation
(A.1)
$$\tau(x \pm a_i \delta)[(a_j \pm a_k | x)] + \tau(x \pm a_j \delta)[(a_k \pm a_i | x)] + \tau(x \pm a_k \delta)[(a_i \pm a_j | x)] = 0$$

for any triple $i, j, k \in \{1, \ldots, l\}$. We also assume that $[(a_i \pm a_j | x)] \neq 0$ on the domain of definition of τ for any distinct pair $i, j \in \{1, \ldots, l\}$. Then (A.1) can be written as

(A.2) $$\tau(x \pm a_k \delta) = \frac{\tau(x \pm a_i \delta)[(a_k \pm a_j | x)] - \tau(x \pm a_j \delta)[(a_k \pm a_i | x)]}{[(a_i \pm a_j | x)]}$$

for any $k \in \{1, \ldots, l\}$. In view of this expression, for each $u \in V$ we define

(A.3) $$f_{ij}(x; u) = \frac{\tau(x \pm a_i)[(u \pm a_j | x)] - \tau(x \pm a_j)[(u \pm a_i | x)]}{[(a_i \pm a_j | x)]}$$

for each distinct pair $i, j \in \{1, \ldots, l\}$, so that $f_{ij}(x; a_k) = \tau(x \pm a_k \delta)$ $(k \in \{1, \ldots, l\})$. A simple but important observation is that the three-term relation (2.1) of the function $[z]$ implies the functional equation
(A.4)
$$f_{ij}(x; u_0)[(u_1 \pm u_2 | x)] + f_{ij}(x; u_1)[(u_2 \pm u_0 | x)] + f_{ij}(x; u_2)[(u_0 \pm u_1 | x)] = 0$$

for any $u_0, u_1, u_2 \in V$. From this, for any distinct pair $r, s \in \{1, \ldots, l\}$ we obtain

(A.5)
$$\begin{aligned}
f_{ij}(x; u)[(a_r \pm a_s | x)] &= f_{ij}(x; a_r)[(u \pm a_s | x)] - f_{ij}(x; a_s)[(u \pm a_r | x)] \\
&= \tau(x \pm a_r \delta)[(u \pm a_s | x)] - \tau(x \pm a_s \delta)[(u \pm a_r | x)] \\
&= f_{rs}(x; u)[(a_r \pm a_s | x)],
\end{aligned}$$

and hence $f_{ij}(x; u) = f_{rs}(x; u)$. Summarizing these arguments, we have

Lemma A.1. *Let* $A = \{\pm a_1, \ldots, \pm a_l\}$ *be a* C_l-*frame and suppose that* $\tau(x)$ *satisfies the Hirota equation* (A.1) *for any triple* $i, j, k \in \{1, \ldots, l\}$. *Then the function*

(A.6)
$$f(x; u) = \frac{\tau(x \pm a_i)[(u \pm a_j|x)] - \tau(x \pm a_j)[(u \pm a_i|x)]}{[(a_i \pm a_j|x)]} \quad (x, u \in V)$$

does not depend on the choice of distinct $i, j \in \{1, \ldots, l\}$. *Furthermore, it satisfies*

(A.7)
$$f(x; u_0)[(u_1 \pm u_2|x)] + f(x; u_1)[(u_2 \pm u_0|x)] + f(x; u_2)[(u_0 \pm u_1|x)] = 0$$

for any $u_0, u_1, u_2 \in V$, *and*

(A.8)
$$f(x; a_k) = \tau(x \pm a_k \delta) \quad (k = 1, \ldots, l).$$

Returning to the setting of Theorem 3.3, we suppose furthermore that $\tau^{(n-1)}(x)$ on $H_{c+(n-1)\delta}$ and $\tau^{(n)}(x)$ on $H_{c+n\delta}$ are $W(E_7)$-invariant. For a function $\varphi = \varphi(x)$, the action $w.\varphi$ of $w \in W(E_7)$ is defined by $(w.\varphi)(x) = \varphi(w^{-1}.x)$. We say that φ is invariant with respect to w if $w.\varphi = \varphi$, namely, $\varphi(w^{-1}.x) = \varphi(x)$. Note also that, for the function ψ defined by $\psi(x) = \varphi(x+v)$ $(v \in V)$, we have $(w.\psi)(x) = (w.\varphi)(x+w.v)$, and hence $(w.\psi)(x) = \varphi(x + w.v)$ if φ is w-invariant.

A.2.   Definition of $\tau^{(n+1)}$: $W(E_7)$-invariance and $(\mathrm{II}_1)_n$

We first show that there exists a unique $W(E_7)$-invariant (meromorphic) function $\tau^{(n+1)}(x)$ on $H_{c+(n+1)\delta}$ that satisfies the bilinear equations of type $(\mathrm{II}_1)_n$.

We consider the C_8-frame $A = \{\pm a_0, \pm a_1, \ldots, \pm a_7\}$ defined by

(A.9)
$$a_0 = \tfrac{1}{2}(v_0 - v_1 + \phi), \quad a_1 = \tfrac{1}{2}(v_1 - v_0 + \phi),$$
$$a_j = v_j + \tfrac{1}{2}(v_0 + v_1 - \phi) \quad (j = 2, \ldots, 7).$$

Note that $(\phi|a_0) = (\phi|a_1) = 1$ and $(\phi|a_j) = 0$ $(j = 2, \ldots, 7)$. Then, from the assumption that $\tau^{(n)}(x)$ satisfies the bilinear equations of type $(\mathrm{II}_0)_n$, by Lemma A.1 it follows that the function

(A.10) $f(x; u) = \dfrac{\tau^{(n)}(x \pm a_i\delta)[(u \pm a_j|x)] - \tau^{(n)}(x \pm a_j\delta)[(u \pm a_i|x)]}{[a_i \pm a_j]}$

does not depend on the choice of distinct $i, j \in \{2, \ldots, 7\}$. In view of the bilinear equations of type $(\mathrm{II}_1)_n$, we define the function $\tau^{(n+1)}(x)$ on $H_{c+(n+1)\delta}$ by the equation

(A.11) $\tau^{(n+1)}(x + a_0\delta)\tau^{(n-1)}(x - a_0\delta) = f(x; a_0).$

This implies that $\tau^{(n+1)}(x)$ satisfies

(A.12)
$$\tau^{(n+1)}(x + a_0\delta)\tau^{(n-1)}(x - a_0\delta)[(a_i \pm a_j|x)]$$
$$= \tau^{(n)}(x \pm a_i\delta)[(a_0 \pm a_j|x)] - \tau^{(n)}(x \pm a_j\delta)[(a_0 \pm a_i|x)]$$

for any distinct $i, j \in \{2, \ldots, 7\}$. By replacing x with $x - a_0\delta$, these equations can be written as

(A.13)
$$\tau^{(n+1)}(x)$$
$$= \frac{\tau^{(n)}(x-(a_0\pm a_i)\delta)[(a_0\pm a_j|x)-\delta]-\tau^{(n)}(x-(a_0\pm a_j)\delta)[(a_0\pm a_i|x)-\delta]}{\tau^{(n-1)}(x - 2a_0\delta)[(a_i\pm a_j|x)]}$$

for any distinct $i, j \in \{2, \ldots, 7\}$. In terms of the basis $v_0, v_1, \ldots, v_7$ for V, we have

(A.14)
$$\tau^{(n+1)}(x) = \frac{1}{\tau^{(n-1)}(x - (\phi + v_0 - v_1)\delta)[(v_j - v_i|x)][(\phi - v_{01ij}|x)]}$$
$$\cdot \Big(\tau^{(n)}(x - v_{0i}\delta)\tau^{(n)}(x - (\phi - v_{1i})\delta)[(v_{0j}|x) - \delta][(\phi - v_{1j}|x) - \delta]$$
$$- \tau^{(n)}(x - v_{0j}\delta)\tau^{(n)}(x - (\phi - v_{1j})\delta)[(v_{0i}|x) - \delta][(\phi - v_{1i}|x) - \delta]\Big)$$

for any distinct $i, j \in \{2, \ldots, 7\}$, where $v_{ab} = v_a + v_b$ and $v_{abcd} = v_a + v_b + v_c + v_d$.

We next show that this function $\tau^{(n+1)}(x)$ is invariant under the action of the Weyl group

(A.15)
$$W(E_7) = \langle s_0, s_1, \ldots, s_6 \rangle; \quad s_0 = r_{\phi - v_{0123}}, \ s_j = r_{v_j - v_{j+1}} \ (j = 1, \ldots, 6).$$

Since $\tau^{(n-1)}(x)$ and $\tau^{(n)}(x)$ are $W(E_7)$-invariant, from the fact that the expression (A.14) does not depend on the choice of $i, j \in \{2, \ldots, 7\}$, it follows that $\tau^{(n+1)}(x)$ is invariant with respect to $s_2, \ldots, s_6$. To see the invariance with respect to s_0, we take $(i, j) = (2, 3)$:

(A.16)
$$\tau^{(n+1)}(x) = \frac{1}{\tau^{(n-1)}(x - (\phi + v_0 - v_1)\delta)[(v_3 - v_2|x)][(\phi - v_{0123}|x)]}$$
$$\cdot \Big(\tau^{(n)}(x - v_{02}\delta)\tau^{(n)}(x - (\phi - v_{12})\delta)[(v_{03}|x) - \delta][(\phi - v_{13}|x) - \delta]$$
$$- \tau^{(n)}(x - v_{03}\delta)\tau^{(n)}(x - (\phi - v_{13})\delta)[(v_{02}|x) - \delta][(\phi - v_{12}|x) - \delta]\Big).$$

Since $s_0(v_{ij}) = \phi - v_{kl}$ for $\{i, j, k, l\} = \{0, 1, 2, 3\}$, this expression is manifestly invariant with respect to s_0. It remains to show that $\tau^{(n+1)}(x)$ is

invariant with respect to s_1. The s_1-invariance of $\tau^{(n+1)}(x)$ is equivalent to the equality of

(A.17)
$$L = \tau^{(n-1)}(x - (\phi + v_0 - v_2)\delta)[(v_3 - v_1|x)]$$
$$\cdot \left(\tau^{(n)}(x - v_{02}\delta)\tau^{(n)}(x - (\phi - v_{12})\delta)[(v_{03}|x) - \delta][(\phi - v_{13}|x) - \delta] \right.$$
$$\left. - \tau^{(n)}(x - v_{03}\delta)\tau^{(n)}(x - (\phi - v_{13})\delta)[(v_{02}|x) - \delta][(\phi - v_{12}|x) - \delta] \right)$$

and

(A.18)
$$R = \tau^{(n-1)}(x - (\phi + v_0 - v_1)\delta)[(v_3 - v_2|x)]$$
$$\cdot \left(\tau^{(n)}(x - v_{01}\delta)\tau^{(n)}(x - (\phi - v_{12})\delta)[(v_{03}|x) - \delta][(\phi - v_{23}|x) - \delta] \right.$$
$$\left. - \tau^{(n)}(x - v_{03}\delta)\tau^{(n)}(x - (\phi - v_{23})\delta)[(v_{01}|x) - \delta][(\phi - v_{12}|x) - \delta] \right).$$

Expanding these as $L = L_1 - L_2$ and $R = R_1 - R_2$, we look at

(A.19)
$$L_1 - R_1$$
$$= \tau^{(n)}(x - (\phi - v_{12})\delta)[(v_{03}|x) - \delta]$$
$$\cdot \left(\tau^{(n-1)}(x - (\phi + v_0 - v_2)\delta)\tau^{(n)}(x - v_{02}\delta)[(v_3 - v_1|x)][(\phi - v_{13}|x) - \delta] \right.$$
$$\left. - \tau^{(n-1)}(x - (\phi + v_0 - v_1)\delta)\tau^{(n)}(x - v_{01}\delta)[(v_3 - v_2|x)][(\phi - v_{23}|x) - \delta] \right).$$

Setting $u_i = \frac{1}{2}\phi - v_i$ and $y = x - (\frac{1}{2}\phi + v_0)$, we compute the last factor as

(A.20)
$$\tau^{(n-1)}(y - u_2\delta)\tau^{(n)}(y + u_2\delta)[(u_1 \pm u_3|y)]$$
$$- \tau^{(n-1)}(y - u_1\delta)\tau^{(n)}(y + u_1\delta)[(u_2 \pm u_3|y)]$$
$$= \tau^{(n-1)}(y - u_3\delta)\tau^{(n)}(y + u_3\delta)[(u_1 \pm u_2|y)]$$
$$= \tau^{(n-1)}(x - (\phi + v_0 - v_3)\delta)\tau^{(n)}(x - v_{03}\delta)[(v_2 - v_1|x)][(\phi - v_{12}|x) - \delta]$$

by the bilinear equation of type $(I)_{n-1/2}$ for the C_3-frame $\{\pm u_1, \pm u_2, \pm u_3\}$. Hence we have

(A.21)
$$L_1 - R_1 = \tau^{(n-1)}(x - (\phi + v_0 - v_3)\delta)\tau^{(n)}(x - v_{03}\delta)\tau^{(n)}(x - (\phi - v_{12})\delta)$$
$$\cdot [(v_{03}|x) - \delta][(v_2 - v_1|x)][(\phi - v_{12}|x) - \delta].$$

On the other hand, we have

(A.22)
$$L_2 - R_2$$
$$= \tau^{(n)}(x - v_{03}\delta)[(\phi - v_{12}|x) - \delta]$$
$$\cdot \left(\tau^{(n-1)}(x - (\phi + v_0 - v_2)\delta)\tau^{(n)}(x - (\phi - v_{13}))[(v_3 - v_1|x)][(v_{02}|x) - \delta]\right.$$
$$\left. -\tau^{(n-1)}(x - (\phi + v_0 - v_1)\delta)\tau^{(n)}(x - (\phi - v_{23})\delta)[(v_3 - v_2|x)][(v_{01}|x) - \delta]\right).$$

Setting $b_i = \frac{1}{2}v_{0123} - v_i$ $(i = 0, 1, 2, 3)$ and $z = x - (\phi - b_0)\delta$, we compute the last factor as

(A.23)
$$\tau^{(n-1)}(z - b_2\delta)\tau^{(n)}(z + b_2\delta)[(b_1 \pm b_3|z)]$$
$$- \tau^{(n-1)}(z - b_1\delta)\tau^{(n)}(z + b_1\delta)[(b_2 \pm b_3|z)]$$
$$= \tau^{(n-1)}(z - b_3\delta)\tau^{(n)}(z + b_3\delta)[(b_1 \pm b_2|z)]$$
$$= \tau^{(n-1)}(x - (\phi + v_0 - v_3)\delta)\tau^{(n)}(x - (\phi - v_{12})\delta)[(v_2 - v_1|x)][(v_{03}|x) - \delta]$$

by the bilinear equation of type $(I)_{n-1/2}$ for the C_3-frame $\{\pm b_1, \pm b_2, \pm b_3\}$. Hence

(A.24)
$$L_2 - R_2 = \tau^{(n-1)}(x - (\phi + v_0 - v_3)\delta)\tau^{(n)}(x - v_{03}\delta)\tau^{(n)}(x - (\phi - v_{12})\delta)$$
$$\cdot [(v_{03}|x) - \delta][(v_2 - v_1|x)][(\phi - v_{12}|x) - \delta]$$
$$= L_1 - R_1,$$

which implies $L = R$ as desired.

Recall that $W(E_7)$ acts transitively on the set of all C_3-frames of type II_1. Since $\tau^{(n+1)}(x)$ is $W(E_7)$-invariant and satisfies (A.12), it readily satisfies the bilinear equations of type $(II_1)_n$ for all C_3-frames of type II_1.

A.3. $(II_1)_n \implies (II_2)_n$

Since $\tau^{(n+1)}(x)$ is $W(E_7)$-invariant, we have only to show that it satisfies the bilinear equation of type $(II_2)_n$ for a particular C_3-frame of type II_2.

Taking the C_8-frame (A.9) of type II as before, we look at the bilinear relations

$$\tau^{(n+1)}(x + a_k\delta)\tau^{(n-1)}(x - a_k\delta)$$

(A.25)
$$= \frac{\tau^{(n)}(x \pm a_2\delta)[(a_k \pm a_3|x)] - \tau^{(n)}(x \pm a_3\delta)[(a_k \pm a_2|x)]}{[(a_2 \pm a_3|x)]}$$

of type $(II_1)_n$ for $k = 0, 1$. From these we have

(A.26)
$$\tau^{(n+1)}(x + a_0\delta)\tau^{(n-1)}(x - a_0\delta)[(a_2 \pm a_1|x)]$$
$$- \tau^{(n+1)}(x + a_1\delta)\tau^{(n-1)}(x - a_1\delta)[(a_2 \pm a_0|x)]$$
$$= \frac{\tau^{(n)}(x \pm a_2\delta)}{[(a_2 \pm a_3|x)]}\left([(a_0 \pm a_3|x)][(a_2 \pm a_1|x)] - [(a_1 \pm a_3|x)][(a_2 \pm a_0|x)]\right)$$
$$= \tau^{(n)}(x \pm a_2\delta)[(a_0 \pm a_1|x)],$$

which is the bilinear equation of type $(II_2)_n$ for the C_3-frame $\{\pm a_0, \pm a_1, \pm a_2\}$ of type II_2.

A.4. $(II_1)_n \implies (I)_{n+1/2}$

We have only to show that the bilinear equation $(I)_{n+1/2}$ holds for some particular C_3-frame of type I.

Taking the C_8-frame of (A.9), we look at the bilinear relation

(A.27)
$$\tau^{(n+1)}(x)\tau^{(n-1)}(x - (\phi + v_0 - v_1)\delta)[(v_3 - v_2|x)][(\phi - v_{0123}|x)]$$
$$= \tau^{(n)}(x - (\phi - v_{12})\delta)\tau^{(n)}(x - v_{02}\delta)[(v_{03}|x) - \delta][(\phi - v_{13}|x) - \delta]$$
$$- \tau^{(n)}(x - (\phi - v_{13})\delta)\tau^{(n)}(x - v_{03}\delta)[(v_{02}|x) - \delta][(\phi - v_{12}|x) - \delta]$$

of type $(II_1)_n$. Replacing x with $x + (v_0 - v_1)\delta$, we rewrite this formula into

(A.28)
$$\tau^{(n+1)}(x + (v_0 - v_1)\delta)\tau^{(n-1)}(x - \phi\delta)[(v_3 - v_2|x)][(\phi - v_{0123}|x)]$$
$$= \tau^{(n)}(x - (\phi - v_{02})\delta)\tau^{(n)}(x - v_{12}\delta)[(v_{03}|x)][(\phi - v_{13}|x)]$$
$$- \tau^{(n)}(x - (\phi - v_{03})\delta)\tau^{(n)}(x - (v_{13})\delta)[(v_{02}|x)][(\phi - v_{12}|x)].$$

Multiplying this by $\tau^{(n)}(x + (v_{01} - \phi)\delta)[(\phi - v_{23}|x)]$, we obtain

(A.29)
$$\tau^{(n+1)}(x + (v_0 - v_1)\delta)\tau^{(n)}(x + (v_{01} - \phi)\delta)[(v_3 - v_2|x)][(\phi - v_{23}|x)]$$
$$\cdot \tau^{(n-1)}(x - \phi\delta)[(\phi - v_{0123}|x)]$$
$$= \tau^{(n)}(x - u_{01}\delta)\tau^{(n)}(x - u_{02}\delta)\tau^{(n)}(x - v_{12}\delta)[(u_{13}|x)][(u_{23}|x)][(v_{03}|x)]$$
$$- \tau^{(n)}(x - u_{01}\delta)\tau^{(n)}(x - u_{03}\delta)\tau^{(n)}(x - v_{13}\delta)[(u_{12}|x)][(u_{23}|x)][(v_{02}|x)]$$

where $u_i = \frac{1}{2}\phi - v_i$ and $u_{ij} = u_i + u_j = \phi - v_{ij}$. Then, applying the cyclic permutation (123) to this formula, we see that the sum of those

three vanishes term by term. It means that

(A.30)
$$\tau^{(n+1)}(x + (v_0 - v_1)\delta)\tau^{(n)}(x + (v_{01} - \phi)\delta)[(v_3 - v_2|x)][(\phi - v_{23}|x)]$$
$$+ \cdots = 0,$$

namely

(A.31)
$$\tau^{(n+1)}(x + (u_1 - u_0)\delta)\tau^{(n)}(x - (u_0 + u_1)\delta)[(u_2 \pm u_3|x)] + \cdots = 0.$$

Replacing x by $x + u_0\delta$, we obtain the bilinear equation

(A.32) $\qquad \tau^{(n+1)}(x + u_1\delta)\tau^{(n)}(x - u_1\delta)[(u_2 \pm u_3|x)] + \cdots = 0$

of type $(\mathrm{I})_{n+1/2}$ for the C_3-frame $\{\pm u_1, \pm u_2, \pm u_3\}$ of type I.

A.5. $(\mathrm{II}_1)_n \implies (\mathrm{II}_0)_{n+1}$

In order to show that $\tau^{(n+1)}(x)$ satisfies the bilinear equations of type $(\mathrm{II})_{n+1}$, we take the C_8-frame $A = \{\pm a_0, \pm a_1, \ldots \pm a_7\}$ of type II defined by

(A.33)
$$a_j = v_j + \tfrac{1}{2}(v_{67} - \phi) \quad (j = 0, 1, 2, 3, 4, 5);$$
$$a_6 = \tfrac{1}{2}(v_6 - v_7 + \phi), \quad a_7 = \tfrac{1}{2}(v_7 - v_6 + \phi).$$

We prove that $\tau^{(n+1)}(x)$ satisfies the bilinear equation

(A.34) $\qquad \tau^{(n+1)}(x + (a_0 \pm a_3)\delta)[(a_4 \pm a_5|x)] + \cdots = 0$

of type $(\mathrm{II}_0)_{n+1}$ for the C_3-frame $\{\pm a_3, \pm a_4, \pm a_5\}$. In terms of the basis $v_0, v_1, \ldots, v_7$ for V, this equation is expressed as

(A.35)
$$\tau^{(n+1)}(x + (v_0 - v_3)\delta)\tau^{(n+1)}(x + (\phi - v_{1245})\delta)[(v_4 - v_5|x)][(\phi - v_{0123}|x)]$$
$$+ \cdots = 0.$$

We now look at the bilinear equation

(A.36)
$$\tau^{(n+1)}(x + (v_0 - v_4)\delta)\tau^{(n-1)}(x - \phi\delta)[(v_2 - v_1|x)][(\phi - v_{0124}|x)]$$
$$= \tau^{(n)}(x - (\phi - v_{01})\delta)\tau^{(n)}(x - v_{14}\delta)[(v_{02}|x)][(\phi - v_{24}|x)]$$
$$- \tau^{(n)}(x - (\phi - v_{02})\delta)\tau^{(n)}(x - v_{24}\delta)[(v_{01}|x)][(\phi - v_{14}|x)]$$

of type $(\mathrm{II}_1)_n$. Applying s_0 to this formula, we obtain

(A.37)
$$\tau^{(n+1)}(x + (\phi - v_{1234})\delta)\tau^{(n-1)}(x - \phi\delta)[(v_2 - v_1|x)][(v_3 - v_4|x)]$$
$$= \tau^{(n)}(x - v_{23}\delta)\tau^{(n)}(x - v_{14}\delta)[(\phi - v_{13}|x)][(\phi - v_{24}|x)]$$
$$- \tau^{(n)}(x - v_{13})\delta)\tau^{(n)}(x - v_{24}\delta)[(\phi - v_{23}|x)][(\phi - v_{14}|x)].$$

Using these formulas, we compute

(A.38)
$$\tau^{(n+1)}(x + (v_0 - v_3)\delta)\tau^{(n+1)}(x + (\phi - v_{1245})\delta)[(v_4 - v_5|x)][(\phi - v_{0123}|x)]$$
$$\cdot \tau^{(n-1)}(x - \phi)^2[(v_2 - v_1|x)]^2$$
$$= \big(\tau^{(n)}(x - u_{01}\delta)\tau^{(n)}(x - v_{13}\delta)[(v_{02}|x)][(u_{23}|x)]$$
$$\quad - \tau^{(n)}(x - u_{02}\delta)\tau^{(n)}(x - v_{23}\delta)[(v_{01}|x)][(u_{13}|x)]\big)$$
$$\cdot \big(\tau^{(n)}(x - v_{24}\delta)\tau^{(n)}(x - v_{15}\delta)[(u_{14}|x)][(u_{25}|x)]$$
$$\quad - \tau^{(n)}(x - v_{14}\delta)\tau^{(n)}(x - v_{25}\delta)[(u_{24}|x)][(u_{15}|x)]\big)$$
$$= \tau^{(n)}(x - u_{01}\delta)[(v_{02}|x)]$$
$$\cdot \big\{\tau^{(n)}(x - v_{13}\delta)\tau^{(n)}(x - v_{24}\delta)\tau^{(n)}(x - v_{15}\delta)[(u_{23}|x)][(u_{14}|x)][(u_{25}|x)]$$
$$\quad - \tau^{(n)}(x - v_{13}\delta)\tau^{(n)}(x - v_{14}\delta)\tau^{(n)}(x - v_{25}\delta)[(u_{23}|x)][(u_{24}|x)][(u_{15}|x)]\big\}$$
$$+ \tau^{(n)}(x - u_{02}\delta)[(v_{01}|x)]$$
$$\cdot \big\{\tau^{(n)}(x - v_{23}\delta)\tau^{(n)}(x - v_{14}\delta)\tau^{(n)}(x - v_{25}\delta)[(u_{13}|x)][(u_{24}|x)][(u_{15}|x)]$$
$$\quad - \tau^{(n)}(x - v_{23}\delta)\tau^{(n)}(x - v_{24}\delta)\tau^{(n)}(x - v_{15}\delta)[(u_{13}|x)][(u_{14}|x)][(u_{25}|x)]\big\}$$

Applying the cyclic permutation (345) to this formula, we can directly observe that the sum of those three vanishes term by term. This completes the proof of Theorem 3.3 in the case where $\tau^{(n-1)}(x)$ and $\tau^{(n)}(x)$ are $W(E_7)$-invariant.

A.6.   General case

We now consider the general case where $\tau^{(n-1)}(x)$ and $\tau^{(n)}(x)$ are not necessarily $W(E_7)$-invariant. In such a situation, we need to deal with all the transforms $w.\tau^{(n-1)}$ and $w.\tau^{(n)}$ by $w \in W(E_7)$ simultaneously.

For each hyperplane $H_\kappa = \{x \in V \mid (\phi|x) = \kappa\}$ $(\kappa \in \mathbb{C})$ perpendicular to ϕ, we introduce the covering space

(A.39)
$$\widetilde{H}_\kappa = W(E_7) \times H_\kappa,$$

and define the action of $w \in W(E_7)$ and the translation T_v ($v \in H_0$) by
(A.40)
$$w.(g,x) = (wg, w.x), \quad T_v.(g,x) = (g, x + v\delta) \quad (g \in W(E_7),\ x \in H_\kappa),$$

so that $wT_v = T_{w.v}w$. For each function ψ on $\widetilde{H}_\kappa$, the induced actions
$w.\psi$ ($w \in W(E_7)$) and $T_v.\psi$ ($v \in H_0$) are described as

(A.41) $(w.\psi)(g,x) = \psi(w^{-1}g, w^{-1}x), \quad (T_v.\psi)(g,x) = \psi(g, x - v\delta).$

For each function φ on H_κ, we define a function $\widetilde{\varphi}$ on $\widetilde{H}_\kappa$ by

(A.42) $\widetilde{\varphi}(g,x) = (g.\varphi)(x) = \varphi(g^{-1}.x) \quad (g \in W(E_7),\ x \in H_\kappa).$

Then this function $\widetilde{\varphi}$ is $W(E_7)$-invariant, and φ is recovered from $\widetilde{\varphi}$ by
$\varphi(x) = \widetilde{\varphi}(1,x)$ ($x \in H_\kappa$). Conversely, a function ψ on $\widetilde{H}_\kappa$ is $W(E_7)$-
invariant if and only if it is obtained as the lift $\psi = \widetilde{\varphi}$ of a function φ
on H_κ.
   Returning to the setting of Theorem 3.3, for the functions $\tau^{(k)}$ on
$H_{c+k\delta}$ we consider the lifts $\widetilde{\tau}^{(k)}$ on $\widetilde{H}_{c+k\delta}$ ($k = n-1, n$). Note that, for
each $g \in W(E_7)$, $g.\tau^{(n-1)}$ and $g.\tau^{(n)}$ also satisfy the bilinear equations
$(\text{I})_{n-1/2}$ and $(\text{II}_0)_n$. Hence, the lifted functions $\widetilde{\tau}^{(k)}$ ($k = n-1, n$) satisfy
the bilinear equations corresponding to $(\text{I})_{n-1/2}$ and $(\text{II}_0)_n$. Formally,
those bilinear equations can be written as

(A.43) $(\text{I})_{n-1/2}$: $[(a_1 \pm a_2|x)]\,\widetilde{\tau}^{(n-1)}(g, x - a_0)\widetilde{\tau}^{(n)}(g, x + a_0) + \cdots = 0$

and

(A.44) $(\text{II})_n$: $[(a_1 \pm a_2|x)]\,\widetilde{\tau}^{(n)}(g, x \pm a_0) + \cdots = 0.$

As in (A.12), we can define a function $\psi = \psi(g,x)$ on $\widetilde{H}_{c+(n+1)\delta}$ so that

(A.45)
$$\psi(g, x + a_0\delta)\widetilde{\tau}^{(n-1)}(g, x - a_0\delta)[(a_i \pm a_j|x)]$$
$$= \widetilde{\tau}^{(n)}(g, x - (a_0 \pm a_i)\delta)[(a_0 \pm a_j|x)] - \widetilde{\tau}^{(n)}(g, x \pm a_j\delta)[(a_0 \pm a_i|x)]$$

for any distinct $i, j \in \{2, \ldots, 7\}$. Since $\widetilde{\tau}^{(k)}$ are $W(E_7)$-invariant on
$\widetilde{H}_{c+k\delta}$ ($k = n-1, n$), applying the previous arguments to the lifted
functions, we see that ψ is also $W(E_7)$-invariant and satisfies the bilinear
equations corresponding to $(\text{II}_1)_n$, $(\text{II}_2)_n$, $(\text{I})_{n+1/2}$, $(\text{II}_0)_{n+1}$. Since ψ is
$W(E_7)$-invariant, it is expressed as $\psi = \widetilde{\tau}^{(n+1)}$ with a function on $\tau^{(n+1)}$
on $H_{c+(n+1)\delta}$:

(A.46) $\psi(g,x) = \widetilde{\tau}^{(n+1)}(g,x) = (g.\tau^{(n+1)})(x), \quad \tau^{(n+1)}(x) = \psi(1,x).$

Then the bilinear equations $(II_1)_n$, $(II_2)_n$, $(I)_{n+1/2}$, $(II_0)_{n+1}$ for $\widetilde{\tau}^{(n-1)}$, $\widetilde{\tau}^{(n)}$ and $\psi = \widetilde{\tau}^{(n+1)}$ means that $\tau^{(n+1)}$ satisfies the corresponding bilinear equations of four types and that the recursive construction of $\tau^{(n+1)}$ from $\tau^{(n-1)}$, $\tau^{(n)}$ is equivariant with respect to the action of $W(E_7)$.

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