Advanced Studies in Pure Mathematics 75, 207
Algebraic Varieties and Automorphism Groups
pp. 465–474

Amalgamations and automorphism groups

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Abstract.

Many types of automorphism groups in algebra have nice structures arising from actions on combinatoric spaces. We recount some examples including Nagao's Theorem, the Jung-Van der Kulk Theorem, and a new structure theorem for the tame subgroup $\mathrm{TA}_3(K)$ of the group $\mathrm{GA}_3(K)$ of polynomial automorphisms of $\mathbb{A}^3_K$, for K a field of characteristic zero. We also ask whether a larger collection of automorphism groups possess a similar kind of structure.

§1. Introduction

This paper is intended to be only an overview. The proofs of the results summarized here are all published in other places that will be referenced.

§2. Amalgamated products of two groups

We start with a classical construction. Given two groups A_1 and A_2 containing a common subgroup B, we can form the free product G of A_1 and A_2 amalgamated along B:

$$G = A_1 *_B A_2$$

In this situation the two groups inject into the amalgamated product and a very strong factorization holds. Moreover the Bass-Serre tree theory of groups acting on trees [17] provides a tree on which G acts without inversion, having a fundamental domain consisting of a single edge with

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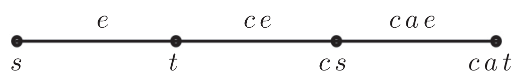
Received June 29, 2015.
Revised June 14, 2016.
2010 *Mathematics Subject Classification.* Primary 14R20, 05E18; Secondary 13A50.
Key words and phrases. Polynomial automorphism, tame automorphism, polynomial ring, affine space, amalgamated product, simplicial complex.

its end vertices, the stabilizers of the vertices being A_1 and A_2 and the stabilizer of the edge the common subgroup B.

According to the Bass-Serre tree theory, a group G having an amalgamated free product structure $G = A *_B C$ is equivalent to G acting without inversion of a tree $\mathcal{T}$ with fundamental domain consisting of single edge:

with A and C being the stabilizers in G of the vertices s and t, respectively, and B the stabilizer of the edge e. The quotient of the tree by the group action is again a single edge, onto which e maps.

Upon choosing nontrivial left coset representatives of A and C mod B, a path corresponds to a factorization using these representatives. For example, if $a \in A$ and $c \in C$ are such representatives, we have the path:



This gives a kind of unique factorization for elements of G. Strong theorems can be proved for groups having such an amalgamated free product structure using the tree action. For example, it is fairly easy to prove that any finite subgroup of G is conjugate to a subgroup of either A or C. This is tantamount to proving that a finite group acting on a tree has a fixed point.

The Bass-Serre theory provides the tree abstractly, but often the tree can be seen in a natural way. We recount some examples.

Example 2.1. The special linear group over $\mathbb{Z}$, the integer has the structure

$$\mathrm{SL}_2(\mathbb{Z}) = (\mathbb{Z}/4\mathbb{Z}) *_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/6\mathbb{Z})\,.$$

The generators of $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$ can be taken to be $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, respectively. This result can be derived from the action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half plane. Here the translates of the circular arc $z = e^{i\theta}$ with $\pi/3 \le \theta \le \pi/2$ form a tree with this arc as a fundamental domain, and this is the tree (up to isomorphism) provided by the Bass-Serre theory.

Example 2.2. Let K be a local field, i.e., the field of fractions of a discrete valuation ring $\mathcal{O}$ with uniformizing parameter π. Let V be a rank two vector space over K. Form a graph whose vertices are equivalence classes Λ of rank two $\mathcal{O}$-lattices in V under the multiplicative action of K^*. Connect Λ to Λ' if they are represented by lattices L and L', respectively, having $\mathcal{O}$-bases $\{e_1, e_2\}$ and $\{\pi e_1, e_2\}$. This forms a tree with a single edge serving as a fundamental domain, yielding Ihara's Theorem:

$$\mathrm{SL}_2(K) = \mathrm{SL}_2(\mathcal{O}) *_\Gamma \mathrm{SL}_2(\mathcal{O})$$

where $\Gamma = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}) \,|\, c \equiv 0 \pmod{\pi} \right\}$. The two injections $\Gamma \hookrightarrow \mathrm{SL}_2(\mathcal{O})$ are the identity map and the map given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & \pi b \\ \pi^{-1} c & d \end{smallmatrix}\right)$. The details of this are laid out in [17].

Example 2.3 (Nagao's Theorem [15])**.** For K an arbitrary field, the general linear group $\mathrm{GL}_2(K[T])$ has the structure of an amalgamated free product. (Here T represents a single variable.) This structure can be derived from the above example as follows: $\mathrm{SL}_2(K[T])$ acts on $\mathcal{O}$-lattices in the rank two vector space over $K(T)$, where $\mathcal{O}$ is the DVR of $K(T)$ with uniformizing parameter $1/T$. Here the quotient of the $\mathrm{SL}_2(K[T])$-action on the tree is not just an edge, but an edge connected to a "directed geodesic."

The stabilizer of v_0 is $\mathrm{SL}_2(K)$, and for $n \geq 1$, the stabilizer of v_d is

$$B^{(d)} = \left\{ \begin{pmatrix} \alpha & 0 \\ f(T) & \beta \end{pmatrix} \in \mathrm{B}_2(K[T]) \,|\, \deg\, f \leq d \right\}.$$

Writing B_2 for the lower triangular group, and noting that $\mathrm{B}_2(K[T])$ is the ascending union of $B^{(d)}, d \geq 1$, one can derive Nagao's Theorem:

$$\mathrm{SL}_2(K[T]) = \mathrm{SL}_2(K) *_{\mathrm{B}_2(K)} \mathrm{B}_2(K[T])\,.$$

This argument is easily modifiable to obtain a similar amalgamated product structure for $\mathrm{GL}_2(K[T])$, namely $\mathrm{GL}_2(K[T]) = \mathrm{GL}_2(K) *_{\mathrm{B}_2(K)} \mathrm{B}_2(K[T])$.

Example 2.4 (Jung-Van der Kulk Theorem [11], [12])**.** For K a field, the group $\mathrm{GA}_2(K)$ of polynomial automorphisms of the affine plane has the structure

$$\mathrm{GA}_2(K) = \mathrm{Af}_2(K) *_{\mathrm{Bf}_2(K)} \mathrm{BA}_2(K)\,. \tag{1}$$

Here $\mathrm{Af}_2(K)$ is the affine group, i.e., those automorphisms having coordinate functions of the type $(\gamma_{11}X_1 + \gamma_{12}X_2 + \delta_1, \gamma_{21}X_1 + \gamma_{22}X_2 + \delta_2)$ with $\gamma_{ij}, \delta_i \in K$, $\left(\begin{smallmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{smallmatrix}\right) \in \mathrm{GL}_2(K)$, and $\mathrm{BA}_2(K)$ is the group of automorphisms of the form $(\alpha X_1 + \gamma, \beta X_2 + f(X_1))$, $\alpha, \beta, \gamma \in K$, $\alpha\beta \neq 0$, $f(X_1) \in K[X_1]$; $\mathrm{Bf}_2(K)$ is then the intersection $\mathrm{Af}_2(K) \cap \mathrm{BA}_2(K)$. Again, this structure arises from the action of $\mathrm{GA}_2(K)$ on a tree whose vertices are certain complete algebraic surfaces realized as collections of local rings ("models") inside the function field $K(X_1, X_2)$ (see [21]).

Remark 2.5. An analog of (1) in Example holds when K is an integral domain, replacing the full automorphism group by its tame subgroup. This and other interesting results can be found in [5].

Remark 2.6. It can be easily derived from the Jung-Van der Kulk Theorem that the subgroup of $\mathrm{GA}_2(K)$ consisting of automorphisms having Jacobian determinant 1, sometimes denotes $\mathrm{SA}_2(K)$, also has an amalgamated product structure, namely the same as 1 replacing $\mathrm{Af}_2(K)$, $\mathrm{BA}_2(K)$, and $\mathrm{B}_2(K)$ by their intersections with $\mathrm{SA}_2(K)$. In [4] it was proved that $\mathrm{SA}_2(K)$ is not a simple group. In [9] this result is refined using results of [16] to describe which elements of $\mathrm{SA}_2(K)$ have normal closure strictly smaller than $\mathrm{SA}_2(K)$ using the "length" of the elements in the amalgamated product structure. This approach allows one to see more clearly why $\mathrm{SA}_2(K)$ is not simple.

§3. Generalized amalgamations

We begin with the definition of a generalized amalgamation of groups. We refer the reader to [14] for a comprehensive discussion of combinatorial group theory.

Definition 3.1. Suppose we are given groups A_i for each $i \in \{1, \cdots, n\}$ and for each $i, j \in \{1, \cdots, n\}$ with $i \neq j$ we have groups $B_{ij} = B_{ji}$ with injective homomorphisms $\varphi_{ij} : B_{ij} \to A_i$ which are compatible, meaning if i, j, k are distinct then $\varphi_{ij}^{-1}(\varphi_{ik}(B_{ik})) = \varphi_{ji}^{-1}(\varphi_{jk}(B_{jk}))$ and on this group $\varphi_{ik}^{-1}\varphi_{ij} = \varphi_{jk}^{-1}\varphi_{ji}$. This gives set-theoretic gluing data by which we can compatibly glue A_i to A_j along B_{ij} via $\varphi_{ij}^{-1}\varphi_{ji}$ forming an amalgamated union S of the sets $A_1, \ldots, A_n$. We then form the free group $\mathcal{F}$ on S, denoting the group operation on $\mathcal{F}$ by $*$. For $i \in \{1, \cdots, n\}$ and $x, y \in A_i \subset S$, we let $r_{x,y} = x * y * (xy)^{-1} \in \mathcal{F}$ (where xy is the product in A_i). Finally we let $\mathcal{G}$ be the quotient of $\mathcal{F}$ by all the relations $r_{x,y}$. The group $\mathcal{G}$ is called the *generalized amalgamated product* of the groups[1] A_i, $i \in \{1, \ldots, n\}$ along the groups B_{ij}, $i, j \in \{1, \ldots, n\}$. There are natural group homomorphisms $\iota_i : A_i \to \mathcal{G}$ with $\iota_i\varphi_{ij} = \iota_j\varphi_{ji}$ on B_{ij}.

[1]We use the term *generalized* amalgamated product to distinguish it from the usual amalgamated product of two or more groups along a single common subgroup.

The group $\mathcal{G}$ has the following universal property: Given a group H and maps $\rho_i : A_i \to H$ for $i \in \{1, \ldots, n\}$ such that $\rho_i\varphi_{ij} = \rho_j\varphi_{ji}$ on B_{ij} for all $i, j \in \{1, \ldots, n\}$, then there is a unique map $\Phi : \mathcal{G} \to H$ with $\rho_i = \Phi\iota_i$ for all i.

When there are only two subgroups A_1 and A_2 containing a common subgroup B, $\mathcal{G}$ is the usual amalgamated free product discussed above. In this case the two groups inject into the amalgamated product and, as was noted, a very strong factorization theorem holds. Such a property does not hold in general for generalized amalgamations of three or more groups along pairwise intersections. Worse, the groups A_i may not even map injectively into $\mathcal{G}$; in fact $\mathcal{G}$ may be the trivial group when none of the groups A_i are trivial, as the following example from [19] shows.

Example 3.2. For $\{i, j, k\} = \{1, 2, 3\}$ let B_{ij} be the infinite cyclic group generated by b_k. Let

$$A_1 = \langle\, b_2, b_3 \mid b_2b_3b_2^{-1} = b_3^2 \,\rangle$$
$$A_2 = \langle\, b_3, b_1 \mid b_3b_1b_3^{-1} = b_1^2 \,\rangle$$
$$A_3 = \langle\, b_1, b_2 \mid b_1b_2b_1^{-1} = b_2^2 \,\rangle$$

Then B_{ij} is a common subgroup of A_i and A_j and we can form the generalized amalgamation $\mathcal{G}$ of the groups A_i along the groups B_{ij}. It can be shown that in this case $\mathcal{G}$ is the trivial group.

Whether such amalgamation data gives rise to the group acting on a simplicial complex is not easy to detect (see, for example, [19], [10], and [3]). It occurs precisely when each of the groups A_i maps injectively to $\mathcal{G}$, and in this situation, the amalgamated union S maps injectively to $\mathcal{G}$ as well. The n-simplex of groups arising from this data is called *developable* by Haefliger ([10]) in case of this occurence.

However, if the groups A_i are subgroups of a given group G and if we take B_{ij} to be $A_i \cap A_j$ and φ_{ij} the inclusion map within G, then clearly there exists a homomorphism $\Phi : \mathcal{G} \to G$ restricting to the identity on each A_i, which shows that in this case the amalgamated union S maps injectively to $\mathcal{G}$. The map Φ will be surjective precisely when G is generated by the subgroups $A_1, \ldots, A_n$. If Φ is an isomorphism, then the structure of $\mathcal{G}$ arises from the action of G on an n-dimensional simply connected simplicial complex, with a single simplex serving as a fundamental domain.

In this case the combinatoric generalization of the Bass-Serre theory holds, providing an $(n-1)$-dimensional simply connected simplicial complex $\mathcal{D}$ on which $\mathcal{G}$ acts without rotation[2], and for which a single simplex serves as a fundamental domain. There are many unanswered questions about $\mathcal{D}$. For example, how does one determine connectivity properties it possesses, and when does it have infinite diameter (i.e., does the 1-skeleton of $\mathcal{D}$ have infinite diameter as a graph)? It is not necessarily true that a finite group of a generalized amalgamation is conjugate to one of the subgroups A_i, as will be seen from Example 3.3 below, but one can ask what condition(s) (e.g., 2-connectivity?) might guarantee that any finite subgroup of $\mathcal{G}$ is conjugate to one of the groups A_i?

Example 3.3. Let $\mathcal{F} = \mathbb{Z}/2\mathbb{Z}$, and let G be the vector space of rank three over $\mathcal{F}$, with basis $\{e_1, e_2, e_3\}$, viewed as an additive group. Set $A_i = \mathcal{F}e_j \oplus \mathcal{F}e_k$ and $B_{ij} = B_{ji} = \mathcal{F}e_k$, where $\{i, j, k\} = \{1, 2, 3\}$. Following the recipe above using the natural inclusions $B_{ij} \hookrightarrow A_i$, we obtain G as the generalized amalgamated product of the groups A_1, A_2, A_3. The associated simplicial complex $\mathcal{D}$ is an octahedron of triangles. In this case $\mathcal{D}$ is not 2-connected, and the finite group G fixes no vertex in $\mathcal{D}$.

Below are two examples of automorphism groups that can be realized as generalized amalgamations of groups.

Example 3.4. The full Cremona group $\mathrm{Cr}_2(K)$ over an algebraically closed field K is the generalized amalgamation of three groups: the automorphism group of $\mathbb{P}^2_K$ (which is $\mathrm{PGL}_2(K)$), the automorphism group of $\mathbb{P}^1_K \times \mathbb{P}^1_K$, and thirdly the K-automorphism group of $\mathbb{P}^1_L$ where $L = K(t)$, with t transcendental over K. There is a naturally realizable simplicial complex of triangles $\mathcal{C}$ on which $\mathrm{Cr}_2(K)$ acts which yields this structure and also contains the tree of Example 2.4 with the action of $\mathrm{GA}_2(K)$ being the restriction of the action of $\mathrm{Cr}_2(K)$ on $\mathcal{C}$. See [21] for details. In private correspondence Stéphane Lamy has shown that this complex is not 2-connected; in fact it has a simplicial 2-sphere comprising twelve triangles and a finite subgroup of $\mathrm{Cr}_2(K)$ acting on this sphere with no fixed point therein.

Remark 3.5. The task of understanding the Cremona group $\mathrm{Cr}_2(K)$ seems formidable. In [2], Bisi, Furter, and Lamy begin this undertaking by establishing some facts about certain subgroups.

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[2]meaning that if a group element fixes a simplex, it acts as the identity on that simplex.

Example 3.6. In this example K is a field of characteristic zero. The tame automorphism group in dimension three over such a field has now been shown to be the generalized amalgamated product of three groups. This recent result appears in [22]. To understand the structure theorem we will need some preliminaries. Generalizing the notation of Example 2.4, we denote by $\mathrm{GA}_n(K)$ the group of polynomial automorphisms of affine n-space over K. We consider the subgroup $\mathrm{EA}_n(K)$ generated by the elementary automorphisms, i.e., those of the form

$$e_i(f) = (X_1, \dots, X_{i-1}, X_i + f, X_{i+1}, \dots, X_n)$$

for some $i \in \{1, \dots, n\}$, $f \in K[X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n]$. We also have a natural containment of the general linear group $\mathrm{GL}_n(K)$ in $\mathrm{GA}_n(K)$. The subgroup of *tame* automorphisms is defined to be the subgroup generated by $\mathrm{EA}_n(K)$ and $\mathrm{GL}_n(K)$, i.e.,

$$\mathrm{TA}_n(K) = \langle \mathrm{GL}_n(K), \mathrm{EA}_n(K) \rangle\,.$$

The classical result of Example 2.4 contains the fact that $\mathrm{TA}_2(K) = \mathrm{GA}_2(K)$. The famous result of Shestakov and Umirbaev ([18]) asserts that $\mathrm{TA}_3(K) \neq \mathrm{GA}_3(K)$. We do not know whether $\mathrm{TA}_n(K) = \mathrm{GA}_n(K)$ for $n \geq 4$.

The group $\mathrm{GA}_n(K)$ acts on the polynomial ring $K[X_1, \dots, X_n]$ in an obvious way. For $i \in \{1, \dots, n\}$, let V_i be the sub-vector space of $K[X_1, \dots, X_n]$ generated by K and the variables $X_1, \dots, X_i$, i.e.,

$$V_i = K \oplus KX_1 \oplus \cdots \oplus KX_i\,. \tag{2}$$

Let H_i be the stabilizer of V_i in $\mathrm{GA}_n(K)$, i.e.,

$$H_i = \{\varphi \in \mathrm{GA}_n(K) \mid \varphi(V_i) = V_i\}\,. \tag{3}$$

These subgroups are defined in [7], p. 23, where it is conjectured that together they generate $\mathrm{GA}_n(K)$ (Conjecture 14.1) and that (whether or not that conjecture is true) the subgroup generated by $H_1, \dots, H_n$ is the generalized amalgamated product of these groups along pairwise intersections (Conjecture 14.2). It should be noted that Freudenburg produced an example (see [8], p. 171) of an automorphism in $\mathrm{GA}_3(K)$ which has not been shown to lie in this subgroup.[3]

Furthermore the groups $\widetilde{H}_i$ are defined by

$$\widetilde{H}_i = H_i \cap \mathrm{TA}_n(K)\,, \tag{4}$$

[3] This example is also of interest because it has not been shown to be *stably tame.* See [1] for the definition of this concept.

which are easily seen to generate $\mathrm{TA}_n(K)$. It is also easy to see that $\widetilde{H}_n = H_n$, both being equal to the affine group $\mathrm{Af}_n(K)$, and that $\widetilde{H}_{n-1} = H_{n-1}$.

There is one case where the containment $\widetilde{H}_n \subseteq H_n$ is known to be a proper containment: For $n = 3$ we have $\widetilde{H}_1 \subsetneq H_1$. This follows from one of the deep results, Corollary 10, of [18]. By the above paragraph, however, we have $\widetilde{H}_2 = H_2$ and $\widetilde{H}_3 = H_3$. We can now state the main result of [22], which is based on generators and relations for $\mathrm{TA}_n(K)$ given in [20]:

Theorem 3.7. *For K a field of characteristic zero, $TA_3(K)$ is the generalized amalgamated product of the three groups $\widetilde{H}_1, H_2, H_3$ along their pairwise intersections.*

As explained previously, this means that $\mathrm{TA}_3(K)$ acts on a simply connected complex of triangles, with a single triangle serving as a fundamental domain whose three vertices have stabilizers $\widetilde{H}_1$, H_2, and H_3. It is not known whether this complex is 2-connected, nor whether all finite subgroups of $\mathrm{TA}_3(K)$ are conjugate to a subgroup of one of the three groups $\widetilde{H}_1, H_2, H_3$.

Remark 3.8. Referring to (3) in the case $n = 2$, the groups H_1 and H_2 (which coincide with $\widetilde{H}_1$ and $\widetilde{H}_2$ for reasons given above) are precisely $\mathrm{BA}_2(K)$ and $\mathrm{Af}_2(K)$, respectively, of Example 2.4. This gives $\mathrm{GA}_2(K)$, which equals $\mathrm{TA}_2(K)$, as the amalgamated product $H_1 *_{H_1 \cap H_2} H_2$. Thus Example 3.6 can be viewed as an extension to $n = 3$ of the Jung-Van der Kulk theorem, for characteristic zero.

Remark 3.9. We note that the preprint [13] offers another proof of Theorem 3.7, using combinatoric methods.

Remark 3.10. It turns out that amalgamations play a role in the study of subgroups of $\mathrm{GA}_3(K)$. In [6] the authors show that any subgroup lying strictly between $\mathrm{Af}_3(K)$ and $\mathrm{GA}_3(K)$ has the structure of an amalgamated product of $\mathrm{Af}_3(K)$ with a finite group, over the intersection.

§4. Concluding remarks

The author suspects that a wider collection of automorphisms groups can be realized as generalized amalgamations. Examples might include a generalization of Nagao's theorem (Example 2.3) to $\mathrm{GL}_n(K[T])$, or even $\mathrm{GL}_n(K[X_1, \ldots, X_n])$ for $n \geq 3$.[4] This, in turn,

[4]The case of $\mathrm{GL}_2(K[X_1, \ldots, X_n])$ is more complicated.

might result from a generalization of Example 2.2 which would provide a structure theorem for $\mathrm{SL}_n(K)$, for K a local field.

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