On deformations and rigidity
of the Godbillon–Vey class

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Abstract.

   The Godbillon–Vey class is the most significant secondary characteristic class for foliations. We will throw a glance at deformations and rigidity of the Godbillon–Vey class especially from a viewpoint of transverse structures of foliations.

§1.   Definitions

   Throughout the article, we work on the C^∞ category. Let M be a manifold and $\mathcal{F}$ a foliation of M. We assume for simplicity that M is without boundary. In addition, we assume that foliations are regular (non-singular), namely, the dimension of the leaves is constant. Let q be the codimension of $\mathcal{F}$. If we assume that $\mathcal{F}$ is transversely orientable, then there is a q-form, say ω, on M such that if we set

$$\operatorname{Ker}\omega = \{X \in TM \mid \iota_X\omega = 0\},$$
$$T\mathcal{F} = \{X \in TM \mid X \text{ is tangent to a leaf}\},$$

where ι_X denotes the interior product with X, then

$$\operatorname{Ker}\omega = T\mathcal{F}.$$

By the Frobenius theorem, there is a 1-form, say θ, such that

$$(1) \qquad\qquad d\omega + \theta \wedge \omega = 0.$$

By virtue of the Bott vanishing theorem [16], the $(2q + 1)$-form

$$(2) \qquad\qquad \mathrm{gv} = \left(\frac{-1}{2\pi}\right)^{q+1} \theta \wedge (d\theta)^q$$

is closed, and the cohomology class represented by gv is independent of the choice of ω and θ [24].

Definition 1.1. *The cohomology class represented by* gv *is called the* Godbillon–Vey class *of $\mathcal{F}$ and denoted by* $\mathrm{GV}(\mathcal{F})$.

Definition 1.2. *We set* $Q(\mathcal{F}) = TM/T\mathcal{F}$. *We call* $T\mathcal{F}$ *the* tangent bundle *of $\mathcal{F}$, and $Q(\mathcal{F})$ the* normal bundle *of $\mathcal{F}$. The set of sections of a vector bundle, say E, is denoted by $\Gamma(E)$.*

Definition 1.3. *A connection ∇^b on $Q(\mathcal{F})$ is said to be a* Bott connection *or a* basic connection *if*

$$(3) \qquad\qquad \nabla^b_X Y = \pi[X, \widetilde{Y}]$$

holds for $X \in \Gamma(T\mathcal{F})$ and $Y \in \Gamma(Q(\mathcal{F}))$, where $\pi\colon TM \to Q(\mathcal{F})$ denotes the projection and $\widetilde{Y}$ is any lift of Y to $\Gamma(TM)$. Note that the right hand side of (3) is equal to $\mathcal{L}_X Y$, where $\mathcal{L}_X$ denotes the Lie derivative with respect to X.

The Lie derivative on $\bigwedge^q Q(\mathcal{F})$ is defined in the standard way. Suppose that $Y \in \Gamma\big(\bigwedge^q Q(\mathcal{F})\big)$ is locally represented as $Y = Y_1 \wedge \cdots \wedge Y_q$, where $Y_1, \ldots, Y_q \in \Gamma(Q(\mathcal{F}))$. If $X \in \Gamma(T\mathcal{F})$, then the Lie derivative of Y with respect to X is by definition

$$\mathcal{L}_X Y = (\mathcal{L}_X Y_1) \wedge Y_2 \wedge \cdots \wedge Y_q + \cdots + Y_1 \wedge \cdots \wedge Y_{q-1} \wedge (\mathcal{L}_X Y_q),$$

where each $\mathcal{L}_X Y_k$, $1 \leq k \leq q$, is defined as in Definition 1.3. Then, $\mathcal{L}_X$ is extended to the whole $\Gamma(\bigwedge^q Q(\mathcal{F}))$ by linearity.

Definition 1.4. *A connection $\mathcal{D}$ on $\bigwedge^q Q(\mathcal{F})$ is said to be a* Bott connection *if*

$$\mathcal{D}_X Y = \mathcal{L}_X Y$$

holds for $X \in \Gamma(T\mathcal{F})$ and $Y \in \Gamma\big(\bigwedge^q Q(\mathcal{F})\big)$.

Note that Bott connections on $Q(\mathcal{F})$ induce Bott connections on $\bigwedge^q Q(\mathcal{F})$. Bott connections always, even in the holomorphic category,

exist as partial connections defined on $T\mathcal{F}$ in the sense that they are defined only on $T\mathcal{F} \subset TM$, however, we need partitions of unity in order to guarantee the existence of Bott connections as affine connections.

The differential form θ in (1) is essentially the connection form of a Bott connection on $\bigwedge^q Q(\mathcal{F})$. Indeed, Definition 1.1 of the Godbillon–Vey class can be reformulated as follows. We fix a Riemannian metric on $Q(\mathcal{F})$ which is not necessarily holonomy invariant. Let ∇^h be a metric connection, i.e., a connection which preserves the metric. If we set

$$
(4) \qquad
\begin{cases}
h_1 = \dfrac{-1}{2\pi}\operatorname{tr}(\nabla^b - \nabla^h), \\[2mm]
c_1 = \dfrac{-1}{2\pi} d\operatorname{tr}(\nabla^b),
\end{cases}
$$

then $h_1 c_1^q$ represents the Godbillon–Vey class. Note that $\operatorname{tr}\nabla^b$ and $\operatorname{tr}\nabla^h$ are connections on $\bigwedge^q Q(\mathcal{F})$ induced by ∇^b and ∇^h, respectively.

Secondary characteristic classes which are generalizations of the Godbillon–Vey classes are defined as follows.

Definition 1.5. *We set* $\deg c_i = 2i$, *and denote by* I_q *the ideal of* $\mathbb{R}[c_1,\ldots,c_q]$ *generated by elements of degree greater than* $2q$. *We set*

$$
\mathbb{R}_q[c_1,\ldots,c_q] = \mathbb{R}[c_1,\ldots,c_q]/I_q,
$$

and DGA's (differential graded algebras) WO_q *and* W_q *by*

$$
\mathrm{WO}_q = \bigwedge[h_1, h_3, \ldots, h_{[q]}] \otimes \mathbb{R}_q[c_1,\ldots,c_q],
$$
$$
\mathrm{W}_q = \bigwedge[h_1, h_2, \ldots, h_q] \otimes \mathbb{R}_q[c_1,\ldots,c_q],
$$

where $[q]$ *denotes the largest odd integer not greater than* q. *We set* $\deg h_i = 2i - 1$. *Let* $I = \{i_1,\ldots,i_r\}$, *where* i_k *are integers with* $1 \le i_1 < \cdots < i_r \le q$, *or* $I = \varnothing$, *and let* $J = (j_1,\ldots,j_q)$ *be a* q-tuple *of non-negative integers. We set* $h_I = h_{i_1} \wedge \cdots \wedge h_{i_r}$, $c_J = c_1^{j_1} \cdots c_q^{j_q}$ *and* $|J| = j_1 + 2j_2 + \cdots + qj_q$. *If* $I = \varnothing$, *then we set* $h_I = 1$. *Finally we set* $dh_i = c_i$ *and* $dc_j = 0$.

We have the following

Theorem 1.6 (Bott, et al. [16])**.** *Once a Bott connection and a metric connection on* $Q(\mathcal{F})$ *are fixed, there is a well-defined homomorphism from* WO_q *to* $\Omega^*(M)$, *where* $\Omega^*(M)$ *denotes the algebra of differential forms on* M. *The induced mapping from* $H^*(\mathrm{WO}_q)$ *to* $H^*(M)$ *does not depend on the choice of metrics and connections. If the normal bundle of* $\mathcal{F}$ *is trivial, then fixing a Bott connection and a trivialization, we have a*

homomorphism from W_q *to* $\Omega^*(M)$. *The induced mapping from* $H^*(\mathrm{W}_q)$ *to* $H^*(M)$ *depends only on the homotopy type of the trivialization.*

Indeed, the Godbillon–Vey class is represented by $h_1 c_1^q$ calculated by (4).

Definition 1.7. *Elements of* $H^*(\mathrm{WO}_q)$ *and* $H^*(\mathrm{W}_q)$ *which involve* h_i's *are called* secondary characteristic classes. *Their images in* $H^*(M)$, *where* M *is equipped with a foliation* $\mathcal{F}$, *are called* secondary character-*istic classes for* $\mathcal{F}$. *Especially, the elements of* $H^*(\mathrm{WO}_q)$ *and* $H^*(\mathrm{W}_q)$ *represented by* $h_1 c_1^q$ *as well as their images are called the* Godbillon–*Vey class.*

We remark that the class represented by c_{2j} is the j-th Pontrjagin class of $Q(\mathcal{F})$. Of course, several definitions of the Godbillon–Vey class in this article coincide each other.

Secondary characteristic classes are functorial with respect to mor-phisms of foliations. Indeed, they are characteristic classes of classifying spaces of foliations such as $B\Gamma_q$, etc. (see for example [25]). We do not work on classifying spaces in this article, however, quite many of subjects essentially concern them.

If we work on transversely holomorphic foliations, then there are DGA's WU_q and $\mathrm{W}_q^{\mathbb{C}}$ which play the role of WO_q and W_q, respectively.

Definition 1.8. *We set* $\deg c_i = 2i$, *and denote by* I_q *the ideal of* $\mathbb{C}[c_1, \ldots, c_q]$ *generated by elements of degree greater than* $2q$. *We set*

$$\mathbb{C}_q[c_1, \ldots, c_q] = \mathbb{C}[c_1, \ldots, c_q]/I_q$$

and define $\mathbb{C}_q[\bar{c}_1, \ldots, \bar{c}_q]$ *in a similar way. We define DGA's* WU_q *and* $\mathrm{W}_q^{\mathbb{C}}$ *by*

$$\mathrm{WU}_q = \bigwedge[\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_q] \otimes \mathbb{C}_q[c_1, \ldots, c_q] \otimes \mathbb{C}_q[\bar{c}_1, \ldots, \bar{c}_q],$$
$$\mathrm{W}_q^{\mathbb{C}} = \bigwedge[u_1, u_2, \ldots, u_q] \otimes \mathbb{C}_q[c_1, \ldots, c_q]$$
$$\wedge \bigwedge[\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_q] \otimes \mathbb{C}_q[\bar{c}_1, \ldots, \bar{c}_q].$$

We set $\deg \widetilde{u}_i = \deg u_i = \deg \bar{u}_i = 2i - 1$. *Finally, we set* $d\widetilde{u}_i = c_i - \bar{c}_i$, $du_i = c_i$, $d\bar{u}_i = \bar{c}_i$ *and* $dc_i = d\bar{c}_i = 0$.

Complex secondary characteristic classes for transversely holo-morphic foliation are defined by means of $H^*(\mathrm{WU}_q)$ and $H^*(\mathrm{W}_q^{\mathbb{C}})$ and a version of Theorem 1.6 holds.

Theorem 1.9 (Bott, et al. [16])**.** *Once a complex Bott connection and a metric connection on* $Q(\mathcal{F})$ *with respect to a Hermitian metric*

on $Q(\mathcal{F})$ are fixed, there is a well-defined homomorphism from WU_q to $\Omega^*(M)$, where $\Omega^*(M)$ denotes the algebra of $\mathbb{C}$-valued differential forms on M. The induced mapping from $H^*(\mathrm{WU}_q)$ to $H^*(M)$ does not depend on the choice of metrics and connections. If the complex normal bundle of $\mathcal{F}$ is trivial, then fixing a Bott connection and a trivialization, we have a homomorphism from $\mathrm{W}_q^{\mathbb{C}}$ to $\Omega^*(M)$. The induced mapping from $H^*(\mathrm{W}_q^{\mathbb{C}})$ to $H^*(M)$ depends only on the homotopy type of the trivialization.

There are characteristic classes 'analogous' to the Godbillon–Vey class.

Definition 1.10. *The class in $H^*(\mathrm{W}_q^{\mathbb{C}})$ represented by $u_1 c_1^q$ is called the* Bott class *and denoted by* Bott. *The class in $H^*(\mathrm{WU}_q)$ represented by $\sqrt{-1}\widetilde{u}_1(c_1^q + c_1^{q-1}\bar{c}_1 + \cdots + \bar{c}_1^q)$ is called the* imaginary part of the Bott class *and denoted by ξ. In general, classes which involve $\widetilde{u}_i$'s, u_i's or $\bar{u}_i$'s are called* complex secondary classes.

We note that the Bott class is *not* an analogue of the Godbillon–Vey class. In fact, the Bott class was found prior to the Godbillon–Vey class (see comments of Bott in [15]). The Bott class is defined for transversely holomorphic foliations with trivialized normal bundles, although it is independent of the choice of trivializations. In general, we cannot assume the triviality of the complex normal bundle, because different from the real case, it is the assumption on the triviality of the first Chern class, so that the Bott class is defined to be a class in $H^{2q+1}(M; \mathbb{C}/\mathbb{Z})$ (it has been a 'common sense'. See [4] for a formulation). If the complex normal bundle is trivial, then the Bott class in $H^{2q+1}(M; \mathbb{C}/\mathbb{Z})$ coincides with the image of the Bott class defined as an element of $H^{2q+1}(M; \mathbb{C})$. We also remark that the both c_j and $\bar{c}_j$ represent the j-th Chern class of $Q(\mathcal{F})$.

We will end this section by introducing the following

Theorem 1.11 ([3], cf. Rasmussen [49])**.** *The Godbillon–Vey class is represented by $\sqrt{-1}\frac{(2q)!}{q!\,q!}\widetilde{u}_1 c_1^q \bar{c}_1^q$ and is equal to $\frac{(2q)!}{q!\,q!}\xi c_1^q$, a non-zero multiple of the imaginary part of the Bott class and the q-th power of the first Chern class c_1.*

§2. Non-triviality and deformations

The Godbillon–Vey class was found to be non-trivial soon after its discovery.

Example 2.1 (Roussarie [24])**.** Let H be the subgroup of $\mathrm{SL}(2;\mathbb{R})$ which consists of upper triangular matrices. Let $\widetilde{\mathcal{F}} = \{gH\}_{g \in \mathrm{SL}(2;\mathbb{R})}$ be

the foliation of $\mathrm{SL}(2;\mathbb{R})$ by left cosets. Suppose that Γ is a cocompact (uniform) lattice of $\mathrm{SL}(2;\mathbb{R})$, namely, $M = \Gamma\backslash\mathrm{SL}(2;\mathbb{R})$ is a closed manifold. As $\widetilde{\mathcal{F}}$ is invariant under the left action of $\mathrm{SL}(2;\mathbb{R})$ on itself, a foliation is induced on M, which we denote by $\mathcal{F}$. The Godbillon–Vey class of $\mathcal{F}$ is a non-zero multiple of the volume form of M and hence is non-trivial.

One of the most significant properties of the Godbillon–Vey class is that it admits continuous variations (or continuous deformations).

Definition 2.2. *A characteristic class χ for foliations is said to admit* continuous variations *or* continuous deformations *if there is a family, or a deformation $\{\mathcal{F}_s\}_{s\in S}$ of foliations, where S denotes the parameter space of the deformation, such that $\chi(\mathcal{F}_s)$ varies continuously as s varies in S.*

The parameter space S is often chosen to be an interval. The regularities, e.g. the class C^r, holomorphic, etc., of deformations are chosen according to purposes.

Thurston showed that the Godbillon–Vey class admits continuous variations.

Theorem 2.3 (Thurston [53], see also [45])**.** *For each q, there is a smooth family $\{\mathcal{F}_s\}_{s\in\mathbb{R}}$ of codimension-q foliations of a closed manifold M such that $\mathrm{GV}(\mathcal{F}_s)$ varies continuously. If $q = 1$, then such a family exists on S^3.*

This was quite striking and attracted many people.

On the other hand, it is easy to see that the Bott class is not only non-trivial, but admits continuous variations. This has been known since the discovery of the Bott class.

Example 2.4 ([14])**.** Let $(z^0,\ldots,z^q)$ be the standard coordinates on $\mathbb{C}^{q+1}$. Let $\lambda_0,\ldots,\lambda_q$ be non-zero complex numbers and

$$X = \sum_{i=0}^{q} \lambda_i z^i \frac{\partial}{\partial z^i}.$$

The integral (complex) curves of X form a *holomorphic* foliation of $\mathbb{C}^{q+1} \setminus \{o\}$ which is invariant under homothecies, where o denotes the origin. Therefore, if we set for example that $M = (\mathbb{C}^{q+1} \setminus \{o\})/ \times 2$, then a foliation of $M \cong S^1 \times S^{2q+1}$ is defined. Normalizing the volume of M, we can show that

$$(5) \qquad \mathrm{Bott}(\mathcal{F}) = \frac{(\lambda_0 + \cdots + \lambda_q)^{q+1}}{\lambda_0 \cdots \lambda_q} [S^{2q+1}],$$

where $[S^{2q+1}]$ denotes the cohomology class naturally defined by the unit sphere in $\mathbb{C}^{q+1}$.

Many studies on continuous variations of the Godbillon–Vey class have been done. Among those, Heitsch clarified the theory of *residues*, which can be seen as a foliation version of the Poincaré–Hopf theorem, and showed the following

Theorem 2.5 (Heitsch [29], cf. [14])**.** *There are smooth families of foliations of sphere bundles over products of surfaces of which some of secondary classes vary continuously.*

The above statement is of course a quite reduced form. The construction and proof are quite parallel to those of Example 2.4, and the evaluations of characteristic classes in Example 2.4 and Theorem 2.5 are done by means of residues. We refer to the original articles for details. On the other hand, Thurston's deformation of the Godbillon–Vey class makes use of the hyperbolic geometry and is an adroit modification of Example 2.1. On this line, there is a study of Rasmussen in the codimension-two case [50]. However, these approaches can be unified under the notion of residues. We remark moreover that, as Hurder mentioned in [33], it seems that the non-triviality of the Godbillon–Vey class has been only recognized via residues.

One can ask what the (non-)triviality of the Godbillon–Vey class means, or conversely, which condition demands the Godbillon–Vey class to be (non-)trivial. This question is quite intensively studied for codimension-one foliations. Tsuboi showed that the triviality of the Godbillon–Vey class of codimension-one foliations is almost equivalent to the null-cobordance of the foliation in the following sense.

Theorem 2.6 (Tsuboi [55])**.** *We assume that foliations are transversely oriented. Let α be a real number with $\frac{1}{2} < \alpha \le 1$ and $\mathcal{F}$ a codimension-one foliation of class $C^{1+\alpha}$, of a 3-manifold M. The Godbillon–Vey class $\mathrm{GV}(\mathcal{F})$ is trivial if and only if $\mathcal{F}$ is cobordant to a codimension-one foliation of a 3-manifold N which is a limit of foliations of N which are null-cobordant.*

In order to be precise, we need to clarify the notion of limits and the regularity of foliations of N. We refer to the original article for details. The (non-)triviality of the Godbillon–Vey class is also closely related to the growth of leaves. The following theorem is one of the most significant ones.

Theorem 2.7 (Duminy [19])**.** *Let M be a closed manifold and $\mathcal{F}$ a codimension-one foliation of M. If $\mathrm{GV}(\mathcal{F})$ is non-trivial, then there exists a resilient leaf.*

In short, a leaf L of $\mathcal{F}$ is said to be *resilient* if L admits a holonomy which contracts L to itself. The existence of a resilient leaf implies that the set of leaves with exponential growth is of positive measure. Some generalizations are studied by Hurder, et al. For example, the following is known.

Theorem 2.8 (Hurder–Katok [34]). *Suppose that almost every leaf of $\mathcal{F}$ is of subexponential growth. Then, residual classes, i.e., classes of the form $h_I c_J$ with $|J| = q$, vanish.*

The notion of the Weil measures was relevant in the proof of Theorem 2.8. See also Heitsch–Hurder [31].

Thus non-triviality of the Godbillon–Vey class is deeply related with the holonomy of foliations. It is clearly seen in the holomorphic setting by Example 2.4. Suppose that $q = 1$ and that λ_0/λ_1 is not a negative real number in Example 2.4. Then, the foliation $\mathcal{F}$ is transversal to (the image of) the unit sphere in $\mathbb{C}^2$. The formula (5) becomes to, modulo $\mathbb{Z}$,

$$\mathrm{Bott}(\mathcal{F}) = \left(\frac{\lambda_0}{\lambda_1} + \frac{\lambda_1}{\lambda_0} \right) [S^3].$$

It is not difficult to see that the ratios λ_0/λ_1 and λ_1/λ_0 coincide with the multiples by $1/2\pi\sqrt{-1}$ of the logarithms of the linear holonomy along the closed orbits which form the Hopf link in S^3. This can be justified by applying a variant of the theory of residues [14], [4].

§3.   Triviality and Rigidity

Some secondary classes certainly admit continuous variations (deformations) but some others do not.

Definition 3.1. *Let $\rho\colon \mathrm{WO}_{q+1} \to \mathrm{WO}_q$ be the homomorphism which satisfies*

$$\rho(h_i) = \begin{cases} h_i, & i \leq q, \\ 0, & i = q+1, \end{cases}$$

$$\rho(c_j) = \begin{cases} c_j, & j \leq q, \\ 0, & j = q+1. \end{cases}$$

The homomorphism induces a homomorphism on the cohomology, which we denote again by ρ.

There are also similar homomorphisms $\rho^{\mathbb{C}}\colon H^*(\mathrm{WU}_{q+1}) \to H^*(\mathrm{WU}_q)$ and so on.

Definition 3.2. *An element of $H^*(\mathrm{WO}_q)$ is said to be* variable *if it admits continuous variations. Otherwise the element is said to be* rigid.

The following is known [28] (see also [7]).

Theorem 3.3. *The image of ρ consists of rigid classes. The same holds for $\rho^{\mathbb{C}}$, etc.*

Some rigid classes are studied by Baker [13], Kamber–Tondeur [35], Hurder [33], Enatsu [20] and [9], et al. The Godbillon–Vey class is one of the variable classes. However, there are some typical cases where it is trivial.

Definition 3.4. *If $\mathcal{F}$ admits a structure* (*e.g., a geometric one*) *invariant under the holonomy, then $\mathcal{F}$ is said to have a* transverse structure.

Transverse structures are often easy to describe in terms of local holonomies. Namely, let $\{U_i\}$ be a foliation atlas which satisfies certain technical conditions such as simplicity. Let γ_{ji} be the transition function in the transversal direction from U_i to U_j. We call γ_{ji}'s *local holonomies* in this article. A foliation $\mathcal{F}$ admits a transverse structure if and only if there is a structure invariant under local holonomies. These notions become clear if we introduce *pseudogroups* of holonomy. We refer to [25] for pseudogroups and foliations.

Example 3.5.   1)   If $Q(\mathcal{F})$ admits a Riemannian metric invariant under the holonomy, then $\mathcal{F}$ is said to be (transversely) *Riemannian*.

2)   If local holonomies are biholomorphic diffeomorphisms, then $\mathcal{F}$ is said to be *transversely holomorphic*.

There is a Čech–de Rham cocycle which represents the Godbillon–Vey class.

Theorem 3.6 (Bott [15], Mizutani [44])**.** *The Čech–de Rham cochain*

$$(\log J\gamma) \cup (d\log J\gamma)^q$$

is a cocycle which represents $\mathrm{GV}(\mathcal{F})$ *up to multiplications of non-zero constants, where $J\gamma$ is a Čech–de Rham $(1,0)$-cochain defined by $(J\gamma)_{ij} = \det D\gamma_{ij}$.*

The following is a quite well-known

Lemma 3.7.   1)   *If $\mathcal{F}$ is Riemannian, then we can find a Bott connection which is also a metric connection.*

2) If $\mathcal{F}$ is transversely affine, then we may assume that
    $d \log J\gamma = 0$.

Therefore, we have the following

Theorem 3.8.   1)   If $\mathcal{F}$ is Riemannian, then $\mathrm{GV}(\mathcal{F})$ is trivial.
    More generally, secondary classes defined by means of $H^*(\mathrm{WO}_q)$
    are trivial.
2) If $\mathcal{F}$ is transversely affine, then $\mathrm{GV}(\mathcal{F})$ is trivial.

Theorem 3.8 implies that there is a room for defining characteristic
classes other than those which defined by means of $H^*(\mathrm{WO}_q)$ if we
consider foliations with transverse structures. There are indeed such
ones. We refer to Lazarov–Pasternack [40], [41], Morita [46] (see also
Nishikawa–Sato [47]).

In view of Theorem 3.8, it is natural to expect transverse structures
of foliations also have something to do with the rigidity of the Godbillon–
Vey class. Indeed, we have the following

Theorem 3.9 ([7]).   1)   The Godbillon–Vey class is non-trivial
    in the category of transversely holomorphic foliations. That
    is, for each q, there is a transversely holomorphic foliation
    of complex codimension-q of which the Godbillon–Vey class is
    non-trivial.
2) The Godbillon–Vey class is rigid in the category of transversely
    holomorphic foliations of any complex codimension.

The first part of Theorem 3.9 is shown by constructing examples.
Some of them are modifications of the example of Roussarie (Ex-
ample 2.1). The rigidity is shown as follows. In the category of
transversely holomorphic foliations, there is a natural homomorphism
from $H^*(\mathrm{WO}_{2q})$ to $H^*(\mathrm{WU}_q)$ which corresponds to forgetting transverse
holomorphic structures [49], [3]. The Godbillon–Vey class does not
belong to the image of ρ defined in Definition 3.1 of course, but to the
image of $\rho^{\mathbb{C}}$. This is shown by using Theorem 1.11.

If we study transversely holomorphic foliations of closed manifolds,
of complex codimension-one, then we can define Julia sets which are
analogous to those of rational mappings. There are at least two defin-
itions. One is due to Ghys, Gomez-Mont and Saludes [22] (see also [26])
which makes use of deformations of foliations. Another is in [8] defined
by means of normal families. If we denote the former by $J_{\mathrm{GGS}}(\mathcal{F})$ and
the latter $J(\mathcal{F})$, then they satisfy $J(\mathcal{F}) \subset J_{\mathrm{GGS}}(\mathcal{F})$, where the equality
fails for example if there are holonomies which are non-trivial rotations.
We have the following

Theorem 3.10 ([8], cf. Theorem 2.7). *Let M be a closed manifold and $\mathcal{F}$ a transversely holomorphic foliation of M, of complex codimension-one. If $J(\mathcal{F}) = \varnothing$, then the imaginary part of the Bott class and Godbillon–Vey class of $\mathcal{F}$ are trivial.*

Since $J(\mathcal{F}) = \varnothing$ if $J_{\mathrm{GGS}}(\mathcal{F}) = \varnothing$, Theorem 3.10 is also valid for $J_{\mathrm{GGS}}(\mathcal{F})$. The most reason for which Theorem 3.10 holds is that $\mathcal{F}$ is transversely Hermitian (and hence Riemannian) if $J(\mathcal{F})$ is empty. More precisely, we can show that $\mathcal{F}$ is transversely Hermitian on the Fatou set which is the complement of the Julia set. Thus the Julia set in the sense of [8] is closely related with the transversal metric properties of $\mathcal{F}$. On the other hand, the Julia sets of Ghys, Gomez-Mont and Saludes naturally appear when we study deformations. Before giving an example (Example 3.16), we introduce some definitions.

Definition 3.11 (Heitsch [27], see also [38]). *An* infinitesimal deformation *of $\mathcal{F}$ is an element of $H^1(M; \Theta_{\mathcal{F}})$, where $\Theta_{\mathcal{F}}$ denotes the sheaf of the germs of foliated sections to $Q(\mathcal{F})$.*

If $\{\mathcal{F}_s\}$ is a smooth 1-parameter family of codimension-q foliations with $\mathcal{F}_0 = \mathcal{F}$, then it is known that $\{\mathcal{F}_s\}$ determines an infinitesimal deformation of $\mathcal{F}$, which is indeed the derivative of the family at $s = 0$.

Theorem 3.12 (Heitsch [27], [28]). *There is a well-defined bilinear mapping $D \colon H^1(M; \Theta_{\mathcal{F}}) \times H^*(\mathrm{WO}_q) \to H^*(M)$ such that if $\dot{\omega} \in H^1(M; \Theta_{\mathcal{F}})$ is induced by a 1-parameter family $\{\mathcal{F}_s\}$ and if $\alpha \in H^*(\mathrm{WO}_q)$, then $D_{\dot{\omega}}\alpha = \frac{\partial}{\partial s}\alpha(\mathcal{F}_s)\big|_{s=0}$.*

We call $D_{\dot{\omega}}\alpha$ the *derivative of α with respect to $\dot{\omega}$*, or *infinitesimal derivative of α* for short. We refer to [18], [23] and [7] for the transversely holomorphic case.

Definition 3.13. *A class $\alpha \in H^*(\mathrm{WO}_q)$ is said to be* infinitesimally rigid *if $D_{\dot{\omega}}\alpha = 0$ for any M, $\mathcal{F}$ and $\dot{\omega} \in H^*(M; \Theta_{\mathcal{F}})$. If we restrict $\mathcal{F}$ to a certain category, then we say that α is infinitesimally rigid in that category.*

Rigidity as well as infinitesimal rigidity are often discussed in a fixed category, e.g., transversely holomorphic foliations, etc. However, as infinitesimal derivatives are kind of directional derivatives, we can also discuss infinitesimal rigidity with respect to infinitesimal deformations towards a large category by appropriately choosing $\Theta_{\mathcal{F}}$. For example, we can discuss infinitesimal rigidity of the Godbillon–Vey class of transversely projective foliations with respect to infinitesimal deformations as real foliations which need not admit transversal projective structures. See also Corollary 3.17 and succeeding remarks. We also remark that

given a differentiable one-parameter family of foliations, we can obtain an infinitesimal deformation by differentiation. Therefore, if the infinitesimal rigidity is shown in a category of foliations, then the rigidity follows from it by integration. We have the following

Theorem 3.14 ([43], [6]). *If $\mathcal{F}$ admits transverse projective structures, then $\mathrm{GV}(\mathcal{F})$ is infinitesimally rigid. Therefore, the Godbillon–Vey class is rigid in the category of transversely projective foliations.*

Note that it is well-known that the Godbillon–Vey class is non-trivial in the category of transversely projective foliations [17], [30], [13]. See also Example 2.1.

Theorem 3.14 is firstly shown for flat projective structures by Maszczyk [43] for $q = 1$ and then in [6] for $q \geq 2$. The proofs consist of calculations in the Čech–de Rham complexes.

Recently, the following theorem is shown, from which Theorem 3.14 also follows [12].

Theorem 3.15 ([10]). *We can introduce transverse projective connections on $Q(\mathcal{F})$ in a natural way, and represent the infinitesimal derivative of the Godbillon–Vey class in terms of transverse projective connections.*

It is classical that there are *normal projective connections* associated with projective structures on manifolds [37] (see also [51]). If we denote by Γ^i_{jk} the Christoffel symbols, then the coefficients of the projective normal connection involve the terms of the form $\frac{\partial \Gamma^i_{jk}}{\partial x^l}$, where $(x^1, \ldots, x^q)$ are local coordinates. When foliations are considered, $(x^1, \ldots, x^q)$ become local coordinates in the transversal direction, and the operators $\frac{\partial}{\partial x^i}$ are not well-defined in general. In order to avoid the difficulty, we can make use of Bott connections on $\bigwedge^q Q(\mathcal{F})$. As a result, thus obtained connections are no longer Ricci flat (note that the Ricci curvature does not make sense in general, either). We note that a similar construction was found by Hlavatý [32].

We now give an example which shows the Julia set in the sense of Ghys, Gomez-Mont and Saludes is related with deformations of foliations.

Example 3.16 ([10, Example 4.17]). We study Example 2.4 again. Let (z, w) be the standard coordinates on $\mathbb{C}^2$. If we define a holomorphic vector field X by $X = \lambda z \frac{\partial}{\partial z} + \mu w \frac{\partial}{\partial w}$, where $\lambda, \mu \in \mathbb{C}^*$, then the integral curves of X form a holomorphic foliation of $\mathbb{C}^2 \setminus \{o\}$, where o denotes the origin. We set $M = (\mathbb{C}^2 \times \{o\})/ \times 2 \cong S^1 \times S^3$. We also set $L_z = \{(z, w) \in S^3 \mid w = 0\}$ and $L_w = \{(z, w) \in S^3 \mid z = 0\}$. Note

that $L = L_z \cup L_w$ is the Hopf link. The foliation $\mathcal{F}_s$ naturally induces a foliation of M which we denote by $\mathcal{F}_s$, where $s = \lambda/\mu$. If s is not a negative real number, then $\mathcal{F}_s$ is transversal to the image of the unit sphere in $\mathbb{C}^2$ so that a transversely holomorphic foliation, indeed a flow, is induced on S^3, which we denote by $\mathcal{G}_s$. It is known that the complex normal bundle of $\mathcal{F}_s$ is trivial and that $\mathrm{Bott}_1(\mathcal{F}_s) = \left(s + 2 + \frac{1}{s}\right)[S^3]$, where $[S^3]$ denotes the class in $H^3(M;\mathbb{C})$ induced by the fundamental class of the unit sphere (see [14] and [16]). We also have $\mathrm{Bott}_1(\mathcal{G}_s) = \left(s + 2 + \frac{1}{s}\right)[S^3]$ if $\mathcal{G}_s$ is defined. We have $\frac{d}{ds}\mathrm{Bott}_1(\mathcal{F}_s) = \left(1 - \frac{1}{s^2}\right)[S^3]$. On the other hand, it can be shown that $J(\mathcal{F}_s) = J_{\mathrm{GGS}}(\mathcal{F}_s) = S^1 \times L$ if $s \neq 1$ and $J(\mathcal{F}_s) = J_{\mathrm{GGS}}(\mathcal{F}_s) = \varnothing$ if $s = 1$. In what follows, we mostly work on $\mathcal{G}_s$ so that we assume that $s \notin \mathbb{R}_{<0}$. If $s \notin \mathbb{R}$, then we have $J(\mathcal{G}_s) = J_{\mathrm{GGS}}(\mathcal{G}_s) = L$. If $s \in \mathbb{R} \setminus \left(\mathbb{N} \cup \left\{\frac{1}{n} \,\middle|\, n \in \mathbb{N},\, n > 1\right\}\right)$, then $J(\mathcal{G}_s) = \varnothing$ and $J_{\mathrm{GGS}}(\mathcal{G}_s) = L$. If $s \in \mathbb{N}$ and if $s > 1$, then $J(\mathcal{G}_s) = \varnothing$ and $J_{\mathrm{GGS}}(\mathcal{G}_s) = L_w$. In order to verify the latter equality, note that $z^s \frac{\partial}{\partial w}$ induces a section of $Q(\mathcal{G}_s)$ which is holonomy invariant and non-vanishing on L_z (in fact, it vanishes only on L_w). Similarly, $J(\mathcal{G}_{1/s}) = \varnothing$ and $J_{\mathrm{GGS}}(\mathcal{G}_{1/s}) = L_z$. Finally if $s = 1$, then $J(\mathcal{G}_s) = J_{\mathrm{GGS}}(\mathcal{G}_s) = \varnothing$. We remark that Julia sets can be reduced under pull-backs. By [10, Corollary 4.13], which is a generalization of Theorem 3.14, $\mathcal{F}_s$ admits invariant complex transverse projective structures only if $s = \pm 1$. If $s = 1$, then the leaves of $\mathcal{F}_1$ are the direct product of S^1 and the fibers of the Hopf fibration. Therefore, the complex projective structure of $\mathbb{C}P^1$ gives an invariant transverse projective structure. If $s = -1$, then diffeomorphisms

$$(u, z) \in \mathbb{C}^* \times \mathbb{C} \mapsto \left(u, \frac{z}{u}\right) \in \mathbb{C}^* \times \mathbb{C} \subset \mathbb{C}^2 \setminus \{o\},$$

$$(v, w) \in \mathbb{C}^* \times \mathbb{C} \mapsto \left(\frac{w}{v}, v\right) \in \mathbb{C} \times \mathbb{C}^* \subset \mathbb{C}^2 \setminus \{o\}$$

give rise to a foliation atlas for $\mathcal{F}_{-1}$. The points (u, z) and (v, w) correspond to the same point of M if and only if we have $2^k u = \frac{w}{v}$ and $2^k \frac{z}{u} = v$ for some $k \in \mathbb{Z}$. If this is the case, we have $2^{2k} z = w$. Therefore, the standard complex projective structure on $\mathbb{C} \sqcup \mathbb{C}$, namely, the disjoint union of the z-plane and the w-plane, is invariant under the holonomy of $\mathcal{F}_{-1}$. If $s \in \mathbb{R}_{>0}$, then $\mathcal{G}_s$ admits an invariant transverse Hermitian metric, i.e. a Hermitian metric on $Q(\mathcal{G}_s)$ invariant under the holonomy. Therefore $\mathcal{G}_s$ admits an invariant *real* transverse projective structure. However, $\mathcal{G}_s$ does not admit any invariant *complex* transverse projective structure if $s \neq 1$. An explanation, not a proof, can be given as follows.

Let s be a positive real number. If we define mappings by

$$(\theta, z) \in \mathbb{R} \times \mathbb{C} \mapsto \left(\frac{e^{2\pi\sqrt{-1}\theta}}{\sqrt{1+|z|^2}}, \frac{e^{2\pi\sqrt{-1}s\theta}z}{\sqrt{1+|z|^2}} \right) \in (\mathbb{C}^* \times \mathbb{C}) \cap S^3,$$

$$(\varphi, w) \in \mathbb{R} \times \mathbb{C} \mapsto \left(\frac{e^{2\pi\sqrt{-1}\varphi}|w|^{s-1}w}{\sqrt{1+|w|^{2s}}}, \frac{e^{2\pi\sqrt{-1}s\varphi}}{\sqrt{1+|w|^{2s}}} \right) \in (\mathbb{C} \times \mathbb{C}^*) \cap S^3,$$

then they give rise to a foliation atlas for $\mathcal{G}_s$ because they are covering maps. If (θ, z) and (φ, w) correspond to the same point of S^3, then $zw^s = 1$, where we choose a branch of $\log w$. This relation suggests that $\mathcal{G}_s$ does not admit any invariant complex projective structure unless $s \neq 1$. Note that if $\mathcal{F}_s$ admits an invariant complex projective structure, then it is also the case for $\mathcal{G}_s$ because we can pull back the structure. Working on $\mathcal{F}_s$, we can make use of mappings

$$(u, z) \in \mathbb{C}^* \times \mathbb{C} \mapsto (u, u^s z) \in \mathbb{C}^* \times \mathbb{C},$$
$$(v, w) \in \mathbb{C}^* \times \mathbb{C} \mapsto (vw, v^s) \in \mathbb{C} \times \mathbb{C}^*$$

by choosing branches of $\log u$ and $\log v$, and repeat arguments in a parallel way as above.

Projective structures are geometric structures of second order [36]. From this point of view, Theorem 3.14 can be seen as a version of Theorem 3.8, and we can ask if the Godbillon–Vey class of transversely conformal foliations are rigid or not. Theorem 3.9 for complex codimension-one foliations implies the following

Corollary 3.17. *The Godbillon–Vey class is non-trivial and rigid in the category of real codimension-two, transversely conformal foliations.*

Theorem 3.14 and Corollary 3.17 are slightly different. That is, we need not restrict ourselves in Theorem 3.14 to infinitesimal deformations in the category of transversely projective foliations but we need to stay in the category of transversely conformal foliations in Corollary 3.17.

The Godbillon–Vey class of transversely conformal foliations become trivial if some additional conditions are satisfied. For example, the following is known.

Theorem 3.18 (Tarquini [52])**.** *Let $\mathcal{F}$ be a transversely conformal foliation of a closed manifold, of codimension greater than two. Suppose in addition that $\mathcal{F}$ is transversely real analytic. Then, $\mathcal{F}$ is either transversely flat conformal (Möbius), or Riemannian.*

Corollary 3.19. *Under the same assumptions as in Theorem 3.18, the Godbillon–Vey class is either infinitesimally rigid or trivial.*

It seems at present that Corollary 3.19 is the most general result for transversely conformal foliations. If we restrict ourselves to transversely flat conformal structure, another study is found in [1]. We do not know if there is a foliation of which the Godbillon–Vey class is non-trivial under the assumption of Theorem 3.18. It should be non-Riemannian by Theorem 3.8.

If an infinitesimal deformation of a foliation is given, then infinitesimal derivatives of elements of $H^*(\mathrm{WO}_q)$ are defined as in Theorem 3.12. There are some other classes than derivatives of elements of $H^*(\mathrm{WO}_q)$. For example, there is a class defined by $\left(\frac{-1}{2\pi}\right)^{q+2}(q+1)\dot\theta\wedge\theta\wedge(d\theta)^q$, where $\dot\theta$ is the derivative of θ with respect to the given infinitesimal deformation [21], [42], [39]. We call this class the *Fuks–Lodder–Kotschick class* in [7]. It seems that the (non-)triviality of the class is still unknown. We remark that the Fuks–Lodder–Kotschick class can be defined for transversely holomorphic foliations with trivialized complex normal bundles and that this class is known to be non-trivial and depends on the homotopy type of the trivialization [7]. The infinitesimal derivatives and other classes such as the Fuks–Lodder–Kotschick class can be defined also by means of a certain DGA and connections on 2-normal bundles.

Theorem 3.20 ([11])**.** *There is a DGA which we call* DWO_q, *and a characteristic mapping from* $H^*(\mathrm{DWO}_q)$ *to* $H^*(M)$ *which can be calculated by means of connections on 2-normal bundles. The image of the characteristic mapping contains secondary classes defined by means of* $H^*(\mathrm{WO}_q)$ *as well as their infinitesimal derivatives. The Fuks–Lodder–Kotschick class also belongs to the image.*

Finally we make some remarks on the regularity of foliations and their conjugacies. First, it is known that the Godbillon–Vey class does not make sense in the category of C^1-foliations as shown by Tsuboi [54]. On the other hand, once the Godbillon–Vey class is defined, then it is invariant under C^1-conjugacies [48], [2]. It seems still unknown if the Godbillon–Vey class is invariant under foliation preserving homeomorphisms. In the holomorphic setting, it seems unknown if the Bott class is invariant under smooth conjugacies. If the complex codimension is equal to one, then it is known that the Bott class is *not* invariant under transversely quasiconformal homeomorphisms which preserves foliations. Indeed, if we assume $q = 1$ and λ_0/λ_1 is not a real number in Example 2.4, then we can construct a quasiconformal deformation (which are not smooth) from one to another [5].

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