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# Character sheaves on exotic symmetric spaces and Kostka polynomials

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**Abstract.**

This paper is a survey on a joint work with K. Sorlin concerning the theory of character sheaves on the exotic symmetric space. After explaining the historical background, we introduce character sheaves on this variety. By using those character sheaves, we show that modified Kostka polynomials, indexed by a pair of double partitions, can be interpreted in terms of the intersection cohomology of the orbits in the exotic nilpotent cone, as conjectured in Achar-Henderson.

## §1. Introduction

This paper is a survey on a joint work with K. Sorlin [SS1], [SS2] concerning the theory of character sheaves on the exotic symmetric space. The theory of character sheaves on reductive groups was established by Lusztig [L2] in 1980's for computing irreducible characters of finite reductive groups in a uniform way. The starting point of this theory is an article [L1] of Lusztig in 1981, where he gave a geometric interpretation of Kostka polynomials in terms of the intersection cohomology of nilpotent orbits. In recent years, it was noticed by several people that certain varieties with group action enjoy the theory of character sheaves quite similar to the case of $GL_n$. In those cases, some variant of Kostka polynomials appear, and an analogue of Luszitg's result on geometric realization of Kostka polynomials holds.

The exotic symmetric space is a variety $G/H \times V$ on which $H$ acts diagonally, where $G = GL(V) \simeq GL_{2n}$ and $H = Sp(V) = Sp_{2n}$. We show that there exists a well-satisfied theory of character sheaves on it. Note that Ginzburg [Gi] developed the theory of character sheaves on symmetric spaces in general, as a generalization of reductive groups.

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Achar-Henderson [AH], and Finkelberg-Ginzburg-Travkin [FGT] studied the case of $G\times V$ with diagonal $G$-action, where $G = GL(V) \simeq GL_n$. The exotic symmetric space is a generalization of both of $G \times V$ and $GL_{2n}/Sp_{2n}$. Our point of view is that the most appropriate generalization of $GL_n$ is the exotic symmetric space, and $G \times V$ or $GL_{2n}/Sp_{2n}$ should be regarded just as an intermediate step.

In the former half of this paper, we give a historical background on the theory of character sheaves in connection with Kostka polynomials, and in the latter half, we explain the theory of character sheaves on the exotic symmetric space, in particular, the Springer correspondence and a geometric realization of Kostka polynomials.

## §2. Geometric realization of Kostka polynomials

**2.1.** Let $\mathcal{P}_n$ be the set of partitions of $n$. For each partition $\lambda \in \mathcal{P}_n$, Schur function $s_\lambda(x)$ can be defined, which is a homogeneous symmetric function of $x : x_1, x_2, \dots$ of degree $n$. The set $\{s_\lambda(x) \mid \lambda \in \mathcal{P}_n\}$ gives a $\mathbf{Z}$-basis of the $\mathbf{Z}$-module $\Lambda_n$ of homogeneous symmetric functions of degree $n$. For each $\mu \in \mathcal{P}_n$, the Hall-Littlewood function $P_\mu(x;t)$ can be defined, which is a symmetric function of $x$ with a parameter $t$. The set $\{P_\mu(x;t) \mid \mu \in \mathcal{P}_n\}$ gives rise to a $\mathbf{Z}[t]$-basis of the $\mathbf{Z}[t]$-module $\mathbf{Z}[t]\otimes_{\mathbf{Z}}\Lambda_n$. *Kostka polynomials* $K_{\lambda,\mu}(t) \in \mathbf{Z}[t]$ are defined as the coefficients of the transition matrix between two bases, consisting of Shcur functions and of Hall-Littlewood functions, namely we have

$$s_\lambda(x) = \sum_{\mu\in\mathcal{P}_n} K_{\lambda,\mu}(t)P_\mu(x;t).$$

For each $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathcal{P}_n$, we define $n(\lambda) \in \mathbf{Z}$ by $n(\lambda) = \sum_{i\ge 1}(i-1)\lambda_i$. Put $\widetilde{K}_{\lambda,\mu}(t) = t^{n(\mu)}K_{\lambda,\mu}(t^{-1})$. One can show that $\widetilde{K}_{\lambda,\mu}(t)$ are again polynomials in $t$, and we call $\widetilde{K}_{\lambda.\mu}(t)$ *modified Kostka polynomials.*

**2.2.** In 1981 Lusztig gave a geometric realization of Kostka polynomials in terms of the intersection cohomology associated to the closure of nilpotent orbits. Let $V$ be an $n$-dimensional vector space over $\mathbf{C}$, and $G = GL(V) \simeq GL_n$. Let $\mathcal{N} = \mathcal{N}_G$ be the nilpotent cone of $G$, namely, the set of nilpotent transformations in $\operatorname{End} V$. The conjugation action of $G$ on $\operatorname{End} V$ preserves $\mathcal{N}$. It is well-known that the set $\mathcal{N}/G$ of $G$-orbits in $\mathcal{N}$ is naturally in bijection with $\mathcal{P}_n$ via the Jordan's normal form of elements in $\mathcal{N}$, namely, we have

$$\mathcal{N}/G \leftrightarrow \mathcal{P}_n.$$

We denote by $\mathcal{O}_\lambda$ th $G$-orbit containing $x \in \mathcal{N}$ whose Jordan type is $\lambda$. Let $\overline{\mathcal{O}}_\lambda$ be the Zariski closure of $\mathcal{O}_\lambda$ in $\mathcal{N}$. We have

$$\overline{\mathcal{O}}_\lambda = \coprod_{\mu \le \lambda} \mathcal{O}_\mu,$$

where $\mu \le \lambda$ is the dominance order on $\mathcal{P}_n$ defined as follows; $\mu \le \lambda$ if $\sum_{i=1}^j \mu_i \le \sum_{i=1}^j \lambda_i$ for each $j$, here we write $\lambda$ as $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ and $\mu = (\mu_1, \mu_2, \cdots, \mu_k)$ with $\lambda_k \ge 0, \mu_k \ge 0$.

We consider the intersection cohomology $K = \mathrm{IC}(\overline{\mathcal{O}}_\lambda, \mathbf{C})$ of the closure of $\mathcal{O}_\lambda$. We denote by $\mathcal{H}^i K$ the $i$th cohomology sheaf of $K$, and denote by $\mathcal{H}^i_x K$ the stalk of $\mathcal{H}^i K$ at $x \in \overline{\mathcal{O}}_\lambda$. Then $\mathcal{H}^i_x K$ is a finite dimensional vector space over $\mathbf{C}$. Lusztig proved the following result, which gives a geometric realization of Kostka polynomials.

**Theorem 2.3** (Lusztig [L1]). *For $\lambda \in \mathcal{P}_n$, put $K = \mathrm{IC}(\overline{\mathcal{O}}_\lambda, \mathbf{C})$.*

(i) *$\mathcal{H}^i K = 0$ for odd $i$.*

(ii) *For each $x \in \mathcal{O}_\mu \subset \overline{\mathcal{O}}_\lambda$, we have*

$$\widetilde{K}_{\lambda,\mu}(t) = t^{n(\lambda)} \sum_{i \ge 0} (\dim \mathcal{H}^{2i}_x K) t^i. \tag{2.3.1}$$

## §3. Springer correspondence for $GL_n$ and Kostka polynomials

**3.1.** We pass from $\mathbf{C}$ to a field $k$, an algebraic closure of a finite field $\mathbf{F}_q$ with characteristic $p$. Let $V$ be a vector space of dimension $n$ over $k$. We consider an algebraic group $G = GL(V)$ and the nilpotent cone $\mathcal{N}_G$. Then the discussion on nilpotent orbits in Section 2 can be transported to this case without change. Lusztig's theorem also holds if one replaces the intersection cohomology with $\mathbf{C}$-sheaves by the intersection cohomology with $\bar{\mathbf{Q}}_l$-sheaves. Here $\bar{\mathbf{Q}}_l$ is an algebraic closure of the $l$-adic number field $\mathbf{Q}_l$ for $l \ne p$.

**3.2.** Let $G_{\mathrm{uni}}$ be the set of unipotent elements in $G$. Then $G_{\mathrm{uni}}$ is isomorphic to $\mathcal{N}_G$ with $G$-action. We denote by the same symbol $\mathcal{O}_\lambda$ the unipotent class of $G$ corresponding to $\lambda \in \mathcal{P}_n$. The Weyl group of $G$ is isomorphic to the symmetric group $S_n$ of degree $n$. Let $S_n^\wedge$ be the set of irreducible representations of $S_n$ over $\bar{\mathbf{Q}}_l \simeq \mathbf{C}$, up to isomorphism. It is well-known that $S_n^\wedge$ is parametrized by $\mathcal{P}_n$. We denote by $V_\lambda$ the irreducible $S_n$-module corresponding to $\lambda$ (here we use the parametrization so that $V_{(n)}$ is the identity representation and $V_{(1^n)}$ is the sign representation). Let $G_{\mathrm{uni}}/G$ be the set of unipotent classes in $G$. Then $G_{\mathrm{uni}}/G$ and $S_n^\wedge$ are both parametrized by $\mathcal{P}_n$. The Springer correspondence for

$GL_n$, which was first established by Lusztig [L1], gives a natural bijective correspondence between $G_{\text{uni}}/G$ and $S_n^\wedge$. (Later it was extended to the case of connected reductive groups in general by Springer [Sp].) We will explain below the case of $GL_n$ following the formulation of Borho-MacPherson [BM].

Let $B = TU$ be a Borel subgroup of $G$, where $T$ is a maximal torus of $B$ and $U$ is the unipotent radical of $B$. We identify the Weyl group of $G$ with $N_G(T)/T$. Let $T_{\text{reg}}$ be the set of regular semisimple elements in $T$, and $G_{\text{reg}} = \bigcup_{g\in G} gT_{\text{reg}}g^{-1}$ the set of regular semisimple elements in $G$. Then $G_{\text{reg}}$ is a smooth open subset of $G$. Let $\mathcal{B} = G/B$ be the flag variety of $G$. Put

$$\widetilde{G} = \{(x, gB) \in G \times \mathcal{B} \mid g^{-1}xg \in B\},$$
$$\widetilde{G}_{\text{uni}} = \{(x, gB) \in G_{\text{uni}} \times \mathcal{B} \mid g^{-1}xg \in U\},$$

and let $\pi : \widetilde{G} \to G, \pi_1 : \widetilde{G}_{\text{uni}} \to G_{\text{uni}}$ be the projections to the first factors. $\pi, \pi_1$ are proper maps. Since $\widetilde{G}_{\text{uni}} \simeq G \times^B U$, $\widetilde{G}_{\text{uni}}$ is a smooth irreducible variety of the same dimension as $G_{\text{uni}}$. It can be proved that $\pi_1$ gives a resolution of singularities of $G_{\text{uni}}$, and is called the Springer resolution of $G_{\text{uni}}$. Let $\pi_0 : \pi^{-1}(G_{\text{reg}}) \to G_{\text{reg}}$ be the restriction of $\pi$ on $\pi^{-1}(G_{\text{reg}})$. Then $\pi_0$ is a finite Galois covering with group $S_n$, and the direct image $(\pi_0)_*\bar{\mathbf{Q}}_l$ ($\bar{\mathbf{Q}}_l$ is the constant sheaf on $\pi^{-1}(G_{\text{reg}})$) is a semisimple local system equipped with $S_n$-action, and is decomposed into simple components as

$$(\pi_0)_*\bar{\mathbf{Q}}_l \simeq \bigoplus_{\lambda\in\mathcal{P}_n} V_\lambda \otimes \mathcal{L}_\lambda,$$

where $\mathcal{L}_\lambda$ is a simple local system on $G_{\text{reg}}$ corresponding to $\lambda$. Lusztig proved that the intersection cohomology $\text{IC}(G, (\pi_0)_*\bar{\mathbf{Q}}_l)$ is isomorphic to $\pi_*\bar{\mathbf{Q}}_l$ as objects in the derived category of $\bar{\mathbf{Q}}_l$-sheaves on $G$. It follows that $\pi_*\bar{\mathbf{Q}}_l[\dim G]$ is a semisimple perverse sheaf equipped with $S_n$-action, and is decomposed as

$$\pi_*\bar{\mathbf{Q}}_l[\dim G] \simeq \bigoplus_{\lambda\in\mathcal{P}_n} V_\lambda \otimes \text{IC}(G, \mathcal{L}_\lambda)[\dim G]. \tag{3.2.1}$$

Since $(\pi_1)_*\bar{\mathbf{Q}}_l \simeq \pi_*\bar{\mathbf{Q}}_l|_{G_{\text{uni}}}$, the $S_n$-action on $\pi_*\bar{\mathbf{Q}}_l$ is inherited to $(\pi_1)_*\bar{\mathbf{Q}}_l$. The following result is due to Lusztig, Springer, Borho-MacPherson and others.

**Theorem 3.3** (Springer correspondence for $GL_n$)**.** *Under the notation as above,*

(i) $(\pi_1)_*\bar{\mathbf{Q}}_l[\dim G_{\text{uni}}]$ *is a semisimple perverse sheaf equipped with* $S_n$*-action, and is decomposed into simple components as*

$$(\pi_1)_*\bar{\mathbf{Q}}_l[\dim G_{\text{uni}}] \simeq \bigoplus_{\lambda\in\mathcal{P}_n} V_\lambda \otimes \mathrm{IC}(\overline{\mathcal{O}}_\lambda, \bar{\mathbf{Q}}_l)[\dim \mathcal{O}_\lambda]. \tag{3.3.1}$$

(ii) *For each* $\lambda \in \mathcal{P}_n$, $\mathrm{IC}(G, \mathcal{L}_\lambda)|_{G_{\text{uni}}} \simeq \mathrm{IC}(\overline{\mathcal{O}}_\lambda, \bar{\mathbf{Q}}_l)$ *up to shift of complexes.*

**3.4.** For each $x \in G_{\text{uni}}$, put $\mathcal{B}_x = \{gB \in \mathcal{B} \mid g^{-1}xg \in B\}$. $\mathcal{B}_x$ is a closed subvariety of $\mathcal{B}$, and is isomorphic to $\pi_1^{-1}(x)$. Hence $\mathcal{B}_x$ is called the *Springer fibre* of $x$. Since $K = (\pi_1)_*\bar{\mathbf{Q}}_l$ is a complex with $S_n$-action, its stalk $\mathcal{H}^i_xK$ turns out to be an $S_n$-module. Thus

$$\mathcal{H}^i_xK \simeq H^i(\pi_1^{-1}(x), \bar{\mathbf{Q}}_l) \simeq H^i(\mathcal{B}_x, \bar{\mathbf{Q}}_l) \tag{3.4.1}$$

has a structure of $S_n$-modules. The $S_n$-actions on the cohomologies $H^i(\mathcal{B}_x, \bar{\mathbf{Q}}_l)$ are called the *Springer representations* of $S_n$, which were first defined by Springer [Sp]. Let $d_x = \dim \mathcal{B}_x$. We consider the top cohomology group $H^{2d_x}(\mathcal{B}_x, \bar{\mathbf{Q}}_l)$. The following result is the original form of the Springer correspondence proved by Springer [Sp].

**Corollary 3.5.** *Assume that* $x \in \mathcal{O}_\lambda$. *Then* $H^{2d_x}(\mathcal{B}_x, \bar{\mathbf{Q}}_l) \simeq V_\lambda$. *In this way, any irreducible* $S_n$*-module* $V_\lambda$ *is realized as the Springer representation of the top cohomology of the Springer fibre.*

**3.6.** We will now give an alternate description of Kostka polynomials in terms of the Springer representations. Note that only the top cohomology groups were concerned with the Springer correspondence. However, the lower cohomology groups also play a role for the description of Kostka polynomials. Let $K = (\pi_1)_*\bar{\mathbf{Q}}_l$. Since $\dim \mathcal{O}_\lambda$ and $\dim G_{\text{uni}}$ are even, $\mathcal{H}^iK = 0$ for odd $i$ by Theorem 2.3 (i). By taking the stalk of the $i$-th cohomologies on the both sides of (3.3.1), and by comparing with (2.3.1), we have

$$\sum_{i\ge 0}(\mathcal{H}^{2i}_xK)t^i = \bigoplus_{\lambda\in\mathcal{P}_n} V_\lambda \otimes \widetilde{K}_{\lambda,\mu}(t)$$

for $x \in \mathcal{O}_\mu$ (here we regard this formula as an equality in the Grothendieck group of the representations of $S_n$, tensored by $\mathbf{Z}[t]$). Taking the inner product with $V_\lambda$ on both sides, and by taking (3.4.1) into account, we have the following result.

**Proposition 3.7.** *Assume that $x \in \mathcal{O}_\mu$. Then we have*

$$\widetilde{K}_{\lambda,\mu}(t) = \sum_{i \ge 0} \langle V_\lambda, H^{2i}(\mathcal{B}_x, \bar{\mathbf{Q}}_l)\rangle_{S_n} t^i,$$

*where* $\langle \cdot, \cdot \rangle_{S_n} = \dim \mathrm{Hom}_{S_n}(\cdot, \cdot)$.

## §4. The enhanced nilpotent cone and Kostka polynomials

**4.1.** The results in previous sections are already classical. They are all obtained in 1980's. Recently, however, interesting variants of this theme were found by several people. In this section, we shall explain about the enhanced nilpotent cone introduced by Achar-Henderson [AH] and, independently, by Finkelberg-Ginzburg-Travkin [FGT].

**4.2.** We follow the notation in 3.1, so $G = GL(V)$ and $\mathcal{N}$ is the nilpotent cone. We consider the product variety $\mathcal{N} \times V$, on which $G = GL(V)$ acts diagonally. We denote by $(\mathcal{N} \times V)/G$ the set of $G$-orbits on $\mathcal{N} \times V$. Let $\mathcal{P}_{n,2}$ be the set of pair of partitions $(\lambda, \mu)$ such that $|\lambda| + |\mu| = n$. The following interesting fact was found by Achar-Henderson and by Travkin independently.

**Proposition 4.3.** *$\mathcal{N} \times V$ has a finitely many $G$-orbits. There exists a natural bijection*

$$(\mathcal{N} \times V)/G \leftrightarrow \mathcal{P}_{n,2}.$$

**4.4.** The variety $\mathcal{N} \times V$ is considered as a certain generalization of nilpotent cone. Following [AH], we call $\mathcal{N} \times V$ the *enhanced nilpotent cone* . The correspondence in the proposition is given as follows; for a given $(x, v) \in \mathcal{N} \times V$, put $E^x = \{y \in \mathrm{End}(V) \mid xy = yx\}$. Then $E^x$ is a subalgebra of $\mathrm{End}(V)$, and $W = E^x v$ is an $x$-stable subspace of $V$. Let $\lambda^{(1)}$ be the Jordan type of $x|_W$, and $\lambda^{(2)}$ be the Jordan type of $x|_{V/W}$. Then $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}) \in \mathcal{P}_{n,2}$, and the correspondence $(x, v) \mapsto \boldsymbol{\lambda}$ gives the required correspondence. We denote by $\mathcal{O}_{\boldsymbol{\lambda}}$ the $G$-orbit in $\mathcal{N} \times V$ corresponding to $\boldsymbol{\lambda} \in \mathcal{P}_{n,2}$.

**4.5.** The closure relations on the $G$-orbits in $\mathcal{N} \times V$ were described by Achar-Henderson. We define a partial order $\boldsymbol{\mu} \le \boldsymbol{\lambda}$ on $\mathcal{P}_{n,2}$ as follows; for $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$, we write them as

$$\lambda^{(1)} = (\lambda^{(1)}_1, \lambda^{(1)}_2, \dots, \lambda^{(1)}_k), \quad \lambda^{(2)} = (\lambda^{(2)}_1, \lambda^{(2)}_2 \dots, \lambda^{(2)}_k)$$

for some $k$, by allowing 0 as parts. We define a composition $c(\boldsymbol{\lambda})$ of $n$ by

$$c(\boldsymbol{\lambda}) = (\lambda^{(1)}_1, \lambda^{(2)}_1, \lambda^{(1)}_2, \lambda^{(2)}_2, \dots, \lambda^{(1)}_k, \lambda^{(2)}_k).$$

Then we define $\boldsymbol{\mu} \le \boldsymbol{\lambda}$ by the condition that $c(\boldsymbol{\mu}) \le c(\boldsymbol{\lambda})$, where the latter is the dominance order on compositions of $n$ (defined by a similar rule as partitions). The closure relations for $\mathcal{O}_{\boldsymbol{\lambda}}$ are given as follows;

$$\overline{\mathcal{O}}_{\boldsymbol{\lambda}} = \coprod_{\boldsymbol{\mu} \le \boldsymbol{\lambda}} \mathcal{O}_{\boldsymbol{\mu}}.$$

**4.6.** A generalization of Kostka polynomials $K_{\lambda,\mu}(t)$ was introduced by [S] in 2004. One can define polynomials $K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) \in \mathbf{Q}[t]$ indexed by $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_{n,2}$ as coefficients of the transition matrix between the bases of Schur functions $s_{\boldsymbol{\lambda}}(x,y)$ and of Hall-Littlewood functions $P_{\boldsymbol{\mu}}(x,y;t)$ suitably defined as symmetric functions on two types of variables, $x : x_1, x_2, \dots, y : y_1, y_2, \dots$. Here the definition of Schur functions is standard, but the construction of Hall-Littlewood functions is rather complicated. As in the case of Kostka polynomials, we define a modified Kostka polynomial $\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) \in \mathbf{Z}[t]$ by $\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = t^{a(\boldsymbol{\mu})} K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t^{-1})$, where $a(\boldsymbol{\lambda})$ is defined for $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}) \in \mathcal{P}_{n,2}$ as

$$a(\boldsymbol{\lambda}) = 2n(\lambda^{(1)}) + 2n(\lambda^{(2)}) + |\lambda^{(2)}|. \tag{4.6.1}$$

As in the case of nilpotent orbits, we consider the intersection cohomology $\mathrm{IC}(\overline{\mathcal{O}}_{\boldsymbol{\lambda}}, \bar{\mathbf{Q}}_l)$ associated to the closure $\overline{\mathcal{O}}_{\boldsymbol{\lambda}}$ of a $G$-orbit $\mathcal{O}_{\boldsymbol{\lambda}}$. The following result, which is an analogue of the geometric realization of Kostka polynomials, was proved by Achar-Henderson [AH] in 2008.

**Theorem 4.7** (Achar-Henderson)**.** *For $\boldsymbol{\lambda} \in \mathcal{P}_{n,2}$, put $K = \mathrm{IC}(\overline{\mathcal{O}}_{\boldsymbol{\lambda}}, \bar{\mathbf{Q}}_l)$.*

(i) *$\mathcal{H}^i K = 0$ for odd $i$.*

(ii) *For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_{n,2}$, and $(x,v) \in \mathcal{O}_{\boldsymbol{\mu}} \subset \overline{\mathcal{O}}_{\boldsymbol{\lambda}}$, we have*

$$\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = t^{a(\boldsymbol{\lambda})} \sum_{i \ge 0} (\dim \mathcal{H}^{2i}_{(x,v)} K) t^{2i}. \tag{4.7.1}$$

**Remarks 4.8.** (i) It is known by Green [Gr] that Kostka polynomials $K_{\lambda,\mu}(t)$ are closely related to the representation theory of $GL_n(\mathbf{F}_q)$. Thus it is natural to expect their relationship with the geometry of $GL_n$. However, the construction of Kostka polynomials $K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$ is purely combinatorial. Although it is related to the representation theory of the Weyl group of type $C_n$, there exists no direct relationship with groups such as $GL_n$ or $Sp_{2n}$. So the theorem connecting $K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$ with the geometry of enhanced nilpotent cone seems to be very interesting.

(ii) The special feature in the case of $GL_n$ is that the isotropy group $Z_G(x)$ is connected for each $G$-orbit $\mathcal{O}_\lambda$ containing $x$. Hence only the constant sheaf $\bar{\mathbf{Q}}_l$ on $\mathcal{O}_\lambda$ is concerned, and the twisted local systems on $\mathcal{O}_\lambda$ do not appear in the results in §2, §3. A similar phenomenon occurs for $\mathcal{N} \times V$, namely the isotropy group $Z_G(x,v)$ is connected for each $(x,v)$, and only the constant sheaf $\bar{\mathbf{Q}}_l$ on $\mathcal{O}_{\boldsymbol{\lambda}}$ are involved in the theorem.

(iii) The remarkable property in Theorem 4.7 compared to Theorem 2.3 is that the $t^i$-term corresponding to the stalk of $2i$-th cohomology sheaf in the case of $GL_n$ is replaced by $t^{2i}$-term. As is explained later in 8.1, the characteristic function of $K = \mathrm{IC}(\overline{\mathcal{O}}_{\boldsymbol{\lambda}}, \bar{\mathbf{Q}}_l)$ is closely related to the sum $\sum_{i \ge 0} (\dim \mathcal{H}^{2i}_{(x,v)} K) t^i$. Then the formula (4.7.1) is understood that the characteristic function of $K$ corresponds, up to the factor $t^{a(\boldsymbol{\lambda})}$, to the modified Kostka polynomial $\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t^{1/2})$.

**4.9.** As in the case of $GL_n$, the enhanced nilpotent cone $\mathcal{N} \times V$ is isomorphic to $G_{\mathrm{uni}} \times V$, and the latter is embedded in $G \times V$. $G$ acts diagonally on $\mathcal{Z} = G \times V$, and $\mathcal{Z}_1 = G_{\mathrm{uni}} \times V$ is an $G$-stable subset of $\mathcal{Z}$, which has a role of unipotent variety in the case of $GL_n$. We denote by the same symbol $\mathcal{O}_{\boldsymbol{\lambda}}$ the $G$-orbit in $\mathcal{Z}_1$ corresponding to $\boldsymbol{\lambda} \in \mathcal{P}_{n,2}$. Finkelberg-Ginzburg-Travkin [FGT] introduced character sheaves on $\mathcal{Z}$ as certain $G$-equivariant simple perverse sheaves (mirabolic character sheaves in their terminology), and proved the following result, which is an analogue of Theorem 3.3. For $0 \le m \le n$, define a variety $\widetilde{\mathcal{Z}}_1^{(m)}$ by

$$\widetilde{\mathcal{Z}}_1^{(m)} = \{(x, v, gB) \in G_{\mathrm{uni}} \times V \times G/B \mid g^{-1}xg \in U, g^{-1}v \in V_m\},$$

where $V_1 \subset V_2 \subset \cdots \subset V_n = V$ is a total flag in $V$ stabilized by $B$. We define a map $\pi_1^{(m)} : \widetilde{\mathcal{Z}}_1^{(m)} \to \mathcal{Z}_1$ by $(x, v, gB) \mapsto (x, v)$. Then $\pi_1^{(m)}$ is a proper map, and $\operatorname{Im} \pi_1^{(m)}$ is a closed subvariety of $\mathcal{Z}_1$, which we denote by $\mathcal{Z}_{1,m}$. Let $\mathcal{P}(m)$ be the subset of $\mathcal{P}_{n,2}$ consisting of $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ such that $|\lambda^{(1)}| = m, |\lambda^{(2)}| = n - m$. Then irreducible $S_m \times S_{n-m}$-modules are given by $V_{\boldsymbol{\lambda}} = V_{\lambda^{(1)}} \otimes V_{\lambda^{(2)}}$ for $\boldsymbol{\lambda} \in \mathcal{P}(m)$ under the notation of 3.2.

**Theorem 4.10** (Finkelberg-Ginzburg-Travkin [FGT]). *Under the notation as above,* $(\pi_1^{(m)})_* \bar{\mathbf{Q}}_l[\dim \mathcal{Z}_{1,m}]$ *is a semisimple perverse sheaf on* $\mathcal{Z}_{1,m}$ *equipped with* $S_m \times S_{n-m}$*-action, and is decomposed into simple components as*

$$(\pi_1^{(m)})_* \bar{\mathbf{Q}}_l[\dim \mathcal{Z}_{1,m}] \simeq \bigoplus_{\boldsymbol{\lambda} \in \mathcal{P}(m)} V_{\boldsymbol{\lambda}} \otimes \mathrm{IC}(\overline{\mathcal{O}}_{\boldsymbol{\lambda}}, \bar{\mathbf{Q}}_l)[\dim \mathcal{O}_{\boldsymbol{\lambda}}].$$

**Remark 4.11.** Theorem 4.10 is regarded as a kind of Springer correspondence, which gives a bijection

$$\coprod_{0 \le m \le n} (S_m \times S_{n-m})^\wedge \leftrightarrow \mathcal{Z}_1/G.$$

But this is not so satisfactory. $\mathcal{Z}_1/G$ is parameterized by $\mathcal{P}_{n,2}$, and $\mathcal{P}_{n,2}$ is a parameter set for irreducible representations of the Weyl group $W_n$ of type $C_n$. So it will be nicer if one could replace the left hand side by $W_n^\wedge$. In §7, we will show that this certainly occurs in the case of the exotic symmetric space.

## §5. Symmetric space $GL_{2n}/Sp_{2n}$

**5.1.** Let $k$ be an algebraic closure of a finite field $\mathbf{F}_q$ with $\operatorname{ch} k \neq 2$. Let $V$ be a $2n$-dimensional vector space over $k$, and put $G = GL_{2n} \simeq GL(V)$. We denote by $\theta$ the involutive automorphism on $G$ defined by $\theta(g) = J^{-1}({}^tg^{-1})J$, where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ with $I$: the identity matrix of degree $n$. Let $H = G^\theta$ be the group of fixed points by $\theta$, namely $G^\theta = \{g \in G \mid \theta(g) = g\}$. Then $H = Sp_{2n}$, and $G/H$ is a symmetric space over $k$. Let $F : G \to G$ be the (standard) Frobenius map on $G$. Then $H$ is stable by $F$, and we have $G^F = G(\mathbf{F}_q) = GL_{2n}(\mathbf{F}_q)$, $H^F = H(\mathbf{F}_q) = Sp_{2n}(\mathbf{F}_q)$. We define a Hecke algebra $\mathcal{H} = \mathcal{H}(G^F, H^F)$ by $\mathcal{H} = \operatorname{End}_{G^F}(\operatorname{Ind} 1_{H^F})$, where $1_{H^F}$ is the trivial representation of $H^F$, and $\operatorname{Ind} 1_{H^F}$ is the induced representation to $G^F$ (here we consider representations on $\bar{\mathbf{Q}}_l \simeq \mathbf{C}$). Bannai-Kawanaka-Song [BKS] described completely the character table of $\mathcal{H}$ in 1990, by showing that the character table of $\mathcal{H}$ is closely related to the character table of $GL_n(\mathbf{F}_q)$.

**5.2.** In view of the results of Green [Gr] concerning the character theory of $GL_n(\mathbf{F}_q)$, Theorem 2.3 implies that the character values at unipotent elements of certain irreducible characters of $GL_n(\mathbf{F}_q)$ can be described in terms of the intersection cohomology of unipotent classes. Motivated by this fact, Lusztig developed the theory of character sheaves on connected reductive groups in 1980's. In the case of $GL_n$, the character value at any element of any irreducible character of $GL_n(\mathbf{F}_q)$ can be interpreted in terms of a certain intersection cohomology related to $GL_n$. Thus the theory of character sheaves is regarded as a geometric reconstruction of the result of Green for irreducible characters of $GL_n(\mathbf{F}_q)$.

As a generalization of character sheaves on connected reductive groups, Ginzburg [Gi] introduced the character sheaves on the symmetric spaces. In his Ph.D. thesis [Gro] at MIT, Grojnowski tried to reconstruct geometrically the result of Bannai-Kawanaka-Song on irreducible characters of $\mathcal{H}$ based on the theory of character sheaves on the symmetric space $GL_{2n}/Sp_{2n}$ due to Ginzburg. Grojnowski's idea was made more precise by Henderson. He proved in [H] the counter-part of Theorem 2.3 in the setting for $G/H$, which will be explained below.

**5.3.** Let $\mathfrak{g} = \operatorname{Lie} G$. Then $\theta$ induces a linear automorphism on $\mathfrak{g}$ of order 2, which we denote also by $\theta$. $\mathfrak{g}$ is decomposed as $\mathfrak{g} = \mathfrak{g}^{\theta} \oplus \mathfrak{g}^{-\theta}$, where $\mathfrak{g}^{\pm\theta} = \{X \in \mathfrak{g} \mid \theta(X) = \pm X\}$. Here $\mathfrak{g}^{\theta} \simeq \operatorname{Lie} H$, and $\mathfrak{g}^{-\theta}$ is isomorphic to the tangent space of $G/H$ at $H$. Both $\mathfrak{g}^{\pm\theta}$ are $H$-stable subspaces in $\mathfrak{g}$. We define an $H$-stable subset $\mathfrak{g}^{-\theta}_{\mathrm{nil}}$ of $\mathfrak{g}$ by $\mathfrak{g}^{-\theta}_{\mathrm{nil}} = \mathfrak{g}^{\theta} \cap \mathcal{N}_G$, where $\mathcal{N}_G$ is the nilpotent cone for $G$. It is known that the set of $H$-orbits is in bijection with $\mathcal{P}_n$, namely, we have

$$\mathfrak{g}^{-\theta}_{\mathrm{nil}}/H \leftrightarrow \mathcal{P}_n.$$

We denote by $\mathcal{O}_\lambda$ the $H$-orbit in $\mathfrak{g}^{-\theta}_{\mathrm{nil}}$ corresponding to $\lambda \in \mathcal{P}_n$. Note that $\mathfrak{g}^{-\theta}_{\mathrm{nil}} \subset \mathfrak{g}_{\mathrm{nil}} = \mathcal{N}_G$. In order to distinguish $H$-orbits in $\mathfrak{g}^{-\theta}_{\mathrm{nil}}$ from $G$-orbits in $\mathfrak{g}_{\mathrm{nil}}$, we denote by $\mathbb{O}_\xi$ the $G$-orbit in $\mathfrak{g}$ corresponding to $\xi \in \mathcal{P}_{2n}$. Then we have $\mathcal{O}_\lambda \subset \mathbb{O}_{\lambda\cup\lambda}$, and closure relations for $\mathfrak{g}^{-\theta}_{\mathrm{nil}}$ are compatible with closure relations for $\mathfrak{g}_{\mathrm{nil}}$ under this embedding. In particular, closure relations for $H$-orbits in $\mathfrak{g}^{-\theta}_{\mathrm{nil}}$ are given by the dominance order on $\mathcal{P}_n$, as in the case of nilpotent cone.

Let $\overline{\mathcal{O}}_\lambda$ be the closure of $\mathcal{O}_\lambda$ in $\mathfrak{g}^{-\theta}_{\mathrm{nil}}$, and we consider the intersection cohomology $\mathrm{IC}(\overline{\mathcal{O}}_\lambda, \bar{\mathbf{Q}}_l)$ on $\mathfrak{g}^{-\theta}_{\mathrm{nil}}$. Note that in this case also, the isotropy group of each orbit is connected, and only the constant sheaf $\bar{\mathbf{Q}}_l$ is involved. The following result is due to Henderson [H] in 2008. However, his proof depends in part on the result of [BKS], so it would be incomplete as a geometric reconstruction of [BKS].

**Theorem 5.4** (Henderson). *For $\lambda \in \mathcal{P}_n$, put $K = \mathrm{IC}(\overline{\mathcal{O}}_\lambda, \bar{\mathbf{Q}}_l)$.*

(i) $\mathcal{H}^i K = 0$ *unless* $i \equiv 0 \pmod 4$.

(ii) *For $x \in \mathcal{O}_\mu \subset \overline{\mathcal{O}}_\lambda$, we have*

$$\widetilde{K}_{\lambda,\mu}(t^2) = t^{2n(\lambda)} \sum_{i \geq 0} (\dim \mathcal{H}^{4i}_x) t^{2i}. \tag{5.4.1}$$

**Remarks 5.5.** (i) The modulo 4 vanishing of the cohomology sheaf of $\mathrm{IC}(\overline{\mathcal{O}}_\lambda, \bar{\mathbf{Q}}_l)$ was first discovered by Grojnowski [Gro].

(ii) In view of the characteristic functions (see Remark 4.8 (iii)), the formula (5.4.1) can be understood that the characteristic function of the intersection cohomology $\mathrm{IC}(\overline{\mathcal{O}}_\lambda, \bar{\mathbf{Q}}_l)$ corresponds, up to the $t^{2n(\lambda)}$ factor, to the modified Kostka polynomial $\widetilde{K}_{\lambda,\mu}(t^2)$.

## §6. Exotic symmetric space $GL_{2n}/Sp_{2n} \times V$

**6.1.** So far, we have discussed a generalization or a variant of the geometry related to Kostka polynomials. They are summarized as follows;

Green / Lusztig — Bannai-Kawanaka-Song / Henderson

$$\begin{array}{ccccc}
\begin{matrix}\widetilde{K}_{\lambda,\mu}(t) \\ (\lambda, \mu \in \mathcal{P}_n)\end{matrix} & GL_n & \longrightarrow & GL_{2n}/Sp_{2n} & \begin{matrix}\widetilde{K}_{\lambda,\mu}(t^2) \\ (\lambda, \mu \in \mathcal{P}_n)\end{matrix} \\
 & \downarrow & & \downarrow & \\
\begin{matrix}\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t^{1/2}) \\ (\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_{n,2})\end{matrix} & GL_n \times V_n & \longrightarrow & GL_{2n}/Sp_{2n} \times V & \textbf{???}
\end{array}$$

Achar-Henderson
Finkelberg-Ginzburg-Travkin

In the above diagram, the left vertical arrow means a generalization from $GL_n$ to $GL_n \times V_n$, where $V_n$ is the $n$-dimensional vector space on which $GL_n$ acts, and the upper horizontal arrow means a generalization from $GL_{2n}$ to the symmetric space $GL_{2n}/Sp_{2n} = G/H$. So, it seems natural to consider the variety $G/H \times V$ as a generalization for both directions, where $V$ is a $2n$ dimensional vector space on which $G$ acts. The role of the nilpotent cone $\mathcal{N}$ for $GL_n$ was played by $\mathcal{N} \times V_n$ for $GL_n \times V_n$, and by $\mathfrak{g}^{-\theta}_{\mathrm{nil}} = \mathfrak{g}^{-\theta} \cap \mathcal{N}_G$ for $G/H$. Hence in the case of $G/H \times V$, one can expect the role of the nilpotent cone for the variety $\mathfrak{g}^{-\theta}_{\mathrm{nil}} \times V$. In fact, this variety is isomorphic to the exotic nilpotent cone introduced by Kato [K1]. So we call $G/H \times V$ as the *exotic symmetric space.* $H$ acts diagonally on $\mathfrak{g}^{-\theta}_{\mathrm{nil}} \times V$, and Kato proved the following.

**Proposition 6.2** (Kato[K1])**.** *The exotic nilpotent cone $\mathfrak{g}^{-\theta}_{\mathrm{nil}} \times V$ has a finitely many $H$-orbits. There exists a natural bijection*

$$(\mathfrak{g}^{-\theta}_{\mathrm{nil}} \times V)/H \leftrightarrow \mathcal{P}_{n,2}.$$

*Moreover, the isotropy group of each $H$-orbit is connected.*

**6.3.** We denote by $\mathcal{O}_{\boldsymbol{\lambda}}$ the $H$-orbit in $\mathfrak{g}^{-\theta}_{\mathrm{nil}} \times V$ corresponding to $\boldsymbol{\lambda} \in \mathcal{P}_{n,2}$. In view of Proposition 6.2, it will be interesting to consider the intersection cohomology of the closure of $\mathcal{O}_{\boldsymbol{\lambda}}$. As in the previous cases, this intersection cohomology will be related to the Kostka polynomials $\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$. In fact, by investigating the effects on Kostka polynomials via the vertical and horizontal transfer, one can expect that exactly Kostka polynomials of this form (without replacing $t$ by $t^2$ or $t^{1/2}$) will appear. Achar-Henderson formulated in [AH] the precise conjecture about this. One of the aim in this article is to draw a road for the proof of their conjecture based on the theory of character sheaves on $G/H \times V$.

**6.4.** In order to introduce the character sheaves smoothly, we adopt a slightly different formulation. Let $\iota : G \to G$ be the anti-automorphism on $G$ defined by $g \mapsto g^{-1}$. We consider the fixed point set $G^{\iota\theta}$ by the map $\iota\theta$. We have

$$G^{\iota\theta} = \{g \in G \mid \theta(g) = g^{-1}\} = \{g\theta(g)^{-1} \mid g \in G\}.$$

Note that for a symmetric space in general, the third set coincides with a connected component of the second set. In our setting, the equality holds. The map $G \to G^{\iota\theta}, g \mapsto g\theta(g)^{-1}$ gives an isomorphism $G/H \xrightarrow{\sim} G^{\iota\theta}$. $G^{\iota\theta}$ is a subset of $G$ which is stable by the conjugation action of $H$ on $G$, and under the above identification, the left multiplication of $H$ on $G/H$ corresponds to the action of $H$ on $G^{\iota\theta}$. We consider an exotic symmetric space $\mathcal{X} = G^{\iota\theta} \times V \simeq G/H \times V$, on which $H$ acts diagonally. Let $G^{\iota\theta}_{\mathrm{uni}} = G^{\iota\theta} \cap G_{\mathrm{uni}}$, and define a subset $\mathcal{X}_{\mathrm{uni}}$ of $\mathcal{X}$ by $\mathcal{X}_{\mathrm{uni}} = G^{\iota\theta}_{\mathrm{uni}} \times V$. Then $\mathcal{X}_{\mathrm{uni}}$ is an $H$-stable subset of $\mathcal{X}$, isomorphic to $\mathfrak{g}^{-\theta}_{\mathrm{nil}} \times V$. Thus by Proposition 6.2, $\mathcal{X}_{\mathrm{uni}}/H$ is canonically in bijection with $\mathcal{P}_{n,2}$. We denote by the same symbol $\mathcal{O}_{\boldsymbol{\lambda}}$ the $H$-orbit corresponding to $\boldsymbol{\lambda} \in \mathcal{P}_{n,2}$. Let $\mathbb{O}_{\boldsymbol{\xi}}$ be the $G$-orbit in $G_{\mathrm{uni}} \times V$ corresponding to $\boldsymbol{\xi} \in \mathcal{P}_{2n,2}$. It is known by Achar-Henderson [AH] that $\mathcal{O}_{\boldsymbol{\lambda}} \subset \mathbb{O}_{\boldsymbol{\lambda}\cup\boldsymbol{\lambda}}$ for each $\boldsymbol{\lambda} \in \mathcal{P}_{n,2}$, and this embedding is compatible with closure relations. It follows that the closure relations for $H$-orbits in $G_{\mathrm{uni}}$ can be described by the partial order introduced in 4.5.

**6.5.** Let $T \subset B$ be a pair of a $\theta$-stable maximal torus and a $\theta$-stable Borel subgroup of $G$. We denote by $M_n$ the maximal isotropic subspace of $V$ stable by $B^\theta$. Put

$$\widetilde{\mathcal{X}} = \{(x, v, gB^\theta) \in G^{\iota\theta} \times V \times H/B^\theta \mid g^{-1}xg \in B^{\iota\theta}, g^{-1}v \in M_n\},$$

and consider the diagram

$$T^{\iota\theta} \xleftarrow{\ \alpha\ } \widetilde{\mathcal{X}} \xrightarrow{\ \pi\ } \mathcal{X},$$

where the maps $\pi, \alpha$ are defined by $\pi : (x, v, gB^\theta) \mapsto (x, v), \alpha : (x, vgB^\theta) \mapsto p(g^{-1}xg)$ (here $p : B^{\iota\theta} \to T^{\iota\theta}$ is the projection).

Let $\mathcal{E}$ be a tame local system on $T^{\iota\theta}$, i.e., the local system such that $\mathcal{E}^{\otimes m} \simeq \bar{\mathbf{Q}}_l$ for some integer $m$ prime to $p$. We define a complex $K_{T,\mathcal{E}}$ on $\mathcal{X}$ by $K_{T,\mathcal{E}} = \pi_*\alpha^*\mathcal{E}[\dim \mathcal{X}]$. Then one can prove that $K_{T,\mathcal{E}}$ is an $H$-equivariant semisimple perverse sheaf on $\mathcal{X}$. (An example of the explicit decomposition to simple factors will be given in §7). We define $\widehat{\mathcal{X}}$ the set of $H$-equivariant simple perverse sheaves on $\mathcal{X}$ appearing as a direct summand of various $K_{T,\mathcal{E}}$. We call perverse sheaves isomorphic to elements in $\widehat{\mathcal{X}}$ as *character sheaves* on $\mathcal{X}$.

## §7. Springer correspondence for exotic symmetric space

**7.1.** Consider a special case where $\mathcal{E}$ is a constant sheaf $\bar{\mathbf{Q}}_l$ on $T^{\iota\theta}$. In this case $K_{T,\bar{\mathbf{Q}}_l} \simeq \pi_*\bar{\mathbf{Q}}_l[\dim \mathcal{X}]$ for a constant sheaf $\bar{\mathbf{Q}}_l$ on $\widetilde{\mathcal{X}}$. Let $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_n$ be the isotropic flag stable by $B^\theta$. For each $0 \le m \le n$, put $\mathcal{X}_m = \bigcup_{g\in H} g(B^{\iota\theta} \times M_m)$. Then $\mathcal{X}_m$ is a closed subvariety of $\mathcal{X}$, and we have a filtration of $\mathcal{X}$ by closed subsets $\mathcal{X}_0 \subset \mathcal{X}_1 \subset \cdots \subset \mathcal{X}_n = \mathcal{X}$. Let $W_n = N_H(T^\theta)/T^\theta$ be the Weyl group of type $C_n$. Then $W_n^\wedge$ is parametrized by $\mathcal{P}_{n,2}$. For each $\boldsymbol{\lambda} \in \mathcal{P}_{n,2}$, we denote by $\widetilde{V}_{\boldsymbol{\lambda}}$ the irreducible $W_n$-module corresponding to $\boldsymbol{\lambda} \in \mathcal{P}_{n,2}$. We have the following result.

**Proposition 7.2.** *$K_{T,\bar{\mathbf{Q}}_l}$ is a semisimple perverse sheaf on $\mathcal{X}$, equipped with $W_n$ action, and is decomposed as*

$$K_{T,\bar{\mathbf{Q}}_l} \simeq \bigoplus_{\boldsymbol{\lambda}\in\mathcal{P}_{n,2}} \widetilde{V}_{\boldsymbol{\lambda}} \otimes \mathrm{IC}(\mathcal{X}_{m(\boldsymbol{\lambda})}, \mathcal{L}_{\boldsymbol{\lambda}})[\dim \mathcal{X}_{m(\boldsymbol{\lambda})}],$$

*where $m(\boldsymbol{\lambda}) = |\lambda^{(1)}|$ for $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$, and $\mathcal{L}_{\boldsymbol{\lambda}}$ is a simple local system on a smooth open subset of $\mathcal{X}_{m(\boldsymbol{\lambda})}$.*

**Remark 7.3** The construction of $W_n$-action in Proposition 7.2 is more complicated than the case of $GL_n$. The role of the Weyl group is played by $\mathcal{W} = N_H(T^{\iota\theta})/Z_H(T^{\iota\theta}) \simeq S_n$, and one cannot expect the Galois covering with group $W_n$. We note also that in the decomposition in (3.2.1), all the simple summands have the same support $G$, which never holds for $K_{T,\bar{\mathbf{Q}}_l}$. This makes the arguments more complicated.

**7.4.** We define a variety $\widetilde{\mathcal{X}}_{\mathrm{uni}}$ by

$$\widetilde{\mathcal{X}}_{\mathrm{uni}} = \{(x, v, gB^\theta) \in \widetilde{\mathcal{X}} \mid (x, v) \in \mathcal{X}_{\mathrm{uni}}\},$$

and define a map $\pi_1 : \widetilde{\mathcal{X}}_{\mathrm{uni}} \to \mathcal{X}_{\mathrm{uni}}$ by $(x, v, gB^\theta) \mapsto (x, v)$. Then by the base change theorem, we have $\pi_*\bar{\mathbf{Q}}_l|_{\mathcal{X}_{\mathrm{uni}}} \simeq (\pi_1)_*\bar{\mathbf{Q}}_l$. It follows, by

Proposition 7.2, that $(\pi_1)_*\bar{\mathbf{Q}}_l$ turns out to be a complex equipped with $W_n$-action. The following result gives the Springer correspondence for the exotic symmetric space, as an analogue of Theorem 3.3. Note that this gives a more satisfactory form compared to Theorem 4.10.

**Theorem 7.5** (Springer corresp. for exotic symmetric space [SS1])**.** *Under the notation as above,*

(i) $(\pi_1)_*\bar{\mathbf{Q}}_l[\dim \mathcal{X}_{\mathrm{uni}}]$ *is a semisimple perverse sheaf on* $\mathcal{X}_{\mathrm{uni}}$ *equipped with* $W_n$*-action, and is decomposed into simple components as*

$$(\pi_1)_*\bar{\mathbf{Q}}_l[\dim \mathcal{X}_{\mathrm{uni}}] \simeq \bigoplus_{\boldsymbol{\lambda}\in\mathcal{P}_{n,2}} \widetilde{V}_{\boldsymbol{\lambda}} \otimes \mathrm{IC}(\overline{\mathcal{O}}_{\boldsymbol{\lambda}}, \bar{\mathbf{Q}}_l)[\dim \mathcal{O}_{\boldsymbol{\lambda}}].$$

(ii) *For each* $\boldsymbol{\lambda} \in \mathcal{P}_{n,2}$*,* $\mathrm{IC}(\mathcal{X}_{m(\boldsymbol{\lambda})}, \mathcal{L}_{\boldsymbol{\lambda}})|_{\mathcal{X}_{\mathrm{uni}}} \simeq \mathrm{IC}(\overline{\mathcal{O}}_{\boldsymbol{\lambda}}, \bar{\mathbf{Q}}_l)$ *up to shift of complexes.*

**7.6.** For each $z = (x, v) \in \mathcal{X}_{\mathrm{uni}}$, put $\mathcal{B}_z = \{gB^\theta \in H/B^\theta \mid g^{-1}xg \in B^{\iota\theta}, g^{-1}v \in M_n\}$. Then $\pi_1^{-1}(z) \simeq \mathcal{B}_z$. Put $K = (\pi_1)_*\bar{\mathbf{Q}}_l$. By a similar argument as in 3.4, $\mathcal{H}^i_zK \simeq H^i(\pi_1^{-1}(z), \bar{\mathbf{Q}}_l) \simeq H^i(\mathcal{B}_z, \bar{\mathbf{Q}}_l)$ has a structure of $W_n$-module. We call this *Springer representations* of $W_n$ on $H^i(\mathcal{B}_z, \bar{\mathbf{Q}}_l)$. Put $d_z = \dim \mathcal{B}_z$, and we consider the top cohomology group $H^{2d_z}(\mathcal{B}_z, \bar{\mathbf{Q}}_l)$. The following result is an analogue of Corollary 3.5.

**Corollary 7.7.** *Assume that* $z \in \mathcal{O}_{\boldsymbol{\lambda}}$*. Then we have* $H^{2d_z}(\mathcal{B}_z, \bar{\mathbf{Q}}_l) \simeq \widetilde{V}_{\boldsymbol{\lambda}}$*. Hence any irreducible* $W_n$*-module is realized as the Springer representation of the top cohomology of the Springer fibre, which gives a natural bijection*

$$W_n^\wedge \leftrightarrow \mathcal{X}_{\mathrm{uni}}/H.$$

**Remark 7.8.** The Springer correspondence (for the exotic nilpotent cone) was first proved by Kato [K1] by making use of Ginzburg theory on affine Hecke algebras (the *exotic Springer correspondence* in his terminology). Our result is an alternate approach based on the theory of character sheaves.

## §8. $\mathbf{F}_q$-structure on character sheaves

**8.1.** Let $X$ be an algebraic variety defined over $\mathbf{F}_q$ with Frobenius map $F : X \to X$. For a complex $K$ on $X$ (bounded, with constructible cohomology sheaves), we consider the pull back $F^*K$ of $K$. Assume that $F^*K \simeq K$, and we fix an isomorphism $\varphi : F^*K \xrightarrow{\sim} K$. We define a characteristic function $\chi_{K,\varphi} : X^F \to \bar{\mathbf{Q}}_l$ by

$$\chi_{K,\varphi}(x) = \sum_i (-1)^i \operatorname{Tr}(\varphi, \mathcal{H}^i_x K)$$

for $x \in X^F$. Note that if a group $G$ acts on $X$, compatible with $\mathbf{F}_q$-structure, and $K$ is a $G$-equivariant perverse sheaf on $X$, then $\chi_{K,\varphi}$ turns out to be a $G^F$-invariant function on $X^F$. So $G$-equivariant perverse sheaf on $G$ (with respect to the conjugation action of $G$ on $G$) produces a class function of $G^F$ by considering the characteristic function.

**Remark 8.2.** In the setting in 8.1, if $\mathcal{H}^i K = 0$ for odd $i$, and the eigenvalues of $\varphi$ on $\mathcal{H}^{2i}K$ are given by $q^i$, then $\chi_{K,\varphi}(x)$ can be expressed as $\chi_{K,\varphi}(x) = \sum_i (\dim \mathcal{H}^{2i}_x K) q^i$. Actually the intersection cohomologies appeared in Theorem 2.3, and in other theorems all enjoy this property.

**8.3.** Returning to our original setting, we consider the exotic symmetric space $\mathcal{X}$, which has a natural $\mathbf{F}_q$-structure. Let $T \subset B$ be a $\theta$-stable pair as before. We assume that $T$ is $F$-stable, but $B$ is not necessarily $F$-stable. Let $\mathcal{E}$ be a tame local system on $T^{\iota\theta}$ such that $F^*\mathcal{E} \simeq \mathcal{E}$. We fix $\varphi_0 : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ so that $\varphi_0$ induces an identity map on the stalk $\mathcal{E}_e$ at the identity element $e \in T^{\iota\theta}$. Then one can construct a canonical isomorphism $\varphi : F^*K_{T,\mathcal{E}} \xrightarrow{\sim} K_{T,\mathcal{E}}$ induced from $\varphi_0$. We denote its characteristic function $\chi_{K_{T,\mathcal{E}},\varphi}$ by $\chi_{T,\mathcal{E}}$, which is an $H^F$-invariant function on $\mathcal{X}^F$. It can be proved that the restriction of $\chi_{T,\mathcal{E}}$ on $\mathcal{X}^F_{\mathrm{uni}}$ is independent of the choice of $\mathcal{E}$, hence is isomorphic to $\chi_{T,\bar{\mathbf{Q}}_l}|_{\mathcal{X}^F_{\mathrm{uni}}}$. We define functions $Q_T : \mathcal{X}^F_{\mathrm{uni}} \to \bar{\mathbf{Q}}_l$ by $Q_T = \chi_{T,\mathcal{E}}|_{\mathcal{X}^F_{\mathrm{uni}}}$, and call them *Green functions* on $\mathcal{X}^F_{\mathrm{uni}}$.

The following result is an analogue of the character formula for character sheaves of connected reductive groups. Note that the Jordan decomposition $g = su = us$, where $s$: semisimple, $u$: unipotent, works also for $G^{\iota\theta}$. For $s \in G^{\iota\theta}$ semisimple, $Z_G(s)$ is a $\theta$-stable subgroup of $G$ such that $Z_G(s)^\theta$ is a product of various $Sp_{2n_i}$, and $Z_G(s)^{\iota\theta} \times V$ can be regarded as a product of various exotic symmetric space $\mathcal{X}_i = GL^{\iota\theta}_{2n_i} \times V_i$ (with $\dim V_i = 2n_i$). Hence the Green function $Q^{Z_G(s)}_T$ can be defined for a maximal torus $T \subset Z_G(s)$, by replacing $G$ by $Z_G(s)$. Also note that the characteristic function $\chi_{\mathcal{E},\varphi_0}$ for a tame local system $\mathcal{E}$ on $T^{\iota\theta}$ gives rise to an element in $(T^{\iota\theta,F})^\wedge = \operatorname{Hom}((T^{\iota\theta})^F, \bar{\mathbf{Q}}_l^*)$, which we denote by $\vartheta$.

**Theorem 8.4** (Character formula)**.** *Let $s, u \in (G^{\iota\theta})^F$ such that $su = us$, $s$: semisimple, $u$: unipotent. Let $\vartheta \in (T^{\iota\theta,F})^\wedge$ be the character* *corresponding to $\mathcal{E}$. Then we have*

$$\chi_{T,\mathcal{E}}(su,v) = |Z_H(s)^F|^{-1} \sum_{\substack{x\in H^F \\ x^{-1}sx\in (T^{\iota\theta})^F}} Q^{Z_G(s)}_{xTx^{-1}}(u,v)\vartheta(x^{-1}sx).$$

By using the character formula, one can prove the following orthogonality relations. The outline of the proof is quite similar to the case of characters sheaves of connected reductive groups. Note that what we need later is the orthogonality relations for Green functions. But for the proof, we have to discuss both theorems simultaneously. Also we need to consider the "generic" tame local systems.

**Theorem 8.5** (Orthogonality relations for $\chi_{T,\mathcal{E}}$)**.** *Assume that $T, T'$ are $F$-stable, $\theta$-stable maximal tori, $\mathcal{E}, \mathcal{E}'$ are tame local systems on $T^{\iota\theta}, T'^{\iota\theta}$, respectively. Let $\vartheta \in (T^{\iota\theta,F})^\wedge, \vartheta' \in (T'^{\iota\theta,F})^\wedge$ be the characters corresponding to $\mathcal{E}, \mathcal{E}'$, respectively. Then we have*

$$\begin{aligned}|H^F|^{-1} &\sum_{z\in\mathcal{X}^F} \chi_{T,\mathcal{E}}(z)\chi_{T',\mathcal{E}'}(z) \\ &= |(T^\theta)^F|^{-1}|(T'^\theta)^F|^{-1} \sum_{\substack{n\in N_H(T^\theta,T'^\theta)^F \\ t\in(T^{\iota\theta})^F}} \vartheta(t)\vartheta'(n^{-1}tn).\end{aligned}$$

**Theorem 8.6** (Orthogonality relations for Green functions)**.**

$$|H^F|^{-1} \sum_{z\in\mathcal{X}^F_{\mathrm{uni}}} Q_T(z)Q_{T'}(z) = \frac{|N_H(T^\theta, T'^\theta)^F|}{|(T^\theta)^F||(T'^\theta)^F|}.$$

**8.7.** Let $T_1$ be the group of diagonal matrices in $GL_{2n}$, which is an $F$-stable and $\theta$-stable maximal torus. We consider $W_n = N_H(T_1^\theta)/T_1^\theta$ and $S_{2n} = N_G(T_1)/T_1$. Then $W_n$ is identified with the subgroup of $S_{2n}$ whcih is the centralizer of the element $(1, n+1)(2, n+2)\cdots(n, 2n)$. For $w \in W_n \subset S_{2n}$, we denote by $T_w$ an $F$-stable maximal torus of $G$ twisted by $w$. Then $T_w$ is $\theta$-stable.

By the Springer correspondence in Theorem 7.5 (i), the map $\varphi : F^*K_{T_1,\bar{\mathbf{Q}}_l} \xrightarrow{\sim} K_{T_1,\bar{\mathbf{Q}}_l}$ induces an isomorphism $\varphi_{\boldsymbol\lambda} : F^*A_{\boldsymbol\lambda} \xrightarrow{\sim} A_{\boldsymbol\lambda}$ for $A_{\boldsymbol\lambda} = \mathrm{IC}(\overline{\mathcal{O}}_{\boldsymbol\lambda}, \bar{\mathbf{Q}}_l)[\dim\mathcal{O}_{\boldsymbol\lambda}]$. We can show that the map $\varphi_w : F^*K_{T_w,\bar{\mathbf{Q}}_l} \xrightarrow{\sim} K_{T_w,\bar{\mathbf{Q}}_l}$ is obtained from $\varphi$ by twisting the Frobenius map by the Springer action $w$. It follows that

$$Q_{T_w} = (-1)^{\dim\mathcal{X}-\dim\mathcal{X}_{\mathrm{uni}}} \sum_{\boldsymbol\lambda\in\mathcal{P}_{n,2}} \chi^{\boldsymbol\lambda}(w)\chi_{A_{\boldsymbol\lambda},\varphi_{\boldsymbol\lambda}}, \tag{8.7.1}$$

where $\chi^{\boldsymbol{\lambda}}$ is the character of the irreducible module $\widetilde{V}_{\boldsymbol{\lambda}}$.

For each $\boldsymbol{\lambda} \in \mathcal{P}_{n,2}$, we define a function $Q_{\boldsymbol{\lambda}}$ by

$$(8.7.2) \qquad Q_{\boldsymbol{\lambda}} = |W_n|^{-1} \sum_{w \in W_n} \chi^{\boldsymbol{\lambda}}(w) Q_{T_w}.$$

By using the orthogonality relations of irreducible characters of $W_n$, (8.7.1) implies that

$$(8.7.3) \qquad Q_{\boldsymbol{\lambda}} = (-1)^{\dim \mathcal{X} - \dim \mathcal{X}_{\mathrm{uni}}} \chi_{A_{\boldsymbol{\lambda}}, \varphi_{\boldsymbol{\lambda}}}.$$

**8.8.** Let $\mathcal{C}_q(\mathcal{X}_{\mathrm{uni}})$ be the $\bar{\mathbf{Q}}_l$-space of all $H^F$-invariant $\bar{\mathbf{Q}}_l$-functions on $\mathcal{X}^F_{\mathrm{uni}}$. We define a bilinear form $\langle f, h \rangle$ on $\mathcal{C}_q(\mathcal{X}_{\mathrm{uni}})$ by

$$\langle f, h \rangle = \sum_{z \in \mathcal{X}^F_{\mathrm{uni}}} f(z) h(z).$$

Substituting (8.7.2) into the formula in Theorem 8.6, we obtain the following orthogonality relations for $Q_{\boldsymbol{\lambda}}$'s.

$$(8.8.1) \qquad \langle Q_{\boldsymbol{\lambda}}, Q_{\boldsymbol{\mu}} \rangle = |H^F||W_n|^{-1} \sum_{w \in W_n} |(T^\theta_w)^F|^{-1} \chi^{\boldsymbol{\lambda}}(w) \chi^{\boldsymbol{\mu}}(w).$$

## §9. The exotic symmetric space and Kostka polynomials

**9.1.** Here we give a characterization of Kostka polynomials $K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$. For any character $\chi$ of $W_n$, put

$$R(\chi) = \frac{\prod_{i=1}^n (t^{2i} - 1)}{|W_n|} \sum_{w \in W_n} \frac{\varepsilon(w)\chi(w)}{\det V_0(t - w)},$$

where $\varepsilon$ is the sign character of $W_n$, and $V_0$ is the reflection representation of $W_n$. Note that $R(\chi)$ coincides with the graded multiplicity of $\chi$ in the coinvariant algebra $R(W_n)$. Hence $R(\chi) \in \mathbf{Z}[t]$. We define a matrix $\Omega = (\omega_{\boldsymbol{\lambda},\boldsymbol{\mu}})_{\boldsymbol{\lambda},\boldsymbol{\mu} \in \mathcal{P}_{n,2}}$ by

$$\omega_{\boldsymbol{\lambda},\boldsymbol{\mu}} = t^N R(\chi^{\boldsymbol{\lambda}} \otimes \chi^{\boldsymbol{\mu}} \otimes \varepsilon),$$

where $N$ is the number of positive roots of the root system of type $C_n$. Recall the partial order $\boldsymbol{\mu} \le \boldsymbol{\lambda}$ and the value $a(\boldsymbol{\lambda})$ as in 4.5 and 4.6. The following theorem was proved in [S].

**Theorem 9.2.** *There exist unique matrices $P = (p_{\boldsymbol{\lambda},\boldsymbol{\mu}}), \varLambda = (\xi_{\boldsymbol{\lambda},\boldsymbol{\mu}})$ over $\mathbf{Q}[t]$ satisfying the equation*

$$P\varLambda\,{}^tP = \Omega$$

*subject to the condition that $\varLambda$ is a diagonal matrices and that*

$$p_{\boldsymbol{\lambda},\boldsymbol{\mu}} = \begin{cases} 0 & \text{unless } \boldsymbol{\mu} \le \boldsymbol{\lambda} \\ t^{a(\boldsymbol{\lambda})} & \text{if } \boldsymbol{\lambda} = \boldsymbol{\mu}. \end{cases}$$

*Then the entry $p_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ of the matrix $P$ coincides with $\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$.*

By making use of Theorem 9.2, we can prove the following result, which was conjectured by Achar-Henderson in [AH]. We consider the intersection cohomology $\mathrm{IC}(\overline{\mathcal{O}}_{\boldsymbol{\lambda}}, \bar{\mathbf{Q}}_l)$ associated to the closure of an $H$-orbit $\mathcal{O}_{\boldsymbol{\lambda}}$ in $\mathcal{X}_{\mathrm{uni}}$.

**Theorem 9.3** ([SS2]). *For each $\boldsymbol{\lambda} \in \mathcal{P}_{n,2}$, Put $K = \mathrm{IC}(\overline{\mathcal{O}}_{\boldsymbol{\lambda}}, \bar{\mathbf{Q}}_l)$.*

(i) $\mathcal{H}^iK = 0$ *unless* $i \equiv 0 \pmod 4$.

(ii) *For $z \in \mathcal{O}_{\boldsymbol{\mu}} \subset \overline{\mathcal{O}}_{\boldsymbol{\lambda}}$, we have*

$$\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = t^{a(\boldsymbol{\lambda})}\sum_{i\ge 0}(\dim \mathcal{H}^{4i}_zK)t^{2i}. \tag{9.3.1}$$

**9.4.** The outline of the proof of Theorem 9.3 is as follows. We consider the characteristic function $\chi_{A_{\boldsymbol{\lambda}},\varphi_{\boldsymbol{\lambda}}}$ for $A_{\boldsymbol{\lambda}} = \mathrm{IC}(\overline{\mathcal{O}}_{\boldsymbol{\lambda}}, \bar{\mathbf{Q}}_l)[\dim \mathcal{O}_{\boldsymbol{\lambda}}]$. Let $c_{\boldsymbol{\mu}} \in \mathcal{C}_q(\mathcal{X}_{\mathrm{uni}})$ the characteristic function of the orbit $\mathcal{O}^F_{\boldsymbol{\mu}}$. Since $\{c_{\boldsymbol{\mu}}\}$ gives a basis of $\mathcal{C}_q(\mathcal{X}_{\mathrm{uni}})$, $\chi_{A_{\boldsymbol{\lambda}},\varphi_{\boldsymbol{\lambda}}}$ can be written uniquely as $\chi_{A_{\boldsymbol{\lambda}},\varphi_{\boldsymbol{\lambda}}} = \sum_{\boldsymbol{\mu}} p_{\boldsymbol{\lambda},\boldsymbol{\mu}}c_{\boldsymbol{\mu}}$ with $p_{\boldsymbol{\lambda},\boldsymbol{\mu}} \in \bar{\mathbf{Q}}_l$. Put $P = (p_{\boldsymbol{\lambda},\boldsymbol{\mu}})$ and $\varLambda = (\langle\!\langle c_{\boldsymbol{\lambda}}, c_{\boldsymbol{\mu}}\rangle\!\rangle)$. Then we have

$$(\langle\!\langle \chi_{A_{\boldsymbol{\lambda}},\varphi_{\boldsymbol{\lambda}}}, \chi_{A_{\boldsymbol{\mu}},\varphi_{\boldsymbol{\mu}}}\rangle\!\rangle) = P\varLambda\,{}^tP.$$

By (8.7.3), $\langle \chi_{A_{\boldsymbol{\lambda}},\varphi_{\boldsymbol{\lambda}}}, \chi_{A_{\boldsymbol{\mu}},\varphi_{\boldsymbol{\mu}}}\rangle$ coincides with $\langle Q_{\boldsymbol{\lambda}}, Q_{\boldsymbol{\mu}}\rangle$, hence its value is given by the right hand side of (8.8.1). On the other hand, by a direct computation on $\omega_{\boldsymbol{\lambda},\boldsymbol{\mu}}$, one can show that the right hand side of (8.8.1) coincides with $\omega_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)|_{t=q}$. This implies, by Theorem 9.2, that $p_{\boldsymbol{\lambda},\boldsymbol{\mu}} = \widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(q)$. On the other hand, one can prove that the eigenvalues of the Frobenius action on $\mathcal{H}^i_zA_{\boldsymbol{\lambda}}$ has the absolute value $q^{i/2}$. The theorem follows from those facts.

**Remarks 9.5.** (i) As in 3.6, we obtain a description of Kostka polynomials in terms of the Springer representations as follows; for $z \in \mathcal{O}_{\boldsymbol{\mu}}$, we have

$$\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = \sum_{i\ge 0}\langle \widetilde{V}_{\boldsymbol{\lambda}}, H^{2i}(\mathcal{B}_z, \bar{\mathbf{Q}}_l)\rangle_{W_n}\, t^i. \tag{9.5.1}$$

Conversely, if (9.5.1) is proved, one can deduce (9.3.1) from (9.5.1). In fact, Kato [K2] proved the conjecture of Achar-Henderson by showing (9.5.1) by a totally different method.

(ii) The discussion employed for the proof of Theorem 9.3 is also applicable to the case of $GL_{2n}/Sp_{2n}$. Then we can reprove Theorem 5.4 without consulting the result from [BKS].

## §10. $H^F$-invariant functions on $\mathcal{X}^F$

**10.1.** The motivation of the theory of character sheaves was to compute the irreducible characters of finite reductive groups in terms of the characteristic functions of character sheaves. In the case of $GL_n$, characteristic functions of $F$-stable character sheaves give all the irreducible characters of $GL_n(\mathbf{F}_q)$. In the case of $G \times V$, Finkelberg-Ginzburg-Travkin [FGT] studies $G^F$-invariant functions of $(G \times V)^F$, and conjectures that characteristic functions of $F$-stable mirabolic character sheaves give rise to a basis of the space of all such functions.

**10.2.** We consider a similar problem for $\mathcal{X} = G^{\iota\theta} \times V$. Let $\mathcal{C}_q(\mathcal{X})$ be the $\bar{\mathbf{Q}}_l$-space of all $H^F$-invariant $\bar{\mathbf{Q}}_l$-functions on $\mathcal{X}^F$. Let $\widehat{\mathcal{X}}$ be the set of character sheaves on $\mathcal{X}$ as defined in 6.5. We denote by $\widehat{\mathcal{X}}^F$ the set of $F$-stable character sheaves, namely, the set of $A \in \widehat{\mathcal{X}}$ such that $F^*A \simeq A$. Since $A$ is a simple perverse sheaf, for $A \in \widehat{\mathcal{X}}^F$, the isomorphism $\varphi_A : F^*A \xrightarrow{\sim} A$ is unique up to scalar. We fix one isomorphism $\varphi_A$ for each $A$, and consider the characteristic function $\chi_{A,\varphi_A}$, which is an element in $\mathcal{C}_q(\mathcal{X})$. We have the following result.

**Theorem 10.3** ([SS2]). (i) *The set* $\{\chi_{A,\varphi_A} \mid A \in \widehat{\mathcal{X}}^F\}$ *gives rise to a basis of* $\mathcal{C}_q(\mathcal{X})$.

(ii) *Assume that* $A = \mathrm{IC}(\mathcal{X}_{m(\boldsymbol{\lambda})}, \mathcal{L}_{\boldsymbol{\lambda}})[\dim \mathcal{X}_{m(\boldsymbol{\lambda})}]$ *for* $\boldsymbol{\lambda} \in \mathcal{P}_{n,2}$ *(see Proposition 7.2). Then under a suitable choice of* $\varphi_A$, $\chi_{A,\varphi_A}(z) = \widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(q)$ *for* $z \in \mathcal{O}_{\boldsymbol{\mu}}$.

(iii) *There exists an algorithm of computing the characteristic functions* $\chi_{A,\varphi_A}$.

**10.4.** (i) is proved by computing the inner product of various $\chi_{A,\varphi_A}$ by using Theorem 8.5. Note that the significance in the case of character sheaves of reductive groups is that the characteristic functions give orthogonal system. Unfortunately the orthogonality does not hold in our case, but the inner product provides us enough information for proving the linearly independence. (ii) is nothing but the Springer correspondence (Theorem 7.5). (iii) follows from the character formula

(Theorem 8.4) together with the computation of Kostka polynomials (Theorem 9.2).

**Remark 10.5.** In a similar way, we obtain a similar result as Theorem 10.3 for the case of $G \times V$, which proves the conjecture of [FGT] on mirabolic character sheaves stated in 10.1.

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