Advanced Studies in Pure Mathematics 75, 2017
Algebraic Varieties and Automorphism Groups
pp. 183–188

One more proof of the Abhyankar-Moh-Suzuki Theorem

Shulim Kaliman

Abstract.

We extract the Abhyankar-Moh-Suzuki theorem from the Lin-Zaidenberg theorem.

§ Introduction

Let Γ_0 be the zero locus of a primitive polynomial $p \in \mathbb{C}[x, y]$. Suppose that Γ_0 is a smooth irreducible simply connected curve (i.e. $\Gamma_0 \simeq \mathbb{C}$). Then the Abhyankar-Moh-Suzuki theorem [1], [16] states that

in a suitable polynomial coordinate system on $\mathbb{C}^2$ the curve Γ_0 is a coordinate axis, i.e. p may be viewed as a coordinate function.

The Lin-Zaidenberg theorem [9] states that if Γ_0 is not isomorphic but only homeomorphic to $\mathbb{C}$ as a topological space then

in a suitable polynomial coordinate system on $\mathbb{C}^2$ the polynomial $p(x, y)$ is of the form $x^k + y^l$ where $k, l \geq 2$ are relatively prime.

These theorems are extremely important tools of the modern affine algebraic geometry and it is not a surprise that different methods of proving them were suggested after the original papers. In the case of the Abhyankar-Moh-Suzuki theorem there is at least a dozen of proofs (e.g., see [11], [14], [15], [12], [7], [2], [3], [5], [4], [18], [8], [10] [13]). Some of these papers contain actually simultaneous proofs of the Abhyankar-Moh-Suzuki and Lin-Zaidenberg theorems (e.g., see [12], [8], [13]) but none of them used the beautiful technique developed in [9] (and refined later in [17]). At first glance this technique (based of the theory of Teichmüller spaces and hyperbolic analysis) does not imply the Abhyankar-Moh-Suzuki theorem but we show below that it is applicable

Received September 6, 2015.
Revised December 23, 2015.
2010 *Mathematics Subject Classification.* 14R10.
Key words and phrases. simply connected curve, zero locus of a primitive polynomial.

and there is a simple reduction of the Abhyankar-Moh-Suzuki theorem to the Lin-Zaidenberg one.

Acknowledgements. In the first version of this text written about 30 years ago the author expressed his gratitude to V. Ya. Lin and M. Zaidenberg for useful discussions. Though none of us remember these discussions now the gratitude still stands. The author is also indebted to the referee for his essential contribution to the quality of this paper.

§1. Reduction

Definition 1.1. Recall that every pair (k,l) of relatively prime natural numbers defines a weighted degree function d on the ring of polynomials in x and y such that for a nonzero polynomial $r(x,y) = \sum_{i,j} a_{ij}x^iy^j$ one has $d(r) = \max_{(i,j)\in I}(li + kj)$ where $I = \{(i,j) \in (\mathbb{Z}_{\geq 0})^2 | a_{ij} \neq 0\}$. We let $\hat{I} = \{(i,j) \in I | li + kj = d(p)\}$ and we call $\hat{r}(x,y) = \sum_{(i,j)\in\hat{I}} a_{ij}x^iy^j$ the leading quasi-homogeneous part of r with respect to the grading induced by d.

Lemma 1.2. *Let the zero locus of a primitive polynomial $p(x,y)$ be isomorphic to $\mathbb{C}$. Then there is an automorphism α of $\mathbb{C}^2$ such that either $q := p\circ\alpha$ is linear or the leading quasi-homogeneous part of q with respect to some weighted degree function d is of the form $\hat{q}(x,y) = (x^k + y^l)^n$ where $k,l \geq 2$ are relatively prime.*

Proof. For d as in Definition 1.1 the fact that k and l are relatively prime implies that the leading quasi-homogeneous part of any nonzero polynomial (and in particular p) is automatically of the form

$$ax^{n_0}\prod_{i=1}^{m}(c_ix^k + y^l)^{n_i}$$

where $n_i \geq 0$, $a \neq 0$, $c_1,\ldots,c_m \in \mathbb{C}$ are distinct, and $d(p) = n_0 + kl\sum_{i=1}^m n_i$. Furthermore, unless p depends on one variable only (in which case it is automatically linear) the pair (k,l) can be chosen so that $m \geq 1$.

However, in this case after the natural embedding of $\mathbb{C}^2$ into the weighted projective plane, associated with the grading induced by d, one can see that the irreducible curve $\Gamma_0 = p^{-1}(0)$ must have $m+1$ (resp. m) points at infinity for $n_0 > 0$ (resp. $n_0 = 0$) and, therefore, at least the same number of punctures. Since $\Gamma_0 \simeq \mathbb{C}$ has one puncture we conclude that

$$\hat{p}(x,y) = a(cx^k + y^l)^n$$

where a and $c \neq 0$. If either k or l (say l) is equal to 1 then applying a de Joinquère's transformation $(x, y) \to (x, y + cx^k)$ we can reduce the standard degree of p. Thus, choosing α so that $q = p \circ \alpha$ has the minimal possible standard degree we see that unless q is linear one can suppose that $\hat{q}(x, y) = (x^k + y^l)^n$ with relatively prime $k, l \geq 2$.

Q.E.D.

Proposition 1.3. *In Lemma 1.2 one has $n = 1$, i.e.unless q is linear the leading quasi-homogeneous part of q is $\hat{q}(x, y) = x^k + y^l$ with relatively prime $k, l \geq 2$.*

Proof. Consider the surface S given by

$$z^{nkl} q(xz^{-l}, yz^{-k}) = 1$$

in $\mathbb{C}^3$ together with the family of curves $z|_S : S \to \mathbb{C}$. Then for $a \neq 0$ the curve $S_a = S \cap \{z = a\}$ is isomorphic to the fiber $\Gamma_c = \{q(x, y) = c\}$ where $c = a^{-nkl}$. By [9] all nonzero fibers of q are pairwise isomorphic, which yields a pairwise isomorphism of curves S_a for $a \neq 0$. Note that S_0 is given by equation $\hat{q}(x, y) = 1$ in the plane $\{z = 0\} \subset \mathbb{C}^3$ and thus S_0 consists of n disjoint components isomorphic to the curve $x^k + y^l = 1$. However, in the case of isomorphic irreducible general fibers any degenerate fiber may contain at most one hyperbolic component[1] [17, Theorem 5.3]. Thus $n = 1$. Q.E.D.

Corollary 1.4. *If q in Proposition 1.3 is not linear then general fibers of q have negative Euler characteristic.*

Proof. Indeed, for relatively prime $k, l \geq 2$ the Riemann-Hurwitz formula implies that the Euler characteristic of the curve $x^k + y^l = 1$ (and, therefore, S_0) is at most -1. The Suzuki formula for Euler characteristic [16] implies that the same is true for general curves S_a. Q.E.D.

Lemma 1.5. *Let Δ_r be the disc $\{x \in \mathbb{C} | \, |x| < r\}$ and $\pi : \mathbb{C}^2 \to \mathbb{C}$ be the projection to the x-axis. Then under the assumption of Lemma 1.2 for every $\delta > 0$ there exists $r > 0$ such that π maps the set*

$$R = \{(x, y) \in \mathbb{C}^2 | \, \frac{\partial q(x, y)}{\partial y} = 0 \ \& \ |q(x, y)| < \delta\}$$

into Δ_r.

[1]More precisely, the smooth part of such a fiber contains at most one hyperbolic component. Furthermore, this component is isomorphic to a quotient of the general fiber with respect to a finite group action.

Proof. By Proposition 1.3 $\hat{q}(x,y) = x^k + y^l$. Hence in the polynomial polyhedron $\{(x,y) \in \mathbb{C}^2 | \, |q(x,y)| < \delta\}$ for x with large absolute value one has $|x|^l \approx |y|^k$. Furthermore, looking at the quasi-leading part of the polynomial $\frac{\partial q(x,y)}{\partial y}$ one can see now that for such values of x the absolute value of $\frac{\partial q(x,y)}{\partial y}$ must be also large, which yields the desired conclusion.

Q.E.D.

Lemma 1.6. *For δ and r from Lemma 1.5 and any c with $0 < |c| < \delta$ the set $\Gamma_c \cap \pi^{-1}(\mathbb{C} \setminus \bar{\Delta}_r)$ is a disjoint union of components biholomorphic to a punctured disc.*

Proof. Note that the solutions of the system $\frac{\partial q(x,y)}{\partial y} = q(x,y) - c = 0$ are ramification points of the finite morphism $\pi|_{\Gamma_c} : \Gamma_c \to \mathbb{C}$ that is the restriction of π. Hence by Lemma 1.5 $\Gamma_c \cap \pi^{-1}(\mathbb{C} \setminus \bar{\Delta}_r)$ is the union of components biholomorphic to a punctured disc.

Q.E.D.

Corollary 1.7. *The Riemann surfaces Γ_c and $\Gamma_c \cap \pi^{-1}(\Delta_r)$ are diffeomorphic.*

1.1. Proof of the Abhyankar-Moh-Suzuki theorem

Assume to the contrary that q in Lemma 1.2 and Proposition 1.3 is not linear. Then the leading quasi-homogeneous part of q is $x^k + y^l$ with $k, l \geq 2$. Note that for large r the set $\Gamma_0 \cap \pi^{-1}(\Delta_r)$ is biholomorphic to a disc. The smoothness of Γ_0 implies that $\Gamma_c \cap \pi^{-1}(\Delta_r)$ is also biholomorphic to a disc for sufficiently small $|c|$. By Corollary 1.7 Γ_c is contractible in contradiction with Corollary 1.4. Hence q must be linear, which concludes the proof.

Remark 1.8. The Abhyankar-Moh-Suzuki theorem is valid over any algebraically closed field K of characteristic zero while the proof before is done over the field of complex numbers. However, the Lefschetz Principle enables us to reduce the general case to the complex one.

Indeed, the description of the polynomial p and an isomorphism between $p^{-1}(0)$ and a line involves a finite number of elements of K. These elements generate an algebraically closed subfield K_0 of K which can be also embedded into $\mathbb{C}$. We can consider p as a polynomial over K_0 and, consequently, over $\mathbb{C}$. If it is known already that every fiber $\Gamma_c = p^{-1}(c)$ of p over $\mathbb{C}$ is isomorphic to a line then the same is true for the fibers of p over K_0 (and, therefore, over K) since the genus and the number of punctures of any affine curve survive the field extention $\mathbb{C} : K_0$. Hence one can extract the general form of the Abhyankar-Moh-Suzuki theorem from the fact that a polynomial with general fibers isomorphic to a line is a variable in a suitable polynomial coordinate system [6].

References

[1] S. Abhyankar, T.-T. Moh, Embeddings of the line in the plane, J. Reine Angew. Math., **276** (1975), 148-166.

[2] E. Artal-Bartolo, Une démonstration géométrique du théorème d'Abhyankar-Moh, J. Reine Angew. Math., **464** (1995), 97-108.

[3] N. A′Campo, M. Oka, Geometry of plane curves via Tschirnhausen resolution tower, Osaka J. Math., **33** (1996), no. 4, 1003-1033.

[4] R.V. Gurjar, A new proof of the Abhyankar-Moh-Suzuki theorem, Transform. Groups, **7** (2002), no. 1, 61-66.

[5] R. V. Gurjar, M. Miyanishi, On contractible curves in the complex affine plane, Tohoku Math. J., (2) **48** (1996), no. 3, 459-469.

[6] A. Gutwirth, An inequality for certain pencils of plane curves, Proc. Amer. Math. Soc., **12** (1961), 631-638.

[7] M. C. Kang, On Abhyankar-Moh's epimorphism theorem, Amer. J. Math., **113** (1991), no. 3, 399-421.

[8] M. Koras, On contractible plane curves, In: Affine Algebraic Geometry, Osaka Univ. Press, Osaka, 2007, pp. 275-288.

[9] V.Ya. Lin, M. Zaidenberg, An irreducible, simply connected algebraic curve in $\mathbb{C}^2$ is equivalent to a quasihomogeneous curve, (Russian) Dokl. Akad. Nauk SSSR, **271** (1983), no. 5, 1048-1052.

[10] L. Makar-Limanov, A new proof of the Abhyankar-Moh-Suzuki theorem via a new algorithm for the polynomial dependence, J. Algebra Appl., **14** (2015), no. 9, 1540001, 12 pp.

[11] M. Miyanishi, Analytic irreducibility of certain curves on a nonsingular affine rational surface, In: Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977), Kinokuniya Book Store, Tokyo, 1978, pp. 575-587.

[12] W. D. Neumann, Complex algebraic plane curves via their links at infinity, Invent. Math., **98** (1989), no. 3, 445-489.

[13] P. A. Palka, A new proof of the theorems of Lin-Zaidenberg and Abhyankar-Moh-Suzuki, J. Algebra Appl., **14** (2015), no. 9, 1540012, 15 pp.

[14] L. Rudolph, Embeddings of the line in the plane, J. Reine Angew. Math., **337** (1982), 113-118.

[15] D. R. Richman, On the computation of minimal polynomials, J. Algebra, **103** (1986), no. 1, 1-17.

[16] M. Suzuki, Propiétés topologiques des polynomes de deux variables complexes, et automorphismes algéarigue de lespace $\mathbb{C}^2$, J. Math. Soc. Japan, **26** (1974), 241-257.

[17] M. Zaidenberg, Isotrivial families of curves on affine surfaces, and the characterization of the affine plane (Russian), Izv. Akad. Nauk SSSR Ser.

Mat., **51** (1987), no. 3, 534-567, 688; translation in Math. USSR-Izv., **30** (1988), no. 3, 503-532.

[18] H. A. Zoladek, A new topological proof of the Abhyankar-Moh theorem, Math. Z., **244** (2003), no. 4, 689-695.

Department of Mathematics
University of Miami
Coral Gables, FL 33124
USA
E-mail address: kaliman@math.miami.edu