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Fundamental groups of symplectic singularities

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§1. Introduction

Let (X, ω) be an affine symplectic variety. By definition (cf. [2]), X is an affine normal variety and ω is a holomorphic symplectic 2-form on the regular locus X_{reg} of X such that it extends to a holomorphic (not necessarily symplectic) 2-form on a resolution $\tilde{X}$ of X. In this article we also assume that X has a $\mathbf{C}^*$-action with positive weights and that ω is homogeneous with respect to the $\mathbf{C}^*$-action. More precisely, the affine ring R of X is positively graded: $R = \oplus_{i \geq 0} R_i$ with $R_0 = \mathbf{C}$ and there is an integer l such that $t^*\omega = t^l \cdot \omega$ for all $t \in \mathbf{C}^*$. Since X has canonical singularities, we have $l > 0$ ([8], Lemma (2.2)). Affine symplectic varieties are constructed in various ways such as nilpotent orbit closures of a semisimple complex Lie algebra (cf. [4]), Slodowy slices to nilpotent orbits ([9]) or symplectic reductions of holomorphic symplectic manifolds with Hamiltonian actions. These varieties come up with $\mathbf{C}^*$-actions and the above assumption of the $\mathbf{C}^*$-action is satisfied in all examples we know.

In the previous article [8] we posed a question:

Problem. *Is the fundamental group $\pi_1(X_{reg})$ finite ?*

Such fundamental groups are explicitly calculated by a group-theoretic method when X is a nilpotent orbit closure (cf. [4]). However no general results are known.

In this short note we give a partial answer to this question. Namely we have

Theorem 1.1. *The algebraic fundamental group $\hat{\pi}_1(X_{reg})$ is a finite group.*

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Notice that a symplectic variety X has canonical singularities. In particular, the log pair $(X, 0)$ has klt (Kawamata log terminal) singularities. The theorem is, in fact, a corollary to the more general result:

Main Theorem. *Let* $X := \mathrm{Spec}R$ *be an affine variety where* R *is positively graded:* $R = \oplus_{i \geq 0} R_i$ *with* $R_0 = \mathbf{C}$. *Assume that the log pair* $(X, 0)$ *has klt singulatities. Then* $\hat{\pi}_1(X_{reg})$ *is a finite group.*

Recently C. Xu [11] proved that, for a klt pair (X, Δ) and a point $p \in X$, the algebraic fundamental group $\hat{\pi}_1(U - \{p\})$ is finite for a small complex analytic neighborhood U of p. By using the Kollár component he obtained it from the finiteness of the algebraic fundamental group of the regular part of a log Fano variety. Since X has a $\mathbf{C}^*$-action in our case, $\hat{\pi}_1(X_{reg}) \cong \hat{\pi}_1(U_{reg})$ for a small complex analytic neighborhood U of the origin $p \in X$. The argument in [Xu] is also valid for $\hat{\pi}_1(U_{reg})$ and one can prove Main Theorem.

In this article we introduce another approach to Main Theorem by using the orbifold fundamental group.

To explain the basic idea of the proof we first assume that R is generated by R_1 as a $\mathbf{C}$-algebra and X has only isolated singularity. Put $\mathbf{P}(X) := \mathrm{Proj}R$. By the assumption $\mathbf{P}(X)$ is a projective manifold. Since $(X, 0)$ has klt singularities, we also see that $\mathbf{P}(X)$ is a Fano manifold (cf. [5], Prposition 4.38). Let L be the tautological line bundle on $\mathbf{P}(X)$ and denote by $(L^{-1})^{\times}$ the $\mathbf{C}^*$-bundle on $\mathbf{P}(X)$ obtained from L^{-1} by removing the 0-section. Then the projection map $p : X - \{0\} \to \mathbf{P}(X)$ can be identified with $(L^{-1})^{\times} \to \mathbf{P}(X)$. There is a homotopy exact sequence

$$\pi_1(\mathbf{C}^*) \to \pi_1(X - \{0\}) \to \pi_1(\mathbf{P}(X)) \to 1.$$

Here $\pi_1(\mathbf{P}(X)) = 1$ because $\mathbf{P}(X)$ is a Fano manifold. We want to show that the first map $\pi_1(\mathbf{C}^*) \to \pi_1(X - \{0\})$ has a nontrivial kernel. Suppose to the contrary that it is an injection. Then $\pi_1(X - \{0\}) = \mathbf{Z}$ and one has a surjective map $\pi_1(X - \{0\}) \to \mathbf{Z}/l\mathbf{Z}$ for any $l > 1$. This determines an étale covering $f : Y \to X - \{0\}$, which extends to a finite surjective map $\bar{f} : \bar{Y} \to L^{-1}$, where $\bar{Y}$ contains Y as a Zariski open subset and $\bar{f}$ is a cyclic covering branched along the 0-section Σ of L^{-1}. The direct image $\bar{f}_* O_{\bar{Y}}$ can be written as $O_{\bar{Y}} \oplus M \oplus M^{\otimes 2} \oplus ... \oplus M^{\otimes l-1}$ with a line bundle M on L^{-1}. Here $M^{\otimes l} \cong O_{L^{-1}}(-\Sigma)$. Restrict this isomorphism to $\Sigma (\cong \mathbf{P}(X))$. Then we have $(M|_{\Sigma})^{\otimes l} \cong L$. This shows that $L \in \mathrm{Pic}(\mathbf{P}(X))$ is divisible by any $l > 1$. But this is absurd because L is an ample line bundle. Therefore $\pi_1(X - \{0\})$ is finite.

In a general situation $\mathbf{P}(X)$ is no more smooth and the projection map $X - \{0\} \to \mathbf{P}(X)$ is not a $\mathbf{C}^*$-bundle. We take a smooth open set $\mathbf{P}(X)^\sharp$ of $\mathbf{P}(X)$ in such a way that $X^\sharp := p^{-1}(\mathbf{P}(X)^\sharp)$ is smooth and $\mathrm{Codim}_{\mathbf{P}(X)}(\mathbf{P}(X) - \mathbf{P}(X)^\sharp) \geq 2$. The map $X^\sharp \to \mathbf{P}(X)^\sharp$ is not still a $\mathbf{C}^*$-bundle, but if we introduce a suitable orbifold structure on $\mathbf{P}(X)$, then it can be regarded as a $\mathbf{C}^*$-bundle on the orbifold $\mathbf{P}(X)^{\sharp, orb}$. Moreover we have a homotopy exact sequence

$$\pi_1(\mathbf{C}^*) \to \pi_1(X^\sharp) \to \pi_1^{orb}(\mathbf{P}(X)^{\sharp, orb}) \to 1.$$

The orbifold structure on $\mathbf{P}(X)$ determines an effective $\mathbf{Q}$-divisor Δ with standard coefficients. By the assumption that $(X, 0)$ has klt singularities, we see that $(\mathbf{P}(X), \Delta)$ is a log Fano variety (§1, Lemma). C. Xu [Xu] has proved that $\hat{\pi}_1(\mathbf{P}(X)_{reg})$ is finite for such a variety. It turns out that his proof can be used to prove that $\hat{\pi}_1^{orb}(\mathbf{P}(X)^{\sharp, orb})$ is finite. We take a finite étale covering $Y^{orb} \to \mathbf{P}(X)^{\sharp, orb}$ such that $\hat{\pi}_1^{orb}(Y^{orb}) = 1$ and define Z to be the normalization of $X^\sharp \times_{\mathbf{P}(X)^\sharp} Y$. Then we see that $Z \to X^\sharp$ is an étale covering in the usual sense. Moreover, we have an exact sequence

$$\pi_1(\mathbf{C}^*) \to \pi_1(Z) \to \pi_1^{orb}(Y^{orb}) \to 1$$

by replacing $X^\sharp$ and $\mathbf{P}(X)^{\sharp, orb}$ by Z and Y^{orb}. Assume that there exists a surjection from $\pi_1(Z)$ to a finite group Γ. Since $\pi_1^{orb}(Y^{orb})$ has no nontrivial finite quotient, the composition map $\pi_1(\mathbf{C}^*) \to \pi_1(Z) \to \Gamma$ is surjective. The orbifold line bundle associated with the orbifold $\mathbf{C}^*$-bundle $Z \to Y^{orb}$ is negative. We prove that the order of Γ cannot be arbitrary large by using this fact; hence $\hat{\pi}_1(Z)$ is finite. Since $\hat{\pi}_1(Z)$ is a finite index subgroup of $\hat{\pi}_1(X^\sharp)$, $\hat{\pi}_1(X^\sharp)$ is also finite. As there is a surjection map $\hat{\pi}_1(X^\sharp) \to \hat{\pi}_1(X_{reg})$, we have Main Theorem.

The argument above also shows that $\pi_1(X_{reg})$ is finite if and only if $\pi_1^{orb}(\mathbf{P}(X)^{\sharp, orb})$ is finite.

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§2. Algebraic orbifolds

In the remainder of this article $X := \mathrm{Spec} R$ is an affine normal variety with a positively graded ring $R = \oplus_{i \geq 0} R_i$, $R_0 = \mathbf{C}$ such that $(X, 0)$ is a klt pair. Take minimal homogeneous generators of R and consider the surjection

$$\mathbf{C}[x_0, ..., x_n] \to R$$

which sends each x_i to the homogeneous generator. Correspondingly X is embedded in $\mathbf{C}^{n+1}$. To x_i we give the same weight as the minimal generator. Put $a_i := wt(x_i)$. We may assume that $GCD(a_0, ..., a_n) = 1$. The quotient variety $\mathbf{C}^{n+1} - \{0\}/\mathbf{C}^*$ by the $\mathbf{C}^*$-action $(x_0, ..., x_n) \to (t^{a_0}x_0, ..., t^{a_n}x_n)$ is the weighted projective space $\mathbf{P}(a_0, ..., a_n)$. We put $\mathbf{P}(X) := X - \{0\}/\mathbf{C}^*$. By definition $\mathbf{P}(X)$ is a closed subvariety of $\mathbf{P}(a_0, ..., a_n)$. Put $W_i := \{x_i = 1\} \subset \mathbf{C}^{n+1}$. Then the projection map $p : \mathbf{C}^{n+1} - \{0\} \to \mathbf{P}(a_0, ..., a_n)$ induces a map $p_i : W_i \to \mathbf{P}(a_0, ..., a_n)$, which is a finite Galois covering of the image. The collection $\{p_i\}$ defines a smooth orbifold structure on $\mathbf{P}(a_0, ..., a_n)$ in the sense of [7], §2. More exactly, the following are satisfied

(i) For each i, W_i is a smooth variety and $p_i : W_i \to p_i(W_i)$ is a finite Galois covering[1]. $\cup \mathrm{Im}(p_i) = \mathbf{P}(a_0, ..., a_n)$.

(ii) Let $(W_i \times_{\mathbf{P}(a_0,...,a_n)} W_j)^n$ denote the normalization of the fibre product $W_i \times_{\mathbf{P}(a_0,...,a_n)} W_j$. Then the maps $(W_i \times_{\mathbf{P}(a_0,...,a_n)} W_j)^n \to W_i$ and $(W_i \times_{\mathbf{P}(a_0,...,a_n)} W_j)^n \to W_j$ are both étale maps.

The orbifold $\mathbf{P}(a_0, ..., a_n)$ admits an orbifold line bundle $O_{\mathbf{P}(a_0,...,a_n)}(1)$. Put $D_i := \{x_i = 0\} \subset \mathbf{P}(a_0, ..., a_n)$ and $D := \cup D_i$. Since x_i are minimal generators, $\bar{D} := \mathbf{P}(X) \cap D$ is a divisor of $\mathbf{P}(X)$. Define

$$\mathbf{P}(X)^\sharp := \mathbf{P}(X) - \mathrm{Sing}(\bar{D}) - \mathrm{Sing}(\mathbf{P}(X)),$$

and

$$X^\sharp := p^{-1}(\mathbf{P}(X)^\sharp).$$

Let

$$\bar{D} = \cup \bar{D}_\alpha$$

be the decomposition into irreducible components [2]. By definition $\bar{D}^\sharp := \bar{D} \cap \mathbf{P}(X)^\sharp$ is a smooth divisor of $\mathbf{P}(X)^\sharp$. Put $\bar{D}^\sharp_\alpha := \bar{D}_\alpha \cap \mathbf{P}(X)^\sharp$. Then $\bar{D}^\sharp$ is the disjoint union of irreducible smooth divisors $\bar{D}^\sharp_\alpha$.

In general $p^{-1}(\mathbf{P}(X)_{reg})$ is not smooth; but if we shrink $\mathbf{P}(X)_{reg}$ to $\mathbf{P}(X)^\sharp$, then its inverse image $X^\sharp$ is smooth.

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[1]The precise definition of an orbifold only needs a slightly weaker condition: $p_i : W_i \to \mathbf{P}(a_0, ..., a_n)$ factorizes as $W_i \stackrel{q_i}{\to} W_i/G_i \stackrel{r_i}{\to} \mathbf{P}(a_0, ..., a_n)$ where G_i is a finite group and r_i is an étale map.

[2]The index α is usually different from the original index i of D_i because $D_{i_1} \cap ... \cap D_{i_k} \cap \mathbf{P}(X)$ may possibly become an irreducible component of $\bar{D}$ or $D_i \cap \mathbf{P}(X)$ may split into more than two irreducible components of $\bar{D}$.

Notice that every fibre of $p^\sharp (:= p|_{X^\sharp}) : X^\sharp \to \mathbf{P}(X)^\sharp$ is isomorphic to $\mathbf{C}^*$, but the fibre over a point of $\bar{D}^\sharp_\alpha$ may possibly be a multiple fibre. We denote by m_α[3] the multiplicity of a fibre over a point of $\bar{D}^\sharp_\alpha$.

The map $p^\sharp$ is a $\mathbf{C}^*$-bundle if we restrict it to the open set $\mathbf{P}(X)^\sharp - \bar{D}^\sharp$. Notice that $\mathbf{P}(X) - \mathbf{P}(X)^\sharp$ has at least codimension 2 in $\mathbf{P}(X)$.

By putting $U_i := X \cap W_i$ and $\pi_i := p_i|_{U_i}$, the collection $\{\pi_i : U_i \to \mathbf{P}(X)\}$ of covering maps induces a (not necessarily smooth) orbifold structure on $\mathbf{P}(X)$. Namely, we have

(i) For each i, U_i is a normal variety and $\pi_i : U_i \to \pi_i(U_i)$ is a finite Galois covering. $\cup \mathrm{Im}(\pi_i) = \mathbf{P}(X)$

(ii) The maps $(U_i \times_{\mathbf{P}(X)} U_j)^n \to U_i$ and $(U_i \times_{\mathbf{P}(X)} U_j)^n \to U_j$ are both étale maps.

We put $\mathcal{L} := O_{\mathbf{P}(a_0,\ldots,a_n)}(1)|_{\mathbf{P}(X)}$, which is an orbifold line bundle on $\mathbf{P}(X)$. We call $\mathcal{L}$ the tautological line bundle. Then $X - \{0\} \to \mathbf{P}(X)$ can be regarded as an orbifold $\mathbf{C}^*$-bundlle $(\mathcal{L}^{-1})^\times$

Notice that, if we restrict this orbifold structure to $\mathbf{P}(X)^\sharp$, then it is a smooth orbifold structure.

Lemma 2.1. *Assume that the log pair $(X, 0)$ has klt singularities. Put $\Delta := \sum (1 - 1/m_\alpha)\bar{D}_\alpha$. Then $(\mathbf{P}(X), \Delta)$ is a log Fano variety, that is, $(\mathbf{P}(X), \Delta)$ has klt singularities and $-(K_{\mathbf{P}(X)} + \Delta)$ is an ample $\mathbf{Q}$-divisor.*

Proof. Take a positive integer d in such a way that the subring $R^{(d)} := \oplus_{i \geq 0} R_{id}$ is generated by R_d as a $\mathbf{C}$-algebra. We put $V := \mathrm{Spec} R^{(d)}$. Then there is a finite surjective map $\mu : X \to V$. Notice that $\mathrm{Proj}(R) = \mathrm{Proj}(R^{(d)})$. Hence there is a natural projection map $q : V - \{0\} \to \mathbf{P}(X)$ and the composition map $X - \{0\} \to V - \{0\} \to \mathbf{P}(X)$ coincides with the natural projection map $p : X - \{0\} \to \mathbf{P}(X)$. Since $R^{(d)}$ is generated by R_d as a $\mathbf{C}$-algebra, the projection map q is a $\mathbf{C}^*$-bundle. We put $V^\sharp := q^{-1}(\mathbf{P}(X)^\sharp)$ and put $q^\sharp := q|_{V^\sharp} : V^\sharp \to \mathbf{P}(X)^\sharp$. For a point $t \in \mathbf{P}(X)^\sharp - \bar{D}^\sharp$, the fibres $(p^\sharp)^{-1}(t)$ and $(q^\sharp)^{-1}(t)$ are both isomorphic to $\mathbf{C}^*$ and μ induces an etale covering between them of the same degree as $\deg(\mu)$. On the other hand, for a point $t \in \bar{D}^\sharp_\alpha$, the fibre $(p^\sharp)^{-1}(t)$ is a multiple fibre with multiplicity m_α and $(p^\sharp)^{-1}(t)_{red} \cong \mathbf{C}^*$.

In this case μ induces an etale covering $(p^\sharp)^{-1}(t)_{red} \to (q^\sharp)^{-1}(t)$ of degree $\deg(\mu)/m_\alpha$. In other words, $X^\sharp \to V^\sharp$ is a finite cover, which is branched along $(q^\sharp)^{-1}(\cup_{\alpha; m_\alpha > 1} \bar{D}^\sharp_\alpha)$. Let B be the $\mathbf{Q}$-divisor of V obtained as the closure of the $\mathbf{Q}$-divisor $q^*\Delta$ of $V - \{0\}$. Here notice

[3]The multiplicity m_α may possibly be one.

that $\mathrm{Codim}_X(X - X^\sharp) \geq 2$ and $\mathrm{Codim}_V(V - V^\sharp) \geq 2$. Then we have

$$K_X = \mu^*(K_V + B).$$

Since $(X, 0)$ is a klt pair, (V, B) is a klt pair by [6], Proposition 5.20. By [5], Proposition 4.38, we conclude that $(\mathbf{P}(X), \Delta)$ is a log Fano variety. Q.E.D.

Remark 2.2. *The lemma is rephrased as: if* $(X, 0)$ *has klt singularities, then* $\mathbf{P}(X)^{orb}$ *is a Fano orbifold. When* X *is an affine symplectic variety, this can be proved directly by using the fact that* $\mathbf{P}(X)$ *has a contact orbifold structure (*[8]*, Theorem 4.4.1).*

§3. Algebraic orbifold fundamental group of $\mathbf{P}(X)^\sharp$

In the previous section we observed that $\mathbf{P}(X)^\sharp$ has a smooth orbifold structure. Namely, if we put $U_i^\sharp := \pi_i^{-1}(\mathbf{P}(X)^\sharp)$, and $\pi_i^\sharp := \pi_i|_{U_i^\sharp}$, then $\mathcal{U} := \{\pi_i^\sharp : U_i^\sharp \to \mathbf{P}(X)^\sharp\}_{i \in I}$ give the orbifold charts of $\mathbf{P}(X)^\sharp$. Assume that $\mathbf{P}(X)^\sharp$ has another orbifold charts $\mathcal{U}' := \{\pi'_j : U'_j \to \mathbf{P}(X)^\sharp\}_{j \in J}$. Then $\mathcal{U}$ and $\mathcal{U}'$ are equivalent if, for each $i \in I$ and $j \in J$, two maps $U_i^\sharp \to \mathbf{P}(X)^\sharp$ and $U'_j \to \mathbf{P}(X)^\sharp$ are *admissible* to each other: in other words, $(U_i^\sharp \times_{\mathbf{P}(X)^\sharp} U'_j)^n \to U_i^\sharp$ and $(U_i^\sharp \times_{\mathbf{P}(X)^\sharp} U'_j)^n \to U'_j$ are both étale maps. An orbifold structure on $\mathbf{P}(X)^\sharp$ is precisely an equivalence class of orbifold charts of $\mathbf{P}(X)^\sharp$. In the remainder we will denote by $\mathbf{P}(X)^{\sharp,orb}$ the orbifold structure defined in the previous section. Let Y^{orb} be a smooth orbifold; namely it is a pair of a normal algebraic variety Y and an equivalence class of orbifold charts $\mathcal{V} = \{\nu_k : V_k \to Y\}_{k \in K}$. Let $f : Y \to \mathbf{P}(X)^\sharp$ be a finite surjective morphism of algebraic varieties. We say that f is an étale covering map from Y^{orb} to $\mathbf{P}(X)^{\sharp,orb}$ if the following property holds:

For any $k \in K$ and $i \in I$, two maps $f \circ \nu_k : V_k \to \mathbf{P}(X)^\sharp$ and $\pi_i : U_i^\sharp \to \mathbf{P}(X)^\sharp$ are admissible to each other.

Notice that f is not necessarily an étale covering map in the usual sense even if f is an étale covering map of orbifolds. An étale covering map is said to be Galois if the underlying morphism is Galois in the usual sense.

Lemma 3.1. *For any finite étale covering* $f : Y^{orb} \to \mathbf{P}(X)^{\sharp,orb}$*, there exists an étale Galois covering* $g : Z^{orb} \to \mathbf{P}(X)^{\sharp,orb}$ *such that* g *factorizes as* $Z^{orb} \to Y^{orb} \xrightarrow{f} \mathbf{P}(X)^{\sharp,orb}$*.*

Proof. Let K and L be the function fields of $\mathbf{P}(X)^\sharp$ and Y. Let M be the Galois closure of L/K and take the normalization Z of Y

in M. We shall give an orbifold structure on Z in such a way that $Z^{orb} \to \mathbf{P}(X)^{\sharp, orb}$ is an étale covering and it factorizes through Y^{orb}. Let $V \to Y$ be an orbifold chart of Y^{orb}. By definition V is smooth and the composition map $V \to Y \to \mathbf{P}(X)^{\sharp}$ is admissible for any orbifold chart $U \to \mathbf{P}(X)^{\sharp}$. In other words, $(V \times_{\mathbf{P}(X)^{\sharp}} U)^n \to V$ and $(V \times_{\mathbf{P}(X)^{\sharp}} U)^n \to U$ are both étale maps. Let L' be the function field of V and let M' be the Galois closure of L'/K. Let W be the normalization of V in M'. Then, since $M \subset M'$, there is a natural map $q_V : W \to Z$. It is clear that $\cup \mathrm{Im}(q_V) = Z$ when V runs through all orbifold charts of Y^{orb}. We prove that the map $W \to V$ is étale.

Here we recall an explicit construction of W. Assume that $V \to \mathbf{P}(X)^{\sharp}$ is not Galois. Then there is an element $\sigma \in G := \mathrm{Gal}(M'/K)$ such that $(L')^{\sigma} \neq L'$, where $(L')^{\sigma} := \sigma(L')$. We take an irreducible component V_1 of $(V \times_{\mathbf{P}(X)^{\sharp}} V^{\sigma})^n$ in such a way that the induced map $V_1 \to V$ is a finite covering with deg ≥ 2. Let L_1 be the function field of V_1 and if L_1/K is not still a Galois extension, we take the Galois closure M_1 of L_1/K. There exists an element $\sigma_1 \in G_1 := \mathrm{Gal}(M_1/K)$ such that $(L_1)^{\sigma_1} \neq L_1$. We take an irreducible component V_2 of $(V_1 \times_{\mathbf{P}(X)^{\sharp}} (V_1)^{\sigma_1})^n$ in such a way that the induced map $V_2 \to V_1$ has degree ≥ 2. When we repeat this process, we finally reach the W.

Thus, to prove that W is étale over V, we only have to show that $(V \times_{\mathbf{P}(X)^{\sharp}} V^{\sigma})^n \to V$ and $(V \times_{\mathbf{P}(X)^{\sharp}} V^{\sigma})^n \to V^{\sigma}$ are both étale maps. In fact, if this is proved, then $V_1 \to V$ is an étale map and hence $V_1 \to \mathbf{P}(X)^{\sharp}$ and $U \to \mathbf{P}(X)^{\sharp}$ are admissible to each other. We may then replace V by V_1 and continue.

Before starting the proof we notice that σ induces a $\mathbf{P}(X)^{\sharp}$-isomorphism $U \cong U^{\sigma}$. Since $V \to \mathbf{P}(X)^{\sharp}$ and $U \to \mathbf{P}(X)^{\sharp}$ are admissible to each other, $(V \times_{\mathbf{P}(X)^{\sharp}} U)^n \to U$ and $(V \times_{\mathbf{P}(X)^{\sharp}} U)^n \to V$ are both étale maps. Since $V^{\sigma} \to \mathbf{P}(X)^{\sharp}$ and $U^{\sigma} \to \mathbf{P}(X)^{\sharp}$ are also admissible to each other, $(V^{\sigma} \times_{\mathbf{P}(X)^{\sharp}} U^{\sigma})^n \to U^{\sigma}$ and $(V^{\sigma} \times_{\mathbf{P}(X)^{\sharp}} U^{\sigma})^n \to V^{\sigma}$ are étale maps. Here, identifying U^{σ} with U by the above isomorphism, we get two maps $(V^{\sigma} \times_{\mathbf{P}(X)^{\sharp}} U)^n \to U$ and $(V^{\sigma} \times_{\mathbf{P}(X)^{\sharp}} U)^n \to V^{\sigma}$. Now we have a commutative diagram

$$\begin{array}{ccc} (V \times_{\mathbf{P}(X)^{\sharp}} U \times_{\mathbf{P}(X)^{\sharp}} V^{\sigma})^n & \longrightarrow & (V^{\sigma} \times_{\mathbf{P}(X)^{\sharp}} U)^n \\ \downarrow & & \downarrow \\ (V \times_{\mathbf{P}(X)^{\sharp}} U)^n & \longrightarrow & U, \end{array} \tag{3.1}$$

where all maps are étale. Let us consider the map $(V \times_{\mathbf{P}(X)^{\sharp}} U \times_{\mathbf{P}(X)^{\sharp}} V^{\sigma})^n \to V$. Since this map factorizes through $(V \times_{\mathbf{P}(X)^{\sharp}} U)^n$, it is an étale map. On the other hand, this map also factorizes as

$$(V \times_{\mathbf{P}(X)^\sharp} U \times_{\mathbf{P}(X)^\sharp} V^\sigma)^n \to (V \times_{\mathbf{P}(X)^\sharp} V^\sigma)^n \to V.$$

As the first map is étale, the second map $(V \times_{\mathbf{P}(X)^\sharp} V^\sigma)^n \to V$ is an étale map. By a similar reasoning we see that $(V \times_{\mathbf{P}(X)^\sharp} V^\sigma)^n \to V^\sigma$ is an étale map.

We finally prove that $\{q_V : W \to Z\}$ gives an orbifold structure to Z. Since M' is a Galois extension of M, we see that $q_V(W)$ is the quotient variety of W by $\mathrm{Gal}(M'/M)$. Moreover, W is a smooth variety because it is an étale cover of a smooth variety V. In the remainder we shall check that $q_{V'} : W' \to Z$ and $q_V : W \to Z$ are admissible to each other. We have a commutative diagram

$$\begin{array}{ccc} (W \times_Y W')^n & \longrightarrow & W \\ \downarrow & & \downarrow \\ (V \times_Y V')^n & \longrightarrow & V. \end{array} \tag{3.2}$$

Here two vertical maps are étale because $W \to V$ and $W' \to V'$ are étale, and the second horizontal map is also étale because $V \to Y$ and $V' \to Y$ are admissible to each other. Hence the first horizontal map $(W \times_Y W')^n \to W$ is étale by the commutative diagram. The map $(W \times_Z W')^n \to W$ is factorizes as

$$(W \times_Z W')^n \to (W \times_Y W')^n \to W.$$

Since first map is an open immersion, we see that $(W \times_Z W')^n \to W$ is an étale map. Similarly, $(W \times_Z W')^n \to W'$ is an étale map. Q.E.D.

Take a point $x \in \mathbf{P}(X)^\sharp$ in such a way that $x \notin \bar{D}^\sharp$. Then $f^{-1}(x)$ consists of exactly $\deg(f)$ points. Consider all pairs (Y^{orb}, y) of étale coverings Y^{orb} of $\mathbf{P}(X)^{\sharp,orb}$ and $y \in Y$ lying on $x \in \mathbf{P}(X)^\sharp$. A morphism $h : (Z^{orb}, z) \to (Y^{orb}, y)$ is a $\mathbf{P}(X)^\sharp$-morphism $h : Z \to Y$ with $h(z) = y$ such that it is an étale covering map from Z^{orb} to Y^{orb}. When Z and Y are both Galois coverings of $\mathbf{P}(X)^\sharp$, h induces a surjective map $\mathrm{Aut}(Z/\mathbf{P}(X)^\sharp) \to \mathrm{Aut}(Y/\mathbf{P}(X)^\sharp)$. As in the usual situation, we can define the algebraic orbifold fundamental group $\hat{\pi}_1^{orb}(\mathbf{P}(X)^{\sharp,orb}, x)$ as the profinite group $\lim \mathrm{Aut}(Y/\mathbf{P}(X)^\sharp)$, where f runs through all finite étale Galois coverings of $\mathbf{P}(X)^{\sharp,orb}$.

Theorem 3.2. [11] $\hat{\pi}_1^{orb}(\mathbf{P}(X)^{\sharp,orb}, x)$ is a finite group.

Proof. Write $\Delta^\sharp$ for $\Delta|_{\mathbf{P}(X)^\sharp}$. Let Y^{orb} be a finite étale covering map of $\mathbf{P}(X)^{\sharp,orb}$ and let $f : Y \to \mathbf{P}(X)^\sharp$ be the underlying map. We shall prove that $K_Y + \Delta_Y = f^*(K_{\mathbf{P}(X)^\sharp} + \Delta^\sharp)$ for some effective divisor Δ_Y on Y. Let $\mathcal{V} = \{\nu_k : V_k \to Y\}_{k \in K}$ be orbifold covering charts. Let Z be an irreducible component of the normalization $(V_k \times_{\mathbf{P}(X)^\sharp} U_i^\sharp)^n$ of the fibre product of the diagram

$$V_k \stackrel{f \circ \nu_k}{\to} \mathbf{P}(X)^\sharp \leftarrow U_i^\sharp.$$

Then we have a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{p_2} & U_i^\sharp \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi_i} \\ V_k & \xrightarrow{f \circ \nu_k} & \mathbf{P}(X)^\sharp. \end{array} \tag{3.3}$$

Here p_1 and p_2 are both étale maps. Since $K_{U_i^\sharp} = \pi_i^*(K_{\mathbf{P}(X)^\sharp} + \Delta^\sharp)$ and $K_Z = p_2^* K_{U_i^\sharp}$, we have $K_Z = (\pi_i \circ p_2)^*(K_{\mathbf{P}(X)^\sharp} + \Delta^\sharp)$. On the other hand, since $K_Z = p_1^* K_{V_k}$, we see that $K_{V_k} = (f \circ \nu_k)^*(K_{\mathbf{P}(X)^\sharp} + \Delta^\sharp)$. Then one can write $K_Y + \Delta_Y = f^*(K_{\mathbf{P}(X)^\sharp} + \Delta^\sharp)$ with some divisor $\Delta_Y \geq 0$.

The finite covering $f : Y \to \mathbf{P}(X)^\sharp$ can be compactified to a finite covering $\bar{f} : \bar{Y} \to \mathbf{P}(X)$. Let $\Delta_{\bar{Y}}$ be the closure of Δ_Y in $\bar{Y}$. Since $\mathbf{P}(X) - \mathbf{P}(X)^\sharp$ has codimension at least 2, one can write $K_{\bar{Y}} + \Delta_{\bar{Y}} = \bar{f}^*(K_{\mathbf{P}(X)} + \Delta)$. Then, by [11], Proposition 1, the degree of such $\bar{f}$ is bounded by a constant only depending on $(\mathbf{P}(X), \Delta)$. Thus $\deg(f)$ is bounded above. Q.E.D.

§4. Complex analytic orbifolds

We can define a complex analytic orbifold structure just by replacing the algebraic orbifold charts in §1 with the complex analytic orbifold charts (cf. [10], Chapter 13). Let U be a sufficiently small open neighborhood of $0 \in \mathbf{C}^m$ where a finite group Γ acts on U fixing the origin. Then a holomorphic map $\pi : U \to \mathbf{P}(X)^\sharp$ is called an orbifold chart if it is factorized as $U \to U/\Gamma \subset \mathbf{P}(X)^\sharp$. We say that two charts $\pi : U \to \mathbf{P}(X)^\sharp$ and $\pi' : U' \to \mathbf{P}(X)^\sharp$ are admissible if $(U \times_{\mathbf{P}(X)^\sharp} U')^n \to U$ and $(U \times_{\mathbf{P}(X)^\sharp} U')^n \to U'$ are both étale maps. Orbifold covering charts of $\mathbf{P}(X)^\sharp$ is a collection $\{\pi : U \to \mathbf{P}(X)^\sharp\}$ of mutually admissible charts such that $\cup \mathrm{Im}(\pi) = \mathbf{P}(X)^\sharp$. An orbifold structure on $\mathbf{P}(X)^\sharp$ is nothing but an equivalence class of such collections.

It is easily checked that a smooth algebraic orbifold structure on an algebraic variety Y naturally induces a complex analytic orbifold structure on Y. Conversely, if an algebraic variety Y has a smooth complex analytic orbifold structure, then Y admits a smooth algebraic orbifold structure. In fact, let $\nu : V \to V/G \subset Y$ be a complex analytic orbifold chart, where V is an open neighborhood of $0 \in \mathbf{C}^m$ and G fixes the origin. We put $y := \nu(0)$. By the local linearization of a finite group action ([3], p.97) we may assume that G-action on V is induced from linear transformations of $\mathbf{C}^m$. By Artin's approximation theorem ([1], Corollary (2.6)) we may take a common étale neighborhood $w \in W$ of $y \in Y$ and $\bar{0} \in \mathbf{C}^m/G$:

$$Y \leftarrow W \to \mathbf{C}^m/G.$$

Take the connected component W' of $W \times_{\mathbf{C}^m/G} \mathbf{C}^m$ containing the point $(w, 0)$. Then one can write $W = W'/G'$ with a suitable subgroup G' of G and the composition map $W' \to W \to Y$ gives a smooth algebraic orbifold chart.

Let Y be a connected complex analytic space with an orbifold structure. Then a *covering map* $f : Y^{orb} \to \mathbf{P}(X)^{\sharp, orb}$ is a holomorphic map $f : Y \to \mathbf{P}(X)^{\sharp}$ of the underlying spaces such that

(i) for any point $x \in \mathbf{P}(X)^{\sharp}$, there exists an admissible orbifold chart $\pi : U \to \mathbf{P}(X)^{\sharp}$ with $x \in \mathrm{Im}(\pi)$ and each connected component V_i of $f^{-1}(\pi(U))$ can be written as U/Γ_i where Γ_i is some subgroup of Γ,

(ii) the map $U \to U/\Gamma_i \cong V_i \subset Y$ is an admissible orbifold chart of Y^{orb}.

Let $f : Y^{orb} \to \mathbf{P}(X)^{\sharp, orb}$ be an étale covering of algebraic orbifolds. Then it induces a covering map of complex analytic orbifolds. In fact, let $y \in Y$ and $x := f(y) \in \mathbf{P}(X)^{\sharp}$. Choose algebraic orbifold charts $\mu : V \to Y$ and $\pi : U \to \mathbf{P}(X)^{\sharp}$ so that their images contain y and x respectively. We choose points $v \in V$ and $u \in U$ so that $\nu(v) = y$ and $\pi(u) = x$. We have a diagram of étale maps

$$V \leftarrow (V \times_{\mathbf{P}(X)^{\sharp}} U)^n \to U,$$

and it induces an isomorphism of complex analytic germs $(V, v) \to (U, u)$. We have a commutative diagram

$$\begin{array}{ccc} (V, v) & \longrightarrow & (U, u) \\ \downarrow & & \downarrow \\ (Y, y) & \longrightarrow & (\mathbf{P}(X)^{\sharp}, x). \end{array} \tag{4.1}$$

By the assumption $(\mathbf{P}(X)^{\sharp}, x) \cong (U/G, \bar{u})$ with a finite group G. By the commutative diagram (Y, y) can be also written as $(U/G', \bar{u})$ with a subgroup G' of G. This shows that f is a covering map of complex analytic orbifolds.

Conversely, if $Y^{orb} \to \mathbf{P}(X)^{\sharp,orb}$ is a covering of complex analytic orbifolds with finite degree, then Y is an algebraic variety with a smooth algebraic orbifold structure and the map $Y^{orb} \to \mathbf{P}^{\sharp,orb}$ is an étale covering of algebraic orbifolds.

Notice however that covering maps of complex analytic orbifolds generally have infinite degrees.

As in §2 take a point $x \in \mathbf{P}(X)^\sharp - \bar{D}^\sharp$ and consider all pairs (Y^{orb}, y) of coverings $Y^{orb} \to \mathbf{P}(X)^\sharp$ and $y \in Y$ lying over x. If (Y^{orb}, y) and (Y'^{orb}, y') are among them, then we can take a unique irreducible component Z of $(Y \times_{\mathbf{P}(X)^{orb}} Y')^n$ passing through the point $z := (y, y')$. Moreover there exists an orbifold structure on Z such that the induced map $Z^{orb} \to \mathbf{P}(X)^{\sharp,orb}$ is a covering map of orbifolds. Such constructions enable us to take the inverse limit $(Y^{*,orb}, y^*)$ of the inductive system $\{(Y^{orb}, y)\}$. Thurston ([10], 13.2.4) has defined the orbifold fundamental group $\pi_1^{orb}(\mathbf{P}(X)^{\sharp,orb}, x)$ as the deck transformation group of $Y^* \to \mathbf{P}(X)^\sharp$.

§5. Proof of Main Theorem

We fix a point $x \in \mathbf{P}(X)^\sharp - \bar{D}$ and $x^\sharp \in X^\sharp$ with $p^\sharp(x^\sharp) = x$.

Lemma 5.1. *There exists an exact sequence*

$$\pi_1(\mathbf{C}^*, x^\sharp) \to \pi_1(X^\sharp, x^\sharp) \to \pi_1^{orb}(\mathbf{P}(X)^{\sharp,orb}, x) \to 1.$$

Proof. Define

$$\mathbf{P}(X)^\sharp_0 := \mathbf{P}(X)^\sharp - \cup_{\alpha; m_\alpha > 1} \bar{D}^\sharp_\alpha$$

and

$$X^\sharp_0 := (p^\sharp)^{-1}(\mathbf{P}(X)^\sharp_0).$$

Since $X^\sharp_0 \to \mathbf{P}(X)^\sharp_0$ is a $\mathbf{C}^*$-bundle, we have an exact sequence

$$\pi_1(\mathbf{C}^*, x^\sharp) \to \pi_1(X^\sharp_0, x^\sharp) \to \pi_1(\mathbf{P}(X)^\sharp_0, x) \to 1,$$

where $\mathbf{C}^*$ is regarded as a fibre $(p^\sharp)^{-1}(x)$. Put

$$C := \mathrm{Coker}[\pi_1(\mathbf{C}^*, x^\sharp) \to \pi_1(X^\sharp, x^\sharp)].$$

We want to prove that $C = \pi_1^{orb}(\mathbf{P}(X)^{\sharp,orb}, x)$. The kernel N of the natural map $\pi_1(X^\sharp_0, x^\sharp) \to \pi_1(X^\sharp, x^\sharp)$ is described as follows. Put $E_\alpha := (p^\sharp)^{-1}(\bar{D}_\alpha)^\sharp$ for α with $m_\alpha > 1$. Let β'_α be a small circle in $X^\sharp_0$ around a point of E_α. Take a point $q_\alpha \in \beta'_\alpha$ and choose a path t_α in $X_0^\sharp$ connecting $x^\sharp$ and q_α. We define a loop β_α starting from $x^\sharp$ as $\beta_\alpha := t_\alpha^{-1} \circ \beta'_\alpha \circ t_\alpha$. Then N is the smallest normal subgroup of $\pi_1(X_0^\sharp, x^\sharp)$ containing the elements $[\beta_\alpha]$.

Summing up these facts, one gets an exact commutative diagram

$$
\begin{array}{ccccccc}
 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \\
 & & N & \longrightarrow & p^\sharp_*(N) & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \\
\pi_1(\mathbf{C}^*, x^\sharp) & \longrightarrow & \pi_1(X_0^\sharp, x^\sharp) & \xrightarrow{p^\sharp_*} & \pi_1(\mathbf{P}(X)^\sharp_0, x) & \longrightarrow & 1\\
{\scriptstyle id}\downarrow & & \downarrow & & \downarrow & & \\
\pi_1(\mathbf{C}^*, x^\sharp) & \longrightarrow & \pi_1(X^\sharp, x^\sharp) & \longrightarrow & C & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & &
\end{array}
\tag{5.1}
$$

We next consider $\pi_1^{orb}(\mathbf{P}(X)^{\sharp, orb}, x)$. For each α with $m_\alpha > 1$, let γ'_α be a small circle in $\mathbf{P}(X)_0^\sharp$ around a point of $\bar{D}_\alpha^\sharp$. Take a point $p_\alpha \in \gamma_\alpha$ and choose a path s_α in $\mathbf{P}(X)_0^\sharp$ connecting x and p_α. We define a loop γ_α starting from x as $\gamma_\alpha := s_\alpha^{-1} \circ \gamma'_\alpha \circ s_\alpha$. Let M be the smallest normal subgroup of $\pi_1(\mathbf{P}(X)_0^\sharp, x)$ containing the elements $[\gamma_\alpha^{m_\alpha}]$. Then we have

$$\pi_1^{orb}(\mathbf{P}(X)^{\sharp, orb}, x) \cong \pi_1(\mathbf{P}(X)_0^\sharp, x)/M.$$

Since $p_*^\sharp([\beta_\alpha]) = [\gamma_\alpha^{m_\alpha}]$, we see that $M = p_*^\sharp(N)$. This implies that $C = \pi_1^{orb}(\mathbf{P}(X)^{\sharp, orb}, x)$. Q.E.D.

By 3.2 and 3.1, there is a finite étale Galois covering $(Y^{orb}, y) \to (\mathbf{P}(X)^\sharp, x)$ such that $\hat{\pi}_1^{orb}(Y^{orb}, y) = \{1\}$. Put $Z := (X^\sharp \times_{\mathbf{P}(X)^\sharp} Y)^n$ and choose a point $z \in Z$ lying over $x^\sharp$ and y. Then we have an exact sequence

$$\pi_1(\mathbf{C}^*, z) \to \pi_1(Z, z) \to \pi_1^{orb}(Y^{orb}, y) \to 1.$$

To prove that $\hat{\pi}_1(X^\sharp, x^\sharp)$ is finite, it is enough to show that $\hat{\pi}_1(Z, z)$ is finite because $Z \to X^\sharp$ is a finite étale covering in the usual sense.

Assume that for an arbitrary positive integer m, there is a finite group Γ with $|\Gamma| \geq m$ such that there is a surjection $\pi_1(Z, z) \to \Gamma$. Put

$$K := \mathrm{Im}[\pi_1(\mathbf{C}^*, z) \to \pi_1(Z, z)].$$

Let Γ_K be the image of the composition map $K \to \pi_1(Z,z) \to \Gamma$. Then the surjection $\pi_1(Z,z) \to \Gamma$ induces a surjection $\pi_1^{orb}(Y^{orb}, y) \to \Gamma/\Gamma_K$. But, since $\hat{\pi}_1^{orb}(Y^{orb}, y)$ is trivial, $\Gamma/\Gamma_K = 1$. Therefore the composition map $K \to \pi_1(Z,z) \to \Gamma$ is a surjection. If K is a finite group, then this contradicts the assumption that m can be arbitrary large. Hence the map $\pi_1(\mathbf{C}^*, z) \to \pi_1(Z,z)$ is an injection and $K = \mathbf{Z}$. Since Γ is a quotient group of $\mathbf{Z}$ we have $\Gamma = \mathbf{Z}/l\mathbf{Z}$ for some $l \geq m$.

As explained in Introduction, this leads to a contradiction when Z is a $\mathbf{C}^*$-bundle over Y. In a general case Z is not a $\mathbf{C}^*$-bundle over Y, but it is an orbifold $\mathbf{C}^*$-bundle over Y^{orb}. Thus we can apply a similar argument to the orbifold $\mathbf{C}^*$-bundle Z to get a contradiction:

Let us consider the finite étale covering $Z' \to Z$ determined by the surjection $\pi_1(Z,z) \to \mathbf{Z}/l\mathbf{Z}$. By definition this covering induces a cyclic covering $\mathbf{C}^* \to \mathbf{C}^*$ of degree l for each general fibre of $Z \to Y$. Notice that $Z \to Y^{orb}$ is an orbifold $\mathbf{C}^*$ bundle. Let $\mathcal{L}$ be the associated orbifold line bundle on Y^{orb}. Then Z can be obtained from $\mathcal{L}$ by removing the zero section. The finite étale covering map $Z' \to Z$ induces a cyclic covering $\mathcal{L}' \to \mathcal{L}$ of degree l from an orbifold line bundle $\mathcal{L}'$ on Y^{orb} branched along the zero section and Z' is obtained from $\mathcal{L}'$ by removing the zero section. This fact, in particular, implies that $[\mathcal{L}] \in \mathrm{Pic}(Y^{orb})$ is divisible by l. Note that m can be arbitrary large; but this is impossible because $\mathcal{L}^{-1}$ is an ample orbifold line bundle[4]. Therefore $\hat{\pi}_1(Z,z)$ is finite and so is $\hat{\pi}_1(X^\sharp, x^\sharp)$. Since the natural map $\hat{\pi}_1(X^\sharp, x^\sharp) \to \hat{\pi}_1(X_{reg}, x^\sharp)$ is surjective, we conclude that $\hat{\pi}_1(X_{reg}, x^\sharp)$ is finite.

Remark 5.2. *If $\pi_1^{orb}(\mathbf{P}(X)^\sharp)$ is finite, then the argument above shows that $\pi_1(X_{reg})$ is finite.*

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[4]As $\mathbf{P}(X)^\sharp$is obtained from a projective variety $\mathbf{P}(X)$ by removing a closed subset of codimension ≥ 2, there is a complete curve inside $\mathbf{P}(X)^\sharp$; hence there is a complete curve C inside Y. There is a positive integer N such that $(\mathcal{M}.C) \in 1/N \cdot \mathbf{Z}$ for any orbifold line bundle $\mathcal{M}$ on Y^{orb}. For the orbifold line bundle $\mathcal{L}$ one can write $(\mathcal{L}.C) = -k/N$ with a positive integer k. Then $[\mathcal{L}]$ is divisible at most by k.

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