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Foliations of $\mathbb{S}^3$ by cyclides

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Abstract.

Throughout the last 2–3 decades, there has been great interest in the extrinsic geometry of foliated Riemannian manifolds (see [2], [4] and [22]).

One approach is to build examples of foliations with reasonably simple singularities with leaves admitting some very restrictive geometric condition. For example (see [22], [23] and [17]), consider in particular foliations of $\mathbb{S}^3$ by totally geodesic or totally umbilical leaves with isolated singularities.

The article [14] provides families of foliations of $\mathbb{S}^3$ by Dupin cyclides with only one smooth curve of singularities. Quadrics and other families of cyclides like Darboux cyclides provide other examples. These foliations are built on solutions of a three contacts problem: we show that the surfaces of the considered family satisfying three imposed contact conditions, if they exist, form a one parameter family of surfaces which will be used to construct a foliation.

Finally we will study the four contact condition problem in the realm of Darboux–d'Alembert cyclides.

§1. Introduction

Codimension one smooth foliations of 3-dimensional compact space-form by totally geodesic, totally umbilical surfaces or even flat tori are very rare. Allowing a finite number of singular points does not increase much the set of examples. Allowing smooth singular curves is a compromise which provides more examples of foliations.

We shall first consider the basic case of ruled quadrics, which contains all the notions we will need to study more general examples.

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A *Dupin cyclide* is the conformal image of a (regular or singular) torus. If regular, it is covered by four families of circles: the two families of characteristic circles (since it is a canal surfaces in two ways) and the Villarceau circles, see [17], [14], [15], [16].

An interesting 13-dimensional family of surfaces, which contains the Dupin cyclides is the family of Darboux cyclides. Generically, a Darboux cyclide has the remarkable property of being covered by two families of circles so that each circle of the first family is co-spherical with each circle of the second. We call this the *d'Alembert property.*

We will see that generic Darboux cyclides split into 9-dimensional families that we will call *d'Alembert families* (see Definition 3.2 and Proposition 5.5). The d'Alembert property looks like the classical Monge property of ruled quadrics, which are twice ruled and each line of the first ruling cuts each line of the second. We shall see that each d'Alembert family is closely related to the 9 parameter family of ruled quadrics.

[19] is another study of cyclides; they use the circles on these cyclides to get webs on them.

§2. Foliations by quadrics

In [15] and [16] the authors studied the existence of Dupin cyclides satisfying three contact conditions, i.e. that are tangent to three planes at three points. The solutions, when they exist, form a foliation of $\mathbb{S}^3$ with a singular locus which is a curve where all the solutions are tangent (see [14]).

Our aim is now to do the same using the 9-dimensional family of ruled quadrics.

2.1. The model

In the sequel, if V is a real vector space, $\mathbb{P}(V)$ the projective space of vector lines of V, and we denote by $[v]$ the point of $\mathbb{P}(V)$ corresponding to the non zero vector v, that is $[v] = \mathbb{R}v$.

When no special structure of the underlying vector space is involved we will denote by $\mathbb{P}^1$ the projective space $\mathbb{P}(\mathbb{R}^2)$, by $\mathbb{P}^2$ the projective space $\mathbb{P}(\mathbb{R}^3)$ and by $\mathbb{P}^3$ the projective space $\mathbb{P}(\mathbb{R}^4)$.

We shall consider foliations of $\mathbb{P}^3$ by projective quadrics, and from them construct in three different ways singular foliations of $\mathbb{S}^3$.

Let us consider a regular planar conic Γ in a plane $\Omega \subset \mathbb{P}^3$ and a point $\omega \in \mathbb{P}^3 \setminus \Omega$. The plane Ω, considered with multiplicity 2, and the cone $\mathcal{C}$ of vertex ω on Γ generate a linear pencil of quadrics tangent along Γ.

Proposition 2.1. *This linear system forms a foliation of $\mathbb{P}^3$ singular at ω, at all points of Ω, and nowhere else.*

Proof. In any homogeneous coordinates $[x, y, z, t]$ where we choose $\omega = [0, 0, 0, 1]$, $\mathcal{C}$ admits an equation of the form $q(x, y, z) = 0$, q homogeneous of degree 2 in x, y, z only. As ω is not on Ω, we may choose for Ω the equation $t = 0$. Then, for any $[x, y, z, t]$ not on Ω, we define $\rho([x, y, z, t]) = -\frac{q(x,y,z)}{t^2}$. The map ρ is regular on $\mathbb{P}^3 \setminus \Omega$. Let us show that it is a submersion except at ω, which proves the proposition, since the quadrics of the linear system have equation $\lambda q(x, y, z) + \mu t^2 = 0$, and, except Ω, are the fibres of ρ.

We may take an affine chart where $t = 1$. Then $\rho([x, y, z, 1]) = -q(x, y, z)$. If the three partial derivatives $\frac{\partial q}{\partial x}$, $\frac{\partial q}{\partial y}$, $\frac{\partial q}{\partial z}$ are null at (x, y, z), by Euler relation, $q(x, y, z) = 0$. If $(x, y, z) \neq (0, 0, 0)$, then the point $[x, y, z, 0]$ is singular on the conic Γ, but Γ has no singularities. Otherwise $[x, y, z, 1] = \omega$. Then ρ is a submersion on $\mathbb{P}^3 \setminus (\Omega \cup \{\omega\})$. Q.E.D.

We shall see that the previous foliation is determined by three contact conditions: the data of a triplet of points on the conic Γ and the planes tangent at these points to the cone $\mathcal{C}$. Moreover, all the foliations by quadrics respecting such a triplet of contacts are constructed in this way.

2.2. The Brianchon construction

Conversely, let us give an "algebraic" way to solve a three contacts problem for quadrics.

Let us consider three points m_i, $i = 1, 2, 3$ on a regular quadric $\mathcal{Q} \subset \mathbb{P}^3$. We shall suppose that they are *in general position*, which will mean precisely that no two of them are on a line included in $\mathcal{Q}$.

We first collect simple consequences of this non-alignment on the contacts which are necessary to obtain the quadric from the data of the three contacts.

1) First, the three points m_i are not aligned (otherwise, the line containing them would be drawn on the quadric).
2) The three planes P_i, $i = 1, 2, 3$ tangent to $\mathcal{Q}$ at m_i have a unique intersection point ω which is the pole of the plane $(m_1m_2m_3) = \Omega$ relative to the quadric.
3) The plane Ω does not contain ω, otherwise it would be tangent to $\mathcal{Q}$ at the pole ω and at least two of the three points would be on a line drawn on $\mathcal{Q}$.
4) Finally none of the three points m_i is on a tangent plane P_j, $j \neq i$, otherwise the line (m_im_j) would be contained in $\mathcal{Q}$.

Definition 2.2. *We call* generic *a triplet of contacts satisfying the above four conditions.*

It is known since Brianchon that another one-dimensional condition is necessary for the existence of a quadric satisfying a given generic triplet of contacts.

Indeed, if $(m_i, P_i)_{i=1,2,3}$ is a generic triplet of contacts on a quadric $\mathcal{Q}$, and denoting by Ω the plane $(m_1m_2m_3)$, the lines $\Delta_i = P_i \cap \Omega$ are respectively tangent at m_i to the regular conic $\Gamma = \mathcal{Q} \cap \Omega$.

Brianchon's theorem ([5], 3. p. 301) states that the conic Γ is tangent to the three lines Δ_i (in the same plane) respectively at m_i if and only if for $a_i = \Delta_j \cap \Delta_k$ (i, j, k distinct), the three lines (a_im_i) are concurrent.

This concurrence implies the uniqueness of Γ tangent to each Δ_i at m_i when these points are not an intersection of two of the lines. For a spatial geometric proof (not the original one), see [11] p. 104.

Definition 2.3. *The generic triplet $(m_i, P_i)_{i=1,2,3}$ of contacts in $\mathbb{P}^3$ satisfies the spatial Brianchon condition if, for $a_i = \Delta_j \cap \Delta_k$ (i, j, k distincts), the three lines (a_im_i) of Ω are concurrent. See* Figure 1.

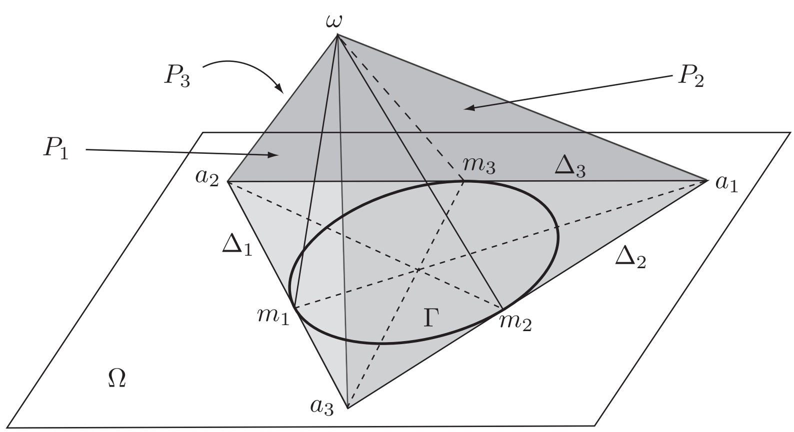


Fig. 1. Spatial Brianchon's condition.

Theorem 2.4. *For a generic triplet of contacts in $\mathbb{P}^3$ satisfying the spatial Brianchon condition, the quadrics satisfying the three contacts (m_i, P_i) form a linear pencil, together with the degenerate solution made of the plane Ω containing the points. This pencil is generated by Ω (counted with multiplicity 2) and the cone $\mathcal{C}$ on the conic Γ in Ω tangent to the planes P_i (or the lines Δ_i) at the points m_i. All the quadrics other than Ω are tangent along Γ. See* Figure 2.

Proof. A quadric $\mathcal{Q}$ satisfying these three contacts must cut Ω in a conic Γ' tangent to each Δ_i at m_i. According to Brianchon there is a unique possible conic, which is Γ. Then all the candidate quadrics have $\Gamma = \mathcal{Q} \cap \Omega$ in common. As ω is the pole common to all of them, they are all tangent at any point of Γ, and all the tangent planes at points of Γ pass through ω. Among them, we find the quadrics of the pencil generated by Ω counted twice and the cone envelope of these tangent planes. We must show that there are no other possible quadrics.

We choose homogeneous coordinates $[x, y, z, t]$ of $\mathbb{P}^3$ in such a way that $\omega = [0, 0, 1, 0]$ and Γ is in the projective plane $z = 0$. In the affine chart $t = 1$, the (part of the) cone $\mathcal{C}$ (which is in the chart) becomes a cylinder of equation $q(x, y) = 0$. Any quadric which cuts (xOy) along Γ has an equation of the form

$$f(x, y, z) = q(x, y) + \alpha(x, y)z + \beta z^2 = 0, \tag{1}$$

where α is an affine function and β a constant. If the quadric $\mathcal{Q}$ is tangent to the cylinder on Γ, at a point $(x, y, 0)$ of Γ,

$$\frac{\partial f}{\partial z} = \alpha(x, y) = 0.$$

This relation is true at the three points m_i. Then if α were not identically zero, the points m_i would be aligned, which is excluded. Then f is of the form $f(x, y, z) = q(x, y) + \beta z^2$, which proves that $\mathcal{Q}$ is in the above pencil of quadrics. Q.E.D.

In the affine Figure 2 (drawn in the chart $t = 1$), ω is not visible, since it is the projective point $[0, 0, 1, 0]$, and Γ, contained in the horizontal plane $z = 0$, is an horizontal ellipse on the left part of Figure 2 and a hyperbola on the right part part of Figure 2.

The ruled quadrics of the pencil form a one parameter family of ruled solutions foliating an open set of $\mathbb{P}^3$ which is one of the two components delimited by the cone $\mathcal{C}$.

To fill-in the space not covered by the foliation by ruled quadrics, we shall need non-ruled quadrics which form, together with the ruled ones, a singular foliation with the same singular locus.

This situation will be the prototype allowing to construct all the foliations of $\mathbb{S}^3$ constructed below with d'Alembert cyclides.

Now we shall retrieve the ruled leaves of the previous foliation by a more "dynamical" construction, without appealing to Brianchon's condition of concurrence of lines (classically relying on Pascal's "mystic hexagram", or on a spatial intersection of three lines as in [11] p. 104 Fig. 109).

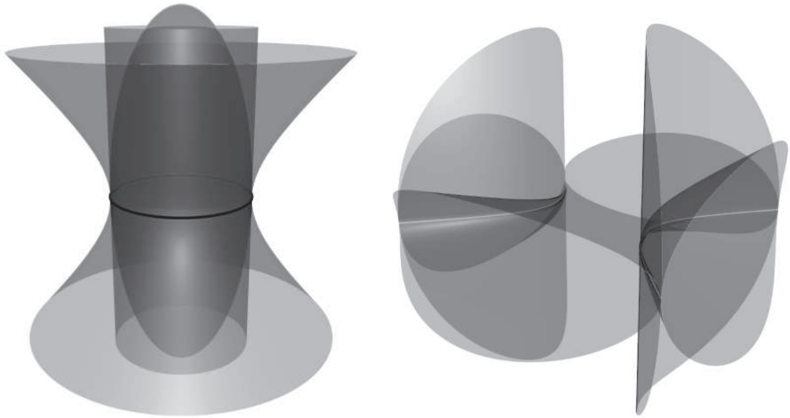


Fig. 2. Foliation of $\mathbb{R}^3$ by quadrics.

2.3. The projective Ping-Pong-Pang

We start from a generic triplet $(m_i, P_i)_{i=1,2,3}$ of contacts in $\mathbb{P}^3$, and we want to construct the ruled quadrics respecting these contacts. As above, we denote by Ω the plane $(m_1 m_2 m_3)$, and by ω the point $P_1 \cap P_2 \cap P_3$.

In a projective plane P, a linear pencil of (projective) lines is always the set of lines passing through a given common point. The set of lines of P passing through $m \in P$ will be called the *pencil of lines in P passing through m*.

For two contacts (m, P) and (m', P') where neither m nor m' are in the line $D = P \cap P'$, there is a natural mapping $\Pi_{L'L}$ from the pencil L of lines in P passing through m to the pencil L' of lines in P' passing through m'. It associates to $d \in L$ the unique line $d' \in L'$ which cuts d, see Figure 3 on the left. For the three contacts (m_i, P_i), whose associate pencils of lines are L_i, this defines three maps: $\mathit{Pang}\colon L_1 \to L_2$, $\mathit{Pong}\colon L_2 \to L_3$, and $\mathit{Ping}\colon L_3 \to L_1$. Form any $d_1 \in L_1$, we define (see Figure 3 on the right):

$$d_2^* = \mathit{Pang}(d_1), \quad d_3 = \mathit{Pong}(d_2^*),$$
$$d_1^* = \mathit{Ping}(d_3), \quad d_2 = \mathit{Pang}(d_1^*), \quad d_3^* = \mathit{Pong}(d_2).$$

On Figure 3, the image $\mathit{Ping}(d_3^*)$ is represented as a (widely) dotted line. When the three contacts are contacts in general position tangent to a ruled quadric $\mathcal{Q}$ (ruled by the two families of lines $\mathcal{F}$ and $\mathcal{F}^*$), with d_1 in $\mathcal{F}$, then all the d_i's and the d_i^*'s are drawn on $\mathcal{Q}$. In detail,

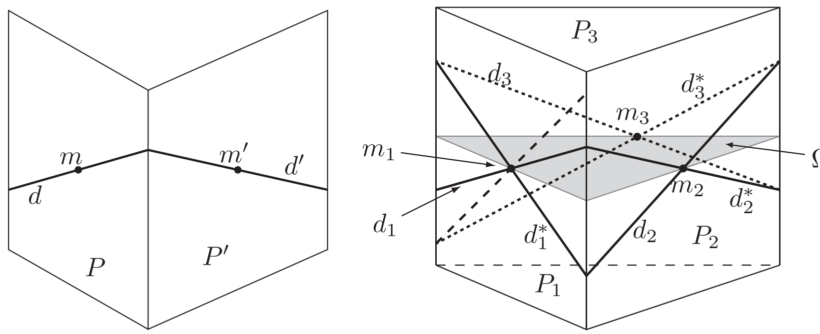


Fig. 3. The map $\Pi_{L'L}$ in $\mathbb{P}^3$.

$Pang(d_1) = d_2^*$ is in $\mathcal{F}^*$, d_3 in $\mathcal{F}$, and d_1^* is in $\mathcal{F}^*$ the second line in $P_1 = T_{m_1}\mathcal{Q}$ on the quadric $\mathcal{Q}$. It implies that the widely dotted line $Ping(d_3^*)$ in Figure 3 is in fact d_1, that is:

$$\begin{aligned} Ping(d_3^*) &= Ping \circ Pong \circ Pang(d_1^*) \\ &= Ping \circ Pong \circ Pang \circ Ping \circ Pong \circ Pang(d_1) = d_1, \end{aligned}$$

like in [11], Fig. 109 p. 104. But our use of this observation will be different from [11].

To obtain a converse statement, we associate to a ruled quadric a curve in the space of lines.

2.4. The space of lines

The set of affine lines of $\mathbb{R}^3$ is a vector bundle of dimension 6 of base $\mathbb{P}^2$ and fiber $\mathbb{R}^2$. The projective space $\mathbb{P}^3$ completes $\mathbb{R}^3$. The set of projective lines of $\mathbb{P}^3$ is isomorphic to the Grassmann manifold $G(4,2)$ of nonoriented planes of $\mathbb{R}^4$.

Let us first show how, using Plücker coordinates, $G(4,2)$ can be seen as a quadric $\mathcal{K} \subset \mathbb{P}\big(\bigwedge^2(\mathbb{R}^4)\big)$ known as *Klein quadric* ([12]).

The condition that a vector U of $\bigwedge^2(\mathbb{R}^4)$ is *pure*, that is of the form $u\wedge v$, $u \in \mathbb{R}^4$, $v \in \mathbb{R}^4$ writes $U \wedge U = 0$; this provides a quadratic form $\mathcal{LP}$: $\mathcal{LP}(U) = U \wedge U$ on $\bigwedge^2(\mathbb{R}^4)$, called the *Plücker form*. It is of index $(3,3)$. The set of pure vectors, called *Plücker cone*, is therefore a quadratic cone of equation $\mathcal{LP}(U) = 0$. The *Klein quadric* $\mathcal{K} \subset \mathbb{P}\big(\bigwedge^2(\mathbb{R}^4)\big)$ is the set of lines of this cone. The map which associates to a (vector) plane in $\mathbb{R}^4$ (or a projective line in $\mathbb{P}^3$) the corresponding point in $\mathcal{K}$ is the *Klein correspondence*.

The incidence relation of two lines of $\mathbb{P}^3$ corresponding to the 2-vectors U and $V \in \bigwedge^2(\mathbb{R}^4)$ is obtained by checking that the corresponding 2-planes of $\mathbb{R}^4$ generate a subspace of dimension at most 3; it writes $\mathcal{LP}(U,V) = U \wedge V = 0$. If U and V are not collinear (i.e. correspond to distinct points in $\mathcal{K}$), and verify $\mathcal{LP}(U,V) = 0$, they generate a totally isotropic vector plane in $\bigwedge^2(\mathbb{R}^4)$. The set of lines passing through 0 of this plane form a projective line ℓ in $\mathbb{P}\big(\bigwedge^2(\mathbb{R}^4)\big)$ called a *light ray*, which is the projective line in $\mathbb{P}\big(\bigwedge^2(\mathbb{R}^4)\big)$ joining the two points of $\mathcal{K}$ corresponding to U and V.

Conversely, a projective line ℓ of $\mathbb{P}\big(\bigwedge^2(\mathbb{R}^4)\big)$ contained in $\mathcal{K}$ is the set of vector lines of a totally isotropic plane of $\bigwedge^2(\mathbb{R}^4)$ for the Plücker form, and is thus a *light-ray*.

Given a projective plane $P \subset \mathbb{P}^3$ and a point $m \in P$, we call the set of projective lines in P containing m a *pencil of lines in* $\mathbb{P}^3$.

The projective points of a light ray ℓ correspond in $\mathbb{P}^3$ to the lines of a unique *pencil of lines in* $\mathbb{P}^3$, since we have seen above that the incidence condition between points of ℓ define a common 3-dimensional subspace of $\mathbb{R}^4$. Then a light ray corresponds to a pencil of lines in $\mathbb{P}^3$. It corresponds thus to a *contact condition*, that is a pair (m,P), $m \in P \subset \mathbb{P}^3$.

2.5. The conics associated to a regular ruled quadric

By Klein correspondence, a ruled surface of $\mathbb{P}^3$ corresponds to a curve in $\mathcal{K}$. A regular ruled quadric Q admits two disjoint families of lines such that each line of one family intersect all the lines of the other family. Therefore, any pair of one line in each family defines a pencil of lines in $\mathbb{P}^3$ corresponding to a projective light-ray contained in $\mathcal{K}$.

To the two families of lines of Q correspond therefore two curves $\mathcal{C}$ and $\mathcal{C}^*$ in $\mathcal{K}$ such that any point of one is joined to any point of the other by a projective light-ray.

For a projective subspace P of $\mathbb{P}(E)$ (set of lines passing through 0 of the vector space E), we shall denote by $\underline{P}$ the subspace of E whose set of lines passing through 0 is P, that is the cone underlying P.

Proposition 2.5. *The correspondence $Q \mapsto \{\mathcal{C},\mathcal{C}^*\}$ is a bijection between the set of regular ruled quadrics $Q \subset \mathbb{P}^3$ and the set of pairs of regular conics $\mathcal{C} = \mathcal{K} \cap P$ and $\mathcal{C}^* = \mathcal{K} \cap P^*$, where the underlying vector spaces $\underline{P}$ and $\underline{P^*}$ are 3-dimensional vector subspaces of $\bigwedge^2(\mathbb{R}^4)$ orthogonal for the Plücker quadratic form $\mathcal{LP}$.*

The conics $\mathcal{C}$ and $\mathcal{C}^*$ are called *sister conics*.

If P is a projective plane in $\mathbb{P}\big(\bigwedge^2(\mathbb{R}^4)\big)$, the following properties are equivalent:

1) the intersection $P \cap \mathcal{K}$ is a regular conic;
2) P cuts $\mathcal{K}$ and the restriction to $\underline{P}$ of $\mathcal{L}P$ is non degenerate;
3) P cuts $\mathcal{K}$ and $\underline{P}$ and $\underline{P}^{\perp}$ are in direct sum;
4) the signature of $\mathcal{L}P$ restricted to $\underline{P}$ is either $(2,1)$ or $(1,2)$.

Indeed, when the restriction to $\underline{P}$ of $\mathcal{L}P$ is non degenerate, its signature is one of the following: $(3,0)$, $(0,3)$, $(2,1)$, $(1,2)$. For the first two, $P \cap \mathcal{K}$ is empty. For the last two, $P \cap \mathcal{K}$ is a regular conic (as is $P^* \cap \mathcal{K}$).

Proof of Proposition 2.5. We choose three distinct points $\delta_i = \mathbb{R}U_i$ in $\mathcal{C}$, $i = 1, 2, 3$. Let us show that the vectors U_1, U_2, U_3 span a 3-dimensional vector space $\underline{P} \subset \bigwedge^2(\mathbb{R}^4)$. As Q is regular, the corresponding lines d_1, d_2, d_3 in the first family on Q are disjoint. If U_1, U_2, and U_3 were not linearly independent, as $\mathcal{K}$ is a quadric and a quadric containing three aligned points contains the projective line joining them, d_1 and d_2 would intersect, which was excluded.

It is known from Monge and Chasles that the set of projective lines cutting d_1, d_2 and d_3 is exactly the set of lines on Q of the second family. In the exterior algebra context, this translate as follows: $\mathcal{C}^*$ is the intersection with $\mathcal{K}$ of the projective plane P^* whose underlying 3-dimensional vector space is $\underline{P^*} = \underline{P}^{\perp}$ orthogonal to $\underline{P}$ for $\mathcal{L}P$.

If $\mathcal{C}^*$ were degenerate, it would be a union of at most two light-rays, which would imply that two lines of the second family of the ruled quadric Q intersect in $\mathbb{P}^3$. Then $\mathcal{C}^*$ is non-degenerate. By symmetry, the same is true for $\mathcal{C}$. Q.E.D.

2.6. Ping-Pong-Pang map in Klein's quadric

When two contacts (m, P) and (m', P') in $\mathbb{P}^3$ are such that neither m and m' are in the intersection $P \cap P'$, we have defined a map from the set L of lines in P through m to the set L' of lines of P' through m' associating to $d \in L$ the unique $d' \in L'$ which cuts d, see Figure 3 on the left. We shall translate it in the Klein's quadric

In Klein's quadric $\mathcal{K}$, the light-ray ℓ of points δ corresponding to the projective lines $d \in L$ is disjoint from the light-ray ℓ' of points δ' corresponding to lines $d' \in L'$. Moreover, no line $d \in L$ cuts all the lines $d' \in L'$, which means in $\mathcal{K}$ that:

(∗) no point of ℓ is conjugate for $\mathcal{L}P$ to all the points of ℓ'.

Then, for any $\delta \in \ell$, the tangent space $T_\delta\mathcal{K}$, identical to the set of points δ' conjugate for $\mathcal{L}P$ to δ, cuts ℓ' at only one point $\delta' = \pi_{\ell'\ell}(\delta)$. This defines a homography $\pi_{\ell'\ell}$ from ℓ to ℓ' which we call a *ping* map.

It means that the map $L \to L'$ considered in Section 2.3 is in fact a homography $\ell \to \ell'$.

Consider now a generic triplet $(m_i, P_i)_{i=1,2,3}$. The corresponding light-rays ℓ_i are disjoint and verify the condition $(*)$. This defines three maps *ping*, *pong*, *pang* which are the translation in $\mathcal{K}$ between the ℓ_i's of the maps *Ping*, *Pong*, *Pang* of Section 2.3:

$$pang = \pi_{\ell_2 \ell_1}, \quad pong = \pi_{\ell_3 \ell_2} \quad \text{and} \quad ping = \pi_{\ell_1 \ell_3}.$$

Starting from $\delta_1 \in \ell_1$, like in Section 2.3 we define:

$$\begin{aligned} &\delta_2^* = pang(\delta_1), \quad \delta_3 = pong(\delta_2^*), \\ &\delta_1^* = ping(\delta_3), \quad \delta_2 = pang(\delta_1^*) \quad \text{and} \quad \delta_3^* = pong(\delta_2), \end{aligned}$$

(see Figure 4, (a)).

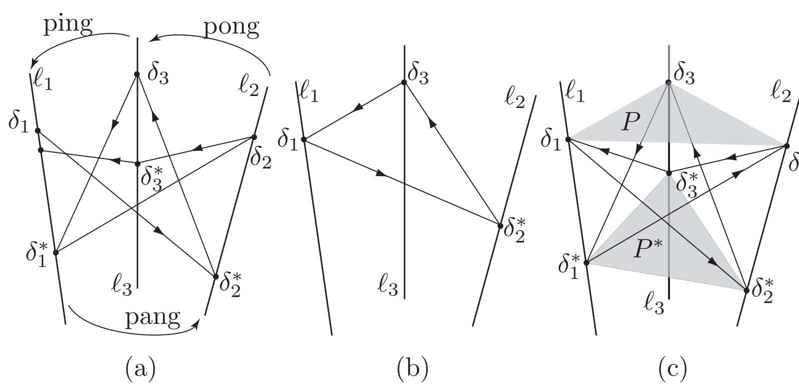


Fig. 4. $\nu = ping \circ pong \circ pang$ map.

As in Section 2.3, when the triplet of contacts is taken on a ruled quadric $\mathcal{Q}$ and if we start from a point $\delta_1 \in \ell_1$ corresponding to a line of the first ruling of $\mathcal{Q}$ at m_1, then all the points $\delta_i \in \ell_i$ are on the conic $\mathcal{C}$ associated to $\mathcal{Q}$, the points $\delta_j^* \in \ell_j$ are on the sister conic $\mathcal{C}^*$ and (see Figure 4 (c)):

$$\begin{aligned} ping(\delta_3^*) &= ping \circ pong \circ pang(\delta_1^*) \\ &= ping \circ pong \circ pang \circ ping \circ pong \circ pang(d_1) = d_1. \end{aligned}$$

Proposition 2.6. *For a generic triplet* $(m_i, P_i)_{i=1,2,3}$*, the homography* $\nu = ping \circ pong \circ pang$: $\ell_1 \to \ell_1$ *has two fixed points and is not the identity. See* Figure 4.

Proof. A first fixed point of ν corresponds to the line joining m_1 to $\omega = P_1 \cap P_2 \cap P_3$ since *Ping*, *Pong* and *Pang* permute the lines $(m_i\omega)$ (vertical lines through m_i in Figure 3).

In the same way, a second fixed point of ν corresponds to the line $P_1 \cap \Omega$ since *Ping*, *Pong* and *Pang* permute the lines $P_i \cap \Omega$.

There is no other fixed point since if δ_1 is a fixed point of ν corresponding to a line d_1 not containing ω, then d_1 and its image by *Ping*, *Pang* and *Pong* are contained in a projective plane which contains also the points m_i, and is identical to Ω. Then $d_1 = P_1 \cap \Omega$, which has already been counted above. Q.E.D.

Theorem 2.7. *Let $(m_i, P_i)_{i=1,2,3}$ be a generic triplet of contacts. A necessary and sufficient condition for the existence of a regular ruled quadric $\mathcal{Q}$ satisfying these contacts is that the map $\nu = ping \circ pong \circ pang\colon \ell_1 \to \ell_1$ is an involution.*

Moreover, in this case, there is a one parameter family of quadrics solutions. All of them are constructed by the ping-pong-pang construction from a point δ_1 on ℓ_1.

Proof. We have already seen if a regular quadric $\mathcal{Q}$ satisfies the three contact condition ℓ_1, ℓ_2, ℓ_3, then the point $\delta_1 = C \cap \ell_1$, where C is one of the curves associated to one ruling of $\mathcal{Q}$ (see Section 2.5), is a fixed point of ν^2. As the quadric is regular, δ_1 is not a fixed point of ν. Then ν^2 has three fixed points and therefore is the identity.

Reciprocally, suppose that ν is involutive. For any point δ_1 of ℓ_1 other than one of the two fixed points of ν, the six points δ_1, $\delta_2^* = pang(\delta_1)$, $\delta_3 = pong(\delta_2^*)$, $\delta_1^* = ping(\delta_3) = \nu(\delta_1)$, $\delta_2 = pang(\delta_1^*)$, $\delta_3^* = pong(\delta_2)$ are distinct, since $\delta_1^* \neq \delta_1$ and $\nu(\delta_1^*) = \delta_1$. As the lines ℓ_i are projectively independent, the planes P generated by δ_1, δ_2, δ_3 and P^* generated by δ_1^*, δ_2^* and δ_3^* are projectively independent. From the ping-pong-pang construction, P and P^* are conjugate, i.e. the underlying vector spaces are orthogonal for the Plücker quadratic form, whose restriction to each of them is thus non degenerate. The above subsections show that the curves $\mathcal{C} = P \cap \mathcal{K}$ and $\mathcal{C}^* = P^* \cap \mathcal{K}$ correspond to the same regular ruled quadric satisfying the three contacts. We find in this way a quadric for each point δ_1 on ℓ_1 other than the two fixed points of ν. Q.E.D.

The involutivity condition of Theorem 2.7 is the dynamic equivalent to Brianchon's spatial condition of Definition 2.3.

Remark 2.8. As the homography ν is one-dimensional and admits two distinct fixed points (Proposition 2.6), it is an involution if and only if the trace of any its matrix representation is zero. This is a one-dimensional constraint as it occurred looking for Dupin cyclides satisfying three contact conditions (see [14] and [16]).

2.7. The common tangency curve

Theorem 2.9. *Let $(m_i, P_i)_{i=1,2,3}$ be a generic triplet of contacts on the ruled quadric $\mathcal{Q}$, associated to the conics $\mathcal{C}$ and $\mathcal{C}^*$, and let $\ell_i = (\delta_i \delta_i^*)$ be the corresponding light rays. Let f be the unique homography $\mathcal{C} \to \mathcal{C}^*$ sending δ_i onto δ_i^*. Then all the light rays joining a point $\delta \in \mathcal{C}$ to its image $f(\delta) \in \mathcal{C}^*$ are common contacts to all the quadrics $\mathcal{Q}'$ associated to the conics $\mathcal{C}'$ constructed by ping-pong-pang from any $\delta_1' \in \ell_1$.*

Moreover, the common contacts are along a conic common to all the quadrics $\mathcal{Q}'$.

Proof. We denote by ${\delta'}_1^*$, δ_2', ${\delta'}_2^*$, δ_3', ${\delta'}_3^*$ the points constructed by *ping-pang-pong* from δ_1'. We shall see that it is sufficient to prove that from any other light ray $\ell = (\delta\, f(\delta))$, the point $\delta'' = \pi_{\ell\ell_1}(\delta_1')$ is conjugate to δ_3', which is the:

Lemma 2.10 (The ping-pong lemma)**.** *With the above notations, the image $\pi_{\ell_3\ell}(\delta'')$ is equal to δ_3'' and is independent on the choice of δ (and thus on ℓ).*

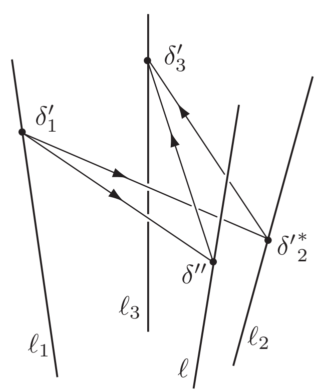


Fig. 5. The ping-pong lemma.

The name *ping-pong* is suggested by $\delta_3' = \pi_{\ell_3\ell_2} \circ \pi_{\ell_2\ell_1}(\delta_1')$.

Proof of Lemma 2.10. To that aim, we shall characterise the relations between δ_1', δ'' and δ_3' in a way where ${\delta'}_2^*$ will not appear.

If $\underline{P}$ and $\underline{P^*}$ are the underlying vector spaces of $\mathcal{C}$ and $\mathcal{C}^*$, let F be a linear mapping $\underline{P} \to \underline{P^*}$ representing f. There exists a constant α such that, for all Z and $Z' \in \underline{P}$, $\mathcal{L}P(F(Z), F(Z')) = \alpha \mathcal{L}P(Z, Z')$.

For any $\ell = (\delta f(\delta))$, $\delta = [X] \in \mathcal{C}$, and any point $\delta' = [\lambda X + \mu F(X)]$, the point $[\lambda, \mu] \in \mathbb{P}^1$ is uniquely determined (whatever may be the choice of the vector X). It defines a homography $\delta' \mapsto [\lambda, \mu]$ from ℓ to $\mathbb{P}^1$. Putting together all the lines $\ell = (\delta f(\delta))$, $\delta \in \mathcal{C}$ this gives a submersion (in fact a fibration) θ from the ruled surface $\mathcal{S} = \bigcup\{(\delta f(\delta)), \delta \in \mathcal{C}\}$ to $\mathbb{P}^1$.

This allows us to read the *ping*, *pong*, *pang* maps directly in $\mathbb{P}^1$ as follows. For the distinct light rays ℓ' and ℓ'' on $\mathcal{S}$ and δ' in ℓ',

$$\theta \circ \pi_{\ell''\ell'}(\delta') = \mathcal{I} \circ \theta(\delta'),$$

where $\mathcal{I}$ is the involution of $\mathbb{P}^1$: $[\lambda, \mu] \mapsto [-\alpha\mu, \lambda]$.

Indeed, if $\delta' = \lambda' X + \mu' F(X)$, $\delta'' = \lambda'' Y + \mu'' F(Y)$ (with $[X] \in \ell' \cap \mathcal{C}$ and $[Y] \in \ell'' \cap \mathcal{C}$),

$$\begin{aligned} \delta'' = \pi_{\ell''\ell'}(\delta') &\Leftrightarrow \mathcal{L}(\lambda' X + \mu' F(X), \lambda'' Y + \mu'' F(Y)) = 0 \\ &\Leftrightarrow \lambda'\lambda'' + \alpha\mu'\mu'' = 0 \\ &\Leftrightarrow [\lambda'', \mu''] = \mathcal{I}([\lambda', \mu']) \Leftrightarrow \theta(\delta'') = \mathcal{I}(\theta(\delta')). \end{aligned}$$

For $\delta_1' \in \ell_1$, let us write $[\lambda, \mu] = \theta(\delta_1')$. Then $\theta({\delta'}_2^*) = [-\alpha\mu, \lambda]$, and

$$\theta(\delta_3') = \theta(\pi_{\ell_3\ell_2}({\delta'}_2^*)) = \mathcal{I}([-\alpha\mu, \lambda]) = [\lambda, \mu] = \theta(\delta_1').$$

Getting back to $\delta'' = \pi_{\ell\ell_1}(\delta_1')$, we see that $\theta(\delta'')$ is also equal to $[-\alpha\mu, \lambda]$, and the above equivalences imply that δ'' is conjugate to all the fibre of θ which contains δ_1', in particular to δ_3'. Q.E.D.

For the same reason, δ'' is also conjugate to δ_2'. It implies that δ'' is conjugate to all $\mathcal{C}'$, and thus is on $\mathcal{C}'^*$, and by symmetry, the light ray ℓ will cut $\mathcal{C}'$ and $\mathcal{C}'^*$. Then the light rays $\ell = (\delta f(\delta))$ are common contacts to the quadrics $\mathcal{Q}'$.

Let us show now that the common curve is a conic. The involution $\nu = \mathit{ping} \circ \mathit{pong} \circ \mathit{pang}$ has two fixed points, as $\theta(\nu(\delta')) = \mathcal{I}^3(\theta(\delta')) = \mathcal{I}(\theta(\delta'))$, and $\mathcal{I}$ has two fixed points since $\alpha < 0$. But we already know these fixed points. The first one corresponds to the tangent to $\mathcal{Q} \cap \Omega$ at m_1, where Ω is the plane $(m_1 m_2 m_3) \subset \mathbb{P}^3$, and the second corresponds to the line $(m_1\omega)$, where $\omega = P_1 \cap P_2 \cap P_3$.

If we choose for δ_1' the first, then $\delta_1' = {\delta'}_1^*$, $\delta_2' = {\delta'}_2^*$ and $\delta_3' = {\delta'}_3^*$. For any light ray $\ell = (\delta f(\delta))$, we have seen that $\delta'' = \pi_{\ell\ell_1}(\delta_1')$ is conjugate to the points δ_i', $i = 1, 2, 3$, which implies that the corresponding line d''

in $\mathbb{P}^3$ cuts the corresponding lines d'_i, that is d'' is in Ω. The common curve is then included in Ω, and is in fact $\mathcal{Q} \cap \Omega$. Q.E.D.

Remark 2.11. We notice that θ gives a trivialisation of the projective line bundle $\mathcal{S} \twoheadrightarrow \mathcal{C}$: $\delta' \in (\delta f(\delta)) \mapsto \delta$, so that $\mathcal{S}$ is topologically the product of two circles.

§3. D'Alembert cyclides

Definition 3.1. *A cyclide is a compact surface of* $\mathbb{S}^3$ *spanned by at least two one parameter families* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *of circles. Moreover exactly one circle of each family goes through each point of the surface, except maybe a finite number of points.*

A one parameter family of circles which spans a compact surface and such that one circle of the family go through each point of the surface, except maybe a finite number of points, will be called a *hooping* of the surface, by analogy with the ruling of a ruled surface.

We will often explicit our examples of surfaces of $\mathbb{S}^3$ *describing them in* $\mathbb{R}^3$*. The true* $\mathbb{S}^3$ *example is obtained composing with the inverse of a stereographic projection.*

Definition 3.2. *Two hoopings of a cyclide satisfy the d'Alembert condition if any pair of circles,* $c_1 \in \mathcal{C}_1$ *and* $c_2 \in \mathcal{C}_2$ *is contained in a sphere* $\Sigma(c_1, c_2)$*. Such a pair of hoopings is called a pair of d'Alembert hoopings. We will call a cyclide admitting a pair of d'Alembert hoopings a d'Alembert cyclide.*

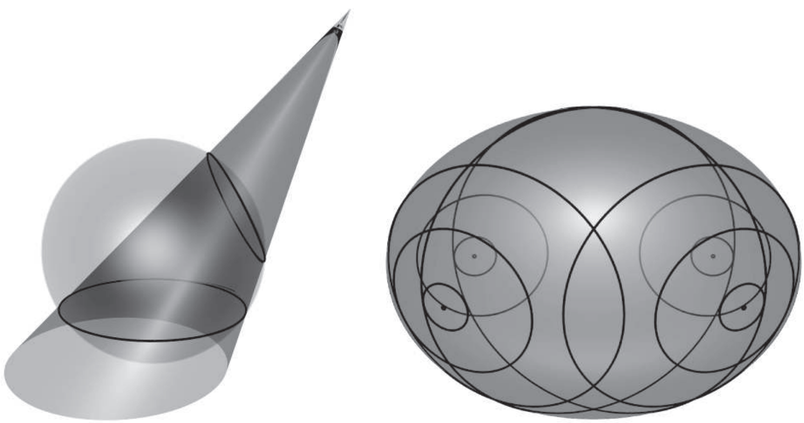
In Figure 6, the d'Alembert sphere $\Sigma(c_1, c_2)$ has been represented only on the left case, since the circles c_1 and c_2 do not intersect. On the right, the sphere $\Sigma(c_1, c_2)$ is evident and need not be drawn. Historically, perhaps the first example of d'Alembert cyclide is a slanted cone. It was known to Apollonius, see Figure 7 (a). Probably the next example of d'Alembert cyclide is an ellipsoid with three principal axes of different lengths, observed by D'ALEMBERT in [1], see Figure 7 (b).

In fact, this property is quite general among quadrics:

Proposition 3.3. *All quadrics admitting an elliptic section are d'Alembert cyclides.*

Proof. We shall give the proof in $\mathbb{R}^3$, and suppose that the plane containing the given elliptic section is (xOy). We begin by observing that if a plane section of a quadric is a circle, all non-empty plane sections parallel to it are points or circle (see [18]).

Fig. 6. Pair of circles (c_1, c_2) on d'Alembert cyclides.



(a) (b)

Fig. 7. Classical examples of d'Alembert cyclides.

Any quadric admitting an elliptic section admits two orthogonal symmetry planes P_1 and P_2. Monge ([18] p. 38) explained that it admits also two families of circles, sections of the quadric by two families of parallel planes orthogonal to, say, P_1. Clearly the directions of these parallel planes are symmetric with respect to P_2. We have to show that two circles, one of each family, are cospherical. This is a consequence of a geometric characterization of four cocyclic points of a conic (as studied by Joachimsthal). We will apply the following lemma to the four points $(c \cap P_1) \cup (c' \cap P_1)$.

Lemma 3.4. *Let D and D' be two lines crossing an axis of a conic with opposite angles. Then the four intersection points of the lines with the conic are cocyclic.*

Proof. Suppose that the symmetry axis is (Ox). The equation of the conic is then

$$Q(x,y) = ax^2 + by^2 + cx + d = 0,$$

The equations of the two lines write $y = \alpha x + \beta$ and $y = -\alpha x + \gamma$. The pencil of conics

$$\lambda Q(x,y) + \mu(y - \alpha x - \beta)(y + \alpha x - \gamma) = 0$$

contains a circle. Q.E.D.

By applying the lemma to the intersection of $\mathcal{Q}$ with P_1 and to the two lines intersections with P_1 of the planes intersecting the quadric in our two circles, we get our proposition. Q.E.D.

Smooth Dupin cyclides are also d'Alembert cyclides. In that case, the two d'Alembert hoopings are the two families of Villarceau circles, see Figure 8 where are represented circles of one of the two hoopings. The characteristic circles of a Dupin cyclide do not form d'Alembert hoopings.

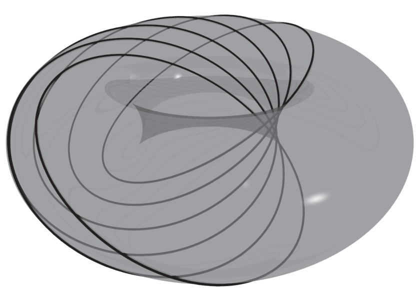


Fig. 8. Foliation of a torus of revolution by Villarceau circles.

§4. The spaces of spheres, and circles

4.1. The space of spheres

It will be convenient for us to realize both our ambient space $\mathbb{S}^3 \subset \mathbb{R}^4$ and the set of oriented 2-spheres by using the Lorentz space $\mathbb{R}^5_1$, that is

$\mathbb{R}^5$ endowed with the Lorentz quadratic form

$$\mathcal{L}(x) = \mathcal{L}(x_0, x_1, x_2, x_3, x_4) = -x_0^2 + \sum_{i=1}^{4} x_i^2.$$

The *light-cone* $\mathcal{L}i$ is the set $\mathcal{L}(x) = 0$. Its generatrices are called *light-rays*. We also call affine lines parallel to a generatrix of the light-cone light-rays.

The light-cone separates vectors of $\mathbb{R}^5_1 \setminus \mathcal{L}i$ in two types: space-like vectors, such that $\mathcal{L}(v) > 0$ and time-like vectors, such that $\mathcal{L}(v) < 0$. A plane will be called space-like if it contains only space-like (non-zero) vectors. It is called time-like if it contains non zero time-like vectors (then it contains vectors of the three types). It is called light-like is it contains non-zero light-like vectors but no time-like vector.

The space of oriented 2-dimensional spheres in $\mathbb{S}^3$ may be parameterized by the *de Sitter quadric* $\Lambda^4 \subset \mathbb{R}^5_1$ defined as the set of points $\sigma = (x_0, x_1, x_2, x_3, x_4)$ such that $\mathcal{L}(\sigma) = 1$, in the following way. The hyperplane $\sigma^\perp$ orthogonal to σ (for the Lorentz quadratic form $\mathcal{L}$) cuts the affine hyperplane $H_0 = \{x_0 = 1\}$ along a 3-dimensional oriented affine hyperplane, which cuts the unit sphere $\mathbb{S}^3_0 = \mathcal{L}i \cap H_0 \subset H_0$ along a 2-dimensional sphere Σ. Let us orient the sphere Σ as boundary of the ball $B_\sigma = \mathbb{S}^3_0 \cap \{\mathcal{L}(x, \sigma) \geq 0\}$.

This correspondence between points σ of Λ^4 and oriented spheres $\Sigma \subset \mathbb{S}^3_0 \subset H_0$ (or $\Sigma \subset \mathbb{S}^3 \subset \mathbb{P}(\mathbb{R}^5_1)$) is bijective.

The sphere $\mathbb{S}^3$ may also be identified with the *sphere at the infinity* of $\mathcal{L}i$, which is the set $\mathbb{S}^3 \subset \mathbb{P}(\mathbb{R}^5_1)$ of generatrices of the cone $\mathcal{L}i$. This is the model of sphere we shall denote by $\mathbb{S}^3$ from now on, since it will appear very useful to see the sphere embedded in $\mathbb{P}(\mathbb{R}^5_1)$. The map sending $x \in \mathbb{S}^3_0 \subset H_0$ to the line $\mathbb{R}x \in \mathbb{S}^3$ is bijective. The quotient $\mathbb{P}(\Lambda^4) = \Lambda^4/(\sigma \sim -\sigma)$ may be used to parametrize the set of non oriented spheres in $\mathbb{S}^3$. In the same way as above, the bijective correspondence between them associates to the class $[\sigma]$ of σ the sphere $\Sigma \subset \mathbb{S}^3$ of lines of the cone $\mathbb{R}\sigma^\perp \cap \mathcal{L}i$.

4.2. The space of circles

A circle $c \subset \mathbb{S}^3$ is the axis of a pencil of 2-spheres in $\mathbb{S}^3$. These spheres may be oriented or not, since if an oriented sphere is in the pencil, the sphere with opposite orientation is in the same pencil.

This pencil corresponds to the points of intersection of the quadric $\Lambda^4 \subset \mathbb{R}^5_1$ and a space-like vector plane $p_c \subset \mathbb{R}^5_1$. This vector plane p_c itself defines a projective line $d_c \subset \mathbb{P}(\mathbb{R}^5_1)$.

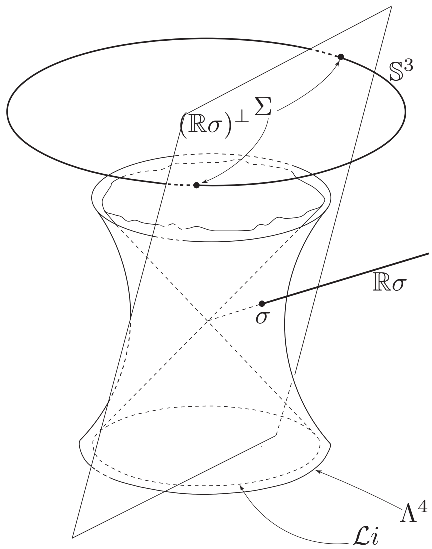


Fig. 9. $\mathbb{S}^3$ and the correspondence between points of Λ^4 and spheres.

A projective subspace R for $\mathbb{P}(\mathbb{R}^5_1)$ will be called *space-like* (resp *time-like* and *light-like*) if the underlying vector subspace $\underline{R} \subset \mathbb{R}^5_1$ has the corresponding property. Then the correspondence $c \mapsto d_c$ is a bijection between the set of circles of $\mathbb{S}^3$ and the set of space-like lines of $\mathbb{P}(\mathbb{R}^5_1)$. The inverse associates to a space-like line $d \subset \mathbb{P}(\mathbb{R}^5_1)$ the circle c of intersection of $\mathbb{S}^3$ with $\mathbb{P}(\underline{d}^\perp)$. If E is a vector space and $\mathcal{E}$ is a union of vector lines of E, we shall denote by $\mathbb{P}(\mathcal{E}) \subset \mathbb{P}(E)$ the set of vector lines whose $\mathcal{E}$ is the union.

Therefore the space of circles of $\mathbb{S}^3$ can be seen as a subset of the set $\mathbb{P}(\mathcal{P}\ell)$ of lines of the cone $\mathcal{P}\ell \subset \bigwedge^2(\mathbb{R}^5_1)$ given by the Plücker relations defining pure 2-vectors.

The wedge product defines a bilinear mapping $\bigwedge\colon \bigwedge^2(\mathbb{R}^5_1) \times \bigwedge^2(\mathbb{R}^5_1) \to \bigwedge^4(\mathbb{R}^5_1)$. The condition $U \bigwedge U = 0$ gives 5 quadratic equations. They are not independent. One can prove that the equality $U \bigwedge U = 0$ defines a 7-dimensional cone $\mathcal{P}\ell \subset \bigwedge^2(\mathbb{R}^5_1)$.

The grassmannian G of projective lines of $\mathbb{P}(\mathbb{R}^5_1)$ is isomorphic to the smooth projective variety $\mathbb{P}(\mathcal{P}\ell)$. In G, the set of space-like lines is open, then the subset $\mathcal{CP}\ell$ of classes of bivectors representing circles of $\mathbb{S}^3$ is also an open set in $\mathbb{P}(\mathcal{P}\ell)$.

Let now U_{c_1} and U_{c_2} be two pure vectors corresponding to the two circles c_1 and c_2. The condition $0 = U_{c_1} \wedge U_{c_2}$ is equivalent to $\dim(p_{c_1} + p_{c_2}) \leq 3$, that is to say: $\exists \sigma \in p_{c_1} \cap p_{c_2} \cap \Lambda^4$. In other terms, the two circles c_1 and c_2 belong to the same sphere Σ if and only if the corresponding 2-vector U_{c_1} and U_{c_2} satisfy $U_{c_1} \wedge U_{c_2} = 0$.

The condition is satisfied in particular when the two circles intersect at two distinct points or are tangent.

§5. Families of d'Alembert cyclides

5.1. The d'Alembert property viewed in Λ^4 and in the space of circles

A cyclide, in general, defines two surfaces of the space of oriented spheres Λ^4: the spheres tangent to the cyclide and containing a circle of the first hooping, and the spheres tangent to the cyclide and containing a circle of the second hooping.

Notice that, as the circles in each hooping are not oriented, the two (points in Λ^4 corresponding to) spheres with opposite orientations are on the same surface in Λ^4.

Then the two surfaces in Λ^4 are ruled by geodesics of Λ^4: arcs of the "circles" (for the Lorentz quadratic form) corresponding to the pencils of spheres defined by the circles of one of the hoopings.

In the case of a d'Alembert cyclide, the two surfaces of spheres corresponding to the two d'Alembert hoopings coincide. Therefore the set of d'Alembert spheres is twice ruled by geodesics of Λ^4. Slightly extending results of Florit (see [9]), we see that the surface of d'Alembert spheres is the intersection of Λ^4 with a quadratic cone contained in a 4-dimensional space. We will provide a more elementary proof of the fact that this surface is contained in a 4-dimensional space.

Proposition 5.1. *The points of Λ^4 corresponding to spheres which contain a pair of circles of a d'Alembert cyclide, one in each family, are contained in a 4-dimensional subspace $\mathcal{H}$ of $\mathbb{R}^5_1$.*

Proof. Let us choose two circles c_1^a, c_1^b of the first family, they are the axis of two pencils of spheres $[c_1^a]$ and $[c_1^b]$. The points corresponding to the spheres of these pencils are intersection of Λ^4 with the planes p_1^a and p_1^b. A circle τ_2 of the second family is the axis of the pencil $[\tau_2] = p_2 \cap \Lambda^4$. The definition of a d'Alembert cyclide implies that a sphere Σ_1 of $[c_1^a] \cap [\tau_2]$ contains c_1^a and τ_2 and that a sphere Σ_2 of $[c_1^b] \cap [\tau_2]$ contains c_1^b and τ_2. Then, if the spheres Σ_1 and Σ_2 are not equal, the plane p_2 is contained in the sum $p_1^a + p_1^b$. In the same way, a plane p_1 corresponding to a circle τ_1 of the first family is contained in the sum

$p_2^a + p_2^b$ of two planes corresponding to two circles of the second family. As the planes p_2^a and p_2^b are contained in the sum $p_1^a + p_1^b$, we conclude that all the planes involved are contained in a 4-dimensional space. Q.E.D.

This proposition proves that the projective lines $d_c \in \mathbb{P}(\mathbb{R}_1^5)$ associated to the circles of the two hoopings on a d'Alembert cyclide are in the projective subspace $\mathbb{P}(\mathcal{H})$ of dimension 3 of $\mathbb{P}(\mathbb{R}_1^5)$. When c varies in the hooping $\mathcal{C}_1$, the union of these lines d_c is a surface in $\mathbb{P}(\mathcal{H})$ to which we shall apply the methods of Section 2.5.

Then we call $\mathcal{L}P$ the restriction to $\bigwedge^2(\mathcal{H})$ of Plücker's quadratic form $\bigwedge$: $\mathcal{LP}(U,U) = U \wedge U$. It is of index $(3,3)$. The totally isotropic subspaces of $\bigwedge^2(\mathcal{H})$ will be called *light-like subspaces*. It is convenient, instead of dealing with planes, 3-dimensional subspaces and the Plücker cone of $\bigwedge^2(\mathcal{H}) \simeq \mathbb{R}^6$ to work in the projective space $\mathbb{P}(\bigwedge^2(\mathcal{H})) \simeq \mathbb{P}(\mathbb{R}^6)$. The *Klein quadric* $\mathcal{K}$ is the image of the Plücker cone of equation (only one in $\bigwedge^2(\mathcal{H})$): $\mathcal{LP}(U,U) = 0$. A *projective light-ray* is the image of a totally isotropic plane and two orthogonal 3-dimensional subspaces provide two conjugate projective planes.

Theorem 5.2. *The sets of points in Klein's quadric $\mathcal{K}$ corresponding to the two families of circles of a d'Alembert cyclide form two open arcs on conics $\mathcal{C}$ and $\mathcal{C}^*$, the intersections of Klein's quadric $\mathcal{K} \subset \mathbb{P}\big(\bigwedge^2(\mathcal{H})\big)$ with two (disjoint) conjugate projective planes.*

Only the points of $\mathcal{C}$ such that the corresponding plane in $\mathcal{H}$ is space-like can correspond to real circles in $\mathbb{S}^3$, which explains the restriction to two arcs of this statement.

One proof of this fact is quite similar to the analogous result obtained in [16] for Dupin cyclides. There is a difference since we are not sure in general to have *all the conic $\mathcal{C}$*.

Proof. Let us choose three distinct points U_1, U_2, U_3 on the Plücker cone corresponding to circles of the first family. If the corresponding planes in $\mathcal{H}$ are linearly independent, U_1, U_2, U_3 generate a projective plane P_1 in $\mathbb{P}(\bigwedge^2(\mathcal{H}))$. Any point U^* corresponding to a circle of the second family is conjugate (for the quadratic form $\mathcal{L}P$) to U_1, U_2 and U_3. Then U^* is in the projective plane P^* conjugate to P. Then P^* contains all the points of $\mathbb{P}\big(\bigwedge^2(\mathcal{H})\big)$ corresponding to the spheres of the second family. By symmetry, P contains all the points of $\mathbb{P}\big(\bigwedge^2(\mathcal{H})\big)$ corresponding to the spheres of the first family. Q.E.D.

Notice that the circles of a pencil containing one circle in each hooping come from a plane of $\bigwedge^2 \mathcal{H} \simeq \mathbb{R}^6$ totally degenerated for $\mathcal{LP}$ and give rise to a projective light-ray joining a pair of points one in each of the two conics.

Conversely we have the

Theorem 5.3. *Let $\mathcal{C}$ and $\mathcal{C}^*$ be two conics, intersections of Klein's quadric $\mathcal{K} \subset \mathbb{P}\big(\bigwedge^2(\mathcal{H})\big)$ with two (disjoint) conjugate projective planes. Let $\mathcal{C}_+$ be the set of points δ of $\mathcal{C}$ such that the corresponding line d in $\mathbb{P}(\mathcal{H})$ is space-like, and $\mathcal{C}^*_+$ the same on $\mathcal{C}^*$. Then the set of circles c on $\mathbb{S}^3$ corresponding to the points $\delta_c \in \mathcal{C}_+$ form a hooping of a d'Alembert cyclide, the points $\mathcal{C}^*_+$ providing the second hooping of the same d'Alembert cyclide.*

In the proof and the sequel, we shall make use of the line η orthogonal to the hyperplane $\mathcal{H}$ for Lorentz's quadratic form. In $\mathbb{P}(\mathbb{R}^5_1)$, η is the conjugate point (also named the *pole*) of $\mathbb{P}(\mathcal{H})$ with respect to the hyperquadric $\mathbb{S}^3 = \mathbb{P}(\mathcal{L}i) \subset \mathbb{P}(\mathbb{R}^5_1)$. We denote by $\mathbb{P}_\eta$ the projective space of dimension 3 of lines of $\mathbb{P}(\mathbb{R}^5_1)$ containing the point η.

We recall that the *conjugate* $R^\perp$ of the projective subspace R of $\mathbb{P}(\mathbb{R}^5_1)$ has underlying vector subspace $\underline{R}^\perp$, orthogonal of $\underline{R}$ for Lorentz's quadratic form. Then $R^\perp$ is the *polar subspace* of R relative to $\mathbb{S}^3$.

For each (projective) line d of $\mathbb{P}(\mathcal{H})$, the conjugate plane $d^\perp \subset \mathbb{P}(\mathbb{R}^5_1)$ contains η, since $d \subset \mathbb{P}(\mathcal{H})$. Then $d^\perp$ defines a line in the projective space $\mathbb{P}_\eta$.

Definition 5.4. *If $\mathcal{Q}$ is a twice ruled quadric in $\mathbb{P}(\mathcal{H})$, the two rulings $\mathcal{D}_1$ and $\mathcal{D}_2$ of $\mathcal{Q}$ give the two rulings of a quadric denoted by $\mathcal{Q}^\perp$ of $\mathbb{P}_\eta$, and called the* conjugate of $\mathcal{Q}$ in $\mathbb{P}_\eta$.

Proof of Theorem 5.3. We must prove that whatever may be the pair $\{\mathcal{C}, \mathcal{C}^*\}$, the surface covered by the (disjoint) circles coming from $\mathcal{C}_+$ is the same as the surface covered by those coming from $\mathcal{C}^*_+$. After that, the fact that the cyclide which is twice hooped and d'Alembert is evident as the construction has been made via $\mathcal{H}$.

The conics $\mathcal{C}$ and $\mathcal{C}^*$ parametrize the two rulings of a ruled quadric $\mathcal{Q}$ in the projective space $\mathbb{P}(\mathcal{H})$. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be the two rulings of $\mathcal{Q}^\perp$ given by Definition 5.4. Then the desired cyclide is (in $\mathbb{S}^3 \subset \mathbb{P}(\mathbb{R}^5_1)$) the union of the circles $d^\perp \cap \mathbb{S}^3$, $d \in \mathcal{D}_1$. Then it is also the union of the circles $d'^\perp \cap \mathbb{S}^3$, $d' \in \mathcal{D}_2$.

Now we have just to observe that $d^\perp \cap \mathbb{S}^3$ is a circle if and only if d is space-like. Q.E.D.

The quadric $\mathcal{Q}$ will be called the quadric *associated* to the d'Alembert cyclide defined by $\{\mathcal{C}, \mathcal{C}^*\}$. This quadric will help us to classify topologically d'Alembert cyclides.

5.2. Characterisations of d'Alembert families

Proposition 5.5. *To each* 4*-dimensional subspace* $\mathcal{H} \subset \mathbb{R}^5_1$ *corresponds a* 9*-dimensional family of d'Alembert cyclides* $\mathcal{A}_{\mathcal{H}}$*; we will call such a* 9*-dimensional family of d'Alembert cyclides a d'Alembert family. The space of d'Alembert families is therefore* 4*-dimensional.*

1) *If* $\mathcal{H}$ *is space-like, there exists a metric of* $\mathbb{S}^3$ *of constant curvature* 1 *such that all the circles of the two families are geodesics.*
2) *If* $\mathcal{H}$ *is light-like, that is tangent to the light-cone along a light-ray* $\mathbb{R} \cdot m$*, then, choosing* m *as the point at infinity, the cyclide becomes a ruled quadric of* $\mathbb{R}^3 \simeq \mathbb{S}^3 \setminus m$.
3) *If* $\mathcal{H}$ *is time-like, then all the circles of the cyclide are orthogonal to the sphere* $\mathcal{S}$ *corresponding to the two points of* $\mathcal{H}^\perp \cap \Lambda^4$.

Proof. Theorem 5.2 and its converse Theorem 5.3 imply the dimensional assertions.

While proving the assertions 1), 2) and 3), we shall describe altogether the cyclides in a d'Alembert family and their topology. The description depends on the position of $\mathcal{H}$ with respect to $\mathcal{L}i$, or, which is equivalent, the position of $\mathbb{P}(\mathcal{H})$ with respect to $\mathbb{S}^3 \subset \mathbb{P}(\mathbb{R}^5_1)$.

First case: $\mathcal{H}$ is space-like. Then, all the generatrices of $\mathcal{Q}$ are space-like, and all the points of $\mathcal{C}$ and $\mathcal{C}^*$ correspond actually to circles of the cyclide. We can be more precise.

The point η is inside the ball of $\mathbb{P}(\mathbb{R}^5_1)$ bounded by $\mathbb{S}^3$. We can choose an affine chart $\mathbb{R}^4 \hookrightarrow \mathbb{P}(\mathbb{R}^5_1)$ whose origin is precisely η, and whose unit sphere in $\mathbb{R}^4$ is $\mathbb{S}^3$. In this chart, the lines passing through η become vector lines, and $\mathbb{P}_\eta$ become the projective space $\mathbb{P}^3$. The unit sphere $\mathbb{S}^3$ is therefore a two-sheeted covering of $\mathbb{P}_\eta$. This implies the existence of a metric of constant curvature of $\mathbb{S}^3$ for which all the circles of the hoopings are geodesics, since they are intersections of planes in $\mathbb{R}^4$ passing trough the chosen origin η with $\mathbb{S}^3$.

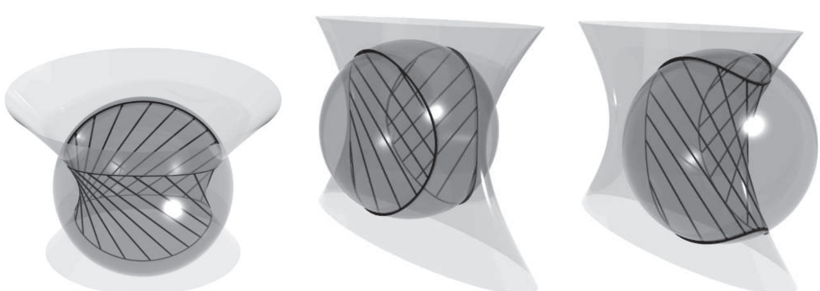
The cyclide defined by $\mathcal{Q}$ in the d'Alembert family is just the lifting of the quadric $\mathcal{Q}^\perp$ of Definition 5.4 over this covering, and it is known to be a topological torus (in particular it is connected). It is not in general a Dupin cyclide, see Figure 6.

Second case: $\mathcal{H}$ is light-like. Then, $\mathbb{P}(\mathcal{H})$ is tangent to $\mathbb{S}^3$ in $\mathbb{P}(\mathbb{R}^5_1)$. One again, $\mathcal{C}$ and $\mathcal{C}^*$ are $\mathcal{C}_+$ and $\mathcal{C}^*_+$. For each line $d \in \mathbb{P}(\mathcal{H})$, $d^\perp$ contains η. Moreover, in $\mathbb{P}(\mathbb{R}^5_1)$, $\mathbb{S}^3$ is a sphere passing by the vertex η of the cone. The map $\mathbb{P}_\eta \to \mathbb{S}^3$ which sends a line of $\mathbb{P}(\mathbb{R}^5_1)$ through η to its other intersection with $\mathbb{S}^3$ is the blowing up of the point η of $\mathbb{S}^3$. If W is any projective hyperplane of $\mathbb{P}(\mathbb{R}^5_1)$ not containing η, the mapping associating to $m \in W$ the line $(m\eta)$ is an isomorphism of projective spaces, and, composed with the blowing up $\mathbb{P}_\eta \to \mathbb{S}^3$, is a stereographic projection. The quadrics $Q^{\perp}$ are images of quadrics in W, and stereographic projections of quadrics in $\mathbb{S}^3$. It means that the study of quadrics in $\mathbb{P}^3$ is a particular case of the study of d'Alembert cyclides in a family $\mathcal{A}_{\mathcal{H}}$.

Third case: $\mathcal{H}$ is time-like. Then $\mathbb{P}(\mathcal{H})$ cuts transversally $\mathbb{S}^3$. In this case, $\mathbb{P}(\mathcal{H}) \cap \mathbb{S}^3$ is a sphere $\mathcal{S}$. This sphere may be interpreted in relation with Λ^4. Indeed, $\eta = \mathcal{H}^{\perp}$ is a space-like line of $\mathbb{R}^5_1$, and $\eta \cap \Lambda^4$ in $\mathbb{R}^5_1$ is a pair $\{\sigma, -\sigma\}$ of points of Λ^4. Then $\mathcal{S}$ is the sphere in $\mathbb{S}^3$ associated to σ or $-\sigma$, that is the set of lines of the cone $\mathbb{R}\sigma^{\perp} \cap \mathcal{L}i$, see end of Section 4.1.

Now, we return to the projective viewpoint. In $\mathbb{P}(\mathbb{R}^5_1)$, the point η is exterior to the ball bounded by $\mathbb{S}^3$ and $\mathbb{P}(\mathcal{H})$ is the hyperplane polar of η with respect to $\mathbb{S}^3$, with $\mathcal{S}$ as the set of points of contact of the lines (or hyperplanes) tangent to $\mathbb{S}^3$ passing by η. In $\mathbb{P}(\mathcal{H})$, the quadric $\mathcal{Q}$ admits a polar reciprocal quadric $\mathcal{Q}^0$ with respect to $\mathcal{S}$. Then $\mathcal{Q}^{\perp}$ is the set of lines joining a point of $\mathcal{Q}^0$ to η. If B is the ball bounded by $\mathbb{S}^3$ in $\mathbb{P}(\mathbb{R}^5_1)$, $B' = B \cap \mathbb{P}(\mathcal{H})$ is a ball bounded by $\mathcal{S}$. The map π_η sending a point m of $\mathbb{S}^3$ to the intersection $(m\eta) \cap \mathbb{P}(\mathcal{H})$ is topologically equivalent to the orthogonal projection of a 3-dimensional sphere onto an equatorial disk. Then the inverse image by π_η of each line of $\mathcal{Q}^0$ which cuts B' along a segment is a circle c.

As c is contained in a plane passing by η, it is orthogonal to the sphere $\mathcal{S}$ intersection of $\mathbb{S}^3$ with the hyperplane $\mathcal{H}$ polar of η, which proves the assertion 3). This gives the topology of the d'Alembert cyclides in the three possible cases, see Figure 10, where are shown segments of $\mathcal{Q}^0 \cap B'$ whose fibre must be thought as a circle.



(a) (b) (c)

Fig. 10. The three types of pairs $(\mathcal{Q}^0, \mathcal{S})$ in $\mathbb{P}(\mathcal{H})$.

If $\mathcal{Q}^0$ is completely outside B', the d'Alembert cyclide is empty. If $\mathcal{Q}^0$ intersects $\mathcal{S}$ like in Figure 10 (a), the d'Alembert cyclide will be a topological torus. If $\mathcal{Q}^0$ intersects $\mathcal{S}$ like in Figure 10 (b), the cyclide will be the disjoint union of two topological spheres.

If the ball has just entered gently into the hyperboloid (like a punch in the belly, see Figure 10 (c)), the d'Alembert cyclide will be a topological sphere.

We retrieve geometrically the topological classification of [20] in the generic cases. Q.E.D.

Remark 5.6. The three transformations between $\mathbb{P}_\eta$ and $\mathbb{S}^3$ used to construct d'Alembert cyclides from quadrics in $\mathbb{P}_\eta$ are very different. The first is the two-sheeted covering of $\mathbb{P}^3$ by $\mathbb{S}^3$. The second is, in the opposite direction, the blowing down from $\mathbb{P}^3$ to $\mathbb{S}^3$ (continuous map which is the rational inverse of stereographic projection). The third is the projection of $\mathbb{S}^3$ onto an equatorial ball. Then $\mathbb{S}^3$ is the source in the first and the third case, not the second one.

A consequence of the proof of Proposition 5.5 is that by any point of a d'Alembert cyclide C, passes a circle of each of the two families.

§6. Cyclides, contact conditions and foliations

6.1. The three contacts problem in a d'Alembert family

In [15] and [16] the authors studied the existence of Dupin cyclides satisfying three contact conditions, that is tangent to three planes at three points. The solutions, when they exist, form a foliation of $\mathbb{S}^3$ with a singular locus which is a curve where all the solutions are tangent (see [14]).

Propositions 5.1 and 5.5 let us hope for a similar result for each d'Alembert family of d'Alembert cyclides.

Now, we turn to the three contacts problem in a d'Alembert family $\mathcal{A}_\mathcal{H}$ attached to the chosen hyperplane $\mathcal{H}$. In fact, all the results obtained are more or less direct consequence of the results obtained for quadrics.

A contact on a d'Alembert cyclide in $\mathcal{A}_\mathcal{H}$ gives two cospherical circles c_1 and c_2 (the containing sphere Σ is called *d'Alembert sphere at this point*), and two lines d_{c_1} and d_{c_2} in $\mathbb{P}(\mathcal{H})$ whose intersection is the line in $\mathbb{R}^5_1$ containing σ, one of the two points of Λ^4 corresponding to Σ.

The pencil of circles c on Σ generated by c_1 and c_2 defines a pencil of lines ℓ_c in $\mathbb{P}(\mathcal{H})$ which generates a plane in $\mathbb{P}(\mathcal{H})$ (whose support, a 3-plane in $\mathcal{H}$, is space-like) containing the projective point $\mathbb{R}\sigma$. This defines a contact in $\mathbb{P}(\mathcal{H})$ at the (projective) intersection point of d_{c_1} and d_{c_2}. If we are given three contacts on a D'Alembert cyclide in $\mathcal{A}_\mathcal{H}$, this gives three contacts in $\mathbb{P}(\mathcal{H})$. These three contacts are obviously space-like.

Theorem 6.1. *Let us choose an hyperplane $\mathcal{H} \subset \mathbb{R}^5_1$. Given three generic contacts in $\mathbb{S}^3$, there is a one parameter family of d'Alembert cyclides in the chosen family $\mathcal{A}_{\mathcal{H}}$ which satisfy these contact conditions if and only if the corresponding three contacts in $\mathbb{P}(\mathcal{H})$ verify Brianchon's condition. A sphere Σ counted twice belong to this family. All the cyclides of the family are tangent along a biquadratic curve drawn on Σ.*

The family does not cover $\mathbb{S}^3$. Nevertheless, there exists a one parameter family of d'Alembert cyclides containing the previously obtained family, which all satisfy the three contact conditions, providing a foliation of $\mathbb{S}^3$ singular along a curve (all leaves tangent along this curve).

These extra d'Alembert cyclides belong to different d'Alembert families $\mathcal{A}_{\mathcal{H}_t}$, $t \in [0,1]$, with $\mathcal{A}_{\mathcal{H}_0} = \mathcal{A}_{\mathcal{H}}$.

The rest of this section is devoted to a (commented) proof of this theorem.

6.2. The solution of the three contacts problem in a given d'Alembert family

Given three contact conditions in $\mathbb{P}(\mathcal{H})$, we ask whether they come in this way from contacts on a d'Alembert cyclide *in* $\mathcal{A}_{\mathcal{H}}$ (since $\mathcal{H}$ is imposed), and we want to construct all these cyclides.

As seen above (just before the statement of Theorem 6.1) the three contacts in $\mathbb{P}(\mathcal{H})$ come from contacts on $\mathbb{S}^3$ if and only if they are space-like, which we shall suppose now.

We shall suppose that this triplet of contacts in $\mathbb{P}(\mathcal{H})$ is *generic* (like in the three contacts problem for quadrics). Theorem 2.4 gives us the pencil $\mathcal{P}$ of all the quadrics in $\mathbb{P}(\mathcal{H})$ which satisfy the three contacts, if the three contacts satisfy the spatial Brianchon condition of Definition 2.3.

The *ruled* quadrics in $\mathcal{P}$ give d'Alembert cyclides, as in the proof of Proposition 5.5. The obtained d'Alembert cyclide will satisfy the imposed three contact conditions.

The d'Alembert cyclides $C(\mathcal{Q})$ in $\mathbb{S}^3$ constructed in the proof of Proposition 5.5 from the ruled quadrics $\mathcal{Q} \in \mathcal{P}$ never cover the whole sphere $\mathbb{S}^3$, and leave always an open hole in $\mathbb{S}^3$. This is not due to the topologies described in Figure 10, since in the first case of Proposition 5.5 the whole quadric $Q^{\perp}$ (see Definition 5.4) is lifted to a toric surface in $\mathbb{S}^3$. The cause of this hole is the presence in $\mathcal{P}$ of non-ruled quadric, necessary to cover $\mathbb{P}(\mathcal{H})$ with quadrics of $\mathcal{P}$. Then we need the non-ruled quadrics of $\mathcal{P}$ to fill in this hole with other d'Alembert cyclides, which cannot be in $\mathcal{A}_{\mathcal{H}}$.

6.3. Completion of a family of cyclides contained in a given $\mathcal{A}_{\mathcal{H}}$ into a foliation of $\mathbb{S}^3$

We shall now extend the construction of $\mathcal{Q}^{\perp}$ (see Definition 5.4) to the non ruled quadrics $\mathcal{Q}$ of $\mathbb{P}(\mathcal{H})$. As degenerate quadrics in $\mathbb{P}(\mathcal{H})$ are ruled, we extend the construction of $\mathcal{Q}^{\perp}$ only for non degenerate quadrics.

Let $\mathcal{Q}$ be a non degenerate quadric in $\mathbb{P}(\mathcal{H})$. We associate to $\mathcal{Q}$ the set of (projective) tangent planes to $\mathcal{Q}$ in $\mathbb{P}(\mathcal{H})$. This defines a regular quadric $\mathcal{Q}^*$ of the dual projective space $\mathbb{P}(\mathcal{H}^*)$.

The projective space $\mathbb{P}_\eta$ of projective lines of $\mathbb{P}(\mathbb{R}^5_1)$ containing η may be identified with the set of (vector) planes of $\mathbb{R}^5_1$ containing the line η, and thus with the set of lines of the quotient vector space $\mathbb{R}^5_1/\eta$.

The bilinear pairing $\mathbb{R}^5_1 \times \mathcal{H} \to \mathbb{R}$ defined by restriction of $\mathcal{L}$ passes to the quotient $\mathbb{R}^5_1/\eta$. If we suppose, as we shall do, that $\mathcal{H}$ is not tangent to the light cone $\mathcal{L}i$ of $\mathcal{L}$, this pairing $\mathbb{R}^5_1/\eta \times \mathcal{H}$ is non degenerate and defines a linear bijection $\Psi\colon \mathbb{R}^5_1/\eta \to \mathcal{H}^*$ such that, for $[v]$ the class modulo η of v and $h \in \mathcal{H}$, $\Psi([v])(h) = \mathcal{L}(v, h)$.

The inverse image in $\mathbb{P}_\eta$ by (the homography associated to) Ψ of $\mathcal{Q}^*$ is the desired $\mathcal{Q}^{\perp}$. This is a non degenerate quadric of $\mathbb{P}_\eta$ such that $[v]$ is in $\mathcal{Q}^{\perp}$ if and only if the linear form $\mathcal{L}(v, \cdot)$ on $\mathcal{H}$ is the equation of some projective tangent plane of $\mathcal{Q}$. When $\mathcal{Q}$ is ruled, this leads to the same $\mathcal{Q}^{\perp} \subset \mathbb{P}_\eta$ as in the preceding subsection.

If we consider now the non degenerate quadrics $\mathcal{Q}$ (ruled or not) in a pencil $\mathcal{P}$ as above, the quadrics $\mathcal{Q}^{\perp}$ will cover the whole $\mathbb{P}_\eta$, and denoting by $\underline{\mathcal{Q}}^{\perp}$ the underlying (degenerate) hyperquadric in $\mathbb{P}(\mathbb{R}^5_1)$ of $\mathcal{Q}^{\perp}$, these hyperquadrics will cover $\mathbb{P}(\mathbb{R}^5_1)$, and their intersections with $\mathbb{S}^3 = \mathbb{P}(\mathcal{L}i)$ will cover $\mathbb{S}^3$.

For that, it is necessary that at least the quadrics $\mathcal{Q}^{\perp}$ form a (eventually) singular foliation of $\mathbb{P}_\eta$.

Remark 6.2. If $\mathcal{P}$ were any pencil of quadrics in $\mathbb{P}(\mathcal{H})$, the quadrics $\mathcal{Q}^{\perp}$, $\mathcal{Q} \in \mathcal{P}$ *would not* in general fit into a linear pencil of quadrics. The reason will appear in the proof of the following proposition.

Proposition 6.3. *If $\mathcal{P}$ is the linear pencil generated by a cone* (*quadric of rank* 2) *and a plane counted twice* (*quadric of rank* 1) *not tangent to this cone, then the quadrics $\mathcal{Q}^{\perp}$, $\mathcal{Q}$ non degenerate in $\mathcal{P}$, are the non degenerate quadrics of a pencil $\mathcal{P}^{\perp}$ of quadrics in $\mathbb{P}_\eta$.*

Proof. We choose a basis $\mathcal{B}$ of $\mathbb{R}^5_1/\eta$. There are coordinates (x, y, z) in $\mathcal{H}$ for which the equation of the generic quadric $\mathcal{Q}$ of $\mathcal{P}$ is

$$\lambda q(x, y) + \mu z^2 = 0.$$

In these coordinates, the matrix of $\mathcal{Q}$ is

$$M(\lambda, \mu) = \begin{pmatrix} \lambda M_0 & 0 \\ 0 & \mu \end{pmatrix}.$$

If A is the matrix of the pairing $\mathcal{L}\colon \mathbb{R}^5_1/\eta \times \mathcal{H} \to \mathbb{R}$ in theses bases, a matrix of $\mathcal{Q}^\perp$ is

$$M' = AM^{-1}\,{}^tA = A \begin{pmatrix} \mu M_0^{-1} & 0 \\ 0 & \lambda \end{pmatrix} {}^tA.$$

This proves our result. Q.E.D.

The tangential pencil $\mathcal{P}^\perp$ of quadrics of $\mathbb{P}_\eta$ gives in $\mathbb{P}(\mathbb{R}^5_1)$ a pencil of quadratic cones (of dimension 4) of vertex η, which gives a singular foliation of $\mathbb{P}(\mathbb{R}^5_1)$. We know the singularities of this foliation. Indeed, the tangential pencil $\mathcal{P}^\perp$ has a singular plane counted twice and a vertex in $\mathbb{P}_\eta$, which give an hyperplane and a line of singularities in $\mathbb{P}(\mathbb{R}^5_1)$.

This singular foliation of $\mathbb{P}(\mathbb{R}^5_1)$ gives, by intersection with $\mathbb{S}^3$ a foliation whose singularities are of two types:

1) the singularities of the foliation of $\mathbb{P}(\mathbb{R}^5_1)$ by quadratic cones;
2) the points where a leaf in $\mathbb{P}(\mathbb{R}^5_1)$ is tangent to $\mathbb{S}^3$.

The second case does not happen in the "first case" of Proposition 5.5 (when $\mathbb{S}^3$ is a covering of $\mathbb{P}_\eta$) since in that case the leaves of the foliation of $\mathbb{P}(\mathbb{R}^5_1)$ cut transversally $\mathbb{S}^3$.

The topology of the leaves may be described with the methods of Proposition 5.5.

The intersections of quadric hypersurfaces in $\mathbb{P}(\mathbb{R}^5_1)$ with $\mathbb{S}^3$ are known under the name of *Darboux cyclides.* They have been classified by Takeuchi [20]. It may easily be deduced from [20] that Darboux cyclides are d'Alembert cyclides, since the formulas given in [20] (pp. 125–129) for her classification give explicit equations for the two d'Alembert hoopings. There is an exception (the case $a_2 = a_3$ in [20] p. 126), which corresponds to cyclides conformally equivalent to revolution surfaces in $\mathbb{R}^3$: they are canal surfaces and may be considered as a limit case of d'Alembert cyclides, when the two hoopings coincide.

The role played here by the quadric $\mathcal{Q}^\perp$ (see Definition 5.4) shows that d'Alembert cyclides are Darboux cyclides, and the topological classification deduced from Section 5.2 indicates in which Takeuchi's class is a given d'Alembert cyclide. Then Darboux cyclides are essentially the same as d'Alembert cyclides. Notice that in the definition of a d'Alembert cyclide, we choose a pair of hoopings.

The Figure 11 shows two cyclides tangent along a curve in a d'Alembert family.

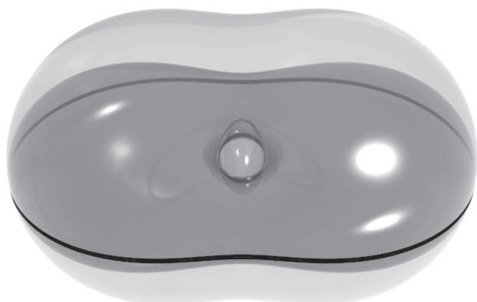


Fig. 11. Darboux cyclides tangent along a curve.

Remark 6.4. In the generic case, it was observed numerically that the poles η of "the other d'Alembert families" move continuously. Then, if we want to complete a one parameter family of d'Alembert cyclides in a given $\mathcal{A}_{\mathcal{H}}$ in a foliation of all $\mathbb{S}^3$ by d'Alembert cyclides, we need an infinity of families $\mathcal{A}_{\mathcal{H}_t}$.

6.4. Comparison with Dupin cyclide foliations

In [14], the authors studied foliations of $\mathbb{S}^3$ by Dupin cyclides. One sees easily that, for the two hoopings by Villarceau circles, a Dupin cyclide is a d'Alembert cyclide, since each Villarceau circle of one hooping intersects at two points any Villarceau circle of the second.

A foliation of $\mathbb{S}^3$ by Dupin cyclides tangent along a common Villarceau circle studied in [14] (called here a *Hopf–Villarceau foliation*) is in a d'Alembert family. Indeed, up to a conformal mapping, the cyclides of the foliation are the pre-images of tangent circles by the Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$. Each circle in $\mathbb{S}^3$ that is the pre-image of a point by the Hopf fibration is a geodesic circle of $\mathbb{S}^3$, and we are precisely in the first case considered in Proposition 5.5, where η is the centre of the sphere $\mathbb{S}^3$ in $\mathbb{C}^2 \simeq \mathbb{R}^4$.

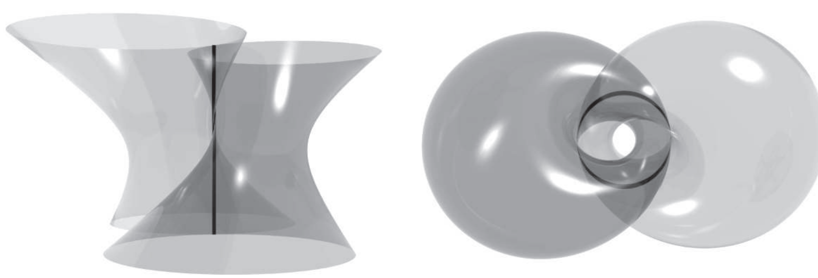
In that case, all the quadrics $\mathcal{Q}^\perp \subset \mathbb{P}_\eta = \mathbb{P}^3$ may be easily determined. Consider $\mathbb{S}^2$ as $\mathbb{P}^1(\mathbb{C})$, singularly foliated by the tangent circles completing the lines $\Re(z) = 1/a$, $a \in \mathbb{R}^*$. The inverse images of these circles of $\mathbb{S}^2$ by the Hopf fibration are, in $\mathbb{S}^3$ identified with the unit sphere in $\mathbb{C}^2$, the tori of equations:

$$|z_1|^2 + |z_2|^2 = 1, \quad \Re(z_2/z_1) = 1/a.$$

Thus, the quadrics $\mathcal{Q}^\perp \subset \mathbb{P}^3$ have, for the homogeneous coordinates (x_1, x_2, x_3, x_4) the equation:

$$x_1^2 + x_2^2 - a(x_1x_3 + x_2x_4) = 0.$$

Figure 12 shows two of them (for $a = \pm 2$) in an affine chart of $\mathbb{P}^3$, and a stereographic projection of the corresponding cyclides. Two of the quadrics of the pencil in $\mathbb{P}_\eta$ are represented in Figure 12 (a). All of them are tangent along a common generatrice, which does not correspond to the generic case studied in this paper (where the points of contact in $\mathbb{P}_\eta$ are not aligned). In fact the linear system of all the quadrics tangent to a given ruled quadric along a generatrice is of projective dimension 3, and is not a pencil of quadrics.



(a) (b)

Fig. 12. Tangent quadrics and the corresponding cyclides for a Hopf–Villarceau foliation.

Some leaves of a Hopf–Villarceau foliation are represented in [14].

Notice that the solution of the three contact problem in the realm of d'Alembert cyclides does not coincide (except in the above case of Hopf–Villarceau foliations, see also [14]) with its solution in the realm of Dupin cyclides since the common curve of the Dupin cyclides in a family is in general not contained in any sphere, see Figure 13. In all cases, the tangency curve(s) of the leaves of one of our foliations by d'Alembert cyclides are contained in a sphere.

Fig. 13. Singular Dupin cyclides tangent along a curve.

§7. The four contacts problem for Darboux cyclides

A simple way to construct foliations of $\mathbb{S}^3$ by Darboux cyclides consists in accepting all these cyclides and imposing more contact conditions. The linear system of Darboux cyclides is of projective dimension 13. Imposing four contacts leaves (for a generic quadruplet of contacts) one degree of freedom in the linear system, and gives a pencil of Darboux cyclides.

This pencil leads to a foliation of the complement of a strict algebraic submanifold, since, according to Bertini's theorem, only the base curve of the pencil and a finite number of cyclides of the pencil may contain a singularities of the foliation.

Generically, the cyclides of the pencil will be tangent along a curve as, among the cyclides of the pencil, we always find the sphere Σ containing the four points of contact (counted twice in the linear system). This implies that if Γ is the intersection of Σ with some cyclide in the pencil, then Γ is a curve of contact of all the cyclides of the pencil (see Figure 14).

Fig. 14. A quadruplet of contacts leading to a pencil of cyclides.

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