Advanced Studies in Pure Mathematics 75, 2017
Algebraic Varieties and Automorphism Groups
pp. 287–392

A variant of Shokurov's criterion of toric surface

Noboru Nakayama

Abstract.

As a variant of Shokurov's criterion of toric surface, we give a criterion of two new classes of normal projective surfaces, called pseudo-toric surfaces of defect one and half-toric surfaces. A typical example of pseudo-toric surface of defect one is a projective toric surface blown up at a non-singular point of the boundary divisor. A half-toric surface is the quotient of a projective toric surface by an almost free involution preserving the boundary divisor. The structure of pseudo-toric surface of defect one and that of half-toric surface are also studied in detail.

Contents

§1. Introduction

We work over the complex number field $\mathbb{C}$. As a surface, we mean a two-dimensional separated integral *scheme* (or *algebraic space*) of finite type over $\operatorname{Spec}\mathbb{C}$. A *normal Moishezon surface* is defined as a two-dimensional normal integral separated algebraic space proper over

Received May 18, 2015.
Revised December 22, 2015.
2010 *Mathematics Subject Classification.* Primary 14J26; Secondary 14M25, 14J10.
Key words and phrases. normal surface, rational surface, toric variety.

Spec $\mathbb{C}$ (cf. Notation and conventions, 1 below). The main purpose of this article is to give a generalization of Shokurov's criterion [51, Th. 6.4] of toric surface in the case of integral divisor, by introducing new surfaces, called *pseudo-toric surfaces* and *half-toric surfaces*. We shall also describe in detail the structures of pseudo-toric surfaces of defect one and of half-toric surfaces, respectively. In the Shokurov criterion, the projective toric surfaces X with boundary divisor D are characterized by a condition on the singularity of (X, D), a numerical property of the divisor $K_X + D$, and by an information on the number of irreducible components of D. More precisely, the following is considered as the Shokurov criterion in the case of integral divisor (for a proof, see also [44, §8.5]).

Theorem 1.1 (cf. [51, Th. 6.4])**.** *Let X be a normal projective surface and D a reduced divisor. Then, the pair (X, D) is toric, i.e., X is a toric variety with boundary divisor D, if and only if*

(i) *(X, D) is log-canonical,*
(ii) *$-(K_X + D)$ is nef, and*
(iii) *$\boldsymbol{n}(D) \geq \hat{\boldsymbol{\rho}}(X) + 2$,*

where $\boldsymbol{n}(D)$ stands for the number of irreducible components of D and $\hat{\boldsymbol{\rho}}(X)$ denotes the Weil–Picard number of X, i.e., the dimension of the vector space $\mathsf{N}(X)$ of $\mathbb{R}$-divisors modulo the numerical equivalence relation (*cf. Definitions* 2.7 *and* 2.23 *below*).

Remark 1.2. (1) The Weil–Picard number $\hat{\boldsymbol{\rho}}(X)$ coincides with the number ρ defined in [51, Th. 6.4].

(2) For a projective toric surface X with boundary divisor D, it is known that the pair (X, D) is log-canonical, $K_X + D \sim 0$, $\boldsymbol{n}(D) = \boldsymbol{\rho}(X) + 2$, and the Picard number $\boldsymbol{\rho}(X)$ is equal to $\hat{\boldsymbol{\rho}}(X)$ (cf. Lemma 3.11 below).

(3) The original criterion [51, Th. 6.4] by Shokurov treats the case where D is only a $\mathbb{Q}$-divisor and $\boldsymbol{n}(D)$ in (iii) is replaced with the sum $\sum d_i$ for the prime decomposition $D = \sum d_i D_i$. Moreover, the original criterion is stated in a relative situation.

(4) In [33], McKernan shows that Theorem 1.1 holds true even if we replace the inequality of (iii) by

$$\boldsymbol{n}(D) \geq \boldsymbol{r}(D) + 2,$$

where $\boldsymbol{r}(D)$ is the dimension of the vector subspace $\mathsf{N}(X)_D$ of $\mathsf{N}(X)$ generated by the numerical equivalence classes of the irreducible components of D (cf. Definition 2.23).

(5) Higher-dimensional generalizations of Shokurov's criterion are studied in [45], [33], [20], etc.

We shall give a generalization of Theorem 1.1 essentially by weakening the condition (iii). Especially, we have a classification of (X, D) satisfying (i), (ii), and $\boldsymbol{n}(D) = \hat{\boldsymbol{\rho}}(X) + 1$. The following is our main theorem.

Theorem 1.3. *Let X be a normal* Moishezon *surface, i.e., a two-dimensional normal integral separated algebraic space proper over $\mathbb{C}$ (cf. Notation and conventions,* 1 *below) and let D be a reduced divisor on X. Here, we define the* defect $\boldsymbol{\delta}(X, D)$ *and the* complexity $\boldsymbol{c}(X, D)$ *by*

$$\boldsymbol{\delta}(X, D) := \hat{\boldsymbol{\rho}}(X) + 2 - \boldsymbol{n}(D) \quad \textit{and} \quad \boldsymbol{c}(X, D) := \boldsymbol{r}(D) + 2 - \boldsymbol{n}(D)$$

(cf. Definition 2.23*). Suppose that*

(i) *(X, D) is log-canonical* along *D (cf. Remark* 3.18(4)*), and*

(ii) *$-(K_X + D)$ is nef.*

Then, $\boldsymbol{\delta}(X, D) \geq \boldsymbol{c}(X, D) \geq 0$. Here, $\boldsymbol{c}(X, D) = 0$ if and only if (X, D) is a projective toric surface, and in this case, $\boldsymbol{\delta}(X, D) = 0$. Furthermore, $\boldsymbol{\delta}(X, D) = 1$ if and only if one of the following holds:

(1) *$(X, B + D)$ is a projective toric surface for a prime divisor $B \not\subset D$;*

(2) *(X, D) is a* pseudo-toric surface of defect one *(cf. Definition* 6.1*);*

(3) *(X, D) is a* half-toric surface *(cf. Definition* 7.1*).*

The pseudo-toric surfaces and half-toric surfaces are defined and studied in Sections 6 and 7 below, respectively. A pair (X, D) is called a pseudo-toric surface if X is a projective rational surface with only rational singularities, (X, D) is log-canonical, $K_X + D \sim 0$, and if D is a big cyclic chain of rational curves (cf. Definitions 6.1 and 4.3, and Lemma 6.3). A pair (X, D) is called a half-toric surface if $K_X + D \not\sim 0$, and if it is obtained as the quotient of a projective toric surface (V, D_V) by an involution which preserves the boundary divisor D_V and which has at most finitely many fixed points (cf. Definition 7.1). Theorem 1.6 (resp. 1.7) below is our structure theorem of pseudo-toric surfaces of defect one (resp. of half-toric surfaces).

Convention 1.4. By abuse of notation, we call (X, D) a *toric surface* when X is a normal algebraic surface and D is a reduced divisor such that X is a two-dimensional toric variety with $X \setminus D$ as an open torus. The divisor D is called the *boundary divisor*. Similarly, the pair (X, D) of a surface X and a divisor D on X is called a *surface* for simplicity.

Remark. (1) Theorem 1.1 and M$^{\mathrm{c}}$Kernan's generalization in Remark 1.2(4), respectively, are derived from Theorem 1.3 in the case where $\boldsymbol{\delta}(X, D) = 0$ and $\boldsymbol{c}(X, D) = 0$.

(2) The defect $\boldsymbol{\delta}(X, D)$ and the complexity $\boldsymbol{c}(X, D)$ are introduced in [33], where the defect is called the *absolute complexity*.

The following is a result only on the complexity but where the condition (ii) of Theorem 1.3 is replaced. This is also a generalization of M$^{\mathrm{c}}$Kernan's version (cf. Remark 1.2(4)) of the Shokurov criterion in the case of integral divisor.

Theorem 1.5. *Let X be a normal Moishezon surface and D a reduced divisor on X. Suppose that*

(i) *(X, D) is log-canonical along D,*
(ii) *D is connected, and*
(iii) *$-(K_X + D)$ is nef* on D *(cf. Definition* 2.14(ii)*).*

Then, $\boldsymbol{c}(X, D) \geq 0$. If $\boldsymbol{c}(X, D) \leq 1$, then X is a projective rational surface with only rational singularities. Moreover, the equality $\boldsymbol{c}(X, D) = 0$ holds if and only if there is a birational morphism $g \colon X \to \overline{X}$ such that

(1) *$(\overline{X}, \overline{D})$ is a projective toric surface for $\overline{D} := g_*(D)$, and*
(2) *the g-exceptional locus is contained in $X \setminus D$.*

We shall prove Theorems 1.3 and 1.5 in Section 8.

Pseudo-toric surfaces

We shall explain some facts and results on pseudo-toric surfaces. As a consequence of Shokurov's criterion (Theorem 1.1), we see that the defect $\boldsymbol{\delta}(X, D)$ of a pseudo-toric surface (X, D) is always non-negative, and $\boldsymbol{\delta}(X, D) = 0$ if and only if (X, D) is a projective toric surface. A typical construction of pseudo-toric surface from a projective toric surface is given by the blowing up at a non-singular point of the boundary divisor: Let (X, D) be a projective toric surface and P a non-singular point of D. Then, X is also non-singular at P. Let $f \colon Y \to X$ be the blowing up at P and let D' be the proper transform of D in Y. Then, (Y, D') is a pseudo-toric surface. In fact, we have $K_Y + D' = f^*(K_X + D) \sim 0$. The operation getting $Y \setminus D'$ from $X \setminus D$ is called a *half-point attachment* in the study of open surfaces (cf. [18, §2], [12, (6.21)]). In this case, we have $\boldsymbol{\delta}(Y, D') = 1$. We can observe that any pseudo-toric surface is essentially obtained from a projective toric surface by successive operations of half-point attachment and followed by contractions of some divisors. But, we can not take the half-point attachment freely, since we have required that the boundary divisor D is big (cf. Definition 6.1(iv)).

Example. Let X be a non-singular projective rational surface admitting an elliptic fibration $\pi\colon X \to T$ such that π has a singular fiber D of type I_a for some $a > 0$ (in Kodaira's notation). Then, D is not big but (X, D) satisfies the other conditions in Definition 6.1 of pseudo-toric surfaces.

Remark. In [32], Looijenga has studied the pairs (X, D) of a normal projective rational surface X and an anti-canonical reduced divisor D satisfying the following conditions:

- X is non-singular along D,
- D is a normal crossing divisor consisting of rational curves,
- D contains no (-1)-curves, and
- the intersection matrix of D is negative semi-definite.

In particular, (X, D) satisfies the conditions in Definition 6.1 except the bigness condition of D. For such (X, D) above, assuming the number $\boldsymbol{n}(D)$ of irreducible components of D to be at most 5, Looijenga has found a natural infinite root system in the Picard group $\mathrm{Pic}(X)$ which describes the classes of (-1)-curves on X. He uses the root systems in order to construct fine moduli spaces of (X, D) above with $\boldsymbol{n}(D) \le 5$.

We introduce the notion of *toroidal blowing up* in Definition 4.19 below. This is étale locally a birational morphism of toric varieties. For a pseudo-toric surface (X, D), if $Y \to X$ is a toroidal blowing up with respect to (X, D), then (Y, D_Y) is also pseudo-toric for $D_Y = f^{-1}(D)$, and $Y \setminus D_Y \simeq X \setminus D$. We introduce the notion of *tangential blowing up* of order m as an m-times operation of half-point attachment at the "same point" followed by the contraction morphism of all the exceptional curves not meeting the proper transform of the boundary divisor (cf. Definition 4.24, Lemma 4.25). In Theorem 6.5 below, we prove that every pseudo-toric surface *of defect one* is obtained from some projective toric surface by a tangential blowing up and by a toroidal blow-down. By this result, we can prove the following fundamental result:

Theorem 1.6. *For any pseudo-toric surface (X, D) of defect one, the following hold*:

(1) *The group $\mathrm{Aut}(X; D)$ of automorphisms of X preserving each irreducible component of D is isomorphic to the multiplicative group $\mathbb{C}^\star := \mathbb{C} \setminus \{0\}$.*

(2) *The open subset $X \setminus D$ is affine and its coordinate ring is isomorphic to*

$$\mathbb{C}[\mathtt{x}, \mathtt{y}, \mathtt{t}, \mathtt{t}^{-1}]/(\mathtt{xy} - (\mathtt{t} - 1)^{k+1})$$

for an integer $k \geq 0$*. Here, the action of* $\theta \in \mathbb{C}^{\star} = \mathrm{Aut}(X; D)$ *on* $X \setminus D$ *is given by* $(\mathtt{x}, \mathtt{y}, \mathtt{t}) \mapsto (\theta\mathtt{x}, \theta^{-1}\mathtt{y}, \mathtt{t})$*. In particular,* $X \setminus D$ *is non-singular when* $k = 0$*, and has a rational double point of type* A_k *as a unique singular point when* $k \geq 1$*. As a consequence,* X *has only cyclic quotient singularities.*

(3) *Let* $\nu \colon N \to X \setminus D$ *be the minimal resolution of singularities. Then, the* logarithmic irregularity (*cf.* [17], [19]) *of* N *is one. Moreover, the* quasi-Albanese map (*cf.* [16], [19]) *of* N *is isomorphic to* $h \circ \nu$ *for the morphism* $h \colon X \setminus D \to \mathbb{G}_{\mathtt{m}}$ *to the one-dimensional algebraic torus* $\mathbb{G}_{\mathtt{m}}$ *corresponding to the natural ring homomorphism*

$$\mathbb{C}[\mathtt{t}, \mathtt{t}^{-1}] \to \mathbb{C}[\mathtt{x}, \mathtt{y}, \mathtt{t}, \mathtt{t}^{-1}]/(\mathtt{xy} - (\mathtt{t} - 1)^{k+1})$$

with respect to the coordinate ring in (2)*.*

The proof of Theorem 1.6 is given at the end of Section 6.2. In the proof of Theorem 1.6, a special linear chain $L_1 + L_2$ of rational curves in Definition 6.7 plays an important role.

Half-toric surfaces

Next, we shall explain some facts and results on half-toric surfaces. By Definition 7.1, giving a half-toric surface (X, D) is equivalent to giving an involution ι of a projective toric surface (V, D_V) such that ι has at most finitely many fixed points, $\iota(D_V) = D_V$, and ι does not preserve a nowhere vanishing global logarithmic two-form $\eta \in \mathrm{H}^0(V, \Omega^2_V(\log D_V))$. Here, (X, D) is the quotient of (V, D_V) by ι, and moreover, the induced involution on the two-dimensional algebraic torus $V \setminus D_V \simeq \mathbb{G}_{\mathtt{m}}^2$ is expressed uniquely up to the choice of coordinates (cf. Lemma 7.17). By this result, we can prove the following fundamental result:

Theorem 1.7. *For any half-toric surface* (X, D)*, the following hold*:

(1) *The normal projective surface* X *is rational with only rational singularities, the pair* (X, D) *is log-canonical,* D *is a big linear chain of rational curves, and* $\boldsymbol{\delta}(X, D) = 1$*.*

(2) *The open subset* $X \setminus D$ *is non-singular and affine, and its coordinate ring is isomorphic to*

$$\mathbb{C}[\mathtt{x}, \mathtt{x}^{-1}, \mathtt{y}, \mathtt{z}]/(\mathtt{x}(\mathtt{y}^2 - 1) - \mathtt{z}^2).$$

In particular, the isomorphism class of $X \setminus D$ *is independent of the choice of* (X, D)*.*

(3) *The fundamental group of the complex manifold* $(X \setminus D)^{\mathrm{an}}$ *associated with* $X \setminus D$ *is generated by two elements* a *and* b *with*

one relation: $aba^{-1} = b^{-1}$. *In other words, the fundamental group is isomorphic to the semi-direct product* $\mathbb{Z} \rtimes \mathbb{Z}$, *where the action of the quotient group* $\mathbb{Z}$ *on the normal subgroup* $\mathbb{Z}$ *is given by* $m \cdot x = (-1)^m x$.

(4) *The group* $\mathrm{Aut}(X; D)$ *of automorphisms of* X *preserving each irreducible component of* D *is isomorphic to* $\mathbb{C}^\star \times (\mathbb{Z}/2\mathbb{Z})$. *Here, the action of* $(\theta, k) \in \mathbb{C}^\star \times (\mathbb{Z}/2\mathbb{Z})$ *on* $X \setminus D$ *is given by*

$$(\mathtt{x}, \mathtt{y}, \mathtt{z}) \mapsto (\theta^2 \mathtt{x}, (-1)^k \mathtt{y}, (-1)^k \theta \mathtt{z})$$

with respect to the coordinate ring in (2).

(5) *For the open subset* $X \setminus D$, *the logarithmic irregularity* $\bar{q}(X \setminus D)$ *is one, and the quasi-Albanese map is isomorphic to the morphism* $X \setminus D \to \mathbb{G}_{\mathtt{m}}$ *corresponding to a natural ring homomorphism*

$$\mathbb{C}[\mathtt{x}, \mathtt{x}^{-1}] \to \mathbb{C}[\mathtt{x}, \mathtt{x}^{-1}, \mathtt{y}, \mathtt{z}]/(\mathtt{x}(\mathtt{y}^2 - 1) - \mathtt{z}^2)$$

with respect to the coordinate ring in (2).

(6) *For the minimal resolution* $\mu \colon M \to X$ *of singularities,* $D_M = \mu^{-1}(D)$ *is a simple normal crossing divisor consisting of rational curves whose dual graph is the extended Dynkin diagram* D_k^* *with* $k + 1 = \boldsymbol{n}(D_M) = \boldsymbol{\rho}(M) + 1 \geq 6$, *in other words, the same dual graph as the singular fiber of type* I_{k-4}^* *of an elliptic surface.*

The proof of Theorem 1.7 is given at the end of Section 7.4. We can also show that the open surface $X \setminus D$ is just the surface having an NC-minimal completion of type $H[-1, 0, -1]$ in Fujita's classification [12] of open surfaces (cf. Remark 7.21). Kojima [28] considers a similar variant of Shokurov's criterion for open surfaces and announces a certain characterization of the surface of type $H[-1, 0, -1]$.

Remark. The referee informed the author of a recent article [46] of Prokhorov in which he has proved a result similar to our Theorem 1.3 in [46, Th. 5.1]. However, this is weaker than Theorem 1.3 combined with Theorems 1.6 and 1.7. For example, when X is projective and (X, D) is log-canonical with $K_X + D \sim 0$, and when $\boldsymbol{\delta}(X, D) = 1$, he proves only that X admits an effective action of $\mathbb{C}^\star$.

Remark. The results in this article hold not only over $\mathbb{C}$ but also over any algebraically closed field of characteristic zero by the Lefschetz principle. Even for an algebraically closed field of characteristic $p > 0$, the same results seem to hold except the results related to double-covers, where we need to assume: $p \neq 2$. Indeed, the vanishing theorem

(Theorem 2.17), the cone and contraction theorems (Theorems 2.19 and 2.21), and the projectivity criterion (Lemma 2.31(1)) are all valid in any characteristic. However, we do not care the positive characteristic case so much.

The organization of this article

In Section 2, we recall basic facts on normal surfaces, especially on Moishezon surfaces, including the intersection theory of divisors, numerical properties of divisors, the cone and contraction theorems, and projectivity criteria. These are studied and explained briefly in Sakai's articles [47], [48], [49], etc., but here, we shall give a unified explanation for the readers' convenience.

In Section 3, we recall some basics on toric varieties and log-canonical pairs of dimension two. The singularities on toric surfaces and the description of projective toric surfaces are explained in Section 3.1. The toroidal singularities are mentioned in Section 3.2, and some general properties on log-canonical pairs are explained in the surface case in Section 3.3. The classification of singularities of a log-canonical pair (X, D) for a surface X and a reduced divisor D is explained briefly in Section 3.4, and as an application, a classification result of singularities of (X, D) lying on a compact irreducible component C of D with $(K_X + D)C \leq 0$ is obtained in Section 3.5.

Some key concepts are introduced and discussed in Section 4. These are: the linear and cyclic chains of rational curves (cf. Section 4.1), the double-covers étale in codimension one (cf. Section 4.2), the toroidal blowing up (cf. Section 4.3), and the tangential blowing up (cf. Section 4.4).

In Section 5, we determine the structure of the pair (X, D) of a normal Moishezon surface X and a reduced connected divisor D such that (X, D) is log-canonical along D, $-(K_X + D)$ is nef on D, there is a $\mathbb{P}^1$-fibration $\pi\colon X \to T$, and that D contains at least two fibers of π. In Section 5.1, we see that there are two possible cases (A) and (B), and the structure is determined in Section 5.2 (resp. 5.3) for the case (A) (resp. (B)).

The pseudo-toric surface and the half-toric surface are introduced and studied in Sections 6 and 7, respectively. The definition and basic properties of pseudo-toric surfaces are given in Section 6.1 as well as the characterization of toric surface as a pseudo-toric surface of defect zero. For pseudo-toric surfaces of defect one, more detailed information is obtained in Section 6.2. The half-toric surface is defined in Section 7.1 with some basic properties, and there is explained a relation with an *H-surface* in Section 7.2. The H-surface is considered as an NC-minimal completion of an open surface of type $H[-1,0,-1]$ in the sense of Fujita (cf. [12, (8.19)]). After giving a description of certain involutions of toric surfaces in Section 7.3, we shall prove Theorem 1.7 in Section 7.4.

Finally in Section 8, we shall prove Theorems 1.3 and 1.5.

Motivation

A motivation for studying pseudo-toric surfaces of defect one comes from the study on the classification of normal projective surfaces admitting non-isomorphic surjective endomorphisms [41]. The classification in [41] has completed for irrational surfaces, and the pseudo-toric surfaces of defect one appear in the possible remaining cases of rational surfaces. Some contents in Sections 2, 3, and 4 of this article are borrowed from [41]. The study of half-toric surface is inspired by the article [28] of Kojima mentioning $H[-1,0,-1]$ in some classification results of open surfaces.

Acknowledgement

Some partial results of this article have been reported as [42] in the proceedings of the symposium "Geometry of Projective Varieties and Related Topics 2012" held at Kochi University. The author is grateful to Professors Yoshiaki Fukuma, Hideo Kojima and Osamu Matsuda for the organization and to younger participants, especially to Professor Yukinori Kitadai, for giving an advice on the name of pseudo-toric surface. The author also expresses appreciation to Professors Kayo Masuda and Masayoshi Miyanishi, organizers of this conference, for giving him a chance to talk. He could generalize and correct the results reported in [42] during the preparation of the conference. The author also expresses his thanks to Professor Yoshio Fujimoto for his continuous encouragement and useful suggestions. The author thanks the referees profoundly for the careful reading and for the invaluable comments.

Notation and conventions

Unless otherwise mentioned, we shall use standard notation and conventions of the classification theory and the minimal model theory of projective varieties. Here, we shall explain some additional things in 1–6 below, but further special notation and conventions on normal surfaces are prepared in Section 2.

1. A *variety* means an integral separated scheme (or algebraic space) of finite type over $\operatorname{Spec}\mathbb{C}$: A *curve* (resp. *surface*) means a variety of dimension one (resp. two). But, as a variety, we sometimes consider the associated analytic space X^{an} instead of the scheme X. For example, a subscheme of X is said to be *compact* if it is proper over $\operatorname{Spec}\mathbb{C}$. By the

functor $X \mapsto X^{\mathrm{an}}$, the category of integral algebraic spaces proper over $\mathbb{C}$ is equivalent to the category of Moishezon varieties (cf. [6, Th. (7.3)]). So, for simplicity, by a *normal Moishezon surface*, we mean a normal integral separated algebraic space of dimension two proper over $\mathbb{C}$.

2. For a compact variety X, a *curve* on X means a compact (irreducible) subvariety of dimension one, by abuse of notation, unless otherwise stated. In particular, when $\dim X = 2$, a curve means a prime divisor. The curves are all projective. For a connected and reduced projective scheme B of dimension one, the *arithmetic genus* $p_a(B)$ is defined as $\dim \mathrm{H}^1(B, \mathcal{O}_B)$.

3. Let X be a normal variety. A *divisor* on X means simply a Weil divisor on X, i.e., a finite linear combination $D = \sum d_i D_i$ of prime divisors D_i on X with coefficients $d_i \in \mathbb{Z}$. If we allow $d_i \in \mathbb{Q}$ (resp. $d_i \in \mathbb{R}$), the sum $D = \sum d_i D_i$ is called a $\mathbb{Q}$-*divisor* (resp. $\mathbb{R}$-*divisor*). The set $\bigcup_{d_i \neq 0} D_i$ is called the *support* of D and is denoted by $\operatorname{Supp} D$. The expression $D = \sum d_i D_i$ is called the *irreducible decomposition* (or the *prime decomposition*) of D. If $\operatorname{Supp} D$ is compact, then D is said to be compact. A $\mathbb{Q}$-divisor D on X is said to be $\mathbb{Q}$-*Cartier* if mD is a Cartier divisor for some positive integer m. If every prime divisor on X is $\mathbb{Q}$-Cartier, then X is said to be $\mathbb{Q}$-*factorial*. The *canonical divisor* of X is denoted by K_X. Note that K_X is not unique as a divisor but unique up to the linear equivalence relation.

4. A *reflexive sheaf* $\mathcal{F}$ on a normal variety X is by definition a coherent $\mathcal{O}_X$-module such that $\mathcal{F}$ is isomorphic to the double-dual $\mathcal{F}^{\vee\vee}$, where $\mathcal{F}^{\vee}$ stands for $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. It is known that a torsion-free coherent $\mathcal{O}_X$-module $\mathcal{F}$ is reflexive if and only if $\mathcal{F}$ satisfies Serre's condition S_2 (cf. [15, Prop. 1.6]). For a divisor D on X, we denote by $\mathcal{O}_X(D)$ the associated reflexive sheaf of rank one: In case D is Cartier, $\mathcal{O}_X(D)$ is the usual associated invertible sheaf, and in general, $\mathcal{O}_X(D)$ is defined by the property that $\mathcal{O}_X(D) \simeq j_*\mathcal{O}_U(D|_U)$ for any open subset $U \subset X$ with $\operatorname{codim}(X \setminus U) \geq 2$, where $D|_U$ is Cartier and j is the open immersion $U \hookrightarrow X$. Here, D is Cartier if and only if $\mathcal{O}_X(D)$ is invertible. The reflexive sheaf $\mathcal{O}_X(K_X)$ is written as ω_X, and is called the *canonical sheaf* or the *dualizing sheaf*. In fact, $\omega_X \simeq j_*(\Omega^n_U)$ for the open immersion $j \colon U \hookrightarrow X$ from the non-singular locus U, where $n = \dim X$. When X is Cohen–Macaulay (e.g., $n = 2$) and compact, we have the Serre duality

$$\mathrm{H}^i(X, \mathcal{F})^{\vee} \simeq \operatorname{Ext}^{n-i}_{\mathcal{O}_X}(\mathcal{F}, \omega_X)$$

for any coherent $\mathcal{O}_X$-module $\mathcal{F}$.

5. A *fibration* is a proper surjective morphism $f\colon X \to Y$ of normal varieties such that all the fibers are connected (equivalently, $\mathcal{O}_Y \simeq f_*\mathcal{O}_X$). A *fiber* of f means a closed fiber with reduced structure, unless otherwise stated. A $\mathbb{P}^1$-*fibration* is a fibration whose general fiber is isomorphic to $\mathbb{P}^1$. For a proper birational morphism $f\colon X \to Y$ of normal varieties, the f-*exceptional locus* (or the exceptional locus for f) is the set of points on X at which f is not an isomorphism. A prime divisor on X is said to be f-*exceptional* (or exceptional for f) if it is contained in the f-exceptional locus. Note that, when $\dim X = 2$, the f-exceptional locus is the union of f-exceptional curves.

6. For a ring R, the group of invertible elements of R is denoted by $R^\star$. For example, $\mathbb{C}^\star = \mathbb{C} \setminus \{0\}$.

§2. On normal Moishezon surfaces

In this section, we explain some basics on normal Moishezon surfaces, such as intersection theory of divisors (Section 2.1), numerical properties of divisors (Section 2.2), the cone and contraction theorems (Section 2.3), and projectivity criteria (Section 2.5). These topics have been studied in Sakai's article [47], [48], [49], etc. In Section 2.4, we define the defect $\boldsymbol{\delta}(X, D)$ and the complexity $\boldsymbol{c}(X, D)$ for a normal Moishezon surface X with a reduced divisor D and we study their properties in connection with the class map.

2.1. Intersection number of two (Weil) divisors

We recall the notion of intersection numbers of two divisors on a normal surface, and recall related properties (cf. [47, Sect. 1]).

Definition 2.1. Let X be a normal surface and let $\mu\colon M \to X$ be a proper birational morphism from a non-singular surface M. For a divisor D on X, the *numerical pullback* of D (due to Mumford [38]) is defined as a $\mathbb{Q}$-divisor

$$\mu^*(D) := D' + \sum\nolimits_{i=1}^{l} a_i E_i$$

such that $\mu^*(D)E_i = 0$ for any $1 \le i \le l$, where D' is the proper transform of D in M, and $E_1, \ldots, E_l$ are the μ-*exceptional curves* (cf. Notation and conventions, 2 and 5). The rational numbers $a_1, \ldots, a_l$ are uniquely determined, since the intersection matrix $(E_iE_j)_{1\le i,j\le l}$ is negative definite (cf. Theorem 2.6 below). For two divisors D_1 and D_2 on X, if D_1 or D_2 is *compact* (cf. Notation and conventions, 1), then the intersection number D_1D_2 is defined by

$$D_1D_2 := \mu^*(D_1)\mu^*(D_2) \in \mathbb{Q}.$$

When $D = D_1 = D_2$, we write D^2 for D_1D_2. The intersections numbers for $\mathbb{Q}$-divisors and $\mathbb{R}$-divisors are defined by linearity.

Remark. (1) For a Cartier divisor D, the numerical pullback $\mu^*(D)$ coincides with the usual pullback as a Cartier divisor.

(2) Let r be the determinant of the intersection matrix (E_iE_j) above. Then, $r\mu^*(D)$ is Cartier. In particular, $rD_1D_2 \in \mathbb{Z}$ for any such divisors D_1 and D_2 as above.

(3) The intersection number D_1D_2 does not depend on the choice of $\mu\colon M \to X$. If D_1 is Cartier, and D_2 is compact, then $D_1D_2 = \deg(\mathcal{O}_X(D_1)|_{D_2})$.

(4) If D_1 and D_2 are effective divisors without common irreducible components and if D_1 or D_2 is compact, then $D_1D_2 \geq 0$, where $D_1D_2 = 0$ if and only if $\operatorname{Supp} D_1 \cap \operatorname{Supp} D_2 = \emptyset$.

The following is well known (cf. [57, Lem. 7.1]).

Lemma 2.2. *On a normal surface, let $D = \sum a_iD_i$ be a finite linear combination of compact $\mathbb{R}$-divisors D_i with real coefficients a_i. Assume that the matrix (D_iD_j) is negative-definite and that $DD_i \leq 0$ for any i. Then $a_i \geq 0$ for any i.*

Definition 2.3. Let $D = \sum_{i=1}^{k} d_iD_i$ be the irreducible decomposition of a compact $\mathbb{R}$-divisor D on a normal surface. If the intersection matrix $(D_iD_j)_{1\leq i,j\leq k}$ is negative definite, we say that D is *negative definite*.

Definition 2.4. Let $f\colon Y \to X$ be a morphism of normal surfaces.

(1) For an $\mathbb{R}$-divisor G on Y, when the restriction $\operatorname{Supp} G \to X$ of f is proper, the *push-forward* $f_*(G)$ is defined to be the $\mathbb{R}$-divisor $\sum d_ib_if(G_i)$, where the summation is taken over all the irreducible components G_i of G with $\dim f(G_i) = 1$, $b_i = \operatorname{mult}_{G_i}(G)$, and d_i is the degree of the finite morphism $G_i \to f(G_i)$.

(2) When f is a proper birational morphism, an $\mathbb{R}$-divisor G is said to be *f-exceptional* if $\operatorname{Supp} G$ is contained in the f-exceptional locus, i.e., if $f_*(G) = 0$.

(3) Assume that f is a dominant morphism. For a divisor D on X, the *numerical pullback* $f^*(D)$ is defined as follows. Let $\mu\colon M \to X$ and $\nu\colon N \to Y$ be proper birational morphisms from non-singular surfaces M and N such that the induced rational map $g = \mu^{-1} \circ f \circ \nu\colon N \to M$ is a morphism. Then, we set

(II-1) $$f^*(D) := \nu_*(g^*(\mu^*(D))),$$

where g^* denotes the pullback of $\mathbb{Q}$-Cartier divisor. Here, $f^*(D)$ is a $\mathbb{Q}$-divisor, and it is independent of the choices of μ and ν. The numerical pullback $f^*(\Delta)$ of an $\mathbb{R}$-divisor Δ is defined by linearity.

(4) In the situation of (3), when D is a reduced divisor, the support of $f^*(D)$ is denoted by $f^{-1}(D)$, and is called the *total transform* of D.

Remark. (1) The projection formula

$$f^*(D)G = Df_*(G) \tag{II-2}$$

holds for any $\mathbb{R}$-divisor D on X and for any $\mathbb{R}$-divisor G on Y such that $\operatorname{Supp} G \to Y$ is proper.

(2) If f is proper and surjective, then another projection formula

$$f_*f^*(D) = (\deg f)D \tag{II-3}$$

holds for any $\mathbb{R}$-divisor D on X, where $\deg f$ denotes the degree of the generically finite morphism f, i.e., the cardinality of a general fiber.

(3) Assume that f is a finite surjective morphism. Then, for a divisor D on X, we can find an open subset U of X such that $D|_U$ is Cartier and that $\operatorname{codim}(X \setminus U, X) \geq 2$. Then, $\operatorname{codim}(Y \setminus f^{-1}(U), Y) \geq 2$, since f is finite. Thus, the Cartier divisor $f^*(D|_U)$ is extended uniquely to a divisor on Y, which is called the *closure* of $f^*(D|_U)$. The numerical pullback $f^*(D)$ is equal to the closure of $f^*(D|_U)$.

(4) Assume that f is a proper birational morphism. If D is an effective $\mathbb{R}$-divisor on X, then $f^*D - D'$ is effective for the proper transform D' of D in Y, and $\operatorname{Supp} f^*(D) = f^{-1}(\operatorname{Supp} D)$. In particular, $f^{-1}(D) = f^{-1}(\operatorname{Supp} D)$ when D is reduced.

Remark 2.5. Let $f \colon Y \to X$ be a proper birational morphism of normal surfaces. If an $\mathbb{R}$-divisor G on Y is f-*nef*, i.e., $GC \geq 0$ for any f-exceptional curve C (cf. Definition 2.14 below), then the difference $\Delta = f^*(f_*(G)) - G$ is an effective $\mathbb{R}$-divisor by Theorem 2.6 and Lemma 2.2. In particular, if G is f-*numerically trivial* (cf. Definition 2.14 below), i.e., $GC = 0$ for any f-exceptional curve C, then $G = f^*(f_*(G))$.

The following theorem on contraction criterion is well known:

Theorem 2.6 (Contraction Criterion)**.** *Let G be a compact reduced divisor on a normal surface Y. Then, the following two conditions are mutually equivalent*:

(i) *The divisor G is negative definite* (*cf. Definition* 2.3).

(ii) *There is a proper birational morphism $f\colon Y \to X$ to a normal surface X such that* $\dim f(G) = 0$, $f^{-1}(f(G)) = G$, *and f induces an isomorphism $Y \setminus G \to X \setminus f(G)$.*

We explain a history on the proof of Theorem 2.6 briefly. The implication (ii) $\Rightarrow$ (i) is shown by Mumford in [38, p. 6]. The other implication (i) $\Rightarrow$ (ii) is proved by Grauert in [14, (e), pp. 366–367] (cf. [38]) in the case where Y is a non-singular complex analytic surface. The same implication is proved for a two-dimensional non-singular algebraic space Y of finite type over $\mathbb{C}$ by Artin in [6, Cor. 6.12(b)]. The general case of normal surface is reduced to the non-singular case by taking resolution of singularities of Y (cf. [47, Th. (1.2)]).

Remark. The morphism f in Theorem 2.6 is called the *contraction morphism* (or the *blowdown*) of G, which is uniquely defined up to isomorphism. Note that if Y is an algebraic space, then so is X, but even if Y is a scheme, X is not necessarily a scheme (cf. [14, (e), p. 366]).

Definition. A prime divisor C on a normal surface X is called a *negative curve* if C is compact and $C^2 < 0$. If C is a non-singular rational curve lying on the non-singular locus of X with $C^2 = -k < 0$, then C is called a $(-k)$*-curve*.

Remark. The contraction morphism f in Theorem 2.6 is written as a succession of contractions of negative curves. The (-1)-curve is just the *exceptional curve of the first kind*. A negative curve C on a non-singular locus of X is a (-1)-curve (resp. (-2)-curve) if and only if $K_X C < 0$ (resp. $K_X C = 0$).

Remark. A proper birational morphism $\mu\colon M \to X$ from a non-singular surface M is called the *minimal resolution of singularities* of X if there is no (-1)-curves in the μ-exceptional locus. This is equivalent to that K_M is μ*-nef* (cf. Definition 2.14(i) below), i.e., $K_M C \geq 0$ for any μ-exceptional curve C. The minimal resolution is unique up to isomorphism over X.

2.2. Numerical properties of divisors

The intersection numbers defined in Section 2.1 give the numerical equivalence relation $\approx$ for $\mathbb{R}$-divisors on a normal *Moishezon* surface (cf. Notation and conventions, 1). We recall basic properties on the real vector space $\mathsf{N}(X)$ of $\mathbb{R}$-divisors modulo $\approx$ for a normal Moishezon surface X, and some results on numerical properties of $\mathbb{R}$-divisors, such as nef, big, and numerically ample, etc. (cf. Definition 2.11 below).

Definition 2.7 ($\mathsf{N}(X)$, $\hat{\boldsymbol{\rho}}(X)$)**.** Let X be a normal Moishezon surface. We denote by $\mathrm{Div}(X)$ the divisor group of X, i.e., the free abelian group generated by prime divisors on X. Note that a $\mathbb{Q}$-*divisor* (resp. an $\mathbb{R}$-*divisor*) is an element of $\mathrm{Div}(X) \otimes \mathbb{Q}$ (resp. $\mathrm{Div}(X) \otimes \mathbb{R}$). The divisor class group $\mathrm{CL}(X)$ is the quotient abelian group $\mathrm{Div}(X)/{\sim}$ by the linear equivalence relation $\sim$. Two $\mathbb{R}$-divisors D_1 and D_2 are said to be *numerically equivalent* to each other if $D_1C = D_2C$ for any (compact) curve C on X. We write the numerical equivalence relation by $\approx$. The numerical equivalence class of an $\mathbb{R}$-divisor D is denoted by $\mathrm{cl}(D)$ or $\mathrm{cl}_X(D)$; it is also called the *numerical class* for simplicity. We define $\mathsf{N}(X)$ to be the group $\mathrm{Div}(X) \otimes \mathbb{R}/{\approx}$ of the numerical classes of $\mathbb{R}$-divisors, which is a real vector space. The intersection numbers for $\mathbb{R}$-divisors induce a non-degenerate bilinear form $\mathsf{N}(X) \times \mathsf{N}(X) \to \mathbb{R}$; $(x, y) \mapsto x \cdot y$, such that $\mathrm{cl}(D) \cdot \mathrm{cl}(E) = DE$ for two $\mathbb{R}$-divisors D and E. The *Weil–Picard number* $\hat{\boldsymbol{\rho}}(X)$ of X is defined as $\dim_{\mathbb{R}} \mathsf{N}(X)$.

Remark 2.8. For the *Néron–Severi group* $\mathrm{NS}(X)$, which is the group of Cartier divisors modulo the algebraic equivalence relation, we have $\mathrm{NS}(X) \otimes \mathbb{R} \subset \mathsf{N}(X)$. In particular, $\hat{\boldsymbol{\rho}}(X) \geq \boldsymbol{\rho}(X)$ for the *Picard number* $\boldsymbol{\rho}(X) = \operatorname{rank} \mathrm{NS}(X)$. If X is non-singular, or more generally, if X is $\mathbb{Q}$-*factorial* (cf. Notation and conventions, 3), then $\hat{\boldsymbol{\rho}}(X) = \boldsymbol{\rho}(X)$.

Remark 2.9. Let $f \colon Y \to X$ be a surjective morphism of normal Moishezon surfaces. Then, the push-forward f_* and the numerical pull-back f^* of divisors induce the linear maps

$$f_\star \colon \mathsf{N}(Y) \to \mathsf{N}(X) \quad \text{and} \quad f^\star \colon \mathsf{N}(X) \to \mathsf{N}(Y),$$

respectively, which satisfy $f_\star(\mathrm{cl}_Y(G)) = \mathrm{cl}_X(f_*(G))$ and $f^\star(\mathrm{cl}_X(D)) = \mathrm{cl}_Y(f^*(D))$ for any $\mathbb{R}$-divisors G on Y and D on X. By the projection formulas (II-2) and (II-3), we have

$$f^\star(x) \cdot y = x \cdot f_\star(y) \quad \text{and} \quad f_\star(f^\star(x)) = (\deg f)x$$

for any $x \in \mathsf{N}(X)$ and $y \in \mathsf{N}(Y)$. In particular, the linear map $f_\star$ is surjective and the other map $f^\star$ is injective.

Lemma 2.10. *Let $f \colon Y \to X$ be a birational morphism of normal Moishezon surfaces. Then, $\hat{\boldsymbol{\rho}}(Y) = \hat{\boldsymbol{\rho}}(X) + k$ for the number k of f-exceptional prime divisors. In particular, $\hat{\boldsymbol{\rho}}(X) \leq \boldsymbol{\rho}(M)$ holds for the minimal resolution $M \to X$ of singularities.*

Proof. Let $C_1, \ldots, C_k$ be the f-exceptional curves, and let $v \colon \mathsf{N}(Y) \to \mathbb{R}^{\oplus k}$ be the homomorphism defined by $v(D) = (DC_1, \ldots, DC_k)$ for an $\mathbb{R}$-divisor D on Y. Then, v is surjective, since $\det(C_iC_j) \neq 0$ (cf.

Theorem 2.6). The kernel of v is just the image of $f^\star \colon \mathsf{N}(X) \to \mathsf{N}(Y)$ by Remark 2.5. Therefore, $\mathsf{N}(Y) \simeq \mathsf{N}(X) \oplus \mathbb{R}^{\oplus k}$, and we have $\hat{\boldsymbol{\rho}}(Y) = \hat{\boldsymbol{\rho}}(X) + k$. Q.E.D.

The following result is called the *Hodge index theorem* as in the non-singular case.

Lemma. *If C and D be $\mathbb{R}$-divisors on a normal Moishezon surface X such that* $\mathrm{cl}(D) \neq 0$ *and $D^2 \geq 0$. If $CD = 0$, then $C^2 \leq 0$, where the equality $C^2 = 0$ holds if and only if* $\mathrm{cl}(C) \in \mathbb{R}\,\mathrm{cl}(D)$. *In particular, if $D^2 > 0$ and $CD = C^2 = 0$, then* $\mathrm{cl}(C) = 0$.

Proof. It is derived from the Hodge index theorem for non-singular projective surfaces, as follows. Let $\mu \colon M \to X$ be a resolution of singularities. Then, M is projective by Fact 2.30 below. Since $\mathrm{cl}(D) \neq 0$, we can take an ample divisor H on M with $\mu^*(D)H \neq 0$. We define a real number r by $(\mu^*(C) - r\mu^*(D))\,H = 0$. Since $H^2 > 0$, by the Hodge index theorem for M, we have

$$0 \geq (\mu^*(C - rD))^2 = C^2 - 2rCD + D^2 = D^2 + C^2 \geq C^2,$$

where $C^2 = 0$ holds if and only if $C - rD \approx 0$. Q.E.D.

Definition 2.11. Let D be an $\mathbb{R}$-divisor on a normal Moishezon surface X.

- (i) D is said to be *numerically trivial* if $D \approx 0$;
- (ii) D is said to be *nef* if $DC \geq 0$ for any curve $C \subset X$;
- (iii) D is said to be *pseudo-effective* if $DB \geq 0$ for any nef divisor B on X;
- (iv) D is said to be *numerically ample* if $D^2 > 0$ and $DC > 0$ for any curve $C \subset X$ (cf. [48, p. 629]);
- (v) D is said to be *big* if $D - A$ is pseudo-effective for a numerically ample $\mathbb{R}$-divisor A.

Remark 2.12. A numerically ample Cartier divisor is ample by the Nakai–Moishezon criterion of ampleness ([39], [34]) when X is projective. This holds true even if X is only a normal Moishezon surface (cf. [35, I, Th. 6]).

Remark 2.13. By the projection formula (II-2), we infer that, for a birational morphism $f \colon Y \to X$ of normal Moishezon surfaces, if an $\mathbb{R}$-divisor B on Y is nef, pseudo-effective, numerically ample, and big, respectively, then so is $f_*(B)$. Similarly, if an $\mathbb{R}$-divisor D on X is nef, pseudo-effective, and big, respectively, then so is $f^*(D)$.

Remark. Every normal Moishezon surface X admits a numerically ample divisor. In fact, by Remark 2.13, $\mu_*(H)$ is numerically ample for the minimal resolution $\mu\colon M \to X$ of singularities and for an ample divisor H on M. In particular, the Hodge index theorem is equivalent to that the signature of the intersection pairing on $\mathsf{N}(X)$ is $(1, \hat{\boldsymbol{\rho}}(X) - 1)$.

On the properties "nef" and "numerically trivial," we introduce some variants:

Definition 2.14. Let X be a normal surface and D an $\mathbb{R}$-divisor.

(i) D is said to be *f-nef* (resp. *f-numerically trivial*) for a proper morphism $f\colon X \to S$, if $DC \geq 0$ (resp. $DC = 0$) for any curve $C \subset X$ mapped to a point of S;

(ii) D is said to be *nef on B* (resp. *numerically trivial on B*) for a compact reduced divisor B on X, if $DB_i \geq 0$ (resp. $DB_i = 0$) for any irreducible component B_i of B.

Remark 2.15. Let $f\colon X \to Y$ be a birational morphism of normal Moishezon surfaces.

- If an $\mathbb{R}$-divisor D on X is f-nef and $f_*D = 0$, then $-D$ is effective by Lemma 2.2, since D is negative definite (cf. Theorem 2.6).
- Let B be a reduced divisor on Y. If an $\mathbb{R}$-divisor L on X is nef on $f^{-1}B$, then f_*L is nef on B, by the projection formula (II-2).

The following result on the properties "big," "pseudo-effective," and "numerically ample" is shown easily by the same argument in the usual case of Cartier divisors. The proof of left to the reader.

Lemma 2.16. *Let X be a normal Moishezon surface with an $\mathbb{R}$-divisor D.*

(1) *When D is nef, D is big if and only if $D^2 > 0$.*

(2) *If $D^2 > 0$* (resp. $D^2 \geq 0$)*, then D or $-D$ is big* (resp. pseudo-effective).

(3) *The numerical ampleness of D is equivalent to that $DE > 0$ for any pseudo-effective $\mathbb{R}$-divisor E which is not numerically trivial.*

The following theorem is a relative version of Kawamata–Viehweg vanishing theorem (cf. [24, Th. 1-2-3]) in the two-dimensional case.

Theorem 2.17 (cf. [47, Th. (6.3)])**.** *Let $f\colon X \to Y$ be a proper surjective morphism between normal surfaces and let D be an f-nef $\mathbb{Q}$-divisor on X. Then,*

$$R^1 f_* \mathcal{O}_X(K_X + \ulcorner D \urcorner) = 0, \tag{II-4}$$

where the round-up $\ulcorner D \urcorner$ *is defined as* $\sum \ulcorner a_i \urcorner D_i$ *for the irreducible decomposition* $D = \sum a_i D_i$*, and the round-up* $\ulcorner r \urcorner$ *of a rational number* r *is defined as the smallest integer not less than* r*.*

Remark. Theorem 2.17 is well known in the case where X is non-singular and $\operatorname{Supp} D$ is a normal crossing divisor: This is shown in [24, Th. 1-2-3] when Y is a scheme, but it is also valid for an algebraic space Y, since it is étale locally a scheme. We can reduce to this case by an argument in [47, Th. (5.1)]. In fact, we have a proper birational morphism $\mu\colon M \to X$ from a non-singular surface M such that $\mu^{-1}(\operatorname{Supp} D) = \operatorname{Supp} \mu^* D$ is normal crossing, and an exact sequence

$$0 \to \mu_* \mathcal{O}_M(K_M + \ulcorner \mu^* D \urcorner) \to \mathcal{O}_X(K_X + \ulcorner D \urcorner) \to \mathcal{T} \to 0$$

on X for a skyscraper sheaf $\mathcal{T}$. Then,

$$R^1 f_*(\mu_* \mathcal{O}_M(K_M + \ulcorner \mu^* D \urcorner)) \subset R^1 (f \circ \mu)_* \mathcal{O}_M(K_M + \ulcorner \mu^* D \urcorner) = 0$$

by [24, Th. 1-2-3], and we have (II-4) by $R^1 f_* \mathcal{T} = 0$. Theorem 2.17 is valid even in the positive characteristic case. In fact, the local vanishing theorem [47, Th. (2.2)] holds in the positive characteristic case by [47, Rem. (2.4)], and we can reduce to the case where X and Y are non-singular and $X \to Y$ is a succession of blowings up at points.

As a corollary of Theorem 2.17, we have the following useful lemma, which is used in proving Propositions 2.29 and 4.8 below.

Lemma 2.18. *For a normal surface* X *and a reduced divisor* D *on* X*, let* C *be a negative curve on* X *such that* $C \not\subset D$ *and* $(K_X + D)C \leq 0$*. Then,* $\sharp C \cap D \leq 1$*.*

Proof. Let $f\colon X \to \overline{X}$ be the contraction morphism of C and set $\overline{D} := f_*(D)$. Then, the structure sheaf $\mathcal{O}_{\overline{D}}$ of the divisor $\overline{D}$ is just the image of $\mathcal{O}_{\overline{X}} \simeq f_* \mathcal{O}_X \to f_* \mathcal{O}_D$. On the other hand, $R^1 f_* \mathcal{O}_X(-D) = 0$ by Theorem 2.17, since $-(D + K_X)$ is f-nef. Hence, $\mathcal{O}_{\overline{D}} \simeq f_* \mathcal{O}_D$, and consequently, every fiber of $f|_D \colon D \to \overline{D}$ is connected. In particular, $C \cap D$ is connected or empty, and thus, $\sharp C \cap D \leq 1$. Q.E.D.

2.3. Cone and contraction theorems

The cone and contraction theorems are important in the study of minimal models and these are stated for log-canonical pairs, usually. Here, we explain a version of the cone theorem valid for any normal Moishezon surface and a version of the contraction theorem valid for any normal projective surface.

Definition. For a normal Moishezon surface X, let $\overline{\mathrm{NE}}(X)$ denote the closure in $\mathsf{N}(X)$ of the cone $\mathrm{NE}(X)$ consisting of the numerical classes $\mathrm{cl}(D)$ of all the effective $\mathbb{R}$-divisors D on X. Then, $\overline{\mathrm{NE}}(X)$ is identical to the set of the numerical classes of all the pseudo-effective $\mathbb{R}$-divisors on X. The dual cone of $\overline{\mathrm{NE}}(X)$ with respect to the intersection pairing $\mathsf{N}(X) \times \mathsf{N}(X) \to \mathbb{R}$ is just the *nef cone* $\mathrm{Nef}(X)$, which is the set of the numerical classes of all the nef $\mathbb{R}$-divisors on X. For an $\mathbb{R}$-divisor B, we set

$$\overline{\mathrm{NE}}(X)_B^{\geq 0} := \{z \in \overline{\mathrm{NE}}(X) \mid \mathrm{cl}(B) \cdot z \geq 0\} \quad \text{and}$$
$$\overline{\mathrm{NE}}(X)_B^{\perp} := \{z \in \overline{\mathrm{NE}}(X) \mid \mathrm{cl}(B) \cdot z = 0\}.$$

An *extremal ray* R of $\overline{\mathrm{NE}}(X)$ is a one-dimensional face of the cone $\overline{\mathrm{NE}}(X)$, i.e., $\mathsf{R} = \mathbb{R}_{\geq 0} v = \overline{\mathrm{NE}}(X)_L^{\perp}$ for a non-zero vector v of $\overline{\mathrm{NE}}(X)$ and a nef $\mathbb{R}$-divisor L.

Remark. (1) An $\mathbb{R}$-divisor D of X is numerically ample (resp. big) if and only if $\mathrm{cl}(D)$ lies in the interior of $\mathrm{Nef}(X)$ (resp. $\overline{\mathrm{NE}}(X)$) (cf. Lemma 2.16).

(2) The cones $\mathrm{Nef}(X)$ and $\overline{\mathrm{NE}}(X)$ are strictly convex closed cones of $\mathsf{N}(X)$, and $\mathrm{Nef}(X) \subset \overline{\mathrm{NE}}(X)$.

(3) The one-dimensional cone $\mathbb{R}_{\geq 0}\, \mathrm{cl}(\Gamma)$ is an extremal ray of $\overline{\mathrm{NE}}(X)$ for any negative curve Γ.

The cone theorem by Mori [36] for non-singular projective surfaces is generalized to the case of normal Moishezon surfaces by Sakai in [48, Prop. 4.8] (cf. [49, Appendix]). As a consequence, we have:

Theorem 2.19. *For a normal Moishezon surface X and for any numerically ample $\mathbb{R}$-divisor A of X, there exist finitely many rational curves C_i with $-3 \leq K_X C_i < 0$ such that $\mathsf{R}_i = \mathbb{R}_{\geq 0}\, \mathrm{cl}(C_i)$ is an extremal ray and*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X+A}^{\geq 0} + \sum \mathsf{R}_i.$$

Corollary 2.20. *Let X be a normal Moishezon surface.*

(1) *If R is an extremal ray of $\overline{\mathrm{NE}}(X)$ with $K_X \mathsf{R} < 0$, then $\mathsf{R} = \mathbb{R}_{\geq 0}\, \mathrm{cl}(C)$ for a rational curve C with $0 > K_X C \geq -3$.*

(2) *For a nef $\mathbb{R}$-divisor L, if $K_X + L$ is not nef, then there is an extremal ray R such that $(K_X + L)\mathsf{R} < 0$.*

Proof. (1): There is a numerically ample $\mathbb{R}$-divisor A such that $(K_X + A)\mathsf{R} < 0$. Since R is extremal, R is one of the extremal rays R_i in Theorem 2.19.

(2): There is a numerically ample $\mathbb{R}$-divisor A such that $K_X + L + A$ is not nef. Then, $K_X + A$ is not nef. Let R_i be the extremal rays in Theorem 2.19. If $(K_X + L)\mathsf{R}_i \geq 0$ for any i, then $K_X + L + A$ is nef, since $\mathrm{cl}(K_X + L + A) \cdot z \geq 0$ for any $z \in \overline{\mathrm{NE}}(X)$ by Theorem 2.19: This is a contradiction. Thus, $(K_X + L)\mathsf{R}_i < 0$ for some R_i. Q.E.D.

The contraction theorem [36, Th. (2.1)] on the extremal rays has been generalized to many situations by [55], [49], etc. The following version is a special case of [48, Th. 4.9], which deals with normal Moishezon surfaces. This seems to hold also in the positive characteristic case (cf. [2, Th. 10.3]).

Theorem 2.21. *Let X be a normal projective surface with an extremal ray R such that $K_X\mathsf{R} < 0$. Then, there exists a fibration $\pi\colon X \to S$ to a normal projective variety S, called the* contraction morphism *of R, such that, for any curve $C \subset X$, its numerical class $\mathrm{cl}(C)$ belongs to R if and only if $\pi(C)$ is a point. Here, $\hat{\boldsymbol{\rho}}(X) = \hat{\boldsymbol{\rho}}(S) + 1$. Moreover, the following hold: Let v be a non-zero vector in R.*

(1) *If $v^2 > 0$, then $\hat{\boldsymbol{\rho}}(X) = 1$, $\overline{\mathrm{NE}}(X) = \mathsf{R}$, X has a rational curve, and π is the constant morphism $X \to \operatorname{Spec}\mathbb{C}$.*
(2) *If $v^2 = 0$, then $\hat{\boldsymbol{\rho}}(X) = 2$ and $\pi\colon X \to S$ is a fibration to a non-singular projective curve S such that every fiber of π is a non-singular rational curve and its numerical class belongs to R.*
(3) *If $v^2 < 0$, then $\mathsf{R} = \mathbb{R}_{\geq 0}\,\mathrm{cl}(\Gamma)$ for a negative rational curve Γ, and π is the contraction morphism of Γ.*

Remark 2.22. In the case (3) above, the projectivity of S is shown as follows (cf. the proof of [4, Th. 2.3]). We can find a very ample divisor H on X and a positive integer r such that $(H + r\Gamma)\Gamma = 0$ and $\mathrm{H}^1(X, \mathcal{O}_X(H)) = 0$. Then, $L = H + r\Gamma$ is a nef and big Cartier on X and $\overline{\mathrm{NE}}(X)^{\perp}_{L} = \mathbb{R}_{\geq 0}\,\mathrm{cl}(\Gamma) = \mathsf{R}$. It is enough to prove that the linear system $|L|$ is base point free. In fact, in this case, the morphism $\Phi_{|L|}\colon X \to |L|^{\vee}$ associated with $|L|$ factors through a finite morphism $S \to |L|^{\vee}$, where $|L|^{\vee}$ is the dual projective space of $|L|$.

We have $R^1\pi_*\mathcal{O}_X = 0$ by Theorem 2.17 applied to the π-nef divisor $-K_X$; hence, $\Gamma \simeq \mathbb{P}^1$ and $\mathcal{O}_X(L)|_{\Gamma} \simeq \mathcal{O}_{\Gamma}$. Since the base locus of $|L|$ is contained in Γ, it is enough to prove that the restriction homomorphism

$$\phi\colon\ \mathrm{H}^0(X, \mathcal{O}_X(L)) \to \mathrm{H}^0(\Gamma, \mathcal{O}_X(L)|_{\Gamma}) \simeq \mathrm{H}^0(\Gamma, \mathcal{O}_{\Gamma})$$

is non-zero. The homomorphism ϕ factors as

$$\mathrm{H}^0(X, \mathcal{O}_X(L)) \xrightarrow{\varphi} \mathrm{H}^0(X, \mathcal{O}_{r\Gamma}(L)) \xrightarrow{\psi} \mathrm{H}^0(X, \mathcal{O}_{\Gamma}(L)),$$

where φ is surjective by $\mathrm{H}^1(X, \mathcal{O}_X(H)) = 0$. The homomorphism ψ is a composition of the restriction homomorphisms

$$\psi_k\colon \mathrm{H}^0(X, \mathcal{O}_{k\Gamma}(L)) \to \mathrm{H}^0(X, \mathcal{O}_{(k-1)\Gamma}(L))$$

for $0 < k \le r$, and each ψ_k is surjective by $\mathrm{H}^1(\Gamma, \mathcal{O}_\Gamma(L - (k-1)\Gamma)) = 0$. Thus, ϕ is surjective, $|L|$ is base point free, and consequently, S is projective.

2.4. The defect and complexity

We shall study basic properties on the defect and the complexity defined as follows:

Definition 2.23. Let X be a normal Moishezon surface and D a reduced divisor on X. We define $\boldsymbol{n}(D)$ to be the number of irreducible components of D. The vector subspace of $\mathsf{N}(X)$ generated by the numerical classes of irreducible components of D is denote by $\mathsf{N}(X)_D$. The dimension of $\mathsf{N}(X)_D$ is denoted by $\boldsymbol{r}(X, D)$ or $\boldsymbol{r}(D)$ for short. We set

$$\boldsymbol{\delta}(X, D) := \hat{\boldsymbol{\rho}}(X) + 2 - \boldsymbol{n}(D) \quad \text{and} \quad \boldsymbol{c}(X, D) := \boldsymbol{r}(D) + 2 - \boldsymbol{n}(D).$$

We call $\boldsymbol{\delta}(X, D)$ the *defect* and $\boldsymbol{c}(X, D)$ the *complexity*.

Remark. By definition, $\boldsymbol{r}(D) \le \hat{\boldsymbol{\rho}}(X) = \dim \mathsf{N}(X)$. If $\boldsymbol{r}(D) = \hat{\boldsymbol{\rho}}(X)$, then D is big. We always have $\boldsymbol{\delta}(X, D) \ge \boldsymbol{c}(X, D)$. The defect $\boldsymbol{\delta}(X, D)$ is called the *absolute complexity* in [33].

Definition 2.24. For (X, D) in Definition 2.23, let $\mathsf{F}(D)$ denote the free abelian group generated by the irreducible components of D. The *class map* is a homomorphism

$$\mathrm{cl}_D\colon \mathsf{F}(D) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathsf{N}(X)$$

of vector spaces which associates with each irreducible component D_i of D the numerical class $\mathrm{cl}(D_i)$. For the (Weil) divisor class group $\mathrm{CL}(X)$ of X, we have another class map

$$\mathrm{cl}_D^{\mathbb{Z}}\colon \mathsf{F}(D) \to \mathrm{CL}(X)$$

which associates with each irreducible component of D the linear equivalence class.

Remark. The complexity $\boldsymbol{c}(X, D)$ is related to the class map. In fact, $\mathsf{N}(X)_D$ is the image of cl_D, and we have

$$\boldsymbol{n}(D) - \boldsymbol{r}(D) = \dim \mathrm{Ker}(\mathrm{cl}_D) \ge 0 \quad \text{and}$$
$$\boldsymbol{c}(X, D) = 2 - \dim \mathrm{Ker}(\mathrm{cl}_D) \le 2.$$

If the numerical equivalence relation $\approx$ coincides with the $\mathbb{Q}$-linear equivalence relation $\sim_{\mathbb{Q}}$ (e.g., the case of Lemma 2.31(4) below), then

$$\operatorname{rank}\mathrm{CL}(X) = \hat{\boldsymbol{\rho}}(X) \quad \text{and} \quad \operatorname{rank}\mathrm{cl}_D^{\mathbb{Z}} = \boldsymbol{r}(D).$$

Lemma 2.25 (cf. [12, Prop. (1.17)])**.** *The kernel of* $\mathrm{cl}_D^{\mathbb{Z}}$ *is isomorphic to* $\mathcal{O}(X \setminus D)^\star/\mathbb{C}^\star$*.*

Proof. By definition, $\mathrm{Ker}(\mathrm{cl}_D^{\mathbb{Z}})$ consists of principal divisors $\mathrm{div}(f)$ associated with non-zero rational functions f on X such that $\mathrm{Supp}\,\mathrm{div}(f) \subset D$; The last condition means that f is invertible on $X \setminus D$. Therefore, we have a surjection $\mathcal{O}_X(X \setminus D)^\star \to \mathrm{Ker}(\mathrm{cl}_D^{\mathbb{Z}})$ by $f \mapsto \mathrm{div}(f)$, and the kernel of this surjection is just $\mathcal{O}(X)^\star = \mathbb{C}^\star$. Q.E.D.

Fact. Let X be a non-singular projective variety of arbitrary dimension and let D be a simple normal crossing divisor on X. In this case, we can also consider the class map $\mathrm{cl}_D\colon \mathsf{F}(D) \otimes \mathbb{R} \to \mathsf{N}(X)$ to the real vector space $\mathsf{N}(X)$ of the numerical equivalence classes of $\mathbb{R}$-divisors on X. Then, the kernel $\mathrm{Ker}(\mathrm{cl}_D)$ is isomorphic to the kernel of $\mathrm{H}^2_{D^{\mathrm{an}}}(X^{\mathrm{an}}, \mathbb{R}) \to \mathrm{H}^2(X^{\mathrm{an}}, \mathbb{R})$, and the equality

$$\dim \mathrm{Ker}(\mathrm{cl}_D) = \bar{q}(X \setminus D) - q(X)$$

holds by [16, Prop. 1] (cf. [12, Prop. (1.15)]), where $\bar{q}$ stands for the logarithmic irregularity and q for the irregularity. Moreover, the following holds true, which seems to be well known.

Proposition 2.26. *Let X be a non-singular projective variety such that $q(X) = 0$ and D a simple normal crossing divisor on X. Then, the* quasi-Albanese variety (*cf.* [16, §3]) *of $X \setminus D$ is an algebraic torus $\mathbb{T}$ of dimension $\bar{q} := \bar{q}(X \setminus D)$ and the* quasi-Albanese map (*cf.* [16, §4]) *is characterized as a morphism $\alpha\colon X \setminus D \to \mathbb{T}$ which induces an isomorphism*

$$\left(\mathbb{C}^\star \times \mathbb{Z}^{\oplus \bar{q}} \simeq\right)\ \mathcal{O}(\mathbb{T})^\star \xrightarrow{\simeq} \mathcal{O}(X \setminus D)^\star.$$

Proof. By the definition of the quasi-Albanese variety in [16, §3], the vanishing $q(X) = 0$ implies that the quasi-Albanese variety is an algebraic torus $\mathbb{T}$ of dimension $\bar{q}(X \setminus D)$. Let $\alpha\colon X \setminus D \to \mathbb{T}$ be the quasi-Albanese map. Then, by the universality of the quasi-Albanese map (cf. [16, Prop. 4]), for any morphism $f\colon X \setminus D \to T$ to another algebraic torus T, there is a unique morphism $u\colon \mathbb{T} \to T$ such that $f = u \circ \alpha$ and u is a group homomorphism of group schemes up to translation. In particular, the group homomorphism $\mathcal{O}(T)^\star \to \mathcal{O}(X \setminus D)^\star$ induced by f^* always factors through the group homomorphism $\mathcal{O}(\mathbb{T})^\star \to \mathcal{O}(X \setminus D)^\star$

induced by α^*. On the other hand, for the d-dimensional algebraic torus $\mathbb{G}_{\mathtt{m}}^d$, giving a morphism $X \setminus D \to \mathbb{G}_{\mathtt{m}}^d$ over $\operatorname{Spec}\mathbb{C}$ is equivalent to giving a group homomorphism $\mathbb{Z}^{\oplus d} \to \mathcal{O}(X \setminus D)^\star$. Therefore, α^* induces an isomorphism $\mathcal{O}(\mathbb{T})^\star \simeq \mathcal{O}(X \setminus D)^\star$, and this property characterizes the quasi-Albanese map α. Q.E.D.

Lemma 2.27. *Let $f\colon X \to \overline{X}$ be a birational morphism of normal Moishezon surfaces. Let D be a reduced divisor on X and set $\overline{D} = f_*(D)$. Then,*

$$\boldsymbol{n}(D) - \boldsymbol{n}(\overline{D}) \le \boldsymbol{r}(D) - \boldsymbol{r}(\overline{D}) \le \hat{\boldsymbol{\rho}}(X) - \hat{\boldsymbol{\rho}}(\overline{X}),$$
$$\text{or equivalently,} \quad 0 \le \boldsymbol{c}(X, D) - \boldsymbol{c}(\overline{X}, \overline{D}) \le \boldsymbol{\delta}(X, D) - \boldsymbol{\delta}(\overline{X}, \overline{D}).$$

Here, the equality $\boldsymbol{n}(D) - \boldsymbol{n}(\overline{D}) = \hat{\boldsymbol{\rho}}(X) - \hat{\boldsymbol{\rho}}(\overline{X})$ holds (equivalently, $\boldsymbol{\delta}(X, D) = \boldsymbol{\delta}(\overline{X}, \overline{D})$ holds) if and only if the f-exceptional locus is contained in D.

Proof. The push-forward of divisors by f defines a homomorphism $f_*\colon \mathsf{F}(D) \to \mathsf{F}(\overline{D})$ for the free abelian groups $\mathsf{F}(D)$ and $\mathsf{F}(\overline{D})$ defined in Definition 2.24, and it also defines the homomorphism $f_\star\colon \mathsf{N}(X) \to \mathsf{N}(\overline{X})$ of Remark 2.9. Let $\mathsf{E}(f)$ (resp. $\mathsf{E}(f)_D$) be the free abelian group generated by the f-exceptional prime divisors on X (resp. f-exceptional irreducible components of D). Then, there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{E}(f)_D \otimes \mathbb{R} \arrow[r] \arrow[d] & \mathsf{F}(D) \otimes \mathbb{R} \arrow[r, "f_* \otimes \mathbb{R}"] \arrow[d, "\mathrm{cl}_D"'] & \mathsf{F}(\overline{D}) \otimes \mathbb{R} \arrow[r] \arrow[d, "\mathrm{cl}_{\overline{D}}"'] & 0 \\
0 \arrow[r] & \mathsf{E}(f) \otimes \mathbb{R} \arrow[r] & \mathsf{N}(X) \arrow[r, "f_\star"] & \mathsf{N}(\overline{X}) \arrow[r] & 0
\end{tikzcd}$$

of exact sequences, where the left vertical homomorphism is induced from the inclusion $\mathsf{E}(f)_D \subset \mathsf{E}(f)$. Hence, for the kernel W of the surjection $\mathsf{N}(X)_D \to \mathsf{N}(\overline{X})_{\overline{D}}$ induced by $f_\star$, we have inclusions

$$\mathsf{E}(f)_D \otimes \mathbb{R} \subset W \subset \mathsf{E}(f) \otimes \mathbb{R}.$$

Comparing the dimensions of these three vector spaces, we have the required inequality, since $\operatorname{rank}\mathsf{E}(f)_D = \boldsymbol{n}(D) - \boldsymbol{n}(\overline{D})$, $\operatorname{rank}\mathsf{E}(f) = \hat{\boldsymbol{\rho}}(X) - \hat{\boldsymbol{\rho}}(\overline{X})$, and $\dim W = \boldsymbol{r}(D) - \boldsymbol{r}(\overline{D})$. Here, the equality holds if and only if $\mathsf{E}(f)_D = \mathsf{E}(f)$, and this proves the last assertion. Q.E.D.

Lemma 2.28. *In the situation of Lemma* 2.27 *above, the following also hold*:

(1) *If the f-exceptional locus is contained in $X \setminus D$, then $\boldsymbol{n}(D) = \boldsymbol{n}(\overline{D})$, $\boldsymbol{r}(D) = \boldsymbol{r}(\overline{D})$, and $\boldsymbol{c}(X, D) = \boldsymbol{c}(\overline{X}, \overline{D})$.*

(2) *If f is the contraction morphism of a negative curve Γ with $\Gamma \not\subset D$, then $\boldsymbol{n}(D) = \boldsymbol{n}(\overline{D})$, $\hat{\boldsymbol{\rho}}(X) = \hat{\boldsymbol{\rho}}(\overline{X}) + 1$, and $\boldsymbol{\delta}(X, D) = \boldsymbol{\delta}(\overline{X}, \overline{D}) + 1$.*

(3) *In the situation of (2), assume that $\Gamma \cap (D - C) = \emptyset$ and $\Gamma \cap C \neq \emptyset$ for an irreducible component C of D. Then, $\boldsymbol{r}(D) = \boldsymbol{r}(\overline{D}) + 1$ (or equivalently, $\boldsymbol{c}(X, D) = \boldsymbol{c}(\overline{X}, \overline{D}) + 1$) if and only if*

$$\mathrm{cl}(\overline{C}) \in \mathsf{N}(\overline{X})_{\overline{D} - \overline{C}},$$

for the curve $\overline{C} = f_*(C)$.

Proof. The assertion (2) is a consequence of Lemma 2.10. For the proof of (1), it is enough to show: $\boldsymbol{r}(D) = \boldsymbol{r}(\overline{D})$. Let Δ be an $\mathbb{R}$-divisor supported on D such that $\Delta \approx G$ for an $\mathbb{R}$-divisor G contained in the exceptional locus. Then, $f_*\Delta \approx f_*G = 0$ and $0 \approx f^*f_*\Delta = \Delta$. Hence, the kernel W in the proof of Lemma 2.27 is zero, and we have $\boldsymbol{r}(D) = \boldsymbol{r}(\overline{D})$. This proves (1). In the situation of (3), the equality $\boldsymbol{r}(D) = \boldsymbol{r}(\overline{D}) + 1$ is equivalent to that $\mathrm{cl}(\Gamma) \in \mathsf{N}(X)_D$. Let Δ be an $\mathbb{R}$-divisor on X supported on D. We write $\Delta = dC + \Delta_1$ for some $d \in \mathbb{R}$ and for an $\mathbb{R}$-divisor Δ_1 supported on $D - C$. If $\Delta \approx r\Gamma$ for some real number $r \neq 0$, then $d \neq 0$ by $dC\Gamma = \Delta\Gamma = r\Gamma^2 \neq 0$, and moreover, $0 \approx f_*\Delta = d\overline{C} + f_*\Delta_1$. Hence, in this case, $\mathrm{cl}(\overline{C}) \in \mathsf{N}(\overline{X})_{\overline{D} - \overline{C}}$. Conversely, if $d \neq 0$ and if $0 \approx f_*\Delta = d\overline{C} + f_*\Delta_1$, then $\Delta \approx r\Gamma$ with $r \neq 0$ by $\Delta\Gamma = dC\Gamma \neq 0$. This proves (3), and we are done. Q.E.D.

The result below is obtained by Lemma 2.18 and by the so-called minimal model program: More precisely, by the cone and contraction theorems (cf. Theorems 2.19 and 2.21) with Corollary 2.20.

Proposition 2.29. *Let X be a normal projective surface and D a reduced divisor on X. Suppose that*

(i) *$-(K_X + D)$ is nef, and*

(ii) *either $\boldsymbol{\delta}(X, D) \leq 1$ or $\boldsymbol{c}(X, D) \leq 0$.*

Then, D is connected and reducible.

Proof. If $D = 0$, then $\boldsymbol{c}(X, D) = 2$. Thus, $D \neq 0$ and $\boldsymbol{r}(D) > 0$. Then, D is reducible by

$$\boldsymbol{n}(D) = \boldsymbol{r}(D) + 2 - \boldsymbol{c}(X, D) \geq \boldsymbol{r}(D) + 1 \geq 2.$$

It remains to prove the connectedness of D. Since $(-D) - K_X$ is nef and $-D$ is not nef, there is an extremal ray R on X such that $(-D)\mathsf{R} < 0$ and $K_X\mathsf{R} < 0$ by Corollary 2.20(2). Let us consider the contraction morphism $\mathrm{cont}_{\mathsf{R}}$ associated with R (cf. Theorem 2.21).

We first consider the case where $\mathrm{cont}_{\mathsf{R}}$ is a birational morphism $f\colon X \to X'$. Then, R is generated by $\mathrm{cl}(\Gamma)$ of a negative curve Γ, and f is just the contraction morphism of Γ. Note that X' is also a normal projective surface (cf. Remark 2.22). We set $D' = f_*(D)$. Then, $-(K_{X'} + D') = f_*(-(K_X + D))$ is nef (cf. Remark 2.15), and the inequalities $\boldsymbol{\delta}(X', D') \leq \boldsymbol{\delta}(X, D)$ and $\boldsymbol{c}(X', D') \leq \boldsymbol{c}(X, D)$ hold by Lemma 2.27. Hence, (X', D') satisfies the same conditions (i) and (ii). If $\Gamma \subset D$, then $D = f^{-1}(D')$, and even if $\Gamma \not\subset D$, we have $\sharp\Gamma \cap D \leq 1$ by Lemma 2.18. As a consequence, if D' is connected, then so is D. Thus, we may replace (X, D) with (X', D').

By the observation above and by Theorem 2.21, taking a succession of birational contractions of extremal rays, we can reduce to the following two cases:

- $\mathrm{cont}_{\mathsf{R}}$ is the structure morphism to a point;
- $\mathrm{cont}_{\mathsf{R}}$ is a fibration $\pi\colon X \to T$ to a non-singular curve T.

In the first case, $\hat{\boldsymbol{\rho}}(X) = 1$, and every non-zero effective divisor is ample and connected. Therefore, D is also connected in this case. In the second case, $\hat{\boldsymbol{\rho}}(X) = 2$, and we have $DF > 0$ and $(K_X + D)F \leq 0$ for a general fiber F of π. Thus, $F \simeq \mathbb{P}^1$ and $1 \leq DF \leq 2$. In particular, D contains at least one irreducible component C_0 which dominates T. Now, we have

$$\begin{aligned} \boldsymbol{n}(D) &= -\boldsymbol{\delta}(X, D) + \hat{\boldsymbol{\rho}}(X) + 2 \geq 3, \quad \text{or} \\ \boldsymbol{n}(D) &= -\boldsymbol{c}(X, D) + \boldsymbol{r}(D) + 2 \geq \boldsymbol{r}(D) + 2 \geq 3. \end{aligned}$$

In particular, D contains at least one fiber F_0 of π, since $DF \leq 2$. Then, the numerical classes of C_0 and F_0 span the two-dimensional vector space $\mathsf{N}(X)$. Thus, D is connected, and we are done. Q.E.D.

2.5. Rationality and projectivity

We shall give some criteria for a normal Moishezon surface to be projective or to be rational. We first note the following well-known:

Fact 2.30. A non-singular Moishezon surface is projective (cf. [10], [27, Th. 3.1], [26, Ch. 4, Th. 3.1; Ch. 5, 4.10]).

Lemma 2.31. *Let X be a normal Moishezon surface.*

(1) *If $\mathrm{H}^2(X, \mathcal{O}_X) = 0$, then X is projective.*
(2) *If X has only rational singularities, then X is $\mathbb{Q}$-factorial and projective.*
(3) *If $\mathrm{H}^2(X, \mathcal{O}_X) = \mathrm{H}^1(M, \mathcal{O}_M) = 0$ for a non-singular projective surface M birational to X, then X has only rational singularities.*

(4) *If X has only rational singularities and if* $\mathrm{H}^1(X, \mathcal{O}_X) = 0$*, then the numerical equivalence relation coincides with the* $\mathbb{Q}$*-linear equivalence relation for* $\mathbb{Q}$*-divisors on* X*.*

Sketch of the proof. The assertion (1) is well known as Brenton's criterion of projectivity (cf. [8, Prop. 7]). Note that this holds also in positive characteristic case by [7]. For the assertion (2), the $\mathbb{Q}$-factoriality of X has been proved in [52, §6, Satz 1], [9, Satz 1.5], and [31, Th. (17.4)], etc. The projectivity of X in this case can be proved by the same argument as in the proof of [4, Th. (2.3)] applied to the minimal resolution $\mu\colon M \to X$ of singularities. We have another proof of projectivity of X which uses the $\mathbb{Q}$-factoriality of X and a strong version of Nakai–Moishezon criterion of ampleness asserting that every numerically ample Cartier divisor is always ample (cf. Remark 2.12). The assertion (3) is shown by considering the Leray spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(X, R^q\mu_*\mathcal{O}_M) \Rightarrow E^{p+q} = \mathrm{H}^{p+q}(M, \mathcal{O}_M)$$

for a resolution of $\mu\colon M \to X$ singularities, and the assertion (4) is reduced to the non-singular case by this spectral sequence. Q.E.D.

Lemma 2.32. *For a normal Moishezon surface X and a reduced divisor D on X, assume that*

(i) *every irreducible component of D is a rational curve,*
(ii) *D is big, and*
(iii) *X has only rational singularities along D.*

Then, $\mathrm{H}^1(M, \mathcal{O}_M) = 0$ *for the minimal resolution M of singularities of X. If* $\mathrm{H}^0(X, \mathcal{O}_X(2K_X)) = 0$ *in addition, then X is a projective rational surface with only rational singularities.*

Proof. Let $\mu\colon M \to X$ be the minimal resolution. Then, every irreducible component of $\mu^*(D)$ is rational. In fact, the μ-exceptional components are rational by (iii) and the non-exceptional components are rational by (i). Thus, every irreducible component of $\mu^*(D)$ is mapped to a point by the Albanese map $\alpha\colon M \to \mathrm{Alb}(M)$. In particular, $\mu^*(D)\alpha^*(H) = 0$ for any ample divisor H of $\mathrm{Alb}(M)$. Then, $\alpha^*(H) \approx 0$ by the Hodge index theorem, since $\mu^*(D)$ is big (cf. Remark 2.13). Therefore, $\alpha(M) = \mathrm{Alb}(M)$ is a point, and hence $\mathrm{H}^1(M, \mathcal{O}_M) = 0$.

Assume in addition that $H^0(X, \mathcal{O}_X(2K_X)) = 0$. Then, X is a projective surface with only rational singularities by (1) and (3) of Lemma 2.31, since we have $\mathrm{H}^2(X, \mathcal{O}_X) \simeq \mathrm{H}^0(X, \mathcal{O}_X(K_X))^\vee = 0$ and $\mathrm{H}^1(M, \mathcal{O}_M) = 0$. Moreover, the canonical injection $\mathrm{H}^0(M, \mathcal{O}_M(2K_M)) \subset \mathrm{H}^0(X, \mathcal{O}_X(2K_X)) = 0$ and the vanishing $\mathrm{H}^1(M, \mathcal{O}_M) = 0$ imply that M is a rational surface, by Castelnuovo's criterion. Q.E.D.

Proposition 2.33. *Let X be a normal Moishezon surface and let $\pi\colon X \to T$ be a $\mathbb{P}^1$-fibration to a non-singular projective curve T (here, a general fiber of π is isomorphic to $\mathbb{P}^1$ (cf. Notation and conventions, 5)). Then, the following hold:*

(1) *The surface X is projective and has only rational singularities. In particular, $\hat{\boldsymbol{\rho}}(X) = \boldsymbol{\rho}(X)$.*
(2) *The higher direct image sheaf $R^i\pi_*\mathcal{O}_X$ is zero for any $i > 0$.*
(3) *Any curve contained in a fiber of π is isomorphic to $\mathbb{P}^1$.*
(4) *If a scheme-theoretic fiber F of π is irreducible and reduced, then π is smooth along F.*
(5) *If an invertible sheaf $\mathcal{L}$ on X is π-numerically trivial (cf. Definition 2.14(i)), then $\mathcal{L}$ is isomorphic to the pullback of an invertible sheaf on T.*
(6) *If any fiber of π is irreducible, then $\boldsymbol{\rho}(X) = 2$.*
(7) *If $F_1, F_2, \ldots, F_k$ are the reducible fibers of π, then*

$$\boldsymbol{\rho}(X) = 2 + \sum\nolimits_{i=1}^{k} (\boldsymbol{n}(F_i) - 1).$$

Proof. (1) and (2): For a general fiber F, we have $K_X F = -2$, since $F \simeq \mathbb{P}^1$. Thus, $\mathrm{H}^0(X, \mathcal{O}_X(K_X)) \simeq \mathrm{H}^2(X, \mathcal{O}_X)^\vee = 0$, and X is projective by Lemma 2.31(1). Let $\mu\colon M \to X$ be the minimal resolution of singularities. Then, there is a proper birational morphism $M \to Y$ to a $\mathbb{P}^1$-bundle Y over T, where $M \to Y$ is a succession of blowdowns of (-1)-curves. Hence, $R^i(\pi \circ \mu)_*\mathcal{O}_M = 0$ for any $i > 0$. By the Leray spectral sequence for π and μ, we have $R^1\pi_*\mathcal{O}_X = 0$ and $\pi_*(R^1\mu_*\mathcal{O}_M) = 0$. Note that $R^i\pi_*\mathcal{O}_X = 0$ for $i \geq 2$, since any fiber of π is one-dimensional. The vanishing of $\pi_*(R^1\mu_*\mathcal{O}_M)$ implies the vanishing of the skyscraper sheaf $R^1\mu_*\mathcal{O}_M$. Thus, X has only rational singularities. The equality $\hat{\boldsymbol{\rho}}(X) = \boldsymbol{\rho}(X)$ follows from Remark 2.8 and Lemma 2.31(2).

(3) and (4): For any effective divisor G contained in a fiber of π, we have $\mathrm{H}^1(G, \mathcal{O}_G) = 0$ by (1), since $0 = R^1\pi_*\mathcal{O}_X \to R^1\pi_*\mathcal{O}_G$ is surjective. In particular, $p_a(\Gamma) = 0$ for any irreducible component Γ of any fiber of π; this proves (3). If a scheme-theoretic fiber F is irreducible and reduced, then $F \simeq \mathbb{P}^1$, and π is smooth along F by the flatness of π; this proves (4).

(5): We have $\deg(\mathcal{L}|_\Gamma) = 0$ for any irreducible component Γ of any fiber of π. Thus, $\mu^*\mathcal{L} \simeq (\pi \circ \mu)^*\mathcal{M}$ for an invertible sheaf $\mathcal{M}$ on T, since $\pi \circ \mu$ is expressed as the composition of the succession $M \to Y$ of blowdowns of (-1)-curves and the $\mathbb{P}^1$-bundle $Y \to T$. Taking μ_*, we have an isomorphism $\mathcal{L} \simeq \pi^*\mathcal{M}$.

(6): Assume that every fiber of π is irreducible. For a fixed fiber F, let us consider a homomorphism $v\colon \operatorname{Pic}(X) \to \mathbb{Z}$ defined by $v(\mathcal{L}) =$

$\deg(\mathcal{L}|_F)$ for invertible sheaves $\mathcal{L}$ on X. If F' is another fiber of π, then $F' \approxeq \alpha F$ for some positive rational number α. Thus, the kernel of v is just $\pi^* \operatorname{Pic}(T)$ by (5). The image of v is not zero, since $v(\mathcal{A}) > 0$ for an ample invertible sheaf $\mathcal{A}$. Therefore, $\boldsymbol{\rho}(X) = 2$.

(7): Let us choose an irreducible component Γ_i of F_i for each $1 \le i \le k$. Since $F_i - \Gamma_i$ is negative-definite, we have the contraction morphism $f\colon X \to \overline{X}$ of $\sum_{i=1}^{k}(F_i - \Gamma_i)$ by Theorem 2.6. Then, $\overline{X}$ is also a normal projective surface with only rational singularities by (1), since there is a $\mathbb{P}^1$-fibration $\bar{\pi}\colon \overline{X} \to T$ with $\pi = \bar{\pi} \circ f$. Moreover, $\hat{\boldsymbol{\rho}}(\overline{X}) = \boldsymbol{\rho}(\overline{X}) = 2$ by (6). On the other hand, $\boldsymbol{\rho}(X) - \boldsymbol{\rho}(\overline{X}) = \sum_{i=1}^{k}(\boldsymbol{n}(F_i) - 1)$ by Lemma 2.10. Thus, (7) follows, and we are done. Q.E.D.

Remark 2.34. In the situation of Proposition 2.33, if X is non-singular, then every fiber is a simple normal crossing divisor. This well-known property is shown by the vanishing $R^1\pi_*\mathcal{O}_X = 0$ as follows. For arbitrary three irreducible components Γ, Γ', and Γ'' of a given fiber, we have

$$\mathrm{H}^1(\Gamma, \mathcal{O}_X(-\Gamma')|_\Gamma) = \mathrm{H}^1(\Gamma, \mathcal{O}_X(-\Gamma' - \Gamma'')|_\Gamma) = 0$$

by the vanishing $\mathrm{H}^1(\mathcal{O}_G) = 0$ for $G = \Gamma + \Gamma'$ and for $G = \Gamma + \Gamma' + \Gamma''$ (cf. the proof of (3) above). This implies $\Gamma\Gamma' \le 1$ and $\Gamma \cap \Gamma' \cap \Gamma'' = \emptyset$. Therefore, the fiber is a simple normal crossing divisor.

§3. Two-dimensional toric varieties and log-canonical pairs

In this section, we recall several properties on toric varieties and log-canonical pairs in the 2-dimensional case. We recall in Section 3.1 some of well-known basics on toric varieties, especially on toric surfaces. A toroidal singularity is a singularity arising at a toric variety. This is defined in Section 3.2 with a few properties in the surface case. The notion of log-canonical has appeared in the study of minimal models of algebraic varieties. In Section 3.3, we discuss the definition and some general properties of log-canonical pairs (X, B) for a normal surface X and an effective $\mathbb{Q}$-divisor B. When B is reduced, we have classification results on the singularities of this log-canonical pair (X, B), which is explained in Section 3.4. In Section 3.5, we classify the singularities of (X, B) along a compact irreducible component C of B such that B is a reduced divisor on a normal surface X and (X, B) is log-canonical along C with $(K_X + B)C \le 0$.

3.1. Projective toric surfaces

We recall here some basic properties on toric varieties, especially on toric surfaces. For details on the theory of toric varieties, the reader refers to the books [25], [43], [13], etc.

An n-dimensional normal algebraic variety X is called a toric variety if there is an action of the n-dimensional algebraic torus $\mathbb{T} = \mathbb{G}_{\mathtt{m}}^n$ on X such that it has an open orbit U which is isomorphic to $\mathbb{T}$ by the action. In other words, $U \subset X$ is an equivariant embedding of $\mathbb{T}$. In particular, X is a rational variety. We call $\mathbb{T}$ the *open torus* of X. The complement $D = X \setminus U$ is a divisor on X, which is called the *boundary divisor*. By abuse of notation (cf. Convention 1.4), the pair (X, D) is also called a toric variety. A two-dimensional toric variety is called a *toric surface*.

The toric variety is determined by the group

$$\mathsf{N} = \mathrm{Hom}_{\mathrm{group}}(\mathbb{G}_{\mathtt{m}}, \mathbb{T})$$

of one-parameter subgroups of $\mathbb{T}$ and by a certain finite collection $\triangle$, called a *fan* in [43] and [13] (or an *f.r.p.p. decomposition* in [25]), of strictly convex rational polyhedral cones in $\mathsf{N} \otimes_{\mathbb{Z}} \mathbb{R}$. We denote by $\mathbb{T}_{\mathsf{N}}(\triangle)$ the toric variety defined by N and $\triangle$ (this is denoted by $T_{\mathsf{N}}\,\mathrm{emb}(\triangle)$ in [43]). The group

$$\mathsf{M} = \mathrm{Hom}_{\mathrm{group}}(\mathbb{T}, \mathbb{G}_{\mathtt{m}})$$

is called the character group of $\mathbb{T}$. There is a natural non-singular bilinear form $\langle\ ,\ \rangle \colon \mathsf{M} \times \mathsf{N} \to \mathbb{Z}$ (cf. [25, Ch. 1, p. 2]). A one-dimensional cone of $\triangle$ is expressed as $\mathbb{R}_{\geq 0} v$ for a *primitive* element v of N, i.e., $\mathsf{N}/\mathbb{Z}v$ is torsion free. The cone $\mathbb{R}_{\geq 0} v$ corresponds to a prime divisor Γ on $X = \mathbb{T}_{\mathsf{N}}(\triangle)$ which is the closure of an orbit of $\mathbb{T}$, and we have

$$\mathrm{ord}_\Gamma(m) = \langle m, v \rangle$$

for any function $m \in \mathsf{M}$, where $\mathrm{ord}_\Gamma(m)$ stands for the order of zeros or the minus of the order of poles along Γ of the rational function m on X. Note that each $m \in \mathsf{M}$ is regarded as a morphism $U \simeq \mathbb{T} \to \mathbb{G}_{\mathtt{m}}$.

For toric varieties X and Y, a morphism $f \colon X \to Y$ of schemes is called a *morphism of toric varieties* (or a *toric morphism*) if f is equivariant with respect to some homomorphism $\phi \colon \mathbb{T}_X \to \mathbb{T}_Y$ between the open tori $\mathbb{T}_X$ and $\mathbb{T}_Y$ of X and Y, respectively: This means symbolically that $f(t \cdot x) = \phi(t) \cdot f(x)$ for any $x \in X$ and $t \in \mathbb{T}_X$. The toric morphism f is also described by a homomorphism between the groups of one-parameter subgroups of $\mathbb{T}_X$ and $\mathbb{T}_Y$ and by an information of fans.

Remark 3.1. For a given toric variety X with an open torus $\mathbb{T}$, there exists a $\mathbb{T}$-equivariant open immersion $X \hookrightarrow \widehat{X}$ to a compact toric variety

$\widehat{X}$ with the same open torus $\mathbb{T}$. This is a consequence of Sumihiro's theorem [54, Th. 3] on equivariant completion. If $X = \mathbb{T}_{\mathsf{N}}(\triangle)$, then $\widehat{X} = \mathbb{T}_{\mathsf{N}}(\hat{\triangle})$ for a fan $\hat{\triangle}$ such that the union $|\hat{\triangle}|$ of cones in $\hat{\triangle}$ is just $\mathsf{N} \otimes \mathbb{R}$ and that every cone on $\triangle$ belongs to $\hat{\triangle}$ (cf. [25, Ch. 1, Th. 8]). The existence of $\hat{\triangle}$ can be seen easily in the two-dimensional case (cf. Example 3.4 below).

Example 3.2. Let X be an affine toric surface with a zero-dimensional orbit. Then, $X \simeq \operatorname{Spec} \mathbb{C}[\boldsymbol{\sigma}^{\vee} \cap \mathsf{M}]$ for a convex cone $\boldsymbol{\sigma} = \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_2$ of $\mathsf{N} \otimes \mathbb{R} \simeq \mathbb{R}^2$, where e_1 and e_2 are primitive elements of N, (e_1, e_2) is a basis of $\mathsf{N} \otimes \mathbb{R}$,

$$\boldsymbol{\sigma}^{\vee} = \{m \in \mathsf{M} \otimes \mathbb{R} \mid \langle m, x \rangle \geq 0 \text{ for any } x \in \boldsymbol{\sigma}\},$$

and $\mathbb{C}[\boldsymbol{\sigma}^{\vee} \cap \mathsf{M}]$ is the semi-group ring defined by the semi-group $\boldsymbol{\sigma}^{\vee} \cap \mathsf{M}$, which is finitely generated. Let Γ_i be the prime divisor associated with the ray $\mathbb{R}_{\geq 0} e_i$ for $i = 1, 2$. Then, $\Gamma_1 \cap \Gamma_2$ is a point O, which is the zero-dimensional orbit corresponding to $\boldsymbol{\sigma}$. By a suitable coordinate change, we may assume that there exist integers $n > q \geq 0$ with $\gcd(n, q) = 1$ such that $\mathsf{N} = \mathbb{Z} e_2 + \mathbb{Z} u$ for $u := (1/n)(e_1 + q e_2)$. If $q = 0$, then $n = 1$, $X \simeq \mathbb{A}^2$, and Γ_1 and Γ_2 are coordinate lines. Assume that $q > 0$. Then, the singularity (X, O) is a cyclic quotient singularity. In fact, for the submodule $\mathsf{N}_0 = \mathbb{Z} e_1 + \mathbb{Z} e_2$, the induced toric morphism

$$X_0 = \mathbb{T}_{\mathsf{N}_0}(\boldsymbol{\sigma}) \simeq \mathbb{A}^2 \to X = \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$$

is regarded as the quotient map for the action of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ on $\mathbb{A}^2$ given by $(\mathtt{x}, \mathtt{y}) \mapsto (\zeta \mathtt{x}, \zeta^q \mathtt{y})$ for an n-th primitive root ζ of unity and for a coordinate $(\mathtt{x}, \mathtt{y})$ of $\mathbb{A}^2$. This (X, O) is called a *cyclic quotient singularity of order* n, or more explicitly, a *cyclic quotient singularity of type* (n, q) (or type $(1/n)(1, q)$ in some literature). Note that this is a rational singularity. It is well known that the minimal resolution $\mu \colon M \to X$ of the cyclic quotient singularity of type (n, q) is given by Hirzebruch–Jung's method and this is described as a toric morphism (cf. [25, Ch. I, §2, pp. 35–38]). For example, the inverse image $\mu^{-1}(\Gamma_1 \cup \Gamma_2)$ is a linear chain of rational curves in the sense of Definition 4.1 in which the proper transforms Γ_1' and Γ_2' of Γ_1 and Γ_2, respectively, in M are the end components of the chain. Furthermore, the self-intersection number $-b_i$ of the i-th irreducible component C_i of the linear chain $\mu^{-1}(O) = C_1 + C_2 + \cdots + C_l$ is determined by the continued fraction

$$\frac{n}{q} = [b_1, b_2, \ldots, b_l] = b_1 - \frac{1}{[b_2, \ldots, b_l]} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{[b_3, \ldots, b_l]}} = \cdots .$$

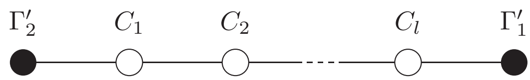

Fig. 1. Dual graph of $\mu^{-1}(\Gamma_1 \cup \Gamma_2)$

The dual graph of $\mu^{-1}(\Gamma_1 \cup \Gamma_2)$ is written as in Figure 1, where ○ (resp. ●) stands for the exceptional (resp. non-exceptional) component.

Remark 3.3. In Example 3.2, the divisor $\Gamma_1 + \Gamma_2$ is isomorphic to the union of two coordinate axes of $\mathbb{A}^2$ as a reduced scheme. In fact, the affine coordinate ring R of $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ is a $\mathbb{C}$-subalgebra of $\mathbb{C}[\mathtt{x}, \mathtt{y}]$ generated by monomials $\mathtt{x}^i\mathtt{y}^j$ such that $i + qj \equiv 0 \bmod n$. Let I be the ideal $I = (\mathtt{xy})\mathbb{C}[\mathtt{x}, \mathtt{y}] \cap R$ and J the ideal of R generated by monomials $\mathtt{x}^i\mathtt{y}^j$ such that i, $j > 0$. Then, $I \supset J$, and $\operatorname{Spec} R/I$ is isomorphic to $\Gamma_1 + \Gamma_2$. Let $P = \mathbb{C}[\mathtt{u}, \mathtt{v}]$ be the polynomial ring of two variables and let $P \to R$ be the $\mathbb{C}$-algebra homomorphism defined by $\mathtt{u} \mapsto \mathtt{x}^n$ and $\mathtt{v} \mapsto \mathtt{y}^n$. Then, the induced homomorphism $P \to R/J$ is surjective and the kernel of the composition

$$P \to R/J \to R/I \to \mathbb{C}[\mathtt{x}, \mathtt{y}]/(\mathtt{xy})$$

is the ideal generated by $\mathtt{uv}$. Since $R/I \to \mathbb{C}[\mathtt{x}, \mathtt{y}]/(\mathtt{xy})$ is injective, we have $I = J$ and $P/(\mathtt{uv})P \simeq R/I$. Therefore, $\Gamma_1 + \Gamma_2$ is isomorphic to the subscheme $\{\mathtt{uv} = 0\}$ of $\operatorname{Spec} P \simeq \mathbb{A}^2$.

Example 3.4. Let $X = \mathbb{T}_{\mathsf{N}}(\triangle)$ be a projective toric surface. Then, the fan $\triangle$ is determined by a collection $(v_1, \ldots, v_k)$ of non-zero elements of $\mathsf{N} \simeq \mathbb{Z}^{\oplus 2}$ satisfying the following conditions:

(i) Each v_i is primitive in N, i.e., $\mathsf{N}/\mathbb{Z}v_i$ is torsion free.

(ii) The pair (v_i, v_{i+1}) is a basis of $\mathsf{N} \otimes \mathbb{R}$ for any $1 \le i \le k-1$ and the same for (v_k, v_1).

(iii) We set $\boldsymbol{\sigma}_i := \mathbb{R}_{\ge 0}v_i + \mathbb{R}_{\ge 0}v_{i+1}$ for $1 \le i \le k-1$, and $\boldsymbol{\sigma}_k = \mathbb{R}_{\ge 0}v_k + \mathbb{R}_{\ge 0}v_0$. Then:
- If $j \equiv i + 1 \bmod k$, then $\boldsymbol{\sigma}_i \cap \boldsymbol{\sigma}_j = \mathbb{R}_{\ge 0}v_j$.
- If $i - j \bmod k$ is not in $\{0, 1, -1\}$, then $\boldsymbol{\sigma}_i \cap \boldsymbol{\sigma}_j = \{0\}$.

Here, the fan $\triangle$ consists of $\boldsymbol{\sigma}_i$, $\mathbb{R}_{\ge 0}v_j$, and $\{0\}$ for all i and j above. Let Γ_i be the prime divisor associated with the one-dimensional cone $\mathbb{R}_{\ge 0}v_i$. Then, $\Gamma_i \simeq \mathbb{P}^1$, and the union $D = \sum_{i=1}^k \Gamma_i$ is the boundary divisor. The two-dimensional cone $\boldsymbol{\sigma}_i$ corresponds to the intersection point $\Gamma_i \cap \Gamma_{i+1}$ for $1 \le i \le k-1$, and $\boldsymbol{\sigma}_k$ to the point $\Gamma_k \cap \Gamma_1$. These points are the zero-dimensional orbits of $\mathbb{T}$. As a consequence, we see that D is a cyclic chain of rational curves in the sense of Definition 4.3 below. Moreover, by Example 3.2, X has only cyclic quotient singularities, and the singularities are lying on $\operatorname{Sing} D$. In particular, X is $\mathbb{Q}$-factorial.

Remark 3.5. Every compact toric surface is projective by Lemma 2.31(2), since it has only rational singularities (cf. Example 3.2).

We list some facts on toric varieties of arbitrary dimension.

Fact 3.6. (1) The toric variety $X = \mathbb{T}_{\mathsf{N}}(\triangle)$ is compact if and only if the union of the cones in $\triangle$ coincides with $\mathsf{N} \otimes_{\mathbb{Z}} \mathbb{R}$ (cf. [43, Th. 1.11]).

(2) For any toric variety X, there is a proper birational toric morphism $M \to X$ giving a resolution of singularities of X (cf. [25, Ch. I, Th. 11]).

(3) For a toric variety X with the boundary divisor D, one has $K_X + D \sim 0$. In fact, this is well known when X is non-singular (cf. [43, Cor. 3.3]). In the general case, it is shown by taking push-forward for the open immersion $X \setminus \operatorname{Sing} X \hookrightarrow X$.

The following is shown in [13, §3.4, p. 63, Proposition]:

Lemma 3.7. *For a toric variety $X = \mathbb{T}_{\mathsf{N}}(\triangle)$ with the boundary divisor D, there is an exact sequence*

$$\mathsf{M} \xrightarrow{u} \mathsf{F}(D) \xrightarrow{\mathrm{cl}_D^{\mathbb{Z}}} \mathrm{CL}(X) \to 0 \tag{III-1}$$

for the class map $\mathrm{cl}_D^{\mathbb{Z}}$ in Definition 2.24 and for the character group $\mathsf{M} = \operatorname{Hom}(\mathsf{N}, \mathbb{Z})$. Here, an element of M is regarded as a semi-invariant rational function on X, and the map u associates with $m \in \mathsf{M}$ the principal divisor $\operatorname{div}(m)$. The map u is injective when the cones in $\triangle$ generate the vector space $\mathsf{N} \otimes \mathbb{R}$.

Now, we return to the two-dimensional case. The projective toric surfaces are described geometrically from some simple examples.

Example 3.8. (1) Let $L = L_1 + L_2 + L_3$ be the union of three lines on the projective plane $\mathbb{P}^2$ such that $L_1 \cap L_2 \cap L_3 = \emptyset$. Then, $(\mathbb{P}^2, L)$ is a toric surface. One can show that any non-singular projective toric surface with exactly three one-dimensional orbits is isomorphic to $(\mathbb{P}^2, L)$.

(2) For the Hirzebruch surface $X_e = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(e))$ of degree $e \geq 0$ with the ruling $\pi \colon X_e \to \mathbb{P}^1$, the pair (X_e, D) is an example of projective toric surfaces for the divisor

$$D := \sigma_0 + \sigma_\infty + F_1 + F_2$$

consisting of sections σ_0 and σ_∞ of π with $\sigma_0^2 = -e$, $\sigma_\infty^2 = e$, and $\sigma_0 \neq \sigma_\infty$, and of two distinct fibers F_1 and F_2 of π. One can show that any non-singular projective toric surface with four one-dimensional orbits is isomorphic to some (X_e, D) above.

(3) In (2) above, $\pi\colon (X_e, D) \to (\mathbb{P}^1, P_1 + P_2)$ is a toric morphism of toric varieties, where $P_i = \pi(F_i)$ for $i = 1, 2$.

(4) Let (X, D) be a non-singular projective toric surface and let $f\colon Y \to X$ be the blowing up at a point P of $\operatorname{Sing} D$. Then, (Y, D_Y) is a projective toric surface for $D_Y = f^{-1}(D)$, and f is a toric morphism. In fact, the action of the open torus of X naturally lifts to Y.

Lemma 3.9. *Let X be a toric surface with boundary divisor D. Let $g\colon X \to Z$ be a proper birational morphism to another normal surface Z such that the g-exceptional locus is contained in D (Here, Z is not assumed to be a scheme). Then, Z is also a toric surface having $g_*(D)$ as a boundary divisor, and g is a toric morphism. If X is compact, then X and Z are both projective.*

Proof. By Remark 3.1, we may assume that X is compact. Then, X is projective by Remark 3.5. Every irreducible component of D is the closure of an one-dimensional orbit of the open torus $\mathbb{T} = X \setminus D$. Hence, the action of $\mathbb{T}$ descends to Z. This implies that Z is a $\mathbb{T}$-equivariant compactification of $\mathbb{T} \simeq Z \setminus g_*(D)$, and g is $\mathbb{T}$-equivariant. It remains to prove that Z is a projective scheme. Note that $g_*(D) \neq 0$, since there is an ample divisor supported on D. Hence, $\mathrm{H}^2(Z, \mathcal{O}_Z) = \mathrm{H}^0(Z, \mathcal{O}_Z(K_Z))^\vee = 0$ by $K_Z + g_*(D) \sim g_*(K_X + D) \sim 0$ (cf. Fact 3.6(3)). Thus, Z is a projective surface by Lemma 2.31(1). Q.E.D.

By Example 3.8 and Lemma 3.9, we have:

Proposition 3.10. (1) *Any non-singular projective toric surface (X, D) is obtained from $(\mathbb{P}^2, L)$ or from (X_e, D) in Example 3.8 by a succession of blowings up at nodes of the boundary divisors (cf.* [43, Th. 1.28], [13, p. 43, Prop.]*).*

(2) *For a normal Moishezon surface X and a reduced divisor D, the pair (X, D) is a projective toric surface if and only if $X \setminus D$ is non-singular and (M, D_M) is a projective toric surface for a minimal resolution $\mu\colon M \to X$ of singularities and for $D_M = \mu^{-1}(D)$. In this case, μ is a toric morphism.*

By Lemma 3.7 or by Proposition 3.10, we have:

Lemma 3.11. *Let (X, D) be a projective toric surface. Then, $\boldsymbol{\rho}(X) = \hat{\boldsymbol{\rho}}(X) = \boldsymbol{r}(D) = \boldsymbol{n}(D) - 2$. In particular, $\boldsymbol{\delta}(X, D) = \boldsymbol{c}(X, D) = 0$.*

Proof. We have $\boldsymbol{\rho}(X) = \hat{\boldsymbol{\rho}}(X)$ by Lemma 2.31 and Example 3.2. There are two proofs of $\hat{\boldsymbol{\rho}}(X) = \boldsymbol{r}(D) = \boldsymbol{n}(D) - 2$. The first proof uses Lemma 3.7: By the exact sequence (III-1), we have $\boldsymbol{n}(D) = \hat{\boldsymbol{\rho}}(X) = \operatorname{rank} \mathrm{CL}(X)$ and $\boldsymbol{n}(D) - \boldsymbol{r}(D) = \operatorname{rank} \mathsf{M} = 2$. In the second proof, by Proposition 3.10 and by Lemma 2.27, we are reduced to the case where (X, D) is isomorphic to $(\mathbb{P}^2, L)$ or (X_e, D) in Example 3.8, and in the case, the equalities hold trivially. Q.E.D.

3.2. Toroidal singularities

Definition 3.12. Let X be a normal variety and B a reduced divisor.

(1) For a closed point P, the pair (X, B) is said to be *toroidal at* P if $X \setminus B \subset X$ is a *toroidal embedding* at P in the sense of [25, Ch. II, §1]. By [5, Cor. (2.6)], this is equivalent to the existence of an affine toric variety V and two étale morphisms $\tau \colon \mathcal{U} \to X$ and $\theta \colon \mathcal{U} \to V$ with a point $Q \in \mathcal{U}$ lying over P such that $\theta^{-1}(\mathbb{T}) = \tau^{-1}(X \setminus B)$ for the open torus $\mathbb{T}$ of V.

(2) The pair (X, B) is said to be *toroidal along* a subset Z of X if (X, B) is toroidal at each closed point of Z. If (X, B) is toroidal along X, then (X, B) is said to be *toroidal*.

The pair $(\mathcal{U}, Q)$ above is a common *étale neighborhood* of (X, P) and $(V, \theta(Q))$ in the following sense:

Definition 3.13 ([5, p. 27])**.** Let X be a scheme (or an algebraic space) and P a point of X. An *étale neighborhood* of (X, P) is defined as a pair $(\mathcal{U}, Q)$ of a scheme (or an algebraic space) $\mathcal{U}$ and a point $Q \in \mathcal{U}$ together with an étale morphism $\tau \colon \mathcal{U} \to X$ such that $P = \tau(Q)$ and τ induces an isomorphism $\Bbbk(P) \simeq \Bbbk(Q)$ of residue fields.

Remark. For a closed point P of an algebraic scheme X over $\mathbb{C}$, an étale neighborhood of (X, P) is an étale morphism $\mathcal{U} \to X$ with a point Q lying over P, since $\Bbbk(P)$ is algebraically closed. So, in this case, frequently, an étale neighborhood of (X, P) is regarded as an étale morphism $\mathcal{U} \to X$ whose image contains P.

By the study of singular affine toric surfaces in Example 3.2 and by Fact 3.6(3), we have:

Lemma 3.14. *Let (X, B) be a pair of normal surface X and a reduced divisor B. For a closed point $P \in B$, assume that (X, B) is* *toroidal at P. Then, $K_X + B$ is Cartier at P, and one of the following holds*:

(i) *X is non-singular at P, and B is also non-singular at P*;

(ii) *X is non-singular at P, and B is a normal crossing divisor at P with $P \in \operatorname{Sing} B$*;

(iii) *(X, P) is a cyclic quotient singularity of type (n, q) for some $n > q > 0$ with $\gcd(n, q) = 1$.*

Here, (iii) *means that (X, P) has a common étale neighborhood with $(\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}), O)$ for the toric surface $(\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}), \Gamma_1 + \Gamma_2)$ and the point $O = \Gamma_1 \cap \Gamma_2$ in Example* 3.2.

Corollary 3.15. *Let X be a normal surface and B a reduced divisor such that (X, B) is toroidal. Let $\mu\colon M \to X$ be the minimal resolution of singularities and set $B_M = \mu^{-1}(B)$. Then, B_M is a normal crossing divisor and $K_M + B_M = \mu^*(K_X + B)$. In particular, (X, B) is* log-canonical (*see Definition* 3.17 *below*).

Proof. We may assume that (X, B) is a singular affine toric surface. Then, the minimal resolution of X has been described in Example 3.2. Hence, (M, B_M) is toric and B_M is a simple normal crossing divisor. By Fact 3.6(3), we have $K_M + B_M \sim 0$ and $K_X + B \sim 0$. Thus, $K_M + B_M = \mu^*(K_X + B)$, and (X, B) is log-canonical by Corollary 3.20. Q.E.D.

Remark 3.16. Let X be a normal surface and B a non-zero effective divisor such that $K_X + B$ is Cartier along B. Then, B is Gorenstein and its dualizing sheaf ω_B is isomorphic to $\mathcal{O}_X(K_X + B)|_B$. In particular, if (X, B) is toroidal along B, then B is Gorenstein. In fact, we have an exact sequence

$$0 \to \omega_X \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(-B), \omega_X) \to \omega_B \to 0$$

from $0 \to \mathcal{O}_X(-B) \to \mathcal{O}_X \to \mathcal{O}_B \to 0$ by taking $\mathcal{H}om(-, \omega_X)$ using the isomorphism

$$\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_B, \omega_X) \simeq \omega_B$$

for the Cohen–Macaulay closed subscheme B of codimension one (cf. [3, Ch. I, Prop. (2.3)]), where the exact sequence is isomorphic to

$$0 \to \mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X + B) \to \mathcal{O}_X(K_X + B)|_B \to 0.$$

Note that every effective divisor B on a normal surface X is Cohen–Macaulay, i.e., satisfies Serre's condition S_1. In fact, for the maximal open subset U of X on which B is Cartier, we know that $B|_U$ is Cohen–Macaulay, and that B satisfies S_1 if the homomorphism r_3 in the commutative diagram of exact sequences below is injective:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_X(-B) \arrow[r] \arrow[d, "r_1"'] & \mathcal{O}_X \arrow[r] \arrow[d, "r_2"'] & \mathcal{O}_B \arrow[r] \arrow[d, "r_3"] & 0 \\ 0 \arrow[r] & j_*(\mathcal{O}_X(-B)|_U) \arrow[r] & j_*\mathcal{O}_U \arrow[r] & j_*(\mathcal{O}_B|_U), & \end{tikzcd}$$

where $j\colon U \hookrightarrow X$ stands for the open immersion. Since X is normal, $\mathcal{O}_X(-B)$ is reflexive, and $\operatorname{codim}(X \setminus U, X) \geq 2$, the vertical homomorphisms r_1 and r_2 are isomorphisms, and it implies that r_3 is injective.

3.3. Log-canonical pairs

Definition 3.17. Let X be a normal surface and B an effective $\mathbb{Q}$-divisor. For a proper birational morphism $\mu\colon M \to X$ from a nonsingular surface M, we have an equality

$$K_M = \mu^*(K_X + B) + \sum a_i E_i,$$

where E_i are the irreducible components of the union E of $\mu^{-1}(\operatorname{Supp} B)$ and the μ-exceptional locus, $a_i \in \mathbb{Q}$, and μ^* stands for the numerical pullback (cf. Definition 2.1). The pair (X, B) is said to be *log-canonical* (resp. *log-terminal*) if there is a proper birational morphism μ above such that E is a normal crossing divisor and that $a_i \geq -1$ (resp. $a_i > -1$) for any i.

Remark 3.18. (1) We can compare K_M and $\mu^*(K_X)$ by a rational two-form on X and its pullback to M. Thus, we can write the equality as above, which is not only a linear equivalence relation.

(2) The definition of log-canonical (resp. log-terminal) above does not depend on the choice of $\mu\colon M \to X$ with E being a normal crossing divisor. This property is generalized to Lemma 3.19 below.

(3) The notion of log-canonical (resp. log-terminal) is étale local on X: For an étale morphism $\mathcal{U} \to X$, if (X, B) is log-canonical (resp. log-terminal), then so is $(\mathcal{U}, B|_{\mathcal{U}})$ for the pullback $B|_{\mathcal{U}}$ of B. Conversely, if $(\mathcal{U}, B|_{\mathcal{U}})$ is log-canonical (resp. log-terminal), then so is $(U, B|_U)$ for the image U of $\mathcal{U} \to X$.

(4) For a subset Z of X, we say that (X, B) is log-canonical (resp. log-terminal) *along* Z if $(U, B|_U)$ is log-canonical (resp. log-terminal) for some open neighborhood U of Z: In case Z is a

point P, we say that (X, B) is log-canonical (resp. log-terminal) *at* P. By (3) above and by [5, Cor. (2.6)], the log-canonicity (resp. log-terminality) of (X, B) at a point P depends only on the completion $\widehat{\mathcal{O}}_{X,P}$ of the local ring $\mathcal{O}_{X,P}$ and on the pullback of B by $\operatorname{Spec}\widehat{\mathcal{O}}_{X,P} \to X$.

(5) The notion of log-canonical (resp. log-terminal) for (X, B) is defined for any dimension in case $K_X + B$ is $\mathbb{Q}$-Cartier. In two-dimensional case, if (X, B) is log-canonical in the sense of Definition 3.17, then $K_X + B$ also $\mathbb{Q}$-Cartier (cf. [23, Cor. 9.5]).

(6) The notion of log-terminal (resp. log-canonical) is introduced in [24]. The log-terminal is called Kawamata log terminal (klt) in [29], [30] when the boundary divisor B is not zero; instead another notion of log terminal is introduced in [29], which is not useful in the study of singularities. Indeed, it is not necessarily étale local and its definition does depend on the choice of good non-singular models M of X.

The following useful lemma is not so mentioned in the literature on birational geometry except in the case where f is a proper birational morphism. This is proved implicitly in [22, Prop. 1.7] or [24, Lem. 0-2-12]. We shall give a proof by tracing the argument there, which uses the logarithmic ramification formula (cf. [17, the formula (R) in p. 180], [19, Th. 11.5]). The same argument of our proof works for higher dimensional case in which $K_X + B$ is $\mathbb{Q}$-Cartier.

Lemma 3.19. *Let X be a normal surface and B an effective $\mathbb{Q}$-divisor. Let $f\colon Y \to X$ be a dominant morphism from a non-singular surface Y. Let G be the $\mathbb{Q}$-divisor on Y such that the ramification formula for f is equivalent to*

$$K_Y = f^*(K_X + B) + G.$$

Let $G = \sum \gamma_i G_i$ be the irreducible decomposition. If (X, B) is log-terminal (resp. log-canonical), then $\gamma_i > -1$ (resp. $\gamma_i \geq -1$) for any i. The converse holds when $\operatorname{Supp} G = \sum G_i$ *is a normal crossing divisor and f is proper surjective.*

Proof. Let $\mu\colon M \to X$ be a proper birational morphism from a non-singular surface M and let $\nu\colon N \to Y$ be a proper birational morphism from a non-singular surface N such that $f \circ \nu = \mu \circ g$ for a morphism

$g\colon N \to M$, i.e., the diagram

$$\begin{array}{ccc} N & \xrightarrow{\nu} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ M & \xrightarrow{\mu} & X \end{array}$$

is commutative, and that the following are satisfied:

- The union E of $\mu^{-1}(\operatorname{Sing} X \cup \operatorname{Supp} B)$ and the μ-exceptional locus is a simple normal crossing divisor.
- There is a normal crossing divisor F on N such that $g^{-1}(E) \subset F$ and that $N \setminus F \to X$ is étale.

We have the logarithmic ramification formula (cf. [17, §4])

$$K_N + F = g^*(K_M + E) + \overline{R} \tag{III-2}$$

for g, in which $\overline{R}$ is an effective divisor supported on F. By considering the ramification along E, we see that every irreducible component of $\overline{R}$ does not dominate any irreducible component of E. Note that $g^*E + \overline{R} - F$ is the ramification divisor for g. In particular, $g^*E + \overline{R} - F$ is effective and $F = \operatorname{Supp}(g^*E + \overline{R})$.

Let Δ be a $\mathbb{Q}$-divisor on M supported on E determined by

$$K_M + E = \mu^*(K_X + B) + \Delta,$$

where μ^* denotes the numerical pullback. Note that (X, B) is log-canonical if and only if Δ is effective and that (X, B) is log-terminal if and only if Δ is effective and $\operatorname{Supp} \Delta = E$. Since $K_Y = f^*(K_X + B) + G$ is equivalent to the logarithmic ramification formula (III-2), we have

$$G = \nu_*(\overline{R} - F + g^*\Delta).$$

Assume that (X, B) is log-canonical, i.e., Δ is effective. Then, $G + \nu_*F$ is effective for the reduced divisor ν_*F, which implies that $G + G_{\mathrm{red}}$ is effective, i.e., $\gamma_i \geq -1$ for any i. Assume that (X, B) is log-terminal, i.e., Δ is effective and $\operatorname{Supp} \Delta = E$. Then, $\operatorname{Supp}(g^*\Delta + \overline{R}) = \operatorname{Supp}(g^*E + \overline{R}) = F$, and it implies that $G + G_{\mathrm{red}}$ is effective with $\operatorname{Supp}(G + G_{\mathrm{red}}) = G_{\mathrm{red}}$: This is equivalent to that $\gamma_i > -1$ for any i. Hence, the first assertion has been proved.

For the second assertion, we assume that G_{red} is normal crossing and that f is proper surjective. In particular, g is surjective. Since $\nu^{-1}(G_{\mathrm{red}}) \subset F$, we have the logarithmic ramification formula

$$K_N + F = \nu^*(K_Y + G_{\mathrm{red}}) + \overline{R}_\nu,$$

where $\overline{R}_\nu$ is an effective divisor. Note also that $\overline{R}_\nu + \nu^* G_{\mathrm{red}} - F$ is effective, since Y is non-singular, and that $F \subset \operatorname{Supp}(\overline{R}_\nu + \nu^* G_{\mathrm{red}})$. Comparing the formulas above, we have

$$g^*\Delta + \overline{R} = \nu^*(G + G_{\mathrm{red}}) + \overline{R}_\nu.$$

If $\gamma_i \geq -1$ for any i, i.e., $G + G_{\mathrm{red}}$ is effective, then $g^*\Delta + \overline{R}$ is effective, and it implies that Δ is effective, since g is surjective and since all the common irreducible components of $\overline{R}$ and g^*E are contracted to points by g. Thus, (X, B) is log-canonical in this case. If $\gamma_i > -1$ for any i, i.e., $G + G_{\mathrm{red}}$ is effective with $\operatorname{Supp}(G + G_{\mathrm{red}}) = G_{\mathrm{red}}$, then Δ is effective and

$$F = \operatorname{Supp}(g^*E + \overline{R}) \supset \operatorname{Supp}(g^*\Delta + \overline{R}) = \operatorname{Supp}(\nu^* G_{\mathrm{red}} + \overline{R}_\nu) \supset F.$$

Hence, in this case, $\operatorname{Supp}\Delta = E$, and (X, B) is log-terminal. Thus, we are done. Q.E.D.

Corollary 3.20. *Let $\tau \colon V \to X$ be a proper surjective morphism of normal surfaces. Let D and Δ be effective $\mathbb{Q}$-divisors on V without common irreducible components and let B be an effective $\mathbb{Q}$-divisor on X such that the ramification formula for τ is equivalent to*:

$$K_V + D = \tau^*(K_X + B) + \Delta.$$

If (V, D) is log-canonical (resp. log-terminal), then so is (X, B). If $\Delta = 0$ and if (X, B) is log-canonical (resp. log-terminal), then so is (V, D).

Proof. This is derived from Lemma 3.19 applied to $Y \to X$ for a proper birational morphism $Y \to V$ from a certain non-singular surface Y. Q.E.D.

Remark. Corollary 3.20 is proved essentially in [30, Prop. 5.20(4)] in the case where f is a finite morphism, which uses the logarithmic ramification formula only for birational morphisms. The proof of [30, Prop. 5.20(4)] is sketchy and there are hidden some arguments like taking Galois closure and equivariant partial resolution, or flattening.

Corollary 3.21. *Let X be a normal surface with an effective $\mathbb{Q}$-divisor B. Let $g \colon X \to \overline{X}$ be a proper birational morphism to another normal surface $\overline{X}$ and set $\overline{B} := g_*(B)$.*

(1) *If (X, B) is log-canonical and $-(K_X + B)$ is g-nef, then $(\overline{X}, \overline{B})$ is log-canonical.*
(2) *For a subset Z of X, if (X, B) is log-canonical along Z, the g-exceptional locus is contained in Z, and if $-(K_X + B)$ is nef on Z, then $(\overline{X}, \overline{B})$ is log-canonical along $g(Z)$.*

Proof. It is enough to prove (1), since (2) is a consequence of (1) applied to the proper birational morphism $g^{-1}(U) \to U$ for an open neighborhood U of $g(Z)$. Let E be a g-exceptional $\mathbb{Q}$-divisor determined by

$$K_X + B = g^*(K_{\overline{X}} + \overline{B}) + E.$$

Then, $-E$ is g-nef, and it implies that E is effective by Lemma 2.2. Let Δ be the maximal effective $\mathbb{Q}$-divisor such that $B \geq \Delta$ and $E \geq \Delta$, and set $B' := B - \Delta$ and $E' := E - \Delta$. Then,

$$K_X + B' = g^*(K_{\overline{X}} + \overline{B}) + E'.$$

Since (X, B) is log-canonical, (X', B') is also log-canonical, and hence, $(\overline{X}, \overline{B})$ is log-canonical by Corollary 3.20. Q.E.D.

3.4. Singularities on boundary curves for log-canonical surfaces

The analytic germs of log-canonical pairs (X, B) of a normal surface X and a reduced divisor B have been classified by Kawamata in [23, Theorem 9.6] by a geometric construction. Alexeev gives the same classification in [1] by a numerical calculation. Note that the case where $B = 0$ has been done by Sakai in [50, Appendix] by another numerical calculation. The numerical classification is also treated implicitly in [21, §2], [56, §3], and [55, §2]. As the classification in case $B \neq 0$, we have:

Theorem 3.22. *Let X be a normal surface, B a reduced divisor on X, and P a* closed *point of B. Then, the pair (X, B) is log-canonical at P if and only if there is an étale neighborhood $(\mathcal{U}, Q)$ of (X, P) satisfying one of the following conditions*:

(i) *$B|_{\mathcal{U}} = B_1 + B_2$ for prime divisors B_1, B_2 with $\{Q\} = B_1 \cap B_2$, and $(\mathcal{U}, B|_{\mathcal{U}})$ is toroidal*;

(ii) *$B|_{\mathcal{U}}$ is non-singular, and there is another prime divisor B' of $\mathcal{U}$ such that $B|_{\mathcal{U}} \cap B' = \{Q\}$ and $(\mathcal{U}, B|_{\mathcal{U}} + B')$ is toroidal*;

(iii) *$B|_{\mathcal{U}}$ is non-singular, and there exist a finite surjective morphism $\tau \colon \mathcal{U}' \to \mathcal{U}$ of degree two and prime divisors B'_1, B'_2 on $\mathcal{U}'$ such that*
- *τ is étale outside Q,*
- *$\tau^*(B|_{\mathcal{U}}) = B'_1 + B'_2$, $B'_1 \cap B'_2 = \tau^{-1}(Q) = \{Q'\}$ for a point Q', and*
- *$(\mathcal{U}', B'_1 + B'_2)$ is toroidal.*

As a consequence, when $P \in \operatorname{Sing} X$, the dual graph of the exceptional divisor on the minimal resolution of X around P is embedded in the graphs in Figure 2 for each case. In the graphs, ◯ stands for the excep-

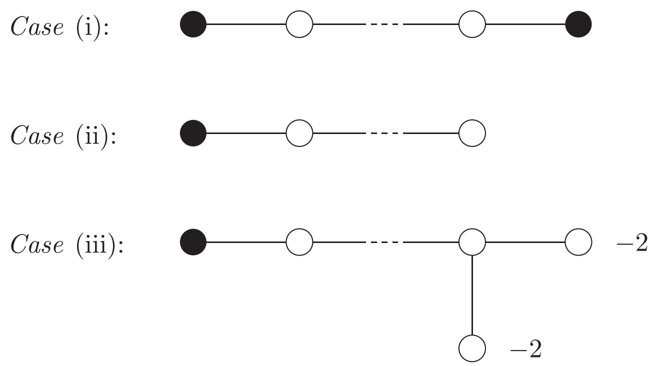

Fig. 2. Dual graphs in Theorem 3.22

tional components, and ● *stands for the proper transforms of irreducible components of* B; *The number* -2 *indicates that the corresponding curve is a* (-2)*-curve.*

Definition 3.23. In the situation of Theorem 3.22, we say that the point P is of type $\mathcal{T}$ (resp. $\mathcal{P}$, resp. $\mathcal{D}$) for (X, B) if the condition (i) (resp. (ii), resp. (iii)) is satisfied.

As a consequence of Theorem 3.22, we have:

Corollary. *Let* X *be a normal surface,* B *a reduced divisor, and* P *a point of* B. *Then, the following conditions are equivalent to each other*:

(i) (X, B) *is toroidal at* P, *and* $P \in \operatorname{Sing} B$;
(ii) (X, B) *is log-canonical at* P, *and* P *is of type* $\mathcal{T}$ *for* (X, B);
(iii) (X, B) *is log-canonical at* P, *and* $P \in \operatorname{Sing} B$.

Remark. For a point $P \in B$, it is of type $\mathcal{P}$ for (X, B) if and only if (X, B) is *purely log terminal* (plt) at P in the sense of [29], [30].

By using Theorem 3.22 and Example 3.2, it is an exercise to prove:

Corollary 3.24. *Let* X, B, *and* P *be as in Theorem* 3.22. *Let* r *be the smallest positive integer such that* $r(K_X + B)$ *is Cartier at* P.

(1) *If* P *is of type* $\mathcal{P}$ *and* $P \in \operatorname{Sing} X$, *then* (X, P) *is a cyclic quotient singularity of order* $r > 1$.

(2) *If P is of type $\mathcal{D}$, then $r = 2$, and (X, P) is either a cyclic quotient singularity with the dual graph of type A_3 or a quotient singularity by a binary dihedral group.*

(3) *Assume that $P \in B \cap \operatorname{Sing} X$. Then, $r = 1$ if and only if $P \in \operatorname{Sing} B$, and it is also equivalent to that P is of type $\mathcal{T}$.*

As a consequence of (1)–(3) above, we have:

Corollary 3.25. *Assume that (X, B) is log canonical along an irreducible component C of B. Then, the following conditions are mutually equivalent*:

- *$K_X + B$ is Cartier along C*;
- *$C \cap \operatorname{Sing} X \subset \operatorname{Sing} B$*;
- *(X, B) is toroidal along C*;
- *there is no singular points of X on C which are of type $\mathcal{P}$ or $\mathcal{D}$ for (X, B).*

Lemma 3.26. *Let X be a normal surface with a unique singular point P and B a reduced divisor containing P. Assume that (X, B) is log-canonical at P. For the minimal resolution $\mu\colon M \to X$ of singularities, let Δ be the μ-exceptional $\mathbb{Q}$-divisor defined by*

$$K_M + B' = \mu^*(K_X + B) - \Delta,$$

where B' is the proper transform of B in M. Then, Δ is effective, and moreover:

- *If P is of type $\mathcal{P}$ for (X, B), then $\Delta B' = 1 - 1/r$ for the smallest positive integer r such that $r(K_X + B)$ is Cartier at P (cf. Corollary 3.24(1)).*
- *If P is of type $\mathcal{D}$ for (X, B), then $\Delta B' = 1$.*

Proof. The μ-exceptional locus is a simple normal crossing divisor $\sum_{i=1}^{N} \Gamma_i$ consisting of non-singular rational curves Γ_i, and the dual graph is the Dynkin diagram A_k or D_{l+2} for some $k \geq 1$ or $l \geq 1$ (cf. Figure 2 of Theorem 3.22). Moreover, B' intersects a unique irreducible component, say Γ_1, and $B'\Gamma_1 = 1$. Let $\Delta = \sum_i \delta_i \Gamma_i$ be the prime decomposition. It is enough to prove that

$$\delta_1 = \begin{cases} 1 - 1/r, & \text{if } P \text{ is of type } \mathcal{P}; \\ 1, & \text{if } P \text{ is of type } \mathcal{D}. \end{cases}$$

Note that $\delta_i \geq 0$ for any i, since $-\Delta$ is μ-nef. By adjunction and by the definition of Δ, we have equalities

$$\text{(III-3)} \quad (K_M + B' + \sum\nolimits_{j=1}^{N} \Gamma_j)\Gamma_i = -2 + B'\Gamma_i + \sharp\{j \mid \Gamma_i \cap \Gamma_j \neq \emptyset\}$$
$$= \mu^*(K_X + B)\Gamma_i + \sum\nolimits_{j=1}^{N} (1-\delta_j)\Gamma_j\Gamma_i = \sum\nolimits_{j=1}^{N} (1-\delta_j)\Gamma_j\Gamma_i$$

for any $1 \leq i \leq N$.

Assume that P is of type $\mathcal{P}$. Then, $N = k$, and by renumbering Γ_i's, we may assume that $\Gamma_i\Gamma_j = 0$ if $|i-j| > 1$, and $\Gamma_i\Gamma_{i+1} = 1$ for any $1 \leq i \leq k-1$. We set $b_i = -\Gamma_i^2$. Then, (X, P) is a cyclic quotient singularity of type (r, q) for integers $0 < q < r$ with $\gcd(r, q) = 1$ determined by $r/q = [b_k, b_{k-1}, \ldots, b_1]$ (cf. Example 3.2). Putting $\varepsilon_i = 1 - \delta_i$, by (III-3), we have equalities

$$\text{(III-4)} \qquad \varepsilon_1 b_1 = \varepsilon_2, \quad \varepsilon_i b_i = \varepsilon_{i-1} + \varepsilon_{i+1}, \quad \varepsilon_k b_k = \varepsilon_{k-1} + 1,$$

where $2 \leq i \leq k-1$. Let m_i for $0 \leq i \leq k+1$ be integers defined inductively by

$$m_0 = 0, \quad m_1 = 1, \quad m_i b_i = m_{i-1} + m_{i+1}$$

for $1 \leq i \leq k$. Then, $\gcd(m_i, m_{i-1}) = 1$ for any $1 \leq i \leq l+1$, and

$$\frac{m_{k+1}}{m_k} = b_k - \frac{m_k}{m_{k-1}} = b_k - \cfrac{1}{b_{k-1} - \cfrac{m_{k-1}}{m_{k-2}}} = \cdots$$
$$= [b_k, b_{k-1}, \ldots, b_1] = r/q.$$

In particular, $m_{k+1} = r$ and $m_k = q$. Moreover, $m_i\varepsilon_1 = \varepsilon_i$ for any $1 \leq i \leq k$, and $m_{k+1}\varepsilon_1 = 1$ by (III-4). Therefore, $\varepsilon_1 = 1/r$ and $\delta_1 = 1 - \varepsilon_1 = 1 - 1/r$.

Assume next that P is of type $\mathcal{D}$. Then, $N = l + 2$. Renumbering Γ_i's, we may assume that Γ_{l+1} and Γ_{l+2} are the end components of self-intersection number -2, and Γ_l is the component intersecting Γ_{l+1} and Γ_{l+2}. Then, by (III-3), we have:

$$(K_M + B' + \sum\nolimits_{j=1}^{l+2} \Gamma_j)\Gamma_i = \begin{cases} 0, & \text{if } i < l; \\ 1, & \text{if } i = l; \\ -1, & \text{if } i > l. \end{cases}$$

On the other hand, we have

$$(1/2)(\Gamma_{l+1} + \Gamma_{l+2})\Gamma_i = \begin{cases} 0, & \text{if } i < l; \\ 1, & \text{if } i = l; \\ -1, & \text{if } i > l. \end{cases}$$

Thus, $\sum(1-\delta_i)\Gamma_i = (1/2)(\Gamma_{l+1}+\Gamma_{l+2})$, and we have $\delta_1 = 1$. Q.E.D.

Definition 3.27. Let (X, B) be a log-canonical pair of a normal surface X and a reduced divisor B. We define $\mathcal{D}(X, B)$ to be the set of points of B which are of type $\mathcal{D}$ for (X, B). We also define $\mathcal{P}(X, B)$ to be the set of points of B which are singular points of X and are of type $\mathcal{P}$ for (X, B). Moreover, for an integer $r > 1$, we set $\mathcal{P}_r(X, B)$ to be the subset of $\mathcal{P}(X, B)$ consisting of points P such that r equals the order of the cyclic quotient singularities (X, P) (cf. Corollary 3.24(1)).

Remark. For (X, B) above, one has

$$\mathcal{P}(X,B) = \bigsqcup\nolimits_{r>1} \mathcal{P}_r(X,B),$$
$$(B \cap \operatorname{Sing} X) \setminus \operatorname{Sing} B = \mathcal{P}(X,B) \sqcup \mathcal{D}(X,B).$$

3.5. On compact boundary curves of log-canonical pairs

For a normal surface X and a reduced divisor D, we shall classify the singularities on a compact irreducible component C of D such that $(K_X + D)C \leq 0$ and (X, D) is log-canonical along C.

Lemma 3.28. *Let X be a normal surface and D a reduced divisor such that (X, D) is log-canonical and $X \setminus D$ is non-singular. Let $\mu\colon M \to X$ be the minimal resolution of singularities and set $D^\flat_M$ to be the union of the proper transform D' of D on M and the μ-exceptional divisors lying over* $\operatorname{Sing} D$. *Then,*

$$K_M + D^\flat_M = \mu^*(K_X + D) - \Delta \tag{III-5}$$

for a μ-exceptional effective $\mathbb{Q}$-divisor Δ lying over $\mathcal{P}(X, D) \cup \mathcal{D}(X, D)$. Moreover, the following equalities hold for any compact irreducible component C of D and its proper transform C' in M:

$$(D^\flat_M - C')C' = \sharp C \cap (D - C) + 2n_C, \tag{III-6}$$

$$\Delta C' = \nu_C(\mathcal{D}) + \sum\nolimits_{r>1} \frac{r-1}{r} \nu_C(\mathcal{P}, r). \tag{III-7}$$

Here, n_C is the number of nodes of $C \setminus (D - C)$ contained in $\operatorname{Sing} X$,

$$\nu_C(\mathcal{D}) := \sharp C \cap \mathcal{D}(X,D), \quad \textit{and} \quad \nu_C(\mathcal{P}, r) := \sharp C \cap \mathcal{P}_r(X,D).$$

Proof. We define Δ by the equality (III-5). First, we shall prove that Δ is effective with $\mu(\operatorname{Supp}\Delta) \subset \mathcal{P}(X, D) \cup \mathcal{D}(X, D)$ and prove (III-7). Let P be a singular point of X. Then, $P \in D$. If $P \in \operatorname{Sing} D$, then (X, D) is toroidal at P by Theorem 3.22, and $\mu^{-1}(P) \cap \operatorname{Supp}\Delta = 0$ by Corollary 3.15. If $P \notin \operatorname{Sing} D$, then P is of type $\mathcal{P}$ or $\mathcal{D}$ for (X, D).

In this case, by Lemma 3.26, Δ is effective on a neighborhood of $\mu^{-1}(P)$, and $\Delta C' = 1 - 1/r$ (resp. $\Delta C' = 1$) if P is of type $\mathcal{P}$ (resp. $\mathcal{D}$), where r equals the order of the cyclic quotient singularity (X, P) in case $P \in \mathcal{P}(X, D)$. Therefore, Δ is effective, $\mu(\operatorname{Supp}\Delta) \subset \mathcal{P}(X, D) \cup \mathcal{D}(X, D)$, and the equality (III-7) holds.

Next, we shall prove (III-6). Let Q be an arbitrary point in $C \cap \operatorname{Sing} D$. Then, (X, D) is toroidal at Q, and it is well known (cf. Corollary 3.15 above) that $\mu^{-1}(Q)$ for the minimal resolution μ is a union of non-singular rational curves whose dual graph is the Dynkin diagram A_k for some k (or a linear chain of rational curves in the sense of Definition 4.1 below) and that $\mu^{-1}(Q)$ intersects C' transversely at end components. Thus, $C' \cap \mu^{-1}(Q)$ consists of one point (resp. two points) if $Q \in C \cap (D - C)$ (resp. Q is a node of $C \setminus (D - C)$). This observation on $\mu^{-1}(Q)$ implies the equality (III-6). Q.E.D.

Proposition 3.29. *Let X be a normal surface and let D be a reduced divisor on X. Let C be a compact irreducible component of D such that (X, D) is log-canonical along C and $(K_X + D)C \leq 0$. Then, one of the following eight cases occurs*:

(A) *C is an elliptic curve and $C \cap (D - C) = C \cap \operatorname{Sing} X = \emptyset$; In this case, $K_X + D$ is Cartier along C and $\mathcal{O}_X(K_X + D)|_C \simeq \mathcal{O}_C$.*

(B) *C is a nodal rational curve with one node, $C \cap (D - C) = \emptyset$, and $C \cap \operatorname{Sing} X \subset \operatorname{Sing} C$; In this case, $K_X + D$ is Cartier along C and $\mathcal{O}_X(K_X + D)|_C \simeq \mathcal{O}_C$.*

(C) *$C \simeq \mathbb{P}^1$, $\sharp C \cap (D - C) = 2$, and $C \cap \operatorname{Sing} X \subset C \cap (D - C)$; In this case, $K_X + D$ is Cartier along C and $\mathcal{O}_X(K_X + D)|_C \simeq \mathcal{O}_C$.*

(D) *$C \simeq \mathbb{P}^1$ and $C \cap (D - C) = \emptyset$; In this case, $-2 \leq (K_X + D)C \leq 0$.*

(E) *$C \simeq \mathbb{P}^1$, $\sharp C \cap (D - C) = 1$, and $C \cap \operatorname{Sing} X \subset C \cap (D - C)$; In this case, $K_X + D$ is Cartier along C and $(K_X + D)C = -1$.*

(F) *$C \simeq \mathbb{P}^1$, $\sharp C \cap (D - C) = 1$, $\sharp C \cap \mathcal{P}(X, D) = 1$, and $C \cap \mathcal{D}(X, D) = \emptyset$; In this case, $r(K_X + D)$ is Cartier along C for the order r of the cyclic quotient singular point in $C \cap \mathcal{P}(X, D)$, and $(K_X + D)C = -1/r$.*

(G) *$C \simeq \mathbb{P}^1$, $\sharp C \cap (D - C) = 1$, $\sharp C \cap \mathcal{P}(X, D) = 2$, and $C \cap \mathcal{D}(X, D) = \emptyset$; In this case, the points in $C \cap \mathcal{P}(X, D)$ are A_1-singularities, $2(K_X + D)$ is Cartier along C, $K_X + D$ is not Cartier at $C \cap \mathcal{P}(X, D)$, and $(K_X + D)C = 0$.*

(H) *$C \simeq \mathbb{P}^1$, $\sharp C \cap (D - C) = 1$, $C \cap \mathcal{P}(X, D) = \emptyset$, and $\sharp C \cap \mathcal{D}(X, D) = 1$; In this case, $2(K_X + D)$ is Cartier along C, $K_X + D$ is not Cartier at $C \cap \mathcal{D}(X, D)$, and $(K_X + D)C = 0$.*

Proof. We may assume that $X \setminus D$ is non-singular. Let $\mu\colon M \to X$, $D^{\flat}_M$, and Δ be as in Lemma 3.28. Then, we have

$$\text{(III-8)} \qquad \begin{aligned} 0 \geq (K_X + D)C &= (K_M + D^{\flat}_M)C' + \Delta C' \\ &= 2p_a(C') - 2 + (D^{\flat}_M - C')C' + \Delta C' \geq 2p_a(C') - 2, \end{aligned}$$

from (III-5) in Lemma 3.28, since Δ is effective. In particular, $p_a(C') \leq 1$. Assume that $p_a(C') = 1$. Then, $C \cap (D - C) = C \cap \operatorname{Sing} X = \emptyset$ by (III-6) in Lemma 3.28. Consequently, $C \simeq C'$ is an elliptic curve or a nodal rational curve, and we have $\mathcal{O}_X(K_X + D)|_C \simeq \omega_C \simeq \mathcal{O}_C$ by adjunction. Thus, we have the case (A) or (B).

Therefore, we may assume that $p_a(C') = 0$, i.e., $C' \simeq \mathbb{P}^1$. Then, by (III-8), we have $(D^{\flat}_M - C')C' + \Delta C' \leq 2$. Assume that $(D^{\flat}_M - C')C' = 2$. Then, $(K_X + D)C = \Delta C' = 0$ by (III-8). In particular, $K_X + D$ is Cartier along C, and $C \cap \operatorname{Sing} X \subset \operatorname{Sing} D$ by (III-7) in Lemma 3.28 and by Corollary 3.25. Here, if C is non-singular, then $C \simeq \mathbb{P}^1$, and $\sharp C \cap (D - C) = (D^{\flat}_M - C')C' = 2$ by (III-6): Thus, the case (C) occurs. Note that in this case, we have $\mathcal{O}_X(K_X + D)|_C \simeq \mathcal{O}_C$ by $(K_X + D)C = 0$. If C is singular, then C is a nodal rational curve with one node and $C \cap (D - C) = \emptyset$ by (III-6); moreover, $C \cap \operatorname{Sing} X \subset \operatorname{Sing} C$ by (III-7), since $\Delta C' = 0$: Thus, the case (B) occurs, where we have $\mathcal{O}_X(K_X + D)|_C \simeq \omega_C \simeq \mathcal{O}_C$ by Remark 3.16.

For the rest, we may assume that $(D^{\flat}_M - C')C' \leq 1$ and $C' \simeq \mathbb{P}^1$. Then, $\sharp C \cap (D - C) \leq 1$ and $C \simeq C' \simeq \mathbb{P}^1$ by (III-6). If $C \cap (D - C) = \emptyset$, then the case (D) occurs, and we have $-2 \leq (K_X + D)C \leq 0$ by (III-8). Thus, we may assume that $\sharp C \cap (D - C) = 1$. Then, $(D^{\flat}_M - C')C' = 1$ by (III-6), and

$$\text{(III-9)} \qquad 0 \geq (K_X + D)C = \Delta C' - 1 \geq -1$$

by (III-8). If $\Delta C' = 0$, then $C \cap \operatorname{Sing} X \subset C \cap (D - C)$ by (III-7) and $(K_X + D)C = -1$: Thus, we have the case (E). The remaining cases are divided into the cases (F)–(H) by (III-7) and (III-9). In fact, if $C \cap \mathcal{D}(X, D) \neq \emptyset$ (resp. $\sharp C \cap \mathcal{P}(X, D) = 1$, resp. $\sharp C \cap \mathcal{P}(X, D) \geq 2$), we have the case (H) (resp. (F), resp. (G)), by (III-7). Therefore, one of the cases (A)–(H) occurs, and we are done. Q.E.D.

§4. Key concepts

In this section, we prepare some concepts playing important roles for proving Theorems 1.3, 1.5, 1.6, 1.7, etc. The notions of linear and cyclic chains of rational curves are introduced in Section 4.1, where, as applications, we can prove some results on log-canonical surfaces (X, D) with reduced divisor D such that $-(K_X + D)$ is nef. There is also proved a result on $\boldsymbol{c}(X, D)$ for a linear (or cyclic) chain D of rational curves. The structure of double-covers étale in codimension one is explained in Section 4.2, and as an application, we obtain a result on the structure of a log-canonical surface (X, D) such that D is a linear chain of rational curves and that $2(K_X + D) \sim 0$. The notion of toroidal blowing up is introduced in Section 4.3, and it is proved that a toroidal blowing up is étale locally a toric birational morphism. We also prove a result on the existence of a toroidal blowing up and a fibration to $\mathbb{P}^1$ for a log-canonical surface (X, D) with $\boldsymbol{c}(X, D) < 2$ and $\mathrm{H}^1(X, \mathcal{O}_X) = 0$. The notion of tangential blowing up is introduced in Section 4.4 and a few properties are mentioned.

4.1. Linear and cyclic chains of rational curves

Definition 4.1. Let D be a compact non-zero connected reduced divisor on a normal surface X. If $D = \sum_{i=1}^{k} C_i$ for irreducible components $C_1, \ldots, C_k$ satisfying the following conditions, then D is called a *linear chain of rational curves*:

(i) Each irreducible component C_i is a non-singular rational curve.
(ii) If $k \geq 2$, then $C_i \cap C_j = \emptyset$ for $|i - j| > 1$ and $\sharp C_i \cap C_{i+1} = 1$ for $1 \leq i \leq k - 1$.

In other words, D is a union of non-singular rational curves whose dual graph is the Dynkin diagram A_k for some $k \geq 1$. The components C_1 and C_k above are called the *end components* of D. The union of non-end component is denoted by $D^\natural$, i.e., $D^\natural = \sum_{1<i<k} C_i$.

Remark 4.2. The linear chain D of rational curves above has the following properties when (X, D) is log-canonical along D:

- D is Gorenstein and $p_a(D) = 0$;
- $\mathrm{Pic}(D) \simeq \bigoplus_{i=1}^{k} \mathrm{Pic}(C_i) \simeq \mathbb{Z}^{\oplus k}$;
- If D is reducible, i.e., if $k > 1$, then $\omega_D|_C \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$ for any end component C, and $\omega_D|_{D^\natural} \simeq \mathcal{O}_{D^\natural}$, where ω_D stands for the dualizing sheaf.

In fact, by Theorem 3.22(i), (X, D) is toroidal at any point of D, and hence, D is locally isomorphic to a normal crossing divisor on a non-singular surface (cf. Remark 3.3). Therefore, we have the properties above by the configuration of the irreducible components C_i of D, and by [4, Th. (1.7)].

Definition 4.3. A compact non-zero connected reduced divisor D on a normal surface X is called a *cyclic chain of rational curves* if it satisfies the following conditions:

(i) Every irreducible component of D is a rational curve.
(ii) If D is irreducible, then D is a nodal rational curve with $p_a(D) = 1$.
(iii) If D is reducible, then any irreducible component C of D is non-singular and $\sharp(D - C) \cap C = 2$.

Remark 4.4. Let D be a cyclic chain of rational curves and let l be the number of irreducible components of D. By Theorem 3.22(i) and by the configuration of the irreducible components, we have the following properties when (X, D) is log-canonical along D:

- The divisor D is Gorenstein, $p_a(D) = 1$, and $\omega_D \simeq \mathcal{O}_D$.
- If $l = 2$, then $D = C_1 + C_2$ for two non-singular rational curves C_1 and C_2 intersecting with each other at two distinct points.
- If $l \geq 3$, then the dual graph of D forms a cycle, i.e., we can write
$$D = \sum\nolimits_{i \in \mathbb{Z}/l\mathbb{Z}} C_i$$
for non-singular rational curves C_i such that $C_i \cap C_j = \emptyset$ for $j \notin \{i-1, i, i+1\}$ and $\sharp C_i \cap C_j = 1$ for $j \in \{i-1, i+1\}$.
- The number l coincides with the topological Euler number $e(D)$.

Lemma 4.5. *Let X be a normal surface and let D be a compact non-zero connected reduced divisor on X such that*

- *(X, D) is log-canonical along D, and*
- *$-(K_X + D)$ is nef* on D (*cf. Definition* 2.14(ii)), *i.e., $(K_X + D)C \leq 0$ for any irreducible component C of D.*

Then, D is an elliptic curve, a linear chain of rational curves, or a cyclic chain of rational curves, and the following hold:

(1) *If D is an elliptic curve, then $D \cap \operatorname{Sing} X = \emptyset$ and $\mathcal{O}_X(K_X + D)|_D \simeq \mathcal{O}_D$.*
(2) *If D is a cyclic chain of rational curves, then $K_X + D$ is Cartier along D and $\mathcal{O}_X(K_X + D)|_D \simeq \mathcal{O}_D$.*
(3) *Assume that D is a reducible linear chain of rational curves. Then, $K_X + D$ is Cartier along $D^\natural \cup \operatorname{Sing} D$, and*
$$\mathcal{O}_X(K_X + D)|_{D^\natural} \simeq \mathcal{O}_{D^\natural},$$
where $D^\natural$ stands for the union of non-end components of D.

Proof. We shall consider the eight cases (A)–(H) of Proposition 3.29 for each irreducible component of D. Assume that an irreducible component C of D is not rational. Then, C is in the case (A). Here, $D = C$, since D is connected. Hence, D is an elliptic curve, $D \cap \operatorname{Sing} X = \emptyset$, and

$\mathcal{O}_X(K_X + D)|_D \simeq \mathcal{O}_D$. Assume next that an irreducible component C of D is singular. Then, C is in the case (B). Here, $D = C$, since D is connected. Hence, D is a nodal rational curve with one node, i.e., an irreducible cyclic chain of rational curve, $K_X + D$ is Cartier along D, and $\mathcal{O}_X(K_X + D)|_D \simeq \mathcal{O}_D$.

Therefore, by Proposition 3.29, we may assume that every irreducible component C is isomorphic to $\mathbb{P}^1$, and $\sharp C \cap (D - C) \leq 2$ for any C. Then, D is a linear chain or a cyclic chain of rational curves.

Assume that D is a reducible cyclic chain of rational curves. Then, every irreducible component C of D belongs to the case (C). In particular, $K_X + D$ is Cartier along D. Moreover, $\mathcal{O}_X(K_X + D)|_D \simeq \omega_D \simeq \mathcal{O}_D$ by Remarks 3.16 and 4.4.

Assume finally that D is a reducible linear chain of rational curves. We know that $K_X + D$ is Cartier along $\operatorname{Sing} D$ (cf. Corollary 3.24(3)). Hence, we may assume that $D^\natural \neq 0$, i.e., $\boldsymbol{n}(D) \geq 3$. Then, every irreducible component C of $D^\natural$ belongs to the case (C), and consequently, $K_X + D$ is Cartier along $D^\natural$, and $K_X + D$ is numerically trivial *on* $D^\natural$ (cf. Definition 2.14(ii)). Since $D^\natural$ is also a linear chain of rational curves, $\operatorname{Pic}(D^\natural)$ is the direct sum of $\operatorname{Pic}(C)$ for all $C \subset D^\natural$ by Remark 4.2, and it implies that $\mathcal{O}_X(K_X + D)|_{D^\natural} \simeq \mathcal{O}_{D^\natural}$. Thus, we are done. Q.E.D.

Corollary 4.6. *Let D be a compact non-zero connected reduced divisor on a normal surface X such that (X, D) is log-canonical along D and $K_X + D$ is Cartier along D. Then, the following three conditions are mutually equivalent*:

(i) $\mathcal{O}_X(K_X + D)|_D \simeq \mathcal{O}_D$;

(ii) $(K_X + D)C = 0$ *for any irreducible component C of D*;

(iii) *D is either an elliptic curve or a cyclic chain of rational curves.*

Proof. If (ii) holds, then every irreducible component C of D satisfies one of the conditions (A), (B), (C) of Proposition 3.29, since $K_X + D$ is Cartier along D. Thus, we have (ii) $\Rightarrow$ (iii) as in the proof of Lemma 4.5. The implication (i) $\Rightarrow$ (ii) is trivial, and the implication (iii) $\Rightarrow$ (i) follows from Remarks 3.16 and 4.4. Q.E.D.

Lemma 4.7. *Let X be a normal Moishezon surface with a reduced divisor D such that*

- *(X, D) is log-canonical along D,*
- *$-(K_X + D)$ is nef on D,*
- *D is connected and big, and*
- *D is not an elliptic curve.*

Then, D is a linear chain or a cyclic chain of rational curves, and X is a projective rational surface with only rational singularities. In particular,

$\hat{\boldsymbol{\rho}}(X) = \boldsymbol{\rho}(X)$. *If D is a cyclic chain of rational curves, then there is an effective divisor G such that $G \sim K_X + D$, $D \cap \operatorname{Supp} G = \emptyset$, and that the intersection matrix of G is negative definite if $G \neq 0$.*

Proof. The divisor D is a linear chain or a cyclic chain of rational curves by Lemma 4.5. We have $\mathrm{H}^1(M, \mathcal{O}_M) = 0$ for a non-singular projective surface M birational to X by Lemma 2.32, since the big divisor D consists of rational curves and X has only rational singularities on D by Theorem 3.22. In particular, $\mathrm{H}^1(X, \mathcal{O}_X) = \mathrm{H}^1(X, \mathcal{O}_X(K_X)) = 0$. On the other hand, K_X is not pseudo-effective. In fact, for the *Zariski-decomposition* $D = P + N$ of D (cf. [57, Th. 7.7], [11, Th. (1.12)], [47, Cor. (7.5)]), the positive part P is nef and big, and $K_X P < (K_X + D)P \leq 0$ by $\operatorname{Supp} P \subset D$. Thus, X is a rational surface with only rational singularities by Lemma 2.32. Then, $\hat{\boldsymbol{\rho}}(X) = \boldsymbol{\rho}(X)$ by Remark 2.8 and Lemma 2.31(2).

Assume that D is a cyclic chain of rational curves. Then, we have an exact sequence

$$0 \to \mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X + D) \to \mathcal{O}_X(K_X + D)|_D \simeq \mathcal{O}_D \to 0$$

by Lemma 4.5 (cf. Remark 3.16). Since $\mathrm{H}^1(X, \mathcal{O}_X(K_X)) = 0$, we can find an effective divisor G such that $G \sim K_X + D$ and $D \cap \operatorname{Supp} G = \emptyset$. Here, G is negative definite by the Hodge index theorem, since $GP = 0$. Thus, we are done. Q.E.D.

Remark. In Lemma 4.7, if $G = 0$, then (X, D) is log-canonical, since $X \setminus D$ has only rational Gorenstein singularities by $K_X + D \sim 0$.

Proposition 4.8. *Let X be a normal Moishezon surface and D a reduced divisor on X.*

(1) *If $\boldsymbol{c}(X, D) \leq 1$ and if D is connected, then D is big.*
(2) *If D is a linear chain of rational curves, then $\boldsymbol{c}(X, D) \geq 1$.*
(3) *If D is a cyclic chain of rational curves, then $\boldsymbol{c}(X, D) \geq 0$.*
(4) *If D is a cyclic chain of rational curves with $\boldsymbol{c}(X, D) = 0$ and if $-(K_X + D)$ is nef, then $\boldsymbol{\delta}(X, D) = 0$.*

Proof. We may assume that $\boldsymbol{c}(X, D) \leq 1$ for the proof. By contracting negative components of D, we have a birational morphism $g\colon X \to \overline{X}$ to another Moishezon surface $\overline{X}$ such that

- the g-exceptional locus is contained in D, and
- every irreducible component of $\overline{D} := g_*(D)$ is nef.

Then, $\boldsymbol{\delta}(X, D) = \boldsymbol{\delta}(\overline{X}, \overline{D})$ and $\boldsymbol{c}(\overline{X}, \overline{D}) = \boldsymbol{c}(X, D)$ by Lemma 2.27. In particular, $\overline{D}$ is a reducible non-zero divisor, since we have $\boldsymbol{n}(\overline{D}) \geq 2$ by

$$\boldsymbol{n}(\overline{D}) = \boldsymbol{r}(\overline{D}) + 2 - \boldsymbol{c}(\overline{X}, \overline{D}) \geq \boldsymbol{r}(\overline{D}) + 1.$$

If D is connected, then so is $\overline{D}$, and now $\overline{D}^2 > 0$ since it is reducible. Therefore, $D = g^{-1}(\overline{D})$ is also big. This proves (1). If D is a linear (resp. cyclic) chain of rational curves, then so is $\overline{D}$. If $-(K_X + D)$ is nef, then $-(K_{\overline{X}} + \overline{D}) = g_*(-(K_X + D))$ is also nef. Hence, in order to prove the remaining assertions, by replacing (X, D) with $(\overline{X}, \overline{D})$, we may assume that every irreducible component of D is nef.

Assume that D is a linear chain of rational curves. Let E and E' be the end components of D. If $\boldsymbol{n}(D) \geq 4$, then $E \cap E' = \emptyset$ and there is an irreducible component C of D such that $C \cap E \neq \emptyset$ and $C \cap E' = \emptyset$. Then, $C + E$ is nef and big, and $(C + D)E' = 0$. Hence, $E'^2 < 0$ by the Hodge index theorem, and this contradicts that E' is nef. Therefore, $\boldsymbol{n}(D) \leq 3$. As a consequence, we have $\boldsymbol{c}(X, D) \geq \boldsymbol{r}(D) - 1$. If $\boldsymbol{c}(X, D) \leq 0$, then $\boldsymbol{r}(D) = 1$ and $\boldsymbol{n}(D) = 3$. However, if $\boldsymbol{n}(D) = 3$, then $EC > 0$, $E'C > 0$, and $EE' = 0$ for the other irreducible component C: This contradicts: $\boldsymbol{r}(D) = 1$. Therefore, $\boldsymbol{c}(X, D) \geq 1$, and we have proved (2).

Assume next that D is a cyclic chain of rational curves. If $\boldsymbol{n}(D) \geq 5$, then we can find three irreducible components C_0, C_2, C_3 of D such that $C_0 \cap (C_2 \cup C_3) = \emptyset$ and $C_2 \cap C_3 \neq \emptyset$. Then, $C_2 + C_3$ is nef and big, and we have $C_0^2 < 0$ by the Hodge index theorem applied to $C_0(C_2 + C_3) = 0$. This contradicts that C_0 is nef. Thus, $\boldsymbol{n}(D) \leq 4$. As a consequence, we have $\boldsymbol{c}(X, D) \geq \boldsymbol{r}(D) - 2$. In particular, $\boldsymbol{c}(X, D) \geq 0$ when $\boldsymbol{r}(D) \geq 2$. If $\boldsymbol{r}(D) = 1$, then $\boldsymbol{n}(D) \leq 3$, since any two irreducible components of D intersect with each other. Thus, $\boldsymbol{c}(X, D) = 3 - \boldsymbol{n}(D) \geq 0$ in this case. Therefore, $\boldsymbol{c}(X, D) \geq 0$ holds in any case, and we have proved (3).

For the proof of (4), we assume that $\boldsymbol{c}(X, D) = 0$ and that $-(K_X + D)$ is nef. Note that K_X is not pseudo-effective by $K_X D < (K_X + D)D \leq 0$, since the nef divisor D is big by (1). Therefore, X is projective by Lemma 2.31(1). Since $4 \geq \boldsymbol{n}(D) = \boldsymbol{r}(D) + 2 \geq 3$, one of the following two cases occurs:

(I) $\boldsymbol{r}(D) = 1$ and $\boldsymbol{n}(D) = 3$.
(II) $\boldsymbol{r}(D) = 2$ and $\boldsymbol{n}(D) = 4$.

There is an extremal ray R on $\overline{\mathrm{NE}}(X)$ such that $D\mathsf{R} > 0$ and $K_X\mathsf{R} < 0$ by Corollary 2.20(2), since $(-D) - K_X$ is nef and $-D$ is not nef. We consider the contraction morphism $\mathrm{cont}_{\mathsf{R}}$ associated with R (cf. Theorem 2.21). If $\mathrm{cont}_{\mathsf{R}}$ is the trivial morphism to a point, then $\hat{\boldsymbol{\rho}}(X) = 1$, and it implies that $\hat{\boldsymbol{\rho}}(X) = \boldsymbol{r}(D) = 1$ and $\boldsymbol{\delta}(X, D) = \boldsymbol{c}(X, D) = 0$.

Assume that $\mathrm{cont}_{\mathsf{R}}$ is a fibration $\pi\colon X \to T$ to a non-singular projective curve T. Then, $(K_X + D)F \leq 0$ and $DF > 0$ for a general fiber F of π. This implies that $\hat{\boldsymbol{\rho}}(X) = 2$, $F \simeq \mathbb{P}^1$, and $1 \leq DF \leq 2$. In the case (I), since $\boldsymbol{r}(D) = 1$, every irreducible component of D dominates

T, and thus $\boldsymbol{n}(D) \le DF = 2$. This is a contradiction. In the case (II), $\boldsymbol{r}(D) = \hat{\boldsymbol{\rho}}(X) = 2$, and hence $\boldsymbol{\delta}(X, D) = \boldsymbol{c}(X, D) = 0$.

Finally, we shall derive a contradiction assuming that $\mathrm{cont}_{\mathsf{R}}$ is a birational morphism $f\colon X \to X'$. This f is just the contraction morphism of a negative curve Γ on X such that $\mathsf{R} = \mathbb{R}_{\ge 0}\,\mathrm{cl}(\Gamma)$. Here, $D\Gamma > 0$ and $(K_X + D)\Gamma < 0$. Since $\Gamma \not\subset D$, we have $\sharp D \cap \Gamma = 1$ by Lemma 2.18. Let C_0 be an irreducible component of D such that $D \cap \Gamma = C_0 \cap \Gamma$. Since $\boldsymbol{n}(D) \ge 3$, we have an irreducible component C_1 such that $C_1 \cap \Gamma = \emptyset$; this implies that $\boldsymbol{r}(D) \ge 3$, and hence, the case (I) does not occur. In the case (II), since $\boldsymbol{n}(D) = 4$, there is an irreducible component C_2 of D such that $C_0 \cap C_2 = \emptyset$. Then, $C_2 \approx rC_0$ for some $r > 0$ by the Hodge index theorem, since C_0 and C_2 are nef with $C_0C_2 = 0$. Then, $C_2\Gamma > 0$, but this contradicts $D \cap \Gamma = C_0 \cap \Gamma$. Thus, we are done. Q.E.D.

4.2. Double-covers étale in codimension one

We recall some basic properties on double-covers étale in codimension one, and apply them to certain log-canonical pairs (X, D) of dimension two such that $2(K_X + D) \sim 0$.

Definition 4.9. Let X be a scheme with a quasi-coherent sheaf $\mathcal{L}$. For a homomorphism $\sigma\colon \mathcal{L}^{\otimes 2} \to \mathcal{O}_X$ which factors through the symmetric tensor product $\mathcal{S}^2(\mathcal{L})$, let $\mathcal{R} = \mathcal{R}(\mathcal{L}, \sigma)$ be the $\mathcal{O}_X$-algebra with $\mathcal{O}_X \oplus \mathcal{L}$ as an underlying $\mathcal{O}_X$-module and with the multiplication map $\mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{R} \to \mathcal{R}$ given by

$$(a, x)(b, y) = (ab + \sigma(x \otimes y), ay + bx)$$

for local sections a and b of $\mathcal{O}_X$ and local sections x and y of $\mathcal{L}$. We define $V = V(\mathcal{L}, \sigma)$ to be the scheme $\mathbf{Spec}_X\, \mathcal{R}(\mathcal{L}, \sigma)$ affine over X, and set $\tau\colon V \to X$ to be the structure morphism. We denote by ι the automorphism of V over X defined by $(a, x) \mapsto (a, -x)$.

It is an exercise to prove the following:

Lemma 4.10. *In the situation above, let $\eta\colon \tau^*\mathcal{L} \to \mathcal{O}_V$ be the homomorphism corresponding to the inclusion $\mathcal{L} \subset \mathcal{O}_X \oplus \mathcal{L} \simeq \tau_*\mathcal{O}_V$ by the adjoint property of τ^* and τ_*. Then, $\eta^{\otimes 2} = \tau^*(\sigma)$. Moreover, for a given morphism $f\colon Y \to X$ of schemes, there is a functorial bijection*

$$\mathrm{Hom}_X(Y, V) \to \{\zeta \in \mathrm{Hom}_{\mathcal{O}_Y}(f^*\mathcal{L}, \mathcal{O}_Y) \mid f^*(\sigma) = \zeta^{\otimes 2}\}$$

which associates the homomorphism $g^*(\eta)\colon f^*\mathcal{L} \simeq g^*(\tau^*\mathcal{L}) \to \mathcal{O}_Y$ with a morphism $g\colon Y \to V$ over X.

In the functorial bijection above, the automorphism $\iota\colon V \to V$ corresponds to $-\eta$. Thus, we have:

Corollary 4.11. *Assume that* 2 *is a regular element of* $\mathrm{H}^0(X, \mathcal{O}_X)$*. Then,* ι *is an involution, i.e., an automorphism of order two. Let* $\mathcal{L}'$ *be another quasi-coherent sheaf and* $\sigma' \colon \mathcal{L}'^{\otimes 2} \to \mathcal{O}_X$ *a homomorphism factoring through* $\mathcal{S}^2(\mathcal{L}')$*. Then, giving a morphism* $V(\mathcal{L}, \sigma) \to V(\mathcal{L}', \sigma')$ *over* X *which is equivariant under the involutions* ι*, is equivalent to giving a homomorphism* $u \colon \mathcal{L}' \to \mathcal{L}$ *such that* $\sigma' = \sigma \circ u^{\otimes 2}$*. In particular,* $V(\mathcal{L}, \sigma) \simeq V(\mathcal{L}, \sigma \circ \varepsilon^{\otimes 2})$ *for any automorphism* $\varepsilon \colon \mathcal{L} \to \mathcal{L}$*.*

Corollary 4.12. *In the situation of Lemma* 4.10*, for a given quasi-coherent sheaf* $\mathcal{M}$ *and a given homomorphism* $\theta \colon \tau^*\mathcal{M} \to \mathcal{O}_V$*, there exist homomorphisms* $\varphi \colon \mathcal{M} \to \mathcal{O}_X$ *and* $\psi \colon \mathcal{M} \to \mathcal{L}$ *such that*

$$\theta = \tau^*(\varphi) + \eta \circ \tau^*(\psi).$$

Proof. Let $\tilde{\theta} \colon \mathcal{M} \to \tau_*\mathcal{O}_V \simeq \mathcal{O}_X \oplus \mathcal{L}$ be the homomorphism corresponding to θ by the adjoint property of τ^* and τ_*. Let $\varphi \colon \mathcal{M} \to \mathcal{O}_X$ (resp. $\psi \colon \mathcal{M} \to \mathcal{L}$) be the composition of $\tilde{\theta}$ with the first (resp. second) projection. Then, φ and ψ satisfy the equality above. In fact, by the adjoint property of τ^* and τ_*, the homomorphism $\tau^*(\varphi)$ (resp. $\eta \circ \tau^*(\psi)$) corresponds to the composition of φ (resp. ψ) with the natural inclusion $\mathcal{O}_X \subset \mathcal{O}_X \oplus \mathcal{L}$ (resp. $\mathcal{L} \subset \mathcal{O}_X \oplus \mathcal{L}$). Q.E.D.

Lemma 4.13. *In the situation of Definition* 4.9*, assume that* $\mathcal{L}$ *is an invertible sheaf and that any residual characteristic of* X *is not two. Then, for a homomorphism* $\sigma \colon \mathcal{L}^{\otimes 2} \to \mathcal{O}_X$*, it is an isomorphism if and only if* $V(\mathcal{L}, \sigma)$ *is étale over* X*.*

Proof. Since $\mathcal{O}_X \oplus \mathcal{L}$ is locally free of rank two, $\tau \colon V \to X$ is a flat finite morphism of degree two. By base change, we may assume that $X = \operatorname{Spec} A$ for a local ring A. Then, $\mathcal{L} \simeq \mathcal{O}_X$, and σ is considered as an element of A. Thus, $V \simeq \operatorname{Spec} A[\mathtt{x}]/(\mathtt{x}^2 - \sigma)$. The A-algebra $A[\mathtt{x}]/(\mathtt{x}^2 - \sigma)$ is étale over A if and only if σ is a unit element, since 2 is invertible in A. Q.E.D.

In what follows in Section 4.2, we assume X to be a normal variety over $\mathbb{C}$, i.e., a normal integral separated *scheme* of finite type over $\mathbb{C}$ (It is possible to treat the case of algebraic spaces, but it is enough to consider only the case of schemes for our purpose).

Definition. By a *double-cover* of a normal variety X, we mean a finite surjective morphism $\tau \colon V \to X$ of degree two from a normal variety V.

Remark. By the purity of branch locus, the double-cover τ is *étale in codimension one* if and only if V is étale over the non-singular locus X_{reg} of X.

For a normal variety X and for a coherent torsion-free sheaf $\mathcal{L}$ of rank one, any homomorphism $\sigma\colon \mathcal{L}^{\otimes 2} \to \mathcal{O}_X$ factors through $\mathcal{S}^2(\mathcal{L})$, since the image of σ is zero or torsion-free of rank one. Thus, we can consider $V(\mathcal{L}, \sigma)$ of Definition 4.9 for such $\mathcal{L}$ and σ. Lemma 4.14 below gives a criterion for $V(\mathcal{L}, \sigma)$ to be a double-cover of X étale in codimension one.

Lemma 4.14. *Let X be a normal variety* (*which is a scheme*) *and let* $(\mathcal{L}, \sigma)$ *be a pair of a reflexive sheaf $\mathcal{L}$ of rank one and a homomorphism $\sigma\colon \mathcal{L}^{\otimes 2} \to \mathcal{O}_X$. We set $V = V(\mathcal{L}, \sigma)$ and consider the following two conditions for* $(\mathcal{L}, \sigma)$:

(i) *There is no homomorphism $u\colon \mathcal{L} \to \mathcal{O}_X$ such that $\sigma = u^{\otimes 2}$.*

(ii) *The homomorphism σ induces an isomorphism $\mathcal{L}^{[2]} \simeq \mathcal{O}_X$ from the double-dual $\mathcal{L}^{[2]} = (\mathcal{L}^{\otimes 2})^{\vee\vee}$. In other words, σ is regarded as a nowhere vanishing section of $\mathcal{L}^{[-2]}$.*

If $(\mathcal{L}, \sigma)$ *satisfies* (ii), *then V is normal and $V \to X$ is a finite morphism of degree two étale in codimension one. If* $(\mathcal{L}, \sigma)$ *satisfies* (i) *in addition, then V is irreducible, and hence V is a double-cover étale in codimension one. Conversely, for any double-cover $W \to X$ étale in codimension one, there exists a pair* $(\mathcal{L}, \sigma)$ *above satisfying* (i) *and* (ii) *such that $W \simeq V(\mathcal{L}, \sigma)$ over X.*

The proof of Lemma 4.14 is well known at least when $\mathcal{L}$ is invertible, by Lemma 4.13 above, and the reduction to the case of invertible sheaf is done by using a property that any reflexive sheaf satisfies Serre's S_2-condition (cf. [15, Prop. 1.6]). We omit the proof.

Remark 4.15. In the situation of Lemma 4.14, assume that $(\mathcal{L}, \sigma)$ satisfies (ii). Then, we have the following for $V = V(\mathcal{L}, \sigma)$ and for the structure morphism $\tau\colon V \to X$.

(1) The homomorphism η of Lemma 4.10 induces an isomorphism $(\tau^*\mathcal{L})^{\vee\vee} \simeq \mathcal{O}_V$. In fact, η is an isomorphism over an open subset U of V on which $\tau^*\mathcal{L}$ is invertible, by the equality $\eta^{\otimes 2} = \tau^*(\sigma)$ in Lemma 4.10: Thus, we have an isomorphism

$$(\tau^*\mathcal{L})^{\vee\vee} \simeq j_*((\tau^*\mathcal{L})|_U) \xrightarrow{j_*(\eta|_U)} j_*\mathcal{O}_U \simeq \mathcal{O}_V,$$

for the open immersion $j\colon U \subset V$, since a reflexive sheaf satisfies Serre's condition S_2 and $\operatorname{codim}(V \setminus U, V) \geq 2$.

(2) For any homomorphism $\sigma'\colon \mathcal{L}^{\otimes 2} \to \mathcal{O}_X$ satisfying (ii), there exists a finite étale surjective morphism $X^{\sim} \to X$ such that

$$V(\mathcal{L}, \sigma) \times_X X^{\sim} \simeq V(\mathcal{L}, \sigma') \times_X X^{\sim}$$

over $X^{\sim}$. In fact, $\sigma' = u\sigma$ for a unit element u of $\mathrm{H}^0(X, \mathcal{O}_X)$, and hence, $X^{\sim} = \mathbf{Spec}_X\, \mathcal{O}_X[\mathtt{x}]/(\mathtt{x}^2 - u)$ satisfies the required condition by Corollary 4.11. As a consequence, $V(\mathcal{L}, \sigma)$ and $V(\mathcal{L}, \sigma')$ have the same singularities.

(3) We have an isomorphism $V \simeq V(\mathcal{L}, \sigma')$ for any homomorphism $\sigma' \colon \mathcal{L}^{\otimes 2} \to \mathcal{O}_X$ satisfying (ii). In fact, $\sigma' = c\sigma$ for a non-zero constant c, and thus $\sigma' = \varepsilon^2\sigma$ for a square-root ε of c (cf. Corollary 4.11).

(4) The variety V is étale over a point P of X if and only if $\mathcal{L}$ is invertible at P (cf. Lemma 4.13).

Lemma 4.16. *Let X be a normal variety (which is a scheme) and let $\tau \colon V = V(\mathcal{L}, \sigma) \to X$ be a double-cover étale in codimension one associated with a reflexive sheaf $\mathcal{L}$ of rank one and a homomorphism $\sigma \colon \mathcal{L}^{\otimes 2} \to \mathcal{O}_X$ inducing an isomorphism $(\mathcal{L}^{\otimes 2})^{\vee\vee} \simeq \mathcal{O}_X$. Then, for any reflexive sheaf $\mathcal{M}$ of rank one, the double-dual $(\tau^*\mathcal{M})^{\vee\vee}$ is an invertible sheaf if and only if, for any point P, either $\mathcal{M}$ or $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{M})$ is invertible at P.*

Proof. It suffices to prove the 'only if' part. By replacing X with an open subset, we may assume that there is a homomorphism $\theta \colon \tau^*\mathcal{M} \to \mathcal{O}_V$ which induces an isomorphism $(\tau^*\mathcal{M})^{\vee\vee} \simeq \mathcal{O}_V$. Let $\varphi \colon \mathcal{M} \to \mathcal{O}_X$ and $\psi \colon \mathcal{M} \to \mathcal{L}$ be the homomorphisms in Corollary 4.12 for θ. Note that the homomorphism $(\varphi, \psi) \colon \mathcal{M} \to \mathcal{O}_X \oplus \mathcal{L} = \tau_*\mathcal{O}_V$ defined by φ and ψ corresponds to θ by the adjoint property of τ^* and τ_*. It is enough to prove that, for any point $P \in X$, either φ or ψ is an isomorphism at P.

Since θ is an isomorphism, the homomorphism

$$\varphi^{\otimes 2} - \sigma \circ \psi^{\otimes 2} \colon \mathcal{M}^{\otimes 2} \to \mathcal{O}_X$$

induces an isomorphism $(\mathcal{M}^{\otimes 2})^{\vee\vee} \simeq \mathcal{O}_X$. In fact, for the involution ι of V, the tensor product $\theta \otimes \iota^*(\theta) \colon \tau^*\mathcal{M}^{\otimes 2} \to \mathcal{O}_V$ is identical to the pullback of $\varphi^{\otimes 2} - \sigma \circ \psi^{\otimes 2}$ by τ. Then, we can construct a homomorphism $k \colon \mathcal{O}_X \oplus \mathcal{L} \to \mathcal{M}$ such that $k \circ (\varphi, \psi) = \mathrm{id}_{\mathcal{M}}$. In particular, $\mathcal{M}$ is a direct summand of $\mathcal{O}_X \oplus \mathcal{L}$. The splitting k is constructed as follows: For the homomorphism $\psi_{\mathcal{L}} \colon \mathrm{id}_{\mathcal{L}} \otimes \psi \colon \mathcal{L} \otimes \mathcal{M} \to \mathcal{L}^{\otimes 2}$, we define a homomorphism $\kappa \colon \mathcal{M} \oplus (\mathcal{L} \otimes \mathcal{M}) \to \mathcal{O}_X$ by

$$\kappa(x, y) = \varphi(x) - \sigma(\psi_{\mathcal{L}}(y))$$

for local sections x and y of $\mathcal{M}$ and $\mathcal{L} \otimes \mathcal{M}$, respectively. Then, the composition

$$\mathcal{M}^{\otimes 2} \xrightarrow{(\varphi,\psi)\otimes \mathrm{id}_{\mathcal{M}}} (\mathcal{O}_X \oplus \mathcal{L}) \otimes \mathcal{M} \simeq \mathcal{M} \oplus (\mathcal{L} \otimes \mathcal{M}) \xrightarrow{\kappa} \mathcal{O}_X$$

is nothing but the homomorphism $\varphi^{\otimes 2} - \sigma \circ \psi^{\otimes 2}$. Thus, we have the splitting k from κ by taking tensor product with $\mathcal{M}^{-1}$ and by taking double-dual.

Let us consider the fibers

$$\varphi(P) \colon \mathcal{M} \otimes \mathbb{C}(P) \xrightarrow{\varphi \otimes \mathbb{C}(P)} \mathbb{C}(P) \quad \text{and}$$
$$\psi(P) \colon \mathcal{M} \otimes \mathbb{C}(P) \xrightarrow{\psi \otimes \mathbb{C}(P)} \mathcal{L} \otimes \mathbb{C}(P)$$

of φ and ψ, respectively, over a point $P \in X$, where $\mathbb{C}(P)$ denotes the residue field. If $\varphi(P) \neq 0$, then φ is an isomorphism at P. Hence, we may assume that $\varphi(P) = 0$. Then, $\mathcal{M} \otimes \mathbb{C}(P)$ is a direct summand of $\mathcal{L} \otimes \mathbb{C}(P)$. In particular, k induces an isomorphism $\mathcal{L} \to \mathcal{M}$ at P, and hence, ψ is an isomorphism at P. Thus, we are done. Q.E.D.

We apply the results above on double-covers étale in codimension one to the study of two-dimensional log-canonical pairs. The following is related to the notion of type $\mathcal{D}$ defined in Definition 3.23:

Lemma 4.17. *Let X be a normal surface (which is a scheme), D a reduced divisor, and P a point of D such that (X, D) is log-canonical at P and $K_X + D$ is not Cartier at P. Let $\tau \colon V \to X$ be a double-cover étale in codimension one such that $K_V + D_V$ is Cartier along $\tau^{-1}(P)$ for $D_V = \tau^{-1}(D)$. Then,*

- *P is a non-singular point of D, more precisely, $P \in \mathcal{P}_2(X, D) \cup \mathcal{D}(X, D)$ (cf. Definition 3.27),*
- *$\tau^{-1}(P)$ consists of one point Q, and*
- *(V, D_V) is log-canonical at Q.*

Here, if $P \in \mathcal{P}_2(X, D)$, then V and D_V are non-singular at Q, and if $P \in \mathcal{D}(X, D)$, then Q is a node of D_V, i.e., Q is of type $\mathcal{T}$ for (V, D_V). Moreover, for any isomorphism $\sigma \colon \mathcal{O}_X(2(K_X + D)) \simeq \mathcal{O}_X$, there is an étale neighborhood $\mathcal{U} \to X$ of (X, P) such that

$$V \times_X \mathcal{U} \simeq V(\mathcal{O}_X(K_X + D), \sigma) \times_X \mathcal{U}.$$

Proof. By Corollary 3.24(3), P is of type $\mathcal{P}$ or $\mathcal{D}$ for (X, D). In particular, D is non-singular at P. Since τ is étale in codimension one, we have $K_V + D_V = \tau^*(K_X + D)$, and (V, D_V) is log-canonical along $\tau^{-1}(P)$ by Corollary 3.20. Since $K_X + D$ is not Cartier at P and $K_V + D_V$ is Cartier along $\tau^{-1}(P)$, the double-cover τ is not étale along $\tau^{-1}(P)$, and hence, $\tau^{-1}(P)$ consists of one point, say Q.

By Lemma 4.14, we have a reflexive sheaf $\mathcal{L}$ of rank one on X and an isomorphism $\sigma_0 \colon \mathcal{L}^{[2]} \xrightarrow{\simeq} \mathcal{O}_X$ such that $V \simeq V(\mathcal{L}, \sigma_0)$. Moreover, by Lemma 4.16, we can find an isomorphism $\mathcal{L}|_U \simeq \mathcal{O}_X(K_X + D)|_U$ for an open neighborhood U of P. In particular, $2(K_X + D)$ is Cartier at P. Hence, the last assertion follows from Remark 4.15(2).

It remains to prove $P \in \mathcal{P}_2(X, D) \cup \mathcal{D}(X, P)$ and prove the assertion on Q for each type of P. Since (V, D_V) is log-canonical at Q and $K_V + D_V$ is Cartier at Q, by Corollary 3.24, we have either

- V and D_V are non-singular at Q, or
- Q is a node of D_V.

In the first case, P is an A_1-singular point, and $P \in \mathcal{P}_2(X, D)$. In the second case, $P \in \mathcal{D}(X, D)$ by Theorem 3.22. Q.E.D.

Remark. By Lemma 4.17, we can take $V(\mathcal{O}_{\mathcal{U}}(K_{\mathcal{U}} + B|_{\mathcal{U}}), \sigma) \to \mathcal{U}$ as the double-cover $\mathcal{U}' \to \mathcal{U}$ in Theorem 3.22(iii), for an isomorphism $\sigma \colon \mathcal{O}_{\mathcal{U}}(2(K_{\mathcal{U}} + B|_{\mathcal{U}})) \xrightarrow{\simeq} \mathcal{O}_{\mathcal{U}}$.

Proposition 4.18. *Let X be a normal surface* (*which is a scheme*) *and D a reduced divisor such that (X, D) is log-canonical along D and that D is a reducible linear chain of rational curves*; *in particular, D is compact and connected. Let $\tau \colon V = V(\mathcal{O}_X(K_X + D), \sigma) \to X$ be the morphism associated with an isomorphism $\sigma \colon \mathcal{O}_X(2(K_X + D)) \xrightarrow{\simeq} \mathcal{O}_X$. Then,*

(1) *τ is a double-cover étale in codimension one,*
(2) *(V, D_V) is log-canonical along D_V with $K_V + D_V \sim 0$ for $D_V = \tau^{-1}(D)$, and*
(3) *D_V is a reducible cyclic chain of rational curves.*

Let E_1 and E_2 be the end components of D and set

$$\Sigma_i := E_i \cap (\operatorname{Sing} X \setminus \operatorname{Sing} D)$$

for $i = 1$, 2. Then, τ is étale along $D_V \setminus \tau^{-1}(\Sigma_1 \cup \Sigma_2)$, and one of the following cases occurs for each $i = 1$, 2:

(a) *The set Σ_i consists of two A_1-singular points of X belonging to $\mathcal{P}_2(X, D)$. The divisor $\tau^{-1}(E_i)$ is irreducible and the induced morphism $\tau^{-1}(E_i) \to E_i$ is a double-cover branched at Σ_i.*
(b) *The set Σ_i consists of one point of type $\mathcal{D}$ for (X, D). The divisor $\tau^{-1}(E_i)$ consists of two irreducible components E_i' and E_i'' which are isomorphic to E_i by τ, and $E_i' \cap E_i''$ is a point identical to $\tau^{-1}(\Sigma_i)$.*

Proof. By assumption, $K_X + D$ is numerically trivial *on D* (cf. Definition 2.14(ii)), and $K_X + D$ is Cartier along $D \setminus (\Sigma_1 \cup \Sigma_2)$ by Lemma 4.5(3). By Proposition 3.29, the sets Σ_1 and Σ_2 are not empty, and the singularity of X around Σ_i for each i is described as in either the case (G) or (H) of Proposition 3.29. In particular, $K_X + D$ is not Cartier on $\Sigma_1 \cup \Sigma_2$ (cf. Corollary 3.24(3)). Hence, τ is a double-cover étale in codimension one by Lemma 4.14. Then, $K_V + D_V = \tau^*(K_X + D)$, and (V, D_V) is log-canonical by Corollary 3.20. Moreover, $\mathcal{O}_V(K_V + D_V) \simeq (\tau^*\mathcal{O}_X(K_X + D))^{\vee\vee} \simeq \mathcal{O}_V$ by Remark 4.15(1). Thus, $K_V + D_V \sim 0$. Each connected component of D_V is a cyclic chain of rational curves or an elliptic curve by Corollary 4.6. Here, the connected components dominate D, since D is connected. Hence, D_V has at most two connected components and these are reducible divisors. If D_V is not connected, then τ is étale along D_V, and it implies that $K_X + D$ is Cartier along D. This is a contradiction. Therefore, D_V is connected and is a reducible cyclic chain of rational curves.

We fix $i = 1$ or 2. Assume first that $\tau^{-1}(E_i)$ is reducible. Then, $\tau^{-1}(E_i) = E_i' + E_i''$ for two irreducible components of D_V such that $E_i' \simeq E_i$ and $E_i'' \simeq E_i$ via τ and that $E_i' \cap E_i''$ consists of one point Q_i. Since τ is étale along $\tau^{-1}(E_i) \setminus \{Q_i\}$, we have $\Sigma_i = \{\tau(Q_i)\}$ and $E_i' \cap E_i'' = \tau^{-1}(\Sigma_i)$. Thus, (X, E_i, Σ_i) belongs to the case (H) of Proposition 3.29, and the case (b) occurs.

Assume next that $\tau^{-1}(E_i)$ is irreducible. Then, it is an irreducible component of D_V, and $\tau^{-1}(E_i) \to E_i$ is a double-cover whose branch locus is just Σ_i. Thus, Σ_i consists of two points and these are belonging to $\mathcal{P}_2(X, D)$ by Lemma 4.17. In particular, (X, E_i, Σ_i) belongs to the case (G) of Proposition 3.29, and the case (a) occurs. Thus, we are done. Q.E.D.

4.3. Toroidal blowing up

We introduce the notion of toroidal blowing up and give characterizations for a birational morphism to be a toroidal blowing up. We also give a sufficient condition for the existence of a fibration from a toroidal embedding.

Definition 4.19. Let X be a normal surface and D a reduced divisor. A proper birational morphism $f\colon Y \to X$ from another normal surface Y is called a *toroidal blowing up* with respect to (X, D) if the following conditions are satisfied:

- The f-exceptional locus Σ is contained in $D_Y = f^{-1}(D)$;
- (X, D) is toroidal along $f(\Sigma)$, and (Y, D_Y) is toroidal along Σ (cf. Definition 3.12);
- $K_Y + D_Y = f^*(K_X + D)$.

Lemma 4.20. *Let X be a normal surface and D a reduced divisor such that (X, D) is toroidal along D.*

(1) *If $f\colon Y \to X$ be a toroidal blowing up with respect to (X, D), then $\sharp(D_Y - E) \cap E = 2$ for any f-exceptional prime divisor*

E, where $D_Y = f^{-1}(D)$. Moreover, if X is compact, then

$$\boldsymbol{n}(D_Y) - \boldsymbol{n}(D) = \boldsymbol{r}(D_Y) - \boldsymbol{r}(D) = \hat{\boldsymbol{\rho}}(Y) - \hat{\boldsymbol{\rho}}(X).$$

(2) *Let Γ be a compact irreducible component of D with $\Gamma^2 < 0$, $(K_X + D)\Gamma \le 0$, and $\sharp(D - \Gamma) \cap \Gamma \ge 2$. Let $h\colon X \to \overline{X}$ be the contraction morphism of Γ. Then, $(\overline{X}, \overline{D})$ is toroidal along $\overline{D} = h_*(D)$, and h is a toroidal blowing up with respect to $(\overline{X}, \overline{D})$.*

Proof. (1): Since $K_X + D$ is Cartier along D (cf. Lemma 3.14, Corollary 3.25), $K_Y + D_Y$ is also Cartier along D_Y, and we have $(K_Y + D_Y)E = 0$ for any f-exceptional prime divisor E. Then, (Y, D_Y) and E satisfy one of the four conditions corresponding to (A), (B), (C), and (D) in Proposition 3.29 stated for (X, D) and C. Now, E is not a connected component of D_Y. For, otherwise, the point $f(E)$ is a connected component of $D = f(D_Y)$; this is a contradiction. Hence, only the case (C) can occur, and thus, $\sharp(D_Y - E) \cap E = 2$. The latter equalities of (1) follow from Lemmas 2.10 and 2.27, since the f-exceptional locus is contained in D_Y.

(2): Applying Proposition 3.29 to (X, D) and Γ, we have $(K_X + D)\Gamma = 0$ and $\sharp(D - \Gamma) \cap \Gamma = 2$, since $K_X + D$ is Cartier along D with $(K_X + D)\Gamma \le 0$ and $\sharp(D - \Gamma) \cap \Gamma \ge 2$. It implies that $K_X + D = h^*(K_{\overline{X}} + \overline{D})$. Note that $(\overline{X}, \overline{D})$ is log-canonical along $\overline{D}$ by Corollary 3.21. Now, $h(\Gamma)$ is a singular point of $\overline{D}$ by $\sharp(D - \Gamma) \cap \Gamma = 2$. Thus, $(\overline{X}, \overline{D})$ is toroidal at $h(\Gamma)$ by Theorem 3.22. As a consequence, h is a toroidal blowing up. Q.E.D.

A proper birational toric morphism of toric surfaces is of course a toroidal blowing up. Conversely, a toroidal blowing up is regarded as an étale localization of the toric morphism of toric surfaces. This is a consequence of the following:

Proposition 4.21. *Let X be a normal surface and D a reduced divisor such that (X, D) is toroidal along D. Let $f\colon Y \to X$ be a proper birational morphism from another normal surface Y such that the induced morphism $Y \setminus D_Y \to X \setminus D$ is an isomorphism, where $D_Y = f^{-1}(D)$. Then, the following conditions are equivalent to each other*:

(i) *The morphism f is a toroidal blowing up with respect to (X, D).*

(ii) *The pair (Y, D_Y) is toroidal along D_Y, and $(K_Y + D_Y)E = 0$ and $\sharp(D_Y - E) \cap E = 2$ for any f-exceptional prime divisor E.*

(iii) *For the minimal resolutions $\mu\colon M \to X$ and $\nu\colon N \to Y$ of singularities, the induced birational morphism $g = \mu^{-1} \circ f \circ$*

$\nu\colon N \to M$ is a succession of blowings up at nodes of the inverse images of $D_M = \mu^{-1}(D)$.

(iv) *For any point $y \in D_Y$, there exist*

- *an étale neighborhood $\mathcal{X}$ of $f(y)$ in X,*
- *a proper birational toric morphism $\phi\colon W \to V$ of two-dimensional toric varieties,*
- *an étale morphism $i\colon \mathcal{X} \to V$ such that $D|_{\mathcal{X}} = i^{-1}(B)$ for the boundary divisor B of the toric variety V, and*
- *a Cartesian diagram*

$$\begin{array}{ccc} f^{-1}(\mathcal{X}) & \xrightarrow{f|_{f^{-1}(\mathcal{X})}} & \mathcal{X} \\ \big\downarrow & & \big\downarrow{\scriptstyle i} \\ W & \xrightarrow{\phi} & V, \end{array}$$

where $f^{-1}(\mathcal{X}) = Y \times_X \mathcal{X}$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) has been shown in Lemma 4.20, and (iv) $\Rightarrow$ (i) follows from that $K_V + B \sim 0$ for any toric pair (V, B) (cf. Fact 3.6(3)). It is enough to prove (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv).

(i) $\Rightarrow$ (iii): The minimal resolution μ (resp. ν) of singularities of X (resp. Y) is a toroidal blowing up by Corollary 3.15. Hence, $g\colon N \to M$ is also a toroidal blowing up. On the other hand, g is a succession blowings up at non-singular points. Then, the non-singular points are nodes of the inverse image of D_M by Lemma 4.20.

(iii) $\Rightarrow$ (iv): For a point y of D_Y, we set $x = f(y)$. If $x \notin \operatorname{Sing} D$, then $x \notin \operatorname{Sing} X$, and f is an isomorphism over x by (iii). Thus, we may assume that $x \in \operatorname{Sing} D$. By replacing X with an étale neighborhood of x, we may assume that there is an étale morphism $i\colon X \to V$ to an affine toric surface V such that $D = i^{-1}(B)$ for the boundary divisor B of V and that $i^{-1}(i(x)) = \{x\}$. In particular, $i(x) \in \operatorname{Sing} B$. Moreover, we may assume that $f\colon Y \to X$ is an isomorphism over $X \setminus \{x\}$.

Let $\widetilde{V} \to V$ be the minimal resolution of singularities. Then, $\widetilde{V}$ is also a toric surface, and $M \simeq \widetilde{V} \times_V X$ for the minimal resolution M of singularities of X. Let G (resp. E) be the exceptional locus for $\widetilde{V} \to V$ (resp. $M \to X$) and let B' (resp. D') be the proper transform of B (resp. D) in $\widetilde{V}$ (resp. M). Then, the induced morphism $E \to G$ is an isomorphism, and in particular, $D' \cap E \to B' \cap G$ is bijective. For the minimal resolution $N \to Y$ of singularities, the birational morphism $g\colon N \to M$ induced by f is a succession of blowings up at nodes of the inverse images of D_M by (iii). Thus, there exists a proper birational morphism $h\colon \widetilde{W} \to \widetilde{V}$ of non-singular surfaces which is a succession of blowings up at nodes of the inverse images of $B' + G$ and which induces g, i.e.,

$$N \simeq M \times_{\widetilde{V}} \widetilde{W}.$$

Then, $\widetilde{W}$ is a non-singular toric surface and h is a toric morphism. Since the exceptional locus for $N \to Y$ is contained in $g^{-1}(E)$, it is the inverse image of a divisor on $\widetilde{W}$ contained in $h^{-1}(G)$. Hence, we have a proper birational morphism $\widetilde{W} \to W$ to a normal surface W and a morphism $Y \to W$ such that

$$N \simeq Y \times_W \widetilde{W}.$$

Then, W is also a toric surface by Lemma 3.9, and the induced birational morphism $W \to V$ is a toric morphism, since it is equivariant for the action of the open torus. By the isomorphisms

$$N \simeq M \times_{\widetilde{V}} \widetilde{W} \simeq X \times_V \widetilde{W} \simeq (X \times_V W) \times_W \widetilde{W},$$

we have $Y \simeq X \times_V W$, and hence, $Y \to W$ is étale. Thus, (iii) $\Rightarrow$ (iv) has been proved, and we are done. Q.E.D.

The argument of (iii) $\Rightarrow$ (iv) in the proof of Proposition 4.21 proves:

Corollary 4.22. *Let X be a toric surface with boundary divisor B. Then, a proper birational morphism $f\colon Y \to X$ from a normal surface Y is a toroidal blowing up with respect to (X, B) if and only if Y is a toric surface with boundary divisor $f^{-1}(B)$ and f is a toric morphism.*

The following gives a sufficient condition for the existence of a fibration from a toroidal blown up surface of a given pair (X, D).

Lemma 4.23. *Let X be a normal projective surface with only rational singularities such that $\mathrm{H}^1(X, \mathcal{O}_X) = 0$. Let D be a reduced divisor on X such that*

- *(X, D) is toroidal along $\operatorname{Sing} D$, and*
- *$\boldsymbol{n}(D) > \boldsymbol{r}(D)$.*

Then, there exist a toroidal blowing up $f\colon Y \to X$ with respect to (X, D) and a fibration $\pi\colon Y \to T \simeq \mathbb{P}^1$ such that $D_Y = f^{-1}(D)$ contains two distinct fibers of π.

Proof. Let $\mathrm{cl}_D\colon \mathsf{F}(D) \otimes \mathbb{R} \to \mathsf{N}(X)$ be the class map defined in Definition 2.24. Then, cl_D has non-trivial kernel by $\boldsymbol{n}(D) > \boldsymbol{r}(D)$. Note that cl_D is defined over $\mathbb{Q}$. Hence, by Lemma 2.31(4), we can find two non-zero effective Cartier divisors Θ_1 and Θ_2 such that

- $\operatorname{Supp} \Theta_1 \cup \operatorname{Supp} \Theta_2 \subset D$,
- $\operatorname{Supp} \Theta_1 \cap \operatorname{Supp} \Theta_2$ is a finite set, and

- $\Theta_1 \sim \Theta_2$.

Let $f\colon Y \to X$ be the blowing up along the scheme-theoretic intersection $\Theta_1 \cap \Theta_2$ followed by the normalization. Then, f is a toroidal blowing up with respect to (X, D), since (X, D) is toroidal along $\Theta_1 \cap \Theta_2$ and since f is étale locally a toric morphism (cf. Proposition 4.21). Then, there is an f-exceptional effective Cartier divisor E such that

$$\Theta_1^{\sim} := f^*(\Theta_1) - E \quad \text{and} \quad \Theta_2^{\sim} := f^*(\Theta_2) - E$$

are mutually disjoint effective divisors. Thus, we have a morphism $Y \to \mathbb{P}^1$ associated with the base-point free pencil generated by $\Theta_1^{\sim}$ and $\Theta_2^{\sim}$. Let $\pi\colon Y \to T$ be the Stein factorization. Then, $T \simeq \mathbb{P}^1$ by $\mathrm{H}^1(X, \mathcal{O}_X) = 0$, and $D_Y = f^{-1}(D)$ contains at least two distinct fibers of π. Q.E.D.

4.4. Tangential blowing up

We introduce the notion of tangential blowing up and explain a few properties. The tangential blowing up is different from the toroidal blowing up but has similar properties.

Definition 4.24. Let X be a normal surface and D a reduced divisor on X. Let P be a point of D such that X and D are non-singular at P. Let (x, y) be a local coordinate of X at P in which D is defined by $y = 0$. For an integer $m \geq 1$, we set $\mathcal{I} \subset \mathcal{O}_X$ to be the ideal defined by (x^m, y), which is defined independently of the choice of the local coordinate (x, y). We define the *tangential blowing up* of (X, D) at P of *order* m to be the blowing up $f\colon Y \to X$ along $\mathcal{I}$.

Remark. The referee pointed out that the notion of tangential blowing up of order m has been introduced by Morrison in [37, Def. 1.1] as "the directed blowup of weight m."

If $m = 1$, then $\mathcal{I}$ is the maximal ideal at P, so the tangential blowing up of order one is just the blowing up at P. In order to explain the description of the tangential blowing up of order $m \geq 2$, let us consider a sequence

$$\cdots \to X_i \to X_{i-1} \to \cdots \to X_1 \to X_0$$

of blowings up, and reduced divisors D_i on X_i with points $P_i \in D_i$ for each $i \geq 0$ determined by the following properties:

- $X_0 = X$, $D_0 = D$, and $P_0 = P$;
- $X_i \to X_{i-1}$ is the blowing up at P_{i-1} for any $i \geq 1$;
- D_i is the proper transform of D_{i-1} in X_i for any $i \geq 1$;
- P_i is the intersection of D_i and the inverse image of P_{i-1} for any $i \geq 1$.

Then, the inverse image of $P = P_0$ in X_i for $i \geq 2$ is a linear chain of rational curves consisting of (-2)-curves and a unique (-1)-curve. Here, the (-1)-curve is the inverse image of P_{i-1} and the (-2)-curves do not meet D_i. It is an easy exercise to prove the following lemma giving a geometric description of the tangential blowing up.

Lemma 4.25. *Let $f \colon Y \to X$ be the tangential blowing up of (X, D) at P of order m as above. Then, there is a birational morphism $\varphi_m \colon X_m \to Y$ over X such that φ_m contracts all the irreducible components of the inverse image of P except the component intersecting D_m.*

Remark. The birational morphism $X_m \to X$ is called in [40, Def. 2.5] the elimination of the 0-dimensional subscheme defined by $\mathcal{I}$.

Remark. The tangential blowing up is a generalization of the so-called "half point attachment" introduced in the theory of open surfaces (cf. [18, §2], [12, (6.21)]): Let (X, D) be a pair of a non-singular surface X and a normal crossing divisor D. Let P be a point of $D \setminus \operatorname{Sing} D$ and let $f \colon Y \to X$ be the blowing up at P. Then, f is not a toroidal blowup with respect to (X, D). In fact, $K_Y + D_Y = f^*(K_X + D) + E$ for the exceptional divisor $E = f^{-1}(P)$, where the total transform $D_Y = f^*(D)$ is expressed as $D' + E$ for the proper transform D' of D in Y. Note here that we have the equality $K_Y + D' = f^*(K_X + D)$ instead. Moreover, D' is also a normal crossing divisor and $D' \simeq D$ via f. The open surface $Y \setminus D'$ is called the *half-point attachment* of $X \setminus D$.

We have the following immediately from Lemma 4.25.

Corollary 4.26. *A tangential blowing up $f \colon Y \to X$ of order m satisfies the following*:

(1) *There is a unique exceptional prime divisor E for f.*
(2) *The proper transform D' of D in Y is isomorphic to D by f.*
(3) *One has $f^*(D) = D' + mE$, and the intersection of D' and E is transversal.*
(4) *If $m \geq 2$, then Y has a unique singular point Q on E, and here, $Q \notin D'$ and Q is a rational double point of type A_{m-1}.*
(5) *The equality $K_Y + D' = f^*(K_X + D)$ holds.*

Remark. For the f above, we have $D' \subsetneq f^{-1}(D)$ and $X \setminus D \simeq Y \setminus f^{-1}(D) \subsetneq Y \setminus D'$. If X is compact, then

$$0 = \boldsymbol{n}(D') - \boldsymbol{n}(D) \leq \boldsymbol{r}(D') - \boldsymbol{r}(D) \leq \hat{\boldsymbol{\rho}}(Y) - \hat{\boldsymbol{\rho}}(X) = 1$$

by Lemma 2.27. Here, $\boldsymbol{r}(D') = \boldsymbol{r}(D)$ if and only if $\boldsymbol{r}(D) > \boldsymbol{r}(D - C)$ for the irreducible component C of D containing P, by Lemma 2.28(3).

§5. Observation on $\mathbb{P}^1$-fibrations

In this section, we study in detail the structure of a pair (X, D) of a normal Moishezon surface X and a reduced divisor D, and a fibration $\pi\colon X \to T$ to a non-singular projective curve T which satisfy Condition 5.1 below. In Section 5.1, it is shown that X is a projective rational surface with only rational singularities, the base curve T and a general fiber of π are all rational, and that D is either a cyclic chain or a linear chain of rational curves. We study the structure of (X, D) in case of cyclic chain (resp. linear chain) in Section 5.2 (resp. 5.3).

5.1. Two possible cases

Throughout Section 5, we consider the triplet (X, D, π) consisting of a normal Moishezon surface X, a reduced divisor D, and a morphism $\pi\colon X \to T$ which satisfy Condition 5.1 below. In Section 5.1, we shall give a rough classification of (X, D, π). Especially, it is shown that (X, D, π) belongs to the case (A) or (B) of Lemma 5.2.

Condition 5.1.

(i) (X, D) is log-canonical along D;
(ii) $-(K_X + D)$ is *nef on D* (cf. Definition 2.14(ii)), i.e., $-(K_X + D)C \geq 0$ for any irreducible component C of D;
(iii) π is a fibration and T is a non-singular projective curve;
(iv) D is connected and contains at least two distinct fibers of π.

Lemma 5.2. *Let $(X, D, \pi\colon X \to T)$ be a triplet satisfying Condition* 5.1. *Then, T and a general fiber of π are rational, and X is a projective rational surface with only rational singularities. Moreover, D is big, and one of the following cases occurs*:

(A) *The divisor D is a cyclic chain of rational curves expressed as $D = C_1 + C_2 + F_1 + F_2$ for two distinct fibers F_1 and F_2 of π and for two sections C_1 and C_2 of π such that $C_1 \cap C_2 = \emptyset$.*
(B) *The divisor D is a linear chain of rational curves expressed as $D = C_0 + F_1 + F_2$ for two distinct fibers F_1 and F_2 and for a non-end component C_0 such that C_0 is a section or a double-section of π, i.e., $C_0F = 1$ or 2 for a general fiber F of π.*

Proof. The divisor D is a linear or cyclic chain of rational curves by Lemma 4.5. Suppose that D is expressed as in the case (A) or (B) above for two sections C_1 and C_2 or for the horizontal component C_0. Then, the expression implies that D is big, and by Lemma 4.7, X is a projective rational surface with only rational singularities. In particular, $T \simeq \mathbb{P}^1$, which follows also from the rationality of C_1 or C_0. Moreover, a general fiber F of π is rational by $K_X F \le -DF < 0$, where $(K_X + D)F \le 0$ follows from $(K_X + D)C \le 0$ for any curve $C \subset F_1$. Thus, it is enough to prove that (A) or (B) occurs.

Assume first that D is a cyclic chain. Then, the fibers F_1 and F_2 are linear chains of rational curves. Since D is connected, $D-(F_1+F_2)$ has an irreducible component C_1 intersecting F_1. Then, $\pi(C_1) = T$, and hence, C_1 intersects also F_2. Hence, $F_1 + C_1 + F_2$ is a linear chain of rational curves. Since D is assumed to be a cyclic chain, $D-(F_1+F_2+C_1)$ has an irreducible component C_2 intersecting F_1. Then, $\pi(C_2) = T$, and C_2 intersects also F_2. Therefore, the cyclic chain D is expressed as $C_1 + C_2 + F_1 + F_2$, in which $C_1 \cap C_2 = \emptyset$. Now, $K_X + D$ is numerically trivial *on* D (cf. Definition 2.14(ii)) by Lemma 4.5. Thus, $(K_X+D)F = 0$ for a general fiber F, and it implies that $F \simeq \mathbb{P}^1$ and $FC_1 = FC_2 = 1$. Hence, C_1 and C_2 are sections of π. Thus, we have the case (A).

Assume next that D is a linear chain. Then, F_1 and F_2 are also linear chains and we can find an irreducible component C_0 of $D-(F_1+F_2)$ intersecting F_1. Then, $\pi(C_0) = T$, and C_0 intersects also F_2. Thus, $F_1 + C_0 + F_2$ is a linear chain in which C_0 is not an end component. We shall show that $D = F_1 + C_0 + F_2$. If Γ is an irreducible component of D not contained in $F_1 + C_0 + F_2$ but intersecting $F_1 + C_0 + F_2$, then $C_0 \cap \Gamma = \emptyset$, and Γ intersects F_1 or F_2. But if so, $\pi(\Gamma) = T$, and Γ intersects both F_1 and F_2, which implies that $F_1+C_0+F_2+\Gamma$ contains a cyclic chain: This is a contradiction. Therefore, $D = F_1+C_0+F_2$. Note that $(K_X+D)F \le 0$ for a general fiber F of π by the same argument as above. Thus, $F \simeq \mathbb{P}^1$ and $0 < DF = C_0F \le -K_X F = 2$. Hence, either $C_0F = 1$ or $C_0F = 2$ holds, and we have the case (B). Q.E.D.

5.2. The case where D is a cyclic chain

We study (X, D, π) in the case (A) of Lemma 5.2. Here, D is a cyclic chain of rational curves expressed as $C_1 + C_2 + F_1 + F_2$ for two mutually disjoint sections C_1 and C_2 and for two distinct fibers F_1 and F_2. We set $P_1 := \pi(F_1)$ and $P_2 := \pi(F_2)$.

Proposition 5.3. *In the case* (A) *of Lemma* 5.2*, the following hold*:

(1) *If* $-(K_X + D)$ *is nef, then* $K_X + D \sim 0$.

(2) *The inequality* $\boldsymbol{n}(D) \le \boldsymbol{\rho}(X)+2$ *holds, where the equality holds if and only if* π *is smooth outside* $F_1 \cup F_2$.

(3) *If* $\boldsymbol{n}(D) = \boldsymbol{\rho}(X) + 2$*, then* (X, D) *is a toric surface and* π *is a toric morphism to the toric curve* $(T, P_1 + P_2)$.

Proof. Since D is big, by Lemma 4.7, there is an effective divisor G on $X \setminus D$ such that $G \sim K_X + D$ and that G is negative definite if $G \neq 0$. If $-(K_X + D)$ is nef, then $G = 0 \sim K_X + D$. This proves (1).

By Proposition 2.33(7), we have an inequality

$$\boldsymbol{\rho}(X) \geq 2 + (\boldsymbol{n}(F_1) - 1) + (\boldsymbol{n}(F_2) - 1) = \boldsymbol{n}(D) - 2,$$

where the equality holds if and only if any fiber of π except F_1 and F_2 is irreducible. Let F_3 be an irreducible fiber different from F_1 and F_2. Then, F_3 is reduced as a scheme-theoretic fiber, since $F_3 \cap (C_1 \cup C_2)$ is contained in the non-singular locus of X (cf. Proposition 3.29). Therefore, π is smooth along F_3 by Proposition 2.33(4). Thus, (2) has been proved.

Assume that $\boldsymbol{\rho}(X) = \boldsymbol{n}(D) - 2$. Then, $G = 0$ by (2), since $X \setminus D$ contains no compact curves. Thus, $K_X + D \sim 0$. In order to show that (X, D) and π are toric, we shall reduce to the non-singular case. Let $\mu\colon M \to X$ be the minimal resolution of singularities. Then, μ is a toroidal blowing up with respect to (X, D), since $X \setminus D$ is non-singular by (2). We see that $D_M = \mu^{-1}(D)$ is a cyclic chain of rational curves with $K_M + D_M = \mu^*(K_X + D) \sim 0$. Moreover, $D_M = C'_1 + C'_2 + F_1^{\sim} + F_2^{\sim}$ for the proper transform C'_i of C_i in M and for the total transform $F_i^{\sim}$ of F_i in M, for $i = 1$ and 2. Thus, the pair (M, D_M) with the fibration $\pi \circ \mu\colon M \to T$ belongs also to the case (A) of Lemma 5.2. Here, (X, D) is a toric surface if and only if (M, D_M) is so, by Lemma 3.9 and Corollary 4.22. Thus, by replacing M with X, we may assume that X is non-singular. We have a birational morphism $\nu\colon X \to \overline{X}$ over T to a $\mathbb{P}^1$-bundle $p\colon \overline{X} \to T$ in which ν contracts only curves contained in $F_1 \cup F_2$. Here, $\overline{D} = \nu_*(D)$ is a cyclic chain consisting of two sections $\nu(C_1)$, $\nu(C_2)$ and two fibers $\nu_*(F_1)$ and $\nu_*(F_2)$ of p. In particular, $(\overline{X}, \overline{D})$ is a toric surface and p is a toric morphism $(\overline{X}, \overline{D}) \to (T, P_1 + P_2)$ (cf. Example 3.8). Moreover, $\nu\colon (X, D) \to (\overline{X}, \overline{D})$ is a toroidal blowing up. Therefore, (X, D) is a toric surface by Corollary 4.22, and we have proved (3). Q.E.D.

Lemma 5.4. *In the case* (A) *of Lemma* 5.2*, assume that* $K_X + D \sim 0$*. Let* F_3 *be a fiber of* π *different from* F_1 *and* F_2*, and assume that* π *is not smooth along* F_3*. Then,* F_3 *is a reducible linear chain of rational curves with end components* Γ_1 *and* Γ_2 *such that*

(1) *the section* C_i *intersects transversally with* Γ_i *for* $i = 1$, 2*, and*
(2) (X, F_3) *is toroidal along* F_3*.*

As a consequence, $X \setminus D$ *has only rational double points of type* A *as singularities.*

Proof. By an argument in the proof of Proposition 5.3, the scheme-theoretic fiber F_3 is reduced at two points $F_3 \cap (C_1 \cup C_2)$. Moreover, F_3 is reducible by Proposition 2.33(4), since π is not smooth along F_3. In particular, every irreducible component of F_3 is a negative curve. Let Γ_1 and Γ_2 be the irreducible components of F_3 which intersect C_1 and C_2, respectively.

First, we shall prove the assertion in the case where X is non-singular. By $K_X\Gamma_i = -D\Gamma_i \leq -C_i\Gamma_i = -1$ for $i = 1, 2$, we see that Γ_1 and Γ_2 are (-1)-curves and $\Gamma_1 \cap C_2 = \Gamma_2 \cap C_1 = \emptyset$. If $F_3 = \Gamma_1 + \Gamma_2$, we have nothing to prove. Assume that $F_3 \neq \Gamma_1 + \Gamma_2$. Then, the irreducible components Γ of F_3 different from Γ_1 and Γ_2 are all (-2)-curves by $K_X\Gamma = -D\Gamma = 0$. In this situation, we can prove that F_3 is a linear chain of rational curves with Γ_1 and Γ_2 as end components. Indeed, this is proved by induction on the number of irreducible components of F_3 and by considering the blowing down of the (-1)-curve Γ_1. Thus, the assertion holds when X is non-singular.

For general X, let us consider the minimal resolution $\mu\colon M \to X$ of singularities. Note that $X \setminus D$ has only rational double points as singularities, since X has only rational singularities and $K_X + D \sim 0$. Hence, μ is a toroidal blowing up with respect to (X, D) along D and is the minimal resolution of rational double points on $X \setminus D$. As a consequence, (M, D_M) is toroidal and $K_M + D_M = \mu^*(K_X + D) \sim 0$. By the same argument as in the proof of Proposition 5.3, we see that the pair (M, D_M) with the fibration $\pi \circ \mu$ satisfies the assumption of Lemma 5.4. Then, the assertion for the non-singular case implies that the total transform $F_3^{\sim} = \mu^{-1}(F_3)$ is a linear chain of rational curves with Γ'_1 and Γ'_2 as end components, where Γ'_i is the proper transform of Γ_i in M for $i = 1, 2$. Thus, $\Gamma_1 \neq \Gamma_2$, and $F_3 = \mu_*(F_3^{\sim})$ is a linear chain of rational curves such that (X, F_3) is toroidal along F_3 by Lemma 4.20(2). Note that X has only cyclic quotient singularities on F_3. Hence, the last assertion holds, since $X \setminus D$ is Gorenstein. Thus, we are done. Q.E.D.

Proposition 5.5. *In the case* (A) *of Lemma* 5.2*, assume that* $-(K_X + D)$ *is nef and* $\boldsymbol{n}(D) = \boldsymbol{\rho}(X) + 1$*. Then,* $K_X + D \sim 0$*, and there exist two rational curves* Γ_1 *and* Γ_2 *on* X *satisfying the following properties*:

(1) $\Gamma_1 + \Gamma_2$ *is a linear chain of rational curves and is a fiber of* π *different from* F_1 *and* F_2;

(2) $(X, \Gamma_1 + \Gamma_2)$ *is toroidal along* $\Gamma_1 + \Gamma_2$;

(3) $\Gamma_1 C_1 = \Gamma_2 C_2 = 1$ *and* $\Gamma_1 \cap C_2 = \Gamma_2 \cap C_1 = \emptyset$;

(4) *if* Γ *is a negative curve on* X *not contained in* D*, then* $\Gamma = \Gamma_1$ *or* Γ_2.

Let $g\colon X \to Z$ *be the contraction morphism of* Γ_1*, and set* $D_Z = g_*(D)$ *and* $Q := g(\Gamma_1)$*. Then, the following also hold*:

(5) (Z, D_Z) *is a toric surface and the induced fibration* $Z \to T$ *by* π *is a toric morphism to the toric curve* $(T, P_1 + P_2)$;

(6) g *is a tangential blowing up* (*cf. Definition* 4.24) *of* (Z, D_Z) *at the point* Q *of order* $k \geq 1$. *Here, if* $k = 1$, *then* $\Gamma_1 \cap \Gamma_2$ *is a non-singular point of* X, *and if* $k > 1$, *then* $\Gamma_1 \cap \Gamma_2$ *is an* A_{k-1}*-singularity of* X.

Proof. We have $K_X + D \sim 0$ by Proposition 5.3(1). There is a unique reducible fiber F_3 of π different from F_1 and F_2 by Proposition 2.33(7), since

$$\boldsymbol{\rho}(X) = \boldsymbol{n}(D) + 1 = 2 + (\boldsymbol{n}(F_1) - 1) + (\boldsymbol{n}(F_2) - 1) + 1,$$

where $\boldsymbol{n}(F_3) = 2$. Let $F_3 = \Gamma_1 + \Gamma_2$ be the irreducible decomposition. Then, applying Lemma 5.4 and assuming $C_1\Gamma_1 = 1$ and $C_2\Gamma_2 = 1$, we have the properties (1)–(3) above.

Let Γ be a negative curve on X not contained in D. Assume that $\pi(\Gamma) = T$. Then, $\Gamma \cap F_1 \neq \emptyset$ and $\Gamma \cap F_2 \neq \emptyset$. Let $\mu\colon M \to X$ be the minimal resolution of singularities. Then, $K_M + D_M = \mu^*(K_X + D) \sim 0$ for $D_M = \mu^{-1}(D)$, and

$$K_M\Gamma' = -D_M\Gamma' \leq -2$$

for the proper transform Γ' of Γ in M; This contradicts $\Gamma'^2 + K_M\Gamma' \geq -2$ and $\Gamma'^2 < 0$. Hence, $\pi(\Gamma) \neq T$. Thus, Γ is an irreducible component of a reducible fiber. Therefore, $\Gamma = \Gamma_1$ or Γ_2, and we have proved (4).

The pair (Z, D_Z) in (5) is log-canonical by Corollary 3.21, and the pair with the induced fibration $Z \to T$ satisfies the conditions in the case (A) of Lemma 5.2. Thus, (5) is a consequence of Proposition 5.3(3). By Lemma 5.4, the inverse image of $\Gamma_1 + \Gamma_2$ by the minimal resolution of X is a linear chain such that the proper transforms of Γ_1 and Γ_2 are (-1)-curves as well as end components and that the non-end components are all (-2)-curves. Therefore, g is a tangential blowing up of (Z, D_Z) at Q of order $k \geq 1$ by Lemma 4.25 and Corollary 4.26. Thus, (6) has been proved, and we are done. Q.E.D.

5.3. The case where D is a linear chain

We study (X, D, π) in the case (B) of Lemma 5.2. Here, D is a linear chain of rational curves expressed as $C_0 + F_1 + F_2$ for a non-end component C_0 which is either a section or a double-section of π. We also set $P_1 := \pi(F_1)$ and $P_2 := \pi(F_2)$.

Lemma 5.6. *In the case* (B) *of Lemma* 5.2, *one has an inequality* $\boldsymbol{n}(D) \leq \boldsymbol{\rho}(X) + 1$, *where the equality holds if and only if* π *is smooth outside* $F_1 \cup F_2$.

Proof. By Proposition 2.33(7), we have $\boldsymbol{\rho}(X) \geq \boldsymbol{n}(F_1) + \boldsymbol{n}(F_2) = \boldsymbol{n}(D) - 1$, where the equality holds if and only if any fiber is irreducible except for F_1 and F_2. Let F_3 be an irreducible fiber different from F_1 and F_2. By Proposition 2.33(4), it is enough to prove that F_3 is reduced as a scheme-theoretic fiber. Now $C_0 \cap F_3$ is contained in the non-singular loci of X and D (cf. Proposition 3.29). Hence, F_3 is reduced if C_0 is a section. If C_0 is a double-section, then the induced double-cover $\tau := \pi|_{C_0} : C_0 \to T$ is étale outside $C_0 \cap (F_1 \cup F_2)$, since $C_0 \cap F_i = \tau^{-1}(P_i)$ is a point for $i = 1, 2$. Thus, in this case, $C_0 \cap F_3$ consists of two points and is reduced. Therefore, F_3 is reduced, and we are done. Q.E.D.

Proposition 5.7. *In the case* (B) *of Lemma* 5.2, *assume that*

- $-(K_X + D)$ *is nef,*
- $\boldsymbol{n}(D) = \boldsymbol{\rho}(X) + 1$, *and*
- C_0 *is a section of* π.

Then, there is a section B of π not contained in D such that $(X, B + D)$ is a toric surface and that $\pi\colon (X, B + D) \to (T, P_1 + P_2)$ is a toric morphism.

Proof. Since π is smooth outside $F_1 \cup F_2$ by Lemma 5.6, we have $\operatorname{Sing} X \subset D$. For $i = 1, 2$, let E_i be the end component of D such that $E_i \subset F_i$. Then, $(K_X + D)E_i < 0$ for any i. In fact, if $(K_X + D)E_1 = 0$ for example, then $(K_X + D)\Gamma = 0$ for any irreducible component Γ of F_1 by Lemma 4.5(3), and it implies that $(K_X + D)F = 0$ for any general fiber F of π. However, this contradicts $(K_X + D)F = (K_X + C_0)F = -1$. Therefore, by Proposition 3.29, $\operatorname{Sing} X \subset E_1 \cup E_2 \cup \operatorname{Sing} D$, and $\Sigma_i := (E_i \setminus \operatorname{Sing} D) \cap \operatorname{Sing} X$ is empty or consisting of one point of type $\mathcal{P}$ for (X, D).

Let $\mu\colon M \to X$ be the minimal resolution of singularities. By the information above of $\operatorname{Sing} X$, we see that the μ-exceptional divisor is a union of linear chains of rational curves, and hence, $D_M = \mu^{-1}(D)$ is a linear chain expressed as $\widetilde{C}_0 + \widetilde{F}_1 + \widetilde{F}_2$ for the proper transform $\widetilde{C}_0$ of C_0 in M and for the fiber $\widetilde{F}_i = \mu^{-1}(F_i) = (\pi \circ \mu)^{-1}(P_i)$. We express the linear chain $\widetilde{F}_i$ as $\Gamma_1 + \Gamma_2 + \cdots + \Gamma_l$ for rational curves Γ_j such that the end component Γ_l intersects $\widetilde{C}_0$. More precisely, we write

$$\Gamma_j^{(i)} = \Gamma_j \quad \text{and} \quad l^{(i)} = l$$

indicating $i = 1, 2$. Let $m_j^{(i)} = m_j$ be the multiplicity of the scheme-theoretic fiber along Γ_i, i.e.,

$$(\pi \circ \mu)^*(P_i) = \sum\nolimits_{j=1}^{l_j} m_j^{(i)} \Gamma_j^{(i)}.$$

Then, $m_1 = m_l = 1$ (cf. Lemma 5.8 below). Note that $\Sigma_i = \emptyset$ if and only if Γ_1 is the proper transform E_i' of E_i in M. If $\Sigma_i \neq \emptyset$, then $E_i' = \Gamma_k$ for some $k > 1$, and $\mu^{-1}(\Sigma_i) = \Gamma_1 + \cdots + \Gamma_{k-1}$; Here, we have $m_j \geq j$ for any $1 < j \leq k$ by Lemma 5.8 below, since μ is the minimal resolution of singularities. We can write

$$K_M + D_M = \mu^*(K_X + D) + \sum\nolimits_{i=1}^{2} \sum\nolimits_{j=1}^{l_i} p_j^{(i)} \Gamma_j^{(i)}, \tag{V-1}$$

with $0 \leq p_j^{(i)} < 1$, where $p_j^{(i)} \neq 0$ if and only if $\Gamma_j^{(i)} \subset \mu^{-1}(\Sigma_i)$ (cf. the proof of Lemma 3.26). Thus, if a section $\widetilde{C}$ of $\pi \circ \mu$ intersects $\Gamma_j^{(i)}$ with $p_j^{(i)} \neq 0$, then $j = 1$. Therefore,

$$(K_M + D_M)\widetilde{C} \leq p_1^{(1)} + p_1^{(2)} < 2 \tag{V-2}$$

for any section $\widetilde{C}$ of $\pi \circ \mu$, since $-(K_X + D)$ is nef.

Let $h\colon M \to Z$ be the contraction morphism of all the irreducible components of $\widetilde{F}_1 + \widetilde{F}_2 - (\Gamma_1^{(1)} + \Gamma_1^{(2)})$. Then, $K_M + D_M$ is h-numerically trivial, since $(K_M + D_M)\Gamma = 0$ for any non-end component Γ of the linear chain D_M of rational curves. Thus,

$$K_M + D_M = h^*(K_Z + D_Z) \tag{V-3}$$

for $D_Z = h_*(D_M)$. We have a fibration $\varpi\colon Z \to T$ with $\varpi \circ h = \pi \circ \mu$. For $i = 1, 2$, the image $\overline{F}_i = h(\widetilde{F}_i) = h(\Gamma_1^{(i)})$ is just the scheme-theoretic fiber $\varpi^{-1}(P_i)$, since $m_1^{(i)} = 1$. Then, $\varpi\colon Z \to T$ is a $\mathbb{P}^1$-bundle by Proposition 2.33(4), since every scheme-theoretic fiber of ϖ is irreducible and reduced. Here, $D_Z = \overline{C}_0 + \overline{F}_1 + \overline{F}_2$ for the section $\overline{C}_0 := h(\widetilde{C}_0)$.

Assume that there is a section $\overline{B}$ of ϖ such that $\overline{B} \cap \overline{C}_0 = \emptyset$. Let $\widetilde{B}$ be the proper transform of $\overline{B}$ in M and set $B := \mu(\widetilde{B})$. Then, $\widetilde{B} + D_M$ is a normal crossing cyclic chain of rational curves, and $K_M + \widetilde{B} + D_M$ is numerically trivial *on* $\widetilde{B} + D_M$ (cf. Definition 2.14(ii)) by Corollary 4.6. Since the μ-exceptional locus is contained in D_M, the divisor $B + D$ is a cyclic chain of rational curves, $K_X + B + D$ is numerically trivial on $B + D$, and $(X, B + D)$ is log-canonical (cf. Corollary 3.21). Thus, $(X, B + D)$ and π satisfy the conditions of Lemma 5.2(A) with $\boldsymbol{\rho}(X) = \boldsymbol{n}(B+D) - 2$. Therefore, $(X, B + D)$ is toric and $(X, B + D) \to (T, P_1 + P_2)$ is a toric morphism by Proposition 5.3(3).

Therefore, it remains to find a section $\overline{B}$ not intersecting $\overline{C}_0$. Assume that there is no such a section. Then, $\overline{C}_0$ is not a minimal section C_Z of the Hirzebruch surface Z. We set $e = -C_Z^2 \geq 0$ and let $\overline{F}$ denote a general fiber of ϖ. Then,

$$\overline{C}_0 \sim C_Z + d\overline{F} \quad \text{and} \quad K_Z + D_Z \sim -C_Z + (d-e)\overline{F}$$

for an integer $d \geq e$. By (V-3) and by (V-2) applied to the proper transform $\widetilde{C}_Z$ of C_Z in M, we have

$$d = (K_Z + D_Z)C_Z = (K_M + D_M)\widetilde{C}_Z \leq p_1^{(1)} + p_1^{(2)} < 2.$$

If $d = 0$, then $d = e = 0$, and $\overline{C}_0$ is a minimal section; this contradicts the assumption. Thus, $d = 1$, and $0 \leq e \leq 1$. If $e = 1$, then we can take $\overline{B}$ as C_Z, since $C_Z \cap \overline{C}_0 = \emptyset$. Hence, we have $(d, e) = (1, 0)$. Therefore, $p_1^{(i)} > 0$ for any $i = 1, 2$, and it implies that $\Sigma_i \neq \emptyset$. The section $\widetilde{C}_Z$ must intersect $\Gamma_1^{(i)}$ for any $i = 1, 2$ by the observation above on irreducible components $\Gamma_j^{(i)}$ with $p_j^{(i)} > 0$. However, we can find another minimal section $C_{Z,1}$ such that $C_{Z,1} \cap \overline{C}_0 = \overline{F}_1 \cap \overline{C}_0$. The proper transform $\widetilde{C}_{Z,1}$ of $C_{Z,1}$ in M does not intersect $B_1^{(1)}$. This is a contradiction. As a consequence, we can find a section $\overline{B}$ not intersecting $\overline{C}_0$, and we are done. Q.E.D.

The following lemma is used in the proof of Proposition 5.7.

Lemma 5.8. *Let M be a non-singular surface with a $\mathbb{P}^1$-fibration $\psi\colon M \to T$ and let F be a reducible fiber of ψ. Assume that F is a linear chain $\Gamma_1 + \Gamma_2 + \cdots + \Gamma_l$ of rational curves Γ_i in this order, and let $\sum_{1 \leq i \leq l} m_i \Gamma_i$ be the scheme-theoretic fiber of ψ.*

(1) *If $m_l = 1$, then $m_1 = 1$.*

(2) *For an integer $k > 1$, assume that $\Gamma_i^2 \leq -2$ for any $i < k$. Then, $m_i \geq i$ for any $1 \leq i \leq k$.*

Proof. Note that F is a simple normal crossing divisor (cf. Remark 2.34). Hence, $\Gamma_i \simeq \mathbb{P}^1$ for any i, and $\Gamma_i \Gamma_{i+1} = 1$ for any $1 \leq i < l$. Then,

$$m_{i-1} + m_i \Gamma_i^2 + m_{i+1} = 0$$

for any $1 \leq i \leq l$, where we set $m_0 = m_{l+1} = 0$. In particular, m_i is divisible by m_1 for any $1 \leq i \leq l$, and this proves (1). Moreover, under the assumption of (2), we have

$$m_{i+1} - m_i = m_i(-\Gamma_i^2 - 2) + m_i - m_{i-1} \geq m_i - m_{i-1}$$

for any $1 \leq i < k$. This implies that $m_i \geq i$ for any $1 \leq i \leq k$. Q.E.D.

Proposition 5.9. *In the case* (B) *of Lemma* 5.2*, assume that* C_0 *is a double-section and* $\boldsymbol{n}(D) = \boldsymbol{\rho}(X) + 1$*. Then,* π *is smooth outside* $F_1 \cup F_2$*, and* $2(K_X + D) \sim 0$*. Let* $\tau\colon V = V(\mathcal{O}_X(K_X + D), \sigma) \to X$ *be the double-cover étale in codimension one associated with an isomorphism* $\sigma\colon \mathcal{O}_X(2(K_X + D)) \xrightarrow{\simeq} \mathcal{O}_X$ *(cf. Definition* 4.9*). Then,* (V, D_V) *is a toric surface for* $D_V = \tau^{-1}(D)$*, and there is a toric morphism* $\pi_V\colon (V, \tau^{-1}(D)) \to (T', P'_1 + P'_2)$ *such that*

- π_V *is the Stein factorization of* $\pi \circ \tau$*,*
- $T' \to T$ *is a double-cover branched at* $\{P_1, P_2\} = \pi(F_1 \cup F_2)$*, and*
- P'_i *is the point of* T' *lying over* P_i *for* $i = 1$*,* 2*.*

Proof. The morphism π is smooth outside $F_1 \cup F_2$ by Lemma 5.6. We shall show that $2(K_X + D) \sim 0$. For $i = 1$, 2, let E_i be the end component of D such that $E_i \subset F_i$. Then, $K_X + D$ is Cartier along $D - E_1 - E_2$ and is numerically trivial on $D - E_1 - E_2$ by Lemma 4.5(3). Since $(K_X + D)F = 0$ for a general fiber F, we have $(K_X + D)E_i = 0$ for $i = 1$, 2. Hence, (X, D, E_i) belongs to either the case (G) or (H) of Proposition 3.29. As a consequence, $2(K_X + D)$ is Cartier and π-numerically trivial. Then, $2(K_X + D) \sim \pi^*L$ for a divisor L on T by Proposition 2.33(5), and now $L = 0$ by $2 \deg L = C_0\pi^*L = 2(K_X + D)C_0 = 0$. Therefore, $2(K_X + D) \sim 0$.

By Proposition 4.18 and Remark 4.15(4), (V, D_V) is log-canonical with $K_V + D_V \sim 0$, D_V is a cyclic chain of rational curves, and $V \setminus \tau^{-1}(F_1 \cup F_2)$ is non-singular and étale over $X \setminus (F_1 \cup F_2)$. Here, $\tau^{-1}(D - E_1 - E_2)$ is a disjoint union of two copies of $D - E_1 - E_2$, and for each $i = 1$, 2, either $\tau^{-1}(E_i)$ is irreducible or is a union of two copies of E_i intersecting at one point. In particular, $\tau^{-1}(F_i)$ is connected and is a fiber of the Stein factorization $\pi_V\colon V \to T'$ of $\pi \circ \tau\colon V \to T$. Thus, (V, D_V) and π_V satisfy the condition of Lemma 5.2(A). Here, $\tau^{-1}(C_0)$ is just the union of two sections of π_V contained in D_V. Hence, $T' \to T$ is a double-cover isomorphic to $C_0 \to T$, which is branched at $\{P_1, P_2\}$, since $C_0F_1 = C_0F_2 = 1$. In particular, $\tau^{-1}(F_i)$ is the fiber over the point $P'_i \in T'$ lying over P_i for $i = 1$, 2. Since $V \setminus \tau^{-1}(F_1 \cup F_2)$ is smooth over T', we have $\boldsymbol{n}(D_V) = \boldsymbol{\rho}(V) + 2$ by Proposition 5.3(2). Therefore, (V, D_V) is a toric surface and π_V is a toric morphism by Proposition 5.3(3). Q.E.D.

§6. Pseudo-toric surfaces

We introduce the notion of pseudo-toric surface in Section 6.1 and explain its basic properties. Especially in Proposition 6.4, it is shown that the defect $\boldsymbol{\delta}(X, D)$ of a pseudo-toric surface (X, D) is always non-negative and that among pseudo-toric surfaces, the toric surfaces are characterized by $\boldsymbol{\delta}(X, D) = 0$. The structure of pseudo-toric surface of *defect one* is studied in detail in Section 6.2, where we shall give a structure theorem as Theorem 6.5 and give a proof of Theorem 1.6.

6.1. Pseudo-toric surfaces and their basic properties

Definition 6.1. A pair (X, D) of a normal projective surface X and a reduced divisor D is called a *pseudo-toric surface* if the following conditions are satisfied:

(i) X is a rational surface with only rational singularities;
(ii) (X, D) is log-canonical along D, and $K_X + D \sim 0$;
(iii) every irreducible component of D is a rational curve;
(iv) D is big.

Remark. Let X be a non-singular projective surface and D an anti-canonical reduced divisor of X. Assume that D is a simple normal crossing divisor consisting of rational curves. Then, (X, D) is pseudo-toric if and only if D is big, by Definition 6.1. It is an exercise to prove that D is big if and only if one of the following holds.

- There is an irreducible component C of D with $C^2 > 0$.
- There is an irreducible component C of D with $C^2 = 0$ and $(D - C)C > 0$.
- There is an irreducible component C of D such that C is a (-1)-curve and that the push-forward $\overline{D} = g_*(D)$ is big for the contraction morphism $g\colon X \to \overline{X}$ of C.

Remark 6.2. Inspired by a comment of the referee, the author found that the condition (i) is superfluous in Definition 6.1. In fact, the conditions (ii) and (iii) imply that each connected component of D is a cyclic chain of rational curves by Corollary 4.6. There is a big connected component $D^\sharp$ of D by (iv). Then, by applying Lemma 4.7 to $(X, D^\sharp)$, we have the condition (i).

Lemma 6.3. *Let (X, D) be a pseudo-toric surface.*

(1) *The Weil-Picard number $\hat{\boldsymbol{\rho}}(X)$ equals the Picard number $\boldsymbol{\rho}(X)$.*
(2) *The open subset $X \setminus D$ has only rational double points as singularities. In particular, (X, D) is log-canonical.*
(3) *The divisor D is connected and is a cyclic chain of rational curves.*
(4) *If $f\colon Y \to X$ is a toroidal blowing up with respect to (X, D), then (Y, D_Y) is also pseudo-toric for $D_Y = f^{-1}(D)$.*

(5) *Let* $f\colon Y \to X$ *be a tangential blowing up of* (X, D) *and let* D' *be the proper transform of* D. *Then* (Y, D') *is pseudo-toric if and only if* D' *is big.*

(6) *Let* $\mu\colon M \to X$ *be the minimal resolution of singularities. Then,* (M, D_M) *is also pseudo-toric for* $D_M = \mu^{-1}(D)$.

(7) *Let* $g\colon X \to \overline{X}$ *be a birational morphism to a normal Moishezon surface* $\overline{X}$ *and set* $\overline{D} = g_*(D)$. *Then,* $(\overline{X}, \overline{D})$ *is a pseudo-toric surface. If the* g*-exceptional locus is contained in* D, *then* g *is a toroidal blowing up with respect to* $(\overline{X}, \overline{D})$.

(8) *There is a birational morphism* $g\colon X \to \overline{X}$ *contracting only curves in* $X \setminus D$ *such that* $\overline{X} \setminus \overline{D}$ *is affine for* $\overline{D} = g_*(D)$.

Proof. (1): This is a consequence of Lemma 2.31.

(2): This follows from that $K_X + D \sim 0$ and that X has only rational singularities. In fact, $X \setminus D$ is Gorenstein with only rational singularities.

(3): By Remark 6.2, we know that each connected component of D is a cyclic chain of rational curves. Suppose that D is not connected. Then, by the Hodge index theorem, a connected component E of D is negative definite, since D is big. Let $g\colon X \to \overline{X}$ be the contraction morphism of E. Then, $K_{\overline{X}} + \overline{D} \sim 0$ for the divisor $\overline{D} = g_*(D) \neq 0$, and

$$\mathrm{H}^2(\overline{X}, \mathcal{O}_{\overline{X}}) = \mathrm{H}^0(\overline{X}, \mathcal{O}(K_{\overline{X}}))^\vee = 0.$$

Hence, $\overline{X}$ has only rational singularities by Lemma 2.31(3). However, the singular point $\pi(E)$ is irrational, since E is a cyclic chain of rational curves. This is a contradiction. Therefore, D is connected and (3) holds.

(4) and (5): Let $f\colon (Y, D_Y) \to (X, D)$ be either a toroidal blowing up or a tangential blowing up. Here, $D_Y = f^{-1}(D)$ in the case of toroidal blowing up, and D_Y is the proper transform of D in the case of tangential blowing up. Then, (Y, D_Y) is also log-canonical, D_Y consists of rational curves, and $K_Y + D_Y = f^*(K_X + D) \sim 0$ by Definition 4.19 and Corollary 4.26. Thus, (Y, D_Y) is also pseudo-toric.

(6): The minimal resolution μ is expressed as a toroidal blowing up along D and is the minimal resolution of $X \setminus D$, which has only rational double points by (2). Thus, $K_M + D_M = \mu^*(K_X + D) \sim 0$, and (M, D_M) is also pseudo-toric by the argument above.

(7): The pair $(\overline{X}, \overline{D})$ is log-canonical by (2) and by Corollary 3.21. We have $K_{\overline{X}} + \overline{D} = g_*(K_X + D) \sim 0$. Therefore, $\overline{D}$ is also a cyclic chain of rational curves by Corollary 4.6. Moreover, $\overline{D}$ is big by Remark 2.13. Hence, $\overline{X}$ is a projective rational surface with only rational singularities by Lemma 4.7. Thus, $(\overline{X}, \overline{D})$ is a pseudo-toric surface. The latter assertion of (7) follows from Definition 4.19.

(8): The union of compact curves in $X \setminus D$ is negative definite by the Hodge index theorem, since D is big. Hence, we have the contraction morphism $X \to \overline{X}$ of the union of these curves by Theorem 2.6. Then, $\overline{X} \setminus \overline{D}$ contains no compact curves for the image $\overline{D}$ of D. Here, $(\overline{X}, \overline{D})$ is pseudo-toric by (7) above. Thus, for the proof of (8), we may assume that $X = \overline{X}$, i.e., $X \setminus D$ contains no compact curves. There is a birational morphism $\pi\colon X \to X'$ to a normal Moishezon surface X' such that the π-exceptional locus is contained in D and that $D' := \pi_*(D)$ contains no negative curves. Then, $X \setminus D \simeq X' \setminus D'$, since $D = \pi^{-1}(D')$. Here, (X', D') is also pseudo-toric by (7). Thus, by replacing X' with X, we may assume furthermore that every irreducible component of D is nef. Then, it is enough to show that D is ample. Now, $DC > 0$ for any irreducible component C of D. In fact, since D is connected by (3), we have $DC = (D - C)C + C^2 > 0$ in case D is reducible, and even in case D is irreducible, we have $D^2 > 0$, since D is big. Thus, if $D\Gamma = 0$ for an irreducible curve Γ on X, then $\Gamma \subset X \setminus D$; this is a contradiction. Hence, D is ample by the Nakai–Moishezon criterion of ampleness (cf. Remark 2.12). Thus, we are done. Q.E.D.

Remark. Let (X, D) be a pseudo-toric surface such that X is non-singular. The structure of (X, D) is studied by considering birational morphisms $f\colon X \to Z$ and $g\colon Z \to S$ satisfying the following conditions:

- The pairs (Z, D_Z) and (S, D_S) are pseudo-toric surfaces for $D_Z = f_*(D)$ and $D_S = (g \circ f)_*(D)$, and Z and S are non-singular.
- Every exceptional divisor for f is not contained in D.
- The exceptional locus of g is contained in D_Z.
- There is no (-1)-curve on Z not contained in D_Z, and there is no (-1)-curve on S.

Here, f is a maximal succession of contractions of (-1)-curves not contained in the images of D, and g is a succession of contractions of (-1)-curves contained in D_Z. Then, (Z, D_Z) and (S, D_S) are pseudo-toric by Lemma 6.3(7). Note that every negative curve Γ on X not contained in D is either a (-1)-curve or a (-2)-curve, since $\Gamma D = -K_X D \geq 0$. Since the pseudo-toric surfaces (S, D_S) are classified easily, we have a detailed structure of (X, D) by investigating the birational morphisms f and g.

Proposition 6.4. *Let (X, D) be a pseudo-toric surface. Then, the* defect $\boldsymbol{\delta}(X, D)$ *and the* complexity $\boldsymbol{c}(X, D)$ *(cf. Definition* 2.23*) are non-negative. Here,* $\boldsymbol{c}(X, D) = 0$ *if and only if (X, D) is a toric surface. In particular, a pseudo-toric surface of defect zero is a toric surface.*

Proof. We have $\boldsymbol{c}(X, D) \geq 0$ by Proposition 4.8(3), and $\boldsymbol{\delta}(X, D) \geq \boldsymbol{c}(X, D)$ by Definition 2.23. If (X, D) is toric, then it is pseudo-toric with $\boldsymbol{\delta}(X, D) = 0$ by Lemma 3.11. For the rest, it is enough to prove that (X, D) is toric when $\boldsymbol{c}(X, D) = 0$. In this situation, we have also $\boldsymbol{\delta}(X, D) = 0$ by Proposition 4.8(4). Since $\boldsymbol{n}(D) > \boldsymbol{\rho}(X) \geq \boldsymbol{r}(D)$, by Lemma 4.23, there exist a toroidal blowing up $f\colon (Y, D_Y) \to (X, D)$ and a fibration $\pi\colon Y \to T$ to a non-singular curve T such that $D_Y = f^{-1}(D)$ contains at least two fibers of π. Here, (Y, D_Y) is also pseudo-toric and $\boldsymbol{\delta}(Y, D_Y) = \boldsymbol{\delta}(X, D) = 0$ by Lemma 2.27. Since D_Y is connected by Lemma 6.3(3), (Y, D_Y) and π satisfy the condition of Lemma 5.2(A). Thus, (Y, D_Y) is toric by Proposition 5.3(3), and (X, D) is toric by Lemma 3.9. Q.E.D.

6.2. The structure of pseudo-toric surfaces of defect one

We first prove a structure theorem as Theorem 6.5 for the pseudo-toric surface (X, D) of defect one. Using a special linear chain of rational curves defined in Definition 6.7 below, we obtain results on the group $\mathrm{Aut}(X; D)$ of automorphisms of X preserving each irreducible component of D, on the coordinate ring of $X \setminus D$, and on the quasi-Albanese map of $X \setminus D$. Finally, we prove Theorem 1.6 gathering these partial results.

Theorem 6.5. *Let X be a normal Moishezon surface with a reduced divisor D. Then, (X, D) is a pseudo-toric surface of defect one if and only if there exist a toroidal blowing up $f\colon Y \to X$ with respect to (X, D) and a tangential blowing up $g\colon Y \to Z$ of a projective toric surface (Z, D_Z) such that $f^{-1}(D)$ is the proper transform of D_Z in Y.*

Proof. If Y is a tangential blowing up of toric surface (Z, D_Z), then (Y, D_Y) is a pseudo-toric surface of defect one for the proper transform D_Y of D_Z in Y by Lemma 6.3(5) and by Proposition 4.8(1), since

$$\boldsymbol{c}(Y, D_Y) \leq \boldsymbol{\delta}(Y, D_Y) = \boldsymbol{n}(D_Y) - (\boldsymbol{\rho}(Y) + 2) = \boldsymbol{n}(D_Z) - (\boldsymbol{\rho}(Z) + 1) = 1.$$

Thus, if there is also a toroidal blowing up $f\colon Y \to X$ with respect to (X, D) such that $D_Y = f^{-1}(D)$, then (X, D) is also pseudo-toric by Lemma 6.3(7), and $\boldsymbol{\delta}(X, D) = \boldsymbol{\delta}(Y, D_Y) = 1$ by Lemma 2.27.

Conversely, if (X, D) is a pseudo-toric surface of defect one, then, by Lemma 4.23, there exist a toroidal blowing up $f\colon (Y, D_Y) \to (X, D)$ and a fibration $\pi\colon Y \to T$ to a non-singular curve T such that $D_Y = f^{-1}(D)$ contains at least two fibers of π. Here, (Y, D_Y) is also a pseudo-toric surface of defect one by Lemmas 6.3(4) and 2.27. Thus, (Y, D_Y) and π satisfy the condition of Lemma 5.2(A), and by Proposition 5.5, there exist a toric surface (Z, D_Z) and a tangential blowing up $g\colon Y \to Z$ of (Z, D_Z) such that D_Y is the proper transform of D_Z. Q.E.D.

Remark. Let (X, D) be a pair of a normal Moishezon surface X and a reduced divisor D on X which satisfies the conditions of Definition 6.1 except (iv). If $\boldsymbol{c}(X, D) \le 1$, then D is big by Proposition 4.8(1), and (X, D) is a pseudo-toric surface. Thus, if $\boldsymbol{\delta}(X, D) = 1$, then (X, D) is a pseudo-toric surface of defect one by Proposition 4.8(4), and in this case, we have $\boldsymbol{\delta}(X, D) = \boldsymbol{c}(X, D)$, or equivalently, $\boldsymbol{\rho}(X) = \boldsymbol{r}(D)$.

Lemma 6.6. *Let (X, D) be a pseudo-toric surface of defect one. Let $f\colon (Y, D_Y) \to (X, D)$ be a toroidal blowing up and $\pi\colon Y \to T \simeq \mathbb{P}^1$ a $\mathbb{P}^1$-fibration such that $D_Y = f^{-1}(D)$ contains the fibers of π over two distinct points P_1 and P_2 of T. The existence of f and π is shown in Lemma* 4.23*. Then, the morphism*

$$h := (\pi \circ f^{-1})|_{X \setminus D} \colon X \setminus D \simeq Y \setminus D_Y \to T \setminus \{P_1, P_2\}$$

induces a group isomorphism

$$h^* \colon \mathcal{O}(T \setminus \{P_1, P_2\})^\star \xrightarrow{\simeq} \mathcal{O}(X \setminus D)^\star.$$

In particular, h is uniquely determined up to isomorphism, the rational map $\pi \circ f^{-1}\colon X \dashrightarrow T$ is independent of the choice of f and π up to birational equivalence, and $\mathcal{O}(X \setminus D)^\star \simeq \mathbb{C}^\star \times \mathbb{Z}$.

Proof. Since $\boldsymbol{c}(X, D) = 1$, the kernel of the class map $\mathrm{cl}_D \colon \mathsf{F}(D) \otimes \mathbb{R} \to \mathsf{N}(X)$ is one-dimensional, and hence, the kernel of $\mathrm{cl}_D^{\mathbb{Z}} \colon \mathsf{F}(D) \to \mathrm{CL}(X)$ is of rank one. Therefore, the divisor $\Theta_1 - \Theta_2$ in the proof of Lemma 4.23 is essentially unique, and indeed, $\pi^*(P_1 - P_2)$ is a generator of $\mathrm{Ker}(\mathrm{cl}_D^{\mathbb{Z}})$. Thus, we have a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{O}(T)^\star \arrow[r] \arrow[d, "\simeq"'] & \mathcal{O}(T \setminus \{P_1, P_2\})^\star \arrow[r] \arrow[d, "h^*"'] & \mathbb{Z} \arrow[r] \arrow[d, "\simeq"] & 0 \\
1 \arrow[r] & \mathcal{O}(X)^\star \arrow[r] & \mathcal{O}(X \setminus D)^\star \arrow[r] & \mathrm{Ker}(\mathrm{cl}_D^{\mathbb{Z}}) \arrow[r] & 0
\end{tikzcd}$$

of exact sequences by Lemma 2.25, and the middle homomorphism h^* is an isomorphism. Q.E.D.

Definition 6.7. For a pseudo-toric surface (X, D) of defect one, let $f\colon (Y, D_Y) \to (X, D)$ be a toroidal blowing up with a $\mathbb{P}^1$-fibration $\pi\colon Y \to T \simeq \mathbb{P}^1$ such that $D_Y = f^{-1}(D)$ contains two fibers of π. Then, by Proposition 5.5, we have two rational curves Γ_1 and Γ_2 on Y such that $\Gamma_1 + \Gamma_2$ is a unique reducible fiber of π outside the two fibers contained in D. For $i = 1$ and 2, we define L_i to be the image $f(\Gamma_i)$. The union $L_1 + L_2$ is denoted by L.

We have the following immediately from Proposition 5.5 and Lemma 6.6:

Lemma 6.8. (1) *The union $L = L_1 + L_2$ is a linear chain of rational curves, and is independent of the choice of f and π.*
(2) *The pair (X, L) is toroidal along $L \setminus D$.*
(3) *The intersection point P_L of L_1 and L_2 is not contained in D.*
(4) *If $X \setminus D$ is singular, then P_L is the unique singular point of $X \setminus D$ and it is a rational double point of type A_k for some $k \geq 1$.*

The rational curves L_1 and L_2 have the following characterization:

Proposition 6.9. *Let (X, D) be a pseudo-toric pair of defect one and let $\nu \colon X' \to X$ be an arbitrary toroidal blowing up with respect to (X, D). If C is a negative curve on X' not contained in $\nu^{-1}(D)$, then $\nu(C) = L_1$ or L_2.*

Proof. The pair (X', D') for $D' = \nu^{-1}(D)$ is also a pseudo-toric surface of defect one by Lemmas 6.3(4) and 2.27. Let $f \colon Y \to X$ be the toroidal blowing up in Definition 6.7. Then, there is a toroidal blowing up $Y' \to X'$ with respect to (X', D') such that the induced rational map $Y' \to Y$ is also a toroidal blowing up with respect to (Y, D_Y). By replacing X' with Y' and replacing C with the proper transform in Y', we may assume that $\nu = f \circ \tau$ for a toroidal blowing up $\tau \colon X' \to Y$ with respect to (Y, D_Y). By Proposition 5.5, $\tau^{-1}(\Gamma_1 + \Gamma_2)$ is a unique reducible fiber of $\pi \circ \tau$ outside $\tau^{-1}(F_1 \cup F_2)$. Then, $C = \tau^{-1}(\Gamma_1)$ or $\tau^{-1}(\Gamma_2)$ by Proposition 5.5(4) applied to $\pi \circ \tau \colon X' \to T$. Therefore, $\nu(C) = L_1$ or L_2. Q.E.D.

We present an example of $L_1 + L_2$ for a simple pseudo-toric surface.

Example 6.10. For the projective plane $X = \mathbb{P}^2$, let $D = D_1 + D_2$ be a union of a line D_1 and a conic D_2 such that $D_1 \cap D_2$ consists of two points P_1 and P_2. Then, (X, D) is pseudo-toric of defect one, and the linear chain $L_1 + L_2$ is just the union of tangent lines of D_2 at the two points P_1 and P_2. In fact, we can take a toroidal blowing up $f \colon Y \to X$ as two-times blowings up at each point of $\{P_1, P_2\}$ so that f resolves the indeterminacy of the pencil generated by $2D_1$ and D_2. For $i = 1$, 2, the proper transform Γ_i in Y of the tangent line L_i of D_2 at P_i is a (-1)-curve, and the union $\Gamma_1 + \Gamma_2$ is a fiber of the fibration $Y \to \mathbb{P}^1$ associated with the pencil.

Proposition 6.11. *For a pseudo-toric surface (X, D) of defect one, the group $\mathrm{Aut}(X; D)$ of automorphisms of X preserving each irreducible component of D is isomorphic to the multiplicative group $\mathbb{C}^\star = \mathbb{C} \setminus \{0\}$.*

Proof. Let $f\colon (Y, D_Y) \to (X, D)$ be a toroidal blowing up. Then, any automorphism in $\mathrm{Aut}(X; D)$ lifts to $\mathrm{Aut}(Y; D_Y)$, and conversely, any automorphism in $\mathrm{Aut}(Y; D_Y)$ descends to $\mathrm{Aut}(X; D)$. Therefore, $\mathrm{Aut}(Y; D_Y) \simeq \mathrm{Aut}(X; D)$. Hence, by replacing (X, D) by (Y, D_Y) in Definition 6.7, we may assume that there is a $\mathbb{P}^1$-fibration $\pi\colon X \to T \simeq \mathbb{P}^1$ such that D contains two fibers $F_1 = \pi^{-1}(P_1)$ and $F_2 = \pi^{-1}(P_2)$. Then, $L_1 + L_2$ is just $\Gamma_1 + \Gamma_2$ in Definition 6.7. For any $\sigma \in \mathrm{Aut}(X; D)$, we have $\sigma(\Gamma_i) = \Gamma_i$ for $i = 1$, 2 by the uniqueness of $L_1 + L_2$ shown in Proposition 6.9 and by the uniqueness of the irreducible component of D meeting Γ_i for each i. Hence, $\mathrm{Aut}(X; D) = \mathrm{Aut}(X; D + \Gamma_1 + \Gamma_2)$.

Let $g\colon X \to Z$ be the contraction morphism of Γ_1. Then, by Proposition 5.5, (Z, D_Z) is a toric surface for $D_Z = g_*(D)$ and the induced fibration $\bar{\pi}\colon Z \to T$ is a toric morphism $(Z, D_Z) \to (T, P_1 + P_2)$. We set $\overline{F} := g(\Gamma_2)$, which is a fiber of $\bar{\pi}$. Since g is also a toroidal blowing up with respect to $(Z, D_Z + \overline{F})$, we have an isomorphism

$$\mathrm{Aut}(X; D + \Gamma_1 + \Gamma_2) \simeq \mathrm{Aut}(Z; D_Z + \overline{F})$$

by the same argument as above. For the toric surface (Z, D_Z), it is well known that $\mathrm{Aut}(Z; D_Z)$ is isomorphic to the group of $\mathbb{C}$-rational points of the open torus $Z \setminus D_Z$, which is isomorphic to $(\mathbb{C}^\star)^2$. Now, the toric morphism $\bar{\pi}$ induces a projection

$$(\mathbb{C}^\star)^2 \simeq \mathrm{Aut}(Z; D_Z) \to \mathrm{Aut}(T; P_1 + P_2) \simeq \mathbb{C}^\star$$

and $\mathrm{Aut}(Z; D_Z + \overline{F})$ is considered as the preimage of $\mathrm{Aut}(T; P_1 + P_2 + P_3) = \{\mathrm{id}_T\}$ for the point $P_3 = \bar{\pi}(\overline{F})$. Hence, $\mathrm{Aut}(Z; D_Z + \overline{F}) \simeq \mathbb{C}^\star$, and consequently, $\mathrm{Aut}(X; D) \simeq \mathbb{C}^\star$. Q.E.D.

Lemma 6.12. *For a pseudo-toric surface (X, D) of defect one, the complement $X \setminus D$ is an affine surface with the coordinate ring isomorphic to*

$$\mathbb{C}[\mathtt{x}, \mathtt{s}, \mathtt{t}, \mathtt{t}^{-1}]/(\mathtt{sx} - (\mathtt{t} - 1)^{k+1})$$

for some $k \geq 0$. Moreover, the following hold:

(1) *Let $f\colon (Y, D_Y) \to (X, D)$ be the toroidal blowing up in Definition 6.7 and let $g\colon Y \to Z$ be the contraction morphism of Γ_1. We set $D_Z^\sharp := g_*(D_Y - C_1)$ for the irreducible component C_1 of D_Y intersecting Γ_1. Then, the morphism $X \setminus D \simeq Y \setminus D_Y \to Z \setminus D_Z^\sharp$ of affine surfaces induced by $g \circ f^{-1}$ is associated with the natural injective ring homomorphism*

$$\mathbb{C}[\mathtt{s}, \mathtt{t}, \mathtt{t}^{-1}] \to \mathbb{C}[\mathtt{x}, \mathtt{s}, \mathtt{t}, \mathtt{t}^{-1}]/(\mathtt{sx} - (\mathtt{t} - 1)^{k}).$$

In particular, the morphism h in Lemma 6.6 *is associated with the natural injective ring homomorphism*

$$\mathbb{C}[\mathtt{t}, \mathtt{t}^{-1}] \to \mathbb{C}[\mathtt{x}, \mathtt{s}, \mathtt{t}, \mathtt{t}^{-1}]/(\mathtt{sx} - (\mathtt{t} - 1)^k).$$

(2) *The action of $\theta \in \mathbb{C}^\star \simeq \mathrm{Aut}(X; D)$ on $X \setminus D$ in Proposition* 6.11 *is given by $(\mathtt{x}, \mathtt{s}, \mathtt{t}) \mapsto (\theta\mathtt{x}, \theta^{-1}\mathtt{s}, \mathtt{t})$.*

Proof. We may assume that $(Y, D_Y) = (X, D)$ and $f = \mathrm{id}_X$. Thus, there is a $\mathbb{P}^1$-fibration $\pi\colon X \to T \simeq \mathbb{P}^1$ such that $D = C_1 + C_2 + F_1 + F_2$ for two sections C_1 and C_2 of π with $C_1 \cap C_2 = \emptyset$ and for two fibers $F_1 = \pi^{-1}(P_1)$ and $F_2 = \pi^{-1}(P_2)$. For the reducible fiber $\Gamma_1 + \Gamma_2$ outside $F_1 \cup F_2$, we may assume that C_i is the unique irreducible component meeting Γ_i for $i = 1$, 2. For the toric surface (Z, D_Z), the coordinate ring of $Z \setminus D_Z$ is written as $\mathbb{C}[\mathtt{s}^{\pm 1}, \mathtt{t}^{\pm 1}]$, where the principal divisors $\mathrm{div}(\mathtt{s})$ and $\mathrm{div}(\mathtt{t})$ on Z are expressed as

$$\mathrm{div}(\mathtt{s}) = g(C_1) - g(C_2) \quad \text{and} \quad \mathrm{div}(\mathtt{t}) = g_*(F_1) - g_*(F_2).$$

In particular, $\mathtt{t}$ is the pullback of a coordinate function of $T \setminus \{P_1, P_2\} \simeq \mathbb{G}_{\mathtt{m}}$. The open subset $Z \setminus D_Z^\sharp$ is also affine and its coordinate ring A is isomorphic to $\mathbb{C}[\mathtt{s}, \mathtt{t}^{\pm 1}]$. We may assume that $g(\Gamma_2)$ is the fiber over the point: $\mathtt{t} = 1$. Then, the contraction morphism $g\colon X \to Z$ of Γ_1 is expressed as the blowing up along an ideal $(\mathtt{s}, (\mathtt{t} - 1)^{k+1})$ for some $k \geq 0$ (cf. Lemma 4.25). Thus, $X \setminus D$ is affine and its coordinate ring R is isomorphic to the degree zero part of the homogeneous coordinate ring

$$A[\mathtt{X}, \mathtt{Y}^{\pm 1}]/(\mathtt{sX} - (\mathtt{t} - 1)^{k+1}\mathtt{Y}),$$

where $\mathtt{X}$, $\mathtt{Y}$ are of degree one, and $\mathtt{s}$, $\mathtt{t}$ are of degree zero. By setting $\mathtt{x} = \mathtt{X}/\mathtt{Y}$, we have

$$R \simeq \mathbb{C}[\mathtt{x}, \mathtt{s}, \mathtt{t}^{\pm 1}]/(\mathtt{sx} - (\mathtt{t} - 1)^{k+1}).$$

The assertions (1) and (2) follow from this description and from the proof of Proposition 6.11. Q.E.D.

Proposition 6.13. *Let (X, D) be a pseudo-toric surface of defect one and let $\mu\colon M \to X$ be the minimal resolution of singularities with $D_M = \mu^{-1}(D)$.*

(1) *The divisor D_M is normal crossing, and*

$$K_M + D_M \sim 0 \quad \textit{and} \quad \bar{q}(M \setminus D_M) = 1$$

for the logarithmic irregularity $\bar{q}$ (cf. [17], [19]*).*

(2) *For the morphism h in Lemma* 6.6, *the composition*

$$h \circ (\mu|_{M \setminus D_M}) \colon M \setminus D_M \to X \setminus D \to T \setminus \{P_1, P_2\} \simeq \mathbb{G}_{\mathtt{m}}$$

is isomorphic to the quasi-Albanese map of $M \setminus D_M$ (cf. [16], [19]*).*

Proof. By Lemma 6.3(6), (M, D_M) is also a pseudo-toric surface. In particular, D_M is a normal crossing anti-canonical divisor, since M is non-singular. Moreover, μ induces an isomorphism $\mathcal{O}(X \setminus D)^{\star} \simeq \mathcal{O}(M \setminus D_M)^{\star}$. Since $q(M) = 0$, the equality $\bar{q}(M \setminus D_M) = 1$ and the assertion (2) are derived from Proposition 2.26 and Lemma 6.6. Q.E.D.

Finally in Section 6.2, we shall prove Theorem 1.6.

Proof of Theorem 1.6. The assertions (1) and (2) have been proved in Proposition 6.11 and Lemma 6.12, respectively. The assertion (3) follows from Proposition 6.13. Q.E.D.

§7. Half-toric surfaces

We introduce the notion of half-toric surface in Section 7.1 and study fundamental properties. In Section 7.2, we introduce the notion of H-surface, which is regarded as an NC-minimal completion of an open surface of type $H[-1, 0, -1]$ in the sense of Fujita [12] (cf. Remark 7.14). The H-surface is unique up to isomorphism and it is useful to describe the structure of half-toric surfaces. We have an explicit description of the involution of the characteristic double-cover of a half-toric surface in Section 7.3. Section 7.4 is devoted to prove Theorem 1.7.

7.1. Definition of half-toric surface

Definition 7.1. Let (X, D) be a pair of a normal projective surface X and a reduced divisor D. It is called a *half-toric surface* if the following conditions are satisfied:

(i) $2(K_X + D) \sim 0$ but $K_X + D \not\sim 0$;

(ii) There is a double-cover $\tau \colon V \to X$ étale in codimension one such that V is a toric surface with $D_V := \tau^{-1}(D)$ as the boundary divisor.

Lemma 7.2. *Let (X, D) and $\tau \colon V \to X$ be as in Definition* 7.1*.*

(1) *The pair (X, D) is log-canonical and $K_V + D_V = \tau^*(K_X + D)$.*

(2) *The divisor D is big and is a linear chain of rational curves.*

(3) *The open subset $X \setminus D$ is non-singular and affine.*

(4) *Let E_1 and E_2 be end components of D and set $\Sigma_i = E_i \cap (\operatorname{Sing} X \setminus \operatorname{Sing} D)$ for $i = 1, 2$. Then, τ is étale over $X \setminus (\Sigma_1 \cup \Sigma_2)$ and one of the cases* (a) *and* (b) *of Proposition* 4.18 *occurs for each Σ_i.*

(5) *For any isomorphism $\sigma \colon \mathcal{O}_X(2(K_X + D)) \xrightarrow{\simeq} \mathcal{O}_X$, one has an isomorphism*

$$V \simeq V(\mathcal{O}_X(K_X + D), \sigma)$$

over X (cf. Definition 4.9*), and an isomorphism $\eta \colon \mathcal{O}_V(K_V + D_V) \xrightarrow{\simeq} \mathcal{O}_V$ such that $\eta^{\otimes 2} = \tau^*(\sigma)$ via the canonical isomorphism $\tau^*\mathcal{O}_X(2(K_X + D)) \simeq \mathcal{O}_V(2(K_V + D_V))$.*

Proof. Since τ is étale in codimension one, (1) is a consequence of Corollary 3.20. The divisor D is big and connected, since so is $D_V = \tau^{-1}(D)$. Hence, D is either a cyclic chain of rational curves or a linear chain of rational curves by Lemma 4.5. If D is a cyclic chain, then $K_X + D \sim 0$ by Lemma 4.7; this is a contradiction. Thus, D is a linear chain of rational curves, and this proves (2). The affineness of $X \setminus D$ follows from that of $V \setminus D_V$ by Chevalley's theorem (cf. EGA II, 6.7.1). We shall prove the rest of (3) assuming (5). If $P \in \operatorname{Sing} X \setminus D$, then P is an A_1-singular point of X, since τ is étale in codimension one. Then, $K_X + D$ is Cartier at P, and it implies that $V(\mathcal{O}_X(K_X + D), \sigma) \to X$ is étale over P by Remark 4.15(4). This is a contradiction. Thus, (3) is proved by assuming (5). The assertion (4) is a consequence of (5) and Proposition 4.18. Hence, it remains to prove (5).

By Lemma 4.14, $V \simeq V(\mathcal{L}, \sigma)$ for a reflexive sheaf $\mathcal{L}$ of rank one on X and a homomorphism $\sigma \colon \mathcal{L}^{\otimes 2} \to \mathcal{O}_X$ satisfying the conditions (i) and (ii) of Lemma 4.14. Thus, in order to prove (5), it suffices to show that $\mathcal{L} \simeq \mathcal{O}_X(K_X + D)$ by Remark 4.15(3), where the existence of η follows from Remark 4.15(1). By (1) and by $K_V + D_V \sim 0$, we have an isomorphism

$$\tau_*\mathcal{O}_V \simeq \tau_*\mathcal{O}_V(K_V + D_V) \simeq (\tau_*\mathcal{O}_V \otimes_{\mathcal{O}_X} \mathcal{O}_X(K_X + D))^{\vee\vee}.$$

Since $\pi_*\mathcal{O}_V \simeq \mathcal{O}_X \oplus \mathcal{L}$, it induces

$$\mathcal{O}_X \oplus \mathcal{L} \simeq \mathcal{O}_X(K_X + D) \oplus (\mathcal{L} \otimes \mathcal{O}_X(K_X + D))^{\vee\vee}.$$

On the other hand, $\mathrm{H}^0(X, \mathcal{O}_X(K_X + D)) = 0$ by Definition 7.1(i). Hence,

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X(K_X + D)) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{L}, (\mathcal{L} \otimes \mathcal{O}_X(K_X + D))^{\vee\vee}) = 0.$$

Therefore, $\mathcal{L} \simeq \mathcal{O}_X(K_X + D)$, and we have proved (5). Q.E.D.

Lemma 7.3. *Let (X, D) be a log-canonical pair of a normal projective surface X and a reduced divisor D such that D is a big reducible linear chain of rational curves and $2(K_X + D) \sim 0$. Let $V = V(\mathcal{O}_X(K_X + D), \sigma) \to X$ be the double-cover étale in codimension one associated with an isomorphism $\sigma\colon \mathcal{O}_X(2(K_X + D)) \xrightarrow{\simeq} \mathcal{O}_X$. Then, (V, D_V) is a pseudo-toric surface for $D_V = \tau^{-1}(D)$. Here, (V, D_V) is a toric surface if and only if (X, D) is a half-toric surface.*

Proof. By Proposition 4.18, we see that (V, D_V) is log-canonical, $K_V + D_V \sim 0$, and D_V is a reducible cyclic chain of rational curves. Here, $D_V = \tau^{-1}(D)$ is big. Thus, X is a rational surface with only rational singularities by Lemma 4.7. Therefore, (V, D_V) is a pseudo-toric surface (cf. Definition 6.1). The last assertion follows from Definition 7.1 and Lemma 7.2(5). Q.E.D.

Definition. For a half-toric surface (X, D), by Lemma 7.2(5), the double-cover $\tau\colon V \to X$ in Definition 7.1 is unique up to isomorphism over X. The double-cover τ or the pair $(V, \tau^{-1}(D))$ is called a *characteristic double-cover* of (X, D).

Lemma 7.4. *For a half-toric surface (X, D), the following hold:*

(1) *Let $f\colon Y \to X$ be a birational morphism from another normal projective surface Y such that (Y, D_Y) is log-canonical and $2(K_Y + D_Y) \sim 0$ for $D_Y = f^{-1}(D)$. Then, (Y, D_Y) is also a half-toric surface.*

(2) *Let $f\colon Y \to X$ be a toroidal blowing up with respect to (X, D). Then, (Y, D_Y) is also a half-toric surface for $D_Y = f^{-1}(D)$.*

(3) *Let $g\colon X \to \overline{X}$ be a birational morphism of normal Moishezon surface $\overline{X}$. If g-exceptional locus is contained in D, then $(\overline{X}, \overline{D})$ is also a half-toric surface for $\overline{D} = g_*(D)$.*

Proof. Let $\tau\colon V \to X$ be a characteristic double cover of (X, D). We fix an isomorphism $\sigma\colon \mathcal{O}_X(2(K_X+D)) \simeq \mathcal{O}_X$. Then, V is isomorphic to $V(\mathcal{O}_X(K_X + D), \sigma)$ by Lemma 7.2(5).

(1) and (2): It suffices to prove (1), since (2) is a special case of (1). Now $2(K_Y + D_Y) = f^*(2(K_X + D)) \sim 0$, but $K_Y + D_Y \not\sim 0$, since $f_*(K_Y + D_Y) \sim K_X + D \not\sim 0$. For the induced isomorphism $f^*(\sigma)\colon \mathcal{O}_Y(2(K_Y + D_Y)) \simeq \mathcal{O}_Y$, we have a double-cover

$$\lambda\colon W := V(\mathcal{O}_Y(K_Y + D_Y), f^*(\sigma)) \to Y$$

étale in codimension one by Lemma 4.14. Here, (W, D_W) is log-canonical for $D_W := \lambda^{-1}(D_Y)$ by Corollary 3.20, and $K_W + D_W \sim 0$ by Remark 4.15(1). Since $K_Y + D_Y$ is f-numerically trivial, $f_*\mathcal{O}_Y(K_Y + D_Y)$

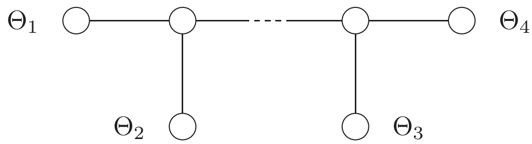

Fig. 3. Dual graph of D_M in Proposition 7.5

is a reflexive sheaf on X, and hence, is isomorphic to $\mathcal{O}_X(K_X + D)$. Thus, we have an isomorphism

$$\tau_*\mathcal{O}_V = \mathcal{O}_X \oplus \mathcal{O}_X(K_X + D) \to f_*\lambda_*\mathcal{O}_W = f_*(\mathcal{O}_Y \oplus \mathcal{O}_Y(K_Y + D_Y))$$

of $\mathcal{O}_X$-algebras, and it induces a morphism $h\colon W \to V$ such that $f \circ \lambda = \tau \circ h$. Then, h is a toroidal blowing up with respect to (V, D_V), since $D_W = h^{-1}(D)$ and $K_W + D_W = h^*(K_V + D_V) \sim 0$. Therefore, (W, D_W) is a toric surface by Corollary 4.22, and consequently, (Y, D_Y) is a half-toric surface.

(3): Now $2(K_{\overline{X}} + \overline{D}) = g_*(2(K_X + D)) \sim 0$, and $K_X + D = g^*(K_{\overline{X}} + \overline{D})$. Hence, $K_{\overline{X}} + \overline{D} \not\sim 0$. Let $V \to \overline{V} \to \overline{X}$ be the Stein factorization of $\tau \circ g$. Then, $(\overline{V}, D_{\overline{V}})$ is a toric surface for the image $D_{\overline{V}}$ of D_V by Lemma 3.9, since the exceptional locus of $V \to \overline{V}$ is contained in $D_V = \tau^{-1}(D)$. The surface $\overline{X}$ is projective, since the induced morphism $\bar{\tau}\colon \overline{V} \to \overline{X}$ is finite and $\overline{V}$ is projective. Moreover, $\bar{\tau}$ is étale in codimension one and $\tau^{-1}(\overline{D}) = D_{\overline{V}}$. Therefore, $(\overline{X}, \overline{D})$ is also a half-toric surface. Q.E.D.

Proposition 7.5. *Let (X, D) be a half-toric surface and let $\mu\colon M \to X$ be the minimal resolution of singularities. Then, $D_M = \mu^{-1}(D)$ is a simple normal crossing divisor consisting of rational curves and its dual graph is expressed as in Figure 3. Here, the four end components Θ_1, $\ldots \Theta_4$ are (-2)-curves satisfying*

$$2(K_M + D_M) \sim \sum\nolimits_{i=1}^{4} \Theta_i.$$

Let $g\colon M \to \overline{M}$ be the contraction morphism of these four end components. Then, $\mu = \bar{\mu} \circ g$ for a birational morphism $\bar{\mu}\colon \overline{M} \to X$, and $(\overline{M}, D_{\overline{M}})$ is a half-toric surface for $D_{\overline{M}} = g_*(D_M) = \bar{\mu}^{-1}(D)$.

Proof. By Lemma 7.2, μ is an isomorphism on $X \setminus D$ and is a toroidal blowing up with respect to (X, D) at least over an open neighborhood of $\operatorname{Sing} X \cap \operatorname{Sing} D$. The set of singular points of X lying on $D \setminus \operatorname{Sing} D$ is $\Sigma_1 \cup \Sigma_2$ for the sets Σ_1 and Σ_2 in Lemma 7.2(4) and the singularities are described as in Proposition 4.18. Therefore, $D_M = \mu^{-1}(D)$ is a simple normal crossing divisor consisting of rational curves with the dual graph above. Here, the end components $\Theta_1, \ldots, \Theta_4$ are (-2)-curves lying over $\Sigma_1 \cup \Sigma_2$. Since μ is a toroidal blowing up over $X \setminus (\Sigma_1 \cup \Sigma_2)$, we can write

$$2(K_M + D_M) - \mu^*(2(K_X + D)) = \sum\nolimits_{i=1}^4 a_i \Theta_i + \sum b_j \Gamma_j$$

for the irreducible components Γ_j in $\mu^{-1}(\Sigma_1 \cup \Sigma_2)$ not contained in $\Theta := \sum_{i=1}^4 \Theta_i$ and for some integers a_i and b_j. By the information of the dual graph of D_M, we have

$$(K_M + D_M)\Theta_i = -1 \quad \text{and} \quad (K_M + D_M)\Gamma_j = \begin{cases} 1, & \text{if } \Gamma_j \cap \Theta \neq \emptyset, \\ 0, & \text{if } \Gamma_j \cap \Theta = \emptyset. \end{cases}$$

for any $1 \le i \le 4$ and j. This implies that $a_i = 1$ for any i and $b_j = 0$ for any j, and hence, we have the required linear equivalence relation on $\sum_{i=1}^4 \Theta_i$.

Since Θ is μ-exceptional, μ factors through the contraction morphism $g \colon M \to \overline{M}$ of Θ. Let $\bar{\mu}$ be the induced birational morphism $\overline{M} \to X$ and set $D_{\overline{M}} := g_*(D_M) = \bar{\mu}^{-1}(D)$. Then, $D_{\overline{M}}$ is a linear chain of rational curves, $(\overline{M}, D_{\overline{M}})$ is log-canonical, and $2(K_{\overline{M}} + D_{\overline{M}}) = \bar{\mu}^*(2(K_X + D)) \sim 0$. Thus, $(\overline{M}, D_{\overline{M}})$ is a half-toric surface by Lemma 7.4(1). Q.E.D.

Proposition 7.6. *In the situation of Lemma* 7.3*, if* $\boldsymbol{\delta}(X, D) = 1$*, then* (X, D) *is a half-toric surface.*

Proof. By Lemma 4.23, there exist a toroidal blowing up $f \colon Y \to X$ with respect to (X, D) and a $\mathbb{P}^1$-fibration $\pi \colon Y \to T \simeq \mathbb{P}^1$ such that $D_Y = f^{-1}(D)$ contains two distinct fibers of π. Here, (Y, D_Y) satisfies the condition of Lemma 7.3, and $\boldsymbol{\delta}(Y, D_Y) = \boldsymbol{\delta}(X, D) = 1$ by Lemma 2.27. By Lemma 7.4(3), we may replace X with Y. Then, (X, D) and $\pi \colon X \to T$ are as in the situation of Section 5, and we have the case (B) in Lemma 5.2. Since $\boldsymbol{\delta}(X, D) = 1$ and $2(K_X + D) \sim 0$, the pair (X, D) is a half-toric surface by Proposition 5.9. Q.E.D.

7.2. An H-surface and a half-toric surface

Definition 7.7. Let S be a non-singular projective rational surface and let D_S be a reduced simple normal crossing divisor. If D_S has an

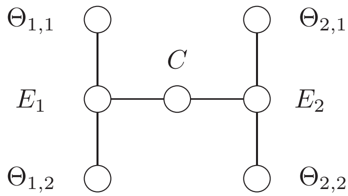

Fig. 4. Dual graph of D_S in Definition 7.7

irreducible decomposition

$$D_S = C + E_1 + E_2 + \Theta_{1,1} + \Theta_{1,2} + \Theta_{2,1} + \Theta_{2,2}$$

with the dual graph in Figure 4 and if the following four conditions are satisfied, then (S, D_S) is called a *pre H-surface*:

(i) C is a non-singular rational curve;
(ii) E_1 and E_2 are (-1)-curves;
(iii) $\Theta_{i,j}$ are all (-2)-curves for any $1 \leq i, j \leq 2$;
(iv) There is no (-1)-curve in $S \setminus D_S$.

If $C^2 = 0$ holds in addition, then (S, D_S) is called an *H-surface.*

Lemma 7.8. *For the pre H-surface (S, D_S) above, there is a linear equivalence relation*

$$\text{(VII-1)} \qquad 2(K_S + D_S) \sim \Theta := \sum\nolimits_{1 \leq i,j \leq 2} \Theta_{i,j}.$$

For $i = 1$, 2, we set

$$F_i := 2E_i + \Theta_{i,1} + \Theta_{i,2}.$$

Then, there exists a $\mathbb{P}^1$-fibration $\pi \colon S \to T \simeq \mathbb{P}^1$ such that

- *F_i is a scheme theoretic fiber of π for any $i = 1$, 2,*
- *C is a double-section of π, and*
- *the double-cover $C \to T$ is branched at two points $P_1 := \pi(F_1)$ and $P_2 := \pi(F_2)$.*

If π is smooth outside $F_1 \cup F_2$, then (S, D_S) is an H-surface.

Proof. By the information of the dual graph of D_S, for each $i = 1$, 2, we have a $\mathbb{P}^1$-fibration $S \to T \simeq \mathbb{P}^1$ such that F_i is the scheme-theoretic fiber. Here, we have the common fibration π, since $F_1 \cap F_2 = \emptyset$ and $CF_1 = CF_2 = 2$. In particular, $\pi|_C \colon C \to T$ is a double-cover branched at $\{P_1, P_2\}$, since $C \cap F_i$ is a point as a set.

Next, we shall show (VII-1). We set

$$L := K_S + C + E_1 + E_2 + \frac{1}{2}\Theta = K_S + C + \frac{1}{2}(F_1 + F_2).$$

Then, (VII-1) is equivalent to the numerical equivalence relation $L \approx 0$, since S is rational. We have $LG = 0$ for any irreducible component G of D_S by a direct calculation from the information of the dual graph of D_S. Since D_S is big, if L is nef, then $L \approx 0$ by the Hodge index theorem. Thus, for the proof of (VII-1), it is enough to derive a contradiction assuming that L is not nef. In this situation, there is an extremal curve Γ with $L\Gamma < 0$ by the cone theorem (cf. Theorem 2.19). Note that $\boldsymbol{\rho}(X) > 2$, since we can contract $E_1 + \Theta_{1,1}$ to a non-singular point. Thus, Γ is a (-1)-curve satisfying

$$(C + E_1 + E_2) \cap \Gamma = \emptyset \quad \text{and} \quad \Theta\Gamma \leq 1.$$

Here, if $\Theta\Gamma > 0$, then $\Theta F_1 = \Theta F_2 > 0$, and it implies that $\Theta\Gamma \geq 2$. Therefore, $\Theta \cap \Gamma = \emptyset$ and this contradicts the condition (iv) of Definition 7.7. Thus, (VII-1) has been proved.

The last assertion is shown as follows. Suppose that π is smooth outside $F_1 \cup F_2$. It is enough to prove: $C^2 = 0$. Let $g\colon S \to Z$ be the contraction morphism of $E_1 + \Theta_{1,1} + E_2 + \Theta_{2,1}$. Then, the induced $\mathbb{P}^1$-fibration $\pi_Z\colon Z \to T$ is smooth. Hence, Z is isomorphic to the Hirzebruch surface $\mathbb{F}_n = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n))$ for some $n \geq 0$. Here, the image $C_Z := g(C)$ is isomorphic to C over T satisfying $C_Z^2 = C^2 + 4$, and the image $F_{i,Z} := g(\Theta_{i,2})$ is the fiber over P_i for $i = 1$, 2. Hence,

$$K_Z + C_Z + F_Z \sim 0$$

for a fiber F_Z of π_Z by (VII-1). This implies that $C_Z^2 = 4$, since $K_Z^2 = 8$ and $K_Z F_Z = -2$. Therefore, $C^2 = 0$, and (S, D_S) is an H-surface. Q.E.D.

We can construct an H-surface from a plane conic with two tangent lines as follows.

Example 7.9. Let C_0 be a non-singular plane conic, and let L_1 and L_2 be two tangent lines to C_0. Let Q_0 be the point $L_1 \cap L_2$, and let Q_i be the point $C_0 \cap L_i$ for $i = 1$, 2. Let $f\colon S_1 \to \mathbb{P}^2$ be the blowing up at the three points Q_i and set $B_i = f^{-1}(Q_i)$ for $i = 1$, 2. Let C_1, L_1', and L_2' be the proper transforms of C_0, L_1, and L_2 in S_1, respectively. Then, L_i' and B_i are (-1)-curves intersecting at a point Q_i', and C_1 intersects $L_i' + B_i$ at Q_i', for each $i = 1$, 2. Moreover, $(L_1' + B_1) \cap (L_2' + B') = \emptyset$ and $C_1^2 = 2$. Let $g\colon S \to S_1$ be the blowing up at $\{Q_1', Q_2'\}$ and let D_S

be the union of the proper transform C of C_1, the exceptional divisor $E_i = g^{-1}(Q'_i)$, the proper transform $\Theta_{i,1}$ of L'_i, and the proper transform $\Theta_{i,2}$ of B_i for $i = 1, 2$. Then, (S, D_S) is an H-surface, since D_S have the same dual graph as in Definition 7.7, $S \setminus D_S$ is affine, and $C^2 = 0$.

We can also construct an H-surface from a certain double-section of a Hirzebruch surface, as follows.

Example 7.10. Let $p \colon \mathbb{F}_n \to \mathbb{P}^1$ be the ruling of the Hirzebruch surface $\mathbb{F}_n = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n))$ and let C_0 be a non-singular curve linearly equivalent to $-K_{\mathbb{F}_n} - F$ for a fiber F of p. Such C_0 exists only when $n \leq 1$, and in this case, we have $C_0 \simeq \mathbb{P}^1$ and $C_0^2 = 4$. Then, there exist exactly two fibers L_1 and L_2 which intersect C_0 tangentially. Let $f \colon S_1 \to \mathbb{F}_n$ be the blowing up at the two points $C_0 \cap (L_1 \cup L_2)$. Note that S_1 is isomorphic to the surface S_1 in Example 7.9 when $n = 1$. Let B_i be the exceptional curve $f^{-1}(C_0 \cap L_i)$ for $i = 1, 2$. Then, the proper transform L'_i of L_i is a (-1)-curve intersecting B_i transversely at a point Q'_i, and the proper transform C_1 of C_0 intersects $L'_i + B_i$ at Q'_i, for $i = 1$, 2. Here, $C_1^2 = 2$. Let $g \colon S \to S_1$ be the blowing up at $\{Q'_1, Q'_2\}$. Then, (S, D_S) is an H-surface for the union D_S of the proper transform C of C_1, the g-exceptional divisor $E_i = g^{-1}(Q'_i)$, the proper transform $\Theta_{i,1}$ of L'_i, and the proper transform $\Theta_{i,2}$ of B_i, for $i = 1, 2$. In fact, D_S has the same dual graph as in Definition 7.7, $S \setminus D_S$ is affine, and $C^2 = 0$.

Lemma 7.11. *Every H-surface (S, D_S) is isomorphic to the H-surface obtained in Example* 7.9. *In particular, $S \setminus D_S$ is affine, the morphism π in Lemma* 7.8 *is smooth outside $F_1 \cup F_2$, and moreover, π induces a $\mathbb{G}_{\mathtt{m}}$-fiber bundle $S \setminus D_S \to T \setminus \{P_1, P_2\}$.*

Proof. Since $C^2 = 0$ and $C \simeq \mathbb{P}^1$, there is a fibration $\varpi \colon S \to E \simeq \mathbb{P}^1$ such that C is a smooth fiber of ϖ. Here, E_1 and E_2 are sections of π and the four (-2)-curves $\Theta_{i,j}$ are all contained in fibers of ϖ, since $E_1C = E_2C = 1$ and $\Theta_{i,j}C = 0$. We can take an irreducible component Δ_1 of the fiber of ϖ containing $\Theta_{1,1}$ such that $\Theta_{1,1} \cap \Delta_1 \neq \emptyset$. We set $d_1 := \Theta_{1,1}\Delta_1 > 0$. Then, $d_1 = F_1\Delta_1 = F_2\Delta_1$, and

$$0 = (K_S + C + \frac{1}{2}(F_1 + F_2))\Delta_1 = K_S\Delta_1 + d_1$$

by (VII-1). Since $\Delta_1^2 < 0$, we see that Δ_1 is a (-1)-curve and $d_1 = 1$. Moreover, $F_2\Delta_1 = 1$ implies that $E_2 \cap \Delta_1 = \emptyset$ and $(\Theta_{2,1} + \Theta_{2,2})\Delta_1 = 1$. By exchanging $\Theta_{2,1}$ and $\Theta_{2,2}$ if necessarily, we may assume that $\Theta_{2,1}\Delta_1 = 1$ and $\Theta_{2,2} \cap \Delta_1 = \emptyset$. Then, $H_1 := 2\Delta_1 + \Theta_{1,1} + \Theta_{2,1}$ is a scheme-theoretic fiber of ϖ.

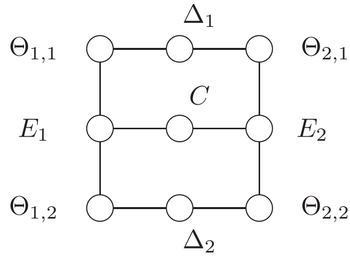

Fig. 5. Dual graph of $D_S + \Delta_1 + \Delta_2$

Let $h\colon S \to Z$ be the contraction morphism of $\Delta_1 + E_1 + \Theta_{1,2} + E_2 + \Theta_{2,2}$. We set $L_i := h_*(\Theta_{i,1})$ for $i = 1, 2$, and $C_0 := h_*(C)$. Then, Z is also a non-singular surface, and C_0, L_1, and L_2 are non-singular rational curves such that

$$L_1^2 = L_2^2 = L_1L_2 = 1, \quad C_0^2 = 4, \quad C_0L_1 = C_0L_2 = 2,$$
$$L_1 \sim L_2 \quad \text{and} \quad 2(K_Z + C_0) + L_1 + L_2 \sim 0.$$

Therefore, $K_Z^2 = 9$, and as a consequence, $Z \simeq \mathbb{P}^2$, C_0 is a conic, and L_1 and L_2 are tangent lines of C_0. Hence, (S, D_S) is obtained as in Example 7.9. We know that $S \setminus D_S$ is affine by Example 7.9. By the construction of (S, D_S) in Example 7.10, we see that π is smooth outside $F_1 \cup F_2$ and that the morphism $S \setminus D_S \to T \setminus \{P_1, P_2\}$ induced by π is a $\mathbb{G}_{\mathtt{m}}$-fiber bundle. Q.E.D.

Corollary 7.12. *For an H-surface (S, D_S), there exist (-1)-curves Δ_1 and Δ_2 on S such that*

(1) *Δ_1 and Δ_2 are sections of π,*
(2) *$(\Delta_1 + \Delta_2) \cap (E_1 + E_2 + C) = \emptyset$, and*
(3) *$(S, D_S^\sharp)$ is a toric surface for $D_S^\sharp := D_S - C + \Delta_1 + \Delta_2$.*

In particular, $D_S + \Delta_1 + \Delta_2 = D_S^\sharp + C$ has the dual graph in Figure 5 after interchanging $\Theta_{2,1}$ and $\Theta_{2,2}$ if necessarily.

Proof. By the proof of Lemma 7.11, after replacing $\Theta_{2,1}$ and $\Theta_{2,2}$ if necessary, we have (-1)-curves Δ_1 and Δ_2 on S such that Δ_i is a section of π and $H_i = 2\Delta_i + \Theta_{i,1} + \Theta_{i,2}$ is a fiber of ϖ for $i = 1, 2$. Let $k\colon S \to Y$ be the contraction morphism of $\Delta_1 + \Theta_{2,1} + \Delta_2 + \Theta_{2,2}$. Then, $\boldsymbol{\rho}(Y) = \boldsymbol{\rho}(S) - 4 = 2$, and as a consequence, the induced morphism $Y \to E \simeq \mathbb{P}^1$ is a $\mathbb{P}^1$-bundle. Let $D_Y^\sharp$ be the image of $D_S^\sharp$. Then, $D_S^\sharp = k^{-1}(D_Y^\sharp)$ and $(Y, D_Y^\sharp)$ is a toric surface. Since k is a toroidal blowing up with respect to $(Y, D_Y^\sharp)$, by Corollary 4.22, $(S, D_S^\sharp)$ is a toric surface. Q.E.D.

Corollary 7.13. *Let (S, D_S) be an H-surface. Then,*

$$\mathrm{CL}(S \setminus D_S) \simeq \mathbb{Z}/2\mathbb{Z},$$

and the $\mathbb{G}_\mathtt{m}$-bundle $S \setminus D_S \to T \setminus \{P_1, P_2\}$ in Lemma 7.11 *is isomorphic to the quasi-Albanese map of $S \setminus D_S$. In particular,*

$$\mathcal{O}(S \setminus D_S)^\star \simeq \mathbb{C}^\star \times \mathbb{Z}.$$

Proof. The class group $\mathrm{CL}(S \setminus D_S)$ is the cokernel of the class map

$$c := \mathrm{cl}^{\mathbb{Z}}_{D_S} : \mathsf{F}(D_S) \to \mathrm{CL}(S)$$

(cf. Definition 2.24). Since $(S, D_S^\sharp)$ is toric for the divisor $D_S^\sharp$ of Corollary 7.12, $\mathrm{CL}(S) \simeq \mathrm{Pic}(S) \simeq \mathbb{Z}^{\oplus 6}$ is generated by the linear equivalence classes of the irreducible components of $D_S^\sharp$. On the other hand, the image of c is generated by the linear equivalence classes of the irreducible components of D_S. There is no divisor B supported on D_S such that $BF = 1$ for a fiber F of $\pi\colon S \to T$. But $\Delta_1 F = 1$ for the divisor Δ_1 of Corollary 7.12. Thus, $\mathrm{CL}(S \setminus D_S) \neq 0$. Since $C \sim H_1 = 2\Delta_1 + \Theta_{1,1} + \Theta_{2,1}$ as in the proof of Lemma 7.11, we have $\mathrm{CL}(S \setminus D_S) \simeq \mathbb{Z}/2\mathbb{Z}$.

The kernel of c consists of principal divisors B supported on D_S. This B is a multiple of $F_1 - F_2$. In fact, $BF_1 = 0$ implies that $\mathrm{Supp}\, B \subset D_S - C = F_1 \cup F_2$, and $BC = 0$ implies that $\mathrm{Supp}(B - m(F_1 - F_2)) \subset \Theta = \sum \Theta_{i,j}$ for some $m \in \mathbb{Z}$. Here, $B = m(F_1 - F_2)$, since Θ is negative definite. By the proof of Lemma 2.25, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{O}(T)^\star \arrow[r] \arrow[d, "="'] & \mathcal{O}(T \setminus \{P_1, P_2\})^\star \arrow[r] \arrow[d, "\phi"'] & \mathsf{F}(P_1 + P_2) \arrow[r, "\mathrm{cl}^{\mathbb{Z}}_{P_1+P_2}"] \arrow[d, "\pi^*"'] & \mathrm{CL}(T) \arrow[d, "\pi^*"] \\
\mathcal{O}(S)^\star \arrow[r] & \mathcal{O}(S \setminus D_S)^\star \arrow[r] & \mathsf{F}(D_S) \arrow[r, "\mathrm{cl}^{\mathbb{Z}}_{D_S}"] & \mathrm{CL}(S)
\end{tikzcd}$$

of exact sequences related to class maps. Since $\pi^*(P_i) = F_i$ for $i = 1$, 2, the homomorphism

$$\mathrm{Ker}(\mathrm{cl}^{\mathbb{Z}}_{P_1+P_2}) \to \mathrm{Ker}(\mathrm{cl}^{\mathbb{Z}}_{D_S})$$

induced by π^* is an isomorphism by the observation above on B. Hence, the second vertical homomorphism ϕ in the diagram is an isomorphism. Note that ϕ is induced from the $\mathbb{G}_\mathtt{m}$-fiber bundle $p\colon S \setminus D_S \to T \setminus \{P_1, P_2\}$. Therefore, p is isomorphic to the quasi-Albanese map of $S \setminus D_S$ by Proposition 2.26, since $q(S) = 0$. Q.E.D.

Remark 7.14. By Lemma 7.11, we see that, for an H-surface (S, D_S), the open subset $S \setminus D_S$ is an open surface of type $H[-1, 0, -1]$ in the sense of Fujita in [12, (8.19)] and (S, D_S) is an NC-minimal completion of $S \setminus D_S$ in his sense (cf. [12, Th. (8.5), (8.9), (8.18), Table (8.64)]).

Proposition 7.15. *For an H-surface (S, D_S), let $S \to \overline{S}$ be the contraction morphism of the four end components $\Theta_{i,j}$ of D_S. Then, $(\overline{S}, D_{\overline{S}})$ is a half-toric surface for the image $D_{\overline{S}}$ of D_S, and the characteristic double-cover of $(\overline{S}, D_{\overline{S}})$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Moreover, $\boldsymbol{\delta}(\overline{S}, D_{\overline{S}}) = 1$ holds.*

Proof. Let $\tau\colon V = V(\mathcal{O}_S(-(K_S + D_S)), \sigma) \to S$ (cf. Definition 4.9) be the double-cover associated with a natural homomorphism

$$\sigma\colon \mathcal{O}_S(-(K_S + D_S))^{\otimes 2} \simeq \mathcal{O}_S(-\Theta) \hookrightarrow \mathcal{O}_S$$

induced by (VII-1). Here, V is non-singular and the branch locus of τ is Θ. Then, we have the following properties:

- $\hat{\Theta}_{i,j} := \tau^{-1}(\Theta_{i,j})$ is a (-1)-curve for any i, j.
- $\hat{E}_i := \tau^{-1}(E_i)$ is a (-2)-curve, and is a double-cover of E_i for $i = 1, 2$.
- $\tau^{-1}(C)$ is a disjoint union $\hat{C}_1 \cup \hat{C}_2$ of two copies of C.

Let $V \to \overline{V}$ be the contraction morphism of the four (-1)-curves $\hat{\Theta}_{i,j}$. Then, the induced morphism $\bar{\tau}\colon \overline{V} \to \overline{S}$ is a double-cover étale in codimension one. We set $G_{i,1} := \tau_*(\hat{E}_i)$ and $G_{i,2} := \tau_*(\hat{C}_i)$ for $i = 1, 2$. Then, $G_{i,j} \simeq \mathbb{P}^1$ and

$$G_{i,j} G_{i',j'} = \begin{cases} 1, & \text{if } i = i'; \\ 0, & \text{if } i \neq i', \end{cases}$$

for any i, i', j, and j'. Thus, $D_{\overline{V}} := \bigcup_{1 \le i,j \le 2} G_{i,j}$ is a cyclic chain of four rational curves with self-intersection number zero. Note that $D_{\overline{V}} = \bar{\tau}^{-1}(D_{\overline{S}})$ and $\overline{V} \setminus D_{\overline{V}}$ is affine, since $\overline{V} \setminus D_{\overline{V}} \simeq \tau^{-1}(S \setminus D_S)$ and $S \setminus D_S$ is affine. Therefore, $\overline{V} \simeq \mathbb{P}^1 \times \mathbb{P}^1$, and $D_{\overline{V}}$ is a union of two fibers of the first projection $\overline{V} \to \mathbb{P}^1$ and of two fibers of the second projection $\overline{V} \to \mathbb{P}^1$. In particular, $(\overline{V}, D_{\overline{V}})$ is a toric surface, and consequently, $(\overline{S}, D_{\overline{S}})$ is a half-toric surface. The equality $\boldsymbol{\delta}(\overline{S}, D_{\overline{S}}) = 1$ is derived from $\boldsymbol{\rho}(\overline{S}) = \boldsymbol{\rho}(S) - 4 = 2$ and $\boldsymbol{n}(D_{\overline{S}}) = 3$. Q.E.D.

Lemma 7.16. *Let (X, D) be a half-toric surface with a $\mathbb{P}^1$-fibration $\pi\colon X \to T \simeq \mathbb{P}^1$ such that $\boldsymbol{\delta}(X, D) = 1$ and that D contains two distinct fibers of π. Let $\mu\colon M \to X$ be the minimal resolution of singularities and set $D_M := \mu^{-1}(D)$. Then, for the H-surface (S, D_S) above, there* *is a toroidal blowing up* $f\colon M \to S$ *with respect to* (S, D_S) *such that* $D_M = f^{-1}(D_S)$.

Proof. By Lemma 5.2 and Propositions 5.7 and 5.9, D is a linear chain of rational curves expressed as $C_0 + F_1 + F_2$ for a double-section C_0 and two fibers F_1 and F_2 of π such that π is smooth outside $F_1 \cup F_2$. By the proof of Proposition 7.5, $D_M = \mu^{-1}(D)$ is a simple normal crossing divisor expressed as $C_M + F_{1,M} + F_{2,M}$ for the proper transform C_M of C_0 in M and two fibers $F_{1,M}$ and $F_{2,M}$ of $\mu \circ \pi\colon M \to T$. Here, for $i = 1$, 2, the fiber $F_{i,M}$ is written as $G_i + \Theta_{i,1} + \Theta_{i,2}$ for a linear chain G_i of rational curves and two (-2)-curves $\Theta_{i,1}$ and $\Theta_{i,2}$ such that, for an end component $G_{i,0}$ of G_i, $\Theta_{i,j} \cap G_i = \Theta_{i,j} \cap G_{i,0}$ for any $j = 1$, 2. We have $2(K_M + D_M) \sim \Theta := \sum_{i,j} \Theta_{i,j}$ by Proposition 7.5. Let $f\colon M \to N$ be the contraction morphism of $G_1 + G_2 - (G_{1,0} + G_{2,0})$. Then, f is a succession of contractions of (-1)-curves. For, if $G_{i,0}$ is a (-1)-curve, then $G_i = G_{i,0}$, since $G_{i,0} + \Theta_{i,1} + \Theta_{i,2}$ is not negative definite. Hence, N is non-singular, and $D_N = f_*(D_M)$ has the same dual graph as that of D_S in Definition 7.7. Thus, (N, D_N) is a pre H-surface. We set $F_{i,N} := f_*(F_{i,M})$ for $i = 1$, 2. Then, the $\mathbb{P}^1$-fibration $\pi_N\colon N \to T$ induced from $\pi\colon X \to T$ is smooth outside $F_{1,N} \cup F_{2,N}$ by construction. On the other hand, π_N is isomorphic to the fibration π in Lemma 7.8 defined for the pre H-surface (N, D_N), where $F_{1,N} \cup F_{2,N}$ corresponds to $F_1 \cup F_2$ in Lemma 7.8. Therefore, (N, D_N) is an H-surface by Lemma 7.8. Q.E.D.

7.3. On certain involutions of toric surfaces

We shall show that a half-toric surface is characterized as the quotient surface of a projective toric surface by a special involution.

Lemma 7.17. *Let* ι *be an involution of the two-dimensional algebraic torus* $\mathbb{T} := \operatorname{Spec} \mathbb{C}[\mathtt{t}_1^{\pm 1}, \mathtt{t}_2^{\pm 1}]$ *such that*

(i) $\iota^* \eta = -\eta$ *for the two-form* $\eta = (\mathtt{t}_1^{-1} d\mathtt{t}_1) \wedge (\mathtt{t}_2^{-1} d\mathtt{t}_2)$, *and*

(ii) *the fixed point set of* ι *contains no prime divisor on* $\mathbb{T}$.

Then, ι *is given by*

$$\iota^*(\mathtt{t}_1) = -\mathtt{t}_1 \quad \textit{and} \quad \iota^*(\mathtt{t}_2) = \mathtt{t}_2^{-1}$$

after changing the coordinate $(\mathtt{t}_1, \mathtt{t}_2)$ *of* $\mathbb{T}$. *In particular,* ι *has no fixed point.*

Proof. The induced involution $\iota^*\colon \mathbb{C}[\mathtt{t}_1^{\pm 1}, \mathtt{t}_2^{\pm 1}] \to \mathbb{C}[\mathtt{t}_1^{\pm 1}, \mathtt{t}_2^{\pm 1}]$ is given by

$$\iota^*(\mathtt{t}_1) = \lambda_1 \mathtt{t}_1^{a_1} \mathtt{t}_2^{a_2} \quad \text{and} \quad \iota^*(\mathtt{t}_2) = \lambda_2 \mathtt{t}_1^{b_1} \mathtt{t}_2^{b_2} \tag{VII-2}$$

for suitable λ_1, $\lambda_2 \in \mathbb{C}^\star = \mathbb{C} \setminus \{0\}$ and suitable integers a_1, a_2, b_1, b_2 such that the matrix

$$A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

has order at most two and

$$\lambda_1^{-1} = \lambda_1^{a_1}\lambda_2^{a_2} \quad \text{and} \quad \lambda_2^{-1} = \lambda_1^{b_1}\lambda_2^{b_2}. \tag{VII-3}$$

Then, $\det A = -1$ by (i). In particular, A has eigenvalues 1 and -1. For a $\mathbb{C}$-scheme Z, let $\mathbb{T}(Z)$ denote the set $\mathrm{Hom}_{\mathrm{Spec}\,\mathbb{C}}(Z, \mathbb{T})$ of the morphisms from Z to the two-dimensional algebraic torus $\mathbb{T}$. Then, $\mathbb{T}(Z)$ is an abelian group. We write $\mathbb{T}(\mathbb{C})$ for $\mathbb{T}(\mathrm{Spec}\,\mathbb{C})$. For an element $u \in \mathbb{T}(Z)$, let σ_u denote the (left) action of u on $\mathbb{T} \times Z = \mathbb{T} \times_{\mathrm{Spec}\,\mathbb{C}} Z$ over Z. Then,

$$\sigma_u \circ \sigma_v = \sigma_{u \cdot v}$$

for any u, $v \in \mathbb{T}(Z)$, where $\cdot$ denotes the multiplication in $\mathbb{T}(Z)$. For $u \in \mathbb{T}(\mathbb{C})$, the σ_u is an automorphism of $\mathbb{T}$, and if $(\mathtt{t}_1(u), \mathtt{t}_2(u)) = (u_1, u_2) \in (\mathbb{C}^\star)^2$, the associated ring homomorphism $\sigma_u^*\colon \mathbb{C}[\mathtt{t}_1^{\pm 1}, \mathtt{t}_2^{\pm 1}] \to \mathbb{C}[\mathtt{t}_1^{\pm 1}, \mathtt{t}_2^{\pm 1}]$ is given by

$$\sigma_u^*(\mathtt{t}_1) = u_1\mathtt{t}_1, \quad \text{and} \quad \sigma_u^*(\mathtt{t}_2) = u_2\mathtt{t}_2.$$

Let ι_A be an involution of $\mathbb{T}$ defined by

$$\iota_A^*(\mathtt{t}_1) = \mathtt{t}_1^{a_1}\mathtt{t}_2^{a_2}, \quad \text{and} \quad \iota_A^*(\mathtt{t}_2) = \mathtt{t}_1^{b_1}\mathtt{t}_2^{b_2}.$$

Then, ι_A is equivariant with respect to the action of $\mathbb{T}$, i.e.,

$$\iota_A(u \cdot v) = \iota_A(u) \cdot \iota_A(v)$$

for any u, $v \in \mathbb{T}(Z)$ for any $\mathbb{C}$-scheme Z. In particular,

$$\iota_A \circ \sigma_\nu = \sigma_{\iota_A(\nu)} \circ \iota_A \quad \text{and} \quad \iota_A \circ \sigma_{\iota_A(\nu)} = \sigma_\nu \circ \iota_A$$

for any $\nu \in \mathbb{T}(\mathbb{C})$. The relations among ι, A, and (λ_1, λ_2) above (cf. (VII-2) and (VII-3)) are translated as

$$\iota = \iota_A \circ \sigma_\lambda \quad \text{and} \quad \iota_A(\lambda) = \lambda^{-1},$$

where λ is an element of $\mathbb{T}(\mathbb{C})$ defined by $(\lambda_1, \lambda_2) = (\mathtt{t}_1(\lambda), \mathtt{t}_2(\lambda))$. In particular, the action of ι on $\mathbb{T}(\mathbb{C})$ is given by

$$\nu \mapsto \iota(\nu) = \iota_A(\lambda \cdot \nu) = \iota_A(\lambda) \cdot \iota_A(\nu) = \lambda^{-1} \cdot \iota_A(\nu).$$

Therefore, the fixed point set $\mathrm{Fix}(\iota)$ of ι is the set of element $\nu \in \mathbb{T}(\mathbb{C})$ satisfying

$$\lambda = \iota_A(\nu) \cdot \nu^{-1}. \tag{VII-4}$$

We shall show that $\mathrm{Fix}(\iota) = \emptyset$. Take an eigenvector ${}^t(p, q)$ of A with eigenvalue 1 such that p and q are integers, we define a morphism $f\colon C := \operatorname{Spec} \mathbb{C}[\mathtt{s}^{\pm 1}] \to \mathbb{T}$ by

$$f^*(\mathtt{t}_1) = \mathtt{s}^p \quad \text{and} \quad f^*(\mathtt{t}_2) = \mathtt{s}^q.$$

Then, f is a morphism of group schemes, and $\iota_A \circ f = f$ by the choice of (p, q). If $\nu \in \mathrm{Fix}(\iota)$, then the image of $\sigma_\nu \circ f\colon C \to \mathbb{T}$ is contained in $\mathrm{Fix}(\iota)$ by (VII-4). Indeed, we have:

$$\iota \circ (\sigma_\nu \circ f) = \iota_A \circ \sigma_{\lambda \cdot \nu} \circ f = \iota_A \circ \sigma_{\iota_A(\nu)} \circ f = \sigma_\nu \circ \iota_A \circ f = \sigma_\nu \circ f.$$

This is a contradiction to (ii). Hence, $\mathrm{Fix}(\iota) = \emptyset$.

Next, we shall show that

$$P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for a matrix $P \in \mathrm{SL}(2, \mathbb{Z})$ by applying Lemma 7.18 below. Assume the contrary. Then, $A \not\equiv I \bmod 2$ by Lemma 7.18(2) for the identity matrix I. By Lemma 7.18(1) applied to the multiplicative abelian group $L = \mathbb{C}^\star$, we see that the image of the homomorphism $\mathbb{T}(\mathbb{C}) \to \mathbb{T}(\mathbb{C})$ given by $\nu \mapsto \nu^{-1}\iota_A(\nu)$ is just the set of elements λ' of $\mathbb{T}(\mathbb{C})$ such that $\iota_A(\lambda') = \lambda'^{-1}$. Therefore, we have an element $\nu \in \mathbb{T}(\mathbb{C})$ satisfying $\lambda = \nu^{-1}\iota_A(\nu)$, which means that $\mathrm{Fix}(\iota) \neq \emptyset$ by (VII-4). This is a contradiction.

Therefore, by changing the coordinates $(\mathtt{t}_1, \mathtt{t}_2)$, we may assume that

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then, $\lambda_1^2 = 1$ by (VII-3). If $\lambda_1 = 1$, then the locus $\{\mathtt{t}_2 = c\} \subset \mathbb{T}$ for a constant c with $c^2 = \lambda_2$ is contained in $\mathrm{Fix}(\iota)$ by (VII-2). Thus, $\lambda_1 = -1$. By changing $\mathtt{t}_2$ again, we may assume that the equalities (VII-2) determining the action of ι is written as

$$\iota^*(\mathtt{t}_1) = -\mathtt{t}_1 \quad \text{and} \quad \iota^*(\mathtt{t}_2) = \mathtt{t}_2^{-1}.$$

Thus, we are done. Q.E.D.

The lemma below is used in the proof of Lemma 7.17 above.

Lemma 7.18. *Let A be an integral 2×2 matrix having eigenvalues 1 and -1. For the 2×2 identity matrix I, the following hold*:

(1) *For an abelian group L, let $(A \pm I)_L$ be the endomorphism $L^{\oplus 2} \to L^{\oplus 2}$ induced from $A \pm I \colon \mathbb{Z}^{\oplus 2} \to \mathbb{Z}^{\oplus 2}$ by taking tensor product with L over $\mathbb{Z}$. If $A \not\equiv I \bmod 2$, then, $\mathrm{Ker}(A - I)_L = \mathrm{Im}(A+I)_L$, where* Ker *and* Im *denote the kernel and the image, respectively.*

(2) *If $A \equiv I \bmod 2$, then*

$$P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for a matrix $P \in \mathrm{SL}(2, \mathbb{Z})$.

Proof. There is a positive integer e such that $\mathrm{Im}(A+I) = e\,\mathrm{Ker}(A-I)$, since A has eigenvalues 1 and -1. Since $A + I \equiv 0 \bmod e$ and $\mathrm{trace}(A) = 0$, we have $e = 1$ or 2. Assume that $A \not\equiv I \bmod 2$. Then $e = 1$, and it implies that

$$0 \to \mathbb{Z} \simeq \mathrm{Ker}(A + I) \to \mathbb{Z}^{\oplus 2} \xrightarrow{A+I} \mathbb{Z}^{\oplus 2} \xrightarrow{A-I} \mathrm{Im}(A - I) \simeq \mathbb{Z} \to 0$$

is an exact sequence. Since this sequence is split, its tensor product with L is also an exact sequence for any abelian group L. Thus, $\mathrm{Ker}(A-I)_L = \mathrm{Im}(A+I)_L$, and we have proved (1). Let ${}^t(p_1, p_2)$ (resp. ${}^t(q_1, q_2)$) be an integral vector generating $\mathrm{Ker}(A - I)$ (resp. $\mathrm{Ker}(A + I)$). Then,

$$P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{for} \quad P := \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix}.$$

By replacing (p_1, p_2) with $(-p_1, -p_2)$ if necessary, we may assume that $\det P > 0$. It suffices to prove $\det P = 1$ in case $A \equiv I \bmod 2$. Note that we have $e = 2$ in this case, since $\mathrm{Im}(A + I) \subset 2\mathbb{Z}^2$. The image of ${}^t(p_1, p_2)$ by $A+I$ is ${}^t(2p_1, 2p_2)$, and it generates $\mathrm{Im}(A+I) = 2\,\mathrm{Ker}(A-I)$. Therefore, ${}^t(p_1, p_2)$ and ${}^t(q_1, q_2)$ generate $\mathbb{Z}^2$, and hence, $\det P = 1$, and we have proved (2). Q.E.D.

Proposition 7.19. *Let (V, D_V) be a projective toric surface and let $\iota \colon V \to V$ be an involution such that $\iota(D_V) = D_V$. If $\iota^*\eta \neq \eta$ for a nowhere vanishing section η of $\mathcal{O}_V(K_V + D_V)$ and if the fixed point set* $\mathrm{Fix}(\iota)$ *of ι contains no prime divisor on $V \setminus D_V$, then* $\mathrm{Fix}(\iota)$ *is a finite set contained in D. In particular, (X, D) is a half-toric surface for the quotient surface X of V by ι and for the image D of D_V in X.*

Proof. Let U be the open torus $V \setminus D_V$. Then, $U \simeq \mathrm{Spec}\,\mathbb{C}[t_1^{\pm 1}, t_2^{\pm}]$, where the coordinate function t_i for $i = 1$, 2, is regarded as a rational function on V which is invertible on U. The restriction of η to U is expressed as $(t_1^{-1}dt_1) \wedge (t_2^{-1}dt_2)$ or its multiple by a non-zero constant. Here, $\iota^*\eta = -\eta$, since ι is an involution. Then, ι has no fixed point in U by Lemma 7.17.

It is enough to prove that any irreducible component of D_V is not contained in the fixed point locus $\mathrm{Fix}(\iota)$ of ι. In fact, if this is true, then the quotient morphism $\tau\colon V \to X$ is étale in codimension one, and $\eta^{\otimes 2}$ descends to a nowhere vanishing section of $\mathcal{O}_X(2(K_X + D))$ with $2(K_V + D_V) = \tau^*(2(K_X + D))$, since $\eta^{\otimes 2}$ is preserved by ι. Furthermore, in this situation, $K_X + D \not\sim 0$. For, a nowhere vanishing section ζ of $\mathcal{O}_X(K_X + D)$ induces a nowhere vanishing section $\tau^*(\zeta)$ of $\mathcal{O}_V(K_V + D_V)$ which satisfies $\iota^*(\tau^*\zeta) = \tau^*\zeta$; this is a contradiction to: $\tau^*\eta = -\eta$. This implies that (X, D) is a half-toric surface. Therefore, we are reduced to show the non-existence of irreducible components of D_V contained in $\mathrm{Fix}(\iota)$.

Let Γ be an irreducible component of D_V. By the description of the toric surface V by a fan (cf. Example 3.4), Γ corresponds to a ray $\mathbb{R}_{\geq 0}v$ in $\mathsf{N} \otimes \mathbb{R}$, where N is the group of one-parameter subgroups of the torus U and v is a primitive element of N. Now, N is identified with $\mathbb{Z}^{\oplus 2}$ in such a way that $(m, n) \in \mathbb{Z}^{\oplus 2}$ corresponds to a one-parameter subgroup $f\colon \operatorname{Spec} \mathbb{C}[s^{\pm 1}] \to U$ defined by

$$f^*(t_1) = s^m \quad \text{and} \quad f^*(t_2) = s^n.$$

Hence, if an element (m, n) of $\mathbb{Z}^2$ corresponds to v, then

$$\gcd(m, n) = 1, \quad \mathrm{ord}_\Gamma(t_1) = m, \quad \text{and} \quad \mathrm{ord}_\Gamma(t_2) = n$$

(cf. [13, p. 61, Lemma], [25, I, Th. 1'], [43, Prop. 1.6, (v)]), where $\mathrm{ord}_\Gamma(\varphi)$ denotes the order of zeros (or the minus of the order of poles) of a rational function φ along Γ. By Lemma 7.17, we may assume that the restriction of ι to U corresponds to an automorphism ι^* of $\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$ given by

$$\iota^*(t_1) = -t_1 \quad \text{and} \quad \iota^*(t_2) = t_2^{-1}.$$

Assume that $\iota(\Gamma) = \Gamma$. Then, $n = 0$ and $m = \pm 1$. Consequently, Γ is an irreducible component of the principal divisor $\mathrm{div}(t_1)$, and the restriction $\bar{t}_2$ of t_2 to Γ is a non-constant rational function on Γ. We have $\iota_\Gamma^*(\bar{t}_2) = \bar{t}_2^{-1}$ for the induced automorphism $\iota_\Gamma = \iota|_\Gamma\colon \Gamma \to \Gamma$. This implies that ι_Γ is not the identity morphism, and hence $\Gamma \not\subset \mathrm{Fix}(\iota)$. Therefore, $\mathrm{Fix}(\iota)$ contains no irreducible component of D_V, and we are done. Q.E.D.

7.4. The structure of a half-toric surface

By applying results in Sections 7.2 and 7.3, we investigate further properties on the half-toric surfaces and prove Theorem 1.7. As a corollary of Lemma 7.17, we have:

Proposition 7.20. *For a half-toric surface (X, D), the open subset $X \setminus D$ is a non-singular affine surface with the coordinate ring isomorphic to*

$$\mathbb{C}[\mathtt{x}, \mathtt{x}^{-1}, \mathtt{y}, \mathtt{z}]/(\mathtt{x}(\mathtt{y}^2 - 1) - \mathtt{z}^2).$$

In particular, the isomorphism class of $X \setminus D$ is independent of the choice of half-toric surfaces (X, D). The fundamental group $\pi_1((X \setminus D)^{\mathrm{an}})$ of the associated complex analytic manifold $(X \setminus D)^{\mathrm{an}}$ is generated by two elements a, b with one relation: $ab = ba^{-1}$. In other words, $\pi_1((X \setminus D)^{\mathrm{an}}) \simeq \mathbb{Z} \rtimes \mathbb{Z}$, where the normal subgroup $\mathbb{Z}$ is regarded as a $\mathbb{Z}$-module by $m \cdot x = (-1)^m x$.

Proof. The open subset $X \setminus D$ is non-singular and affine by Lemma 7.2(3). This is derived also from Lemma 7.17. It implies that $V \setminus D_V \to X \setminus D$ is a finite étale morphism from an affine surface for the characteristic double-cover (V, D_V) of (X, D). The coordinate ring R of $X \setminus D$ is isomorphic to the ι^*-invariant ring of the coordinate ring of $V \setminus D_V$ for the Galois involution ι. By Lemma 7.17, for a suitable coordinate $(\mathtt{t}_1, \mathtt{t}_2)$ of $V \setminus D_V$ and for a monomial $\mathtt{t}_1^m \mathtt{t}_2^n$, we have

$$\iota^*(\mathtt{t}_1^m \mathtt{t}_2^n) = (-1)^m \mathtt{t}_1^m \mathtt{t}_2^{-n}.$$

Hence, the invariant ring of $\mathbb{C}[\mathtt{t}_1^{\pm 1}, \mathtt{t}_2^{\pm 1}]$ is generated by

$$\mathtt{x} := \mathtt{t}_1^2, \quad \mathtt{x}^{-1} = \mathtt{t}_1^{-2}, \quad \mathtt{y} := \frac{1}{2}(\mathtt{t}_2 + \mathtt{t}_2^{-1}), \quad \mathtt{z} := \frac{1}{2}\mathtt{t}_1(\mathtt{t}_2 - \mathtt{t}_2^{-1}).$$

By writing $\mathtt{t}_2$ and $\mathtt{t}_2^{-1}$ in terms of $\mathtt{t}_1$, $\mathtt{y}$, $\mathtt{z}$, we have one relation: $\mathtt{x}\mathtt{y}^2 - \mathtt{z}^2 = \mathtt{x}$. Since $\mathtt{x}(\mathtt{y}^2 - 1) - \mathtt{z}^2$ is irreducible in $\mathbb{C}[\mathtt{x}, \mathtt{y}, \mathtt{z}]$, we have the description above of R.

Let $\mathbb{C}^2 \to (\mathbb{C}^\star)^2$ be the map defined by $(z_1, z_2) \mapsto (\mathtt{e}(z_1), \mathtt{e}(z_2))$, where $\mathtt{e}(z) = \exp(2\pi\sqrt{-1}z)$. This map is a universal covering map of $(V \setminus D_V)^{\mathrm{an}}$. We may assume that $\mathtt{e}(z_1) = \mathtt{t}_1$ and $\mathtt{e}(z_2) = \mathtt{t}_2$ for the coordinate (z_1, z_2) of $\mathbb{C}^2$ and for the coordinate $(\mathtt{t}_1, \mathtt{t}_2)$ above of $V \setminus D_V$. The fundamental group $\pi_1((V \setminus D_V)^{\mathrm{an}}) \simeq \mathbb{Z}^{\oplus 2}$ acts on $\mathbb{C}^2$ by $(z_1, z_2) \mapsto (z_1 + m, z_2 + n)$ for $(m, n) \in \mathbb{Z}^{\oplus 2}$. The involution $\iota\colon V \to V$ lifts to an automorphism $b\colon \mathbb{C}^2 \to \mathbb{C}^2$ defined as $(z_1, z_2) \mapsto (z_1 + 1/2, -z_2)$. Thus, $\pi_1((X \setminus D)^{\mathrm{an}})$ is isomorphic to the automorphism subgroup of $\mathbb{C}^2$ generated by b and $\mathbb{Z}^{\oplus 2}$. Since $b^2(z_1, z_2) = (z_1 + 1, z_2)$, the group $\pi_1((X \setminus D)^{\mathrm{an}})$ is generated by b and an automorphism a defined by

$a(z_1, z_2) = (z_1, z_2 + 1)$. Here, we have a relation: $a \circ b = b \circ a^{-1}$, and this determines $\pi_1((X \setminus D)^{\mathrm{an}})$. Q.E.D.

Remark 7.21. For an H-surface (S, D_S) (cf. Definition 7.7), we have a half-toric surface $(\overline{S}, D_{\overline{S}})$ with $S \setminus D_S \simeq \overline{S} \setminus D_{\overline{S}}$ in Proposition 7.15. By Proposition 7.20 above, we see that $X \setminus D$ is isomorphic to the open surface $S \setminus D_S$ of type $H[-1, 0, -1]$ in the sense of Fujita [12, (8.19)] (cf. [12, Th. (8.5), Table (8.64)]) for any half-toric surface (X, D). The topological fundamental group of the open surface of type $H[-1, 0, -1]$ is also calculated in [12, Table (8.64), Example (7.24)], but its method is different from ours.

Proposition 7.22. *For a half-toric surface* (X, D)*, let* $\mathrm{Aut}(X; D)$ *be the group of automorphisms of* X *preserving each irreducible components of* D*. Then,* $\mathrm{Aut}(X; D) \simeq \mathbb{C}^\star \times \mathbb{Z}/2\mathbb{Z}$*, and the action of* $\mathrm{Aut}(X; D)$ *on the open subset* $X \setminus D$ *is given by*

$$(\mathtt{x}, \mathtt{y}, \mathtt{z}) \mapsto (\lambda^2 \mathtt{x}, (-1)^k \mathtt{y}, (-1)^k \lambda \mathtt{z})$$

with respect to the expression of the coordinate ring of $X \setminus D$ *in Proposition* 7.20 *for* $\lambda \in \mathbb{C}^\star$ *and* $k \in \{0, 1\}$*.*

Proof. Let $\tau \colon V \to X$ be the characteristic double-cover of (X, D). An automorphism σ of X lifts to an automorphism σ_V of V commuting with the Galois involution ι of τ, since τ is étale in codimension one. Assume that $\sigma \in \mathrm{Aut}(X; D)$. Then σ_V preserves $\tau^* D_i$ for any irreducible component D_i of D, and in particular, σ_V preserves $D_V = \tau^{-1}(D)$. Hence, σ_V acts on $V \setminus D_V$. The action of σ_V on $V \setminus D_V$ is determined by an automorphism σ_V^* of the coordinate ring $\mathbb{C}[\mathtt{t}_1^{\pm 1}, \mathtt{t}_2^{\pm 2}]$, where we may assume that the coordinate $(\mathtt{t}_1, \mathtt{t}_2)$ satisfies $\iota^*(\mathtt{t}_1) = -\mathtt{t}_1$ and $\iota^*(\mathtt{t}_2) = \mathtt{t}_2^{-1}$ by Lemma 7.17. The automorphism σ_V^* is given by

$$\sigma_V^*(\mathtt{t}_1) = \lambda \mathtt{t}_1^{a_1} \mathtt{t}_2^{a_2} \quad \text{and} \quad \sigma_V^*(\mathtt{t}_2) = \nu \mathtt{t}_1^{b_1} \mathtt{t}_2^{b_2}$$

for some λ, $\nu \in \mathbb{C} \setminus \{0\}$ and for some integers a_1, a_2, b_1, b_2 such that

$$\det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \pm 1.$$

The lift σ_V satisfies $\iota \circ \sigma_V = \sigma_V \circ \iota$. Thus,

$$a_2 = b_1 = 0, \quad a_1 = \pm 1, \quad b_2 = \pm 1, \quad \nu = \pm 1,$$

and σ_V^* is given by

$$\sigma_V^*(\mathtt{t}_1) = \lambda \mathtt{t}_1^{\varepsilon_1} \quad \text{and} \quad \sigma_V^*(\mathtt{t}_2) = \varepsilon_3 \mathtt{t}_2^{\varepsilon_2}$$

for some constants $\lambda \in \mathbb{C} \setminus \{0\}$ and ε_1, ε_2, $\varepsilon_3 \in \{\pm 1\}$. As in the proof of Proposition 7.19, an irreducible component Γ of D_V corresponds to a pair (m, n) of integers with $\gcd(m, n) = 1$ defined by

$$\mathrm{ord}_\Gamma(\mathtt{t}_1) = m \quad \text{and} \quad \mathrm{ord}_\Gamma(\mathtt{t}_2) = n.$$

The irreducible component $\iota(\Gamma)$ corresponds to $(m, -n)$. Since $\Gamma + \iota(\Gamma)$ is preserved by σ_V, we have

$$(\mathrm{ord}_\Gamma(\sigma_V^*(\mathtt{t}_1)), \mathrm{ord}_\Gamma(\sigma_V^*(\mathtt{t}_2))) = (\varepsilon_1 m, \varepsilon_2 n) = (m, n) \text{ or } (m - n).$$

Thus, $\varepsilon_1 = 1$, since $m \neq 0$ for some Γ. Conversely, if an automorphism σ_V^* of $\mathbb{C}[\mathtt{t}_1^\pm, \mathtt{t}_2^\pm]$ is given by

$$\sigma_V^*(\mathtt{t}_1) = \lambda \mathtt{t}_1 \quad \text{and} \quad \sigma_V^*(\mathtt{t}_2) = \varepsilon_3 \mathtt{t}_2^{\varepsilon_2}$$

for some $\lambda \in \mathbb{C}^\star = \mathbb{C} \setminus \{0\}$ and ε_2, $\varepsilon_3 \in \{\pm 1\}$, then σ_V^* is induced from an automorphism σ_V of V such that σ_V commutes with ι and that σ_V preserves $\tau^*(D_i)$ for any irreducible component D_i of D. The subgroup of $\mathrm{Aut}(V)$ consisting of such σ_V is isomorphic to $\mathbb{C}^\star \times (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$, and we have an exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\phi} \mathbb{C}^\star \times (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Aut}(X; D) \to 1$$

in which $\phi(1) = (-1, (1, 0))$ corresponds to ι. Therefore, $\mathrm{Aut}(X; D) \simeq \mathbb{C}^\star \times \mathbb{Z}/2\mathbb{Z}$. For an element $(\lambda, k) \in \mathbb{C}^\star \times \mathbb{Z}/2\mathbb{Z}$ (where $k = 0$ or 1), let σ be the associated automorphism in $\mathrm{Aut}(X; D)$. Then, the action of σ on $X \setminus D$ lifts to an automorphism on $\mathbb{C}[\mathtt{t}_1^\pm, \mathtt{t}_2^\pm]$ given by

$$(\mathtt{t}_1, \mathtt{t}_2) \mapsto (\lambda \mathtt{t}_1, (-1)^k \mathtt{t}_2).$$

Hence, the induced automorphism σ^* of the coordinate ring of $X \setminus D$ is given by

$$\sigma^*(\mathtt{x}) = \lambda^2 \mathtt{x}, \quad \sigma^*(\mathtt{y}) = (-1)^k \mathtt{y} \quad \text{and} \quad \sigma^*(\mathtt{z}) = \lambda(-1)^k \mathtt{z}.$$

Thus, we are done. Q.E.D.

Lemma 7.23. *The equality $\boldsymbol{\delta}(X, D) = 1$ holds for any half-toric surface (X, D).*

Proof. We consider the class map $\mathrm{cl}_D^{\mathbb{Z}} \colon \mathsf{F}(D) \to \mathrm{CL}(X)$ (cf. Definition 2.24). The cokernel of $\mathrm{cl}_D^{\mathbb{Z}}$ is isomorphic to the divisor class group $\mathrm{CL}(X \setminus D)$, and the kernel of $\mathrm{cl}_D^{\mathbb{Z}}$ is isomorphic to $\mathcal{O}(X \setminus D)^\star/\mathbb{C}^\star$ by Lemma 2.25. By Proposition 7.20 and Remark 7.21, we have an isomorphism $X \setminus D \simeq S \setminus D_S$ for an H-surface (S, D_S). Thus, $\mathrm{CL}(X \setminus D) \simeq \mathbb{Z}/2\mathbb{Z}$ and $\mathcal{O}(X \setminus D)^\star/\mathbb{C}^\star \simeq \mathbb{Z}$ by Corollary 7.13. Therefore, $\boldsymbol{\delta}(X, D) = \boldsymbol{\rho}(X) + 2 - \boldsymbol{n}(D) = 1$. Q.E.D.

The isomorphism $X \setminus D \simeq \overline{S} \setminus D_{\overline{S}}$ in Remark 7.21 is extended to a suitable birational map $X \cdots\to \overline{S}$ as follows.

Proposition 7.24. *Let (X, D) be a half-toric surface. Then, there exist birational morphisms $\nu\colon Y \to X$ and $h\colon Y \to \overline{S}$ satisfying the following conditions*:

(1) *(Y, D_Y) is a half-toric surface for $D_Y = \nu^{-1}(D)$ and $Y \setminus D_Y \simeq X \setminus D$ by ν*;

(2) *$(\overline{S}, D_{\overline{S}})$ is the half-toric surface associated with an H-surface (S, D_S) in Proposition 7.15, where $D_Y = h^{-1}(D_{\overline{S}})$ and h is a toroidal blowing up with respect to $(\overline{S}, D_{\overline{S}})$.*

Proof. We have $\boldsymbol{n}(D) = \boldsymbol{\rho}(X) + 1 > \boldsymbol{r}(D)$ by Lemma 7.23, and (X, D) is toroidal along $\operatorname{Sing} D$ by Lemma 7.2. Thus, we can apply Lemma 4.23 to (X, D). As a consequence, by replacing X by a toroidal blowing up, we may assume that X admits a $\mathbb{P}^1$-fibration $\pi\colon X \to T$ such that D has two distinct fibers of π. Let $\mu\colon M \to X$ be the minimal resolution of singularities. Then, there is a toroidal blowing up $f\colon M \to S$ with respect to (S, D_S) such that $\mu^{-1}(D) = f^{-1}(D_S)$ by Lemma 7.16. Let $g\colon M \to \overline{M}$ be the contraction morphism of the four end components of $\mu^{-1}(D)$ in Proposition 7.5. Then, $(Y, D_Y) := (\overline{M}, g_*(\mu^{-1}(D)))$ is a half-toric surface and the morphism $\nu\colon Y \to X$ induced by μ satisfies (1) by the proof of Proposition 7.5. Moreover, the morphism $h\colon Y \to \overline{S}$ induced by f is just a toroidal blowing up with respect to $(\overline{S}, D_{\overline{S}})$ and $D_Y = h^{-1}(D)$ by Lemma 7.16. Q.E.D.

Lemma 7.25. *For a half-toric surface (X, D), let $\mu\colon M \to X$ be the minimal resolution of singularities of X and set $D_M = \mu^{-1}(D)$. Then, $\bar{q}(M \setminus D_M) = 1$ for the logarithmic irregularity $\bar{q}$. The quasi-Albanese map of $M \setminus D_M$ is a smooth morphism $\alpha\colon M \setminus D_M \to \mathbb{G}_{\mathtt{m}}$ which is described in the following two ways*:

(1) *The morphism α is the composition of the isomorphism $M \setminus D_M \simeq X \setminus D \simeq S \setminus D_S$ in Remark 7.21 for an H-surface (S, D_S) and the $\mathbb{G}_{\mathtt{m}}$-fiber bundle $S \setminus D_S \to T \setminus \{P_1, P_2\}$ induced from the $\mathbb{P}^1$-fibration $\pi\colon S \to T$ in Lemma 7.8 (cf. Lemma 7.11).*

(2) *The morphism α of affine varieties is isomorphic to the morphism associated with the natural ring homomorphism*

$$\mathbb{C}[\mathtt{x}, \mathtt{x}^{-1}] \to \mathbb{C}[\mathtt{x}, \mathtt{x}^{-1}, \mathtt{y}, \mathtt{z}]/(\mathtt{x}(\mathtt{y}^2 - 1) - \mathtt{z}^2)$$

for the description of the coordinate ring of $X \setminus D \simeq M \setminus D_M$ in Proposition 7.20.

Proof. By Lemma 7.12, Proposition 7.20, and Remark 7.21, we have isomorphisms

$$\mathbb{C}^\star \times \mathbb{Z} \simeq \mathcal{O}(T \setminus \{P_1, P_2\})^\star \simeq \mathcal{O}(S \setminus D_S)^\star \simeq \mathcal{O}(X \setminus D)^\star \simeq \mathcal{O}(M \setminus D_M)^\star.$$

Then, the equality $\bar{q}(M \setminus D_M) = 1$ and the assertion (1) are derived from Proposition 2.26, since $q(M) = 0$. The remaining assertion (2) follows from the description of $\mathcal{O}(X \setminus D)$ in Proposition 7.20. Q.E.D.

Finally in Section 7.4, we shall prove Theorem 1.7.

Proof of Theorem 1.7. For a half-toric surface (X, D), from Definition 7.1, we see that X is a projective rational surface with only rational singularities. The pair (X, D) is log-canonical and D is a big linear chain of rational curves by Lemma 7.2. The equality $\boldsymbol{\delta}(X, D) = 1$ is proved in Lemma 7.23. This completes the proof of the first assertion (1) of Theorem 1.7. The assertions (2) and (3) have been proved in Proposition 7.20. Similarly, the assertions (4), (5), and (6) have been proved in Proposition 7.22, Lemma 7.25, and Proposition 7.5, respectively. Thus, we are done. Q.E.D.

§8. Proofs of Theorems 1.3 and 1.5

Finally, we shall prove Theorems 1.3 and 1.5. Note that the proofs below do not use the results on pseudo-toric surfaces and half-toric surfaces obtained in Sections 6 and 7 except Lemma 7.23 on the defect of half-toric surface.

Proof of Theorem 1.3. We may assume that $\boldsymbol{\delta}(X, D) \le 1$ or $\boldsymbol{c}(X, D) \le 0$. For, otherwise, the assertions of Theorem 1.3 hold trivially. Under the assumption, we have $D \neq 0$, since $\boldsymbol{\delta}(X, 0) = \hat{\boldsymbol{\rho}}(X) + 2 \ge 3$ and $\boldsymbol{c}(X, 0) = 2$. Moreover, X is projective by Lemma 2.31(1), since we have $\mathrm{H}^2(X, \mathcal{O}_X) \simeq \mathrm{H}^0(X, \mathcal{O}_X(K_X))^\vee = 0$ by the assumption that $-(K_X + D)$ is nef. Furthermore, we can prove the following:

(a) D is a big reducible linear (or cyclic) chain of rational curves;
(b) X is a projective rational surface with only rational singularities;
(c) $0 \le \boldsymbol{\delta}(X, D) = \boldsymbol{c}(X, D) \le 1$;
(d) $\boldsymbol{\delta}(X, D) = \boldsymbol{c}(X, D) = 1$ when D is a linear chain.

In fact, D is connected and reducible by Proposition 2.29. Hence, D is a linear chain or a cyclic chain of rational curves by Lemma 4.5, and D is big by Proposition 4.8(1). Thus, we have (a). The assertion (b) follows from Lemma 2.32, and the assertions (c) and (d) from Proposition 4.8. In particular, we have proved the inequality $\boldsymbol{\delta}(X, D) \ge \boldsymbol{c}(X, D) \ge 0$.

For the rest of Theorem 1.3, "if" part follows from Lemma 3.11 on toric surface, the definition of defect for pseudo-toric surface, and from Lemma 7.23 on half-toric surface. Thus, it suffices to show the "only if" part.

Let $f\colon Y \to X$ be a toroidal blowing up with respect to (X, D) and set $D_Y = f^{-1}(D)$. Then, (Y, D_Y) is also log-canonical along D_Y, and $-(K_Y + D_Y) = f^*(-(K_X + D))$ is nef. Moreover, $\boldsymbol{\delta}(X, D) = \boldsymbol{c}(Y, D_Y) = \boldsymbol{\delta}(X, D) = \boldsymbol{c}(X, D)$ by Lemma 2.27. In particular, (Y, D_Y) also satisfies the same conditions in Theorem 1.3. We shall show that if Theorem 1.3 holds on (Y, D_Y), then the same holds on (X, D). In fact, Lemma 3.9 implies that if (Y, D_Y) is toric, then (X, D) is also toric, and that if $(Y, B_Y + D_Y)$ is toric for a prime divisor $B_Y \not\subset D_Y$, then $(X, B + D)$ is toric for $B = f_*(B_Y) \not\subset D$. Moreover, if (Y, D_Y) is a pseudo-toric surface of defect one (resp. a half-toric surface), then so is (X, D) by Lemma 6.3(7) (resp. Lemma 7.4(3)). Thus, we can replace (X, D) with (Y, D_Y).

Since $\boldsymbol{n}(D) - \boldsymbol{r}(D) = 2 - \boldsymbol{c}(X, D) \geq 1$, by applying Lemma 4.23 and by the replacement above, we may assume that there is a fibration $\pi\colon X \to T \simeq \mathbb{P}^1$ and that D contains at least two fibers. Then, (X, D, π) belongs to the case (A) or (B) of Lemma 5.2.

Suppose that (X, D, π) belongs to the case (B). Then, D is a linear chain of rational curves containing a section or a double-section of π, and $\boldsymbol{\delta}(X, D) = 1$ by (d) above or by Lemma 5.6. If D contains a section, then $(X, B + D)$ is a toric surface for a prime divisor $B \not\subset D$ by Proposition 5.7. If D contains a double-section, then (X, D) is a half-toric surface by Proposition 5.9 (cf. Definition 7.1).

Suppose next that (X, D, π) belongs to the case (A). Then, D is a cyclic chain of rational curves, and $K_X + D \sim 0$ by Proposition 5.3(1). If $\boldsymbol{\delta}(X, D) = 0$, then (X, D) is a projective toric surface by Proposition 5.3(3). If $\boldsymbol{\delta}(X, D) = 1$, then (X, D) is a pseudo-toric surface of defect one by Proposition 5.5 (cf. Definition 6.1).

These arguments complete the proof of Theorem 1.3. Q.E.D.

Proof of Theorem 1.5. We may assume that $\boldsymbol{c}(X, D) \leq 1$ for the proof. Then, D is reducible by

$$\boldsymbol{n}(D) = \boldsymbol{r}(D) + 2 - \boldsymbol{c}(X, D) \geq \boldsymbol{r}(D) + 1 \geq 2.$$

Moreover, D is big by Proposition 4.8(1). Then, Lemma 4.7 implies that X is a normal projective rational surface with only rational singularities and that D is a linear chain or a cyclic chain of rational curves. Hence, we have $\boldsymbol{c}(X, D) \geq 0$ by (2) and (3) of Proposition 4.8.

Here, assume that $\boldsymbol{c}(X, D) = 0$. Then, D is a cyclic chain of rational curves by Proposition 4.8(2). Thus, by Lemma 4.7, we have an effective divisor G on $X \setminus D$ such that $K_X + D \sim G$. Let $g\colon X \to \overline{X}$ be the contraction morphism of G: This exists because G is negative definite when $G \neq 0$ (cf. Lemma 4.7). We set $\overline{D} = g_*(D)$. Then, $(\overline{X}, \overline{D})$ satisfies the same assumptions (i), (ii), and (iii) of Theorem 1.5, and $0 \leq \boldsymbol{c}(\overline{X}, \overline{D}) \leq \boldsymbol{c}(X, D) = 0$ by Lemma 2.27. Moreover, $K_{\overline{X}} + \overline{D} \sim 0$. Therefore, $(\overline{X}, \overline{D})$ is a projective toric surface by Theorem 1.3.

Conversely, assume that there is a morphism $g\colon X \to \overline{X}$ satisfying (1) and (2) of Theorem 1.5. Then, $D \simeq \overline{D}$ and $\boldsymbol{n}(D) = \boldsymbol{n}(\overline{D})$. For the rest of the proof, it suffices to show: $\boldsymbol{c}(X, D) = 0$. Let Δ be a divisor on X supported on D such that $g_*\Delta \sim 0$. Then, $\Delta = g^*(g_*\Delta) \sim 0$. This argument implies that the kernel of the class map $\mathrm{cl}_D^{\mathbb{Z}}\colon \mathsf{F}(D) \to \mathrm{CL}(X)$ (cf. Definition 2.24) is isomorphic to the kernel of the class map $\mathrm{cl}_{\overline{D}}^{\mathbb{Z}}\colon \mathsf{F}(\overline{D}) \to \mathrm{CL}(\overline{X})$. Hence, $\boldsymbol{r}(D) = \boldsymbol{r}(\overline{D})$, and $\boldsymbol{c}(X, D) = \boldsymbol{c}(\overline{X}, \overline{D}) = 0$. Thus, we are done. Q.E.D.

References

[1] V. Alexeev, Classification of log canonical surface singularities: Arithmetical proof, Chapter 3 of [29], pp. 47–58.

[2] V. Alexeev, Boundedness and K^2 for log surfaces, Internat. J. Math., **5** (1994), 779–810.

[3] A. Altman and S. Kleiman, Introduction to Grothendieck Duality Theory, Lecture Notes in Math. **146**, Springer-Verlag, 1970.

[4] M. Artin, Some numerical criteria for contractibility of curves on algebraic surfaces, Amer. J. of Math., **84** (1962), 485–496.

[5] M. Artin, Algebraic approximation of structures over complete local rings, Publ. Math. I.H.É.S., **36** (1969), 23–58.

[6] M. Artin, Algebraization of formal moduli: II. Existence of modifications, Ann. Math., **91** (1970), 88–135.

[7] L. Bădescu, On some contractibility criteria of curves on surfaces, In: Sitzungsberichte der Berliner Mathematischen Gesellschaft, Berliner Math. Gesellschaft, Berlin, 2001, pp. 41–51.

[8] L. Brenton, Some algebraicity criteria for singular surfaces, Invent. Math., **41** (1977), 129–147.

[9] E. Brieskorn, Rationale Singularitäten komplexer Flächen, Invent. Math., **4** (1968), 336–358.

[10] W. L. Chow and K. Kodaira, On analytic surfaces with two independent meromorphic functions, Proc. Nat. Acad. Sci. U.S.A., **38** (1952), 319–325.

[11] T. Fujita, On Zariski Problem, Proc. Japan Acad., **55**, Ser. A (1979), 106–110.

[12] T. Fujita, On the topology of non-complete algebraic surfaces, J. Fac. Sci. Univ. Tokyo, **29** (1982), 503–566.
[13] W. Fulton, Introduction to Toric Varieties, Ann. of Math. Studies, **131**, Princeton Univ. Press, 1993.
[14] H. Grauert, Über Modifikationen und exzeptionelle analytische Mengen, Ann. Math., **146** (1962), 331–368.
[15] R. Hartshorne, Stable reflexive sheaves, Math. Ann., **254** (1980), 121–176.
[16] S. Iitaka, Logarithmic forms of algebraic varieties, J. Fac. Sci. Univ. Tokyo, **23** (1976), 525–544.
[17] S. Iitaka, On logarithmic Kodaira dimension of algebraic varieties, In: Complex Analysis and Algebraic Geometry (eds. W. L. Baily, Jr. and T. Shioda), Iwanami-Shoten Publishers and Cambridge Univ. Press, 1977, pp. 175–189.
[18] S. Iitaka, A numerical criterion of quasi-abelian surfaces, Nagoya Math. J., **73** (1979), 99–115.
[19] S. Iitaka, Algebraic Geometry. An Introduction to Birational Geometry of Algebraic Varieties, Grad. Texts in Math. **76**, Springer-Verlag, 1982.
[20] I. Karzhemanov, On characterization of toric varieties, preprint (arXiv: 1306.4131v3), 2013.
[21] Y. Kawamata, On the classification of non-complete algebraic surfaces, In: Algebraic Geometry (Copenhagen 1978, ed. K. Lønsted), Lecture Notes in Math. **732**, Springer-Verlag, 1979, pp. 215–232.
[22] Y Kawamata, The cone of curves of algebraic varieties, Ann. of Math., **199** (1984), 606–633.
[23] Y. Kawamata, Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces, Ann. of Math., **127** (1988), 93–163.
[24] Y. Kawamata, K. Matsuda, and K. Matsuki, Introduction to the minimal model problem, In: Algebraic geometry, Sendai, 1985 (ed. T. Oda), Adv. Stud. Pure Math., **10**, Kinokuniya and North-Holland, 1987, pp. 283–360.
[25] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat, Toroidal Embeddings, I, Lecture Notes in Math., **339**, Springer-Verlag, 1973.
[26] D. Knutson, Algebraic Spaces, Lecture Notes in Math. **203**, Springer-Verlag, 1971.
[27] K. Kodaira, On compact complex analytic surfaces, I, Ann. of Math., **71** (1960), 111–152.
[28] H. Kojima, On open algebraic surfaces of logarithmic Kodaira dimension zero (in Japanese), In: The proceedings of mini-conference of algebraic geometry held at Saitama Univ., organized by F. Sakai and M. Ebihara, March, 2013, pp. 71–80.
[29] J. Kollár et al, Flips and Abundance for Algebraic Threefolds, Astérisque **211**, Soc. Math. de France, 1992.
[30] J. Kollár and S. Mori, Birational geometry of algebraic varieties, Cambridge Tracts in Math., **134**, Cambridge Univ. Press, 1998.

[31] J. Lipman, Rational singularities with applications to algebraic surfaces with unique factorization, Publ. Math. I.H.É.S. **36** (1969), 195–279.

[32] E. Looijenga, Rational surfaces with an anti-canonical cycle, Ann. of Math., **114** (1981), 267–322.

[33] J. M^{c}Kernan, A simple characterization of toric varieties, In: The proceedings of Kinosaki symposium 2001, pp. 59–72.

[34] B. G. Moĭšezon, A criterion for projectivity of complete algebraic abstract varieties, Izv. Akad. Nauk SSSR Ser. Mat., **28** (1964), 179–224, (Amer. Math. Soc. Translations Ser. 2, **63** (1967), 1–50).

[35] B. G. Moĭšezon, On n-dimensional compact varieties with n algebraically independent meromorphic functions, I, II, III, Izv. Akad. Nauk SSSR Ser. Mat., **30** (1966), 133–174, 345–386, 621–656, (Amer. Math. Soc. Translations Ser. 2, **63** (1967), 51–177).

[36] S. Mori, Threefolds whose canonical bundles are not numerically effective, Ann. of Math., **116** (1982), 133–176.

[37] D. R. Morrison, The birational geometry of surfaces with rational double points, Math. Ann., **271** (1985), 415–438.

[38] D. Mumford, The topology of normal surface singularities of an algebraic surface and a criterion for simplicity, Publ. Math. I.H.É.S. **9** (1961), 5–22.

[39] Y. Nakai, A criterion of an ample sheaf on a projective scheme, Amer. J. Math., **85** (1963), 14–26.

[40] N. Nakayama, Classification of log del Pezzo surfaces of index two, J. Math. Sci. Univ. Tokyo, **14** (2007), 293–498.

[41] N. Nakayama, On complex normal projective surfaces admitting non-isomorphic surjective endomorphisms, in preparation.

[42] N. Nakayama, On almost toric surfaces (in Japanese), In: The proceedings of the symposium "Geometry of Projective Varieties and Related Topics, 2012" held at Kochi Univ., pp. 33–50.

[43] T. Oda, Convex Bodies and Algebraic Geometry – An Introduction to the Theory of Toric Varieties, Ergebnisse der Math., (3) **15**, Springer-Verlag, 1988.

[44] Y. G. Prokhorov, Lectures on Complements on Log Surfaces, MSJ Memoirs vol. **10**, Math. Soc. Japan 2001.

[45] Y. G. Prokhorov, On a conjecture of Shokurov: Characterization of Toric varieties, Tohoku Math. J., **53** (2001), 581–592.

[46] Y. G. Prokhorov, A note on degenerations of del Pezzo surfaces, Ann. Inst. Fourier, Grenoble, **65** (2015), 369–388.

[47] F. Sakai, Weil divisors on normal surfaces, Duke Math. J., **51** (1984), 877–888.

[48] F. Sakai, The structure of normal surfaces, Duke Math. J., **52** (1985), 627–648.

[49] F. Sakai, On polarized normal surfaces, Manuscripta Math., **59** (1987), 109–127.

[50] F. Sakai, Classification of normal surfaces, In: Algebraic Geometry, Bowdoin 1985 (ed. S. Bloch), Proc. of Sympo. in Pure Math., **46**, Amer. Math. Soc., 1987, pp. 451–465.
[51] V. V. Shokurov, Complements on surfaces, J. Math. Sci. New York, **102**, No. 2, (2000), 3876–3932.
[52] U. Storch, Fastfaktorielle Ringe, Schr. Math. Inst. Univ. Münster **36** (1967), iv+42 pp.
[53] U. Storch, Über die Divisorklassengruppen normaler komplexe-analytischer Algebren, Math. Ann., **183** (1969), 93–104.
[54] H. Sumihiro, Equivariant completion, J. Math. Kyoto Univ., **14** (1974), 1–28.
[55] S. Tsunoda and M. Miyanishi, The structure of open algebraic surfaces II, In: Classification of Algebraic and Analytic Manifolds (ed. K. Ueno), Progress in Math. **39**, Birkhäuser 1983, pp. 499–544.
[56] K. Watanabe, On plurigenera of normal isolated singularities. I, Math. Ann., **250** (1980), 65–94.
[57] O. Zariski, The theorem of Riemann–Roch for high multiples of an effective divisor on an algebraic surface, Ann. Math., **76** (1962), 560–615.

Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-8502
Japan
E-mail address: nakayama@kurims.kyoto-u.ac.jp