Advanced Studies in Pure Mathematics 72, 2017
Geometry, Dynamics, and Foliations 2013
pp. 441–451

Actions of groups of diffeomorphisms on one-manifolds by C^1 diffeomorphisms

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Abstract.

Denote by $\mathrm{Diff}_c^r(M)_0$ the identity component of the group of the compactly supported C^r diffeomorphisms of a connected C^∞ manifold M. We show that if $\dim(M) \geq 2$ and $r \neq \dim(M)+1$, then any homomorphism from $\mathrm{Diff}_c^r(M)_0$ to $\mathrm{Diff}^1(\mathbb{R})$ or $\mathrm{Diff}^1(S^1)$ is trivial.

§1. Introduction

É. Ghys [G] asked if the group of diffeomorphisms of a manifold admits a nontrivial action on a lower dimensional manifold. A break through towards this problem was obtained by K. Mann [M] for one dimensional target manifolds. Subsequently, satisfactory results were obtained by S. Hurtado [H] for higher dimensional target manifolds. Surprizingly enough, his argument is an induction on the dimension of the target manifolds, based upon the following result of Mann (Theorem 1.1).

Let M be a connected C^∞ manifold without boundary, compact or not. For $r = 0, 1, 2, \ldots, \infty$, denote by $\mathrm{Diff}_c^r(M)_0$ the identity component of the group of the compactly supported C^r diffeomorphisms (homeomorphisms for $r = 0$) of M.

Theorem 1.1 (K. Mann)**.** *Any homomorphism from* $\mathrm{Diff}_c^r(M)_0$ *to* $\mathrm{Diff}^2(S^1)$ *or to* $\mathrm{Diff}^2(\mathbb{R})$ *is trivial, provided* $\dim(M) \geq 2$ *and* $r \neq \dim(M)+1$.

Received April 22, 2014.
Revised September 13, 2014.
2010 *Mathematics Subject Classification.* Primary 57S05, secondary 22F05.
Key words and phrases. group of diffeomorphisms, action on the real line.
The author is partially supported by Grant-in-Aid for Scientific Research (C) No. 25400096.

For a simpler proof of this fact, see also [Ma2]. A natural question is whether it is possible to lower the differentiability of the target group. In fact for $r = 0$, E. Militon [Mi] obtained the final result.

Theorem 1.2 (E. Militon)**.** *Any homomorphism from* $\mathrm{Diff}^0_c(M)_0$ *to* $\mathrm{Diff}^0(S^1)$ *is trivial if* $\dim(M) \geq 2$.

Notice that $\mathrm{Diff}^0(\mathbb{R})$ can be considered to be a subgroup of $\mathrm{Diff}^0(S^1)$. So we do not mention in the above theorem the case where the target group is $\mathrm{Diff}^0(\mathbb{R})$.

Even for $r \geq 1$, we have:

Conjecture 1.3. *Any homomorphism from* $\mathrm{Diff}^r_c(M)_0$ *to* $\mathrm{Diff}^0(S^1)$ *is trivial if* $\dim(M) \geq 2$.

The purpose of this paper is to mark one step forward towards this conjecture.

Theorem 1.4. *If* $\dim(M) \geq 2$ *and* $r \neq \dim(M) + 1$, *any homomorphism from* $\mathrm{Diff}^r_c(M)_0$ *to* $\mathrm{Diff}^1(S^1)$ *or* $\mathrm{Diff}^1(\mathbb{R})$ *is trivial.*

Frequent use of the simplicity of the group $\mathrm{Diff}^r_c(M)_0$ is made in the proof. The condition $r \neq \dim(M)+1$ is needed for it. As for Theorem 1.1, the proof is built upon a theorem of Kopell and Szekeres about C^2 actions of abelian groups on a compact interval, while for Theorem 1.4, upon a theorem of Bonatti, Monteverde, Navas and Rivas about C^1 actions of solvable Baumslag–Solitar groups on a compact interval.

By virtue of the fragmentation lemma, Theorem 1.4 reduces to:

Theorem 1.5. *For* $n \geq 2$ *and* $r \neq n + 1$, *any homomorphism from* $\mathrm{Diff}^r_c(\mathbb{R}^n)_0$ *to* $\mathrm{Diff}^1(S^1)$ *or* $\mathrm{Diff}^1(\mathbb{R})$ *is trivial.*

In Section 2, we show that the case of target group $\mathrm{Diff}^1(S^1)$ can be reduced to the case $\mathrm{Diff}^1(\mathbb{R})$. In Sections 3 and 4, we establish fixed point results for certain subgroups of $\mathrm{Diff}^\infty_c(\mathbb{R}^n)_0$. In Section 5, we prove Theorem 1.5 following an argument of E. Militon [Mi]. Finally we give some sporadic results for $\mathrm{Diff}^0(S^1)$ target in Section 6.

Acknowledgement. The author is greatful to Kathryn Mann and Andres Navas for helpful comments and conversations.

§2. Reduction to the case $\mathrm{Diff}^1(\mathbb{R})$

In this section, we show that Theorem 1.5 for the target group $\mathrm{Diff}^1(S^1)$ is reduced to the case of $\mathrm{Diff}^1(\mathbb{R})$.

Proposition 2.1. *Let* $r \neq n+1$ *and* $n \geq 1$. *Assume that* $\Phi\colon \mathrm{Diff}^r_c(\mathbb{R}^n)_0 \to \mathrm{Diff}^0(S^1)$ *is a nontrivial homomorphism. then the global fixed point set is nonempty*: $\mathrm{Fix}(\Phi(\mathrm{Diff}^r_c(\mathbb{R}^n)_0)) \neq \emptyset$.

This proposition enables us to conclude that the image of Φ is contained in the group of the homeomorphisms of $\mathbb{R}$. In particular, Theorem 1.5 for the target group $\mathrm{Diff}^1(S^1)$ is reduced to the case of $\mathrm{Diff}^1(\mathbb{R})$.

Denote $\mathcal{G} = \mathrm{Diff}^r_c(\mathbb{R}^n)_0$. By the simplicity of the group $\mathcal{G}$, the homomorphism Φ in the proposition is injective and its image is contained in $\mathrm{Diff}^0_+(S^1)$, the group of the orientation preserving homeomorphisms.

Let B_0 be the closed unit ball in $\mathbb{R}^n$ centered at the origin. Define a family $\mathcal{B}$ of the closed balls in $\mathbb{R}^n$ by

$$\mathcal{B} = \{g(B_0) \mid g \in \mathcal{G}\}.$$

Also for $B \in \mathcal{B}$, let

$$\mathcal{G}(B) = \{g \in \mathcal{G} \mid \mathrm{Supp}(g) \subset \mathrm{Int}(B)\}.$$

To show Proposition 2.1, it is sufficent to show the following.

Proposition 2.2. *For any* $B \in \mathcal{B}$, *the fixed point set* $\mathrm{Fix}(\Phi(\mathcal{G}(B)))$ *is nonempty.*

In fact, choose an increasing sequence of balls, $\{B_k\}_{k\in\mathbb{N}} \subset \mathcal{B}$ such that $\bigcup_k B_k = \mathbb{R}^n$. Then we have $\mathcal{G} = \cup_k \mathcal{G}(B_k)$ and $\mathrm{Fix}(\Phi(\mathcal{G})) = \bigcap_k \mathrm{Fix}(\Phi(\mathcal{G}(B_k)))$. Therefore by the compactness of S^1, Propositon 2.1 follows from Proposition 2.2.

Now for any $B_1, B_2 \in \mathcal{B}$, the groups $\mathcal{G}(B_1)$ and $\mathcal{G}(B_2)$ are conjugate in $\mathcal{G}$. Therefore their images $\Phi(\mathcal{G}(B_1))$ and $\Phi(\mathcal{G}(B_2))$ are conjugate in $\mathrm{Diff}^0_+(S^1)$. They are simple. Moreover if B_1 and B_2 are disjoint, any element of $\Phi(\mathcal{G}(B_1))$ commutes with any element of $\Phi(\mathcal{G}(B_2))$. Therefore Proposition 2.2 reduces to the following.

Proposition 2.3. *Let* G_1 *and* G_2 *be simple nonabelian subgroups of* $\mathrm{Diff}^0_+(S^1)$. *Assume that* G_2 *is conjugate to* G_1 *in* $\mathrm{Diff}^0_+(S^1)$ *and that any element of* G_1 *commutes with any element of* G_2. *Then there is a global fixed point of* G_1: $\mathrm{Fix}(G_1) \neq \emptyset$.

Proof. Let $X_2 \subset S^1$ be a minimal set of G_2. The set X_2 is either a finite set, a Cantor set or the whole of S^1. If X_2 is a singleton, then G_2 admits a fixed point. Since G_1 is conjugate to G_2, we have $\mathrm{Fix}(G_1) \neq \emptyset$, as is required. So assume for contradiction that X_2 is not a singleton.

First if X_2 is a finite set which is not a singleton, we get a nontrivial homomorphism from G_2 to a finite abelian group, contrary to the assumption of the simplicity. In the remaining case, it is well known, easy to show, that the minimal set is unique. That is, X_2 is contained in any nonempty G_2 invariant closed subset.

Let F_1 be the subset of G_1 formed by the elements g such that $\mathrm{Fix}(g) \neq \emptyset$. Let us show that there is a nontrivial element in F_1. Assume the contrary. Then G_1 acts freely on S^1. Consider the group $\tilde{G}_1$ formed by any lift of any element of G_1 to the universal covering space $\mathbb{R} \to S^1$. Now $\tilde{G}_1$ acts freely on $\mathbb{R}$. A theorem of Hölder asserts that $\tilde{G}_1$ is abelian. See [N] for a short proof, or [Th] for an even shorter proof. The canonical projection $\pi\colon \tilde{G}_1 \to G_1$ is a group homomorphism, and $G_1 = \pi(\tilde{G}_1)$ would be abelian, contrary to the assumption of the proposition.

Since G_1 and G_2 commute, the fixed point set $\mathrm{Fix}(g)$ of any element $g \in F_1$ is G_2 invariant. Therefore we have

$$X_2 \subset \mathrm{Fix}(g) \text{ for any } g \in F_1. \tag{1}$$

This shows that F_1 is in fact a subgroup. By the very definition, F_1 is normal. Since G_1 is simple and F_1 is nontrivial, $F_1 = G_1$. Finally again by (1), $\mathrm{Fix}(G_1) \neq \emptyset$, as is required. Q.E.D.

§3. Fixed point set of $\Phi(G)$

Again consider $\mathcal{G} = \mathrm{Diff}^r_c(\mathbb{R}^n)_0$, where $n \geq 1$ and $r \neq n+1$. We shall show Theorem 1.5 for the target group $\mathrm{Diff}^1(\mathbb{R})$ by a contradiction. The condition $n \geq 2$ will be used only in Section 5. Let us assume that $\Phi\colon \mathcal{G} \to \mathrm{Diff}^1(\mathbb{R})$ is a nontrivial homomorphism. By the simplicity of $\mathcal{G}$, Φ is injective and its image is contained in $\mathrm{Diff}^1_+(\mathbb{R})$. For the purpose of showing Theorem 1.5, it is no loss of generality to assume the following.

Assumption 3.1. *There is no global fixed point of* $\Phi(\mathcal{G})$: $\mathrm{Fix}(\Phi(\mathcal{G})) = \emptyset$.

In fact, we only have to pass from $\mathbb{R}$ to a connected component of $\mathbb{R} \setminus \mathrm{Fix}(\Phi(\mathcal{G}))$. This assumption will be made all the way until the end of the proof of Theorem 1.5.

We consider an embedding of Baumslag–Solitar group $\mathrm{BS}(1,2)$ into the group $\mathcal{G}(B)$. See Section 2 for the definition of $\mathcal{G}(B)$. Recall that

$$\mathrm{BS}(1,2) = \langle a, b \mid aba^{-1} = b^2 \rangle.$$

This group is a subgroup of GA, the group of the orientation preserving affine transformations of $\mathbb{R}$, where a corresponds to $x \mapsto 2x$, and b to $x \mapsto x+1$. The group GA is a subgroup of $PSL(2,\mathbb{R})$. The group $PSL(2,\mathbb{R})$ acts on the circle at infinity S^1_∞ of the Poincaré upper half plane, where GA is the isotropy subgroup of $\infty \in S^1_\infty$. Cutting S^1_∞ at

∞, we get a C^∞ action of $\mathrm{BS}(1,2)$ on a compact interval, say $[-1,1]$. This is called the *affine* action of $\mathrm{BS}(1,2)$.

T. Tsuboi [Ts] showed that there is a homeomorphism h of $[-1,1]$ which is a C^∞ diffeomorphism on $(-1,1)$ such that the conjugate by h of any element of $\mathrm{BS}(1,2)$ is C^∞ tangent to the identity at the end points. Then the conjugated action extends to an C^∞ action on $[-2,2]$ in such a way that it is trivial on $[-2,-1] \cup [1,2]$. Consider a subset $S^{n-1} \times [-2,2]$ embedded in B. The group GA acts on $S^{n-1} \times [-2,2]$, trivially on the first factor. This way we obtain a subgroup of $\mathcal{G}(B)$ isomorphic to $\mathrm{BS}(1,2)$, which we shall denote by G.

The key fact for the proof of Theorem 1.5 is the following result of [BMNR], which improves the semiconjugacy result in [GL].

Theorem 3.1 (C. Bonatti, I. Monteverde, A. Navas and C. Rivas)**.** *Assume* $\mathrm{BS}(1,2)$ *acts faithfully on a compact interval by* C^1 *diffeomorphisms in such a way that there is no interior global fixed poit. Then the action is topologically conjugate to the affine action. In particular, all the interior orbits are dense.*

The topological conjugacy in the theorem may be orientation reversing. The compactness assumption on the interval is indispensible. In fact, there is a C^∞ exotic action of $\mathrm{BS}(1,2)$ on $\mathbb{R}$. See [CC]. All the actions of $\mathrm{BS}(1,2)$ by homeomorphisms of $\mathbb{R}$ are classified in [DNR].

In order to apply the above theorem, we need the following fixed point result in the first place.

Proposition 3.2. *The fixed point set* $\mathrm{Fix}(\Phi(G))$ *is nonempty.*

Proof. We assume for contradiction that $\mathrm{Fix}(\Phi(G)) = \emptyset$. The proof follows the same line as Proposition 2.2. But since our target manifold is $\mathbb{R}$ and is noncompact, extra care will be needed.

Since $\mathrm{Fix}(\Phi(G)) = \emptyset$, any orbit of $\Phi(G)$ is unbounded towards both directions. Since G is finitely generated, $\Phi(G)$ has a compact cross section I in $\mathbb{R}$, that is, a compact interval I which intersects any $\Phi(G)$ orbit. In fact, choose any point $x_0 \in \mathbb{R}$ and let x_1 be the supremum of $g(x_0)$, where g runs over a finite symmetric generating set. Then clearly any orbit intersects the interval $I = [x_0, x_1]$. Since $G \subset \mathcal{G}(B)$, I is also a cross section for $\Phi(\mathcal{G}(B))$. That is, any $\Phi(\mathcal{G}(B))$ orbit intersects the compact interval I.

Now we follow the proof of Proposition 6.1 in [DKNP], to show that there is a unique minimal set X for $\Phi(\mathcal{G}(B))$. In fact we shall show a bit more: there is a nonempty $\Phi(\mathcal{G}(B))$ invariant closed subset X in $\mathbb{R}$ which has the property that any nonempty $\Phi(\mathcal{G}(B)))$ invariant closed subset contains X.

The proof goes as follows. Let F be the family of nonempty $\Phi(\mathcal{G}(B))$ invariant closed subsets of $\mathbb{R}$, and F_I the family of nonempty closed subsets Y in I such that $\Phi(\mathcal{G}(B))(Y) \cap I = Y$, where we denote

$$\Phi(\mathcal{G}(B))(Y) = \bigcup_{g \in \mathcal{G}(B)} \Phi(g)(Y).$$

Define a map $\phi\colon F \to F_I$ by $\phi(X) = X \cap I$, and $\psi\colon F_I \to F$ by $\psi(Y) = \Phi(\mathcal{G}(B))(Y)$. They satisfiy $\psi \circ \phi = \phi \circ \psi = \mathrm{id}$.

Let $\{Y_\alpha\}$ be a totally ordered set in F_I. Then the intersection $\bigcap_\alpha Y_\alpha$ is nonempty. Let us show that it belongs to F_I, namely,

$$\Phi(\mathcal{G}(B))\left(\bigcap_\alpha Y_\alpha\right) \cap I = \bigcap_\alpha Y_\alpha. \tag{2}$$

For the inclusion $\subset$, we have

$$\begin{aligned}\Phi(\mathcal{G}(B))\left(\bigcap_\alpha Y_\alpha\right) \cap I &\subset \left(\bigcap_\alpha \Phi(\mathcal{G}(B))(Y_\alpha)\right) \cap I \\ &= \bigcap_\alpha (\Phi(\mathcal{G}(B))(Y_\alpha) \cap I) \\ &= \bigcap_\alpha Y_\alpha.\end{aligned}$$

For the other inclusion, notice that

$$\bigcap_\alpha Y_\alpha \subset \Phi(\mathcal{G}(B))\left(\bigcap_\alpha Y_\alpha\right) \quad \text{and} \quad \bigcap_\alpha Y_\alpha \subset I.$$

Therefore by Zorn's lemma, there is a minimal element Y in F_I. The set Y is not finite. In fact, if it is finite, the set $X = \psi(Y)$ in F is discrete, and there would be a nontrivial homomorphism from $\Phi(\mathcal{G}(B))$ to $\mathbb{Z}$, contrary to the fact that $\mathcal{G}(B)$, and hence $\Phi(\mathcal{G}(B))$, is simple.

Now the correspondence ϕ and ψ preserve the inclusion. This shows that there is no nonempty $\Phi(\mathcal{G}(B))$ invariant closed proper subset of $X = \psi(Y)$. In other words, any $\Phi(\mathcal{G}(B))$ orbit contained in X is dense in X. Therefore X is either $\mathbb{R}$ itself or a locally Cantor set. In the former case, any nonempty $\Phi(\mathcal{G}(B))$ invariant closed subset must be $\mathbb{R}$ itself.

Let us show that in the latter case, X satisfies the desired property: X is contained in any nonempty $\Phi(\mathcal{G}(B))$ invariant closed subset. For this, we only need to show that the $\Phi(\mathcal{G}(B))$ orbit of any point x in $\mathbb{R} \setminus X$ accumulates to a point in X. In fact, if this is true, then any nonempty $\Phi(\mathcal{G}(B))$ invariant closed subset must intersects X. But the intersection must be the whole X by the above remark.

Let (a, b) be the connected component of $\mathbb{R} \setminus X$ that contains x. (If $x \in X$, there is nothing to prove.) There is a sequence $g_k \in \mathcal{G}(B)$ ($k \in \mathbb{N}$) such that $\Phi(g_k)(a)$ accumulates to a and that $\Phi(g_k)(a)$'s are mutually distinct. Then the intervals $\Phi(g_k)((a, b))$ are mutually disjoint, and consequently $\Phi(g_k)(x)$ converges to a. This concludes the proof that X is contained in any nonempty $\Phi(\mathcal{G}(B)))$ invariant closed subset.

Choose $B' \in \mathcal{B}$ such that $B' \cap B = \emptyset$. Any element of $\mathcal{G}(B')$ commutes with any element of $\mathcal{G}(B)$. Define $\mathcal{F}(B')$ to be the subset of the group $\mathcal{G}(B')$ consisting of those elements g such that $\mathrm{Fix}(\Phi(g)) \neq \emptyset$. By a theorem of Hölder, there is a nontrivial element in $\mathcal{F}(B')$. For any $g \in \mathcal{F}(B')$, the set $\mathrm{Fix}(\Phi(g))$ is closed, nonempty and invariant by $\Phi(\mathcal{G}(B))$ by the commutativity. Therefore we have

$$X \subset \mathrm{Fix}(\Phi(g)) \quad \text{for any} \quad g \in \mathcal{F}(B'). \tag{3}$$

This shows that $\mathcal{F}(B')$ is a subgroup of $\mathcal{G}(B')$, normal and nontrivial. Since $\mathcal{G}(B')$ is simple, we have $\mathcal{F}(B') = \mathcal{G}(B')$. Finally again by (3), we get $\mathrm{Fix}(\Phi(\mathcal{G}(B'))) \neq \emptyset$. Since $\mathcal{G}(B)$ is conjugate to $\mathcal{G}(B')$ and G is a subgroup of $\mathcal{G}(B)$, we have $\mathrm{Fix}(\Phi(G)) \neq \emptyset$, contrary to the assumption. The contradiction concludes the proof of Proposition 3.2. Q.E.D.

§4. Fixed point set of $\Phi(\mathcal{G}_B)$

For $B \in \mathcal{B}$, define a subgroup $\mathcal{G}_B$ of $\mathcal{G}$ by

$$\mathcal{G}_B = \{g \in \mathcal{G} \mid g = \mathrm{id} \text{ in a neighbourhood of } B\}.$$

Let $\Phi \colon \mathcal{G} \to \mathrm{Diff}^1_+(\mathbb{R})$ be a homomorphism satisfying Assumption 3.1. The purpose of this section is to show the following.

Proposition 4.1. *For any $B \in \mathcal{B}$, the fixed point set* $\mathrm{Fix}(\Phi(\mathcal{G}_B))$ *is nonempty.*

Proof. Any element of $\Phi(\mathcal{G}(B))$ commutes with any element of $\Phi(\mathcal{G}_B)$. Let us denote $F = \mathrm{Fix}(\Phi(G))$, which we have shown to be nonempty in Proposition 3.2. Clearly F is invariant by any element of $\Phi(\mathcal{G}_B)$. We shall show that there is a fixed point of $\Phi(\mathcal{G}_B)$ in F. If F is bounded to the left or to the right, then the extremal point will be a fixed point of $\Phi(\mathcal{G}_B)$. So we assume that F is unbounded towards both directions. That is, any connected component U of $\mathbb{R} \setminus F$ is bounded.

Assume that there is $g \in \mathcal{G}_B$ such that $\Phi(g)(U) \cap U = \emptyset$. (Otherwise $\Phi(g)(U) = U$ for any $g \in \mathcal{G}_B$, and the proof will be complete.) There is a subgroup G' of $\mathcal{G}_B$ conjugate to G. By some abuse, denote the generators of G' by a and b. They satisfy $aba^{-1} = b^2$. Notice that finite products of conjugates of $b^{\pm 1}$ by elements of $\mathcal{G}_B$ form a normal subgroup of $\mathcal{G}_B$. Since $\mathcal{G}_B$ is simple, any element of $\mathcal{G}_B$ can be written as such a product. Writing the above element g this way, one finds a conjugate of b whose Φ-image displaces U. We may assume that $\Phi(b)U \cap U = \emptyset$, passing from G' to its conjugate by an element of $\mathcal{G}_B$ if necessary. (The conjugate is still denoted by G'.)

Let V be the component of $\mathbb{R} \setminus \mathrm{Fix}(\Phi(G'))$ that contains U. Since G' is conjugate to G, V is a bounded open interval and $F \cap V$ is a closed nonempty proper subset of V invariant by $\Phi(G')$. It is easy to show that $\Phi(b)|_V \neq \mathrm{id}$ implies that the action $\Phi(G')|_V$ is faithful. By Theorem 3.1, any $\Phi(G')$ orbit in V must be dense in V. This contradicts the fact that $F \cap V$ is invariant by $\Phi(G')$. The proof is now complete. Q.E.D.

§5. Proof of Theorem 1.5

Again we assume that $\Phi\colon \mathcal{G} \to \mathrm{Diff}^1_+(S^1)$ is a homomorphism satisfying Assumption 3.1. Our purpose here is to get a contradiction. We follow an argument in [Mi].

Lemma 5.1. *Assume B and B' are mutually disjoint balls of $\mathcal{B}$. Then any $g \in \mathcal{G}$ can be written as $g = g_1 \circ g_2 \circ g_3$, where g_1 and g_3 belongs to $\mathcal{G}_B$ and g_2 to $\mathcal{G}_{B'}$.*

Proof. Take any $g \in \mathcal{G}$. Then there is an element $g_1 \in \mathcal{G}_B$ such that $g_1^{-1} \circ g(B)$ is disjoint from B'. Next, there is an element $g_2 \in \mathcal{G}_{B'}$ such that $g_2^{-1} \circ g_1^{-1} \circ g$ is the identity in a neighbourhood of B. Thus $g_3 = g_2^{-1} \circ g_1^{-1} \circ g$ belongs to $\mathcal{G}_B$ and the proof is complete. Q.E.D.

Lemma 5.2. *Assume B and B' are mutually disjoint elements of $\mathcal{B}$. If two points a and b $(a < b)$ belong to $\mathrm{Fix}(\Phi(\mathcal{G}_B))$, then $\mathrm{Fix}(\Phi(\mathcal{G}_{B'})) \cap [a, b] = \emptyset$.*

Proof. Assume a point c in $[a, b]$ belongs to $\mathrm{Fix}(\Phi(\mathcal{G}_{B'}))$. Choose an arbitrary element $g \in \mathcal{G}$. There is a decomposition $g = g_1 \circ g_2 \circ g_3$ as in Lemma 5.1. Now $\Phi(g_3)(a) = a$. Since $\Phi(g_2)(c) = c$ and $a \leq c$, we have $\Phi(g_2) \circ \Phi(g_3)(a) \leq c$. Likewise $\Phi(g)(a) = \Phi(g_1) \circ \Phi(g_2) \circ \Phi(g_3)(a) \leq b$. Since $g \in \mathcal{G}$ is arbitrary, the $\Phi(\mathcal{G})$ orbit of a is bounded from the right. Then the supremum of the orbit must be a global fixed poit, which is against Assumption 3.1: $\Phi(\mathcal{G})$ has no global fixed point. Q.E.D.

For any point $x \in \mathbb{R}^n$, define a subgroup $\mathcal{G}_x$ of $\mathcal{G}$ by

$$\mathcal{G}_x = \{g \in \mathcal{G} \mid g \text{ is the identity in a neighbourhood of } x\}.$$

Lemma 5.3. *For any $x \in \mathbb{R}^n$, the fixed point set* $\mathrm{Fix}(\Phi(\mathcal{G}_x))$ *is nonempty.*

Proof. Notice that for any $x \in \mathbb{R}^n$, there is an decreasing sequence $\{B_k\}$ $(k \in \mathbb{N})$ in $\mathcal{B}$ such that $\{x\} = \bigcap_k B_k$. Then $\mathcal{G}_{B_k}$ is an increasing senquence of subgroups of $\mathcal{G}$ such that $\bigcup_k \mathcal{G}_{B_k} = \mathcal{G}_x$. Therefore the closed subsets $\mathrm{Fix}(\Phi(\mathcal{G}_{B_k}))$ is decreasing and we have

$$\mathrm{Fix}(\Phi(\mathcal{G}_x)) = \bigcap_k \mathrm{Fix}(\Phi(\mathcal{G}_{B_k})).$$

Now it suffices to prove that $\mathrm{Fix}(\Phi(\mathcal{G}_B))$ is compact for $B \in \mathcal{B}$. Recall that we have already shown that $\mathrm{Fix}(\Phi(\mathcal{G}_B))$ is nonempty. Assume in way of contradiction that $\sup \mathrm{Fix}(\Phi(\mathcal{G}_B)) = \infty$. (The other case can be dealt with similarly.) Choose $B' \in \mathcal{B}$ such that $B \cap B' = \emptyset$. Notice that $\Phi(\mathcal{G})$ consists of orientation preserving diffeomorphisms and $\Phi(\mathcal{G}_{B'})$ is conjugate to $\Phi(\mathcal{G}_B)$ by such a diffeomorphism. Therefore we also have that $\sup \mathrm{Fix}(\Phi(\mathcal{G}_{B'})) = \infty$. Now one can find points $a, b \in \mathrm{Fix}(\Phi(\mathcal{G}_B))$ and a point $c \in \mathrm{Fix}(\Phi(\mathcal{G}_{B'})$ such that $a < c < b$. This is contrary to Lemma 5.2. Q.E.D.

We use the assumption $n \geq 2$ only in the sequel. Let D_0 be the unit compact disc centered at 0 in $\mathbb{R}^{n-1} \subset \mathbb{R}^n$. Define a family $\mathcal{D}$ of closed subsets of $\mathbb{R}^n$ by

$$\mathcal{D} = \{g(D_0) \mid g \in \mathcal{G}\}.$$

For any $D \in \mathcal{D}$, define a subgroup $\mathcal{G}_D$ of $\mathcal{G}$ by

$$\mathcal{G}_D = \{g \in \mathcal{G} \mid g \text{ is the identity in a neighbourhood of } D\}.$$

Lemma 5.3 implies that $\mathrm{Fix}(\Phi(D)) \neq \emptyset$ for any $D \in \mathcal{D}$.

Lemma 5.4. *For any $D \in \mathcal{D}$, the set* $\mathrm{Fix}(\Phi(D))$ *is a singleton.*

Proof. First of all notice that for any $D, D' \in \mathcal{D}$ such that $D \cap D' = \emptyset$, we have $\mathrm{Fix}(\Phi(\mathcal{G}_D)) \cap \mathrm{Fix}(\Phi(\mathcal{G}_{D'})) = \emptyset$. In fact, as is easily shown, $\mathcal{G}_D$ and $\mathcal{G}_{D'}$ generate $\mathcal{G}$. Thus a point of the above intersection would be a global fixed point of $\mathcal{G}$, against Assumption 3.1. This shows that the interior $\mathrm{Int}(\mathrm{Fix}(\Phi(\mathcal{G}_D)))$ is empty. In fact, there are uncountably many mutually disjoint elements of $\mathcal{D}$, while mutually disjoint open subsets of $\mathbb{R}$ are at most countable.

Assume that $\mathrm{Fix}(\Phi(\mathcal{G}_D))$ contains more than one point. Since $\mathrm{Int}(\mathrm{Fix}(\Phi(\mathcal{G}_D)))$ is empty, $\mathrm{Fix}(\Phi(\mathcal{G}_D))$ is not connected. To any $D \in \mathcal{D}$, assign a bounded component I_D of $\mathbb{R} \setminus \mathrm{Fix}(\Phi(\mathcal{G}_D))$ in an arbitrary way. This is possible by the axiom of choice. Notice that Lemmata 5.1 and 5.2 for the family $\mathcal{B}$ are valid for $\mathcal{D}$ as well. (No changes of the proofs are needed.) Consequently $I_D \cap I_{D'} = \emptyset$ if $D \cap D' = \emptyset$. Again this is contrary to the fact that there are uncountably many mutually disjoint elements of $\mathcal{D}$. Q.E.D.

Finally let us prove Theorem 1.5. Choose any element $D \in \mathcal{D}$ and distinct two ponits $x_1, x_2 \in D$ that are contained in D. Then since $\mathrm{Fix}(\Phi(\mathcal{G}_D))$ is a singleton and $\mathrm{Fix}(\Phi(\mathcal{G}_{x_i}))$ is nonempty, we have $\mathrm{Fix}(\Phi(\mathcal{G}_{x_1})) = \mathrm{Fix}(\Phi(\mathcal{G}_{x_2}))$. But $\mathcal{G}_{x_1}$ and $\mathcal{G}_{x_2}$ generate $\mathcal{G}$, and there would be a global fixed point of $\Phi(\mathcal{G})$, against Assumption 3.1. The contradiction shows that the homomorphism Φ must be trivial.

§6. Sporadic results for $\mathrm{Diff}^0(S^1)$ target

Let $M = L \times S^m$ be a closed n-dimensional manifold such that $1 \le m \le n$, where S^m is the m-dimensionla sphere. Then we have the following result.

Theorem 6.1. *If $n \ge 2$ and $r \ne n+1$, there is no nontrivial homomorphism from $\mathrm{Diff}^r_c(M)_0$ to $\mathrm{Diff}^0(S^1)$.*

Proof. Assume that $\Phi\colon \mathrm{Diff}^r_c(M)_0 \to \mathrm{Diff}^0(S^1)$ is a nontrivial homomorphism. The Lie group $PSL(2,\mathbb{R}) < PSO(m+1,1)$ acts on S^m as Moebius transformations. So it acts on $M = L \times S^m$, trivially on L-coordinates. Denote the inclusion by $\iota\colon PSL(2,\mathbb{R}) \to \mathrm{Diff}^r_c(M)_0$. The simplicity of the group $\mathrm{Diff}^r_c(M)_0$ shows that the homomorphism

$$\Phi \circ \iota\colon PSL(2,\mathbb{R}) \to \mathrm{Diff}^0(S^1)$$

is nontrivial.

Now Theorem 5.2 in [Ma2] asserts that the homomorphism $\Phi \circ \iota$ is the conjugation of the standard embedding $\iota_0\colon PSL(2,\mathbb{R}) \to \mathrm{Diff}^0(S^1)$ by a homeomorphism of S^1. It is no loss of generality to assume that $\Phi \circ \iota = \iota_0$, by changing Φ if necessary. If the dimension of L is positive, then $\mathrm{Diff}^r_c(L)_0$ also acts on M, trivially on S^m-coordinates. Any element of the group $\Phi(\mathrm{Diff}^r_c(L))$ must commute with any element of $PSL(2,\mathbb{R})$. But there is no nontrivial element in $\mathrm{Diff}^0_+(S^1)$ which commutes with all the element of $PSL(2,\mathbb{R})$. A contradiction.

Let us consider the case where L is a singleton. Then there is an element g in $\mathrm{Diff}^r(S^n)$, $n \ge 2$, which commutes with all the elements of $\iota(SO(2))$ and is not contained in $\iota(SO(2))$. But $\Phi \circ \iota(SO(2)) = SO(2)$, and any element in $\mathrm{Diff}^0(S^1)$ which commutes with all the elements of $SO(2)$ must be an element of $SO(2)$, contradicting the injectivity of Φ. Q.E.D.

Remark 6.2. There are a wider class of manifolds for which the above argument holds. For example, if M is the unit tangent bundle of a closed hyperbolic surface and if $r \neq 4$, then any homomorphism from $\mathrm{Diff}^r_c(M)_0$ to $\mathrm{Diff}^0(S^1)$ is trivial.

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