Advanced Studies in Pure Mathematics 75, 2017
Algebraic Varieties and Automorphism Groups
pp. 189–206

Irrational open surfaces of non-negative logarithmic Kodaira dimension

Hideo Kojima

Abstract.

We study irrational open algebraic surfaces of non-negative logarithmic Kodaira dimension in any characteristic. We give a structure theorem for the irrational open surfaces of logarithmic Kodaira dimension zero. Then, by using this result and the results in [7], we prove that, for an irrational ruled open surface, its logarithmic Kodaira dimension is non-negative if and only if its logarithmic bigenus is positive.

§0. Introduction

Let k be an algebraically closed field, which we fix as the ground field.

In the case of char $(k) = 0$, classification theory for open algebraic surfaces has been developed by Kawamata, Fujita, Miyanishi, Tsunoda, etc. In particular, Kawamata [6] gave structure theorems for open algebraic surfaces of non-negative logarithmic Kodaira dimension. For more details, we refer to [9] and [11]. Some of the results on open algebraic surfaces are valid also in the case of char $(k) > 0$. For example, the minimal model theory for open algebraic surfaces due to Miyanishi and Tsunoda [12] (see also [11, Chapter 2]) works in any characteristic. Miyanishi [10] proved that every irrational open algebraic surface of logarithmic Kodaira dimension $-\infty$ is affine ruled. Recently, the author [7] gave a structure theorem for open algebraic surfaces of logarithmic Kodaira dimension one in any characteristic.

Received September 3, 2015.

Revised March 5, 2016.

2010 *Mathematics Subject Classification.* Primary 14J26; Secondary 14R25.

Key words and phrases. Open algebraic surface, logarithmic Kodaira dimension, logarithmic plurigenus.

The author was supported by Grant-in-Aid for Scientific Research (No. 26400042 and 24340006) from JSPS.

In the present article, we study irrational open algebraic surfaces of non-negative logarithmic Kodaira dimension in any characteristic. After recalling the minimal model theory for open algebraic surfaces (see, e.g., [11, Chapter 2] and [7, Section 1]), we classify the strongly minimal irrational open algebraic surfaces of logairhtmic Kodaira dimension zero (cf. Theorem 2.1). In the case of char $(k) = 0$, the irrational open algebraic surfaces of logarithmic Kodaira dimension zero were studied in Iitaka [3], Sakai [14, Section 2], Miyanishi [11, Theorem 2.6.4.1 (p. 184)], etc. In particular, [11, (1) and (2) of Theorem 6.4.1 (p. 184)] is the same as Theorem 2.1 in the case of char $(k) = 0$. In Section 3, by using the results in Section 2 and the structure theorem for open algebraic surfaces of logarithmic Kodaira dimension one in [7], we study logarithmic plurigenera of irrational open algebraic surfaces of non-negative logarithmic Kodaira dimension. We prove the following theorem.

Theorem 0.1. *Let S be a smooth irrational open algebraic surface. Then $\overline{\kappa}(S) \geq 0$ if and only if $\overline{P}_4(S) > 0$ or $\overline{P}_6(S) > 0$. Moreover, if S is ruled, then $\overline{\kappa}(S) \geq 0$ if and only if $\overline{P}_2(S) > 0$.*

In the case of char $(k) = 0$, Theorem 0.1 follow from the results of Kuramoto [8] and Tsunoda [15]. Kuramoto [ibid] and Tsunoda [ibid] considered the problem finding the smallest positive integer m such that $\overline{P}_m(S) > 0$ for any smooth open algebraic surface S of $\overline{\kappa}(S) \geq 0$ and gave various interesting results. However, the problem has not yet been solved completely when S is a rational surface of $\overline{\kappa}(S) \geq 1$ even in the case of char $(k) = 0$.

Terminology. A reduced effective divisor D is called an SNC-divisor if it has only simple normal crossings. We employ the following notations. For the definitions of $\overline{P}_m$ and $\overline{\kappa}$, see [4] (see also [5] for the definitions in any characteristic).

K_V: the canonical divisor on V.
$\kappa(V)$: the Kodaira dimension of V.
$\overline{P}_m(S)$ $(m \geq 1)$: the logarithmic m-genus of S.
$\overline{\kappa}(S)$: the logarithmic Kodaira dimension of S.
$\lfloor Q \rfloor$: the integral part of a $\mathbb{Q}$-divisor Q.
$\lceil Q \rceil := -\lfloor -Q \rfloor$: the roundup of a $\mathbb{Q}$-divisor Q.
$D_1 \sim D_2$: D_1 and D_2 are linearly equivalent.
$D_1 \equiv D_2$: D_1 and D_2 are numerically equivalent.

§1. Preliminary results

We recall some basic notions in the theory of peeling. For more details, see [11, Chapter 2] or [12, Chapter 1]. Let X be a smooth projective surface and B an SNC-divisor on X. We call such a pair (X, B) an SNC-pair. A connected curve consisting only of irreducible components of B is called a connected curve in B for shortness. A connected curve T in B is *admissible* (resp. *rational*) if there are no (-1)-curves in $\operatorname{Supp}(T)$ and the intersection matrix of T is negative definite (resp. it consists only of rational curves). A connected curve T in B is a *twig* if its dual graph is a linear chain and T meets $B - T$ in a single point at one of the end components of T. A connected curve R (resp. F) in B is a *rational rod* (resp. *rational fork*) if it is rational and its dual graph is a linear chain (resp. the dual graph of the exceptional curves of a minimal resolution of a log terminal singular point and is not a linear chain). An admissible rational twig T in B is *maximal* if it is not extended to an admissible rational twig with more irreducible components of B. By a (-2)-rod (resp. a (-2)-fork), we mean a rational rod (resp. a rational fork) consisting only of (-2)-curves.

Let $\{T_\lambda\}$ (resp. $\{R_\mu\}$, $\{F_\nu\}$) be the set of all admissible rational maximal twigs (resp. all admissible rational rods, all admissible rational forks). Then there exists a unique decomposition of B as a sum of effective $\mathbb{Q}$-divisors $B = B^{\#} + \operatorname{Bk}(B)$ such that the following conditions are satisfied:

(a) $\operatorname{Supp}(\operatorname{Bk}(B)) = (\cup_\lambda T_\lambda) \cup (\cup_\mu R_\mu) \cup (\cup_\nu F_\nu)$.

(b) $(B^{\#} + K_X) \cdot Z = 0$ for every irreducible component Z of $\operatorname{Supp}(\operatorname{Bk}(B))$.

We call the divisor $\operatorname{Bk}(B)$ the *bark* of B.

Lemma 1.1. *With the same notations as above, each connected component of* $B - \lceil B^{\#} \rceil$ *is a* (-2)*-rod or a* (-2)*-fork.*

Proof. See [11, p. 94]. Q.E.D.

Definition 1.2. An SNC-pair (X, B) is *almost minimal* if, for every irreducible curve C on X, either $(B^{\#} + K_X) \cdot C \geq 0$ or $(B^{\#} + K_X) \cdot C < 0$ and the intersection matrix of $C + \operatorname{Bk}(B)$ is not negative definite.

Lemma 1.3. *Let* (X, B) *be an SNC-pair. Then there exists a birational morphism* $\mu : X \to \tilde{X}$ *onto a smooth projective surface* $\tilde{X}$ *such that the following four conditions* (i) – (iv) *are satisfied*:

(i) $\tilde{B} := \mu_*(B)$ *is an SNC-divisor.*

(ii) $\mu_*(\operatorname{Bk}(B)) \leq \operatorname{Bk}(\tilde{B})$ *and* $\mu_*(B^{\#} + K_X) \geq \tilde{B}^{\#} + K_{\tilde{X}}$.

(iii) $\overline{P}_n(X - B) = \overline{P}_n(\tilde{X} - \tilde{B})$ *for every integer* $n \geq 1$*. In particular,* $\overline{\kappa}(X - B) = \overline{\kappa}(\tilde{X} - \tilde{B})$.

(iv) *The pair* $(\tilde{X}, \tilde{B})$ *is almost minimal.*

Proof. See [11, Theorem 2.3.11.1 (p. 107)], which is the same as [12, Theorem 1.11]. Q.E.D.

In Lemma 1.3, we call the pair $(\tilde{X}, \tilde{B})$ an *almost minimal model* of (X, B).

Lemma 1.4. *Let (X, B) be an almost minimal SNC-pair. Then the following assertions hold true.*

(1) $\overline{\kappa}(X - B) \geq 0$ *if and only if* $B^{\#} + K_X$ *is nef.*

(2) *If* $\overline{\kappa}(X - B) \geq 0$, *then* $B^{\#} + K_X$ *is semiample. Moreover, we have the following.*

(2-1) $\overline{\kappa}(X - B) = 0 \iff B^{\#} + K_X \equiv 0$.

(2-2) $\overline{\kappa}(X - B) = 1 \iff (B^{\#} + K_X)^2 = 0$ *and* $B^{\#} + K_X \not\equiv 0$.

(2-3) $\overline{\kappa}(X - B) = 2 \iff (B^{\#} + K_X)^2 > 0$.

Proof. See [7, Lemma 1.4]. Q.E.D.

In order to study an SNC-pair (X, B) of $\overline{\kappa}(X - B) \geq 0$, it is convenient to consider its strongly minimal model. We recall the following lemma.

Lemma 1.5. *Let (X, B) be an almost minimal SNC-pair of $\overline{\kappa}(X - B) \geq 0$. Assume that there exists a (-1)-curve E such that $E \cdot (B^{\#} + K_X) = 0$, $E \not\subset \mathrm{Supp}\,(\lfloor B^{\#} \rfloor)$ and the intersection matrix of $E + \mathrm{Bk}\,(B)$ is negative definite. Let $\sigma : X \to Y$ be a composite of the contraction of E and the contractions of all subsequently contractible components of $\mathrm{Supp}\,(\mathrm{Bk}\,(B))$. Set $B_Y := \sigma_*(B)$. Then the following assertions hold.*

(1) *The divisor B_Y is an SNC-divisor and each connected component of $\sigma(\mathrm{Supp}\,(\mathrm{Bk}\,(B)))$ is an admissible rational twig, an admissible rational rod or an admissible rational fork of B_Y.*

(2) *The pair (Y, B_Y) is an almost minimal SNC-pair.*

(3) *For every integer $n \geq 1$, $\overline{P}_n(X - B) = \overline{P}_n(Y - B_Y)$. In particular, $\overline{\kappa}(Y - B_Y) = \overline{\kappa}(X - B)$.*

Proof. All the assertions follow from [11, (4), (6) and (7) of Lemma 2.4.4.1 (p. 123)]. Q.E.D.

Let E be a (-1)-curve on X. Then E is called a *superfluous exceptional component* of B if $E \subset \mathrm{Supp}\,(\lfloor B^{\#} \rfloor)$, $E \cdot (B - E) = E \cdot (\lfloor B^{\#} \rfloor - E) = 2$ and E meets two irreducible components of $\lfloor B^{\#} \rfloor - E$. Assume that E is a superfluous exceptional component of B. Let $\mu : X \to Y$ be the contraction of E and set $B_Y := \mu_*(B)$. It is then clear that (Y, B_Y) is an SNC-pair and $B^{\#} + K_X \equiv \mu^*(B_Y^{\#} + K_Y)$. Further, $\overline{P}_n(X - B) = \overline{P}_n(Y - B_Y)$ for every integer $n \geq 1$. So, when we construct an almost minimal model, we assume that there exist no superfluous exceptional components.

By using the argument as above and Lemmas 1.3 and 1.5, we have the following result.

Lemma 1.6. *Let* (X, B) *be an SNC-pair of* $\overline{\kappa}(X - B) \geq 0$. *Then there exists a birational morphism* $f : X \to V$ *onto a smooth projective surface* V *such that the following conditions are satisfied*:

(1) *Set* $D := f_*(B)$. *Then* (V, D) *is an almost minimal SNC-pair with* $\overline{P}_n(V - D) = \overline{P}_n(X - B)$ *for every* $n \geq 1$. *In particular,* $\overline{\kappa}(V - D) = \overline{\kappa}(X - B)$.

(2) *There exist no superfluous exceptional components of* D.

(3) *There exist no* (-1)*-curves* E *such that* $E \cdot (D^{\#} + K_V) = 0$, $E \not\subset \mathrm{Supp}\,(\lfloor D^{\#} \rfloor)$ *and the intersection matrix of* $E + \mathrm{Bk}\,(D)$ *is negative definite.*

Definition 1.7. (1) In Lemma 1.6, we call the pair (V, D) a *strongly minimal model* of a given SNC-pair (X, B) of $\overline{\kappa}(X - B) \geq 0$. An SNC-pair (V, D) of $\overline{\kappa}(V - D) \geq 0$ is said to be *strongly minimal* if (V, D) becomes a strongly minimal model of itself.

(2) Let S be a smooth open algebraic surface of $\overline{\kappa}(S) \geq 0$. It is then clear that there exists an SNC-pair (V, D) such that $S \cong V - D$. Let (V', D') (resp. (V'', D'')) be an almost minimal (resp. strongly minimal) model of (V, D). We call the surface $V' - D'$ (resp. $V'' - D''$) an almost minimal model of S (resp. a strongly minimal model of S).

Here, we recall a structure theorem for open algebraic surfaces of $\overline{\kappa} = 1$.

Lemma 1.8. (cf. [7, Theorem 2.1]) *Let* (V, D) *be a strongly minimal SNC-pair of* $\overline{\kappa}(V - D) = 1$. *Then, for a sufficiently large integer* n, *the complete linear system* $|n(D^{\#} + K_V)|$ *defines a fibration* $\rho : V \to B$ *from* V *onto a smooth projective curve* B *such that* ρ *is an elliptic fibration, a quasi-elliptic fibration or a* $\mathbb{P}^1$*-fibration. Moreover, let* $h : V \to W$ *be a birational morphism such that* $\pi := \rho \circ h^{-1}$ *is a relatively minimal model of the fibration* ρ, *let* $C := h_*(D^{\#})$ *and let* F *be a general fiber of* π. *Then the following assertions hold.*

(1) *Assume that* π *is an elliptic or quasi-elliptic fibration. Then we have*:

(1-1) $C = \sum_i d_i F_i$, *where* $0 < d_i \leq 1$ *and* $m_i F_i$ *is a scheme-theoretic fiber for some integer* $m_i \geq 1$.

(1-2) *Write* $R^1\pi_*\mathcal{O}_W = \mathcal{L} \oplus \mathcal{T}$, *where* $\mathcal{L}$ *is a locally free* $\mathcal{O}_B$*-module and* $\mathcal{T}$ *is a torsion* $\mathcal{O}_B$*-module. Then the divisor*

$C+K_W$ can be expressed as follows:

$$C+K_W=\pi^*(K_B+\delta)+\sum_s a_sE_s+\sum_i d_iF_i,$$

where a_sE_s ranges over all multiple fibers of π with multiplicity m_s, $0\le a_s<m_s$, $a_s=m_s-1$ if m_sE_s is not a wild fiber of π, and δ is a divisor on B with $\deg\delta=\chi(\mathcal{O}_W)+\mathrm{length}\mathcal{T}$.

(2) *Assume that π is a $\mathbb{P}^1$-fibration. Then we have:*

(2-1) *We set as $C=H+\sum_i d_iF_i$, where H is the sum of the horizontal components of C and the F_i's are fibers of π. Then H is an SNC-divisor and consists of either two sections or an irreducible 2-section of π.*

(2-2) *The divisor $C+K_W$ can be expressed as follows:*

$$C+K_W=\pi^*(K_B+\delta)+\sum_i d_iF_i,$$

where δ is a divisor on B such that $\deg\delta$ equals $H_1\cdot H_2$ (resp. one half of the number of the branch points of $\pi|_H$, $1-g(B)$) if $H=H_1+H_2$ with sections H_1 and H_2 (resp. H is irreducible and $\pi|_H$ is not purely inseparable, H is irreducible and $\pi|_H$ is purely inseparable) and

$$d_i=\begin{cases}\frac{1}{2}\left(1-\frac{1}{m_i}\right) & \text{if } \#(F_i\cap H)=1,\\ 1-\frac{1}{m_i} & \text{if } \#(F_i\cap H)=2,\end{cases}$$

where m_i is a positive integer or $+\infty$.

Proof. See [7, Section 2]. Q.E.D.

§2. Irrational open surfaces of $\overline{\kappa}=0$

In this section, we study smooth irrational open algebraic surfaces of $\overline{\kappa}=0$. The main result of this section is the following theorem, which contains [11, (1) and (2) of Theorem 2.6.4.1 (p. 184)].

Theorem 2.1. *Let (V,D) be an SNC-pair of $\overline{\kappa}(V-D)=0$. Then the following assertions hold.*

(1) *If $\kappa(V)\ge 0$ and (V,D) is almost minimal (see Section 1), then V is a minimal surface of $\kappa(V)=0$ and each connected component of D is a (-2)-rod or a (-2)-fork.*

(2) *Assume that V is an irrational ruled surface. Then V is an elliptic ruled surface and $\overline{P}_2(V - D) = 1$. Furthermore, if the pair (V, D) is strongly minimal* (*see Section* 1), *the following assertions hold.*

(2-1) *If V is relatively minimal, then either* (a) $D + K_V \sim 0$, $V = \mathbb{P}_B(\mathcal{O}_B \oplus \mathcal{L})$, *where B is an elliptic curve and* $\mathcal{L} \in \operatorname{Pic}(B)$, *and $D = D_1 + D_2$ is a sum of two disjoint sections D_1 and D_2 of the ruling $\pi : V \to B$, or* (b) *D is an elliptic curve with $D \equiv -K_V$ and $V = \mathbb{P}_B(\mathcal{E})$, where B is an elliptic curve and $\mathcal{E}$ is an indecomposable vector bundle of rank two over B.*

(2-2) *If V is not relatively minimal, then* $\operatorname{char}(k) = 2$ *and the pair (V, D) is one of the pairs constructed in Example* 2.2.

Here we give the pairs as in (2-2) of Theorem 2.1.

Example 2.2. (cf. [10, 2.1 and 2.2]) Assume that $\operatorname{char}(k) = 2$. Let B be an elliptic curve and let $F : B \to B$ be the absolute Frobenius morphism. Then we have an exact sequence

$$0 \to \mathcal{O}_B \to F_*\mathcal{O}_B \to \mathcal{L} \to 0,$$

where $F_*\mathcal{O}_B$ is a vector bundle of rank two over B and $\mathcal{L}$ is an invertible sheaf of $\deg \mathcal{L} = 0$. By [10, Lemmas 2.4 and 2.6], $2\mathcal{L} \sim 0$. The vector bundle $F_*\mathcal{O}_B$ defines a $\mathbb{P}^1$-bundle $\pi : V_{\mathrm{Frob}} := \mathbb{P}(F_*\mathcal{O}_B) \to B$ and the surjection $F_*\mathcal{O}_B \to \mathcal{L}$ defines a section M. Moreover, the $\mathcal{O}_B$-algebra $F_*\mathcal{O}_B$ defines a smooth projective curve D_{Frob} on V_{Frob} such that $\pi|_{D_{\mathrm{Frob}}} : D_{\mathrm{Frob}} \to B$ is identified with $F : B \to B$ (cf. [10, 2.1]). The pair $(V_{\mathrm{Frob}}, D_{\mathrm{Frob}})$ is called the Frobenius pair over B.

Let (V', D') be the Frobenius pair $(V_{\mathrm{Frob}}, D_{\mathrm{Frob}})$ or the pair obtained from the Frobenius pair $(V_{\mathrm{Frob}}, D_{\mathrm{Frob}})$ by an elementary transformation at a point on D_{Frob}. Let $P_1, \ldots, P_r$ $(r \geq 0)$ be points on D' and let F_i $(1 \leq i \leq r)$ be the fiber of the ruling, say π', on V' passing through P_i. Then $F_i \cdot D' = 2$ and $F_i \cap D' = \{P_i\}$ for $i = 1, \ldots, r$. Let $f : V \to V'$ be a composite of blowing-ups over the points $P_1, \ldots, P_r$ such that the fiber f^*F_i $(i = 1, \ldots, r)$ of $\pi' \circ f : V \to B$ has the dual graph in Figure 1, where $f^*F_i = 2(E_i + D_1^i + \cdots + D_{s_i-2}^i) + D_{s_i-1}^i + D_{s_i}^i$ $(s_i \geq 2)$ and the integer is the self-intersection number of the corresponding curve. Set $D := f'(D') + \sum_{i=1}^{r}(\sum_{j=1}^{s_i} D_j^i)$. Since $\overline{P}_2(V' - D') = 1$ by [10, Lemmas 2.4 and 2.6], we have $\overline{P}_2(V - D) = 1$. In fact, we know that $D^{\#} = f'(D')$ and $2(D^{\#} + K_V) \sim 0$ by [10, Lemmas 2.4 and 2.6]. We can easily see that $\overline{\kappa}(V - D) = 0$ and (V, D) is strongly minimal.

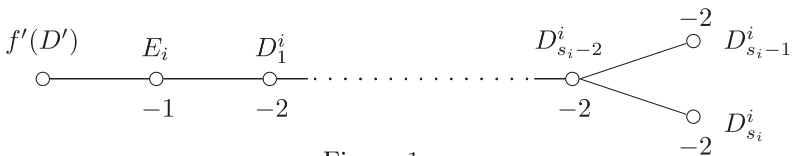

Figure 1

In what follows, we prove Theorem 2.1.

In Lemma 2.3, we consider the case $\kappa(V) \geq 0$. It is then clear that $\kappa(V) = 0$ because $\kappa(V) \leq \overline{\kappa}(V - D) = 0$.

Lemma 2.3. *Let (V, D) be an almost minimal pair of $\overline{\kappa}(V - D) = \kappa(V) = 0$. Then the following assertions hold.*

(1) *V is minimal.*

(2) *If $D \neq 0$, then each connected component of D is a (-2)-rod or a (-2)-fork.*

Proof. By (2) of Lemma 1.4, $D^{\#} + K_V \equiv 0$. Let H be an ample divisor on V. Then $H \cdot D^{\#} = H \cdot K_V = 0$ since $D^{\#}$ is effective and $\kappa(V) = 0$. Hence V is minimal, which proves the assertion (1). Since $D^{\#} = 0$, the assertion (2) follows from Lemma 1.1. Q.E.D.

In the subsequent argument, we consider the case $\kappa(V) = -\infty$. Then V is an irrational ruled surface and so there exists a ruling $\pi : V \to B$ over a smooth projective curve B of genus $g(B) = h^1(V, \mathcal{O}_V) \geq 1$.

Lemma 2.4. *With the same notations and assumptions as above, assume further that the pair (V, D) is strongly minimal. Let $D_1, \dots, D_s$ be all the irrational components of D. Then the following assertions hold.*

(1) *$s = 1$ or 2 and $(\sum_{i=1}^{s} D_i) \cdot F = 2$, where F is a general fiber of π.*

(2) *Each D_i $(1 \leq i \leq s)$ is an elliptic curve and a connected component of* Supp(D). *In particular, V is an elliptic ruled surface.*

Proof. (1) If $s = 0$, then each irreducible component of D is a fiber component of π. Then $\overline{\kappa}(V - D) = -\infty$, which is a contradiction. Hence $s \geq 1$. Let F be a general fiber of π. Since $D^{\#} + K_V \equiv 0$ by (2) of Lemma 1.4 and (1) of Lemma 1.6, we have $D^{\#} \cdot F = -K_V \cdot F = 2$. The coefficient of D_i $(1 \leq i \leq s)$ in $D^{\#}$ equals one (see the definition of $D^{\#}$ in Section 1). Hence $s = 1$ or 2 and $(\sum_{i=1}^{s} D_i) \cdot F = 2$.

(2) Since $D^{\#} + K_V \equiv 0$ and the coefficient of D_i $(1 \leq i \leq s)$ in $D^{\#}$ equals one, we have

$$\begin{aligned} 0 = D_i \cdot (D^{\#} + K_V) &= D_i \cdot (D^{\#} - D_i) + D_i \cdot (D_i + K_V) \\ &\geq D_i \cdot (D_i + K_V) \\ &\geq 0 \end{aligned}$$

for $i = 1, \ldots, s$. Then D_i is an elliptic curve and $D_i \cdot (D^{\#} - D_i) = 0$ for $i = 1, \ldots, s$. Hence the assertion (2) follows. Q.E.D.

Lemma 2.5. *With the same notations and assumptions as in Lemma* 2.4, *assume further that either $s = 2$ or $s = 1$ and the morphism $\pi|_{D_1} : D_1 \to B$ is not purely inseparable. Then V is relatively minimal and $D = D_1 + \cdots + D_s$.*

Proof. We prove that V is relatively minimal. Suppose to the contrary that V is not relatively minimal. Let G be a singular fiber (i.e., a reducible fiber) of π. Since $D_1 \cdot G > 0$, there exists an irreducible component E of G meeting D_1. By (2) of Lemma 2.4, E is not a component of D. Then

$$0 = E \cdot (D^{\#} + K_V) \geq E \cdot D_1 + E \cdot K_V \geq 0.$$

So, E is a (-1)-curve and $E \cdot D_1 = 1$.

Let $\mathcal{S} := \{D'_1, \ldots, D'_t\}$ be the set of all connected components of $\mathrm{Supp}\,(D - D_1)$ meeting E. If $\mathcal{S} = \emptyset$, then $E \cdot D = E \cdot D_1 = 1$. This implies that $E \not\subset \mathrm{Supp}\,(\lfloor D^{\#} \rfloor)$, $E \cdot (D^{\#} + K_V) = 0$ and the intersection matrix of $E + \mathrm{Bk}\,(D)$ is negative definite. This is a contradiction because (V, D) is strongly minimal. So, $\mathcal{S} \neq \emptyset$.

Claim.

(1) $t = 1$.
(2) D'_1 is a (-2)-rod.
(3) E meets one of the terminal components of D'_1.

Proof. Let $\overline{D}_i$ $(i = 1, \ldots, t)$ be an irreducible component of D'_i that meets E. Let $\overline{\beta}_i$ be the coefficient of $\overline{D}_i$ in $D^{\#}$. Since $D^{\#} + K_V \equiv 0$ and $E \cdot D_1 = 1$, we have $E \cdot (D^{\#} - D_1) = 0$. So $\overline{\beta}_i = 0$ for $1 \leq i \leq t$. Lemma 1.1 then implies that each D'_i is a (-2)-rod or a (-2)-fork. Since $E + \sum_{i=1}^{t} D'_i$ is connected, $\mathrm{Supp}\,(E + \sum_{i=1}^{t} D'_i) \subset \mathrm{Supp}\,(G)$.

Suppose that the intersection matrix of $E + \sum_{i=1}^{t} D'_i$ is not negative definite. Then $\mathrm{Supp}\,(E + \sum_{i=1}^{t} D'_i) = \mathrm{Supp}\,(G)$. Let $m(E)$ be the coefficient of E in G. Since $E \cdot D_1 = 1$ and D_1 is a section or a 2-section of π by (1) of Lemma 2.4, we know that $m(E) = 1$ or 2. If

$m(E) = 1$, then Supp$\,(G)$ contains a (-1)-curve other than E, which is a contradiction. So $m(E) = 2$. Then D_1 is a 2-section of π and so $s = 1$. Further, the morphism $\pi|_{D_1}$ of degree two is branching at $\pi(G)$. This is a contradiction because D_1 and B are elliptic curves and $\pi|_{D_1}$ is separable by the hypothesis. Hence the intersection matrix of $E + \sum_{i=1}^{t} D'_i$ is negative definite. This proves the assertions (1) – (3) of Claim. Q.E.D.

We infer from Claim that $E \not\subset$ Supp$\,(\lfloor D^{\#} \rfloor)$, $E \cdot (D^{\#} + K_V) = 0$ and the intersection matrix of $E + \mathrm{Bk}\,(D)$ is negative definite. This is a contradiction because (V, D) is strongly minimal. Thus, we know that V is relatively minimal.

Suppose that D contains a rational curve F. Since V is relatively minimal and D_1 is a section or a 2-section, $F \cdot D_1 = 1$ or 2. This contradicts (2) of Lemma 2.4. Hence $D = \sum_{i=1}^{s} D_i$. Q.E.D.

We consider the cases $s = 1$ and $s = 2$ separately.

Lemma 2.6. *With the same notations and the assumptions as in Lemma* 2.5*, assume further that $s = 2$. Then the pair (V, D) is the one* (a) *in* (2-1) *of Theorem* 2.1.

Proof. Lemma 2.5 implies that $D = D_1 + D_2$. Moreover, by the definition of $D^{\#}$, we have $D^{\#} = D$. By [9, Lemma 2.1.1 (p. 4)], we have

$$\overline{P}_1(V - D) \geq p_a(D_1) + p_a(D_2) + p_g(V) - q(V) = 1.$$

So $\overline{P}_1(V - D) = 1$ because $\overline{\kappa}(V - D) = 0$. Since $D^{\#} + K_V = D + K_V \equiv 0$ and $\overline{P}_1(V - D) = h^0(V, D + K_V) = 1$, we have $D + K_V \sim 0$. Since V is relatively minimal, we know that $V \cong \mathbb{P}_B(\mathcal{E})$, where $\mathcal{E}$ is a vector bundle of rank two over B. Since V contains two disjoint sections, it follows that $\mathcal{E}$ is decomposable (see [2, Chapter V, Section 2]). Hence, the pair (V, D) is the one (a) in (2-1) of Theorem 2.1. Q.E.D.

Lemma 2.7. *With the same assumptions as in Lemma* 2.5*, assume further that $s = 1$ and $\alpha := \pi|_{D_1} : D_1 \to B$ is not purely inseparable. Then $V = \mathbb{P}_B(\mathcal{E})$, where $\mathcal{E}$ is a vector bundle of rank two over B such that* $\deg(\det \mathcal{E}) \geq 0$*, $D = D_1$, and* $2(D + K_V) \sim 0$.

Proof. (cf. Proof of [11, Lemma 2.6.4.3 (p. 186)]) Lemma 2.5 implies that $D = D_1$. Note that $\alpha := \pi|_{D_1} : D_1 \to B$ is an étale covering of degree two. Here we may assume that α is a homomorphism of abelian varieties of dimension one. Let $V' := V \times_B D$, let $\pi' : V' \to D$ be the base change of π, and let $\alpha' : V' \to V$ be the base change of α. Then V' is smooth, π' is a $\mathbb{P}^1$-fibration, and α' is a finite étale morphism. Furthermore, π' has a section $D'_1 := \{(P, P); P \in D\}$, and ${\alpha'}^*(D) = D'_1 + D'_2$, where $D_2' := \{(P, P+Q); P \in D\}$ is another section of π' with $Q \in \mathrm{Ker}\,\alpha$ that is not the origin of D. In fact, V' has an involution ψ induced by the involution $P \mapsto P + Q$ on B, and V is the quotient $V'/ < \psi >$. Since $D_1' \cap D_2' = \emptyset$, we have $D_1' + D_2' + K_{V'} \sim 0$ by Lemma 2.6. Since $\alpha'^*(D + K_V) = D_1' + D_2' + K_{V'}$, we conclude that $2(D + K_V) \sim 0$.

It follows from Lemma 2.5 that V is a $\mathbb{P}^1$-bundle over B, i.e., $V = \mathbb{P}_B(\mathcal{E})$ for some vector bundle $\mathcal{E}$ of rank two over B. Set $e := -\deg(\det \mathcal{E})$. Let F be a fiber of $\pi : V \to B$ and M a section of π with $M^2 = -e$. Then $D \equiv -K_V \equiv 2M + eF$. Since $M \cdot D = -2e + e = -e \geq 0$, we have $e \leq 0$. Q.E.D.

The pair (V, D) as in Lemma 2.7 is the one (b) in (2-1) of Theorem 2.1.

Lemma 2.8. *With the same assumptions as in Lemma* 2.4*, assume further that* $s = 1$ *and* $\alpha := \pi|_{D_1} : D_1 \to B$ *is purely inseparable. Then* (V, D) *is the one in Example* 2.2*. In particular,* $\overline{P}_2(V - D) = 1$.

Proof. Let $F_1, \ldots, F_t$ $(t \geq 0)$ be all the singular fibers of π. Then, by using the same argument as in the proof of Lemma 2.5, we know that the weighted dual graph of F_i $(i = 1, \ldots, t)$ looks like that in Figure 1 and that

$$D = D_1 + \sum_{i=1}^{t} ((F_i)_{\mathrm{red}} - E_i),$$

where E_i is a unique (-1)-curve in $\mathrm{Supp}\,(F_i)$. Since each connected component of $D - D_1$ is a (-2)-rod or a (-2)-fork, $D^{\#} = D_1$. So $\overline{P}_n(V - D) = \overline{P}_n(V - D_1)$ for every positive integer n. In particular, $\overline{\kappa}(V - D_1) = \overline{\kappa}(V - D) = 0$.

Let $g : V \to V'$ be a birational morphism from V onto a relatively minimal surface V' and set $D' := g(D_1)$. Then D' is an elliptic curve, $\pi \circ g^{-1}|_{D'} : D' \to B$ is a purely inseparable morphism of degree two, and $\overline{\kappa}(V' - D') = \overline{\kappa}(V - D) = 0$. Then [10, Theorem 2] implies that (V', D') is the Frobenius pair $(V_{\mathrm{Frob}}, D_{\mathrm{Frob}})$ over B or the pair obtained from the Frobenius pair $(V_{\mathrm{Frob}}, D_{\mathrm{Frob}})$ by an elementary transformation with center at a point on D_{Frob} (see Example 2.2). Hence the assertions follow. Q.E.D.

The proof of Theorem 2.1 is thus completed.

§3. Logarithmic plurigenera of irrational open algebraic surfaces

In this section, we give some results on logarithmic plurigenera of irrational open algebraic surfaces of $\overline{\kappa} \geq 0$ in any characteristic. The main result of this section is the following:

Theorem 3.1. *Let S be a smooth irrational ruled oepn algebraic surface. Then $\overline{\kappa}(S) \geq 0$ if and only if $\overline{P}_2(S) > 0$.*

In what follows, we prove Theorem 3.1. The "if" part of Theorem 3.1 is clear. So we prove the "only if" part.

Let (V, D) be an SNC-pair with $V - D \cong S$. By Lemmas 1.3 and 1.6, we may assume that (V, D) is strongly minimal. Since V is an irrational ruled surface, there exists a ruling $p : V \to B$ onto a smooth projective curve B of genus $g(B) = h^1(V, \mathcal{O}_V) > 0$. Let D' be the sum of the fiber components of D. It is clear that $\overline{P}_n(S) \geq \overline{P}_n(V - (D - D'))$ for every positive integer n. Since $\overline{\kappa}(V - D) \geq 0$, by using [10, Theorem 1], we konw that $2 \leq F \cdot D = F \cdot (D - D')$ for a fiber F of π. By using [10, Theorem 1] again, we have $\overline{\kappa}(V - (D - D')) \geq 0$. Therefore, in order to prove Theorem 3.1, we may assume further that $D' = 0$. We consider the following three cases separately.

Case: $\overline{\kappa}(V - D) = 0$. In this case, it follows from Theorem 2.1 that $\overline{P}_2(V - D) = 1$.

Case: $\overline{\kappa}(V - D) = 2$. We note that $D^{\#} = D$ because D contains no rational curves. Since $\overline{\kappa}(V - D) = 2$ and (V, D) is almost minimal, it follows from (2-3) of Lemma 1.4 that $D + K_V$ is nef and big. Then we have

$$h^2(V, n(D + K_V)) = h^0(V, K_V - n(D + K_V)) = 0$$

for every positive integer n. By using the Riemann–Roch theorem, we have

$$\begin{aligned} h^0(V, n(D + K_V)) &\geq \frac{n}{2}(D + K_V) \cdot (n(D + K_V) - K_V) + \chi(\mathcal{O}_V) \\ &= \frac{n(n-1)}{2}(D + K_V)^2 + \frac{n}{2}(D + K_V) \cdot D \\ &\qquad + 1 - g(B) \end{aligned}$$

for every positive integer n. Since every component of D has genus $\geq g(B)$, we have

$$\frac{n}{2}(D + K_V) \cdot D \geq n(g(B) - 1).$$

Then,

$$h^0(V, n(D+K_V)) \geq \frac{n(n-1)}{2}(D+K_V)^2 + (n-1)(g(B)-1).$$

Since $(D+K_V)^2 > 0$ and $g(B) \geq 1$, we conclude that $\overline{P}_n(S) \geq h^0(V, n(D+K_V)) > 0$ for every integer $n \geq 2$.

Case: $\overline{\kappa}(V-D) = 1$**.** We use Lemma 1.8. Let the assumptions and notations be the same as in Lemma 1.8.

We consider the case where the fibration π is either an elliptic fibration or a quasi-elliptic fibration.

Lemma 3.2. *With the same notations and assumptions as in* (1) *of Lemma* 1.8*, assume further that* V *is an irrational ruled surface. Then* $\overline{P}_2(V-D) > 0$.

Proof. Since V is an irrational ruled surface, W is an elliptic ruled surface and π is an elliptic fibration. Moreover, since π is relatively minimal, so is W.

Claim. With the same notations as above, the following assertions hold.

(1) $B \cong \mathbb{P}^1$.
(2) $t := \deg \delta = \mathrm{length} \mathcal{T}$.
(3) For every i with $d_i \neq 0$, $d_i = 1$.

Proof. Since W is an elliptic ruled surface and $\deg \delta = \mathrm{length} \mathcal{T}$, the assertions (1) and (2) are clear. Moreover, we see that every fiber of π is a multiple of a smooth elliptic curve. Since d_i is the coefficient of $h'(F_i)$ in $D^{\#}$, the assertion (3) holds because $h'(F_i)$ is not a rational curve. Q.E.D.

By (1-2) of Theorem 1.8, we have

$$C + K_W = (t-2)\pi^*(P) + \sum_{r=1}^{s} a_r E_r + \sum_{i=1}^{j} F_i,$$

where P is a point of $B \cong \mathbb{P}^1$, $s, j \geq 0$, $a_r E_r$ ranges over all multiple fibers of π with multiplicity m_r, $0 \leq a_r < m_r$, and $a_r = m_r - 1$ if $m_r E_r$ is not a wild fiber of π. If $t \geq 2$, then $C + K_W \geq 0$ and so $\overline{P}_1(V-D) > 0$. From now on, we assume $t \leq 1$. Since $\kappa(W) = -\infty$ and $\overline{\kappa}(V-D) = 1$, we have $C = \sum_{i=1}^{j} F_i > 0$.

Since $(C + K_W) \cdot A > 0$ for any ample divisor A on W, we have

$$t - 2 + \sum_{r=1}^{s} \frac{a_r}{m_r} + \sum_{i=1}^{j} \frac{1}{n_i} > 0, \tag{3.1}$$

where n_iF_i is the scheme-theoretic fiber of π containing F_i.

Case 1: $t = 1$. We may assume that m_1E_1 is the unique wild fiber of π. By (3.1), we have

$$\frac{a_1}{m_1} + \sum_{r=2}^{s} \frac{m_r - 1}{m_r} + \sum_{i=1}^{j} \frac{1}{n_i} > 1. \tag{3.2}$$

We consider the following subcases separately.

Subcase 1-1: π has the unique multiple fiber, i.e., $s = 1$. Then we have

$$C + K_W = -\pi^*(P) + a_1E_1 + \sum_{i=1}^{j} F_i.$$

If $n_i = 1$ for some i, $1 \le i \le j$, then $C + K_W \ge 0$ and so $\overline{P}_1(V - D) > 0$. Suppose that $n_i \ge 2$ for any $i = 1, \ldots, j$. By the assumption, we know that $j = 1$, $n_1 = m_1$ and $n_1F_1 = m_1E_1$. So

$$\frac{a_1}{m_1} + \sum_{i=1}^{j} \frac{1}{n_i} = \frac{a_1 + 1}{m_1} \le 1.$$

This contradicts (3.2). Therefore, we know that $\overline{P}_1(V - D) > 0$.

Subcase 1-2: π has just two multiple fibers. Then m_1E_1 and m_2E_2 exhaust the multiple fibers of π and $a_2 = m_2 - 1$. By (3.2), we have

$$\frac{a_1}{m_1} + \sum_{i=1}^{j} \frac{1}{n_i} > \frac{1}{m_2}.$$

By the canonical bundle formula (cf. [1, Theorem 2]), we know that either $a_1 = m_1 - 1$ or $a_1 = m_1 - \nu_1 - 1$, where ν_1 is a positive integer satisfying $\nu_1 | m_1$. If $a_1 = m_1 - 1$, then

$$2K_W = -2\pi^*(P) + 2(m_1 - 1)E_1 + 2(m_2 - 1)E_2 \ge 0.$$

This is a contradiction because $\kappa(W) = -\infty$. Hence, $a_1 = m_1 - \nu_1 - 1$ and $\nu_1 \ge 1$.

If $n_i = 1$ for some i, $1 \le i \le j$, then $C + K_W \ge -\pi^*(P) + \sum_{i=1}^{j} F_i \ge 0$ and so $\overline{P}_1(V - D) > 0$. So we may assume that $n_i \ge 2$ for $i = 1, \ldots, j$. Then $(1 \le) j \le 2$ and $\{n_1F_1, \ldots, n_jF_j\} \subset \{m_1E_1, m_2E_2\}$. If $n_iF_i = m_2E_2$ for some i, $1 \le i \le j$, then $C + K_W \ge -\pi^*(P) + (m_2 - 1)E_2 + F_i \ge 0$ and so $\overline{P}_1(V - D) > 0$. Hence we may assume further that $j = 1$ and

$F_1 = E_1$. Then $C + K_W = -\pi^*(P) + (m_1 - \nu_1)E_1 + (m_2 - 1)E_2$ and $\nu_1 | m_1$. Since $m_1 - 2\nu_1, m_2 - 2 \geq 0$, we have

$$\begin{aligned} 2(C + K_W) &= -2\pi^*(P) + 2(m_1 - \nu_1)E_1 + 2(m_2 - 1)E_2 \\ &= (m_1 - 2\nu_1)E_1 + (m_2 - 2)E_2 \\ &\geq 0. \end{aligned}$$

Therefore, we conclude that $\overline{P}_2(V - D) > 0$.

Subcase 1-3: π has at least three multiple fibers. Then we see that $\overline{\kappa}(W) \geq 0$ by using the canonical bundle formula (cf. [1, Theorem 2]). This is a contradiction. So this subcase does not take place.

Case 2: $t = 0$. In this case, π has no wild fibers. So $a_s = m_s - 1$ for any $s = 1, \ldots, t$. By (3.1), we have

$$\sum_{r=1}^{s} \frac{m_r - 1}{m_r} + \sum_{i=1}^{j} \frac{1}{n_i} > 2. \tag{3.3}$$

We consider the following subcases separately.

Subcase 2-1: π has no multiple fibers. By (3.3), $n_i = 1$ for any $i = 1, \ldots, j$ and $j > 2$. Then $C + K_W = -2\pi^*(P) + \sum_{i=1}^{j} F_i > 0$ and so $\overline{P}_1(V - D) > 0$.

Subcase 2-2: π has the unique multiple fiber m_1E_1. By (3.3),

$$\sum_{i=1}^{j} \frac{1}{n_i} > 1 + \frac{1}{m_1}.$$

Since $n_i = 1$ or m_1 for $i = 1, \ldots, j$, we may assume that $j \geq 2$ and $n_1 = n_2 = 1$. Then $C + K_W \geq -2\pi^*(P) + F_1 + F_2 \geq 0$ and so $\overline{P}_1(V - D) > 0$.

Subcase 2-3: π has just two multiple fibers m_1E_1 and m_2E_2. By (3.3),

$$\sum_{i=1}^{j} \frac{1}{n_i} > \frac{1}{m_1} + \frac{1}{m_2}.$$

Since $n_i \in \{1, m_1, m_2\}$ for $i = 1, \ldots, j$, the above inequality implies that $n_i = 1$ for some i, $1 \leq i \leq j$. We may assume that $n_1 = 1$. Then

$$C + K_W = -\pi^*(P) + (m_1 - 1)E_1 + (m_2 - 1)E_2 + \sum_{i=2}^{j} F_i.$$

Since $m_1, m_2 \geq 2$, we obtain

$$2(C + K_W) = (m_1 - 2)E_1 + (m_2 - 2)E_2 + \sum_{i=2}^{j} 2F_2 \geq 0.$$

Hence $\overline{P}_2(V - D) > 0$.

Subcase 2-4: π has just three multiple fibers m_1E_1, m_2E_2, and m_3E_3. If $n_i = 1$ for some i, $1 \leq i \leq j$, then, by using the same argument as in Subcase 2-3, we see that $\overline{P}_2(V - D) > 0$. Suppose that $n_i > 1$ for every $i = 1, \ldots, j$. We note that $\sum_{i=1}^{j} F_i \neq 0$, i.e., $j \geq 1$. So we may assume that $F_1 = E_1$. Then

$$C + K_W = -\pi^*(P) + (m_2 - 1)E_2 + (m_3 - 1)E_2 + \sum_{i=2}^{j} F_i.$$

Since $m_2, m_3 \geq 2$, we obtain

$$2(C + K_W) = (m_2 - 2)E_2 + (m_3 - 2)E_2 + \sum_{i=2}^{j} 2F_i \geq 0.$$

Hence $\overline{P}_2(V - D) > 0$.

Subcase 2-5: π has at least four multiple fibers. Then $\kappa(W) \geq 0$ by using the canonical bundle formula (cf. [1, Theorem 2]). This is a contradiction.

The proof of Lemma 3.2 is thus completed. Q.E.D.

We consider the case where the fibration π is a $\mathbb{P}^1$-fibration.

Lemma 3.3. *With the same notations and assumptions as in* (2) *of Lemma* 1.8, *assume further that* V *is an irrational ruled surface. Then* $\overline{P}_2(V - D) > 0$.

Proof. Since V is an irrational ruled surface, $g(B) > 0$. If $g(B) \geq 2$, then it follows from [7, Lemma 3.1] that $\overline{P}_n(V - D) > 0$ for every integer $n \geq 2$.

Suppose that $g(B) = 1$. As seen from the proof of [7, Lemma 3.2] in [7], we see that $\overline{P}_n(V - D) > 0$ for every integer $n \geq 2$ if either H is reducible or H is irreducible and the morphism $\pi|_H : H \to B$ is separable.

Suppose further that H is irreducible and $\pi|_H : H \to B$ is not separable. Since $\deg \pi|_H = 2$, $\pi|_H$ is then a purely inseparable covering of degree two. Then we infer from [10, Lemma 2.5] that the pair

(W, H) is isomorphic to either the Frobenius pair $(V_{\mathrm{Frob}}, D_{\mathrm{Frob}})$ or the pair obtained from the Frobenius pair by an elementary transformation with center at a point on D_{Frob} (see Example 2.2). By [10, Lemma 2.6], $2(H + K_W) \sim 0$. Since $C \geq H$, we have $|2(\lfloor C \rfloor + K_W)| \neq \emptyset$. Hence $\overline{P}_2(V - D) > 0$. Q.E.D.

The proof of Theorem 3.1 is thus completed.

Finally, we prove Theorem 0.1.

Proof of Theorem 0.1. The last assertion is Theorem 3.1. We prove the first assertion. Let S be a smooth irrational open algebraic surface and (V, D) be an SNC-pair with $V - D \cong S$. The "if" part is clear. We prove the "only if" part. If $\kappa(V) = -\infty$, then V is an irrational ruled surface. So $\overline{P}_2(S) > 0$ by Theorem 3.1. Assume that $\kappa(V) \geq 0$. Since $\overline{P}_n(S) = h^0(V, n(D + K_V)) \geq P_n(V)$ for every positive integer n, it follows from the structure theorems on smooth projective surfaces (cf. [13]) that $\overline{P}_4(S) > 0$ or $\overline{P}_6(S) > 0$. Q.E.D.

References

[1] E. Bombieri and D. Mumford, Enriques' classification of surfaces in char. p, II, In: Complex analysis and algebraic geometry, Iwanami Shoten, Tokyo, 1977, pp. 23–42.

[2] R. Hartshorne, Algebraic Geometry, Graduate Texts in Mathematics 52, Springer-Verlag, New York, Heiderberg, Berlin, 1977.

[3] S. Iitaka, On logarithmic Kodaira dimension of algebraic varieties, In: Complex analysis and algebraic geometry, Iwanami Shoten, Tokyo, 1977, pp. 175–189.

[4] S. Iitaka, Algebraic Geometry, Graduate Texts in Mathematics 76, Springer-Verlag, New York, Heiderberg, Berlin, 1981.

[5] T. Kambayashi, On Fujita's strong cancellation theorem for the affine plane, J. Fac. Sci. Univ. Tokyo, **27** (1980), 535–548.

[6] Y. Kawamata, On the classification of non-complete algebraic surfaces, In: Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), Lecture Notes in Mathematics 732, Springer, Berlin, 1979, pp. 215–232.

[7] H. Kojima, Open algebraic surfaces of logarithmic Kodaira dimension one, In: Affine Algebraic Geometry (eds. K. Masuda, H. Kojima and T. Kishimoto), World Scientific, 2013, pp. 135–159.

[8] Y. Kuramoto, On the logarithmic plurigenera of algebraic surfaces, Compos. Math., **43** (1981), 343–364.

[9] M. Miyanishi, Non-complete algebraic surfaces, Lecture Notes in Mathematics 857, Springer-Verlag, Berlin-New York, 1981.

[10] M. Miyanishi, On affine-ruled irrational surfaces, Invent. Math., **70** (1982), 27–43.

[11] M. Miyanishi, Open algebraic surfaces, CRM Monograph Series 12, American Mathematical Society, Providence, RI, 2001.

[12] M. Miyanishi and S. Tsunoda, Non-complete algebraic surfaces with logarithmic Kodaira dimension $-\infty$ and with non-connected boundaries at infinity, Japan. J. Math., **10** (1984), 195–242.

[13] D. Mumford, Enriques' classification of surfaces in char. p, I, In: Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, pp. 325–339.

[14] F. Sakai, Semi-stable curves on algebraic surfaces and logarithmic pluricanonical maps, Math. Ann., **254** (1980), 89–120.

[15] S. Tsunoda, Structure of open algebraic surfaces, I, J. Math. Kyoto Univ., **23** (1983), 95–125.

Department of Mathematics
Faculty of Sicence, Niigata University
8050 Ikarashininocho, Nishi-ku, Niigata 950-2181
Japan
E-mail address: `kojima@math.sc.niigata-u.ac.jp`