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Unipotent group actions on projective varieties

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Abstract.

The correspondence between G_a-actions on affine varieties and locally nilpotent derivations of the coordinate algebras is generalized in the projective case to the correspondence between stratified G_a-actions on smooth projective varieties V and regular vector fields on V which are effectively locally nilpotent with stratification. These notions with stratifications are inspired by explicit computations of G_a-actions on the projective space $\mathbb{P}^n$ as well as the Hirzebruch surface $\mathbb{F}_n$ and the associated regular vector fields. Using partly these observations, we investigate the existence of $\mathbb{A}^1$-cylinders in Fano threefolds with rank one.

§ Introduction

In studying algebraic varieties of higher dimension, one effective approach is to decompose a given variety into algebraic varieties of lower dimension via a fibration. To find a fibration via an algebraic group action on the variety, it is expected that there exists the (algebraic) quotient variety and the quotient morphism is a fibration whose general fibers are the orbits of the algebraic group. The quotient variety exists for a nice group like a reductive algebraic group, although the fiber tends to have a complicated structure as a homogeneous space.

An algebraic group action on an algebraic variety X can be detected if one knows the automorphism group $\mathrm{Aut}(X)$ and its subgroups. If X is a projective variety, $\mathrm{Aut}(X)$ is given a group scheme structure, and the connected component $\mathrm{Aut}^0(X)$ of the identity element is an algebraic

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group. If the variety X is not projective, say affine, the automorphism group does not necessarily have any algebraic group structure possibly except an ind-affine group structure. But the structure of $\mathrm{Aut}(X)$ is helpful to find elementary algebraic subgroups like the additive group G_a or the multiplicative group G_m.

Let X be a smooth projective variety and let $\mathcal{T}_{X/k}$ be the tangent bundle. Then it is known that the Lie algebra of the algebraic group $\mathrm{Aut}^0(X)$ is $\Gamma(X, \mathcal{T}_{X/k})$. For a smooth algebraic variety X, an element v of $\Gamma(X, \mathcal{T}_{X/k})$ is a regular vector field on X. If $X = \mathrm{Spec}\, A$ is an affine variety, this field corresponds to an element Δ of $\mathrm{Der}_k(A)$, which is a k-derivation of A. If Δ is locally nilpotent, $\exp(t\Delta)$ with $t \in k$ defines a G_a-action on X. If an affine variety $X = \mathrm{Spec}\, A$ has a G_a-action defined in this way, one can think about the "algebraic quotient" $X//G_a$ and the "quotient morphism" $q : X \to X//G_a$ if $A^\Delta := \mathrm{Ker}\, \Delta$ is finitely generated over k. Then $X//G_a$ is defined as $\mathrm{Spec}\, A^\Delta$ and q is the morphism associated to the inclusion $A^\Delta \hookrightarrow A$. If $\dim X \le 3$, by a theorem of Zariski, A^Δ is finitely generated over k. Hence $X//G_a$ and q exist, and the morphism q is an $\mathbb{A}^1$-fibration. Here an $\mathbb{A}^1$-fibration is a dominant morphism $f : X \to Y$ of algebraic varieties such that general closed fibers as well as the generic fiber are isomorphic to $\mathbb{A}^1$. We discussed $\mathbb{A}^1$-fibrations on affine threefolds in [16]. The correspondence between G_a-actions on affine varieties and locally nilpotent derivations on the coordinate algebras has been successfully used in affine algebraic geometry. Meanwhile, the results on G_a-actions on projective varieties are not abundant except for some basic ones in [2, 7, 20].

In the present article, we look into unipotent group actions on projective varieties. In the later sections, we restrict ourselves to G_a-actions. In order to develop some meaningful theory about this subject, we need leading models (or examples) and we take the projective space $\mathbb{P}^n$, the Hirzebruch surface $\mathbb{F}_n$, etc. Throughout the article, keywords are regular vector fields and stratifications on a given projective variety V, which is a sequence of closed subsets

$$V_0 = V \supset V_1 \supset \cdots \supset V_{n-1} \supset V_n,\ \dim V_i = n - i \qquad (*)$$

In the sequence $(*)$, strata consist of $V_0 \setminus V_1, V_1 \setminus V_2, \ldots, V_{n-1} \setminus V_n$ and each stratum satisfies some property varying from one situation to the other.

In Section one, we determine explicitly the Lie algebra $\Gamma(X, \mathcal{T}_{X/k})$ for $X = \mathbb{P}^n, X = \mathbb{F}_n$ and a Danielewski surface $X = \{xy = z^2 - 1\}$. In the last case which treats an affine surface, $\Gamma(X, \mathcal{T}_{X/k})$ is seen to have more complicated structure than in the first two cases. Furthermore, we observe the behavior of $\Gamma(X, \mathcal{T}_{X/k})$ under the blowing-up. Thus this section is for preliminary results for the later developments.

In Section two, we introduce the notion of unipotent group orbit stratification on a smooth projective variety V which is the sequence $(*)$ such that each stratum is the finite union of orbits under the given action of a unipotent group U (see Definition 2.5). In particular, since $V_0 \setminus V_1$ is a single unipotent group orbit, the variety V itself is very restrictive. In fact, if we simply assume that each V_i is smooth and $\overline{\kappa}(V_i - V_{i+1}) = -\infty$, then V is isomorphic to $\mathbb{P}^n$ (Theorem 2.8). So, the unipotent group orbit stratification is thought to be a prototype of stratifications of other kinds to be introduced in later sections. In fact, the assumption that each stratum consists of finite unipotent orbits is too strong, and some algebraic or topological substitutes are desirable. For example, we may consider the condition that each stratum has as many independent G_a-actions as the dimension of the stratum or the condition that each stratum is simply connected. In Theorem 2.9, we look into the relationship between these conditions.

Section three deals with generalities of G_a-actions on smooth projective varieties. Let V be a smooth projective variety with a nontrivial G_a-action. Let H be a very ample divisor. Since H is G_a-linearizable, G_a acts on the linear system $|H|$ and hence there exists a G_a-stable member H_1 in $|H|$. The G_a-action induces a locally nilpotent homogeneous derivation of degree 0 on the graded domain $\oplus_{n\geq 0} H^0(V, \mathcal{O}(nH))$, which in turn determines the G_a-action on V (Theorem 3.3). The stratum $V \setminus H_1$ has the induced G_a-action, and the regular vector field Δ on V corresponding to the G_a-action restricts to a locally nilpotent derivation on the coordinate algebra of $V \setminus H_1$. This leads to the notion of stratified G_a-action (Definition 3.7) and the notion of a regular vector field being effectively locally nilpotent with stratification (Definition 3.9). Theorem 3.10 shows that these two notions are dual to each other.

Section four is devoted to a study of a smooth projective threefold such that Pic $(V) = \mathbb{Z}[H]$ for a smooth ample divisor H and V has a G_a-action making H stable. Then V is a Fano threefold of Picard rank one, and the structures of such threefolds are known (see [22]), but we are interested in the structure or properties of the principal stratum $X := V \setminus H$. We will treat basically the case V is $\mathbb{P}^3$ or a quadric hypersurface in $\mathbb{P}^4$. Especially noteworthy is Theorem 4.5, the assertion (2). It gives a characterization of $\mathbb{P}^3$ in terms of a G_a-action and the topological properties of X. Once X becomes isomorphic to $\mathbb{A}^3$ which is a conclusion of (2), the quotient surface $X/\!/G_a$ is isomorphic to $\mathbb{A}^2$ and the quotient morphism $q : X \to X/\!/G_a$ is surjective. But, in Theorem 4.9, this conclusion $X/\!/G_a \cong \mathbb{A}^2$ is derived, without the topological properties on X, from the existence of the above G_a-ation and the assumption that the index of V is greater than one and H is smooth. But in the case where V is a smooth quadric hypersurface in $\mathbb{P}^4$, though the same conclusion is obtained, the quotient morphism $q : X \to X//G_a$ is not surjective (Theorem 4.6). Therefore these properties reflect subtle differences of Fano threefolds of rank one equipped with G_a-actions.

The article is partly meant to free G_a-actions from the framework of affine varieties and to consider them in more general settings. As explained above, a key is a regular vector field on an algebraic variety. Though there are many results related to vector fields, most of them are not written to fit our purpose and scattered in various references. So, we chose our way to exhibit the idea by giving concrete examples (though elementary). This might cause an impression that promising or original ideas are buried in isolated examples. A task to develop the details is perhaps left to our subsequent works and possibly to the interested readers.

We assume throughout the article that the ground field k is an algebraically closed field of characteristic zero. Whenever topological arguments are employed, we assume tacitly that k is the complex field $\mathbb{C}$. If V is an algebraic variety and V_1 a closed subvariety of V, the complement $V \setminus V_1$ is also denoted by $V - V_1$, especially when V_1 is a divisor of V.

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§1. Preliminary results on vector fields on projective varieties

In this section, we consider global vector fields on $\mathbb{P}^n$ or on the Hirzebruch surface $\mathbb{F}_n$ $(n \geq 0)$ in terms of vector fields on the affine space $\mathbb{A}^n$ naturally embedded into $\mathbb{P}^n$ or $\mathbb{A}^2$ into $\mathbb{F}_n$. Perhaps these computations are well-known but buried in the various references. We will give them for our conveniences.

First of all, in the case where $\mathbb{A}^n \hookrightarrow \mathbb{P}^n$, we consider a system of homogeneous coordinates $(X_0, X_1, \ldots, X_n)$ on $\mathbb{P}^n$ and set $x_i = X_i/X_0$ for $1 \leq i \leq n$.

Lemma 1.1. *Let Δ be a regular vector field on $\mathbb{A}^n$ and write*

$$\Delta = f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n} \text{ with } f_1, \ldots, f_n \in k[x_1, \ldots, x_n] \, .$$

Then Δ *extends to a regular vector field on* $\mathbb{P}^n$ *if and only if*

$$\begin{aligned}
f_1 &= a_1x_1^2 + a_2x_1x_2 + \cdots + a_nx_1x_n + \sum_{i=1}^n b_{1i}x_i + c_1 \\
f_2 &= a_1x_1x_2 + a_2x_2^2 + \cdots + a_nx_2x_n + \sum_{i=1}^n b_{2i}x_i + c_2 \\
&\cdots\cdots\cdots \\
f_n &= a_1x_1x_n + a_2x_2x_n + \cdots + a_nx_n^2 + \sum_{i=1}^n b_{ni}x_i + c_n \ .
\end{aligned}$$

where $a_i(1 \le i \le n), b_{ij}(1 \le i,j \le n), c_i(1 \le i \le n)$ *are elements of* k. *Hence* $\dim \Gamma(\mathbb{P}^n, \mathcal{T}_{\mathbb{P}^n}) = n(n+2)$.

Proof. To avoid complicated computations, we exhibit the idea in the case $n = 2$. We set $x = X_1/X_0$ and $y = X_2/X_0$. Let $U_i = \{X_i \neq 0\}$ $(i = 0,1,2)$ be the open sets of $\mathbb{P}^2$ isomorphic to $\mathbb{A}^2$. Hence $U_0 = \operatorname{Spec} k[x,y]$. Let $U_1 = \operatorname{Spec} k[u,v]$ with $u = X_0/X_1 = x^{-1}$ and $v = X_2/X_1 = yx^{-1}$. Assume that Δ is a regular vector field on $\mathbb{P}^2$. Write

$$\Delta = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} = \xi\frac{\partial}{\partial u} + \eta\frac{\partial}{\partial v}, \quad x,y \in k[x,y],\ \xi,\eta \in k[u,v].$$

Since $\xi = \Delta(u) = -u^{2-d}u^d f\left(\frac{1}{u},\frac{v}{u}\right)$ with $d = \deg_{x,y} f$, we have $d \le 2$. Similarly, on $U_2 = \operatorname{Spec} k[z,w]$ with $z = X_0/X_2 = y^{-1}$ and $w = X_1/X_2 = xy^{-1}$. Writing $\Delta = \varphi\frac{\partial}{\partial z} + \psi\frac{\partial}{\partial w}$, we have $\varphi = \Delta(z) = -z^{2-e}z^e g\left(\frac{w}{z},\frac{1}{z}\right)$ with $e = \deg_{x,y} g$. Hence $e \le 2$. Hence we can write

$$\begin{aligned}
f &= a_0x^2 + a_1xy + a_2y^2 + c_0x + c_1y + c_2 \\
g &= b_0x^2 + b_1xy + b_2y^2 + d_0x + d_1y + d_2.
\end{aligned}$$

Then we have

$$\begin{aligned}
\xi &= -u^2\left(\frac{a_0}{u^2}+a_1\frac{v}{u^2}+a_2\frac{v^2}{u^2}+\frac{c_0}{u}+c_1\frac{v}{u}+c_2\right)\\
&= -\left(a_0+a_1v+a_2v^2+c_0u+c_1uv+c_2u^2\right)\\
\eta &= -\frac{v}{u}\cdot u^2\left(\frac{a_0}{u^2}+a_1\frac{v}{u^2}+a_2\frac{v^2}{u^2}+\frac{c_0}{u}+c_1\frac{v}{u}+c_2\right)\\
&\quad +\frac{1}{u}\cdot u^2\left(\frac{b_0}{u^2}+b_1\frac{v}{u^2}+b_2\frac{v^2}{u^2}+\frac{d_0}{u}+d_1\frac{v}{u}+d_2\right)\\
&= -a_0\frac{v}{u}-a_1\frac{v^2}{u}-a_2\frac{v^3}{u}-v(c_0+c_1v+c_2u)\\
&\quad +\frac{b_0}{u}+b_1\frac{v}{u}+b_2\frac{v^2}{u}+d_0+d_1v+d_2u
\end{aligned}$$

Hence $b_0=0, b_1=a_0, b_2=a_1$ and $a_2=0$. Then it is easy to show that Δ is regular on U_2 as well. So, f and g are as stated above for $n=2$ and $x=x_1, y=x_2$. Q.E.D.

Remark 1.2. There is an exact sequence of $\mathcal{O}_{\mathbb{P}^n}$-Modules

$$0\to\mathcal{O}_{\mathbb{P}^n}\to\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)}\to\mathcal{T}_{\mathbb{P}^n}\to 0.$$

Since $\mathrm{H}^1(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n})=0$, we have $\dim\mathrm{H}^0(\mathbb{P}^n,\mathcal{T}_{\mathbb{P}^n})=n(n+2)$.

Let $V=\mathbb{F}_n$ be the Hirzebruch surface of degree n and let M be a minimal section of the canonical $\mathbb{P}^1$-fibration $p: V\to\mathbb{P}^1$. The affine plane $\mathbb{A}^2$ can be embedded into V as the complement $V\setminus(M\cup\ell_\infty)$, where ℓ_∞ is the fiber at infinity of p. We consider a regular vector field Δ on $\mathbb{A}^2$ and look for a condition with which Δ is extendable to a regular vector field on V.

Write $V=\mathrm{Proj}\,(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(n))$ and let M be defined by the projection $\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(n)\to\mathcal{O}_{\mathbb{P}^1}$. To be more precise, let $\mathbb{P}^1=U_0\cup U_1$, where $U_0=\mathrm{Spec}\,k[x]$ and $U_1=\mathrm{Spec}\,k[x^{-1}]$. Then $\mathcal{O}_{\mathbb{P}^1}(n)|_{U_0}=\mathcal{O}_{U_0}e_1$ and $\mathcal{O}_{\mathbb{P}^1}(n)|_{U_1}=\mathcal{O}_{U_1}e_1'$, where $e_1'=x^ne_1$. We write the direct summand $\mathcal{O}_{\mathbb{P}^1}$ as $\mathcal{O}_{\mathbb{P}^1}e_0$ to give a base e_0. Then V is covered by four open sets $V=V_0\cup V_1\cup V_2\cup V_3$, where

$$\begin{aligned}
V_0 &= \operatorname{Spec} k[x,y], \quad y = \frac{e_0}{e_1} \\
V_1 &= \operatorname{Spec} k[u,v], \quad u = \frac{1}{x},\ v = \frac{e_0}{e_1'} = \frac{y}{x^n} \\
V_2 &= \operatorname{Spec} k[x,z], \quad z = \frac{1}{y} = \frac{e_1}{e_0} \\
V_3 &= \operatorname{Spec} k[u,t], \quad u = \frac{1}{x},\ t = \frac{1}{v} = \frac{e_1'}{e_0} = \frac{x^n}{y}
\end{aligned}$$

Write a regular vector field Δ on the open set V_0 as

$$\Delta = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$$

with $f, g \in k[x,y]$. Express Δ on the open sets V_1, V_2, V_3 in terms of the above respective coordinate systems and find the condition for Δ to be regular on each of the above open sets. The computations show the following result.

Lemma 1.3. *Embed $\mathbb{A}^2$ into $\mathbb{F}_n$ $(n \geq 0)$ as $\mathbb{A}^2 = \mathbb{F}_n \setminus (M \cup \ell_\infty)$. Let $\Delta = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$ be a regular vector field on $\mathbb{A}^2$. Then Δ is extendable to a regular vector field on $\mathbb{F}_n$ if and only if*

(1) $f(x,y) = a_{20}x^2 + a_{10}x + a_{00}$,

(2)

$$g(x,y) = \begin{cases} b_{n0}x^n + \cdots + b_{10}x + b_{00} + b_{01}y + b_{11}xy \ (b_{11} = na_{20}) & (n > 0) \\ b_{02}y^2 + b_{01}y + b_{00} & (n = 0) \end{cases}$$

Hence $\dim \mathrm{H}^0(\mathbb{F}_n, \mathcal{T}_{\mathbb{F}_n})$ *is equal to* $n+5$ *if* $n > 0$ *and* 6 *if* $n = 0$.

Remark 1.4. Let V be $\mathbb{P}^2$ or $\mathbb{F}_n$. In Lemma 1.1 and Lemma 1.3, we tacitly used the coincidence of two k-vector spaces $\mathrm{H}^0(V, \mathcal{T}_V)$ and

$$\Gamma = \left\{ \Delta \in \Gamma(\mathbb{A}^2, \mathcal{T}_{\mathbb{A}^2}) \;\middle|\; \begin{array}{l} \text{a regular vector field on } \mathbb{A}^2 \text{ which is ex-} \\ \text{tendable to a regular vector field on } V \end{array} \right\}$$

Since a given vector field on $\mathbb{A}^2$ is uniquely extendable to a rational vector field on V, where only the coefficients are restricted if it is regular on V, there is a natural correspondence which assigns Δ to itself

$$\theta : \Gamma \to \mathrm{H}^0(V, \mathcal{T}_V).$$

Then the correspondence is an isomorphism. In fact, an element $\Delta \in \mathrm{H}^0(\mathbb{A}^2, \mathcal{T}_{\mathbb{A}^2})$ is identified with a k-derivation of the function field

$k(\mathbb{A}^2)$. So, $\mathrm{H}^0(V, \mathcal{T}_V)$ is a k-derivation of $k(\mathbb{A}^2)$ which is regular on V. The extendability of a given vector field on $\mathbb{A}^2$ onto V depends on the embedding $\mathbb{A}^2 \hookrightarrow V$. The above remark applies if one replaces $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$ by $\mathbb{A}^n \hookrightarrow \mathbb{P}^n$.

Remark 1.5. The dimension of $\mathrm{H}^0(\mathbb{F}_n, \mathcal{T}_{\mathbb{F}^n})$ can be also computed by an exact sequence

$$0 \to \mathrm{H}^0(V, \mathcal{O}_V(2M + n\ell)) \to \mathrm{H}^0(V, \mathcal{T}_V) \to \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) \to 0,$$

where $h^0(V, \mathcal{O}_V(2M + n\ell)) = n + 2$ if $n > 0$ and $= 3$ if $n = 0$. This sequence is obtained from the exact sequence

$$0 \to \mathcal{O}_V(2M + n\ell) \to \mathcal{T}_V \to p^*\mathcal{O}_{\mathbb{P}^1}(2) \to 0,$$

which is dual to

$$0 \to p^*\Omega^1_{\mathbb{P}^1/k} \to \Omega^1_{V/k} \to \Omega^1_{V/\mathbb{P}^1} \to 0.$$

Determination of regular vector fields on a smooth algebraic surface is not so easy as for $\mathbb{A}^2$. As an example, we determine those for a Danielewski surface $X = \{xy = z^2 - 1\} \subset \mathbb{A}^3$. For a similar direction of research, one can refer to [28]. Let $K = k(X)$ and let Δ be a regular vector field on X. Then, as a derivation of $k(X)/k$, Δ is written as

$$\Delta = g\frac{\partial}{\partial x} + h\frac{\partial}{\partial z},$$

where $g = \Delta(x)$ and $h = \Delta(z)$. Since $y = x^{-1}(z^2 - 1)$, we have

$$f := \Delta(y) = -\frac{1}{x^2} \cdot xyg + \frac{2z}{x}h,$$

whence $xf + yg = 2zh$. Since Δ corresponds to $\delta \in \mathrm{Hom}_R(\Omega^1_{R/k}, R)$ by $\Delta = \delta \cdot d$ with $d : R \to \Omega^1_{R/k}$ being the universal derivation of R, where R is the coordinate ring of X [19, Definition, p.172]. We have $f = \delta(dy)$, $g = \delta(dx)$ and $h = \delta(dz)$, whence $f, g, h \in R$.

Lemma 1.6. *Since $R = k[x, y] + k[x, y]z$ is a free $k[x, y]$-module, write*

$$f = f_0 + f_1 z,\ g = g_0 + g_1 z,\ h = h_0 + h_1 z,$$

where $f_i, g_i, h_i \in k[x, y]$ for $i = 0, 1$. Then we have:

(1) $h_0 = \frac{1}{2}(xf_1 + yg_1)$.

(2) *There exist* $L, M, F \in k[x, y]$ *such that*

$$f_0 = 2(xy+1)L + yF,\ \ g_0 = 2(xy+1)M - xF,\ \ h_1 = xL + yM,$$

where L, M, F, f_1 *and* g_1 *are chosen arbitrarily.*

(3) *With the choice of these elements,* Δ *is written as*

$$\Delta = \{2(xy+1)M - xF + g_1 z\}\frac{\partial}{\partial x}$$
$$+\{(\frac{1}{2}f_1 + Lz)x + (\frac{1}{2}g_1 + Mz)y\}\frac{\partial}{\partial z}.$$

Proof. Since $xf + yg = 2zh$, we have

$$(xf_0 + yg_0) + z(xf_1 + yg_1) = 2h_1(xy+1) + 2h_0 z,$$

whence

$$xf_0 + yg_0 = 2h_1(xy+1) \tag{i}$$

and

$$h_0 = \frac{1}{2}(xf_1 + yg_1). \tag{ii}$$

Let (x, y) be the maximal ideal in $k[x, y]$. Since $xy + 1 \notin (x, y)$, (i) implies $h_1 \in (x, y)$. Hence we may write

$$h_1 = xL + yM. \tag{iii}$$

By (i), we have an equality in $k[x, y]$,

$$x\{f_0 - 2(xy+1)L\} = y\{2(xy+1)M - g_0\}.$$

Since $\gcd(x, y) = 1$, we have

$$f_0 = 2(xy+1)L + yF, \quad g_0 = 2(xy+1)M - xF$$

for some $F \in k[x, y]$. Tracing the above computations backward, it is clear that the choice of L, M, F, f_1, g_1 in $k[x, y]$ is arbitrary. Q.E.D.

Let $\sigma : W \to V$ be the blowing-up of a smooth algebraic variety V with a smooth center Z. Let Δ be a vector field on V which is regular along Z. If V' is a variety birational to V, then Δ is viewed as a rational vector field on V'. So, we use the same symbol Δ to denote the rational vector field on W. The regularity of Δ near the exceptional subvariety $\sigma^{-1}(Z)$ is given by the following.

Lemma 1.7. *With the above notations, let P be a point of Z and let $\{x_1, x_2, \ldots, x_n\}$ be a system of local parameters of V at P such that Z is defined by $x_1 = x_2 = \cdots = x_d = 0$, where $d = \mathrm{codim}\,_V(Z)$. Write Δ near P as*

$$\Delta = f_1 \frac{\partial}{\partial x_1} + f_2 \frac{\partial}{\partial x_2} + \cdots + f_n \frac{\partial}{\partial x_n},$$

where $f_1, \ldots, f_n \in \mathcal{O}_{V,P}$. Then Δ is regular near $\sigma^{-1}(P)$ if and only if $f_1(P) = \cdots = f_d(P) = 0$.

Proof. Since $\sigma^{-1}(Z)$ is a $\mathbb{P}^{d-1}$-bundle over Z, $\sigma^{-1}(P)$ is a projective space $\mathbb{P}^{d-1}$ with a system of homogeneous coordinates $\{X_1, \ldots, X_d\}$. Fix i with $1 \le i \le d$. Then, on the open set $U_i = \{X_i \neq 0\}$, it holds that $X_j/X_i = x_j/x_i$ for $1 \le j \le d$ and $j \neq i$. Set $u_j = x_j/x_i$ if $1 \le j \le d$ and $j \neq i$ and $u_i = x_i$. For any point $Q \in \sigma^{-1}(P)$, the set $\{u_1 - u_1(Q), \ldots, u_d - u_d(Q), x_{d+1}, \ldots, x_n\}$ is a system of local parameters of W at Q. Hence we can write

$$\Delta = \xi_1 \frac{\partial}{\partial u_1} + \cdots + \xi_d \frac{\partial}{\partial u_d} + f_{d+1} \frac{\partial}{\partial x_{d+1}} + \cdots + f_n \frac{\partial}{\partial x_n},$$

where $\xi_i = f_i$. If $1 \le j \le d$ and $j \neq i$, we have

$$\begin{aligned} \xi_j &= \Delta(u_j) = \Delta\left(\frac{x_j}{x_i}\right) = -\frac{x_j}{x_i^2} f_i + \frac{1}{x_i} f_j \\ &= \frac{1}{x_i}\{f_j(u_1 x_i, \ldots, x_i, \ldots, u_d x_i, x_{d+1}, \ldots, x_n) \\ &\qquad -u_j f_i(u_1 x_i, \ldots, x_i, \ldots, u_d x_i, x_{d+1}, \ldots, x_n)\}\ . \end{aligned}$$

If $f_j(P) \neq 0$, then ξ_j has a simple pole along $\sigma^{-1}(Z)$. So, ξ_j is regular only if $f_j(P) = 0$. This implies that Δ is regular along $\sigma^{-1}(Z)$ only if $f_1(P) = \cdots = f_d(P) = 0$. The converse is clear by the above expression of the ξ_j. Q.E.D.

Example 1.8. Let $\sigma : \mathbb{F}_1 \to \mathbb{P}^2$ be the blowing-up with center P. By Lemma 1.7, $\Gamma(\mathbb{F}_1, \mathcal{T}_{\mathbb{F}^1})$ is identified with

$$\Gamma = \left\{ \Delta \,\middle|\, \begin{array}{l} \text{a regular vector field on } \mathbb{P}^2 \text{ which van-} \\ \text{ishes at } P \end{array} \right\}.$$

Hence $\dim \Gamma(\mathbb{F}_1, \mathcal{T}_{\mathbb{F}_1}) = \dim \Gamma(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}) - 2 = 8 - 2 = 6$.

Lemma 1.7 implies the following result.

Lemma 1.9. *Let $\sigma : W \to V$ be the blowing-up of a smooth projective variety V with center P. Assume that $\dim V = n > 1$. Then we have*

$$\dim \Gamma(V, \mathcal{T}_V) \ge \dim \Gamma(W, \mathcal{T}_W) \ge \dim \Gamma(V, \mathcal{T}_V) - n.$$

Proof. Let $\widetilde{\Delta}$ be a regular vector field on W. Let $E = \sigma^{-1}(P)$. Then $\widetilde{\Delta}|_{W \setminus E}$ is a regular vector field on $V \setminus \{P\}$. Hence it extends to a regular vector field Δ on V such that $\Delta = 0$ at P. Indeed, let $\{x_1, \ldots, x_n\}$ be a system of local parameters of V at P. As a rational vector field on V, we can write $\widetilde{\Delta} = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$. Then $f_1, \ldots, f_n$ are elements of $k(V)$ which are regular on an open neighborhood of P punctured the point P. Then $f_1, \ldots, f_n$ are regular at P as well because V is smooth at P and $n \geq 2$. Hence Δ is regular at P and $\widetilde{\Delta}$ is the extension of Δ on W. Then $\Delta = 0$ at P by Lemma 1.7. The condition that Δ vanishes at P imposes on $\Gamma(V, \mathcal{T}_V)$ the condition of dimension by at most n. Hence we obtain the stated inequalities. Q.E.D.

Example 1.10. Consider $\mathbb{F}_n$ $(n \geq 0)$ as V. With the notations of Lemma 1.3, let P be defined by $u = t = 0$ (the point of origin of the open set V_3). Let Δ be a regular vector field in Lemma 1.3. Then the condition that Δ vanishes at P imposes the condition of dimension 1 (resp. 2) if $n > 0$ (resp. $n = 0$), i.e., $a_{20} = 0$ (resp. $a_{20} = b_{02} = 0$) if $n > 0$ (resp. $n = 0$). Furthermore, by the computation in Lemma 1.7 repeated for the blowing-up at P, we conclude that

$$\dim \Gamma(\mathbb{F}_{n+1}, \mathcal{T}_{\mathbb{F}_{n+1}}) = \begin{cases} \dim \Gamma(\mathbb{F}_n, \mathcal{T}_{\mathbb{F}_n}) - 1 + 2 & (n > 0) \\ \dim \Gamma(\mathbb{F}_n, \mathcal{T}_{\mathbb{F}_n}) - 2 + 2 & (n = 0) \end{cases}$$

Hence $\dim \Gamma(\mathbb{F}_{n+1}, \mathcal{T}_{\mathbb{F}_{n+1}}) = n + 6$.

We shall give one more result (see also [25, pp. 225–226]).

Example 1.11. Let V be a del Pezzo surface of degree d. If $d \leq 5$, then there are no regular vector fields on V.

Proof. The surface V is obtained by blowing up $(9 - d)$ points $P_1, \ldots, P_m$ $(m = 9 - d)$ on $\mathbb{P}^2$ in general position, i.e., no three of them lie on a line and no five of them lie on a conic. We can choose the line at infinity ℓ_∞ so that none of $P_1, \ldots, P_m$ lies on ℓ_∞. Suppose there exists a nonzero regular vector field $\widetilde{\Delta}$ on V. Then, by the proof of Lemma 1.9, there exists a regular vector field Δ on $\mathbb{P}^2$ such that Δ vanishes at points $P_1, \ldots, P_m$. We may choose a system of coordinates $\{x, y\}$ on $\mathbb{A}^2 = \mathbb{P}^2 \setminus \ell_\infty$ so that $P_1 = (0, 0), P_2 = (1, 0)$ and $P_3 = (0, 1)$. With the notations in Lemma 1.1 (the proof in the case $n = 2$), it follows that $c_2 = d_2 = 0, d_0 = 0, a_1 + d_1 = 0, a_0 + c_0 = 0$ and $c_1 = 0$. Hence we have $f = a_0(x^2 - x) + a_1 xy$ and $g = a_0 xy + a_1(y^2 - y)$. Suppose that $P_4 = (\alpha, \beta)$ is involved. Then $\alpha\beta \neq 0$ because no three of P_1, P_2, P_3, P_4

lie on a line. Since $f(\alpha,\beta) = g(\alpha,\beta) = 0$, we have

$$\begin{vmatrix} \alpha^2 - \alpha & \alpha\beta \\ \alpha\beta & \beta^2 - \beta \end{vmatrix} = 0.$$

For otherwise $a_0 = a_1 = 0$ and $\Delta = 0$ everywhere. The above determinant gives $\alpha + \beta = 1$. Then P_2, P_3 and P_4 are colinear, which is a contradiction. So, P_4 cannot be involved, and $9 - d \leq 3$. Q.E.D.

This implies that there is no G_a-action on V if $d = 5$. The last result follows from the following two facts.

(i) $\mathrm{H}^0(V, \mathcal{T}_V)$ is the Lie algebra of the algebraic group $\mathrm{Aut}^0(V)$.

(ii) Let φ be an element of $\mathrm{Aut}^0(V)$. Then φ comes from an automorphism of $\mathbb{P}^2$ fixing the points $P_1, \ldots, P_m$. Hence if $m \geq 4$ then $\varphi = \mathrm{id}$. In fact, no three of $P_1, \ldots, P_m$ lie on a line. Hence any automorphism of $\mathbb{P}^2$ fixing four of them is the identity automorphism.

§2. Unipotent group orbit stratifications

Let X be a smooth algebraic variety with an algebraic group G acting on it, whence there is a group homomorphism $\sigma : G \to \mathrm{Aut}(X)$. Taking the Lie algebra homomorphism, we have

$$d\sigma : \mathfrak{g} \to \Gamma(X, \mathcal{T}_X),$$

where $\mathcal{T}_X$ is the tangent bundle of X and $\mathfrak{g}$ is the Lie algebra of G. If (X, D) is a pair of a G-variety X and a G-stable effective divisor D with simple normal crossings and further if G is connected, then G stabilizes each of the irreducible components $D = D_1 + \cdots + D_m$. Let $x \in X$ and let $\{t_1, \ldots, t_n\}$ be a system of local parameters at $x \in X$ such that D is defined by $t_1 \cdots t_r = 0$. Then an infinitesimal automorphism $\exp(\varepsilon\delta)$ with $\delta \in \mathcal{T}_x$ acts on $\widehat{\mathcal{O}}_{X,x} = k[[t_1, \ldots, t_n]]$ as

$$\exp(\varepsilon\delta)(a) = a + \delta(a)\varepsilon + \frac{1}{2!}\delta^2(a)\varepsilon^2 + \cdots + \frac{1}{j!}\delta^j(a)\varepsilon^j + \cdots$$

Hence that $\exp(\varepsilon\delta)$ preserves each irreducible component D_i means $\delta(t_i a) \in t_i\widehat{\mathcal{O}}_{X,x}$. If we write δ as

$$\delta = c_1\partial_1 + c_2\partial_2 + \cdots + c_i\partial_i + \cdots + c_n\partial_n \quad \text{with} \quad \partial_i = \frac{\partial}{\partial t_i},$$

the above condition is that $t_1 \mid c_1, \ldots, t_r \mid c_r$. Thus, δ belongs to the stalk

$$\mathcal{T}_X(-\log D)_x = \mathcal{O}_x(t_1\partial_1) + \cdots + \mathcal{O}_x(t_r\partial_r) + \mathcal{O}_x(\partial_{r+1}) + \cdots + \mathcal{O}_x(\partial_n),$$

where $\mathcal{T}_X(-\log D)$ is the dual bundle $\mathcal{H}om(\Omega^1_X(\log D), \mathcal{O}_X)$ of the bundle of logarithmic 1-differential forms $\Omega^1_X(\log D)$ along D and hence a subbundle of the tangent bundle $\mathcal{T}_X$ [1]. Hence the above Lie algebra automorphism $d\sigma$ factors through a homomorphism

$$\mathfrak{g} \to \Gamma(X, \mathcal{T}_X(-\log D)) \to \Gamma(X, \mathcal{T}_X).$$

We consider an orbit of a unipotent group U, which is a homogeneous space U/H, where H is the isotropy group of a base point of the orbit. We recall first the following well-known results [37, Corollary, p. 1043] and [41].

Lemma 2.1. *Let U be a unipotent group and $X = U/H$ be a homogeneous space. Then X is isomorphic to $\mathbb{A}^n$ with $n = \dim X$. In particular, the underlying scheme of U is the affine space $\mathbb{A}^d$ with $d = \dim U$.*

The following result follows from the closedness of orbits of unipotent group actions on quasi-affine varieties (see [4, Prop. 4.10, Chap. 1]).

Corollary 2.2. *Let U be a unipotent group and let X be a U-variety containing an open U-orbit. Then the following assertions hold.*

(1) *Assume that X is affine. Then X coincides with the open U-orbit and hence isomorpbic to the affine space $\mathbb{A}^n$.*

(2) *Let Y be a U-stable open set of X. Then Y contains the open orbit.*

Let V be a smooth projective variety which is a U-variety and let X be an open U-orbit. Since X is affine by Lemma 2.1, the complement $D = V \setminus X$ is a U-stable subvariety of pure codimension one. Let $D = D_1 + \cdots + D_r$ be the irreducible decomposition. Since U is connected, each irreducible component is U-stable. We shall see in concrete examples what takes place in the boundary D.

Example 2.3. (1) Embed $\mathbb{A}^2 = \operatorname{Spec} k[x, y]$ into $\mathbb{P}^2$ in the standard way $(x, y) \mapsto (1, x, y)$. Let $D = \ell_\infty$ be the line at infinity. Then $\Gamma(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(\log D))$ is a k-module generated by the elements

$$\Delta = (c_0x + c_1y + c_2)\frac{\partial}{\partial x} + (d_0x + d_1y + d_2)\frac{\partial}{\partial y},$$

[1]See [23] for the definition, where it is denoted by $\mathcal{T}_X(\log D)$.

where $c_i, d_j \in k$. Hence $\dim \Gamma(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(\log D)) = 6$. Let U be the upper triangular unipotent subgroup of SL(3), which consists of matrices of the form

$$\begin{pmatrix} 1 & s_1 & s_2 \\ 0 & 1 & s_3 \\ 0 & 0 & 1 \end{pmatrix}, \quad s_1, s_2, s_3 \in k.$$

Then U acts from the right on $\mathbb{P}^2$ as $(X_0, X_1, X_2) \mapsto (X_0, X_1 + s_1X_0, X_2 + s_3X_1 + s_2X_0)$ with the line at infinity $\ell_\infty = \{X_0 = 0\}$ stabilized under this action. With the inhomogeneous coordinates $x = X_1/X_0, y = X_2/X_0$, the action is given as $(x, y) \mapsto (x + s_1, y + s_3x + s_2)$. The Lie algebra $\mathfrak{u}$ of U is generated by the matrices

$$\delta_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \delta_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \delta_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The Lie algebra homomorphism $d\sigma : \mathfrak{u} \to \Gamma(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(\log D))$ is given by

$$(\delta_1, \delta_2, \delta_3) \mapsto \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}\right)$$

(2) Let $O = (1, 0, 0)$. Since U acts from the right on $\mathbb{P}^2$, the orbit $O \cdot U$ is $\{(1, s_1, s_2) \mid s_1, s_2 \in k\}$ and the isotropy group at O is

$$H = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & s_3 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; s_3 \in k \right\}.$$

The homogeneous space $H \backslash U$ is the ordinary (x, y)-plane. The line at infinity ℓ_∞ has a U-action

$$(0, X_1, X_2) \mapsto (0, X_1, X_2 + s_3X_1).$$

Hence ℓ_∞ contains an U-orbit $O_1 \cdot U = \{(0, 1, s_3) \mid s_3 \in k\}$, where $O_1 = (0, 1, 0)$ and the isotropy group is

$$H_1 = \left\{ \begin{pmatrix} 1 & s_1 & s_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; s_1, s_2 \in k \right\}.$$

The point $O_2 = (0, 0, 1)$ is a unique U-fixed point. So, there exist a decomposition of $\mathbb{P}^2$ into strata of U-orbits $\mathbb{P}^2 = O \cdot U \cup O_1 \cdot U \cup \{O_2\}$.

(3) The observation made in (2) above can be easily generalized to the case of $\mathbb{P}^n$ and the group U_n of upper triangular unipotent matrices in $\mathrm{SL}(n+1)$. The decomposition into U_n-orbits

$$\mathbb{P}^n = O \cdot U_n \cup O_1 \cdot U_n \cup \cdots \cup O_{n-1} \cdot U_n \cup \{O_n\}$$

is also the decomposition into the B_n-orbits, where B_n is the Borel subgroup of $\mathrm{SL}(n+1)$ consisting of upper triangular matrices.

We prove only the first statement of the assertion (1). The rest are obvious. By Lemma 1.1, $\Gamma(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2})$ is a k-module generated by the elements

$$\Delta = (a_0x^2+a_1xy+c_0x+c_1y+c_2)\frac{\partial}{\partial x}+(a_0xy+a_1y^2+d_0x+d_1y+d_2)\frac{\partial}{\partial y},$$

where $a_i, c_j, d_\ell \in k$. Meanwhile, ℓ_∞ is defined by $x^{-1} = 0$ near the point $(0,1,0)$ and by $y^{-1} = 0$ near the point $(0,0,1)$. Hence $\Delta \in \Gamma(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(\log D))$ if and only if $\Delta(x^{-1})$ (resp. $\Delta(y^{-1})$) is divisible by x^{-1} (resp. y^{-1}). Hence we obtain the above expression of Δ.

Example 2.4. (1) With the notations before Lemma 1.3, identify $\mathbb{A}^2 = \operatorname{Spec} k[x,y]$ with the open set V_0 of $\mathbb{F}_n$. Let $D = \ell_\infty + M$. Then $\Gamma(\mathbb{F}_n, \mathcal{T}_{\mathbb{F}_n}(\log D))$ is a k-module consisting of vector fields

$$\Delta = \begin{cases} (a_{10}x + a_{00})\dfrac{\partial}{\partial x} + (b_{n0}x^n + \cdots + b_{10}x + b_{00} + b_{01}y)\dfrac{\partial}{\partial y} & (n > 0) \\ (a_{10}x + a_{00})\dfrac{\partial}{\partial x} + (b_{01}y + b_{00})\dfrac{\partial}{\partial y} & (n = 0) \end{cases}$$

Hence $\dim \Gamma(\mathbb{F}_n, \mathcal{T}_{\mathbb{F}_n}(\log D)) = n + 4$ for $n \geq 0$.

(2) Assume that $n > 0$. By [29], the automorphism group $G := \mathrm{Aut}^0(\mathbb{F}_n)$ satisfies an exact sequence

$$(1) \to H \to \mathrm{Aut}^0(\mathbb{F}_n) \to \mathrm{PGL}(2) \to (1),$$

where H consists of automorphisms

$$(x,y) \mapsto (x, cy + d_0 + d_1x + \cdots + d_nx^n)$$

with $c \in k^*$ and $d_0, d_1, \ldots, d_n \in k$. Hence $H = U_0 \rtimes G_m$, where U_0 is the unipotent subgroup with $c = 1$ in the above expression and $U_0 \cong G_a^{\times(n+1)}$. Let $\mathfrak{u}_0$ be the Lie algebra of U_0. Then the natural G-action σ on $\mathbb{F}_n$ induces a Lie algebra isomorphism

$$d\sigma : \mathfrak{u}_0 \to \left\{ \Delta \,\middle|\, \Delta = (d_0 + d_1x + \cdots + d_nx^n)\frac{\partial}{\partial y},\ d_0, d_1, \ldots, d_n \in k \right\}.$$

Let U be a maximal unipotent subgroup of G containing U_0. Then U/U_0 is a unipotent subgroup of PGL(2). For the Lie algebra $\mathfrak{u}$ of U, $(d\sigma)(\mathfrak{u}) = k\frac{\partial}{\partial x} + (d\sigma)(\mathfrak{u}_0)$. So, $\mathbb{F}_n$ is a G-variety and V_0 is a U-orbit. However G is not a reductive algebraic group if $n > 0$.

(3) The U_0-action on $\ell_\infty \cup M$ is given by

$$\begin{array}{lcl} \ell_\infty & : & u \mapsto u, \quad v \mapsto v + d_0u^n + \cdots + d_n \\ M & : & x \mapsto x, \quad z \mapsto \dfrac{z}{1 + d_0z + d_1xz + \cdots + d_nx^nz} \end{array}$$

This shows that $\ell_\infty \setminus \{P_\infty\}$ with $P_\infty = (u = 0, t = 0)$ is an U_0-orbit and all points on M are U_0-fixed points. However since U/U_0 moves the x-coordinate, $M \setminus \{P_\infty\}$ is a U-orbit and P_∞ is the unique U-fixed point. Thus the decomposition into U-orbits is

$$\mathbb{F}_n = V_0 \cup (\ell_\infty \setminus \{P_\infty\}) \cup (M \setminus \{P_\infty\}) \cup \{P_\infty\}$$

These examples suggest the following definition.

Definiton 2.5. Let U be a unipotent group and let V be a smooth projective variety of dimension n equipped with a nontrivial U-action. A sequence of closed subsets

$$V_0 = V \supset V_1 \supset \cdots \supset V_{n-1} \supset V_n \qquad (*)$$

is the *unipotent group orbit stratification* if the following conditions are satisfied.

(1) Each V_i is a (possibly reducible) subvariety of pure dimension $n - i$ and $V_i - V_{i+1}$ is affine.

(2) Each irreducible component of V_i is the closure of a single U-orbit.

Note that V_n consists of a single point because the U-fixed point locus of V is connected (cf. [7]).

If a smooth projective variety V has the unipotent group orbit stratification $(*)$, we may ask if the stratification determines the ambient variety V. In order to answer the question, we first recall a result of C.P. Ramanujam [38].

Lemma 2.6. *Let V be a smooth projective variety of dimension $n > 2$ and let H be a divisor of V isomorphic to $\mathbb{P}^{n-1}$ such that $V - H$ is affine and $H_1(V - H; \mathbb{Z})$ is torsion free. Then V is isomorphic to $\mathbb{P}^n$.*

We can easily obtain the following result.

Corollary 2.7. *Let V be a smooth projective variety of dimension n equipped with a nontrivial action of a unipotent group U. Assume that V has the unipotent group orbit stratification $(*)$ such that every V_i is smooth and irreducible. Then V is isomorphic to $\mathbb{P}^n$.*

Proof. If $n = 1$, it is clear that $V \cong \mathbb{P}^1$. If $n = 2$, then $V_1 \cong \mathbb{P}^1$ and $V - V_1 \cong \mathbb{A}^2$. Then $V \cong \mathbb{P}^2$ by the classification of minimal normal completions of $\mathbb{A}^2$ [33]. Assume now that $n \geq 3$. By induction on n, we may assume that $V_1 \cong \mathbb{P}^{n-1}$ and $V - V_1$ is an open U-orbit. Then $V - V_1 \cong \mathbb{A}^n$ by Corollary 2.2. Hence $H_1(V - V_1; \mathbb{Z}) = 0$. Then $V \cong \mathbb{P}^n$ by Lemma 2.6. Q.E.D.

In the unipotent group orbit stratification $(*)$ with smooth and irreducible V_i, we have the pair (V_i, V_{i+1}) of logarithmic Kodaira dimension $\overline{\kappa}(V_i - V_{i+1}) = -\infty$. If we assume this property instead of a unipotent group action, we can obtain a similar characterization of the projective space $\mathbb{P}^n$ as we will see in Theorem 2.8 below. A stratification like $(*)$ is obtained from a very ample divisor H on a smooth projective variety V of dimension n. Namely, V and H satisfy the following two conditions.

(1) The linear system $|H|$ has no base points and contains a smooth member.

(2) There exist members $H_1, H_2, \ldots, H_n$ of $|H|$ such that $V_i = H_1 \cap H_2 \cap \cdots \cap H_i$ is a smooth and irreducible subvariety of dimension $n - i$, where $0 \leq i \leq n$ and $V_0 = V$. In particular, $H^n = 1$.

We consider a descending chain of log pairs (V_i, V_{i+1}) for $0 \leq i \leq n-1$.

Theorem 2.8. *With the above notations and conditions (1) and (2), we assume that the pair (V_i, V_{i+1}) has log Kodaira dimension $-\infty$ for $0 \leq i \leq n-1$. Then V is isomprphic to $\mathbb{P}^n$ and H is a hyperplane.*

Proof. We consider first the case $n = 2$. Then V_1 is a smooth irreducible curve with $\overline{\kappa}(V_1 - V_2) = -\infty$. Hence $V_1 \cong \mathbb{P}^1$ and V_2 is a point. This implies that the self-intersection number H^2 is equal to one. Let $X = V \setminus V_1$. Since H is ample, X is an affine surface with $\overline{\kappa}(X) = -\infty$. Hence there exists an $\mathbb{A}^1$-fibration $\rho : X \to C$, where C is a smooth curve. Suppose that ρ extends to a $\mathbb{P}^1$-fibration $p : V \to \overline{C}$. Then $V_1 \cong \overline{C}$, and hence V is a rational ruled surface and V_1 is a cross-section of p. If V is not minimal, there exists an irreducible fiber component which is disjoint from V_1. This is absurd because H is ample. So, V is a Hirzebruch surface $\mathbb{F}_d$ of degree d. Let M be a minimal section of $\mathbb{F}_d$. Then $H \sim M + s\ell$, where ℓ is a fiber of p. Since $1 = H^2 = -d + 2s$, we have $d = 2s - 1$. Meanwhile, $H \cdot M = s - d = -s + 1 \geq 0$ and $d \geq 0$, whence $s = 1$ and $d = 1$. Then $H \cdot M = 0$, which contradicts the ampleness of H. This implies that the closures of the fibers of ρ in V form a linear pencil Λ with a base point, say P. The curve V_1 is a member of Λ. Since $H^2 = 1$, the pencil Λ becomes free of base point after a single blowing-up with center P. Then the blown-up surface becomes a Hirzebruch surface $\mathbb{F}_1$ and the exceptional curve is the minimal section. Here we note that if there is a reducible member, say F, of Λ, then $F \cdot V_1 = 1$ and hence there exists an irreducible component of F which is disjoint from V_1. This contradicts the ampleness of H. By contracting the exceptional curve back to the point P, we know that $V \cong \mathbb{P}^2$ and H is a line.

Suppose that $n > 2$. We assume by induction that V_1 is isomorphic to $\mathbb{P}^{n-1}$ and V_2 is a hyperplane. We consider a $\mathbb{Z}$-cohomology exact sequence for a pair (V, V_1):

$$H^{2n-2}(V;\mathbb{Z}) \xrightarrow{i^*_{2n-2}} H^{2n-2}(V_1;\mathbb{Z}) \longrightarrow H^{2n-1}(V, V_1;\mathbb{Z}) \longrightarrow H^{2n-1}(V;\mathbb{Z})$$

where $i : V_1 \to V$ is the canonical inclusion. Since $V_2 = V_1 \cap H_2$ for a general member H_2 of $|H|$ and V_2 is a hyperplane of $\mathbb{P}^{n-1}$, H_2 is also $\mathbb{P}^{n-1}$ and V_2 is a hyperplane of H_2. Then there exists a line L on H_2 such that L intersects V_1 transversally in one point P. By the Poincaré duality, $H^{2n-2}(V_1;\mathbb{Z}) \cong H_0(V_1;\mathbb{Z}) \cong \mathbb{Z}$. We may assume that $H_0(V_1;\mathbb{Z})$ is generated by the point P. Similarly, $H^{2n-2}(V;\mathbb{Z}) \cong H_2(V;\mathbb{Z})$. The class $[L]$ in $H_2(V;\mathbb{Z})$ gives rise to an element α of $H^{2n-2}(V;\mathbb{Z})$ which is mapped by i^*_{2n-2} to the element of $H^{2n-2}(V_1;\mathbb{Z})$ corresponding to $[P]$. By the Poincaré duality, this mapping is simply the intersection $L \cap V_1 = \{P\}$. Hence the mapping i^*_{2n-2} is surjective. On the other hand, $H^{2n-1}(V;\mathbb{Z})$ is isomorphic to $H_1(V;\mathbb{Z})$ by the Poincaré duality, and $H_1(V;\mathbb{Z}) \cong H_1(V_1;\mathbb{Z})$ by the Lefschetz hyperplane section theorem. Then $H_1(V_1;\mathbb{Z}) = (0)$ because $V_1 \cong \mathbb{P}^{n-1}$. Hence $H^{2n-1}(V;\mathbb{Z}) = (0)$. The above exact sequence shows that $H^{2n-1}(V, V_1;\mathbb{Z}) = (0)$. By the Lefschetz duality, it follows that $H_1(V - V_1;\mathbb{Z}) = (0)$. Now we can use Lemma 2.6 to conclude that $V \cong \mathbb{P}^n$ and V_1 is a hyperplane. Q.E.D.

In Lemma 2.6, the condition that $H_1(V - H;\mathbb{Z})$ is torsion free is crucial and the condition is satisfied if $V - H$ is simply connected. Instead of an open orbit of a unipotent group in $V - H$, we can think of independent G_a-actions $\sigma_1, \ldots, \sigma_n$ on $V - H$. We shall make the situation more precise.

Let X be a smooth affine variety of dimension n. Let $\sigma_i : G_a \times X \to X$ be an action of G_a for $1 \le i \le m$. We say that the actions σ_i $(1 \le i \le m)$ are *independent* if there exists a point P such that the vector fields Δ_i associated with σ_i span a vector subspace of dimension m in the tangent space $\mathcal{T}_{X,P}$. It is clear that we can choose as P any point from an open set U of X. In fact, let Δ_i be the vector field associated with the action σ_i. Then the mapping $P \mapsto (\Delta_1)_P \wedge \cdots \wedge (\Delta_m)_P \in \wedge_{i=1}^m \mathcal{T}_{X,P}$ defines a section

$$\Delta_1 \wedge \cdots \wedge \Delta_m : X \to \wedge_{i=1}^m \mathcal{T}_X,$$

which is nonzero at the point P by the assumption. Then it is non-zero in an open neighborhood of P. We then prove the following result.

Theorem 2.9. *Let X be a smooth affine variety of dimension n defined over $\mathbb{C}$ with n independent G_a-actions σ_i $(1 \le i \le n)$. Then the following assertions hold.*

(1) *The fundamental group of X is a finite group and $\Gamma(X, \mathcal{O}_X)^* = \mathbb{C}^*$.*

(2) *Assume that $n = 2$ and X is factorial. Then $X \cong \mathbb{A}^2$ and hence X is simply connected.*

(3) *If $n = 3$, X is factorial and the (algebraic) quotient surface of X with respect to one of the actions σ_i, say σ_1, is smooth, then X is simply connected.*

Proof. (1) We denote the group scheme G_a by G_i if G_a acts on X by σ_i. The corresponding vector field Δ_i is a locally nilpotent derivation of the coordinate ring A of X. Let $B_1 = \operatorname{Ker} \Delta_1$. By slice theorem, $A_b := A[b^{-1}]$ is a polynomial ring in one variable over $B_{1,b} := B_1[b^{-1}]$ for some nonzero element $b \in B_1$. Then the inclusion $B_{1,b} \hookrightarrow A_b$ defines an $\mathbb{A}^1$-bundle morphism $q_b : X_b \to Y_b := \operatorname{Spec} B_{1,b}$.

Let P be a general point of X_b. Hence $G_1 P \cong \mathbb{A}^1$ is a fiber of q_b. Let $\tau : G_2 \times \cdots \times G_n \to Y_b$ be a rational mapping defined by $(g_2, \ldots, g_n) \mapsto q_b(g_n(\cdots(g_2 P)\cdots))$. Then τ is a dominant mapping and holomorphic in a small analytic neighborhood of the point of origin $(e_2, \ldots, e_n)$ of $G_2 \times \cdots \times G_n$. In fact, if ε_i $(2 \le i \le n)$ moves a complex number with $|\varepsilon_i|$ small, then $\exp(\varepsilon_n \Delta_n) \cdots \exp(\varepsilon_2 \Delta_2) P$ is considered to be a transversal section of a tubular neighborhood of the orbit $G_1 P$. This implies that $\sigma : G_1 \times \cdots \times G_n \to X$ defined by $(g_1, \ldots, g_n) \mapsto g_n(\cdots(g_2(g_1 P))\cdots)$ is a dominant morphism. In fact, if $|\varepsilon_1|$ is small, $\exp(\varepsilon_1 \Delta_1) P$ is a disc neighborhood of P in the orbit $G_1 P$. Hence $\exp(\varepsilon_n \Delta_n) \cdots \exp(\varepsilon_2 \Delta_2) \cdot \exp(\varepsilon_1 \Delta_1)(P)$ gives a ball-like, analytic, open neighborhood of the point P. Since $G_1 \times \cdots \times G_n$ has $\mathbb{A}^n$ as the underlying space, σ gives a dominant morphism $\sigma : \mathbb{A}^n \to X$. Let $q : Z \to X$ be a connected topological covering. Then the fiber product $q_{\mathbb{A}^n} : Z \times_X \mathbb{A}^n \to \mathbb{A}^n$ is a topological covering. Since $\pi_1(\mathbb{A}^n) = (1)$, $Z \times_X \mathbb{A}^n$ contains $\mathbb{A}^n$ as a connected component. Then the morphism σ splits as $\sigma : \mathbb{A}^n \longrightarrow Z \xrightarrow{q} X$. In fact, Z has locally the same complex structure as X since Z is a topological covering of X. Hence Z is a smooth connected complex manifold of dimension n. Since $Z \times_X \mathbb{A}^n$ is a disjoint union of the connected components isomorphic to $\mathbb{A}^n$, the canonical projection $pr_Z : Z \times_X \mathbb{A}^n \to Z$ yields a morphism of complex manifolds $\mathbb{A}^n \to Z$ which factors $\sigma : \mathbb{A}^n \to X$. By counting the number of points in $\mathbb{A}^n$ (resp. Z) lying over a general point P of X, we infer that $\deg q \le \deg \sigma$. Hence $q : Z \to X$ is a finite covering of X and Z is a smooth affine algebraic variety of dimension n. Hence $|\pi_1(X)| \le \deg \sigma$. Since $\sigma : \mathbb{A}^n \to X$ is a dominant morphism, it follows that $A^* = \Gamma(X, \mathcal{O}_X)^* = \mathbb{C}^*$.

(2) By an algebraic characterization of the affine plane, X is isomorphic to $\mathbb{A}^2$ because X is factorial, $\Gamma(X, \mathcal{O}_X)^* = \mathbb{C}^*$ and X has a G_a-action.

(3) Since $n = 3$, $B_1 = \operatorname{Ker} \Delta_1$ is a normal affine domain of dimension 2. Hence we can think of the algebraic quotient $Y = X/\!/G_1$ and the quotient morphism $q : X \to Y$ instead of the morphism q_b. In fact, the restriction of q onto X_b is the morphism q_b. Further, the mapping $\tau : G_2 \times G_3 \to Y_b$ extends to a dominant morphism $\tau : \mathbb{A}^2 \to Y$. Assume that X is factorial. Then Y is factorial because $\Gamma(Y, \mathcal{O}_Y)$ is factorially closed in $\Gamma(X, \mathcal{O}_X)$ as the kernel of Δ_1. Since $\tau : \mathbb{A}^2 \to Y$ is dominant, it follows that $\overline{\kappa}(Y) = -\infty$ and there are no non-constant units in $\Gamma(Y, \mathcal{O}_Y)$. In fact, Y is smooth by the assumption, and hence we have $\overline{\kappa}(Y) \le \overline{\kappa}(\mathbb{A}^2) = -\infty$. Then Y is isomorphic to $\mathbb{A}^2$. Since the general fiber of q is isomorphic to $\mathbb{A}^1$ and the factorial closedness of $\Gamma(Y, \mathcal{O}_Y)$ in $\Gamma(X, \mathcal{O}_X)$ implies that the fibers over codimension one points of Y are all reduced, we can apply Nori's lemma [35, Lemma 1.5] to obtain an exact sequence

$$\pi_1(\mathbb{A}^1) \to \pi_1(X) \to \pi_1(Y) \to (1) \ .$$

This implies that $\pi_1(X) = (1)$. In fact, if there is a point $Q \in Y$ such that the fiber $q^{-1}(Q)$ has all non-reduced irreducible components, let $Y^\circ = Y \setminus S$, where S is the finite set

$$\{Q \in Y \mid \text{ every component of } q^{-1}(Q) \text{ is non-reduced}\}$$

and let $X^\circ = q^{-1}(Y^\circ)$. Since $\pi_1(Y^\circ) = \pi_1(Y) = (1)$, we can apply Nori's lemma to $q^\circ : X^\circ \to Y^\circ$ to obtain $\pi_1(X^\circ) = (1)$. Since $\operatorname{codim}_X(X \setminus X^\circ) \ge 2$, we have $\pi_1(X) = \pi_1(X^\circ) = (1)$. Q.E.D.

For the X in Theorem 2.9, the universal covering space $\widetilde{X}$ of X is a smooth affine variety of dimension n such that $\widetilde{X}$ is simply connected and has n independent G_a-actions. In fact, any locally nilpotent derivation of an affine domain extends uniquely to a locally nilpotent derivation of a finite étale extension of the domain (cf. [31]). It is an interesting problem to find out what kind of structure the variety $\widetilde{X}$ has. In the surface case, let X be a smooth affine surface which is an ML_0-surface (i.e., whose coordinate ring has trivial Makar-Limanov invariant; see [13] for the definition and relevant results) and simply connected. In general, $\pi_1(X)$ for an ML_0-surface X has order bounded by the intertwining number of two G_a-orbits corresponding to two independent G_a-actions [30, Lemma 1.3]. If X is further a $\mathbb{Q}$-homology plane, or equivalently if X has the Picard number 0, then X is isomorphic to $\mathbb{A}^2$. If the Picard number is positive, the structure of such surface X is described in [13].

In the threefold case, let X be the product of a Danielewski surface $\{xy = z^2 - 1\}$ and the affine line $\mathbb{A}^1$. Then X has three independent G_a-actions because the Danielewski surface has two independent G_a-actions and the direct product factor $\mathbb{A}^1$ has a third G_a-action making the Danielewski surface invariant. Furthermore, X is simply connected because the Danielewski surface is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 - \{\text{diagonal}\}$. Meanwhile, the Picard number of X is 1. A similar example is a hypersurface $X = \{xyz = u^3 - 1\}$ in $\mathbb{A}^4$. X has three G_a-actions which are defined by locally nilpotent derivations δ_i $(i = 1, 2, 3)$, where $\delta_1(x) = \delta_1(y) = 0$, $\delta_1(z) = 3u^2$ and $\delta_1(u) = xy$ with δ_2 and δ_3 defined in a similar fashion by changing the roles of x, y, z. X is simply connected, but the Picard number is nonzero. So, we can ask if a smooth factorial affine threefold X is isomorphic to $\mathbb{A}^3$ provided it has three independent G_a-actions. But there are again many counterexamples to this question. A typical one is SL(2) which is the hypersurface $xz - yu = 1$ in $\mathbb{A}^4$ (see [8, Remark 5.15]).

§3. G_a-actions on projective varieties

Let V be a smooth projective variety of dimension n defined over k. Let Δ be a regular vector field. We say that Δ has a *locally nilpotent stratification* if V has a decomposition $V = \coprod_i W_i$ into locally closed subsets satisfying the following conditions:

(1) W_0 is an affine open set of V, W_i is an affine open set of the closure $\overline{W}_i$ and $\overline{W}_i \setminus W_i$ is the union of closures $\overline{W}_j$ of several W_j with $j > i$.

(2) For every i, $\Delta|_{W_i}$ is a locally nilpotent derivation on $\Gamma(W_i, \mathcal{O}_{W_i})$ unless W_i is a point.

The locally nilpotent stratifications used in the subsequent arguments are constructed in such a way that $V \setminus W_0$ is supported by an ample divisor, and for $i > 0$, $\overline{W}_i \setminus W_i$ is also supported by an ample divisor on $\overline{W}_i$ provided $\overline{W}_i$ is smooth (cf. Definition 3.9).

Lemma 3.1. *Let $U = \operatorname{Spec} R$ be an affine open set. Then $\Delta|_U$, the restriction of Δ onto U, is a k-derivation D of R.*

Proof. Since $\Delta \in \Gamma(V, \mathcal{T}_{V/k})$, it follows that $\Delta|_U \in \Gamma(U, \mathcal{T}_{U/k}) \cong \mathrm{Der}_k(R)$. Q.E.D.

Lemma 3.2. *Let $U = \operatorname{Spec} R$ and let D be as above. Let $U' = \operatorname{Spec} R'$ be an affine open set such that $U' \subseteq U$ and let D' be the k-derivation corresponding to $\Delta|_{U'}$. Then D is the restriction of D' to R. If D' is locally nilpotent, then so is D on R.*

Proof. Note that R and R' are subalgebras in the function field $k(V)$ and hence that $R \subseteq R'$ in $k(V)$. Define a k-algebra homomorphism $\Phi_D : R \longrightarrow R[[t]]$ by

$$\Phi_D(a) = \sum_{i \geq 0} \frac{1}{i!} D^i(a) t^i.$$

Then we have a commutative diagram

$$\begin{tikzcd} R \arrow[r, "\Phi_D"] \arrow[d, "i"] & R[[t]] \arrow[d, "i[[t]]"] \\ R' \arrow[r, "\Phi_{D'}"'] & R'[[t]] \end{tikzcd}$$

where i and $i[[t]]$ are the canonical inclusions. If D' is locally nilpotent, $\Phi_{D'}$ splits via $R'[t]$. Then Φ_D splits via $R[t]$ because $R[t] = R'[t] \cap R[[t]]$. Hence D is locally nilpotent. Q.E.D.

This result implies that if Δ is locally nilpotent on a non-empty affine open set, there exists a maximal affine open set $U_{\max}$ of V such that Δ induces a locally nilpotent derivation on $U_{\max}$.

Let $\widetilde{A} = \oplus_{n \geq 0} A_n$ be a graded affine domain over k with $A_0 = k$ and generated by A_1 and let $\widetilde{\Delta}$ be a nonzero locally nilpotent derivation of $\widetilde{A}$ which is homogeneous of degree 0. Let $V = \operatorname{Proj}(\widetilde{A})$ which is identified with $(\operatorname{Spec}(\widetilde{A}) \setminus \{\mathfrak{M}\})//G_m$, where $\mathfrak{M}$ is the irrelevant ideal of $\widetilde{A}$ and G_m acts on $\operatorname{Spec}(\widetilde{A})$ via the grading. Then the G_a-action on $\operatorname{Spec}(\widetilde{A})$ induced by $\widetilde{\Delta}$ commutes with the G_m-action and hence induces a G_a-action on V. This G_a-action is described as follows. Since $\widetilde{\Delta}$ restricted on A_1 is a nilpotent linear endomorphism, there exists an element $s_0 \neq 0$ of A_1 such that $\widetilde{\Delta}(s_0) = 0$. Then $\widetilde{\Delta}$ induces a locally nilpotent derivation Δ on $\widetilde{A}[s_0^{-1}]_0$ and it gives rise to a G_a-action on the affine open set $V \setminus V_+(s_0)$. This G_a-action coincides with the restriction of the above-obtained G_a-action on V restricted to $V \setminus V_+(s_0)$. The G_a-action on $V_+(s_0)$ is described by the locally nilpotent homogeneous derivation $\widetilde{\Delta}$ (mod (s_0)) induced on $\widetilde{A}/(s_0)$.

Conversely, a G_a-action on a smooth projective variety is obtained via G_a-linealization from this construction.

Theorem 3.3. *Let V be a smooth projective variety which has an algebraic G_a-action and let Δ be the regular vector field associated with the G_a-action. The following assertions hold:*

(1) *Let H be an effective ample divisor such that the subset H_{red} of codimension one is G_a-stable and let $W_0 = \operatorname{Spec} R_0$ be the complement of H_{red}. Then $D_0 = \Delta|_{W_0}$ is a locally nilpotent derivation.*

(2) *Let H be an effective ample divisor. Then H is G_a-linearizable. Hence there exists a member of $|H|$ which is G_a-stable. If H_0 is a G_a-stable member of $|H|$, then $V \setminus H_0$ is a G_a-stable affine open set.*

(3) *Let H be a G_a-stable effective very ample divisor and let $\widetilde{A} = \oplus_{n \geq 0} H^0(V, \mathcal{O}(nH))$. Then there exists a locally nilpotent, homogeneous derivation $\widetilde{\Delta}$ of degree 0 on $\widetilde{A}$ such that $\widetilde{\Delta}$ induces the G_a-action on V.*

Proof. (1) Since H is ample, the complement $W_0 = V \setminus H_{\mathrm{red}}$ is an affine open set and has the induced G_a-action. Hence the restriction $\Delta|_{W_0}$ gives rise to the induced G_a-action on W_0. This implies that D_0 is locally nilpotent.

(2) Since Pic $(G_a) = (0)$, H is G_a-linearizable by [24, Prop. 2.4 and its remark]. Hence G_a acts linearly on $H^0(V, \mathcal{O}_V(H))$, and hence acts on the projective space $|H|$. Since the fixed point locus on $|H|$ is connected and non-empty (see [7]), there exists an element of $|H|$ which is G_a-stable.

(3) Since H is very ample, $H^0(V, \mathcal{O}(nH))$ is generated by $H^0(V, \mathcal{O}(H))$. Hence the extended G_a-coaction on $H^0(V, \mathcal{O}(H))$ (cf. [34, p. 32]) extends to a locally nilpotent homogeneous derivation $\widetilde{\Delta}$ on the graded domain $\widetilde{A}$. Since $\widetilde{\Delta}(H^0(V, \mathcal{O}(H))) \subseteq H^0(V, \mathcal{O}(H))$, $\widetilde{\Delta}$ has degree 0. Let H_0 be a G_a-stable member of $|H|$ and let $\{s_0, s_1, \ldots, s_N\}$ be a basis of $H^0(V, \mathcal{O}(H))$ such that $H_0 = \{s_0 = 0\}$. Then $\widetilde{\Delta}$ induces the G_a-action on $V - H_0 = \operatorname{Spec} k[s_1/s_0, \ldots, s_N/s_0]$ which is the given G_a-action on V by construction. By repeating the same argument to H_0 and $H|_{H_0}$, it is now easy to conclude the assertion. Q.E.D.

Remark 3.4. Let V be a smooth projective variety with a nontrivial G_a-action. Let H be a very ample divisor. If we assume that the irregularity $q := h^1(V, \mathcal{O}_V) = 0$ and the G_a-fixed point locus consists of a single point, we can take a sequence of closed subvarieties

$$V_0 = V \supset V_1 \supset \cdots \supset V_{n-1} \supset V_n$$

as considered before Theorem 2.8 in a G_a-equivariant way provided the smoothness condition on V_i is guaranteed. In fact, by Theorem 3.3, (2), there is a G_a-stable open set W_0 such that $W_0 = V \setminus H_1$ with a G_a-stable member $H_1 \in |H|$. If $V_1 := H_1$ is smooth, then we consider a very ample divisor $H|_{H_1}$. Since the induced G_a-action on V_1 is nontrivial if $\dim V_1 > 0$, we find a G_a-stable member of $|H|_{V_1}|$. Suppose further that the irregularity $q := h^1(V, \mathcal{O}_V) = 0$. Then the exact sequence

$$0 \longrightarrow \mathcal{O}_V \longrightarrow \mathcal{O}_V(H) \longrightarrow \mathcal{O}_{H_1}(H) \longrightarrow 0$$

induces the surjection $\mathrm{H}^0(V, \mathcal{O}_V(H)) \to \mathrm{H}^0(H_1, \mathcal{O}_{H_1}(H))$. So, the G_a-stable member of $|H|_{H_1}|$ is written as $H_1 \cap H_2$. Note that the irregularity of H_1 vanishes. In fact, if $\dim V \geq 3$, then the exact sequence

$$\mathrm{H}^1(V, \mathcal{O}_V) \longrightarrow \mathrm{H}^1(H_1, \mathcal{O}_{H_1}) \longrightarrow \mathrm{H}^2(V, \mathcal{O}_V(-H))$$

implies the assertion because $\mathrm{H}^2(V, \mathcal{O}_V(-H)) = 0$ by the Kodaira vanishing theorem. If $\dim V = 2$, then $\mathrm{H}^2(V, \mathcal{O}_V(-H)) \cong \mathrm{H}^0(V, \mathcal{O}(K_V + H)) = 0$ because the existence of a G_a-action implies $\overline{\kappa}(V \setminus H_1) = -\infty$. Since the G_a-fixed point locus on V consists of a single point by assumption, we proceed the above construction inductively under the assumption that $H_1 \cap H_2 \cap \cdots \cap H_i$ is smooth for $1 \leq i \leq \dim V - 1$. Then the G_a-action on $V_i := H_1 \cap \cdots \cap H_i$ is nontrivial if $i \leq \dim V - 1$. Thus we reach to the set-up of Theorem 2.8. Since the nontrivial G_a-action on $V_i - V_{i+1}$ implies that $\overline{\kappa}(V_i - V_{i+1}) = -\infty$, the theorem shows that $V \cong \mathbb{P}^n$. If $V \not\cong \mathbb{P}^n$, it fails to hold that $H_1 \cap \cdots \cap H_i$ is smooth for some $1 \leq i \leq \dim V - 1$.

Given a regular vector field Δ on a smooth projective variety V, an irreducible subvariety W of codimension one is called *integral* if for every smooth point P of W and for a system of local parameters $\{u_1, \ldots, u_n\}$ of V at P such that W is defined by $u_1 = 0$, we have $\Delta(u_1) \in u_1 \mathcal{O}_{V,P}$. If dimV$= 2$, we call W an *integral curve* of Δ.

Example 3.5. With the notations in Lemma 1.1, we consider the case $V = \mathbb{P}^2 = U_0 \coprod H_0$ with $H_0 = \{X_0 = 0\}$, where $U_0 = \operatorname{Spec} k[x, y]$.

Write a regular vector field Δ on $\mathbb{P}^2$ as $\Delta = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$, where

$$\begin{aligned} f &= a_0x^2 + a_1xy + c_0x + c_1y + c_2 \\ g &= a_0xy + a_1y^2 + d_0x + d_1y + d_2 \ . \end{aligned}$$

In terms of the coordinates $\{u, v\}$ on $U_1 = \{X_1 \neq 0\}$ and $\{z, w\}$ on $U_2 = \{X_2 \neq 0\}$ (cf. Lemma 1.1), we compute $\Delta|_{U_1}$ and $\Delta|_{U_2}$ as follows.

$$\begin{aligned} \Delta|_{U_1} &= -(a_0 + a_1v + c_0u + c_1uv + c_2u^2)\frac{\partial}{\partial u} \\ &\quad +(d_0 + d_1v + d_2u - c_0v - c_1v^2 - c_2uv)\frac{\partial}{\partial v} \\ \Delta|_{U_2} &= -(a_1 + d_1z + a_0w + d_2z^2 + d_0zw)\frac{\partial}{\partial z} \\ &\quad +(c_1 + c_2z + (c_0 - d_1)w - d_2zw - d_0w^2)\frac{\partial}{\partial w} \ . \end{aligned}$$

We assume that H_0 is an integral curve of Δ. By the above expression of $\Delta|_{U_1}$ and $\Delta|_{U_2}$, where H_0 is defined by $u = 0$ and $z = 0$ respectively, H_0 is an integral curve if and only if $a_0 = a_1 = 0$. Hence we have

$$\Delta = (c_0x + c_1y + c_2)\frac{\partial}{\partial x} + (d_0x + d_1y + d_2)\frac{\partial}{\partial y} \ .$$

Namely we have

$$\Delta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \Delta(x) \\ \Delta(y) \end{pmatrix} = \begin{pmatrix} c_0 & c_1 \\ d_0 & d_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c_2 \\ d_2 \end{pmatrix}$$

Furthermore, we have

$$\Delta^n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_0 & c_1 \\ d_0 & d_1 \end{pmatrix}^n \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c_0 & c_1 \\ d_0 & d_1 \end{pmatrix}^{n-1} \begin{pmatrix} c_2 \\ d_2 \end{pmatrix}$$

for all $n \geq 1$. This implies that Δ is locally nilpotent if and only if the matrix $\begin{pmatrix} c_0 & c_1 \\ d_0 & d_1 \end{pmatrix}$ is a nilpotent matrix. Hence there exists a matrix $P \in \mathrm{GL}(2, k)$ such that

$$P^{-1} \begin{pmatrix} c_0 & c_1 \\ d_0 & d_1 \end{pmatrix} P = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} .$$

Let $\widetilde{P} = \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix}$. By a change of coordinates ${}^t(X_0, X_1, X_2) \mapsto \widetilde{P}^{-1\,t}(X_0, X_1, X_2)$, we may assume that

$$\Delta = (\alpha y + c_2)\frac{\partial}{\partial x} + d_2\frac{\partial}{\partial y} \ ,$$

whence $\Delta(x) = \alpha y + c_2$ and $\Delta(y) = d_2$. Now we define a G_a-action on $\mathbb{P}^2$ by

$$t \cdot \begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ c_2 t + \frac{1}{2!}\alpha d_2 t^2 & 1 & \alpha t \\ d_2 t & 0 & 1 \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix} .$$

It is now clear that Δ is the vector field associated to the G_a-action.

Consider a regular vector field $\Delta = (c_1 y + c_2)\frac{\partial}{\partial x}$ with $c_1 \neq 0$. Then $\Delta|_{U_2 \cap H_0} = c_1 \frac{\partial}{\partial w}$. This implies that the G_a-action is effective on $U_2 \cap H_0$ and the fixed point locus Γ consists of a single point $(0,1,0)$. In fact, the decomposition $U_0 \coprod (U_2 \cap H_0) \coprod \Gamma$ is a locally nilpotent stratification of Δ.

A similar thing holds in the case $n > 2$. With the notations in Lemma 1.1, we have the following.

Remark 3.6. Assume that the hyperplane H_0 is integral for Δ as in Lemma 1.1. This implies that $\Delta(\frac{X_0}{X_i})$ is divisible by $\frac{X_0}{X_i}$ for every $i \neq 0$. This condition is equivalent to $a_1 = a_2 = \cdots = a_n = 0$. Hence we have

$$\Delta \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = B \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} .$$

Since

$$\Delta^n \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = B^n \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + B^{n-1} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} ,$$

Δ is locally nilpotent if and only if B is nilpotent. Hence, after a suitable base change of $(x_1, \ldots, x_n)$, we may assume that B is an upper triangular matrix (c_{ij}) with all the diagonal entries zero. Then the vector field Δ is associated with a G_a-action on $\mathbb{P}^n$ which stabilizes the hyperplane H_0.

Modeled after the above examples, we introduce the following two definitions, where we note that the stratifications stop at the second strata.

Definiton 3.7. Let V be a smooth projective variety of dimension n. A G_a-action on V is called a *stratified action* if there exists a reduced effective divisor $H = H_1 + \cdots + H_r$ (irreducible decomposition) supporting an ample divisor satisfying the following two conditions.

(1) H is G_a-stable, whence there exists the induced effective G_a-action on $X = V \setminus H$. Further, there is an induced G_a-action on each H_i.

(2) For each irreducible component H_i, there exists a reduced effective divisor K_i supporting an ample divisor on H_i such that K_i is G_a-stable and the induced G_a-action on $H_i \setminus K_i$ is effective.

Remark 3.8. In view of Remark 3.4 and the argument therein, if V is a smooth projective variety with a nontrivial G_a-action and if the G_a-fixed point locus has codimension greater than or equal to 2, then the G_a-action is a stratified G_a-action.

Definiton 3.9. Let V be a smooth projective variety of dimension n and let Δ be a regular vector field. We call Δ *effectively locally nilpotent with stratification* if there exists a reduced effective divisor $H = H_1 + \cdots + H_r$ supporting an ample divisor satisfying the following two conditions.

(1) Δ induces a nontrivial locally nilpotent derivation on $X = V \setminus H$ and each irreducible component H_i is Δ-integral in the sense that, for each smooth point P of H_i, $\Delta(u_i)$ is divisible by u_i in $\mathcal{O}_{V,P}$, where $u_i = 0$ is a local defining equation of H_i.

(2) For each H_i, there exists a reduced effective divisor K_i on H_i supporting an ample divisor such that $\Delta|_{H_i \setminus K_i}$ induces a nontrivial locally nilpotent derivation.

It is well known that a G_a-action on an affine scheme corresponds bijectively to a locally nilpotent derivation on the coordinate algebra of the scheme. The following result will correspond partly to this result for affine schemes and explain when a given regular vector field comes from a G_a-action on a smooth projective variety.

Theorem 3.10. *Let V be a smooth projective variety of dimension $n \geq 2$. Then the following assertions hold.*

(1) *A stratified G_a-action σ on V induces the regular vector field Δ on V which is effectively locally nilpotent with stratification.*

(2) *Let Δ be a regular vector field on V which is effectively locally nilpotent with stratification. Then there exists a stratified G_a-action on V which induces the vector field Δ.*

Proof. (1) By the hypothesis, the G_a-action σ on X (see the notations in Definition 3.7) is effective, whence the associated vector field is nontrivial on X. Similarly, we can extend this vector field, say Δ, to $H_i \setminus K_i$ for each i because it is associated to the induced G_a-action on $H_i \setminus K_i$. Then Δ is an element of $\Gamma(V \setminus (\cup_{i=1}^r K_i), \mathcal{T}_{V/k})$, where

$\mathrm{codim}\,_V(\cup_{i=1}^r K_i) \geq 2$. Then Δ is defined on V as a regular vector field. It is now clear that Δ is effectively locally nilpotent with stratification because the G_a-action on $H_i \setminus K_i$ is effective.

(2) Let $R = \Gamma(X, \mathcal{O}_V)$. Then Δ is considered as a locally nilpotent derivation of R. Let σ_X be the induced G_a-action, which is given by the coaction

$$\Phi : R \to R[t], \qquad \Phi(z) = \sum_{i \geq 0} \frac{1}{i!} \Delta^i(z) t^i \ .$$

For $\alpha \in k$, define the automorphism φ_α of R by $\varphi_\alpha(z) = \Phi(z)|_{t=\alpha}$. Then $\sigma_\alpha := {}^a\varphi_\alpha$ is the automorphism of X such that $\sigma_\alpha \cdot \sigma_\beta = \sigma_{\alpha+\beta}$ for $\alpha, \beta \in k$. The k-algebra homomorphism Φ extends to a k-homomorphism $\Phi_{k(V)} : k(V) \to k(V)(t)$ such that $\Phi_{k(V)}(\frac{z_2}{z_1}) = \Phi(z_2)/\Phi(z_1)$, where $k(V) = Q(R)$ is the function field of V over k and $z_1, z_2 \in R$. Then σ_α for $\alpha \in k$ is viewed as a birational automorphism of V. Although σ_α is biregular on $X = V \setminus H$, it may not be biregular on the irreducible component $H_1, \ldots, H_r$. Suppose that σ_α induces a correspondence between H_i and a curve or a point. However, the correspondence induces an automorphism on the affine open set $H_i \setminus K_i$ by the hypothesis. Hence σ_α is biregular on all codimension one points of V. This implies that σ_α is a biregular automorphism of V. Since $\sigma_\alpha \cdot \sigma_\beta = \sigma_{\alpha+\beta}$ for $\alpha, \beta \in k$, the collection $\{\sigma_\alpha \mid \alpha \in k\}$ defines a G_a-action on V. It is clear that this G_a-action is stratified and induces the vector field Δ. Q.E.D.

Example 3.11. Let $V = \mathbb{F}_n$ with $n \geq 0$. With the notations in Lemma 1.3, a regular vector field $\Delta = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$ makes the divisor $M + \ell_\infty$ integral if and only if $a_{20} = 0$ if $n > 0$ and $a_{20} = b_{02} = 0$ if $n = 0$. Δ is locally nilpotent on $\mathbb{F}_n \setminus (M + \ell_\infty)$ if and only if $a_{10} = b_{01} = 0$ for $n \geq 0$. If the latter condition is satisfied, Δ is associated with a stratified G_a-action on $\mathbb{F}_n$ provided other constants a_{00} and the b_{i0} are nonzero.

Remark 3.12. Dubouloz-Liendo [6] introduced the notion of rationally integrable k-derivation and showed that regular G_a-actions on a semi-affine variety X are in one-to-one correspondence with rationally integrable k-derivations $\widetilde{\partial} : \mathcal{O}_X \to \mathcal{O}_X$ such that the derivation $\Gamma(X, \widetilde{\partial}) : \Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X)$ is locally nilpotent, where an algebraic variety X is said to be semi-affine if the canonical morphism $p : X \to \mathrm{Spec}\,\Gamma(X, \mathcal{O}_X))$ is a proper morphism. Hence a complete variety or an affine variety is semi-affine.

§4. G_a-actions on Fano varieties

Our objective in this section is to describe the structure of a smooth projective variety V with a stratified G_a-action, mostly in the case where

V is a Fano variety of rank 1. Our result is very restrictive since we only know of few examples for which all possible G_a-actions are known together with the fixed-point loci and the behaviors of orbits. We shall see first what is the situation with the simplest example $\mathbb{P}^2$.

Lemma 4.1. (1) The standard form of a G_a-action on $\mathbb{P}^2$ is given by

$$t \cdot (X_0, X_1, X_2) = (X_0, X_1 + (bX_2 + c_1X_0)t + \frac{1}{2}bc_2X_0t^2, X_2 + c_2X_0t)$$

where $t \in k$, $b, c_1, c_2 \in k$ with the notations being slightly different from Example 3.5.

(2) The fixed point locus Γ is given by

$$\Gamma = \begin{cases} \text{one point } \{(0,1,0)\} & \text{if} \quad c_2b \neq 0 \\ \text{line } \{X_0 = 0\} & \text{if} \quad c_2 \neq 0,\ b = 0 \\ \text{line } \{bX_2 + c_1X_0 = 0\} & \text{if} \quad c_2 = 0\ . \end{cases}$$

(3) Suppose $c_2b \neq 0$. Then the closure of each G_a-orbit passes through the point $P_0 := (0,1,0)$ and is smooth at P_0, and the intersection multiplicity of the closures of two distinct G_a-orbits is 4. By the blowing-ups with centers P_0 and its 3 more consecutive, infinitely-near points, the proper transforms of the closures of general G_a-orbits are separeted from each other. If either $c_2 \neq 0$ and $b = 0$ or $c_2 = 0$, the closure of each G_a-orbit is a line passing through the point $(0, c_1, c_2)$ or the point P_0 respectively.

(4) If $c_2b \neq 0$ then the G_a-action is a stratified action. If either $c_2 \neq 0$ and $b = 0$ or $c_2 = 0$ then the action is not a stratified action. The third G_a-action with $c_2 = 0$ is brought to the second one by a projective transformation.

Proof. (1) By Theorem 3.3, (2), the system of hyperplanes $|H|$ has an induced G_a-action. Hence it contains a member H_0 which is G_a-stable. Then we can choose a system of homogeneous coordinates (X_0, X_1, X_2) so that H_0 is defined by $X_0 = 0$. By Example 3.5, we can write the associated vector field Δ as

$$\Delta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\ .$$

The expression of the G_a-action in (1) is obtained from this Δ. Then the assertion (2) is straightforward.

(3) Let $P(\alpha, \beta)$ be a point of $U_0 = \{X_0 \neq 0\}$. Then the G_a-orbit is the set of points

$$t \cdot (1, \alpha, \beta) = (1, \alpha + (b\beta + c_1)t + \frac{1}{2}bc_2t^2, \beta + c_2t)\ .$$

Eliminating t from

$$x = \alpha + (b\beta + c_1)t + \frac{1}{2}bc_2t^2, \quad y = \beta + c_2t \ ,$$

we obtain the equation of an affine curve on U_0 which is the G_a-orbit. If $c_2 \neq 0$, the curve is defined by

$$c_2(x - \alpha) = (b\beta + c_1)(y - \beta) + \frac{1}{2}b(y - \beta)^2 \ .$$

The projective closure of the curve is defined by

$$c_2(X_1 - \alpha X_0)X_0 = (b\beta + c_1)(X_2 - \beta X_0)X_0 + \frac{1}{2}b(X_2 - \beta X_0)^2 \ .$$

If $c_2 b \neq 0$, then this curve is irreducible and smooth at the point $P_0 = (0, 1, 0)$. If $c_2 \neq 0$ and $b = 0$, then the projective closure of the curve is defined by

$$c_2(X_1 - \alpha X_0) = c_1(X_2 - \beta X_0) \ .$$

Hence it passes through the point $(0, c_1, c_2)$. If $c_2 = 0$, the orbit through a point (α, β) is defined by $y = \beta$ and its projective closure is $X_2 = \beta X_0$. Hence it passes through the point P_0.

(4) It is easy to verify the assertion. For the last assertion, let $Y_0 = bX_2 + c_1X_0, Y_1 = X_2$ and $Y_2 = X_1$. Then the G_a-action becomes

$$t \cdot (Y_0, Y_1, Y_2) = (Y_0, Y_1, Y_2 + Y_0t)$$

which is the case $c_2 = 1$ and $b = c_1 = 0$. Q.E.D.

We consider next the case of the Hirzebruch surface $V = \mathbb{F}_n$ $(n \geq 0)$.

Example 4.2. (1) Suppose that $V = \mathbb{F}_n$ with $n > 0$. Then the minimal section M is G_a-stable because $(M^2) = -n < 0$. Furthermore, the pencil $|\ell|$ of fibers has an induced G_a-action and contains a G_a-stable member ℓ_∞. By Lemma 1.3, the associated vector field Δ on $\mathbb{A}^2 = \mathbb{F}_n \setminus (M \cup \ell_\infty)$ is locally nilpotent. This implies that $f(x, y) = a_{00}$ and $g(x, y) = b_{n0}x^n + \cdots + b_{10}x + b_{00}$. If $a_{00}b_{n0} \neq 0$, then the point $M \cap \ell_\infty$ is the fixed point locus, and the G_a-action is stratified with respect to $M + \ell_\infty$.

(2) Suppose that $n = 0$. Then the pencils $|\ell|$ and $|M|$ contain G_a-stable members ℓ_∞ and M_∞. The complement $\mathbb{A}^2 = \mathbb{F}_0 \setminus (\ell_\infty \cup M_\infty)$ has the associated vector field

$$\Delta = a_{00}\frac{\partial}{\partial x} + b_{00}\frac{\partial}{\partial y}$$

which is locally nilpotent. If $a_{00}b_{00} \neq 0$ then $\ell_\infty \cap M_\infty$ is the fixed point locus, and the G_a-action is stratified with respect to $M_\infty + \ell_\infty$.

In the case $\dim V = 2$, we have the following result.

Theorem 4.3. *Let V be a smooth projective surface with an effective G_a-action. Then the following assertions hold.*

(1) *V is birationally a ruled surface. If there is a (-1) curve E on V, then E is G_a-stable. Hence we can contract E so that the G_a-action is preserved on the contracted surface.* We assume below that V is relatively minimal.

(2) *Suppose that V is irrational. Then the fixed point locus Γ consists of a cross-section S_0 and a (possibly empty) set of the fibers $\ell_1, \ldots, \ell_r$, where S_0 is not an ample section. Let $\mathcal{L} := \mathcal{O}_{S_0}(S_0)$. Then there exists a non-zero section $s \in H^0(C, \mathcal{L}^{-1})$ such that the zeroes of s defines the fibers $\ell_1, \ldots, \ell_r$.*

(3) *Suppose that V is a rational ruled surface. Then the G_a-action on V is described in Example 4.2.*

(4) *Suppose that $V \cong \mathbb{P}^2$. Then the G_a-action is described in Lemma 4.1.*

Proof. (1) Let H be a very ample divisor. Since H is G_a-linearizable, the linear system $|H|$ contains a G_a-stable member H_0. Since $X = V \setminus H_0$ has a non-trivial G_a-action, V is birationally a ruled surface. The rest of the assertion (1) is clear.

(2) Any G_a-orbit is contained in a fiber of the canonical $\mathbb{P}^1$-fibration $\pi : V \to C$, where C is an irrational smooth projective curve. Hence the fixed point locus Γ contains a corss-section S_0. The other components of Γ is a (possibly empty) set of fibers $\ell_1, \ldots, \ell_r$. The section S_0 is not ample. In fact, if S_0 is ample, the complement $X = V \setminus S_0$ is affine and endowed with a G_a-action. The quotient morphism is the restriction $\pi|_X$ of the $\mathbb{P}^1$-fibration $\pi : V \to C$ such that $\pi(X) = C$. This is a contradiction because $X/\!/G_a$ is an affine curve.

Consider an exact sequence

$$0 \to \mathcal{O}_V \to \mathcal{O}_V(S_0) \to \mathcal{O}_{S_0}(S_0) \to 0$$

whose direct images by π gives an exact sequece

$$0 \to \mathcal{O}_C \to \pi_*\mathcal{O}_V(S_0) \to \mathcal{L} \to 0 \ ,$$

where $\mathcal{L} \cong \mathcal{O}_{S_0}(S_0)$ with $\deg \mathcal{L} = (S_0^2)$. Let $\mathcal{E} = \pi_*\mathcal{O}_V(S_0)$. Then $\mathcal{E}$ is a rank 2 vector bundle over C and $V = \mathrm{Proj}\,(\mathcal{E})$. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of C such that $\mathcal{L}|_{U_i} = \mathcal{O}_{U_i} e_i$. Let $\mathcal{E}|_{U_i} = \mathcal{O}_{U_i} e \oplus \mathcal{O}_{U_i} \widetilde{e}_i$, where $\widetilde{e}_i$ is a lift of e_i in $\mathcal{E}$. Then

$$(\widetilde{e}_j, e) = (\widetilde{e}_i, e) \begin{pmatrix} f_{ji} & 0 \\ g_{ji} & 1 \end{pmatrix}$$

over $U_i \cap U_j$. Then $\pi^{-1}(U_i) = \operatorname{Proj} \Gamma(U_i, \mathcal{O}_C)[\widetilde{e}_i, e]$ and $\pi^{-1}(U_i) \setminus S_0 = \operatorname{Spec} \Gamma(U_i, \mathcal{O}_C)\left[\frac{\widetilde{e}_i}{e}\right]$. Then the G_a-action is given by a locally nilpotent derivation Δ defined by $\Delta\left(\frac{\widetilde{e}_i}{e}\right) = s_i \in \Gamma(U_i, \mathcal{O}_C)$. Since $\frac{\widetilde{e}_j}{e} = f_{ji}\frac{\widetilde{e}_i}{e} + g_{ji}$, we have $s_j = f_{ji}s_i$. Hence $\{s_i\}$ defines a section of $H^0(C, \mathcal{L}^{-1})$. The zeroes of s gives the fiber components of the fixed point locus Γ. Conversely, a section $s \in H^0(C, \mathcal{L}^{-1})$ yields a G_a-action on V. Q.E.D.

Now we consider the case $\dim V \geq 3$. We assume further the condition that the Picard number ρ of V equals one, i.e., V has (Picard) rank one.

Lemma 4.4. *With the above condition, if V has an effective G_a-action, then V is a Fano variety of $\rho = 1$.*

Proof. Let H be a very ample divisor. Then $|H|$ contains a G_a-stable (possibly reducible) member H_0. Then $V \setminus (H_0)_{\mathrm{red}}$ is an affine, G_a-stable open set. There exists an open set U of $V \setminus (H_0)_{\mathrm{red}}$ such that $U \cong U_0 \times \mathbb{A}^1$, where U_0 is an affine variety. Hence V is birationally a ruled variety. In particular, the canonical divisor K_V is not a torsion divisor. Since $\rho = 1$, it follows that $-K_V$ is ample. So, V is a Fano variety of $\rho = 1$. Q.E.D.

The following properties are well known about Fano threefolds (not necessarily of the Picard number one).

(1) A Fano manifold, i.e., a smooth projective variety with ample anti-canonical divisor, is rationally connected and hence simply connected [26]. The Kodaira vanishing theorem implies that $H^i(V, \mathcal{O}_V) = 0$ for every $i > 0$.

(2) Let r be the index of a Fano threefold V, i.e., r is the maximal positive integer such that $-K_V \sim rH$ with $H \in \operatorname{Pic}(V)$. Then the linear system $|H|$ contains a smooth irreducible surface. If $r \geq 2$ then the set of base points of $|H|$ is finite [40]. More precisely, H is very ample if $H^3 \geq 3$, the base point locus $\operatorname{Bs}|H| = \emptyset$ if $H^3 \geq 2$ and $\operatorname{Bs}|H|$ is a one-point set if $H^3 = 1$ (see [39]).

(3) Let V be a smooth projective threefold with $H^0(V, K_V) = 0$. Then the Brauer group $\operatorname{Br}(V)$ is isomorphic to the torsion group T of $H^3(V; \mathbb{Z})$ (see [22, p.166]). If V is a Fano threefold, $\operatorname{Br}(V) = 0$, whence the torsion group T is zero (see p.168, *loc.cit.*). The torsion group T is isomorphic to the torsion group of $H_2(V; \mathbb{Z})$ by the universal coefficient theorem. In fact, if $H_2(V; \mathbb{Z})$ has the torsion group T', then $H^3(V; \mathbb{Z}) \cong \operatorname{Hom}(H_3(V; \mathbb{Z}), \mathbb{Z}) \oplus \operatorname{Ext}(H_2(V; \mathbb{Z}), \mathbb{Z})$ by the universal coefficient theorem, where $\operatorname{Hom}(H_3(V; \mathbb{Z}), \mathbb{Z})$ is a free abelian group and $\operatorname{Ext}(H_2(V; \mathbb{Z}), \mathbb{Z}) \cong T'$. Hence $T \cong T'$. The torsion group T of

$\mathrm{H}^3(V;\mathbb{Z})$ is a birational invariant by [1]. For example, if V is rational, then $H^3(V;\mathbb{Z}) \cong H^3(\mathbb{P}^3;\mathbb{Z}) = 0$. So, $T = 0$.

We are interested in the affine open set $X = V \setminus H$ when V is a Fano threefold and H is an effective ample divisor. X is also the first stratum if V has a G_a-action with H stable. We prove the following result.

Theorem 4.5. *Let V be a smooth Fano threefold with* $\mathrm{Pic}\,(V) = \mathbb{Z}[H]$*, where H is an ample effective divisor*[2]*. Assume that H is smooth. Let $X = V \setminus H$. Then the following assertions hold.*

(1) $H_1(X;\mathbb{Z}) = H_2(X;\mathbb{Z}) = 0$.

(2) *If X is a homology threefold, i.e., $H_3(X;\mathbb{Z}) = 0$, then $H \cong \mathbb{P}^2$ and $V \cong \mathbb{P}^3$.*

(3) *In addition to the assumption in* (2)*, assume further that V has a non-trivial G_a-action and H is G_a-stable. Then the quotient surface $Y := X//G_a$ is isomorphic to $\mathbb{A}^2$ and the quotient morphism $q : X \to Y$ is surjective.*

Proof. The proof of the assertion (1) consists of several steps.

(i) We have $H_1(V;\mathbb{Z}) = 0$ because V is simply connected. The Lefschetz hyperplane theorem implies $H_1(H;\mathbb{Z}) = 0$. In fact, Fujita [10] proved a generalization of the Lefschetz theorem, which we use here.

(ii) Consider the exact sequence

$$H^1(V,\mathcal{O}_V) \to H^1(V,\mathcal{O}_V^*) \to H^2(V;\mathbb{Z}) \to H^2(V,\mathcal{O}_V)$$

associated to

$$0 \to \mathbb{Z} \to \mathcal{O}_V \xrightarrow{\exp} \mathcal{O}_V^* \to 0\ .$$

This implies that $H^2(V;\mathbb{Z}) \cong H^1(V,\mathcal{O}_V^*) = \mathrm{Pic}\,(V) \cong \mathbb{Z}$ by the hypothesis. By the universal coefficient theorem, it follows that $H_2(V;\mathbb{Z}) = \mathbb{Z}$ because $H_2(V;\mathbb{Z})$ has no torsion group.

(iii) Now consider the long exact sequence of singular cohomology groups for a pair (V, H),

$$\begin{aligned} &H^2(V,H;\mathbb{Z}) \to H^2(V;\mathbb{Z}) \to H^2(H;\mathbb{Z}) \to H^3(V,H;\mathbb{Z}) \to \\ &H^3(V;\mathbb{Z}) \to H^3(H;\mathbb{Z}) \to H^4(V,H;\mathbb{Z}) \to H^4(V;\mathbb{Z}) \to \\ &H^4(H;\mathbb{Z}) \to H^5(V,H;\mathbb{Z}) \to H^5(V;\mathbb{Z})\ , \end{aligned}$$

where $H^i(V,H;\mathbb{Z}) \cong H_{6-i}(X;\mathbb{Z})$ by the Lefschetz duality and hence $H^i(V,H;\mathbb{Z}) = 0$ if $i \le 2$ because X is affine. Since $H^4(V;\mathbb{Z}) \cong H_2(V;\mathbb{Z}) \cong \mathbb{Z}$ by (ii), $H^4(H;\mathbb{Z}) \cong H_0(H;\mathbb{Z}) \cong \mathbb{Z}$ and $H^5(V;\mathbb{Z}) \cong$

[2]It is an easy consequence of $\mathrm{Pic}\,(V) = \mathbb{Z}[H]$ that H is irreducible and reduced.

$H_1(V;\mathbb{Z})=0$ by (i), a part of the long exact sequence reads as an exact sequence

$$H_2(V;\mathbb{Z}) \xrightarrow{\alpha} H_0(H;\mathbb{Z}) \to H_1(X;\mathbb{Z}) \to 0 \ ,$$

where $\alpha : H_2(V;\mathbb{Z}) \to H_0(H;\mathbb{Z})$ is non-trivial. In fact, $H_2(V;\mathbb{Z})$ is represented as $\mathbb{Z}[C]$, where C is a curve and $\alpha(C)=H\cdot C$. Since H is an ample divisor, $H\cdot C>0$. So, $H_1(X;\mathbb{Z})$ is a finite cyclic group $\mathbb{Z}/n\mathbb{Z}$. By the universal coefficient theorem, $H^1(X;\mathbb{Z}/n\mathbb{Z}) \cong H_1(X;\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$. On the other hand, the exact sequence of étale sheaves on X

$$0 \to \mu_n \to G_m \xrightarrow{t\mapsto t^n} G_m \to 1$$

yields an exact sequence

$$H^0(X,\mathcal{O}_X^*) \xrightarrow{t\mapsto t^n} H^0(X,\mathcal{O}_X^*) \to H^1(X;\mathbb{Z}/n\mathbb{Z}) \to \mathrm{Pic}\,(X) \xrightarrow{\times n} \mathrm{Pic}\,(X) \ ,$$

where $\mathrm{Pic}\,(X)=\mathrm{Pic}\,(V)/\langle H\rangle=0$ and $H^0(X,\mathcal{O}_X^*)=k^*$. Hence we have $n=1$ and $H_1(X;\mathbb{Z})=0$.

(iv) Since α is an isomorphism, the above long exact sequence gives an exact sequence

$$H^3(V;\mathbb{Z}) \to H^3(H;\mathbb{Z}) \to H_2(X;\mathbb{Z}) \to 0 \ .$$

Since $H^3(H;\mathbb{Z}) \cong H_1(H;\mathbb{Z})=0$, we have $H_2(X;\mathbb{Z})=0$. This completes the proof of the assertion (1).

We prove the assertion (2). Since $H^2(V,H;\mathbb{Z}) \cong H_4(X;\mathbb{Z})=0$, the long exact sequence in the step (iii) above yields an exact sequence

$$0 \to H^2(V;\mathbb{Z}) \to H^2(H;\mathbb{Z}) \to H_3(X;\mathbb{Z}) \to H^3(V;\mathbb{Z}) \to 0 \ .$$

Suppose that $H_3(X;\mathbb{Z})=0$. Then $H^2(H;\mathbb{Z})\cong\mathbb{Z}$ and $H^3(V;\mathbb{Z})=0$ because $H^2(V;\mathbb{Z})\cong\mathbb{Z}$. Let r be the index of V. Then H is a del Pezzo surface if $r\ge 2$ and a K3 surface if $r=1$. Since the second Betti number $b_2(H)=1$ now, H cannot be a K3 surface for which $b_2(H)=22$. Hence $r\ge 2$ and H is a rational surface. Since $H^2(H;\mathbb{Z})\cong\mathbb{Z}$ as above, it follows that $\mathrm{Pic}\,(H)\cong H^2(H;\mathbb{Z})\cong\mathbb{Z}$. This implies that $H\cong\mathbb{P}^2$. Since $H_1(X;\mathbb{Z})=0$ by (1), $V\cong\mathbb{P}^3$ by Lemma 2.6. In fact, H is a hyperplane and $X\cong\mathbb{A}^3$.

(3) Now the assertion (3) that $Y\cong\mathbb{A}^2$ follows from [32], and the surjectivity of $q: X\to Y$ follows from [3]. Q.E.D.

Perhaps we need some explanation on the significance of the assertion (3) of Theorem 4.5. If an affine variety X has a nontrivial G_a-action, the slice theorem shows that X contains an $\mathbb{A}^1$-cylinder. The property that X contains an $\mathbb{A}^1$-cylinder and how big is it is a crucial matter in determining the structure of an affine variety. So, the algebraic quotient $Y := X//G_a$ being isomorphic or not to $\mathbb{A}^2$ is thought of as a measure to know how close a given variety is to a rational threefold like $\mathbb{P}^3$. Furthermore, the quotient morphism $q : X \to Y$ being surjective or not when $Y \cong \mathbb{A}^2$ is another measure. The assertion (3) above is generalized in Theorem 4.9 below by dropping the assumption that X be a homology threefold.

Theorem 4.5 shows that with the above notation X is not, in general, a homology threefold. A simple counter-example is given by a smooth quadric hypersurface Q in $\mathbb{P}^4$. Before stating the result, a smooth quadric hypersurface is given by a nondegenerate quadratic form in the homogeneous coordinates $\{X_0, X_1, X_2, X_3, X_4\}$. By a suitable change of coordinates, we may assume that Q is of Fermat type, i.e., it is defined by $X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = 0$. Again, by a change of coordinates, we may assume that Q is defined by $X_0^2 - X_1X_3 + X_2X_4 = 0$.

Theorem 4.6. *Let Q be a quadric hypersurface in $\mathbb{P}^4$ defined by $F = X_0^2 - X_1X_3 + X_2X_4 = 0$. Then the following assertions hold.*

(1) $\mathrm{Pic}(Q)$ *is generated by a hyperplane section $H_Q = H \cap Q$, where $H = \{X_0 = 0\}$, and hence* $\mathrm{Pic}(Q) \cong \mathbb{Z}[H_Q]$, *and $K_Q \sim -3H_Q$, whence Q is a Fano threefold.*

(2) *Q has the G_a-action induced by a G_a-action on $\mathbb{P}^4$*

$$^t(X_0, X_1, X_2, X_3, X_4) = (X_0, X_1, X_2, X_3 + X_2t, X_4 + X_1t) .$$

The surface H_Q is G_a-stable, and the induced action on $X = Q \setminus H_Q \cong \mathrm{Spec}\, k[x, y, z, u]/(xz - yu - 1)$ is given by a locally nilpotent derivation δ such that $\delta(x) = \delta(y) = 0, \delta(z) = y$ and $\delta(u) = x$.

(3) *X is simply connected, $H_1(X; \mathbb{Z}) = H_2(X; \mathbb{Z}) = 0$ and $H_3(X; \mathbb{Z}) \cong \mathbb{Z}$. Hence X is not a homology threefold.*

(4) *$Y := X//G_a$ is isomorphic to $\mathbb{A}^2$, but the quotient morphism $q : X \to Y$ is not surjective.*

Proof. (1) This follows from [11, Exp. 12, Cor. 3.7]. Hence $X := Q \setminus H_Q$ is factorial and $\Gamma(X, \mathcal{O}_X)^* = k^*$. This also follows from [16, Example 1.12].

(2) It is straightforward. By *loc.cit.*, the quotient surface $Y := X//G_a \cong \mathbb{A}^2 = \mathrm{Spec}\, k[x, y]$ and the quotient morphism $q : X \to Y$ is the projection $(x, y, z, u) \mapsto (x, y)$. Furthermore, $q(X) = \mathbb{A}^2 - \{(0, 0)\}$ and every fiber of q over a point of $q(X)$ is reduced and isomorphic to $\mathbb{A}^1$. This proves the assertion (4).

(3) By Nori's result [35, Lemma 1.5], we have an exact sequence

$$\pi_1(\mathbb{A}^1) \to \pi_1(X) \to \pi_1(\mathbb{A}^2 \setminus \{(0,0)\}) \to 1 \ .$$

Hence $\pi_1(X) = 1$. Namely, X is simply connected. By [42, Theorem 1.2], we have $h^{1,2}(Q) = h^{2,1}(Q) = 0$. Since $b_1(Q) = 0$ and $b_2(Q) = 1$ by Lefschetz theorem, we have $b_0(Q) = b_6(Q) = 1,\ b_1(Q) = b_5(Q) = 0$ and $b_2(Q) = b_4(Q) = 1$. Since $h^{3,0}(Q) = h^{0,3}(Q) = 0$, we have $b_3(Q) = 0$. Hence the Euler number $\chi(Q)$ is equal to 4. On the other hand, H_Q is a quadric surface in $\mathbb{P}^3$. Hence $\chi(H_Q) = 4$. So, $\chi(X) = 0$. If $b_3(X) = 0$ which is equivalent to $H_3(X;\mathbb{Z}) = 0$ because $H_3(X;\mathbb{Z})$ has no torsion by Hamm's theorem [12, Lemma 1.2], then X is contractible and $\chi(X) = 1$. This contradicts the above calculation of $\chi(X)$. Hence, $b_3(X) = 1$ and $H_3(X;\mathbb{Z}) \cong \mathbb{Z}$. [3] Q.E.D.

Remark 4.7. Let V be a cubic hypersurface in $\mathbb{P}^4$. Then V is a Fano threefold and Pic (V) is generated by a hyperplane section. However, V has no G_a-actions because V is irrational and unirational. See [5] and [17]. Similarly, if V is a quartic hypersurface in $\mathbb{P}^4$, it seems that V has no G_a-actions because some of quartic hypersurfaces are not rational, but unirational. See [21].

Concerning the assertion (3) in Theorem 4.6, we have a more general result.

Theorem 4.8. *Let V be a Fano threefold such that* Pic $(V) = \mathbb{Z}[H]$ *for an ample effective divisor H. Assume that the index r of V is greater than one and H is smooth. Let $X := V \setminus H$. Then X is simply connected.*

Proof. Let S be a general member of $|H|$, which is irreducible and smooth. Since $r \geq 2$ by the assumption, S is a del Pezzo surface with $K_S \sim -(r-1)\Gamma$, where $\Gamma = H|_S$. Since $1 \leq K_S^2 \leq 9$, we have $r-1 = 3, 2$ or 1, whence $r = 4, 3$ or 2. By the Lefschetz theorem for affine threefolds (see [35]), we have an isomorphism

$$\pi_1(S \setminus \Gamma) \cong \pi_1(V \setminus H) = \pi_1(X) \ .$$

Hence it suffices to show that $\pi_1(S \setminus \Gamma) = 1$. If $r = 4$ then $S \cong \mathbb{P}^2$ and Γ is a line. Hence $S \setminus \Gamma \cong \mathbb{A}^2$ and $\pi_1(S \setminus \Gamma) = 1$. If $r = 3$ then S is

[3] One of the referees suggested us the following simple argument for the step (3). By the argument in the step (2) above, $q : X \to \mathbb{A}^2 \setminus \{(0,0)\}$ is a locally trivial $\mathbb{A}^1$-bundle (actually a principal G_a-bundle). Hence q gives a homotopy equivalence of topological manifolds, and X has the same homology type as $\mathbb{A}^2 \setminus \{(0,0)\} \approx \mathbb{R}^4 \setminus \{(0,0,0,0)\}$, which is itself homotopy equivalent to the real 3-space S^3.

isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and $\Gamma \sim \ell + M$, where ℓ and M are respective fibers of two projections onto $\mathbb{P}^1$. So, $S \setminus \Gamma$ contains $\mathbb{A}^2$ as an open set and $\pi_1(S \setminus \Gamma) = 1$. Suppose now that $r = 2$ and $1 \le K_S^2 \le 7$. Then we have $K_S^2 = H^3$ and S is obtained from $\mathbb{P}^2$ by blowing up s points in general position lying on a cubic curve C on $\mathbb{P}^2$, where $2 \le s = 9 - H^3 \le 8$. If $s \ne 8$ then $H^3 \ge 2$ and $\mathrm{Bs}|H| = \emptyset$. Hence we can take C to be a smooth curve. In fact, $|H_S|$ is the restriction of $|H|$ onto S, and $|H_S|$ has no base points. If $s = 8$, then $|H_S|$ is the proper transform of a pencil generated by two smooth cubic curves on $\mathbb{P}^2$ intersecting transversally in 9 points, out of which we choose 8 points to blow up. So, for any value of s ($2 \le s \le 8$), we can choose S so that the image of Γ on $\mathbb{P}^2$ is a smooth cubic curve. The divisor Γ on S is the proper transform of C, and $\mathbb{P}^2 \setminus C$ is an open set of $S \setminus \Gamma$. Hence we have a surjection $\pi_1(\mathbb{P}^2 \setminus C) \to \pi_1(S \setminus \Gamma)$. Since $\pi_1(\mathbb{P}^2 \setminus C) \cong \mathbb{Z}/3\mathbb{Z}$, we know that $\pi_1(S \setminus \Gamma)$ is abelian.

On the other hand, by the Lefschetz theorem for affine threefolds, we have

$$H_1(S \setminus \Gamma; \mathbb{Z}) \cong H_1(X; \mathbb{Z}) \ .$$

Since $H_1(X; \mathbb{Z}) = 0$ by Theorem 4.5, we have $\pi_1(S \setminus \Gamma) = 1$. Q.E.D.

We have shown in Theorem 4.5 that Y is isomorphic to $\mathbb{A}^2$ under the assumption that V has a non-trivial G_a-action with G_a-stable H and X is a homology threefold. In the following theorem, we show that the same result holds without assuming that X is a homology threefold.

Theorem 4.9. *Let V be a smooth Fano threefold with* $\mathrm{Pic}\,(V) = \mathbb{Z}[H]$*, where H is an ample effective divisor. Assume that the index r of V is greater than one and H is smooth. Let $X = V \setminus H$. Assume further that V has a non-trivial G_a-action and H is G_a-stable. Then the quotient surface $Y := X//G_a$ is isomorphic to $\mathbb{A}^2$.*

Proof. The proof consists of several steps.

(1) Let $q : X \to Y$ be the quotient morphism. Since X is factorial, so is Y by [16]. Since V is rationally connected and the quotient morphism $q : X \to Y$ extends to a proper morphism $\overline{q} : V' \to \overline{Y}$, where V' is birational to V and $\overline{Y}$ is a smooth completion of Y, the surface $\overline{Y}$ is then rationally connected and hence rational. So, Y is rational and has no non-constant invertible regular functions. Furthermore, every fiber of q is one-dimensional, and Y has at most quotient singularities because there is locally a smooth hyperplane of X dominating the given point of Y. Since Y is factorial and rational, every singularity is E_8-singularity. Set $\overline{Y}$ anew a smooth normal completion of Y. Namely, $\overline{Y}$ is a normal projective surface such that $\overline{Y}$ is smooth at every point of

$D := \overline{Y} - Y$ and D is a divisor with simple normal crossings. Then q defines a rational mapping $q' : V \to \overline{Y}$. We eliminate the indeterminacies of q' by blowing up a G_a-fixed point of H and subsequently blowing up the smooth centers which consist of G_a-fixed points, and obtain a morphism $\widetilde{q} : \widetilde{V} \to \overline{Y}$ such that the divisor $\widetilde{D} = \widetilde{V} - X$ is a divisor with simple normal crossings.

Since $\overline{\kappa}(X) = -\infty$ as X contains an $\mathbb{A}^1$-cylinder and since $K_V + H \sim -(r-1)H$ with the index r, it follows that $r > 1$. Then H is a del Pezzo surface.

(2) Let $\Delta = \widetilde{q}^{-1}(D)$. Then $\Delta \subseteq \widetilde{D}$. Since $\widetilde{q}|_\Delta : \Delta \to D$ is a proper morphism, Δ is a finite connected union of irreducible components which consist of the proper transforms of the divisor H and the exceptional divisors. We shall show that $\pi_1(\Delta) = 1$ by making use of Van Kampen's theorem on the fundamental group of a connected simplicial complex. The proper transform of any exceptional divisor which constitutes Δ is either the blowing-up of $\mathbb{P}^2$ or the blowing-up a $\mathbb{P}^1$-bundle over a smooth curve. We prove $\pi_1(\Delta) = 1$ by induction on the number of irreducible components of Δ which are indexed after the order of blowing-ups $p : \widetilde{V} \to V$ by which they appear. The divisor H is rational since H is a del Pezzo surface. So, $\pi_1(H) = 1$. The exceptional divisor E_1 which appear by the blowing-up of the fixed point on H is isomorphic to $\mathbb{P}^2$ meeting the proper transform H' of H along a smooth rational curve. By Van Kampen's theorem applied to $E_1 \cup H'$, it follows that $\pi_1(E_1 \cup H')$ is the amalgamated product of $\pi_1(E_1)$ and $\pi_1(H')$ over $\pi_1(E_1 \cap H')$. Since $\pi_1(H') = \pi_1(E_1) = 1$, we have $\pi_1(E_1 \cup H') = 1$. After performing blowing-ups at least k times, we obtain the exceptional divisors $E_1, \ldots, E_k$ and the proper transform of H', although some of the exceptional divisors may have the images by $\widetilde{q}$ meeting the open set Y and do not appear in Δ. Here we denote the proper transforms of H' and the exceptional divisors obtained by the earlier blowing-ups by the same letters. By induction, we assume that $\pi_1(H' \cup E_1 \cup \cdots \cup E_{k-1}) = 1$. In order to compute $\pi_1(H' \cup E_1 \cup \cdots E_{k-1} \cup E_k)$, we need more observations on what are the centers of the above blowing-ups.

The indeterminacy of $q' : V \to \overline{Y}$ or of the subsequently induced rational mappings is either a base point where the closures of general G_a-orbits pass through or an irreducible curve C such that each general point of C is a base point of a one-dimensional subfamily of the closures of G_a-orbits. Namely, there exists a morphism $q'' : V'' \to C$, where V'' appears in the course of blowing-ups $p : \widetilde{V} \to V$. In the first case, the center of the blowing-up is a smooth point. In the second case, C is contained in an exceptional divisor and it is possibly singular. The curve C is possibly the intersection curve of a newly born exceptional divisor with the old exceptional divisor, in which case C is smooth. If C is singular, we blow up smooth points in a threefold to eliminate the singularities of C. After making the proper transform of C a smooth curve, we blow up C to obtain the exceptional divisor E which is a $\mathbb{P}^1$-bundle over C. Afterwards, the centers to be blown up are the irreducible smooth intersection curves of two irreducible components, say $F_1 \cap F_2$, where one of F_1, F_2, say F_2, meets a chain of $\mathbb{P}^1$-bundles $F_3, \ldots, F_s$ and where only possible intersections of $F_2, F_3, \ldots, F_s$ with each other and with other components of $\widetilde{D}$ are the intersection curves $F_2 \cap F_3, F_3 \cap F_4, \ldots, F_{s-1} \cap F_s$ which are all isomorphic to C.

Now we return to the computation of $\pi_1 := \pi_1(H' \cup E_1 \cup \cdots \cup E_{k-1} \cup E_k)$.

(i) If E_k is the exceptional divisor of the blowing-up with center at either a point or an irreducible smooth curve, Van Kampen's theorem shows that $\pi_1 = 1$. In fact, the π_1 of a $\mathbb{P}^1$-bundle over a smooth curve C is equal to $\pi_1(C)$ and the π_1 of the intersection curve is also equal to $\pi_1(C)$. Since $\pi_1(H' \cup E_1 \cup \cdots \cup E_{k-1}) = 1$, we obtain $\pi_1(H' \cup E_1 \cup \cdots \cup E_{k-1} \cup E_k) = 1$.

(ii) Suppose that the center is the intersection curve $C = F_1 \cap F_2$ and F_2 meets a chain of $\mathbb{P}^1$-bundles $F_3, \ldots, F_s$. Again, we have $\pi_1(E_k \cup F_2 \cup \cdots \cup F_s) \cong \pi_1(C)$ by Van Kampen's theorem, where E_k is the exceptional divisor arising from the blowing-up of C and $F_2, \ldots, F_s$ are identified with the proper transforms by this blowing-up. It is then easy to see that $\pi_1(H' \cup E_1 \cup \cdots \cup E_{k-1} \cup E_k) \cong \pi_1(H' \cup E_1 \cup \cdots \cup E_{k-1}) = 1$.

Thus we have shown that $\pi_1(\Delta) = 1$.

(3) Let $\overline{q} := \widetilde{q}|_\Delta : \Delta \to D$, where D (resp. Δ) is a reduced effective divisor with simple normal crossings in $\overline{Y}$ (resp. $\widetilde{V}$). Let $Z \to D$ be an unramified connected (topological) covering of D. Then the fiber product $\Delta \times_D Z \to \Delta$ is a connected unramified covering of Δ because the fibers of $\overline{q}$ are connected, and the unramifiedness of $Z \to D$ implies that Z and $\Delta \times_D Z$ are the unions of smooth irreducible components with simple normal crossings. Hence $\pi_1(\Delta) \to \pi_1(D)$ is surjective by the covering space theory. This implies that $\pi_1(D) = 1$.

(4) We shall show that Y is a homology plane. Note that Y has at most E_8-singularities. Let $Y^\circ = Y - \mathrm{Sing} Y$ and $\widetilde{Y}$ be the minimal desingularization of $\overline{Y}$. Then $Y^\circ = \widetilde{Y} \setminus (D \cup E)$, where E is the union of the exceptional curves of the desingularization. Consider an exact

sequence of integral cohomology groups for a pair $(\widetilde{Y}, D \cup E)$,

$$\begin{aligned} &H^1(\widetilde{Y};\mathbb{Z}) \to H^1(D\cup E;\mathbb{Z}) \to H^2(\widetilde{Y}, D\cup E;\mathbb{Z}) \to \\ &H^2(\widetilde{Y};\mathbb{Z}) \to H^2(D\cup E;\mathbb{Z}) \to H^3(\widetilde{Y}, D\cup E;\mathbb{Z}) \to \\ &H^3(\widetilde{Y};\mathbb{Z}) \to 0\ , \end{aligned}$$

where $H^1(\widetilde{Y};\mathbb{Z}) = 0$ and $H^3(\widetilde{Y};\mathbb{Z}) \cong H_1(\widetilde{Y};\mathbb{Z}) = 0$ since $\widetilde{Y}$ is rational, $H^1(D \cup E;\mathbb{Z}) = 0$ because $\pi_1(D) = 1$ and E is a rational tree, and $H^2(\widetilde{Y};\mathbb{Z}) \cong \mathrm{Pic}\,(\widetilde{Y})$ is a free abelian group of rank $\#D + \#E$. Hence $H^2(\widetilde{Y};\mathbb{Z})$ is isomorphic to $H^2(D \cup E;\mathbb{Z})$. So, $H_2(Y^\circ;\mathbb{Z}) \cong H^2(\widetilde{Y}, D \cup E;\mathbb{Z}) = 0$ and $H_1(Y^\circ;\mathbb{Z}) \cong H^3(\widetilde{Y}, D \cup E;\mathbb{Z}) = 0$. Meanwhile, a small open neighborhood (in the Euclidean topology) of an E_8-singularity is homologous to a ball in $\mathbb{C}^2$. This implies that $H_i(Y;\mathbb{Z}) \cong H_i(Y^\circ;\mathbb{Z}) = 0$ for $i = 1, 2$. This shows that Y is a homology plane. Furthermore, Y is topologically contractible. In fact, the quotient morphism $q : X \to Y$ has irreducible and reduced fibers over all codimension one points of Y if X is factorial, q has no multiple fibers over the codimension one points of Y (see the argument in the step (3) of the proof of Theorem 2.9). Hence, by Nori's theorem [35, Lemma 1.5] and since $\pi_1(X) = 1$ by Theorem 4.8, it follows that $\pi_1(Y) = 1$.

(5) We argue by the logarithmic Kodaira dimension of Y°. If $\overline{\kappa}(Y^\circ) = 2$, then the singular point of Y is at most one cyclic singularity by [15]. This is impossible if $\mathrm{Sing} Y \neq \emptyset$ because Y has only E_8-singularities. Suppose that Y has an $\mathbb{A}^1_*$-fibration. Then the singularities are at most cyclic singularities. By the same reason as above, $\mathrm{Sing} Y = \emptyset$. Suppose that Y° has an $\mathbb{A}^1_*$-fibration, but the $\mathbb{A}^1_*$-fibration does not extend to an $\mathbb{A}^1_*$-fibration on Y (hence Y is singular). Then Y has a unique singular point, and the closures (in Y) of general fibers of the $\mathbb{A}^1_*$-fibration form a family of rational curves with one place at infinity which pass through the singular point of Y. Then the desingularization $\widetilde{Y}$ of Y is dominated by a smooth surface with an $\mathbb{A}^1$-fibration. Hence $\overline{\kappa}(\widetilde{Y}) = -\infty$. Then, by Koras-Russell [27, Theorem 1.1], $\overline{\kappa}(Y^\circ) = -\infty$. We treat this case later. If $\overline{\kappa}(Y^\circ) = 0$, then by Palka [36, Theorem 7.2], either Y is smooth or Y has only A_1 or A_2 singularity. So, the singular case does not occur, and the smooth case does not occur either by Fujita [9] and [18].

We consider finally the case $\overline{\kappa}(Y^\circ) = -\infty$. If Y° is affine-ruled, then $Y \cong \mathbb{A}^2$ since Y is factorial and has no non-constant invertible regular functions. If Y° is not affine-ruled, then $Y \cong \mathbb{A}^2/\!/G$, where G is a binary icosahedral group. Then Y is isomorphic to a hypersurface $x^2 + y^3 + z^5 = 0$ in $\mathbb{A}^3$. Then $\pi_1(Y^\circ) \cong G \neq (1)$. Meanwhile, the fiber of $q : X \to Y$ over the singular point is one-dimensional. Hence $\pi_1(X^\circ) = (1)$, where $X^\circ = q^{-1}(Y^\circ)$. Then $\pi_1(Y^\circ) = 1$ by Nori's result, *loc.cit.* This is a contradiction. Hence we have proven that Y is a smooth contractible surface.

We have a very conceptual proof of Y being smooth by using an Affine Mumford Theorem [14]. Since X is simply connected by Theorem 4.8 and the fibers of the quotient morphism are one-dimensional, Nori's theorem [35, Lemma 1.5] implies that $\pi_1(Y - \mathrm{Sing}Y) = 1$. In (4) above, we have shown that $\pi_1(Y) = 1$. Now, Y is smooth by [14, Theorem 3.6].

(6) As in the proof of Theorem 4.8, we take a general member S of $|H|$. Then S is a smooth del Pezzo surface, and $\Gamma = H|_S$ is a smooth curve on S. Since $K_S \sim -(r-1)\Gamma$ with $r \geq 2$, it follows that $\overline{\kappa}(S \setminus \Gamma)$ is 0 or $-\infty$. We may assume that $r = 2$ since we have treated the cases $r = 4$ and $r = 3$ in Theorems 4.5 and 4.6. Then S is horizontal to the quotient morphism. In fact, Γ is then isomorphic to a smooth curve, and $S \setminus \Gamma$ does not contain an $\mathbb{A}^1$-cylinder because $\overline{\kappa}(S \setminus \Gamma) = 0$. If S were not horizontal to the quotient morphism, S would contain a family of G_a-orbits. This is a contradiction. So, the restriction of $q : X \to Y$ onto $S \setminus \Gamma$ is a dominant morphism, and therefore $\overline{\kappa}(Y) = 0$ or $-\infty$. Meanwhile, there is no homology plane in the case $\overline{\kappa}(Y) = 0$ by Fujita's result [9, 18]. This implies that $\overline{\kappa}(Y) = -\infty$. Since Y is factorial and $\Gamma(Y, \mathcal{O}_Y^*) = k^*$, it follows that $Y \cong \mathbb{A}^2$. Q.E.D.

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