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Rotation number and lifts of a Fuchsian action of the modular group on the circle

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Abstract.

We characterize the semi-conjugacy class of a Fuchsian action of the modular group on the circle in terms of rotation numbers of two standard generators and that of their product. We also show that among lifts of a Fuchsian action of the modular group, only 5-fold lift admits a similar characterization. These results indicate similarity and difference between rotation number and linear character.

§1. Introduction

Rotation number of an orientation-preserving homeomorphism of the circle has similar properties to absolute value of the trace of an element in $\mathrm{PSL}(2,\mathbb{R})$. For example, they are invariant under conjugation and furthermore, Jørgensen's criterion of discreteness for subgroups of $\mathrm{PSL}(2,\mathbb{R})$ [11, Theorem 2], which can be described in terms of absolute value of the trace, has an analogue for the group of real analytic diffeomorphisms of the circle (see [13, Theorem 1.2]). In this article, we give another similarity between rotation number and linear character from a viewpoint given by D. Calegari and A. Walker [5].

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1.1. Rotation number

We denote by $\mathrm{Homeo}_+(\mathbb{S}^1)$ the group of orientation-preserving homeomorphisms of the circle. We regard the circle $\mathbb{S}^1$ as the quotient

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$\mathbb{R}/\mathbb{Z}$ and denote by $p\colon \mathbb{R} \to \mathbb{S}^1$ the projection. Let $\widetilde{\mathrm{Homeo}}_+(\mathbb{S}^1)$ be the group of lifts of orientation-preserving homeomorphisms to $\mathbb{R}$, namely, homeomorphisms of $\mathbb{R}$ commuting with integral translations.

For $\tilde{f} \in \widetilde{\mathrm{Homeo}}_+(\mathbb{S}^1)$, we define the translation number $\widetilde{\mathrm{rot}}(\tilde{f}) \in \mathbb{R}$ of $\tilde{f}$ by

$$\widetilde{\mathrm{rot}}(\tilde{f}) = \lim_{n\to\infty} \frac{(\tilde{f})^n(\tilde{x}) - \tilde{x}}{n},$$

where $\tilde{x} \in \mathbb{R}$. Note that the limit exists and does not depend on the choice of a point $\tilde{x} \in \mathbb{R}$. For $f \in \mathrm{Homeo}_+(\mathbb{S}^1)$, we define the rotation number $\mathrm{rot}(f) \in \mathbb{R}/\mathbb{Z}$ of f by

$$\mathrm{rot}(f) = \widetilde{\mathrm{rot}}(\tilde{f}) \mod \mathbb{Z},$$

where $\tilde{f} \in \widetilde{\mathrm{Homeo}}_+(\mathbb{S}^1)$ is a lift of f to $\mathbb{R}$.

Among several properties of rotation number, we recall that $\mathrm{rot}(f) = \frac{p}{q}$, where $\frac{p}{q}$ is a reduced fraction if and only if f has a period point of period q. In particular, $\mathrm{rot}(f) = 0$ if and only if f has a fixed point (see for example [9] in detail and other properties of rotation number).

1.2. Lifts of a group action on the circle

For a group Γ, we denote by $\mathrm{R}(\Gamma)$ the space of homomorphisms from Γ to $\mathrm{Homeo}_+(\mathbb{S}^1)$. We equip $\mathrm{R}(\Gamma)$ with the uniform convergence topology on generators if necessary.

We define a lift of a group action on the circle.

Let $k \geq 2$ be a positive integer and denote by $p_k\colon \mathbb{S}^1 \to \mathbb{S}^1$ the k-fold covering map. For a group Γ, a homomorphism $\phi \in \mathrm{R}(\Gamma)$ is a k-fold lift of a homomorphism $\psi \in \mathrm{R}(\Gamma)$ if $p_k \circ \phi(\gamma) = \psi(\gamma) \circ p_k$ for every $\gamma \in \Gamma$.

We remark that if $\phi \in \mathrm{R}(\Gamma)$ is a k-fold lift of a homomorphism $\psi \in \mathrm{R}(\Gamma)$, then we have $k\,\mathrm{rot}(\phi(\gamma)) = \mathrm{rot}(\psi(\gamma))$ for every $\gamma \in \Gamma$.

1.3. Semi-conjugacy class

Semi-conjugacy between two actions of a group on the circle has been defined in several ways (see [8], [9], [1]). In this paper, we follow the way presented in [3].

For $\phi_1, \phi_2 \in \mathrm{R}(\Gamma)$, we say that ϕ_1 is *semi-conjugate* to ϕ_2 if there exists a continuous degree-one monotone map such that $h \circ \phi_1(\gamma) = \phi_2(\gamma) \circ h$ for every $\gamma \in \Gamma$. Here, a map $h\colon \mathbb{S}^1 \to \mathbb{S}^1$ is called a degree-one monotone map if it admits a lift $\tilde{h}\colon \mathbb{R} \to \mathbb{R}$ commuting with integral translations, and nondecreasing on $\mathbb{R}$.

Note that semi-conjugacy is not symmetric and is not an equivalence relation. We consider the equivalence relation generated by semi-conjugacy, which is called monotone equivalence in [3]. We call the monotone equivalence class of $\phi \in \mathrm{R}(\Gamma)$ the *semi-conjugacy class* of ϕ. Note that if two minimal homomorphisms belong to the same semi-conjugacy class, then they are topologically conjugate. We define the semi-conjugacy class of an orientation-preserving homeomorphism of the circle in a similar way.

A classical result due to H. Poincaré says that two homeomorphisms are in the same semi-conjugacy class if and only if their rotation numbers coincide, which is similar to the fact that two matrices in $\mathrm{SL}(2,\mathbb{R})\setminus\{\pm E\}$ are conjugate if and only if their traces coincide.

As for group actions, however, $\phi_1, \phi_2 \in \mathrm{R}(\Gamma)$ do not belong to the same semi-conjugacy class if we only suppose that $\mathrm{rot}(\phi_1(\gamma)) = \mathrm{rot}(\phi_2(\gamma))$ for every γ. It can be seen by considering Fuchsian actions corresponding to hyperbolic structures on 2-orbifolds (see for example [6] about 2-orbifolds and hyperbolic structures on them).

1.4. Fuchsian actions

Let $\mathcal{O}$ be a compact, connected, oriented 2-orbifold with negative orbifold Euler characteristic $\chi^{orb}(\mathcal{O}) < 0$. For each hyperbolic structure on the interior of $\mathcal{O}$ compatible with the orientation of $\mathcal{O}$, we have a homomorphism from the orbifold fundamental group $\pi_1^{orb}(\mathcal{O})$ to $\mathrm{PSL}(2,\mathbb{R})$ by identifying the universal cover $\tilde{\mathcal{O}}$ with the hyperbolic plane $\mathbb{H}^2$. By considering the action on the ideal boundary $\partial\mathbb{H}^2 \simeq \mathbb{S}^1$, we obtain a homomorphism $\phi_{\mathcal{O}} \in \mathrm{R}(\pi_1^{orb}(\mathcal{O}))$. We call such a homomorphism a *Fuchsian action* associated to $\mathcal{O}$. Note that the semi-conjugacy class of a Fuchsian action associated to a fixed 2-orbifold $\mathcal{O}$ is independent of the choice of a hyperbolic structure and that a Fuchsian action corresponding to a hyperbolic structure with finite area is minimal.

In general, we cannot characterize the semi-conjugacy class of a Fuchsian action only by rotation numbers of all elements. In fact, for a Fuchsian action ϕ_S associated to a compact, connected, oriented surface S with negative Euler characteristic, the homeomorphism $\phi_S(\gamma)$ has a fixed point for every $\gamma \in \Gamma$ but there is no global fixed point. This means that $\mathrm{rot}(\phi_S(\gamma)) = 0$ for every $\gamma \in \Gamma$ but the Fuchsian action ϕ_S does not belong to the semi-conjugacy class of the trivial action.

Now we show, however, that we can characterize the semi-conjugacy classes of a Fuchsian action of a specific 2-orbifold and its certain lift by only rotation numbers of finite elements.

1.5. Main result

We focus on a special 2-orbifold. Let $\mathcal{O}_{2,3}$ be the 2-orbifold which is obtained from a 2-disk by making two cone-points of orders 2, 3. Note that the interior of $\mathcal{O}_{2,3}$ is homeomorphic to $\mathbb{H}^2/\mathrm{PSL}(2;\mathbb{Z})$ and

$\pi_1^{orb}(\mathcal{O}_{2,3})$ is isomorphic to the modular group $\mathrm{PSL}(2,\mathbb{Z})$. We fix a presentation

$$\pi_1^{orb}(\mathcal{O}_{2,3}) = \langle \alpha, \beta \mid \alpha^2 = \beta^3 = 1 \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_3,$$

where $\alpha = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and $\beta = \left[\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. Let $\phi_{\mathcal{O}_{2,3}}$ be a Fuchsian action of $\mathcal{O}_{2,3}$ which is equal to the action by linear fractional transformations on $\mathbb{R} \cup \{\infty\} \simeq \mathbb{S}^1$. It follows that

$$\begin{aligned}
&\phi_{\mathcal{O}_{2,3}}(\alpha)(0) = \infty, \quad \phi_{\mathcal{O}_{2,3}}(\alpha)(\infty) = 0, \\
&\phi_{\mathcal{O}_{2,3}}(\beta)(0) = \infty, \quad \phi_{\mathcal{O}_{2,3}}(\beta)(\infty) = -1, \quad \phi_{\mathcal{O}_{2,3}}(\beta)(-1) = 0 \quad \text{and} \\
&\phi_{\mathcal{O}_{2,3}}(\alpha\beta)(0) = 0.
\end{aligned}$$

Hence we have

$$(\mathrm{rot}(\phi_{\mathcal{O}_{2,3}}(\alpha)), \mathrm{rot}(\phi_{\mathcal{O}_{2,3}}(\beta)), \mathrm{rot}(\phi_{\mathcal{O}_{2,3}}(\alpha\beta))) = \left(\frac{1}{2}, \frac{1}{3}, 0\right).$$

It follows from the presentation of $\pi_1^{orb}(\mathcal{O}_{2,3})$ that there exists a k-fold lift $\phi^{(k)}_{\mathcal{O}_{2,3}}$ of $\phi_{\mathcal{O}_{2,3}}$ if and only if $k \equiv \pm 1 \mod 6$ and that such a lift is unique if it exists. We also have

$$\begin{aligned}
&(\mathrm{rot}(\phi^{(k)}_{\mathcal{O}_{2,3}}(\alpha)), \mathrm{rot}(\phi^{(k)}_{\mathcal{O}_{2,3}}(\beta)), \mathrm{rot}(\phi^{(k)}_{\mathcal{O}_{2,3}}(\alpha\beta))) \\
&= \begin{cases} \left(\dfrac{1}{2}, \dfrac{1}{3}, \dfrac{k-1}{k}\right) & (k \equiv 1 \bmod 6), \\ \left(\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{1}{k}\right) & (k \equiv -1 \bmod 6). \end{cases}
\end{aligned}$$

Now we are ready to state the main result.

Theorem 1.1. *Let* $\phi \in \mathrm{R}(\pi_1^{orb}(\mathcal{O}_{2,3}))$.

(1) *If* $(\mathrm{rot}(\phi(\alpha)), \mathrm{rot}(\phi(\beta)), \mathrm{rot}(\phi(\alpha\beta))) = \left(\frac{1}{2}, \frac{1}{3}, 0\right)$, *then* ϕ *belongs to the semi-conjugacy class of a Fuchsian action* $\phi_{\mathcal{O}_{2,3}}$.

(2) *If* $(\mathrm{rot}(\phi(\alpha)), \mathrm{rot}(\phi(\beta)), \mathrm{rot}(\phi(\alpha\beta))) = \left(\frac{1}{2}, \frac{2}{3}, \frac{1}{5}\right)$, *then* ϕ *belongs to the semi-conjugacy class of the* 5*-fold lift* $\phi^{(5)}_{\mathcal{O}_{2,3}}$ *of a Fuchsian action* $\phi_{\mathcal{O}_{2,3}}$.

Remark 1.2. (1) Theorem 1.1 cannot be generalized to the other lifts of $\phi_{\mathcal{O}_{2,3}}$. Indeed for each positive integer $k \geq 2$ we denote by $\mathcal{O}_{2,3,k}$ a compact, connected, oriented 2-orbifold which is obtained from a 2-sphere by making three cone-points of orders 2, 3, k. Now suppose that $k \equiv \pm 1 \mod 6$ and $k \neq 5$. Then we have $\chi^{orb}(\mathcal{O}_{2,3,k}) < 0$. Let

$\phi_{\mathcal{O}_{2,3,k}} \in \mathrm{R}(\pi_1^{orb}(\mathcal{O}_{2,3,k}))$ be a Fuchsian action of $\mathcal{O}_{2,3,k}$. For a suitable presentation

$$\pi_1^{orb}(\mathcal{O}_{2,3,k}) = \langle \alpha, \beta, \gamma \mid \alpha^2 = \beta^3 = \gamma^k = \alpha\beta\gamma = 1 \rangle,$$

we have

$$\begin{aligned}&(\mathrm{rot}(\phi_{\mathcal{O}_{2,3,k}}(\alpha)), \mathrm{rot}(\phi_{\mathcal{O}_{2,3,k}}(\beta)), \mathrm{rot}(\phi_{\mathcal{O}_{2,3,k}}(\gamma)))\\ &= \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{k}\right)\end{aligned}$$

and hence

$$\begin{aligned}&(\mathrm{rot}(\phi_{\mathcal{O}_{2,3,k}}(\alpha)), \mathrm{rot}(\phi_{\mathcal{O}_{2,3,k}}(\beta)), \mathrm{rot}(\phi_{\mathcal{O}_{2,3,k}}(\alpha\beta)))\\ &= \left(\frac{1}{2}, \frac{1}{3}, \frac{k-1}{k}\right).\end{aligned}$$

Let q be the homomorphism from $\pi_1^{orb}(\mathcal{O}_{2,3})$ onto $\pi_1^{orb}(\mathcal{O}_{2,3,k})$ such that $q(\alpha) = \alpha$ and $q(\beta) = \beta$ and let ι be the automorphism of $\pi_1^{orb}(\mathcal{O}_{2,3})$ such that $\iota(\alpha) = \alpha$ and $\iota(\beta) = \beta^{-1}$. We define a homomorphism $\hat{\phi}_{\mathcal{O}_{2,3,k}} \in \mathrm{R}(\pi_1^{orb}(\mathcal{O}_{2,3}))$ by

$$\hat{\phi}_{\mathcal{O}_{2,3,k}} = \begin{cases} \phi_{\mathcal{O}_{2,3,k}} \circ q & (k \equiv 1 \bmod 6), \\ \phi_{\mathcal{O}_{2,3,k}} \circ q \circ \iota & (k \equiv -1 \bmod 6). \end{cases}$$

Since both $\phi_{\mathcal{O}_{2,3,k}}$ and $\phi_{\mathcal{O}_{2,3}}$ are minimal, it follows that both $\hat{\phi}_{\mathcal{O}_{2,3,k}}$ are $\phi^{(k)}_{\mathcal{O}_{2,3}}$ are also minimal. It follows that

$$\begin{aligned}&(\mathrm{rot}(\hat{\phi}_{\mathcal{O}_{2,3,k}}(\alpha)), \mathrm{rot}(\hat{\phi}_{\mathcal{O}_{2,3,k}}(\beta)), \mathrm{rot}(\hat{\phi}_{\mathcal{O}_{2,3,k}}(\alpha\beta)))\\ &= (\mathrm{rot}(\phi^{(k)}_{\mathcal{O}_{2,3}}(\alpha)), \mathrm{rot}(\phi^{(k)}_{\mathcal{O}_{2,3}}(\beta)), \mathrm{rot}(\phi^{(k)}_{\mathcal{O}_{2,3}}(\alpha\beta))).\end{aligned}$$

Note that if $k \equiv -1 \bmod 6$, then we have

$$\begin{aligned}&\mathrm{rot}(\hat{\phi}_{\mathcal{O}_{2,3,k}}(\alpha\beta))\\ &= \mathrm{rot}(\phi_{\mathcal{O}_{2,3,k}}(\alpha\beta^{-1}))\\ &= \mathrm{rot}(\phi_{\mathcal{O}_{2,3,k}}(\beta)(\phi_{\mathcal{O}_{2,3,k}}(\alpha\beta))^{-1}(\phi_{\mathcal{O}_{2,3,k}}(\beta))^{-1})\\ &= -\,\mathrm{rot}(\phi_{\mathcal{O}_{2,3,k}}(\alpha\beta)).\end{aligned}$$

On the other hand $\hat{\phi}_{\mathcal{O}_{2,3,k}}$ and $\phi^{(k)}_{\mathcal{O}_{2,3}}$ do not belong to the same semi-conjugacy class. Indeed if they belonged the same conjugacy class, then they would be topologically conjugate by minimality. However this contradicts the fact that

$$\hat{\phi}_{\mathcal{O}_{2,3,k}}((\alpha\beta)^k) = \mathrm{id} \neq \phi^{(k)}_{\mathcal{O}_{2,3}}((\alpha\beta)^k).$$

(2) We can prove Theorem 1.1 (1) by generalizing the notion of the bounded Euler number defined in [2] to actions of 2-orbifold groups. It will be indicated in a forthcoming paper together with generalizations of Theorem 1.1 to actions of other 2-orbifold groups.

(3) Theorem 1.1 can be considered as a weak analogue of the following classical theorem about linear character [7], which we write in a specified form. Let $F\langle\alpha,\beta\rangle$ be a free group of rank two with a basis α, β.

Theorem 1.3. *Let $\phi, \psi\colon F\langle\alpha,\beta\rangle \to \mathrm{SL}(2,\mathbb{R})$ be homomorphisms. If we have*

$$\begin{aligned}&(\mathrm{tr}(\phi(\alpha)), \mathrm{tr}(\phi(\beta)), \mathrm{tr}(\phi(\alpha\beta)))\\ &= (\mathrm{tr}(\psi(\alpha)), \mathrm{tr}(\psi(\beta)), \mathrm{tr}(\psi(\alpha\beta)))\\ &= (x, y, z)\end{aligned}$$

with $x^2 + y^2 + z^2 - xyz \neq 4$, then ϕ and ψ are conjugate by an element of $\mathrm{PSL}(2,\mathbb{R})$.

(4) When the author mentioned Theorem 1.1 in his talk given in the conference “Geometry and Foliations 2013”, E. Ghys informed us the following theorem about linear character.

Theorem 1.4 ([10, Example 8.2])**.** *Let F_m be a free group of rank $m \geq 2$. For every positive integer n, there exist mutually non-conjugate elements $w_1, \dots, w_n$ of F_m such that for every homomorphism $\phi\colon F_m \to$* $\mathrm{SL}(2,\mathbb{R})$, *we have*

$$\mathrm{tr}(\phi(w_1)) = \cdots = \mathrm{tr}(\phi(w_n)).$$

After that, he asked the following question.

Question 1.5. Does the following analogue of Theorem 1.4 hold for $\mathrm{Homeo}_+(\mathbb{S}^1)$? Namely, for every positive integer $m \geq 2$ and every positive integer n, does there exist mutually non-conjugate elements $w_1, \dots, w_n$ of F_m such that for every homomorphism $\phi \in \mathrm{R}(F_m)$, we have

$$\mathrm{rot}(\phi(w_1)) = \cdots = \mathrm{rot}(\phi(w_n))?$$

Note that D. Calegari asked this question for the case where $m = 2$, $n = 2$ and w_2 is fixed as the identity element [4].

§2. Proof of Theorem 1.1

For $r_1, r_2, r_3 \in \mathbb{R}/\mathbb{Z}$, we put

$$\mathrm{R}(r_1, r_2, r_3)$$
$$= \{\phi \in \mathrm{R}(\pi_1^{orb}(\mathcal{O}_{2,3})) \mid (\mathrm{rot}(\phi(\alpha)), \mathrm{rot}(\phi(\beta)), \mathrm{rot}(\phi(\alpha\beta))) = (r_1, r_2, r_3)\}.$$

2.1. Proof of (1)

Let $\phi \in \mathrm{R}\big(\frac{1}{2}, \frac{1}{3}, 0\big)$. The following sufficient condition for belonging to the same semi-conjugacy class given in [12] is a corollary of a criterion in [14].

Proposition 2.1 ([12, Corollary 7.5])**.** *Let* Γ *be a group and* $U \subset \mathrm{R}(\Gamma)$ *be connected. Suppose that* $\mathrm{rot}(\phi_1(\gamma)) = \mathrm{rot}(\phi_2(\gamma))$ *for every* $\phi_1, \phi_2 \in U$ *and every* $\gamma \in \Gamma$*, then* U *is contained in a single semi-conjugacy class.*

In view of Proposition 2.1, it suffices to show the following.

Lemma 2.2. $\mathrm{rot}(\phi(\gamma)) = \mathrm{rot}(\phi_{\mathcal{O}_{2,3}}(\gamma))$ *for every* $\gamma \in \pi_1^{orb}(\mathcal{O}_{2,3})$.

Lemma 2.3. *The space* $\mathrm{R}\big(\frac{1}{2}, \frac{1}{3}, 0\big)$ *is path-connected.*

Proof of Lemma 2.2. We denote by $\tilde{a}$ (resp. $\tilde{b}$) the lift of $\phi(\alpha)$ (resp. $\phi(\beta)$) with $\widetilde{\mathrm{rot}}(\tilde{a}) = \frac{1}{2}$ (resp. $\widetilde{\mathrm{rot}}(\tilde{b}) = \frac{1}{3}$). Since $0 < \widetilde{\mathrm{rot}}(\tilde{a}) < 1$, we have

$$\tilde{x} < \tilde{a}(\tilde{x}) < \tilde{x} + 1$$

for every $\tilde{x} \in \mathbb{R}$. Hence we have

$$\tilde{b}(\tilde{x}) < (\tilde{a}\tilde{b})(\tilde{x}) < \tilde{b}(\tilde{x}) + 1$$

for every $\tilde{x} \in \mathbb{R}$. This implies that

$$\frac{1}{3} = \widetilde{\mathrm{rot}}(\tilde{b}) \leq \widetilde{\mathrm{rot}}(\tilde{a}\tilde{b}) \leq \widetilde{\mathrm{rot}}(\tilde{b}) + 1 = \frac{4}{3}.$$

Since $\mathrm{rot}(\phi(\alpha\beta)) = 0$, we have $\widetilde{\mathrm{rot}}(\tilde{a}\tilde{b}) = 1$. Then there exists a point $\tilde{x}_0 \in \mathbb{R}$ such that $(\tilde{a}\tilde{b})(\tilde{x}_0) = \tilde{x}_0 + 1$. Since both $\tilde{a}^2$ and $\tilde{b}^3$ are the translation by one, we have

$$\tilde{x}_0 < \tilde{a}(\tilde{x}_0) = \tilde{b}(\tilde{x}_0) < \tilde{b}^2(\tilde{x}_0) < \tilde{x}_0 + 1.$$

We put

$$I = p([\tilde{x}_0, \tilde{b}(\tilde{x}_0)] \quad \text{and}$$
$$J = p([\tilde{b}(\tilde{x}_0), \tilde{x}_0 + 1]).$$

Then we have

$$\phi(\alpha)(J) = I \quad \text{and}$$
$$\phi(\beta^{\pm 1})(I) \subset J.$$

We claim that if $\gamma \in \Gamma$ is not conjugate to a power of α, β, then there exists a closed interval $K \subset \mathbb{S}^1$ such that $\phi(\gamma)(K) \subset K$. Indeed by taking conjugates if necessary, we may assume that $\gamma = \alpha\beta^{e_1} \cdots \alpha\beta^{e_n}$, where $e_i \in \pm 1$ for $i \in \{1, \dots, n\}$. Then we have $\phi(\gamma)(I) \subset I$.

This implies that if γ is not conjugate to a power of α, β, then $\mathrm{rot}(\phi(\gamma)) = 0$. This finishes the proof of the lemma. Q.E.D.

Proof of Lemma 2.3. Let $\phi_0, \phi_1 \in \mathrm{R}\left(\frac{1}{2}, \frac{1}{3}, 0\right)$. We show that there exists a path in $\mathrm{R}\left(\frac{1}{2}, \frac{1}{3}, 0\right)$ from ϕ_0 to ϕ_1. For $t \in \{0, 1\}$, we denote by $\tilde{a}_t$ (resp. $\tilde{b}_t$) the lift of $\phi_t(\alpha)$ (resp. $\phi_t(\beta)$) with $\widetilde{\mathrm{rot}}(\tilde{a}_t) = \frac{1}{2}$ (resp. $\widetilde{\mathrm{rot}}(\tilde{b}_t) = \frac{1}{3}$). By taking conjugates, we may assume that both $\phi_0(b)$ and $\phi_1(b)$ are the rotation by $\frac{1}{3}$, and that $(\tilde{a}_t\tilde{b}_t)(0) = 1$ for $t \in \{0, 1\}$. We take a path $\{\tilde{a}_t\}_{t \in [0,1]}$ in $\widetilde{\mathrm{Homeo}}_+(\mathbb{S}^1)$ from $\tilde{a}_0$ to $\tilde{a}_1$ such that $(\tilde{a}_t)\left(\frac{1}{3}\right) = 1$ and $(\tilde{a}_t)^2$ is the translation by one. We denote by $a_t \in \mathrm{Homeo}_+(\mathbb{S}^1)$ the projection of $\tilde{a}_t$. Then the path $\{\phi_t\}_{t \in [0,1]}$ in $\mathrm{R}\left(\frac{1}{2}, \frac{1}{3}, 0\right)$ defined by the condition that $\phi_t(\alpha) = a_t$ and $\phi_t(\beta)$ is the rotation by $\frac{1}{3}$ is a desired one. Q.E.D.

2.2. Proof of (2)

Let $\phi \in \mathrm{R}\left(\frac{1}{2}, \frac{2}{3}, \frac{1}{5}\right)$. Then ϕ has no finite orbits. In fact if there were finite orbits, then the map $\mathrm{rot} \circ \phi \colon \mathbb{Z}_2 * \mathbb{Z}_3 \to \mathbb{R}/\mathbb{Z}$ must be a homomorphism, which is impossible since $\mathrm{rot}(\phi(\alpha)) = \frac{1}{2}$, $\mathrm{rot}(\phi(\beta)) = \frac{2}{3}$ and $\mathrm{rot}(\phi(\alpha\beta)) = \frac{1}{5}$. Therefore the action ϕ admits a unique minimal set, either a Cantor set or the whole circle. Passing to a semi-conjugate action, we may assume the latter, that is, the action is minimal.

By Theorem 1.1 (1), it suffices to show that ϕ is the 5-fold lift of some action, namely, there exists a homeomorphism $\theta \in \mathrm{Homeo}_+(\mathbb{S}^1)$ which is $\phi(\pi_1^{orb}(\mathcal{O}_{2,3}))$-equivariant and periodic of period 5.

We denote by $\tilde{a}$ (resp. $\tilde{b}$) the lift of $\phi(\alpha)$ (resp. $\phi(\beta)$) with $\widetilde{\mathrm{rot}}(\tilde{a}) = \frac{1}{2}$ (resp. $\widetilde{\mathrm{rot}}(\tilde{b}) = \frac{2}{3}$). Since $0 < \widetilde{\mathrm{rot}}(\tilde{a}) < 1$, we have

$$\tilde{x} < \tilde{a}(\tilde{x}) < \tilde{x} + 1$$

for every $\tilde{x} \in \mathbb{R}$. Hence we have

$$\tilde{b}(\tilde{x}) < (\tilde{a}\tilde{b})(\tilde{x}) < \tilde{b}(\tilde{x}) + 1$$

for every $\tilde{x} \in \mathbb{R}$. This implies that

$$\frac{2}{3} = \widetilde{\mathrm{rot}}(\tilde{b}) \leq \widetilde{\mathrm{rot}}(\tilde{a}\tilde{b}) \leq \widetilde{\mathrm{rot}}(\tilde{b}) + 1 = \frac{5}{3}.$$

Since $\mathrm{rot}(\phi(\alpha\beta)) = \frac{1}{5}$, we have $\widetilde{\mathrm{rot}}(\tilde{a}\tilde{b}) = \frac{6}{5}$. We denote by $\widetilde{ab}$ the lift of $\phi(\alpha\beta)$ with $\widetilde{\mathrm{rot}}(\widetilde{ab}) = \frac{1}{5}$. Then there exists a point $\tilde{x}_0 \in \mathbb{R}$ such that $(\widetilde{ab})^5(\tilde{x}_0) = \tilde{x}_0 + 1$. Note that $\tilde{a}\tilde{b}(\tilde{x}) = \widetilde{ab}(\tilde{x}) + 1$ for every $\tilde{x} \in \mathbb{R}$.

Lemma 2.4. *We have the following.*

(1) $\tilde{a}(\tilde{x}) < \tilde{b}(\tilde{x})$ *for every* $\tilde{x} \in \mathbb{R}$.

(2) $(\widetilde{ab})^2\tilde{a}(\tilde{x}) < \tilde{x} + 1$ *for every* $\tilde{x} \in \mathbb{R}$.

(3) $(\widetilde{ab})^l(\tilde{x}_0) < \tilde{b}(\widetilde{ab})^{l+2}(\tilde{x}_0) - 1 < \tilde{b}^2(\widetilde{ab})^{l+4}(\tilde{x}_0) - 2 < (\widetilde{ab})^{l+1}(\tilde{x}_0)$ *for every* $l \in \mathbb{Z}$.

Proof. (1) Since $\widetilde{\mathrm{rot}}(\tilde{a}\tilde{b}) = \frac{6}{5} > 1$, we have

$$\tilde{a}^2(\tilde{x}) = \tilde{x} + 1 < \tilde{a}\tilde{b}(\tilde{x})$$

for every $\tilde{x} \in \mathbb{R}$. This implies the desired inequality.

(2) It follows from (1) that for every $\tilde{x} \in \mathbb{R}$ we have

$$(\widetilde{ab})^2\tilde{a}(\tilde{x}) = (\tilde{a}\tilde{b})^2\tilde{a}(\tilde{x}) - 2 < \tilde{a}\tilde{b}^3\tilde{a}(\tilde{x}) - 2 = \tilde{a}^2(\tilde{x}) = \tilde{x} + 1.$$

(3) By substituting $\tilde{b}(\widetilde{ab})^{l+2}(\tilde{x}_0)$ for $\tilde{x}$ in inequality (2), it follows that

$$(\widetilde{ab})^2\tilde{a}\tilde{b}(\widetilde{ab})^{l+2}(\tilde{x}_0) < \tilde{b}(\widetilde{ab})^{l+2}(\tilde{x}_0) + 1.$$

Since we have

$$(\widetilde{ab})^2\tilde{a}\tilde{b}(\widetilde{ab})^2((\widetilde{ab})^l(\tilde{x}_0)) = (\widetilde{ab})^5((\widetilde{ab})^l(\tilde{x}_0)) + 1 = (\widetilde{ab})^l(\tilde{x}_0) + 2,$$

we obtain the first inequality. Since $l \in \mathbb{Z}$ is an arbitrary integer, it follows that

$$(\widetilde{ab})^{l+2}(\tilde{x}_0) < \tilde{b}(\widetilde{ab})^{l+4}(\tilde{x}_0) - 1.$$

This implies the second inequality. Similarly we have

$$(\widetilde{ab})^{l+4}(\tilde{x}_0) < \tilde{b}(\widetilde{ab})^{l+6}(\tilde{x}_0) - 1 = \tilde{b}(\widetilde{ab})^{l+1}(\tilde{x}_0).$$

This implies the third inequality. Q.E.D.

The following lemma follows from Lemma 2.4 (3) and the equality $\tilde{a}(\widetilde{ab})^l(\tilde{x}_0) = \tilde{b}(\widetilde{ab})^{l+4}(\tilde{x}_0) - 1$.

Lemma 2.5. *For every integer* $l \in \mathbb{Z}$, *we put*

$$\tilde{I}_l = ((\widetilde{ab})^l(\tilde{x}_0), (\tilde{b}(\widetilde{ab})^{l+2})(\tilde{x}_0) - 1] \quad \textit{and}$$
$$\tilde{J}_l = ((\tilde{b}(\widetilde{ab})^{l+2})(\tilde{x}_0) - 1, (\widetilde{ab})^{l+1}(\tilde{x}_0)].$$

Then we have the following.

(1) $\tilde{b}^{-1}((\widetilde{ab})^l(\tilde{x}_0)) \in \mathrm{Int}(\tilde{J}_{l-4})$ *and* $(\tilde{b}\tilde{a})((\widetilde{ab})^l(\tilde{x}_0)) \in \mathrm{Int}(\tilde{J}_{l+5})$.
(2) $\tilde{a}(\tilde{J}_l) = \tilde{I}_{l+3}$, $\tilde{b}(\tilde{I}_l) \subset \tilde{J}_{l+3}$ *and* $\tilde{b}^{-1}(\tilde{I}_l) \subset \tilde{J}_{l-4}$.

We denote by $\widetilde{\phi(\pi_1^{orb}(\mathcal{O}_{2,3}))}$ the subgroup of $\widetilde{\mathrm{Homeo}}_+(\mathbb{S}^1)$ consisting of lifts of elements of $\phi(\pi_1^{orb}(\mathcal{O}_{2,3}))$ to $\mathbb{R}$. We define a map $\tilde{\theta}$ of $\widetilde{\phi(\pi_1^{orb}(\mathcal{O}_{2,3}))}(\tilde{x}_0)$ onto itself by

$$\tilde{\theta}(\widetilde{\phi(\gamma)}(\tilde{x}_0)) = \widetilde{\phi(\gamma)}(\widetilde{ab}(\tilde{x}_0)),$$

where $\gamma \in \pi_1^{orb}(\mathcal{O}_{2,3})$ and $\widetilde{\phi(\gamma)}$ is a lift of $\phi(\gamma)$ to $\mathbb{R}$.

Lemma 2.6. *The map $\tilde{\theta}$ is well-defined and strictly increasing.*

Proof. First we prove that $\tilde{\theta}$ is well-defined. It suffices to show that for $\widetilde{\phi(\gamma)} \in \widetilde{\phi(\pi_1^{orb}(\mathcal{O}_{2,3}))}$ with $\widetilde{\phi(\gamma)}(\tilde{x}_0) = \tilde{x}_0$, we have $\widetilde{\phi(\gamma)}(\widetilde{ab}(\tilde{x}_0)) = \widetilde{ab}(\tilde{x}_0)$.

If $\gamma = \beta^{e_0}\alpha\beta^{e_1}\cdots\alpha\beta^{e_n}$, where $e_0 \in \{0, \pm 1\}$ and $e_i \in \{\pm 1\}$ for $i \in \{1, \dots, n\}$, then we have $e_i \neq -1$ for $i \in \{0, 1, \dots, n\}$. Indeed if $(e_i, e_{i+1}, \dots, e_n) = (-1, 1, \dots, 1)$ for some $i \in \{0, 1, \dots, n\}$, then it would follow from Lemma 2.5 (1) that

$$\begin{aligned} \tilde{b}^{e_i}\cdots\tilde{a}\tilde{b}^{e_n}(\tilde{x}_0) &= \tilde{b}^{-1}(\tilde{a}\tilde{b})^{n-i}(\tilde{x}_0) \\ &= \tilde{b}^{-1}((\widetilde{ab})^{n-i}(\tilde{x}_0)) + (n-i) \in \mathrm{Int}(\tilde{J}_{6(n-i)-4}) \end{aligned}$$

and hence

$$\widetilde{\phi(\gamma)}(\tilde{x}_0) \in \mathrm{Int}(\tilde{I}_l) \cup \mathrm{Int}(\tilde{J}_l)$$

for some $l \in \mathbb{Z}$ by Lemma 2.5 (2), which contradicts the assumption. Therefore we have $\gamma = \beta^{e_0}(\alpha\beta)^n$, where $e_0 \in \{0, 1\}$ and it follows from Lemma 2.4 (3) we have $e_0 \neq 1$. Hence there exists an integer $m \in \mathbb{Z}$ such that

$$\widetilde{\phi(\gamma)}(\tilde{x}) = (\widetilde{ab})^n(\tilde{x}) + m$$

for every $\tilde{x} \in \mathbb{R}$. We have $n = -5m$ by the assumption and hence

$$\widetilde{\phi(\gamma)}(\widetilde{ab}(\tilde{x}_0)) = (\widetilde{ab})^{-5m+1}(\tilde{x}_0) + m = \widetilde{ab}(\tilde{x}_0).$$

If $\gamma = \beta^{e_0}\alpha\beta^{e_1}\cdots\alpha\beta^{e_n}\alpha$, where $e_0 \in \{0, \pm 1\}$ and $e_i \in \{\pm 1\}$ for $i \in \{1, \dots, n\}$, then we have $e_i \neq 1$ for $i \in \{0, 1, \dots, n\}$. Indeed if $(e_i, e_{i+1}, \dots, e_n) = (1, -1, \dots, -1)$ for some $i \in \{0, 1, \dots, n\}$, then it would follow from Lemma 2.5 (1) that

$$\begin{aligned} \tilde{b}^{e_i}\cdots\tilde{a}\tilde{b}^{e_n}(\tilde{x}_0) &= (\tilde{b}\tilde{a})(\tilde{b}^{-1}\tilde{a})^{n-i}(\tilde{x}_0) \\ &= (\tilde{b}\tilde{a})((\widetilde{ab})^{-(n-i)}(\tilde{x}_0)) \in \mathrm{Int}(\tilde{J}_{-(n-i)+5}) \end{aligned}$$

and hence

$$\widetilde{\phi(\gamma)}(\tilde{x}_0) \in \mathrm{Int}(\tilde{I}_l) \cup \mathrm{Int}(\tilde{J}_l)$$

for some $l \in \mathbb{Z}$ by Lemma 2.5 (2), which contradicts the assumption. Therefore we have $\gamma = \beta^{e_0}\alpha(\beta\alpha)^{n-1}$, where $e_0 \in \{0, -1\}$ and it follows from Lemma 2.4 (3) that we have $e_0 \neq 0$. Hence there exists an integer $m \in \mathbb{Z}$ such that

$$\widetilde{\phi(\gamma)}(\tilde{x}) = (\widetilde{ab})^{-(n+1)}(\tilde{x}) + m$$

for every $\tilde{x} \in \mathbb{R}$. We have $n = 5m - 1$ by the assumption and hence

$$\widetilde{\phi(\gamma)}(\widetilde{ab}(\tilde{x}_0)) = (\widetilde{ab})^{-5m+1}(\tilde{x}_0) + m = \widetilde{ab}(\tilde{x}_0).$$

Next we prove that $\tilde{\theta}$ is strictly increasing. It suffices to show that for $\widetilde{\phi(\gamma)} \in \widetilde{\phi(\pi_1^{orb}(\mathcal{O}_{2,3}))}$ with $\tilde{x}_0 < \widetilde{\phi(\gamma)}(\tilde{x}_0)$, we have $\tilde{\theta}(\tilde{x}_0) < \tilde{\theta}(\widetilde{\phi(\gamma)}(\tilde{x}_0))$.

If $\gamma = \beta^{e_0}\alpha\beta^{e_1}\cdots\alpha\beta^{e_n}$, where $e_0 \in \{0, \pm 1\}$ and $e_i \in \{\pm 1\}$ for $i \in \{1, \ldots, n\}$, then it follows from Lemma 2.5 (2) that

$$\widetilde{\phi(\gamma)}(\tilde{I}_0) \subset \tilde{I}_l \cup \tilde{J}_l$$

for some non-negative integer $l \in \mathbb{Z}$. This implies that

$$\widetilde{\phi(\gamma)}(\tilde{I}_1) \subset \tilde{I}_{l+1} \cup \tilde{J}_{l+1}$$

and hence $\tilde{\theta}(\tilde{x}_0) < \tilde{\theta}(\widetilde{\phi(\gamma)}(\tilde{x}_0))$.

If $\gamma = \beta^{e_0}\alpha\beta^{e_1}\cdots\alpha\beta^{e_n}\alpha$, where $e_0 \in \{0, \pm 1\}$ and $e_i \in \{\pm 1\}$ for $i \in \{1, \ldots, n\}$, then it follows from Lemma 2.5 (2) that

$$\widetilde{\phi(\gamma)}(\tilde{J}_{-1}) \subset \tilde{I}_l \cup \tilde{J}_l$$

for some non-negative integer $l \in \mathbb{Z}$. This implies that

$$\widetilde{\phi(\gamma)}(\tilde{J}_0) \subset \tilde{I}_{l+1} \cup \tilde{J}_{l+1}$$

and hence $\tilde{\theta}(\tilde{x}_0) < \tilde{\theta}(\widetilde{\phi(\gamma)}(\tilde{x}_0))$. Q.E.D.

The map $\tilde{\theta}$ is $\widetilde{\phi(\pi_1^{orb}(\mathcal{O}_{2,3}))}$-equivariant and we have $\tilde{\theta}^5(\widetilde{\phi(\gamma)}(\tilde{x}_0)) = \widetilde{\phi(\gamma)}(\tilde{x}_0) + 1$ for every element $\widetilde{\phi(\gamma)}$ of $\widetilde{\phi(\pi_1^{orb}(\mathcal{O}_{2,3}))}$. Since ϕ is minimal, $\widetilde{\phi(\pi_1^{orb}(\mathcal{O}_{2,3}))}(\tilde{x}_0)$ is dense in $\mathbb{R}$ and hence $\tilde{\theta}$ can be extended to an element of $\widetilde{\mathrm{Homeo}}_+(\mathbb{S}^1)$, which we also denote by $\tilde{\theta}$. The homeomorphism $\tilde{\theta}$ is $\widetilde{\phi(\pi_1^{orb}(\mathcal{O}_{2,3}))}$-equivariant and we have $\tilde{\theta}^5(\tilde{x}) = \tilde{x} + 1$ for every $\tilde{x} \in \mathbb{R}$. This gives the desired homeomorphism $\theta \in \mathrm{Homeo}_+(\mathbb{S}^1)$.

References

[1] M. Bucher, *An introduction to bounded cohomology*, preprint, available at http://www.unige.ch/math/folks/bucher/pdf/KTHnotesIV.pdf.

[2] M. Burger, A. Iozzi and A. Wienhard, Higher Teichmüller spaces: from $\mathrm{SL}(2,\mathbb{R})$ to other Lie groups, in *Handbook of Teichmüller theory. IV*, IRMA Lect. Math. Theor. Phys. **19**, Eur. Math. Soc., Zürich, 2014, 539–618.

[3] D. Calegari, *Foliations and the geometry of* 3*-manifolds*, Oxford Mathematical Monographs, Oxford Univ. Press, Oxford, 2007.

[4] D. Calegari, Ziggurats and rotation numbers, Talk at *Low-dimensional Geometry and Topology*, Tokyo, September 2012, Slide is available at http://www.is.titech.ac.jp/hyperbolic/index.html.

[5] D. Calegari and A. Walker, Ziggurats and rotation numbers, *J. Mod. Dyn.* **5** (2011), 711–746.

[6] S. Choi, *Geometric structures on* 2*-orbifolds: exploration of discrete symmetry*, MSJ Memoirs **27**, Math. Soc. Japan, Tokyo, 2012.

[7] R. Fricke and F. Klein, *Vorlesungen der Automorphen Funktionen*, Teubner, Leipzig, 1897.

[8] É. Ghys, Groupes d'homéomorphismes du cercle et cohomologie bornée, in *The Lefschetz centennial conference, Part III* (Mexico City, 1984), Contemp. Math. **58**, Amer. Math. Soc., Providence, RI, 81–106.

[9] É. Ghys, Groups acting on the circle, *Enseign. Math.* (2) **47** (2001), 329–407.

[10] R. D. Horowitz, Characters of free groups represented in the two-dimensional special linear group, *Comm. Pure Appl. Math.* **25** (1972), 635–649.

[11] T. Jørgensen, A note on subgroups of $SL(2,\mathbf{C})$, *Quart. J. Math. Oxford Ser.* (2) **28** (1977), 209–211.

[12] K. Mann, Space of surface group representations, *Invent. Math.*, published online: 22 November 2014.

[13] Y. Matsuda, Groups of real analytic diffeomorphisms of the circle with a finite image under the rotation number function, *Ann. Inst. Fourier* (*Grenoble*) **59** (2009), 1819–1845.

[14] S. Matsumoto, Numerical invariants for semiconjugacy of homeomorphisms of the circle, *Proc. Amer. Math. Soc.* **98** (1986), 163–168.

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