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Generalized (co)homology of the loop spaces of classical groups and the universal factorial Schur P- and Q-functions

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Abstract.

In this paper, we study the generalized (co)homology Hopf algebras of the loop spaces on the infinite classical groups, generalizing the work due to Kono-Kozima and Clarke. We shall give a description of these Hopf algebras in terms of symmetric functions. Based on topological considerations in the first half of this paper, we then introduce a *universal* analogue of the factorial Schur P- and Q-functions due to Ivanov and Ikeda-Naruse. We investigate various properties of these functions such as the *cancellation property*, which we call the $\mathbb{L}$*-supersymmetric property*, the *factorization property*, and the *vanishing property*. We prove that the universal analogue of the Schur P-functions form a formal basis for the ring of symmetric functions with the $\mathbb{L}$-supersymmetric property. By using the universal analogue of the Cauchy identity, we then define the *dual* universal Schur P- and Q-functions. We describe the duality of these functions in terms of Hopf algebras.

Contents

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§1. Introduction

Let $SU = SU(\infty)$, $Sp = Sp(\infty)$, and $SO = SO(\infty)$ be the infinite special unitary, symplectic, and special orthogonal group respectively, and ΩSU, ΩSp, and $\Omega_0 SO$ denote its based loop space (Ω_0 means the connected component of the identity). These spaces have natural (homotopy-commutative) H-space structure given by the usual loop multiplication. On the other hand, it is well-known that these loop spaces have no torsion and no odd degree elements in (co)homology with integer coefficients (see e.g., the classical work due to Bott [4], [5]). Therefore the H-space structure on ΩSU (resp. ΩSp, $\Omega_0 SO$) endows $H_*(\Omega SU)$ and $H^*(\Omega SU)$ (resp. $H_*(\Omega Sp)$ and $H^*(\Omega Sp)$, $H_*(\Omega_0 SO)$ and $H^*(\Omega_0 SO)$) with the structure of dual Hopf algebras over the integers $\mathbb{Z}$. These Hopf algebras were intensively studied by Bott [6]. Although Bott uses the technique of symmetric functions very much in that paper (see e.g., [6, §8]), he did not give explicitly the descriptions of these Hopf algebras in terms of symmetric functions. On the other hand, by means of the celebrated Bott periodicity theorem (Bott [7], [6, Proposition 8.3], Switzer [62, 11.60, 16.47]), there exists a homotopy equivalence of H-spaces $BU \simeq \Omega SU$, where BU denotes the classifying space of the infinite unitary group $U = U(\infty)$. The H-space structure of BU is induced from the Whitney sum of complex vector bundles (see e.g., May [47, p.201, Proposition], Switzer [62, p.213]). Therefore they have the isomorphic (co)homology. By the theory of characteristic classes of complex vector bundles, it is well known that the integral cohomology ring of BU is $H^*(BU) \cong \mathbb{Z}[c_1, c_2, \ldots]$, where c_i $(i = 1, 2, \ldots)$ denote the universal Chern classes. The homology ring of BU is also known to be a polynomial ring of the form $H_*(BU) \cong \mathbb{Z}[\beta_1, \beta_2, \ldots]$, where β_i $(i = 1, 2, \ldots)$ is an element of degree $2i$ induced from the natural map $BU(1) \simeq \mathbb{C}P^\infty \longrightarrow BU$ (see e.g., Switzer [62, Corollary 16.11, Theorem 16.17]). Moreover, $H^*(BU)$ is a Hopf algebra which is *self-dual*: $H_*(BU)$ is isomorphic to $H^*(BU)$ as a Hopf algebra. In topology, it is customary to think of Chern classes of complex vector bundles as elementary symmetric functions in certain variables (sometimes called the *Chern roots*). From this, both $H^*(BU) \cong H^*(\Omega SU)$ and $H_*(BU) \cong H_*(\Omega SU)$ can be identified with the *ring of symmetric functions* denoted by Λ. For cohomology, the universal Chern classes

c_i $(i = 1, 2, \ldots)$ correspond to the i-th elementary symmetric functions e_i, and for homology, the elements β_i $(i = 1, 2, \ldots)$ correspond to the i-th complete symmetric functions h_i (for this point of view, see e.g., Baker-Richter [3, §5], Lenart [44, §4], Liulevicius [45]). The starting point of our work is to give descriptions of other Hopf algebras $H^*(\Omega Sp)$, $H_*(\Omega Sp)$, $H^*(\Omega_0 SO)$, and $H_*(\Omega_0 SO)$ in terms of symmetric functions. By the Bott periodicity theorem again, there exist homotopy equivalences of H-spaces $Sp/U \simeq \Omega Sp$, $SO/U \simeq \Omega_0 SO$ (see e.g., Switzer [62, p.409]). Especially we have the isomorphisms $H^*(\Omega Sp) \cong H^*(Sp/U)$ and $H^*(\Omega_0 SO) \cong H^*(SO/U)$. On the other hand, by the work of Pragacz [55, §6] and Józefiak [28], we know that the integral cohomology ring $H^*(Sp/U)$ (resp. $H^*(SO/U)$) of the infinite Lagrangian Grassmannian Sp/U (resp. infinite orthogonal Grassmannian SO/U) is isomorphic to the ring of Schur Q-functions (resp. Schur P-functions) denoted by Γ (resp. Γ'). Therefore we have the following isomorphism abstractly: $H^*(\Omega Sp) \cong \Gamma$ and $H^*(\Omega_0 SO) \cong \Gamma'$. Since Γ and Γ' are mutually dual Hopf algebras over $\mathbb{Z}$, we also have $H_*(\Omega Sp) \cong \Gamma'$ and $H_*(\Omega_0 SO) \cong \Gamma$.

Recently, the study of the *affine Grassmannian* Gr_G for a simply-connected simple complex algebraic group G in terms of *Schubert calculus* has been developed extensively (see e.g., Peterson [56], Lam [36], Lam-Schilling-Shimozono [39], [40]). On the one hand, Gr_G admits a cell decomposition by *Schubert cells* from its presentation $\mathrm{Gr}_G = G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$(see e.g., Garland-Raghunathan [19, Theorem 1.9], Mitchell [50, Theorem 1.3], [51, Theorem 1.2], [52, Corollary 3.2]). From this, $H_*(\mathrm{Gr}_G)$ and $H^*(\mathrm{Gr}_G)$ have free $\mathbb{Z}$-module bases consisting of *Schubert classes*. On the other hand, by the result of Garland-Raghunathan [19, Corollary 1.7] and an unpublished work of Quillen (see also Mitchell [51, Theorems 1.1 and 1.4]), [52, Theorem 4.2]), it is known that Gr_G is homotopy equivalent to the based loop space ΩK on the maximal compact subgroup K of G. Therefore their work is closely related to our current work. In particular, Lam studied the affine Grassmannian $\mathrm{Gr}_{SL(n,\mathbb{C})} \simeq \Omega SU(n)$, and identified $H_*(\mathrm{Gr}_{SL(n,\mathbb{C})})$ and $H^*(\mathrm{Gr}_{SL(n,\mathbb{C})})$ with a subring $\Lambda_{(n)}$ and a quotient $\Lambda^{(n)}$ of the ring of symmetric functions Λ. Moreover he identified the Schubert classes of $H_*(\mathrm{Gr}_{SL(n,\mathbb{C})})$ and $H^*(\mathrm{Gr}_{SL(n,\mathbb{C})})$ as explicit symmetric functions (see [36, Theorem 7.1]). Also Lam-Schilling-Shimozono [39] considered the affine Grassmannian $\mathrm{Gr}_{Sp_{2n}(\mathbb{C})} \simeq \Omega Sp(n)$ and identified $H_*(\mathrm{Gr}_{Sp_{2n}(\mathbb{C})})$ and $H^*(\mathrm{Gr}_{Sp_{2n}(\mathbb{C})})$ with certain dual Hopf algebras $\Gamma_{(n)}$ and $\Gamma^{(n)}$ of symmetric functions, defined in terms of Schur P- and Q-functions. By taking limit $n \to \infty$, we obtain immediately the above descriptions of

$H_*(\Omega Sp)$ and $H^*(\Omega Sp)$. However, these deep results depend on Peterson's remarkable result ([56]): In his lecture notes, he constructed an isomorphism between the torus equivariant homology $H_*^T(\mathrm{Gr}_G)$ of the affine Grassmannian and a certain subalgebra of the *affine nilHecke ring* $\mathbb{A}_{\mathrm{aff}}$. Thus it seems difficult to find a geometric or topological meaning of their result.

We then turned our attention to topologists' work on the loop spaces on Lie groups (e.g., Clarke [14], [15], [16], Kono-Kozima [31], Kozima [33], [34], [35]). Especially we focused on Kono-Kozima's work. In [31], they considered the following homomorphisms in homology:

(1)
$$\Omega(c\circ q)_* = (\Omega c)_* \circ (\Omega q)_* : H_*(\Omega SU) \overset{(\Omega q)_*}{\longrightarrow} H_*(\Omega Sp) \overset{(\Omega c)_*}{\longrightarrow} H_*(\Omega SU),$$

where $q : SU \longrightarrow Sp$ and $c : Sp \longrightarrow SU$ are induced from the *quaternionification* $SU(n) \hookrightarrow Sp(n)$ and the *complex restriction* $Sp(n) \hookrightarrow SU(2n)$ (see (4) in this paper). It is easy to show that $(\Omega c)_*$ is a split monomorphism (Lemma 2.3) and thus $H_*(\Omega Sp)$ can be regarded as a subalgebra of $H_*(\Omega SU)$. By a topological argument, they fixed elements $z_i \in H_{2i}(\Omega Sp)$ $(i = 1, 2, \ldots)$, and determined the Hopf algebra structure of $H_*(\Omega Sp)$ explicitly in terms of these elements ([31, Theorem 2.18])[1]. In course of consideration, they showed the following formula ([31, Theorem 2.9]):

$$\Omega(c\circ q)_*(\beta(x)) = \beta(x)/\beta(-x), \tag{2}$$

where we used the isomorphism $H_*(\Omega SU) \cong H_*(BU) \cong \mathbb{Z}[\beta_1, \beta_2, \ldots]$ as mentioned before, and $\beta(x) := \sum_{i\geq 0} \beta_i x^i \in H_*(\Omega SU)[[x]]$, a formal power series. This formula is of particular importance in our present work: From (2), we see immediately that under the afore-mentioned isomorphism $H_*(BU) \cong H_*(\Omega SU) \cong \Lambda$, the monomorphism $(\Omega c)_* : H_*(\Omega Sp) \hookrightarrow H_*(\Omega SU)$ can be identified with the natural inclusion $\Gamma' \hookrightarrow \Lambda$ (the similar consideration can be found in Lam [37, §2.3]). In this way, we are able to give a sequence of homomorphisms (1) an interpretation in terms of symmetric functions (for more details, see §3.4). The advantage of our method is that it has immediate application to any generalized homology theory $E_*(-)$ which is *complex oriented* in the sense of Adams [1, p.37] (for the application to the K-homology theory, see Clarke [15]).

[1] In topology, it was known that $H_*(\Omega Sp)$ is a polynomial algebra generated by elements of degrees $4i-2$ $(i = 1, 2, \ldots)$. The problem was to fix algebra generators and to give an explicit description of the coalgebra structure of $H_*(\Omega Sp)$. (cf. Bott [6, §11]).

Let $E^*(-)$ denote a complex oriented generalized (multiplicative) cohomology theory, and $E_*(-)$ the corresponding homology theory in the sense of Adams [1, p.37]. The coefficient rings for these two theories are given by $E_* := E_*(\mathrm{pt})$ (E-homology of a point) and $E^* := E^*(\mathrm{pt})$ (E-cohomology of a point). Our first aim is to describe the E_*-(co)homology of ΩSU, ΩSp, and $\Omega_0 SO$ in terms of symmetric functions. Generalizing the approach due to Kono-Kozima [31] and Clarke [15], we are able to describe the Hopf algebras $E_*(\Omega Sp)$ and $E^*(\Omega Sp)$ (resp. $E_*(\Omega_0 SO)$ and $E^*(\Omega_0 SO)$) as subalgebras of $E_*(\Omega SU)$ and $E^*(\Omega SU)$ (see Section 2). Motivated by the description of $E_*(\Omega_0 SO)$ and $E_*(\Omega Sp)$ (resp. $E^*(\Omega Sp)$ and $E^*(\Omega_0 SO)$), in Section 3, we shall introduce certain subalgebras Γ^E_*, Γ'^E_* of Λ^E_* (resp. Γ^*_E, Γ'^*_E of Λ^*_E). By definition, using the identification $E_*(\Omega SU) \cong E_*(BU)$ with $\Lambda^E_* := E_* \otimes_{\mathbb{Z}} \Lambda$ (scalar extension) (resp. $E^*(\Omega SU) \cong E^*(BU)$ with $\Lambda^*_E := \mathrm{Hom}_{E_*}(\Lambda^E_*, E_*)$ (graded dual)), one sees immediately that $E_*(\Omega_0 SO) \cong \Gamma^E_*$ and $E_*(\Omega Sp) \cong \Gamma'^E_*$ (resp. $E^*(\Omega Sp) \cong \Gamma^*_E$ and $E^*(\Omega_0 SO) \cong \Gamma'^*_E$). Thus our first main result identifies $E_*(\Omega Sp)$ and $E^*(\Omega Sp)$ (resp. $E_*(\Omega_0 SO)$ and $E^*(\Omega_0 SO)$) with certain dual Hopf algebras Γ'^E_* and Γ^*_E (resp. Γ^E_* and Γ'^*_E) of symmetric functions, defined in terms of a generalization of the rings of Schur P- and Q-functions (Propositions 3.11, 3.12).

Having constructed certain Hopf algebras Γ^E_*, Γ'^E_* of Λ^E_* and Γ^*_E, Γ'^*_E of Λ^*_E of symmetric functions as above, our next task is to construct certain symmetric functions which may serve as "nice bases" for these algebras as free E_* (or E^*)-modules. Since the complex cobordism theory $MU^*(-)$ is "universal" among complex oriented generalized cohomology theories (Quillen [57]), it suffices to consider the case $E = MU$. In this case, the coefficient ring $MU_* = MU^{-*}$ is known to be isomorphic to the *Lazard ring* $\mathbb{L}$ (Lazard [42], Quillen [57, Theorem 2], Adams [1, Part II, Theorem 8.2])[2]. We wish to construct certain symmetric functions which constitute bases for Γ'^*_{MU}, Γ^*_{MU}, and their dual bases for Γ^{MU}_*, Γ'^{MU}_*. We also expect them to correspond to *cohomology Schubert bases* for $MU^*(\Omega_0 SO) \cong MU^*(SO/U)$, $MU^*(\Omega Sp) \cong MU^*(Sp/U)$, and *homology Schubert bases* for $MU_*(\Omega_0 SO) \cong MU_*(SO/U)$, $MU_*(\Omega Sp) \cong MU_*(Sp/U)$ respectively. In Sections 4 and 5, we tackle this problem along the following lines: Firstly, at the time of this writing, it is not

[2] $\mathbb{L}$ is known to be a polynomial algebra over the integers $\mathbb{Z}$ on generators of degrees $2, 4, 6, 8, \dots$ (see e.g., Adams [1, Part II, Theorem 7.1], Ravenel [58, Theorem A2.1.10]). One can define $\mathbb{L}$ as the quotient of a polynomial ring P generated by formal symbols $a_{i,j}$ $(i, j \geq 1)$ of degree $2(i+j-1)$ by a certain ideal I (see e.g., Adams [1, Part II, Theorem 5.1], Ravenel [58, Theorem A2.1.8]).

known how to define or characterize geometrically the Schubert classes of general flag varieties G/P or affine Grassmannians Gr_G in *generalized* (co)homology theories $E^*(-)$ and $E_*(-)$ (for the Schubert classes in generalized (co)homology theories, see e.g., Ganter-Ram [18]). Thus we switch our attention from geometry to algebra, and deal with the above problem purely algebraically. Secondly, it seems plausible to define the Schubert classes by way of the *torus equivariant cohomology theory* and the technique of the *localization theory* (see Ganter-Ram [18], Harada-Henriques-Holm [20], Ikeda-Naruse [25], Kostant-Kumar [32]). In torus equivariant theory, various *factorial* analogues of Schur functions play an important role (for the *factorial Schur functions* and the complex Grassmannians, see Knutson-Tao [30], Molev-Sagan [54]; for the *factorial Schur P- and Q-functions* and the maximal isotropic Grassmannians, see Ivanov [27], Ikeda [22], Ikeda-Naruse [23]; for the K-theoretic analogues of factorial Schur P- and Q-functions, see Ikeda-Naruse [25]). Thus we wish to construct the required functions as a natural generalization of these factorial Schur functions. This suggests that our functions will contain multi-parameter $\mathbf{b} = (b_1, b_2, \ldots)$. The definition of our functions in *cohomology*, denoted $P^{\mathbb{L}}_{\lambda}(\mathbf{x}|\mathbf{b})$ and $Q^{\mathbb{L}}_{\lambda}(\mathbf{x}|\mathbf{b})$ with λ a strict partition, $\mathbf{x} = (x_1, x_2, \ldots)$ independent variables, $\mathbf{b} = (b_1, b_2, \ldots)$ parameters, will be given in §4.2, Definition 4.1. We shall call them the *universal factorial Schur P- and Q-functions*. As the name suggests, if we specialize $a_{i,j} = 0$ for all $i, j \geq 1$, we will obtain the usual factorial Schur P- and Q-functions $P_{\lambda}(\mathbf{x}|\mathbf{b})$ and $Q_{\lambda}(\mathbf{x}|\mathbf{b})$ due to Ivanov [27][3], and if we specialize $a_{1,1} = \beta$, "Bott's element", and $a_{i,j} = 0$ for all $(i, j) \neq (1, 1)$, we will obtain the K-theoretic analogue of the factorial Schur P- and Q-functions $GP_{\lambda}(\mathbf{x}|\mathbf{b})$ and $GQ_{\lambda}(\mathbf{x}|\mathbf{b})$ due to Ikeda-Naruse [25]. Further we shall investigate various properties of our functions such as

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[3]In [27, Definitions 2.10, 2.13], Ivanov introduced a multi-parameter generalization of the usual Schur P- and Q-functions denoted by $P_{\lambda;a}$ and $Q_{\lambda;a}$, where λ is a strict partition and $a = (a_k)_{k \geq 1}$ (with $a_1 = 0$) is an arbitrary sequence of complex numbers. By definition, $Q_{\lambda;a} = 2^{\ell(\lambda)} P_{\lambda;a}$, where $\ell(\lambda)$ denotes the length of λ. In this paper, we use the definition of these functions due to Ikeda-Mihalcea-Naruse [24, §4.2]. They denote these functions by $P_{\lambda}(x|a)$ and $Q_{\lambda}(x|a)$, where $a = (a_i)_{i \geq 1}$ is an infinite sequence of variables. By definition, $Q_{\lambda}(x|a) = 2^{\ell(\lambda)} P_{\lambda}(x|0, a)$. Note that $P_{\lambda}(x|a)$ is the *even* limit of the corresponding polynomials $P^{(n)}_{\lambda}(x_1, \ldots, x_n|a)$ of finite variables because of the *mod* 2 *stability* (see Ikeda-Naruse [23, Proposition 8.2]). Also we shall use $[\mathbf{x}|\mathbf{b}]^k := \prod_{i=1}^{k}(x + b_i)$ as a generalization of the ordinary k-th power in place of $(x|a)^k := \prod_{i=1}^{k}(x - a_i)$.

- the $\mathbb{L}$-*supersymmetric property* which is a generalization of the "Q-cancellation property" due to Pragacz [55, p.145], "supersymmetricity" due to Ivanov [27, Definition 2.1], and "K-supersymmetric property (K-theoretic Q-cancellation property)" due to Ikeda-Naruse [25, Definition 1.1];
- the *factorization property* (cf. Pragacz [55, Proposition 2.2] and Ikeda-Naruse [25, Proposition 2.3]);
- the *vanishing property* (cf. Ivanov [27, Theorem 5.3], Ikeda-Naruse [23, Proposition 8.3], Ikeda-Mihalcea-Naruse [24, Proposition 4.2], Ikeda-Naruse [25, Proposition 7.1]);
- the *basis theorem* (cf. Ikeda-Mihalcea-Naruse [24, Proposition 4.2], Ikeda-Naruse [25, Theorem 3.1, Propositions 3.2, 3.4, 3.5]).

In particular, in the non-equivariant case, i.e., $\mathbf{b} = 0$, our functions $\{P^{\mathbb{L}}_{\lambda}(\mathbf{x})\}$ (resp. $\{Q^{\mathbb{L}}_{\lambda}(\mathbf{x})\}$) turn out to constitute a formal $\mathbb{L}$-basis for the ring ${\Gamma'}^{*}_{MU} \cong MU^*(\Omega_0 SO)$ (resp. $\Gamma^*_{MU} \cong MU^*(\Omega Sp)$). Combining these "cohomology bases" $\{P^{\mathbb{L}}_{\lambda}(\mathbf{x})\}$ and $\{Q^{\mathbb{L}}_{\lambda}(\mathbf{x})\}$ with the argument using an analogue of the *Cauchy identity*, we shall define the dual bases. More precisely, we define functions $\widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y})$ and $\widehat{q}^{\mathbb{L}}_{\lambda}(\mathbf{y})$ with λ a strict partition, $\mathbf{y} = (y_1, y_2, \ldots)$ one more set of independent variables, by the following identity (Definition 5.3):

$$\begin{aligned}
\Delta(\mathbf{x};\mathbf{y}) &= \prod_{i,j\geq 1} \frac{1-\overline{x}_i y_j}{1-x_i y_j} = \sum_{\lambda:\ \text{strict}} Q^{\mathbb{L}}_{\lambda}(\mathbf{x})\, \widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y}),\\
\Delta(\mathbf{x};\mathbf{y}) &= \prod_{i,j\geq 1} \frac{1-\overline{x}_i y_j}{1-x_i y_j} = \sum_{\lambda:\ \text{strict}} P^{\mathbb{L}}_{\lambda}(\mathbf{x})\, \widehat{q}^{\mathbb{L}}_{\lambda}(\mathbf{y}).
\end{aligned}$$

We also show the basis theorem for these dual functions, and furthermore we will discuss the duality of these functions in terms of the theory of Hopf algebras. Thus our second main result is the algebraic construction of the universal factorial Schur P- and Q-functions $P^{\mathbb{L}}_{\lambda}(\mathbf{x}|\mathbf{b})$'s, $Q^{\mathbb{L}}_{\lambda}(\mathbf{x}|\mathbf{b})$'s and their duals $\widehat{q}^{\mathbb{L}}_{\lambda}(\mathbf{y})$'s, $\widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y})$'s (with $\mathbf{b} = 0$). At present, the geometric meaning of our functions $P^{\mathbb{L}}_{\lambda}(\mathbf{x}|\mathbf{b})$, $Q^{\mathbb{L}}_{\lambda}(\mathbf{x}|\mathbf{b})$ and $\widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y})$, $\widehat{q}^{\mathbb{L}}_{\lambda}(\mathbf{y})$ is not apparent. However, for instance, the vanishing property of $Q^{\mathbb{L}}_{\lambda}(\mathbf{x}|\mathbf{b})$'s strongly suggests that these functions will provide the "Schubert basis" for the torus equivariant complex cobordism $MU^*_T(LG(n))$ for a Lagrangian Grassmannian $LG(n) \cong Sp(n)/U(n)$, T a maximal torus of $Sp(n)$. We shall discuss this problem elsewhere.

This paper is organized as follows: In Section 2, we review the topologists' work concerning the loop spaces on SU, Sp, and SO. Especially, we shall give descriptions of E-(co)homology Hopf algebras of

ΩSp and $\Omega_0 SO$, where $E^*(-)$ denotes a complex oriented generalized cohomology theory (see Theorems 2.8, 2.13, and 2.18). In Section 3, we introduce the E-theoretic analogues of the rings of Schur P- and Q-functions (see Subsections 3.2, 3.3), and give an interpretation of E-(co)homology Hopf algebras of ΩSp and $\Omega_0 SO$ in terms of symmetric functions (see Propositions 3.11, 3.12). These are the first main result of this paper. In Section 4, we define the universal factorial Schur P- and Q-functions $P^{\mathbb{L}}_\lambda(\mathbf{x}_n|\mathbf{b})$ and $Q^{\mathbb{L}}_\lambda(\mathbf{x}_n|\mathbf{b})$ (Definition 4.1), and establish the fundamental properties of these functions such as $\mathbb{L}$-supersymmetricity (Subsection 4.3), stability property (Subsection 4.4), factorization formula (Subsection 4.6), vanishing property (Subsection 4.8), and basis theorem (Subsections 4.7, 4.10). In this section, we also define the universal factorial Schur functions $s^{\mathbb{L}}_\lambda(\mathbf{x}_n|\mathbf{b})$ and discuss their properties (Subsection 4.5). In Section 5, we define the dual universal Schur P- and Q-functions $\widehat{p}^{\mathbb{L}}_\lambda(\mathbf{y})$ and $\widehat{q}^{\mathbb{L}}_\lambda(\mathbf{y})$, and establish the basis theorem (Theorem 5.6). Moreover, we describe the duality of these functions. Then our second main result is summarized in Theorem 5.8. In the Appendix, Section 6, we discuss another version of the universal factorial Schur functions $s^{\mathbb{L}}_\lambda(\mathbf{x}_n||\mathbf{b}_{\mathbb{Z}})$, which are the universal analogues of Molev's double Schur functions [53] (Subsection 6.1). We also collect the necessary data concerning the Weyl groups, the root systems, etc. of classical types.

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§2. E-(co)homology of ΩSp and $\Omega_0 SO$

2.1. Generalized (co)homology theory

As in the introduction, $E^*(-)$ denotes a complex oriented generalized (multiplicative) cohomology theory, and $E_*(-)$ the corresponding homology theory in the sense of Adams [1, Part II, p.37], Switzer [62, 16.27]. A generator $x^E \in \tilde{E}^2(\mathbb{C}P^\infty)$, where $\tilde{E}^*(-)$ is the corresponding reduced cohomology theory, is specified and it is called the *orientation class*. The coefficient rings for these two theories are given by $E_* = E_*(\mathrm{pt})$ and $E^* = E_{-*}$. In what follows, the coefficient ring E_* is assumed to be torsion free. Then it is known that the cohomology ring of the infinite projective space $\mathbb{C}P^\infty$ is $E^*(\mathbb{C}P^\infty) = E^*[[x^E]]$, a formal power series ring with the given generator $x^E \in \tilde{E}^2(\mathbb{C}P^\infty)$ (see Adams [1, Part II, Lemma 2.5]), and the homology $E_*(\mathbb{C}P^\infty)$ of $\mathbb{C}P^\infty$ is a free E_*-module with a basis $\{\beta^E_i\}_{i\geq 0}$ ($\beta^E_0 = 1$) (Adams [1, Part II, Lemma 2.14]). With respect to the E-theory Kronecker product (pairing), we have

$$\langle (x^E)^i, \beta_j^E \rangle = \delta_{ij},$$

namely, $\{(x^E)^i\}_{i\geq 0}$ and $\{\beta_j^E\}_{j\geq 0}$ are dual bases over E_*.

Let

$$\mu_E(u,v) = u + v + \textstyle\sum_{i,j\geq 1} a_{i,j}^E u^i v^j \in E^*[[u,v]] \qquad (a_{i,j}^E = a_{i,j} \in E^{2(1-i-j)} = E_{2(i+j-1)})$$

be the (one dimensional commutative) formal group law over the graded ring E^* associated with the cohomology theory $E^*(-)$. The formal power series $\mu_E(u,v)$ satisfies the conditions

(i) $\mu_E(u,0) = u$, $\mu_E(0,v) = v$,
(ii) $\mu_E(u,v) = \mu_E(v,u)$,
(iii) $\mu_E(u, \mu_E(v,w)) = \mu_E(\mu_E(u,v), w)$.

It follows from (i), (ii) that

$$a_{i,0} = \begin{cases} 1 & (i=1), \\ 0 & (i \geq 2) \end{cases} \quad \text{and} \quad a_{0,j} = \begin{cases} 1 & (j=1), \\ 0 & (j \geq 2) \end{cases}$$

and $a_{i,j} = a_{j,i}$ $(i,j \geq 1)$. Therefore $\mu_E(u,v)$ is in fact of the form

$$\begin{aligned} \mu_E(u,v) &= u + v + \sum_{i,j\geq 1} a_{i,j} u^i v^j \\ &= u + v + a_{1,1}uv + a_{1,2}u^2v + a_{1,2}uv^2 + \cdots . \end{aligned}$$

We shall use this formal group law to define the *formal sum*, *formal inverse*, and *formal subtraction*. Namely, for two indeterminates X, Y, the formal sum $X +_\mu Y$ is defined as

$$X +_\mu Y := \mu_E(X,Y) = X + Y + \sum_{i,j\geq 1} a_{i,j} X^i Y^j \in E^*[[X,Y]].$$

Denote by

$$[-1]_E(X) = \iota_E(X) = \overline{X} = \sum_{j\geq 1} c_j X^j \in E^*[[X]]$$

the formal inverse series[4]. Namely $[-1]_E(X)$ is the unique formal power series satisfying the condition $\mu_E(X, [-1]_E(X)) \equiv 0$, or equivalently

[4]It might be convenient to use the notation $\overline{X}$ instead of $[-1]_E(X)$ in later sections (see §2.5, §3.2).

$X +_\mu [-1]_E(X) = 0$. It follows directly from the definition that we have

$$
\begin{aligned}
[-1]_E(X) &= -X + a_{1,1}X^2 - a_{1,1}^2X^3 + (a_{1,1}^3 + a_{1,1}a_{1,2} + 2a_{1,3} - a_{2,2})X^4 \\
&\quad +(-a_{1,1}^4 - 3a_{1,1}^2a_{1,2} - 6a_{1,1}a_{1,3} + 3a_{1,1}a_{2,2})X^5 + \cdots .
\end{aligned} \tag{3}
$$

This formal inverse allows us to define the formal subtraction:

$$X -_\mu Y := X +_\mu [-1]_E(Y) = X +_\mu \overline{Y}.$$

Finally, we define $[1]_E(X) := X$, and inductively,

$$[n]_E(X) := [n-1]_E(X) +_\mu X = \mu_E([n-1]_E(X), X) \quad (n \geq 2),$$

and $[-n]_E(X) := [n]_E([-1]_E(X)) = [-1]_E([n]_E(X))$ $(n \geq 1)$. We call $[n]_E(X)$ the *n-series* in the following. In later sections, we often need the 2-series $[2]_E(X) = X +_\mu X = \mu_E(X, X)$. If we put $[2]_E(X) = \sum_{k\geq 1} \alpha_k^E X^k$, then

$$\alpha_1^E = 2,\ \alpha_2^E = a_{1,1},\ \alpha_3^E = 2a_{1,2},\ \alpha_4^E = 2a_{1,3}+a_{2,2},\ \alpha_5^E = 2a_{1,4}+2a_{2,3}, \ldots .$$

Example 2.1.

(1) *For the ordinary cohomology theory (with integer coefficients) $E = H$, the coefficient ring is $H^* = H^*(\mathrm{pt}) = \mathbb{Z}$ $(H^0 = \mathbb{Z}$, $H^k = 0$ $(k \neq 0))$. We choose the standard orientation, namely the class of a hyperplane $x^H \in \tilde{H}^2(\mathbb{C}P^\infty)$. Then the associated formal group law is the additive formal group law $\mu_H(X, Y) = X + Y$. The 2-series is $[2]_H(X) = 2X$, and the formal inverse is $[-1]_H(X) = -X$.*

(2) *For the (topological) K-theory $E = K$, the coefficient ring is $K^* = K^*(\mathrm{pt}) = \mathbb{Z}[\beta, \beta^{-1}]$, with $\beta := 1 - \eta_1^* \in K^{-2}(\mathrm{pt}) \cong \tilde{K}(S^2)$, where η_1 stands for the tautological (or Hopf) line bunlde over $\mathbb{C}P^1 \cong S^2$ and η_1^* its dual. We choose the standard orientation $x^K := \beta^{-1}(1 - \eta_\infty^*) \in \tilde{K}^2(\mathbb{C}P^\infty)$, where η_∞ stands for the tautological line bundle over $\mathbb{C}P^\infty$ [5]. Then the associated formal group law is the multiplicative formal group*

[5] We adopt the convention due to Bott [8, Theorem 7.1], Levine-Morel [43, Example 1.1.5] so that the K-theory first Chern class of a line bundle L (over a space X) is given by $c_1^K(L) = \beta^{-1}(1 - L^*)$, where L^* denotes the dual bundle of L. In this convention, the orientation class x^K is equal to the K-theory first Chern class of the bundle η_∞, namely $c_1^K(\eta_\infty) = \beta^{-1}(1 - \eta_\infty^*)$.

law $\mu_K(X,Y) = X + Y - \beta XY$*. The* 2*-series is* $[2]_K(X) = 2X - \beta X^2$*, and the formal inverse is*

$$[-1]_K(X) = -\frac{X}{1-\beta X} = -X - \beta X^2 - \beta^2 X^3 - \beta^3 X^4 - \cdots .$$

(3) *For the complex cobordism theory* $E = MU$*, the coefficient ring* $MU^* = MU^*(\mathrm{pt})$ *is a polynomial algebra over* $\mathbb{Z}$ *on generators of degrees* $-2, -4, \ldots$ (*see e.g., Adams* [1, Part II, Theorem 8.1])*. As in Adams* [1, Part II, Examples (2.4)]*, Ravenel* [58, Example 4.1.3]*, we take the orientation class* $x^{MU} \in \tilde{MU}^2(\mathbb{C}P^\infty)$ *to be the* (*stable*) *homotopy class of the map* $\mathbb{C}P^\infty \simeq BU(1) \xrightarrow{\sim} MU(1)$*, where* $MU(1)$ *denotes the Thom space of the universal line bundle over* $BU(1)$*. Then the associated formal group law*

$$\mu_{MU}(X,Y) = X + Y + \sum_{i,j\geq 1} a_{i,j}^{MU} X^i Y^j, \quad a_{i,j}^{MU} \in MU^{2(1-i-j)}$$

is a universal formal group law first shown by Quillen [57, Theorem 2]*. Namely, for any formal group law* μ *over a commutative ring* R *with unit, there exists a unique ring homomorphism* $\theta : MU^* \longrightarrow R$ *such that* $\mu(X,Y) = (\theta_*\mu_{MU})(X,Y) := X + Y + \sum_{i,j\geq 1} \theta(a_{i,j}^{MU}) X^i Y^j$*. Quillen also showed that the coefficient ring* MU^* *is isomorphic to the Lazard ring* $\mathbb{L}$ (*see also Section* 4)*.*

2.2. E-(co)homology of the loop space of SU

Let $SU = SU(\infty)$ be the infinite special unitary group, and ΩSU its based loop space. By the celebrated Bott periodicity theorem (see Bott [6, Proposition 8.3], [7, p.314, Theorem II], Switzer [62, 16.47]), there exists a homotopy equivalence $g_\infty : BU \xrightarrow{\sim} \Omega SU$ (for the precise construction of the map g_∞, see Bott [6, Propositions 8.2, 8.3], Switzer [62, 16.47], Kono-Kozima [31, §1]). Therefore they have isomorphic (co)homology for any (co)homology theory. Moreover, both spaces ΩSU and BU are equipped with H-space structures defined by the loop multiplication and the Whitney sum map respectively, and the above homotopy equivalence is actually an equivalence as H-spaces. Since the integral homology of BU has no torsion, the (co)homology of these spaces are isomorphic as Hopf algebras for any (complex oriented) generalized cohomology theory $E^*(-)$. The following facts are well known to topologists (see e.g., Adams [1, Part II, Lemma 4.1, Lemma 4.3], Switzer [62, Theorems 16.31, 16.32]).

Theorem 2.2.

(1) *The E-cohomology ring of BU is given as follows*:

$$E^*(BU) \cong E^*[[c_1^E, c_2^E, \ldots, c_n^E, \ldots]],$$

where $c_n^E \in E^{2n}(BU)$ $(n = 1, 2, \ldots)$ *are the* E*-theory universal Chern classes. The coalgebra structure is given by*

$$\phi(c_n^E) = \sum_{i+j=n} c_i^E \otimes c_j^E \quad (c_0^E := 1),$$

where the coproduct map[6] ϕ *is induced from the multiplication* $\mu : BU \times BU \longrightarrow BU$ *that arises from the Whitney sum of complex vector bundles.*

(2) *The E-homology ring of BU is given as follows*:

$$E_*(BU) \cong E_*[\beta_1^E, \beta_2^E, \ldots, \beta_n^E, \ldots],$$

where $\beta_n^E \in E_{2n}(BU)$ $(n = 1, 2, \ldots)$ *are the images of the elements* $\beta_n^E \in E_{2n}(\mathbb{C}P^\infty)$ *under the induced homomorphism* $E_*(\mathbb{C}P^\infty) \longrightarrow E_*(BU)$ *from the natural map* $\mathbb{C}P^\infty \simeq BU(1) \longrightarrow BU$*. The coalgebra structure is given by*

$$\phi(\beta_n^E) = \sum_{i+j=n} \beta_i^E \otimes \beta_j^E \quad (\beta_0^E := 1),$$

where the coporoduct map ϕ *is induced from the diagonal map* $\Delta : BU \longrightarrow BU \times BU$*.*

Notice that the Kronecker product

$$\langle -, - \rangle : E^n(BU) \times E_m(BU) \longrightarrow E_{m-n} \quad (n, m \in \mathbb{Z})$$

induces the following isomorphism (see e.g., Switzer [62, pp.289–290, p.396])

$$E^n(BU) \xrightarrow{\sim} \mathrm{Hom}^{-n}_{E_*}(E_*(BU), E_*)$$

for each $n \in \mathbb{Z}$, and under this duality, $E^*(BU)$ and $E_*(BU)$ are mutually dual Hopf algebras over E_*. In what follows, we shall use the identification

$$E^*(\Omega SU) \xrightarrow{\sim} E^*(BU) \cong E^*[[c_1^E, c_2^E, \ldots, c_n^E, \ldots]],$$
$$E_*(\Omega SU) \xleftarrow{\sim} E_*(BU) \cong E_*[\beta_1^E, \beta_2^E, \ldots, \beta_n^E, \ldots].$$

[6] In the following, when we refer to the coproduct of a certain Hopf algebra, we shall always denote it by ϕ if there is no fear of confusion.

2.3. E-homology of the loop space of Sp

Let $Sp = Sp(\infty)$ be the infinite symplectic group and ΩSp its based loop space. Let

$$q : SU(n) \hookrightarrow Sp(n), \quad c : Sp(n) \hookrightarrow SU(2n)$$

be the *quaternionification* and the *complexification* (or *complex restriction*) respectively. These maps induce the following sequence of inclusions:

$$SU \xhookrightarrow{q} Sp \xhookrightarrow{c} SU.$$

Further they induce the following maps of based loop spaces:

$$\Omega SU \xrightarrow{\Omega q} \Omega Sp \xrightarrow{\Omega c} \Omega SU.$$

Consider the induced homomorphisms in E-homology:

$$\Omega(c \circ q)_* : E_*(\Omega SU) \xrightarrow{(\Omega q)_*} E_*(\Omega Sp) \xrightarrow{(\Omega c)_*} E_*(\Omega SU). \tag{4}$$

Then one can show the following fact:

Lemma 2.3 (Kono-Kozima [31], Corollary 6.7)**.** *The homomorphism*

$$(\Omega c)_* : E_*(\Omega Sp) \longrightarrow E_*(\Omega SU)$$

is a split monomorphism.

Therefore we can regard the algebra $E_*(\Omega Sp)$ as a subalgebra of $E_*(\Omega SU)$. Following the idea of Kono-Kozima [31] and Clarke [15], we shall describe this algebra explicitly. We extend the algebra homomorphism (4) to the following algebra homomorphism of formal power series rings:

$$\Omega(c \circ q)_* : E_*(\Omega SU)[[T]] \longrightarrow E_*(\Omega SU)[[T]].$$

Let $\beta^E(T) := \sum_{i \geq 0} \beta_i^E T^i \in E_*(\Omega SU)[[T]]$ be the formal power series with coefficients in $E_*(\Omega SU)$. By a topological argument, Kono-Kozima calculated the image of $\beta^E(T)$ under the homomorphism $\Omega(c \circ q)_*$:

Proposition 2.4 (Kono-Kozima [31], Theorem 2.9, Proposition 6.10; Clarke [15], p.18)**.**

$$\Omega(c \circ q)_*(\beta^E(T)) = \frac{\beta^E(T)}{\beta^E([-1]_E(T))}. \tag{5}$$

This formula will be crucial for our study later.

Remark 2.5. *Here we make one important remark which gives a more geometric interpretation about the map* $\Omega(c \circ q)$*. Let* $\gamma \longrightarrow BU$ *be the universal (virtual) bundle over* BU*, and* $\overline{\gamma}$ *its conjugate bundle. Denote by* $h : BU \longrightarrow BU$ *the classifying map of* $\gamma - \overline{\gamma}$*. Then Clarke* [15, Proof of Proposition 2] *showed essentially that the following diagram is homotopy-commutative*:

$$\begin{array}{ccc} BU & \underset{g_\infty}{\xrightarrow{\sim}} & \Omega SU \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \Omega(c\circ q)} \\ BU & \underset{g_\infty}{\xrightarrow{\sim}} & \Omega SU. \end{array} \tag{6}$$

From this, the expression (5) *follows easily* (*see Clarke* [15, p.18])*. The above construction* (*in the case of SO*) *of the map* $h : BU \longrightarrow BU$ *also appears in Baker* [2, p.711]*, Ray* [59, Theorem 7.1]*.*

Example 2.6. *For example, we compute*[7]:

$$\begin{aligned} \Omega(c \circ q)_*(\beta_1) &= 2\beta_1, \\ \Omega(c \circ q)_*(\beta_2) &= 2\beta_1^2 - a_{1,1}\beta_1, \\ \Omega(c \circ q)_*(\beta_3) &= 2(\beta_3 - \beta_2\beta_1 + \beta_1^3) + 2a_{1,1}\beta_2 - 3a_{1,1}\beta_1^2 + a_{1,1}^2\beta_1. \end{aligned}$$

In Clarke [15, pp.16–17], he introduced the elements $\eta_i^E \in E_{2i}(\Omega Sp)$ $(i = 1, 2, \ldots)$ so as to satisfy the following relation

$$(\Omega q)_*(\beta^E(T)) = 1 + [2]_E(T)\eta^E(T), \tag{7}$$

where $\eta^E(T) := \sum_{j \geq 0} \eta_{j+1}^E T^j \in E_*(\Omega Sp)[[T]]$. More explicitly, we have

$$(\Omega q)_*(\beta_l) = \sum_{k=1}^{l} \alpha_k \eta_{l+1-k} = 2\eta_l + \alpha_2\eta_{l-1} + \cdots + \alpha_l\eta_1 \quad (l = 1, 2, \ldots).$$

Then in $E_*(\Omega SU)$, we have

$$\Omega(c \circ q)_*(\beta_l) = 2(\Omega c)_*(\eta_l) + \alpha_2(\Omega c)_*(\eta_{l-1}) + \cdots + \alpha_l(\Omega c)_*(\eta_1) \; (l = 1, 2, \ldots). \tag{8}$$

Using Proposition 2.4 and (8), we can express $(\Omega c)_*(\eta_l)$ $(l = 1, 2, \ldots)$ in terms of β_j $(j = 1, 2, \ldots)$.

[7] Here and in what follows, we often omit the superscript E for simplicity.

Example 2.7. *Using Example* 2.6 *and* (8), *we compute*

$$\begin{aligned}
(\Omega c)_*(\eta_1) &= \beta_1, \\
(\Omega c)_*(\eta_2) &= \beta_1^2 - a_{1,1}\beta_1, \\
(\Omega c)_*(\eta_3) &= \beta_3 - \beta_2\beta_1 + \beta_1^3 + a_{1,1}\beta_2 - 2a_{1,1}\beta_1^2 + (a_{1,1}^2 - a_{1,2})\beta_1.
\end{aligned}$$

Then it can be shown that $E_*(\Omega Sp)$ is multiplicatively generated by $\eta_1^E, \eta_2^E, \ldots$ (Clarke [15, p.16, Corollary 4]). Moreover, one sees that $E_*(\Omega Sp)$ is polynomially generated by $\eta_1^E, \eta_3^E, \eta_5^E, \ldots$ (Clarke [15, p.17]). More precisely, one can express the *even* η_{2i}^E $(i = 1, 2, \ldots)$ in terms of the *odd* η_{2i-1}^E $(i = 1, 2, \ldots)$ in the following way: By Proposition 2.4, we have

$$\begin{aligned}
\frac{\beta^E(T)}{\beta^E([-1]_E(T))} &= \Omega(c \circ q)_*(\beta^E(T)) = (\Omega c)_* \circ (\Omega q)_*(\beta^E(T)) \\
&= (\Omega c)_*(1 + [2]_E(T)\eta^E(T)).
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&(\Omega c)_*((1 + [2]_E(T)\eta^E(T))(1 + [2]_E([-1]_E(T))\eta^E([-1]_E(T)))) \\
&= \frac{\beta^E(T)}{\beta^E([-1]_E(T))} \cdot \frac{\beta^E([-1]_E(T))}{\beta^E(T)} = 1.
\end{aligned}$$

Since $(\Omega c)_*$ is injective, the elements η_i^E's satisty the following relation:

$$(1 + [2]_E(T)\eta^E(T))(1 + [2]_E([-1]_E(T))\eta^E([-1]_E(T))) = 1,$$

or equivalently, we have the following (see Clarke [15, p.19]):

$$\begin{aligned}
(9) \quad & [2]_E(T)\eta^E(T) + [-2]_E(T)\eta^E([-1]_E(T)) \\
& \qquad + [2]_E(T)[-2]_E(T)\eta^E(T)\eta^E([-1]_E(T)) = 0.
\end{aligned}$$

Using the relation (9), it can be shown directly that η_{2i}^E $(i = 1, 2, \ldots)$ can be eliminated (see Clarke [15, p.19], Examples 2.9 (3), 2.10 (3) below). The coproduct $\phi(\eta_n^E)$ can be obtained immediately by the definition of η_n^E (see Clarke pp.17–18). Thus we obtain the following description of $E_*(\Omega Sp)$:

Theorem 2.8 (Clarke [15])**.** *The Hopf algebra structure of* $E_*(\Omega Sp)$ *is given as follows*:

(1) *As an algebra,*

$$\begin{aligned}
E_*(\Omega Sp) &= \frac{E_*[\eta_1^E, \eta_2^E, \ldots, \eta_i^E, \ldots]}{((1 + [2]_E(T)\eta^E(T))(1 + [2]_E([-1]_E(T))\eta^E([-1]_E(T))) = 1)} \\
&= E_*[\eta_1^E, \eta_3^E, \ldots, \eta_{2i-1}^E, \ldots].
\end{aligned}$$

(2) *The coalgebra structure is given by*

$$\begin{aligned}\phi(\eta_1^E) &= \eta_1^E \otimes 1 + 1 \otimes \eta_1^E, \\ \phi(\eta_l^E) &= \eta_l^E \otimes 1 + 1 \otimes \eta_l^E + \sum_{\substack{i+j+k=l,\\ i,j\geq 1}} \alpha_{k+1}^E \eta_i^E \otimes \eta_j^E \\ &= \eta_l^E \otimes 1 + 1 \otimes \eta_l^E + 2\sum_{\substack{i+j=l,\\ i,j\geq 1}} \eta_i^E \otimes \eta_j^E + \alpha_2^E \sum_{\substack{i+j=l-1,\\ i,j\geq 1}} \eta_i^E \otimes \eta_j^E \\ &\quad + \cdots + \alpha_{l-1}^E \eta_1^E \otimes \eta_1^E \quad (l \geq 2).\end{aligned}$$

(3) *The elements η_{2n}^E $(n = 1, 2, \ldots)$ are inductively determined by the recursive formula*:

$$[2]_E(T)\eta^E(T) + [-2]_E(T)\eta^E([-1]_E(T)) \\ +[2]_E(T)[-2]_E(T)\eta^E(T)\eta^E([-1]_E(T)) = 0.$$

Example 2.9 (Kono-Kozima [31], Theorem 2.9)**.** *For the ordinary homology theory, we have*

$$[2]_H(T) = 2T \quad \textit{and} \quad [-1]_H(T) = -T.$$

From Theorem 2.8, *the Hopf algebra structure of $H_*(\Omega Sp)$ is given as follows*:

(1) *As an algebra,*

$$H_*(\Omega Sp) \cong \mathbb{Z}[z_1, z_3, \ldots, z_{2n-1}, \ldots],$$

where we set $z_i = \eta_i^H$ $(i = 1, 2, \ldots)$.

(2) *The coalgebra structure is given by*

$$\phi(z_n) = z_n \otimes 1 + 1 \otimes z_n + 2\sum_{\substack{i+j=n,\\ i,j\geq 1}} z_i \otimes z_j.$$

(3) *The elements z_{2n} $(n = 1, 2, \ldots)$ are determined by the recursive formula*:

$$z_{2n} + \sum_{\substack{i+j=2n,\\ i,j\geq 1}} (-1)^i z_i z_j = 0.$$

Example 2.10 (Clarke [15], Theorem 1)**.** *For $\mathbb{Z}/2\mathbb{Z}$-graded K-theory*[8]*, we have*

$$[2]_K(T) = 2T + T^2 \quad \textit{and} \quad [-1]_K(T) = -\frac{T}{1+T}.$$

From Theorem 2.8*, the Hopf algebra structure of $K_0(\Omega Sp)$ is given as follows*:

(1) *As an algebra,*

$$K_0(\Omega Sp) \cong \mathbb{Z}[\eta_1^K, \eta_3^K, \dots, \eta_{2n-1}^K, \dots].$$

(2) *The coalgebra structure is given by*

$$\phi(\eta_n^K) = \eta_n^K \otimes 1 + 1 \otimes \eta_n^K + 2 \sum_{\substack{i+j=n,\\ i,j\geq 1}} \eta_i^K \otimes \eta_j^K + \sum_{\substack{i+j=n-1,\\ i,j\geq 1}} \eta_i^K \otimes \eta_j^K.$$

(3) *The elements η_{2k}^K $(k = 1, 2, \dots)$ are determined by the recursive formula*:

$$(2\eta_k^K + \eta_{k-1}^K) + \sum_{i=1}^{k-1}(-1)^i(2\eta_{k-i}^K + \eta_{k-i-1}^K)\sum_{j=1}^{i}\left\{2\binom{i-1}{j-1} + \binom{i-1}{j}\right\}\eta_j^K$$
$$+(-1)^k\sum_{j=1}^{k}\left\{2\binom{k-1}{j-1} + \binom{k-1}{j}\right\}\eta_j^K = 0 \quad (\eta_0^K := 0).$$

2.4. E-homology of the loop space of SO

Quite analogously, we can describe the E-homology Hopf algebra of the loop space on an infinite special orthogonal group $SO = SO(\infty)$. Let

$$r : SU(n) \hookrightarrow SO(2n), \quad c : SO(n) \hookrightarrow SU(n)$$

be the *real restriction* and the *complexification* respectively. These maps induce the following sequence of inclusions:

$$SU \overset{r}{\hookrightarrow} SO \overset{c}{\hookrightarrow} SU.$$

Further they induce the following maps on based loop spaces:

$$\Omega SU \xrightarrow{\Omega r} \Omega_0 SO \xrightarrow{\Omega c} \Omega SU.$$

[8]We put formally $\beta = -1$ in Example 2.1.

Consider the induced homomorphisms in E-homology:

$$\Omega(c \circ r)_* : E_*(\Omega SU) \xrightarrow{(\Omega r)_*} E_*(\Omega_0 SO) \xrightarrow{(\Omega c)_*} E_*(\Omega SU).$$

We can show the following facts:

Lemma 2.11.

(1) *The homomorphism* $(\Omega r)_* : E_*(\Omega SU) \longrightarrow E_*(\Omega_0 SO)$ *is surjective.*

(2) *The homomorphism* $(\Omega c)_* : E_*(\Omega_0 SO) \longrightarrow E_*(\Omega SU)$ *is injective.*

Proof. (1) According to Bott [6, p.44], as a generating variety for $SU(n)$ (resp. $SO(2n)$), we can take the complex projective space $\mathbb{C}P^{n-1} \cong U(n)/(U(1) \times U(n-1))$ (resp. the even dimensional complex quadric $Q_{2n-2} \cong SO(2n)/(SO(2) \times SO(2n-2))$). Let $g'_n : \mathbb{C}P^{n-1} \longrightarrow \Omega SU(n)$ (resp. $g''_n : Q_{2n-2} \longrightarrow \Omega_0 SO(2n)$) be the corresponding generating map. Then we have the natural embedding $i_n : \mathbb{C}P^{n-1} \cong U(n)/(U(1) \times U(n-1)) \hookrightarrow Q_{2n-2} \cong SO(2n)/(SO(2) \times SO(2n-2))$, and we have the following commutative diagram:

$$\begin{array}{ccc}
\mathbb{C}P^{n-1} & \xrightarrow{g'_n} & \Omega SU(n) \\
{\scriptstyle i_n}\downarrow & & \downarrow{\scriptstyle \Omega r} \\
Q_{2n-2} & \xrightarrow{g''_n} & \Omega_0 SO(2n).
\end{array}$$

Letting $n \to \infty$, this diagram induces the following commutative diagram:

$$\begin{array}{ccc}
\mathbb{C}P^{\infty} & \xrightarrow{g'_\infty} & \Omega SU \\
{\scriptstyle i_\infty}\downarrow & & \downarrow{\scriptstyle \Omega r} \\
Q_{\infty} & \xrightarrow{g''_\infty} & \Omega_0 SO.
\end{array}$$

Note that the vertical map $i_\infty : \mathbb{C}P^\infty \xrightarrow{\sim} Q_\infty$ is a homotopy equivalence. Passing to homology, we have the following commutative diagram:

$$\begin{array}{ccc}
H_*(\mathbb{C}P^{\infty}) & \xrightarrow{g'_{\infty *}} & H_*(\Omega SU) \\
{\scriptstyle i_{\infty *}}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle (\Omega r)_*} \\
H_*(Q_{\infty}) & \xrightarrow{g''_{\infty *}} & H_*(\Omega_0 SO).
\end{array}$$

By the result of Bott [6, p.36, Theorem 1] again, the Pontrjagin ring $H_*(\Omega_0 SO)$ is generated by $\operatorname{Im} g''_{\infty *}$ as an algebra. From this along with the isomorphism $i_{\infty *}$, the surjectivity of $(\Omega r)_*$ follows. For a complex oriented generalized homology theory $E_*(-)$, we argue as follows: By the universality of the complex bordism homology theory $MU_*(-)$, it is sufficient to prove that the homomorphism $(\Omega r)_* : MU_*(\Omega SU) \longrightarrow MU_*(\Omega_0 SO)$ is surjective. Since both $H_*(\Omega SU)$ and $H_*(\Omega_0 SO)$ are concentrated in even degrees and $MU_* = MU_*(\mathrm{pt})$ is also evenly graded, the homology Atiyah-Hirzebruch spectral sequence

$$E^2 = H_*(X; MU_*) \cong H_*(X) \otimes MU_* \quad \Longrightarrow \quad MU_*(X)$$

collapses for $X = \Omega SU$ and $\Omega_0 SO$. As a consequence, we have an isomorphism $MU_*(X) \cong H_*(X) \otimes MU_*$ for $X = \Omega SU$ and $\Omega_0 SO$. Therefore the surjection $(\Omega r)_* : H_*(\Omega SU) \twoheadrightarrow H_*(\Omega_0 SO)$ induces a surjection $(\Omega r)_* : MU_*(\Omega SU) \cong H_*(\Omega SU) \otimes MU_* \longrightarrow MU_*(\Omega_0 SO) \cong H_*(\Omega_0 SO) \otimes MU_*$ since MU_* is free as a $\mathbb{Z}$-module.

(2) By the Bott periodicity, we have homotopy equivalences

$$SO/U \xrightarrow{\sim} \Omega_0 SO \quad \text{and} \quad g_\infty : BU \xrightarrow{\sim} \Omega SU,$$

and the following diagram is commutative (see e.g., Cartan [12, p.11]):

$$\begin{array}{ccc} SO/U & \xrightarrow{\chi} & BU \\ \simeq \big\downarrow & & g_\infty \big\downarrow \simeq \\ \Omega_0 SO & \xrightarrow[\Omega c]{} & \Omega SU, \end{array} \tag{10}$$

where χ is induced from the inclusion $SO(2n)/U(n) \hookrightarrow U(2n)/(U(n) \times U(n))$ (note that χ is the fiber inclusion of the Borel fibration $SO/U \xhookrightarrow{\chi} BU \longrightarrow BSO$). Therefore we have to show that $\chi_* : H_*(SO/U) \longrightarrow H_*(BU)$ is injective. In cohomology, it is well known that $\chi^* \otimes \mathbb{Q} : H^*(BU) \otimes \mathbb{Q} \longrightarrow H^*(SO/U) \otimes \mathbb{Q}$ is surjective. From this and the fact that $H_*(SO/U)$ is torsion free, the injectivity of χ_* follows. For a complex oriented generalized homology theory $E_*(-)$, the same discussion as in (1) above can be applied, and we obtain the required result. Q.E.D.

By means of the monomorphism $(\Omega c)_*$, we can regard $E_*(\Omega_0 SO)$ as a subalgebra of $E_*(\Omega SU)$. We shall describe this algebra explicitly. Since $(\Omega r)_*$ is surjective, if we define

$$\tau_i^E := (\Omega r)_*(\beta_i^E) \ (i = 1, 2, \ldots), \tag{11}$$

then $E_*(\Omega_0 SO)$ is generated by τ_i^E $(i = 1, 2, \ldots)$ as an algebra. We shall determine the relations that τ_i^E's satisfy. Let $\beta^E(T) := \sum_{i \geq 0} \beta_i^E T^i \in$

$E_*(\Omega SU)[[T]]$ be the formal power series with coefficients in $E_*(\Omega SU)$, and consider the following ring homomorphism

$$\Omega(c \circ r)_* : E_*(\Omega SU)[[T]] \longrightarrow E_*(\Omega SU)[[T]].$$

Then by the similar manner to Proposition 2.4, one can show the following:

Proposition 2.12.

$$\Omega(c \circ r)_*(\beta^E(T)) = \frac{\beta^E(T)}{\beta^E([-1]_E(T))}. \tag{12}$$

We put $\tau^E(T) := \sum_{i \geq 0} \tau_i^E T^i \in E_*(\Omega_0 SO)[[T]]$ ($\tau_0^E := 1$). By definition, we have

$$(\Omega r)_*(\beta^E(T)) = \tau^E(T),$$

and hence

$$\frac{\beta^E(T)}{\beta^E([-1]_E(T))} = \Omega(c{\circ}r)_*(\beta^E(T)) = (\Omega c)_*{\circ}(\Omega r)_*(\beta^E(T)) = (\Omega c)_*(\tau^E(T)).$$

Therefore we have

$$(\Omega c)_*(\tau^E(T)\tau^E([-1]_E(T))) = \frac{\beta^E(T)}{\beta^E([-1]_E(T))} \cdot \frac{\beta^E([-1]_E(T))}{\beta^E(T)} = 1.$$

Since $(\Omega c)_*$ is injective, we obtain the following relation:

$$\tau^E(T)\tau^E([-1]_E(T)) = 1. \tag{13}$$

For $E = H$, the ordinary homology theory, $H_*(\Omega_0 SO)$ can be easily obtained from the result of Bott [6, Propositions 9.1 and 10.1]. It is generated by τ_i^H ($i = 1, 2, \ldots$) and these elements satisfy the relation (13) for $E = H$, namely $\tau^H(T)\tau^H(-T) = 1$. Since E-homology Atiyah-Hirzebruch spectral sequence

$$E^2 = H_*(\Omega_0 SO; E_*) \cong H_*(\Omega_0 SO) \otimes E_* \quad \Longrightarrow \quad E_*(\Omega_0 SO)$$

collapses by degree reasons, no other relations except (13) can arise. Thus we obtain the following description of $E_*(\Omega_0 SO)$:

Theorem 2.13. *The Hopf algebra structure of $E_*(\Omega_0 SO)$ is given as follows*:

(1) *As an algebra,*

$$E_*(\Omega_0 SO) = \frac{E_*[\tau_1^E, \tau_2^E, \dots, \tau_i^E, \dots]}{(\tau^E(T)\tau^E([-1]_E(T)) = 1)},$$

where $(\tau^E(T)\tau^E([-1]_E(T)) = 1)$ *means an ideal in* $E_*[\tau_1^E, \tau_2^E, \dots]$ *generated by the coefficients of the formal power series* $\tau^E(T)\tau^E([-1]_E(T)) - 1$.

(2) *The coalgebra structure is given by*

$$\phi(\tau_n^E) = \sum_{i+j=n} \tau_i^E \otimes \tau_j^E \quad (\tau_0^E = 1).$$

Remark 2.14. *Baker* [2, Proposition 3.3] *described the E-homology $E_*(SO/U)$ of SO/U. By the homotopy equivalence $SO/U \simeq \Omega_0 SO$ derived from the Bott periodicity theorem, his description of $E_*(SO/U)$ is the same as that of Theorem* 2.13.

Example 2.15 (Bott [6], Propositions 9.1 and 10.1)**.** *For the ordinary homology theory, we have*

$$[2]_H(T) = 2T \quad \textit{and} \quad [-1]_H(T) = -T.$$

From Theorem 2.13*, the Hopf algebra structure of $H_*(\Omega_0 SO)$ is given as follows*:

(1) *As an algebra,*

$$H_*(\Omega_0 SO) \cong \mathbb{Z}[\tau_1, \tau_2, \dots, \tau_n, \dots]/(\tau_i^2 + 2\sum_{j=1}^{i}(-1)^j \tau_{i+j}\tau_{i-j} \ (i \geq 1)).$$

(2) *The coalgebra structure is given by*

$$\phi(\tau_n) = \sum_{i+j=n} \tau_i \otimes \tau_j.$$

2.5. E-cohomology of the loop space of Sp

In a similar manner as in §2.3, we can describe the E-cohomology Hopf algebra of ΩSp. We consider the maps of based loop spaces

$$\Omega SU \xrightarrow{\Omega q} \Omega Sp \xrightarrow{\Omega c} \Omega SU,$$

and the induced homomorphisms in E-cohomology:

$$E^*(\Omega SU) \xleftarrow{(\Omega q)^*} E^*(\Omega Sp) \xleftarrow{(\Omega c)^*} E^*(\Omega SU). \tag{14}$$

Then we can show the following fact:

Lemma 2.16.
(1) $(\Omega c)^* : E^*(\Omega SU) \longrightarrow E^*(\Omega Sp)$ *is surjective.*
(2) $(\Omega q)^* : E^*(\Omega Sp) \longrightarrow E^*(\Omega SU)$ *is injective.*

Proof. (1) By the Bott periodicity theorem, we have the homotopy equivalences

$$Sp/U \xrightarrow{\sim} \Omega Sp \quad \text{and} \quad g_\infty : BU \xrightarrow{\sim} \Omega SU,$$

and the following diagram is commutative (see e.g., Cartan [12, p.10]):

$$\begin{array}{ccc} Sp/U & \xrightarrow{\chi} & BU \\ \simeq \downarrow & & g_\infty \downarrow \simeq \\ \Omega Sp & \xrightarrow[\Omega c]{} & \Omega SU, \end{array} \tag{15}$$

where χ is induced from the inclusion $Sp(n)/U(n) \hookrightarrow U(2n)/(U(n) \times U(n))$ (note that χ is the fiber inclusion of the Borel fibration $Sp/U \overset{\chi}{\hookrightarrow} BU \longrightarrow BSp$). Therefore it suffices to show that $\chi^* : E^*(BU) \longrightarrow E^*(Sp/U)$ is surjective. For $E = H$, the ordinary cohomology theory, this follows immediately from the fact that the Leray-Serre spectral sequence (with integer coefficients) for the Borel fibration $Sp/U \xrightarrow{\chi} BU \longrightarrow BSp$ collapses. Then the collapsing of the Atiyah-Hirzebruch spectral sequence

$$E_2 = H^*(X; MU^*) \cong H^*(X) \hat{\otimes} MU^* \quad \Longrightarrow \quad MU^*(X)$$

for $X = BU$ and Sp/U implies that $MU^*(X) \cong H^*(X) \hat{\otimes} MU^*$ [9] for $X = BU$ and Sp/U, and we obtain the desired result (by the universality of $MU^*(-)$).

(2) First we show that $(\Omega q)^* : H^*(\Omega Sp) \longrightarrow H^*(\Omega SU)$ is injective. By a result of Kono-Kozima (see Example 2.9), there exists elements $z_i \in H_{2i}(\Omega Sp)$ such that $(\Omega q)_*(\beta_i) = \frac{1}{2} z_i$ $(i = 1, 2, \ldots)$, and $H_*(\Omega Sp)$ is generated by z_i's as an algebra. Therefore with rational coefficients, $(\Omega q)_* \otimes \mathbb{Q} : H_*(\Omega SU) \otimes \mathbb{Q} \longrightarrow H_*(\Omega Sp) \otimes \mathbb{Q}$ is surjective. This implies that $(\Omega q)^* \otimes \mathbb{Q} : H^*(\Omega Sp) \otimes \mathbb{Q} \longrightarrow H^*(\Omega SU) \otimes \mathbb{Q}$ is injective. Since $H^*(\Omega Sp)$ and $H^*(\Omega SU)$ are torsion free, it follows that $(\Omega q)^* : H^*(\Omega Sp) \longrightarrow H^*(\Omega SU)$ is also injective. Again analogous argument as in (1) above shows that $(\Omega q)^* : E^*(\Omega Sp) \longrightarrow E^*(\Omega SU)$ is injective for any complex oriented generalized cohomology theory. Q.E.D.

[9] We need to use the *completed tensor product* $\hat{\otimes}$ because the coefficient ring $MU^* = \bigoplus_{i \geq 0} MU^{-2i}$ is negatively graded.

By means of the monomorphism $(\Omega q)^*$, we can regard the algebra $E^*(\Omega Sp)$ as a subalgebra of $E^*(\Omega SU)$. Using the same idea as in §2.4, we shall identify this algebra explicitly. We define the elements of $E^*(\Omega Sp)$ to be

$$\mu_i^E := (\Omega c)^*(c_i^E) \ (i = 1, 2, \ldots).$$

We shall determine the relations that μ_i^E's satisfy. As stated in §2.2, the Chern classes c_i^E's can be identified with the elementary symmetric functions in the variables $x_1, x_2, \ldots$. This means that the total Chern class of the universal bundle γ on BU can be written formally as

$$c^E(\gamma) = \sum_{i \geq 0} c_i^E = \prod_{i \geq 1}(1 + x_i).$$

Then the total Chern class of the conjugate bundle $\overline{\gamma}$ can be written as

$$c^E(\overline{\gamma}) = \sum_{i \geq 0} c_i^E(\overline{\gamma}) = \prod_{i \geq 1}(1 + \overline{x}_i).$$

Therefore, by the definition of the map $h : BU \longrightarrow BU$ (see Remark 2.5), we have

$$c^E(\gamma - \overline{\gamma}) = \frac{c^E(\gamma)}{c^E(\overline{\gamma})} = \prod_{i \geq 1} \frac{1 + x_i}{1 + \overline{x}_i}. \tag{16}$$

With these preliminaries, we argue as follows: The algebra homomorphism (14) extends to the algebra homomorphism of formal power series rings:

$$\Omega(c \circ q)^* : E^*(\Omega SU)[[T]] \longrightarrow E^*(\Omega SU)[[T]].$$

We put $c^E(T) := \sum_{i \geq 0} c_i^E T^i \in E^*(\Omega SU)[[T]]$, and we would like to calculate the image of this formal power series. First we can write formally

$$c^E(T) = \sum_{i \geq 0} c_i^E T^i = \prod_{i \geq 1}(1 + x_i T).$$

Define the formal power series $\overline{c^E}(T) := \sum_{i \geq 0} \overline{c_i^E} T^i$ by

$$\overline{c^E}(T) = \sum_{i \geq 0} \overline{c_i^E} T^i = \prod_{i \geq 1}(1 + \overline{x}_i T).$$

Then the homotopy-commutative diagram (6) and the formula (16) imply the following:

Proposition 2.17.

$$\Omega(c\circ q)^*(c^E(T)) = \frac{c^E(T)}{\overline{c^E}(T)}. \tag{17}$$

We also define

$$\overline{\mu_i^E} := (\Omega c)^*(\overline{c_i^E}) \ (i=1,2,\ldots),$$

and we put $\mu^E(T) := \sum_{i\geq 0}\mu_i^E T^i$, $\overline{\mu^E}(T) = \sum_{i\geq 0}\overline{\mu_i^E}T^i \in E^*(\Omega Sp)[[T]]$ ($\mu_0^E := 1$, $\overline{\mu_0^E} := 1$). By definition, we have

$$(\Omega c)^*(c^E(T)) = \mu^E(T),$$

and hence,

$$\begin{aligned}\frac{c^E(T)}{\overline{c^E}(T)} &= \Omega(c\circ q)^*(c^E(T)) = (\Omega q)^*\circ(\Omega c)^*(c^E(T)) = (\Omega q)^*(\mu^E(T)),\\ \frac{\overline{c^E}(T)}{c^E(T)} &= \Omega(c\circ q)^*(\overline{c^E}(T)) = (\Omega q)^*\circ(\Omega c)^*(\overline{c^E}(T)) = (\Omega q)^*(\overline{\mu^E}(T)).\end{aligned}$$

Therefore we have

$$(\Omega q)^*(\mu^E(T)\overline{\mu^E}(T)) = \frac{c^E(T)}{\overline{c^E}(T)}\cdot\frac{\overline{c^E}(T)}{c^E(T)} = 1.$$

Since $(\Omega q)^*$ is injective, the following relation holds:

$$\mu^E(T)\overline{\mu^E}(T) = 1.$$

Thus we obtain the following description of $E^*(\Omega Sp)$:

Theorem 2.18. *The Hopf algebra structure of $E^*(\Omega Sp)$ is given as follows*:

(1) *As an algebra,*

$$E^*(\Omega Sp) = \frac{E^*[[\mu_1^E,\mu_2^E,\ldots,\mu_i^E,\ldots]]}{(\mu^E(T)\overline{\mu^E}(T)=1)}.$$

(2) *The coalgebra structure is given by*

$$\phi(\mu_n^E) = \sum_{i+j=n}\mu_i^E\otimes\mu_j^E \quad (\mu_0^E = 1).$$

Remark 2.19. *In* [14, Proposition 6.1], *Clarke computed the K-theory of a Lagrangian Grassmannian manifold $W_n := Sp(n)/U(n)$. In the limit $n\to\infty$, Sp/U is homotopy equivalent to ΩSp by the Bott periodicity theorem, and therefore his result gives a description of $K^*(\Omega Sp)$.*

2.6. E-cohomology of the loop space of SO

Analogously we can describe the E-cohomology Hopf algebra of $\Omega_0 SO$. We consider the maps of based loop spaces

$$\Omega SU \xrightarrow{\Omega r} \Omega_0 SO \xrightarrow{\Omega c} \Omega SU,$$

and the induced homomorphisms in E-cohomology:

$$E^*(\Omega SU) \xleftarrow{(\Omega r)^*} E^*(\Omega_0 SO) \xleftarrow{(\Omega c)^*} E^*(\Omega SU).$$

Then we can show the following fact:

Lemma 2.20. *The homomorphism* $(\Omega r)^* : E^*(\Omega_0 SO) \longrightarrow E^*(\Omega SU)$ *is a split monomorphism.*

Proof. In Lemma 2.11, we proved that $(\Omega r)_* : E_*(\Omega SU) \twoheadrightarrow E_*(\Omega_0 SO)$ is surjective. Since both $E_*(\Omega SU)$ and $E_*(\Omega_0 SO)$ are free E_*-modules, the result follows. Q.E.D.

By means of the monomorphism $(\Omega r)^*$, we can regard the algebra $E^*(\Omega_0 SO)$ as a subalgebra of $E^*(\Omega SU)$. However, unlike the case of Sp, the homomorphism $(\Omega c)^*$ is not a surjection in general, and we have not been able to find the generators of $E^*(\Omega_0 SO)$ for generalized cohomology theory. Here we only comment the following fact: In the case of ordinary cohomology theory, we know that there exists a homotopy equivalence $SO/U \simeq \Omega_0 SO$ by the Bott periodicity theorem. Therefore we have the isomorphism of algebras: $H^*(SO/U) \cong H^*(\Omega_0 SO)$. The integral cohomology ring $H^*(SO/U)$ is well known since Borel (see e.g., Cartan [13, 11. Homologie et cohomologie de $SO(X)/U(X)$]), and it is a polynomial algebra generated by the elements of degrees $4k+2$ ($k = 0, 1, 2, \ldots$). Then by the collapse of the Atiyah-Hirzebruch spectral sequence

$$E_2 = H^*(\Omega_0 SO; E^*) \cong H^*(\Omega_0 SO) \hat{\otimes} E^* \quad \Longrightarrow \quad E^*(\Omega_0 SO),$$

we know that $E^*(\Omega_0 SO)$ is of the form

$$E^*(\Omega_0 SO) \cong E^*[[y_1^E, y_3^E, \ldots, y_{2k+1}^E, \ldots]],$$

where $y_{2k+1}^E \in E^{4k+2}(\Omega_0 SO)$ ($k = 0, 1, 2, \ldots$).

Finally we state the analogous formula to Proposition 2.17, which might be useful for the description of $E^*(\Omega_0 SO)$. Let $c^E(T) := \sum_{i \geq 0} c_i^E T^i \in E^*(\Omega SU)[[T]]$ be the formal power series ring with coefficients in $E^*(\Omega SU)$, and we write formally

$$c^E(T) = \sum_{i \geq 0} c_i^E T^i = \prod_{i \geq 1} (1 + x_i T).$$

We also put

$$\overline{c^E}(T) = \sum_{i \geq 0} \overline{c_i^E}\, T^i = \prod_{i \geq 1} (1 + \overline{x}_i T).$$

Consider the ring homomorphism

$$\Omega(c \circ r)^* : E^*(\Omega SU)[[T]] \longrightarrow E^*(\Omega SU)[[T]].$$

Then by the similar manner to Proposition 2.17, one can show the following:

Proposition 2.21.

$$\Omega(c \circ r)^*(c^E(T)) = \frac{c^E(T)}{\overline{c^E}(T)}. \tag{18}$$

Remark 2.22. *Baker* [2, Proposition 3.12] *described* $E^*(SO/U)$*. He asserted that*

$$E^*(SO/U) \cong E^*[[y_1^E, y_3^E, \ldots, y_{2k+1}^E, \ldots]],$$

where the elements $y_{2k+1}^E \in E^{4k+2}(SO/U)$ *are dual to a certain* E_**-basis for the module* $PE_*(SO/U)$ *of primitive elements. Notice that the formula in that paper* [2, p.713], *"*$\phi^*\chi^* c^E(T) = c^E(T)c^E([-1](T))^{-1}$*" should be corrected as in* (18).

§3. Rings of E-(co)homology Schur P- and Q-functions

In this section, we introduce the E-theoretic analogues of the rings of Schur P- and Q-functions, and describe the E-(co)homology of ΩSU, ΩSp, and $\Omega_0 SO$ in terms of symmetric functions.

3.1. Ring of symmetric functions

We use standard notations for symmetric functions as in Macdonald [46]. Let $\Lambda = \Lambda(\mathbf{x})$ be the *ring of symmetric functions* with integer coefficients in infinitely many variables $\mathbf{x} = (x_1, x_2, \ldots)$. In the sequel, we provide the variables x_i $(i = 1, 2, \ldots)$ with degree $\deg(x_i) = 1$, and regard Λ as a graded algebra over $\mathbb{Z}$. When we work with the finite set of variables $\mathbf{x}_n = (x_1, \ldots, x_n)$, we denote by $\Lambda(\mathbf{x}_n)$ the ring of symmetric polynomials, namely $\Lambda(\mathbf{x}_n) = \mathbb{Z}[x_1, \ldots, x_n]^{S_n}$, where S_n is the symmetric group on n letters acting on $\mathbf{x}_n = (x_1, \ldots, x_n)$ by permutations. Then $\Lambda(\mathbf{x})$ is the inverse limit of the $\Lambda(\mathbf{x}_n)$ in the category of *graded* rings. Note that it is possible to describe $\Lambda(\mathbf{x})$ by using the formal power series ring instead of the inverse limits (see Hoffman-Humphreys [21, p.92], Stanley [60, §7.1]). Thus $\Lambda(\mathbf{x}) \subset \mathbb{Z}[[\mathbf{x}]] = \mathbb{Z}[[x_1, x_2, \ldots]]$.

We denote by $e_i = e_i(\mathbf{x})$ $(i = 1, 2, \ldots)$ (resp. $h_i = h_i(\mathbf{x})$ $(i = 1, 2, \ldots)$) the i-th *elementary symmetric function* (resp. the i-th *complete symmetric function*). The generating functions are respectively given by

$$E(T) = \sum_{i \geq 0} e_i T^i = \prod_{i \geq 1} (1 + x_i T) \quad \text{and} \quad H(T) = \sum_{i \geq 0} h_i T^i = \prod_{i \geq 1} \frac{1}{1 - x_i T}$$

($e_0 := 1$ and $h_0 := 1$). Note that the relation $H(T)E(-T) = 1$ holds. It is well known that Λ is a polynomial algebra over $\mathbb{Z}$ in both e_i $(i = 1, 2, \ldots)$ and h_i $(i = 1, 2, \ldots)$:

$$\Lambda = \mathbb{Z}[e_1, e_2, \ldots] = \mathbb{Z}[h_1, h_2, \ldots].$$

Furthermore, by the coproduct (or comultiplication, diagonal map),

$$\phi(e_k) = \sum_{i+j=k} e_i \otimes e_j \quad \text{and} \quad \phi(h_k) = \sum_{i+j=k} h_i \otimes h_j, \tag{19}$$

Λ is a commutative and co-commutative Hopf algebra over $\mathbb{Z}$ (see Macdonald [46, I, §5, Examples 25]). Using the Hall inner product $\langle -, - \rangle : \Lambda \times \Lambda \longrightarrow \mathbb{Z}$ (see Macdonald [46, I, §4, (4.5)]), we can identify Λ with its *graded* dual $\Lambda^* = \mathrm{Hom}\,(\Lambda, \mathbb{Z})$.

Next we recall the Schur Q- and P-functions (for details, see Macdonald [46, III, §8]). Define Q_k $(k = 1, 2, \ldots)$ as the coefficients of T^k in the generating function

$$Q(T) = \sum_{k \geq 0} Q_k T^k := \prod_{i \geq 1} \frac{1 + x_i T}{1 - x_i T} = H(T)E(T)$$

($Q_0 := 1$). In other words, Q_k is defined to be $Q_k := \sum_{i+j=k} h_i e_j$. The obvious identity $Q(T)Q(-T) = 1$ yields the following relations:

$$Q_i^2 + 2 \sum_{j=1}^{i} (-1)^j Q_{i+j} Q_{i-j} = 0 \quad (i \geq 1). \tag{20}$$

Define Γ to be the subalgebra of Λ generated by Q_i's. Then we have

$$\Gamma = \mathbb{Z}[Q_1, Q_2, \ldots, Q_i, \ldots]/(Q_i^2 + 2 \sum_{j=1}^{i} (-1)^j Q_{i+j} Q_{i-j} \ (i \geq 1)).$$

The function P_k $(k = 1, 2, \ldots)$ is defined by $P_k := \frac{1}{2} Q_k = \frac{1}{2} \sum_{i+j=k} h_i e_j$. By (20), P_i's satisfy the following relations:

$$P_i^2 + 2 \sum_{j=1}^{i-1} (-1)^j P_{i+j} P_{i-j} + (-1)^i P_{2i} = 0 \quad (i \geq 1). \tag{21}$$

Note that by the above relations (21), we can eliminate P_{2i} $(i \geq 1)$. Define Γ' to be the subalgebra of Λ generated by P_i's. By definition, $\Gamma \subset \Gamma' \subset \Lambda$. Then we have

$$\begin{aligned}\Gamma' &= \mathbb{Z}[P_1, P_2, \ldots, P_i, \ldots]/(P_i^2 + 2\textstyle\sum_{j=1}^{i-1}(-1)^j P_{i+j}P_{i-j} + (-1)^i P_{2i} \ (i \geq 1)) \\ &= \mathbb{Z}[P_1, P_3, \ldots, P_{2n-1}, \ldots].\end{aligned}$$

These two subalgebras Γ and Γ' also have natural Hopf algebra structures (for the Hopf algebra structures on Γ and Γ', readers are referred to e.g., Lam-Schilling-Shimozono [39, §2]). The coproduct is given by

$$\phi(Q_k) = \sum_{i+j=k} Q_i \otimes Q_j \quad \text{and} \quad \phi(P_k) = P_k \otimes 1 + 1 \otimes P_k + 2 \sum_{i+j=k,\, i,j \geq 1} P_i \otimes P_j.$$

One can also define the pairing $[-,-] : \Gamma' \times \Gamma \longrightarrow \mathbb{Z}$ (Lam-Schilling-Shimozono [39, (2.14)], Macdonald [46, III, §8, (8.12)], Stembridge [61, (5.2)]), and then Γ and Γ' are dual Hopf algebras each other with respect to this pairing.

3.2. Rings of E-homology Schur P- and Q-functions

Motivated by the description of $E_*(\Omega_0 SO)$ and $E_*(\Omega Sp)$ in the previous two subsections §2.3 and 2.4, we define E-homology analogues of the rings of Schur P- and Q-functions (Definitions 3.4 and 3.7). First we define

$$\begin{aligned}\Lambda_*^E &:= E_* \otimes_{\mathbb{Z}} \Lambda \quad \text{(extension of coefficients)}, \\ \Lambda_E^* &:= \mathrm{Hom}_{E_*}(\Lambda_*^E, E_*) \quad \text{(graded dual of } \Lambda_*^E).\end{aligned} \tag{22}$$

Then Λ_*^E inherits naturally the Hopf algebra structure from that of Λ. Actually it is obvious from the definition that $\Lambda_*^E = E_*[h_1, h_2, \ldots]$, and the coalgebra structure is given by (19). Dually, $\Lambda_E^* = E^*[[e_1, e_2, \ldots]]$, the formal power series ring in e_i $(i = 1, 2, \ldots)$, and the coalgebra structure is also given by (19). One can verify this latter assertion by purely algebraic manner (see e.g., May-Pont [48, §21.4]).

Definition 3.1 (E*-homology Schur* Q*-functions*)**.** *We define* $\widehat{q}_k^E = \widehat{q}_k^E(\mathbf{y}) \in \Lambda_*^E$ $(k = 1, 2, \ldots)$ *as the coefficients of the generating function*

$$\widehat{q}^E(T) = \sum_{k \geq 0} \widehat{q}_k^E T^k = \frac{H(T)}{H(\overline{T})} = \prod_{i \geq 1} \frac{1 - \overline{T} y_i}{1 - T y_i} \ \in \Lambda_*^E[[T]] \tag{23}$$

$(\widehat{q}_0^E := 1)$.

For $E = H$, the ordinary homology theory, the formal inverse $\overline{T} = [-1]_H(T) = -T$, and hence $\widehat{q}_k^H$ $(k = 1, 2, \ldots)$ coincide with the usual Schur Q-functions $Q_k = \sum_{i+j=k} h_i e_j$ $(k = 1, 2, \ldots)$. By definition,

$$\widehat{q}^E(T) = H(T)E(-\overline{T}) = \left(\sum_{i \geq 0} h_i T^i\right)\left(\sum_{j \geq 0} e_j (-\overline{T})^j\right),$$

and since $\overline{T} = -T +$ higher terms in T, we have

$$\widehat{q}_k^E = Q_k + \text{lower terms in } \mathbf{y} = (y_1, y_2, \ldots).$$

Example 3.2. *By the same calculation as in Example* 2.6*, we have*

$$\begin{aligned}
\widehat{q}_1^E &= 2h_1 = Q_1, \\
\widehat{q}_2^E &= 2h_1^2 - a_{1,1}h_1 = Q_2 - a_{1,1}h_1, \\
\widehat{q}_3^E &= 2(h_3 - h_2h_1 + h_1^3) + 2a_{1,1}h_2 - 3a_{1,1}h_1^2 + a_{1,1}^2 h_1 \\
&= Q_3 + 2a_{1,1}h_2 - 3a_{1,1}h_1^2 + a_{1,1}^2 h_1.
\end{aligned}$$

By Definition 3.1, the relation

$$\widehat{q}^E(T)\widehat{q}^E(\overline{T}) = 1 \tag{24}$$

holds.

Example 3.3. *For example, the relations in low degrees are given by*

$$\begin{aligned}
(\widehat{q}_1^E)^2 &= 2\widehat{q}_2^E + a_{1,1}\widehat{q}_1^E, \\
(\widehat{q}_2^E)^2 &= -2\widehat{q}_4^E + 2\widehat{q}_3^E\widehat{q}_1^E - 3a_{1,1}\widehat{q}_3^E + a_{1,1}\widehat{q}_2^E\widehat{q}_1^E - a_{1,1}^2\widehat{q}_2^E \\
&\quad + (-a_{1,1}a_{1,2} - 2a_{1,3} + a_{2,2})\widehat{q}_1^E.
\end{aligned}$$

Definition 3.4 (*Ring of E-homology Schur Q-functions*)**.** *Define Γ_*^E to be the E_*-subalgebra of $\Lambda_*^E = E_* \otimes_{\mathbb{Z}} \Lambda$ generated by $\widehat{q}_1^E, \widehat{q}_2^E, \ldots$. More explicitly we define*

$$\Gamma_*^E = E_*[\widehat{q}_1^E, \widehat{q}_2^E, \ldots, \widehat{q}_i^E, \ldots]/(\widehat{q}^E(T)\widehat{q}^E(\overline{T}) = 1) \hookrightarrow \Lambda_*^E.$$

For $E = H$, the ordinary homology theory, $\widehat{q}_i^H = Q_i$ $(i = 1, 2, \ldots)$, and hence $\Gamma_*^H = \Gamma = \mathbb{Z}[Q_1, Q_2, \ldots]/(Q(T)Q(-T) = 1)$ is the ring of Schur Q-functions.

Analogously, we make the following definition.

Definition 3.5 (*E-homology Schur P-functions*)**.** *We define* $\widehat{p}_k^E = \widehat{p}_k^E(\mathbf{y}) \in \Lambda_*^E$ $(k = 1, 2, \ldots)$ *by the equation*

$$(25) \qquad 1 + [2]_E(T)\widehat{p}^E(T) = 1 + (T +_\mu T)\widehat{p}^E(T) = \widehat{q}^E(T) \in \Lambda_*^E[[T]],$$

where we put $\widehat{p}^E(T) := \sum_{j\geq 0} \widehat{p}_{j+1}^E T^j$. *Equivalently, since*

$$[2]_E(T)\widehat{p}^E(T) = \left(\sum_{i\geq 1} \alpha_i^E T^i\right)\left(\sum_{j\geq 0} \widehat{p}_{j+1}^E T^j\right) = \sum_{k\geq 1}\left(\sum_{j=1}^{k} \alpha_{k+1-j}^E \widehat{p}_j^E\right) T^k,$$

$\widehat{p}_k^E$*'s are defined by the recursive formulas*:

$$(26) \quad \widehat{q}_k^E = \sum_{j=1}^{k} \alpha_{k+1-j}^E \widehat{p}_j^E = 2\widehat{p}_k^E + \alpha_2^E \widehat{p}_{k-1}^E + \cdots + \alpha_k^E \widehat{p}_1^E \ (k = 1, 2, \ldots).$$

Actually we have to show that each $\widehat{p}_k^E$ $(k = 1, 2, \ldots)$ can be determined uniquely as an element of Λ_*^E by these formulas. By (25), we have

$$(27) \qquad [2]_E(T)\widehat{p}^E(T) = \widehat{q}^E(T) - 1 = \frac{H(T)}{H(\overline{T})} - 1 = \frac{H(T) - H(\overline{T})}{H(\overline{T})}.$$

Since $H(T) - H(\overline{T})$ is divisible by $T - \overline{T}$, and $T - \overline{T}$ is divisible by $[2]_E(T)$, the right hand side of (27) is divisible by $[2]_E(T)$ (see also the argument in §5.1). For the ordinary cohomology theory, we have $\widehat{q}_k^H = Q_k$ $(k = 1, 2, \ldots)$, and therefore $\widehat{p}_k^H$ $(k = 1, 2, \ldots)$ are the usual Schur P-functions P_k $(k = 1, 2, \ldots)$. In general, we have

$$\widehat{p}_k^E = P_k + \text{lower terms in } \mathbf{y} = (y_1, y_2, \ldots).$$

Example 3.6. *By the same calculation as in Example* 2.7*, we have*

$$\begin{aligned}
\widehat{p}_1^E &= h_1 = P_1, \\
\widehat{p}_2^E &= h_1^2 - a_{1,1}h_1 = P_2 - a_{1,1}h_1, \\
\widehat{p}_3^E &= (h_3 - h_2h_1 + h_1^3) + a_{1,1}h_2 - 2a_{1,1}h_1^2 + (a_{1,1}^2 - a_{1,2})h_1 \\
&= P_3 + a_{1,1}h_2 - 2a_{1,1}h_1^2 + (a_{1,1}^2 - a_{1,2})h_1.
\end{aligned}$$

By the definition (25), the following relation holds:

$$(28) \qquad (1 + [2]_E(T)\widehat{p}^E(T))(1 + [2]_E(\overline{T})\widehat{p}^E(\overline{T})) = \widehat{q}^E(T)\widehat{q}^E(\overline{T}) = 1.$$

Using the same argument as in §2.3, with the above relation (28), we can eliminate $\widehat{p}_{2i}^E$ $(i = 1, 2, \ldots)$.

Definition 3.7 (*Ring of E-homology Schur P-functions*)**.** *Define ${\Gamma'}^E_*$ to be the subalgebra of Λ^E_* generated by $\widehat{p}^E_1, \widehat{p}^E_2, \ldots$. More explicitly, we define*

$$\begin{aligned}{\Gamma'}^E_* &= E_*[\widehat{p}^E_1, \widehat{p}^E_2, \ldots, \widehat{p}^E_i, \ldots]/((1+[2]_E(T)\widehat{p}^E(T))(1+[2]_E(\overline{T})\widehat{p}^E(\overline{T})) = 1) \\ &= E_*[\widehat{p}^E_1, \widehat{p}^E_3, \ldots, \widehat{p}^E_{2i-1}, \ldots] \hookrightarrow \Lambda^E_*.\end{aligned}$$

The coalgebra structures of Γ^E_* and ${\Gamma'}^E_*$ are defined by

$$\begin{aligned}\phi(\widehat{q}^E_k) &= \sum_{i+j=k} \widehat{q}^E_i \otimes \widehat{q}^E_j \quad (k \geq 1), \\ \phi(\widehat{p}^E_1) &= \widehat{p}^E_1 \otimes 1 + 1 \otimes \widehat{p}^E_1, \\ \phi(\widehat{p}^E_l) &= \widehat{p}^E_l \otimes 1 + 1 \otimes \widehat{p}^E_l + \sum_{\substack{i+j+k=l, \\ i,j\geq 1}} \alpha^E_{k+1} \widehat{p}^E_i \otimes \widehat{p}^E_j \\ &= \widehat{p}^E_l \otimes 1 + 1 \otimes \widehat{p}^E_l + 2\sum_{\substack{i+j=l, \\ i,j\geq 1}} \widehat{p}^E_i \otimes \widehat{p}^E_j + \alpha^E_2 \sum_{\substack{i+j=l-1, \\ i,j\geq 1}} \widehat{p}^E_i \otimes \widehat{p}^E_j \\ &\quad + \cdots + \alpha^E_{l-1} \widehat{p}^E_1 \otimes \widehat{p}^E_1 \quad (l \geq 2).\end{aligned}$$

These colagebra structures make Γ^E_* and ${\Gamma'}^E_*$ into Hopf algebras over E_*.

3.3. Rings of E-cohomology Schur P- and Q-functions

Motivated by the description of $E^*(\Omega Sp)$ and $E^*(\Omega_0 SO)$ in the previous two subsections §2.5 and 2.6, we define the E-cohomology analogues of the rings of Schur P- and Q-functions. In order to do so, let $\overline{e}_i := e_i(\overline{x}_1, \overline{x}_2, \ldots)$ (resp. $\overline{h}_i := h_i(\overline{x}_1, \overline{x}_2, \ldots)$) be the i-th elementary symmetric function (resp. the i-th complete symmetric function) in the variables $\overline{x}_1, \overline{x}_2, \ldots$. Namely their generating functions are given by

$$\overline{E}(T) = \sum_{i\geq 0} \overline{e}_i T^i = \prod_{i\geq 1}(1+\overline{x}_i T) \quad \text{and} \quad \overline{H}(T) = \sum_{i\geq 0} \overline{h}_i T^i = \prod_{i\geq 1} \frac{1}{1-\overline{x}_i T}.$$

Note that the relation $\overline{H}(T)\overline{E}(-T) = 1$ holds. Then we make the following definition:

Definition 3.8. *We define $\tilde{q}^E_k = \tilde{q}^E_k(\mathbf{x}) \in \Lambda^*_E$ $(k = 1, 2, \ldots)$ as the coefficients of the generating function*

$$\tilde{q}^E(T) = \sum_{k\geq 0} \tilde{q}^E_k T^k = \frac{E(T)}{\overline{E}(T)} = \prod_{i\geq 1} \frac{1+x_i T}{1+\overline{x}_i T} \in \Lambda^*_E[[T]] \tag{29}$$

$(\tilde{q}^E_0 := 1)$.

For $E = H$, the ordinary cohomology theory, we have $\overline{x}_i = -x_i$, and hence $\tilde{q}^H_k$ $(k = 1, 2, \ldots)$ coincide with the usual Schur Q-functions $Q_k = \sum_{i+j=k} h_i e_j$ $(k = 1, 2, \ldots)$. By definition,

$$\begin{aligned} \tilde{q}^E(T) = \overline{H}(-T)E(T) \quad &= \left(\sum_{i\geq 0} \overline{h}_i\,(-T)^i\right)\left(\sum_{j\geq 0} e_j T^j\right) \\ &= \sum_{k\geq 0}\left(\sum_{i+j=k}(-1)^i\overline{h}_i e_j\right)T^k, \end{aligned}$$

and we have $\tilde{q}^E_k = \sum_{i+j=k}(-1)^i\overline{h}_i e_j$. Since $\overline{x}_i = -x_i+$higher terms in x_i, we have $\overline{h}_i = (-1)^i h_i +$ higher terms, and hence we have

$$\tilde{q}^E_k = Q_k + \text{higher terms in } \mathbf{x} = (x_1, x_2, \ldots).$$

We also define

$$(30) \qquad \overline{\tilde{q}^E}(T) = \sum_{k\geq 0}\overline{\tilde{q}^E_k}T^k = \frac{\overline{E}(T)}{E(T)} = \prod_{i\geq 1}\frac{1+\overline{x}_i T}{1+x_i T}$$

$(\overline{\tilde{q}^E_0} := 1)$. By Definition 3.8, the relation

$$(31) \qquad \tilde{q}^E(T)\overline{\tilde{q}^E}(T) = \frac{E(T)}{\overline{E}(T)}\cdot\frac{\overline{E}(T)}{E(T)} = 1$$

holds.

Definition 3.9 (Ring of E-cohomology Schur Q-functions)**.** *Define the subalgebra Γ^*_E of Λ^*_E by*

$$\Gamma^*_E = E^*[[\tilde{q}^E_1, \tilde{q}^E_2, \ldots, \tilde{q}^E_i, \ldots]]/(\tilde{q}^E(T)\overline{\tilde{q}^E}(T) = 1) \hookrightarrow \Lambda^*_E.$$

For $E = H$, the ordinary cohomology theory, $\tilde{q}^H_i = Q_i$ $(i = 1, 2, \ldots)$, and hence $\Gamma^*_H = \Gamma$ is the ring of Schur Q-functions. The coalgebra structure of Γ^*_E is defined by

$$\phi(\tilde{q}^E_k) = \sum_{i+j=k}\tilde{q}^E_i \otimes \tilde{q}^E_j.$$

This makes Γ^*_E into a Hopf algebra over E^*.

As we saw in subsection §2.6, we do not have a good grasp of $E^*(\Omega_0 SO) \cong \mathrm{Hom}_{E_*}(E_*(\Omega_0 SO), E_*)$. So we merely make the following definition.

Definition 3.10 (Ring of E-cohomology Schur P-functions). *Define* Γ'^*_E *to be the graded dual of* Γ^E_* *over* E_**, namely* $\Gamma'^*_E = \mathrm{Hom}_{E_*}(\Gamma^E_*, E_*)$*.*

Γ'^*_E has also a Hopf algebra structure over E^* as a graded dual of the Hopf algebra Γ^E_* over E_*.

3.4. Identifications with symmetric functions

In topology, it is well-known that the ordinary homology and cohomology of BU can be identified with the ring of symmetric functions Λ. This identification can be generalized obviously in generalized (co)homology theory $E^*(-)$ and $E_*(-)$. In the previous subsection, we defined

$$\begin{aligned} \Lambda^E_* &= E_* \otimes_{\mathbb{Z}} \Lambda \quad \text{(extension of coefficients)}, \\ \Lambda^*_E &= \mathrm{Hom}_{E_*}(\Lambda^E_*, E_*) \quad \text{(graded dual of } \Lambda^E_*). \end{aligned}$$

Then by Theorem 2.2, we have the following identifications of Hopf algebras:

$$\begin{aligned} E_*(\Omega SU) = E_*[\beta^E_1, \beta^E_2, \ldots] &\xrightarrow{\sim} \Lambda^E_* = E_*[h_1, h_2, \ldots], \\ \beta^E_i &\longmapsto h_i, \\ E^*(\Omega SU) = E^*[[c^E_1, c^E_2, \ldots]] &\xrightarrow{\sim} \Lambda^*_E = E^*[[e_1, e_2, \ldots]], \\ c^E_i &\longmapsto e_i. \end{aligned} \tag{32}$$

Under these identifications together with Lemmas 2.3, 2.11 (2), 2.16 (2), and 2.20, we can realize $E_*(\Omega Sp)$, $E_*(\Omega_0 SO)$, $E^*(\Omega Sp)$, and $E^*(\Omega_0 SO)$ as algebras of certain symmetric functions. By Theorems 2.13 and 2.8, we have the following:

Proposition 3.11. *There is a natural identification of Hopf algebras over* E_* :

$$\begin{aligned} E_*(\Omega_0 SO) = \frac{E_*[\tau^E_1, \tau^E_2, \ldots]}{(\tau^E(T)\tau^E(\overline{T}) = 1)} \quad &\xrightarrow{\sim} \quad \Gamma^E_* = \frac{E_*[\widehat{q}^E_1, \widehat{q}^E_2, \ldots]}{(\widehat{q}^E(T)\widehat{q}^E(\overline{T}) = 1)}, \\ \tau^E_i \quad &\longmapsto \quad \widehat{q}^E_i. \\ E_*(\Omega Sp) = E_*[\eta^E_1, \eta^E_3, \ldots] \quad &\xrightarrow{\sim} \quad \Gamma'^E_* = E_*[\widehat{p}^E_1, \widehat{p}^E_3, \ldots], \\ \eta^E_{2i-1} \quad &\longmapsto \quad \widehat{p}^E_{2i-1}. \end{aligned}$$

Furthermore,

(1) *the surjective homomorphism*

$$(\Omega r)_* : E_*(\Omega SU) \twoheadrightarrow E_*(\Omega_0 SO), \quad \beta_i^E \longmapsto \tau_i^E \ (i = 1, 2, \ldots)$$

is identified with the surjection

$$\Lambda_*^E \twoheadrightarrow \Gamma_*^E, \quad h_i \longmapsto \widehat{q}_i^E \ (i = 1, 2, \ldots),$$

(2) *the injective homomorphism* $(\Omega c)_* : E_*(\Omega_0 SO) \hookrightarrow E_*(\Omega SU)$ *is identified with the natural inclusion* $\Gamma_*^E \hookrightarrow \Lambda_*^E$*,*

(3) *the homomorphism*

$$\begin{aligned}
&(\Omega q)_* : E_*(\Omega SU) \longrightarrow E_*(\Omega Sp), \\
&\beta_i^E \longmapsto 2\eta_i^E + \alpha_2^E \eta_{i-1}^E + \cdots + \alpha_i^E \eta_1^E \ (i = 1, 2, \ldots)
\end{aligned}$$

is identified with the homomorphism

$$\Lambda_*^E \longrightarrow {\Gamma'}_*^E, \ h_i \longmapsto 2\widehat{p}_i^E + \alpha_2^E \widehat{p}_{i-1}^E + \cdots + \alpha_i^E \widehat{p}_1^E \ (i = 1, 2, \ldots),$$

(4) *the injective homomorphism* $(\Omega c)_* : E_*(\Omega Sp) \hookrightarrow E_*(\Omega SU)$ *is identified with the natural inclusion* ${\Gamma'}_*^E \hookrightarrow \Lambda_*^E$.

Dually from Theorem 2.18 and the argument in subsection §2.6, we have the following:

Proposition 3.12. *There is a natural identification of Hopf algebras over* E^* :

$$\begin{array}{rcl}
E^*(\Omega Sp) = \dfrac{E^*[[\mu_1^E, \mu_2^E, \ldots]]}{(\mu^E(T)\overline{\mu^E}(T) = 1)} & \xrightarrow{\sim} & \Gamma_E^* = \dfrac{E^*[[\tilde{q}_1^E, \tilde{q}_2^E, \ldots]]}{(\tilde{q}^E(T)\overline{\tilde{q}^E}(T) = 1)}, \\
\mu_i^E & \longmapsto & \tilde{q}_i^E. \\
E^*(\Omega_0 SO) & \xrightarrow{\sim} & {\Gamma'}_E^*.
\end{array}$$

Furthermore,

(1) *the surjective homomorphism*

$$(\Omega c)^* : E^*(\Omega SU) \twoheadrightarrow E^*(\Omega Sp), \quad c_i^E \longmapsto \mu_i^E \ (i = 1, 2, \ldots)$$

is identified with the surjection

$$\Lambda_E^* \twoheadrightarrow \Gamma_E^*, \quad e_i \longmapsto \tilde{q}_i^E \ (i = 1, 2, \ldots),$$

(2) *the injective homomorphism* $(\Omega q)^* : E^*(\Omega Sp) \hookrightarrow E^*(\Omega SU)$ *is identified with the natural inclusion* $\Gamma_E^* \hookrightarrow \Lambda_E^*$*,*

(3) *the injective homomorphism* $(\Omega r)^* : E^*(\Omega_0 SO) \hookrightarrow E^*(\Omega SU)$ *is identified with the natural inclusion* ${\Gamma'}^*_E \hookrightarrow \Lambda^*_E$.

Remark 3.13. *For* $E = H$, *the ordinary* (*co*)*homology theory so that* $\Lambda^H_* = \Lambda^*_H = \Lambda$ *and* $\Gamma^H_* = \Gamma^*_H = \Gamma$, *the homomorphisms*

$$\begin{aligned} &(\Omega r)_* : H_*(\Omega SU) \twoheadrightarrow H_*(\Omega_0 SO), && \beta_i \longmapsto \tau_i \ (i = 1, 2, \ldots), \\ &(\Omega c)^* : H^*(\Omega SU) \twoheadrightarrow H^*(\Omega Sp), && c_i \longmapsto \mu_i \ (i = 1, 2, \ldots) \end{aligned}$$

give a geometric interpretation of the ring homomorphism $\varphi : \Lambda \longrightarrow \Gamma$, $\varphi(e_n) = Q_n$ $(n \geq 1)$ (*Macdonald* [46, III, §8, Examples 10]).

§4. Universal factorial Schur P- and Q-functions

In the previous section, we have constructed certain subalgebras Γ^E_*, ${\Gamma'}^E_*$ of Λ^E_* and Γ^*_E, ${\Gamma'}^*_E$ of Λ^*_E. Our next task is to construct certain symmetric functions which may serve as nice bases for these algebras as free E_* (or E^*)-modules. As mentioned in the introduction (see also Example 2.1), Quillen showed that the complex cobordism theory $MU^*(-)$ (with the associated formal group law μ_{MU}) has the following universal property: for any complex oriented cohomology theory $E^*(-)$ (with the associated formal group law μ_E), there exists a homomorphism of rings $\theta : MU^* \longrightarrow E^*$ such that $\mu_E(X, Y) = (\theta_* \mu_{MU})(X, Y) = X + Y + \sum_{i,j \geq 1} \theta(a^{MU}_{i,j}) X^i Y^j$. Thus it will be sufficient to consider the case when $E = MU$, for general case follows immediately from the universal one by the specialization $a^{MU}_{i,j} \longmapsto \theta(a^{MU}_{i,j})$ $(i, j \geq 1)$. Recall that, by Quillen again, the coefficient ring $MU_* = MU^{-*}$ is isomorphic to the Lazard ring $\mathbb{L}$. In this section, we shall construct the "universal factorial Schur P- and Q-functions" $P^{\mathbb{L}}_\lambda(\mathbf{x}|\mathbf{b})$, $Q^{\mathbb{L}}_\lambda(\mathbf{x}|\mathbf{b})$ for λ strict partitions. Then we shall see at the end of Section 5 that these functions constitute required bases for ${\Gamma'}^*_{MU}$, Γ^*_{MU} (when $\mathbf{b} = 0$). Since the functions $P^{\mathbb{L}}_\lambda(\mathbf{x}|\mathbf{b})$'s and $Q^{\mathbb{L}}_\lambda(\mathbf{x}|\mathbf{b})$'s will be of independent interest in terms of, e.g., algebraic combinatorics, so apart from geometry, we shall deal with the above problem purely algebraically in this section.

4.1. Lazard ring $\mathbb{L}$ and the universal formal group law

We begin with collecting the basic facts about the Lazard ring (in Sections 4 and 5, we use the convention as in Levine-Morel's book [43]). In [42], Lazard considered a universal commutative formal group law of rank one $(\mathbb{L}, F_{\mathbb{L}})$, where the ring $\mathbb{L}$, called the *Lazard ring*, is isomorphic to the polynomial ring in countably infinite number of variables with integer coefficients, and $F_{\mathbb{L}} = F_{\mathbb{L}}(u, v)$ is the *universal formal group* *law* (for a construction and basic properties of $\mathbb{L}$, see Levine-Morel [43, §1.1]):

$$F_{\mathbb{L}}(u,v) = u + v + \sum_{i,j\geq 1} a_{i,j}u^i v^j \in \mathbb{L}[[u,v]].$$

This is a formal power series in u, v with coefficients $a_{i,j}$ of formal variables which satisfies the axiom of the formal group law (see §2.1). For the universal formal group law, we shall use the notation (see Levine-Morel [43, §2.3.2])

$$\begin{aligned} a +_{{}_F} b &= F_{\mathbb{L}}(a,b) && \text{(formal sum)}, \\ \overline{a} &= \chi_{{}_{\mathbb{L}}}(a) && \text{(formal inverse of } a). \end{aligned}$$

Note that $\overline{a} \in \mathbb{L}[[a]]$ is a formal power series in a with initial term $-a$, and first few terms appear in Levine-Morel [43, p.41] (see also (3)). In what follows, we regard $\mathbb{L}$ as a *graded* algebra over $\mathbb{Z}$, and the grading of $\mathbb{L}$ is given as follows (see Levine-Morel [43, p.5]): The *homological* (resp. *cohomological*) degree of $a_{i,j}$ $(i,j \geq 1)$ is defined by $\deg_h(a_{i,j}) = i+j-1$ (resp. $\deg^h(a_{i,j}) = 1-i-j$). We will indicate which grading of $\mathbb{L}$ we choose in each time, and sometimes we use the notation $\mathbb{L}^*$ or $\mathbb{L}_*$ to avoid confusion. Be aware that in topology, it is customary to give $a_{i,j}$ the *homological* degree $2(i+j-1)$ or the *cohomological* degree $2(1-i-j)$ (see Section 1).

4.2. Definition of $P_\lambda^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b})$, $Q_\lambda^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b})$

The factorial Schur P- and Q-functions were first introduced by Ivanov [27, Definitions 2.10 and 2.13] (for their definition, see also Ikeda-Mihalcea-Naruse [24, §4.2]), and it turns out that these functions represent the Schubert classes of the torus equivariant cohomology rings of Lagrangian or orthogonal Grassmannians (see a series of works due to Ikeda [22, Theorem 6.2], Ikeda-Naruse [23, Theorem 8.7]). Furthermore, in [25, Definition 2.1], they introduced the K-theoretic analogue of the factorial Schur P- and Q-functions. They showed that these functions also represent the Schubert classes of the torus equivariant K-theory of the above homogeneous spaces.

In this subsection, we shall generalize their definition to the case of *complex cobordism cohomology theory*. Besides the variables $\mathbf{x} = (x_1, x_2, \ldots)$, we prepare another set of variables $\mathbf{b} = (b_1, b_2, \ldots)$. We provide the variables $\mathbf{x} = (x_1, x_2, \ldots)$ and $\mathbf{b} = (b_1, b_2, \ldots)$ with degree $\deg(x_i) = \deg(b_i) = 1$ for $i = 1, 2, \ldots$, and we use $\deg^h(a_{i,j}) = 1-i-j$ $(i,j \geq 1)$. Following §3.2, 3.4, we introduce the rings of symmetric functions

$$\begin{aligned} \Lambda_{\mathbb{L}}(\mathbf{x}) &:= \mathbb{L}_* \otimes_{\mathbb{Z}} \Lambda(\mathbf{x}), \\ \Lambda^{\mathbb{L}}(\mathbf{x}) &:= \mathrm{Hom}_{\mathbb{L}_*}(\Lambda_{\mathbb{L}}(\mathbf{x}), \mathbb{L}_*). \end{aligned}$$

By definition, we have $\Lambda_{\mathbb{L}}(\mathbf{x}) \cong \Lambda_*^{MU}$ and $\Lambda^{\mathbb{L}}(\mathbf{x}) \cong \Lambda^*_{MU}$, and we know that

$$\begin{aligned} \Lambda_{\mathbb{L}}(\mathbf{x}) &\cong \mathbb{L}_*[h_1(\mathbf{x}), h_2(\mathbf{x}), \ldots] \ (\cong MU_*(\Omega SU)), \\ \Lambda^{\mathbb{L}}(\mathbf{x}) &\cong \mathbb{L}^*[[e_1(\mathbf{x}), e_2(\mathbf{x}), \ldots]] \ (\cong MU^*(\Omega SU)). \end{aligned}$$

Moreover we put

$$\begin{aligned} \Lambda_{\mathbb{L}}(\mathbf{x}|\mathbf{b}) &:= \mathbb{L}[[\mathbf{b}]] \,\hat{\otimes}_{\mathbb{L}}\, \Lambda_{\mathbb{L}}(\mathbf{x}), \\ \Lambda^{\mathbb{L}}(\mathbf{x}|\mathbf{b}) &:= \mathbb{L}[[\mathbf{b}]] \,\hat{\otimes}_{\mathbb{L}}\, \Lambda^{\mathbb{L}}(\mathbf{x}), \end{aligned}$$

where $\mathbb{L}[[\mathbf{b}]] = \mathbb{L}[[b_1, b_2, \ldots]]$ is a formal power series ring in the variables $b_1, b_2, \ldots$. In this section, we also consider the ring of symmetric formal power series of finite variables $\mathbf{x}_n = (x_1, \ldots, x_n)$ with coefficients in $\mathbb{L}$ (or $\mathbb{L}[[\mathbf{b}]]$), and use the notation

$$\Lambda^{\mathbb{L}}(\mathbf{x}_n) := \mathbb{L}^*[[e_1(\mathbf{x}_n), \ldots, e_n(\mathbf{x}_n)]] \quad \text{and} \quad \Lambda^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b}) := \mathbb{L}[[\mathbf{b}]] \,\hat{\otimes}_{\mathbb{L}}\, \Lambda^{\mathbb{L}}(\mathbf{x}_n).$$

In what follows, when considering polynomials or formal power series $f(x_1, x_2, \ldots)$ with coefficients in $\mathbb{L}$ (or $\mathbb{L}[[\mathbf{b}]]$), we shall call the degree with respect to $x_1, x_2, \ldots, b_1, b_2, \ldots$, and $a_{i,j}$ $(i, j \geq 1)$ the *total* degree of $f(x_1, x_2, \ldots)$.

For an integer $k \geq 1$, we define a generalization of the ordinary k-th power t^k by

$$[t|\mathbf{b}]^k := \prod_{i=1}^{k} (t +_F b_i) = (t +_F b_1)(t +_F b_2) \cdots (t +_F b_k)$$

and its variant by

$$[[t|\mathbf{b}]]^k := (t +_F t)[t|\mathbf{b}]^{k-1} = (t +_F t)(t +_F b_1)(t +_F b_2) \cdots (t +_F b_{k-1}),$$

where we set $[t|\mathbf{b}]^0 = [[t|\mathbf{b}]]^0 := 1$. For a partition, i.e., a non-increasing sequence of non-negative integers $\lambda = (\lambda_1, \ldots, \lambda_r)$ $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0)$, we set

$$[\mathbf{x}|\mathbf{b}]^\lambda := \prod_{i=1}^{r} [x_i|\mathbf{b}]^{\lambda_i} \quad \text{and} \quad [[\mathbf{x}|\mathbf{b}]]^\lambda := \prod_{i=1}^{r} [[x_i|\mathbf{b}]]^{\lambda_i}.$$

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a strict partition of length $\ell(\lambda) = r$, i.e., a sequence of positive integers such that $\lambda_1 > \cdots > \lambda_r > 0$. We denote by

$\mathcal{SP}$ the set of all strict partitions, and $\mathcal{SP}_n$ the subset of $\mathcal{SP}$ consisting of strict partitions of length $r \leq n$. The following Hall-Littlewood type definition with coefficients in $\mathbb{L}[[\mathbf{b}]]$ was suggested to us by Anatol Kirillov (cf. Kirillov-Naruse [29]), and we thank him for this.

Definition 4.1 (Universal factorial Schur P- and Q-functions)**.** *For a strict partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ with length $\ell(\lambda) = r \leq n$, we define*

(33)
$$\begin{aligned}
P^{\mathbb{L}}_{\lambda}(\mathbf{x}_n|\mathbf{b}) &= P^{\mathbb{L}}_{\lambda}(x_1, \ldots, x_n|\mathbf{b}) \\
&:= \frac{1}{(n-r)!} \sum_{w \in S_n} w \left[[\mathbf{x}|\mathbf{b}]^{\lambda} \prod_{i=1}^{r} \prod_{j=i+1}^{n} \frac{x_i +_F x_j}{x_i +_F \overline{x}_j} \right], \\
Q^{\mathbb{L}}_{\lambda}(\mathbf{x}_n|\mathbf{b}) &= Q^{\mathbb{L}}_{\lambda}(x_1, \ldots, x_n|\mathbf{b}) \\
&:= \frac{1}{(n-r)!} \sum_{w \in S_n} w \left[[[\mathbf{x}|\mathbf{b}]]^{\lambda} \prod_{i=1}^{r} \prod_{j=i+1}^{n} \frac{x_i +_F x_j}{x_i +_F \overline{x}_j} \right],
\end{aligned}$$

where the symmetric group S_n acts only on the x-variables $x_1, \ldots, x_n$ by permutations. If $\ell(\lambda) = r > n$, we set $P^{\mathbb{L}}_{\lambda}(\mathbf{x}_n|\mathbf{b}) = Q^{\mathbb{L}}_{\lambda}(\mathbf{x}_n|\mathbf{b}) = 0$.

Since

$$\begin{aligned}
&x_i +_F \overline{x}_j \\
&= (x_i - x_j)(1 + \text{higher degree terms in } x_i \text{ and } x_j \text{ with coefficients in } \mathbb{L}),
\end{aligned}$$

$P^{\mathbb{L}}_{\lambda}(x_1, \ldots, x_n|\mathbf{b})$ and $Q^{\mathbb{L}}_{\lambda}(x_1, \ldots, x_n|\mathbf{b})$ are well defined elements of $\Lambda^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b})$, namely these are symmetric formal power series with coefficients in $\mathbb{L}[[\mathbf{b}]]$ in the variables $x_1, \ldots, x_n$ and formal power series in $b_1, b_2, \ldots, b_{\lambda_1}$ for $P^{\mathbb{L}}_{\lambda}$ (resp. $b_1, b_2 \ldots, b_{\lambda_1 - 1}$ for $Q^{\mathbb{L}}_{\lambda}$). Notice that these are homogeneous formal power series of *total* degree $|\lambda| = \sum_{i=1}^{r} \lambda_i$, the size of λ. We call these formal power series the *universal factorial Schur P- and Q- functions*. In Definition 4.1, if we put $a_{i,j} = 0$ for all $i, j \geq 1$, the functions $P^{\mathbb{L}}_{\lambda}(\mathbf{x}_n|\mathbf{b})$, $Q^{\mathbb{L}}_{\lambda}(\mathbf{x}_n|\mathbf{b})$ reduce to the usual factorial Schur P- and Q-polynomials $P_{\lambda}(\mathbf{x}_n|\mathbf{b})$, $Q_{\lambda}(\mathbf{x}_n|\mathbf{b})$. If we put $a_{1,1} = \beta$, $a_{i,j} = 0$ for all $(i,j) \neq (1,1)$, then $P^{\mathbb{L}}_{\lambda}(\mathbf{x}_n|\mathbf{b})$, $Q^{\mathbb{L}}_{\lambda}(\mathbf{x}_n|\mathbf{b})$ reduce to the K-theoretic factorial Schur P- and Q-polynomials $GP_{\lambda}(\mathbf{x}_n|\mathbf{b})$, $GQ_{\lambda}(\mathbf{x}_n|\mathbf{b})$ due to Ikeda-Naruse. Thus our functions $P^{\mathbb{L}}_{\lambda}(\mathbf{x}_n|\mathbf{b})$, $Q^{\mathbb{L}}_{\lambda}(\mathbf{x}_n|\mathbf{b})$ are a generalization of these polynomials and hence *universal* in this sense.

Example 4.2.

(1) *For $n = 1$ and $\lambda = (1)$, we have*

$$\begin{aligned}
P^{\mathbb{L}}_{(1)}(x_1|\mathbf{b}) &= [x_1|\mathbf{b}] = x_1 +_F b_1 = x_1 + b_1 + a_{1,1} x_1 b_1 + \cdots, \\
Q^{\mathbb{L}}_{(1)}(x_1|\mathbf{b}) &= [[x_1|\mathbf{b}]] = x_1 +_F x_1 = 2x_1 + a_{1,1} x_1^2 + \cdots.
\end{aligned}$$

(2) *For $n = 2$ and $\lambda = (1) = (1, 0)$, we have*

$$\begin{aligned} P^{\mathbb{L}}_{(1)}(x_1, x_2|\mathbf{b}) &= (x_1 +_F x_2)\left(\frac{x_1 +_F b_1}{x_1 +_F \overline{x}_2} + \frac{x_2 +_F b_1}{x_2 +_F \overline{x}_1}\right), \\ Q^{\mathbb{L}}_{(1)}(x_1, x_2|\mathbf{b}) &= (x_1 +_F x_2)\left(\frac{x_1 +_F x_1}{x_1 +_F \overline{x}_2} + \frac{x_2 +_F x_2}{x_2 +_F \overline{x}_1}\right). \end{aligned}$$

We also set

$$\begin{aligned} P^{\mathbb{L}}_{\lambda}(\mathbf{x}_n) &= P^{\mathbb{L}}_{\lambda}(x_1, \dots, x_n) := P^{\mathbb{L}}_{\lambda}(x_1, \dots, x_n|0), \\ Q^{\mathbb{L}}_{\lambda}(\mathbf{x}_n) &= Q^{\mathbb{L}}_{\lambda}(x_1, \dots, x_n) := Q^{\mathbb{L}}_{\lambda}(x_1, \dots, x_n|0). \end{aligned}$$

These are homogeneous symmetric formal power series in $\Lambda^{\mathbb{L}}(\mathbf{x}_n)$ of total degree $|\lambda|$. By definition, we have

$$\begin{aligned} P^{\mathbb{L}}_{\lambda}(x_1, \dots, x_n) &= P_{\lambda}(x_1, \dots, x_n) + g_{\lambda}(x_1, \dots, x_n), \\ Q^{\mathbb{L}}_{\lambda}(x_1, \dots, x_n) &= Q_{\lambda}(x_1, \dots, x_n) + h_{\lambda}(x_1, \dots, x_n), \end{aligned}$$

where $P_\lambda(x_1, \dots, x_n)$ and $Q_\lambda(x_1, \dots, x_n)$ are the usual Schur P- and Q-polynomials, and $g_\lambda(x_1, \dots, x_n)$ and $h_\lambda(x_1, \dots, x_n)$ are formal power series with coefficients in $\mathbb{L}$ in $x_1, \dots, x_n$ whose degrees with respect to $x_1, \dots, x_n$ are strictly greater than $|\lambda|$. From this, we see that $P^{\mathbb{L}}_{\lambda}(x_1, \dots, x_n)$'s (resp. $Q^{\mathbb{L}}_{\lambda}(x_1, \dots, x_n)$'s), $\lambda \in \mathcal{SP}_n$, are linearly independent over $\mathbb{L}$. For suppose that there exists a homogeneous relation of the form

$$\sum_{\lambda \in \mathcal{A}} c_\lambda P^{\mathbb{L}}_{\lambda}(x_1, \dots, x_n) = 0 \tag{34}$$

with $c_\lambda \in \mathbb{L}$, $\deg(c_\lambda) + |\lambda| = N > 0$, and the summation ranges over a certain subset $\mathcal{A} \subset \mathcal{SP}_n$. Since $\deg(c_\lambda) \leq 0$ by our convention of the grading of $\mathbb{L} = \mathbb{L}^*$, we have $|\lambda| \geq N$ for all λ that appear in the above relation (34). Let $N_0 \geq N$ be the minimum of the size $|\lambda|$ for all $\lambda \in \mathcal{A}$, and $\mathcal{A}_0 := \{\lambda \in \mathcal{A} \mid |\lambda| = N_0\} \subset \mathcal{A}$. Then we have from (34), $\sum_{\lambda \in \mathcal{A}_0} c_\lambda P^{\mathbb{L}}_{\lambda}(x_1, \dots, x_n) = 0$. Thus we have

$$\begin{aligned} 0 &= \sum_{\lambda \in \mathcal{A}_0} c_\lambda P^{\mathbb{L}}_{\lambda}(x_1, \dots, x_n) = \sum_{\lambda \in \mathcal{A}_0} c_\lambda (P_\lambda(x_1, \dots, x_n) + g_\lambda(x_1, \dots, x_n)) \\ &= \sum_{\lambda \in \mathcal{A}_0} c_\lambda P_\lambda(x_1, \dots, x_n) + \sum_{\lambda \in \mathcal{A}_0} c_\lambda g_\lambda(x_1, \dots, x_n). \end{aligned}$$

Considering the degrees with respect to $x_1, \dots, x_n$, we have $\sum_{\lambda \in \mathcal{A}_0} c_\lambda P_\lambda(x_1, \dots, x_n) = 0$. On the other hand, it is known that

$P_\lambda(x_1,\dots,x_n)$'s, $\lambda\in\mathcal{SP}_n$, form a $\mathbb{Z}$-basis for the ring $\Gamma(\mathbf{x}_n)$ of *supersymmetric* polynomials in n variables $x_1,\dots,x_n$ (see Pragacz [55, Theorem 2.11], Ivanov [27, Proposition 2.12]). Therefore they form an $\mathbb{L}$-basis for the ring $\mathbb{L}\otimes_{\mathbb{Z}}\Gamma(\mathbf{x}_n)$. In particular, they are linearly independent over $\mathbb{L}$, and hence we conclude that $c_\lambda=0$ for all $\lambda\in\mathcal{A}_0$. Next we consider the minimum N_1 of the size of $|\lambda|$ for all $\lambda\in\mathcal{A}\setminus\mathcal{A}_0$, and repeat the above argument. In this way, we conclude that $c_\lambda=0$ for all $\lambda\in\mathcal{A}$.

Similarly, by definition, we have

$$\begin{aligned} P^{\mathbb{L}}_\lambda(x_1,\dots,x_n|\mathbf{b}) &= P_\lambda(x_1,\dots,x_n|\mathbf{b})+g_\lambda(x_1,\dots,x_n|\mathbf{b}),\\ Q^{\mathbb{L}}_\lambda(x_1,\dots,x_n|\mathbf{b}) &= Q_\lambda(x_1,\dots,x_n|\mathbf{b})+h_\lambda(x_1,\dots,x_n|\mathbf{b}),\end{aligned}$$

where $P_\lambda(x_1,\dots,x_n|\mathbf{b})$ and $Q_\lambda(x_1,\dots,x_n|\mathbf{b})$ are the usual factorial Schur P- and Q-polynomials, and $g_\lambda(x_1,\dots,x_n|\mathbf{b})$ and $h_\lambda(x_1,\dots,x_n|\mathbf{b})$ are formal power series in $x_1,\dots,x_n,\ b_1,b_2,\dots$ whose degrees with respect to $x_1,\dots,x_n,\ b_1,b_2,\dots$ are strictly greater than $|\lambda|$. The verification of linear independence of $P^{\mathbb{L}}_\lambda(x_1,\dots,x_n|\mathbf{b})$'s (resp. $Q^{\mathbb{L}}_\lambda(x_1,\dots,x_n|\mathbf{b})$'s), $\lambda\in\mathcal{SP}_n$, over $\mathbb{L}[[\mathbf{b}]]$ will be deferred until we show the "vanishing property" of these functions in §4.8.

4.3. $\mathbb{L}$-supersymmetric series

In this subsection, we introduce the notion of the "$\mathbb{L}$-supersymmetricity" which is a generalization of the "Q-cancellation property" due to Pragacz [55, p.145], "supersymmetricity" due to Ivanov [27, Definition 2.1], and "K-supersymmetric property (K-theoretic Q-cancellation property)" due to Ikeda-Naruse [25, Definition 1.1]. $\mathbb{L}$-supersymmetric formal series is defined as follows.

Definition 4.3 ($\mathbb{L}$-supersymmetric series)**.** *A formal power series $f(x_1,\dots,x_n)$ in the variables $x_1,\dots,x_n$ with coefficients in $\mathbb{L}$ is called $\mathbb{L}$-supersymmetric if*

(1) *$f(x_1,\dots,x_n)$ is symmetric in the variables $x_1,\dots,x_n$, and*
(2) *$f(t,\overline{t},x_3,\dots,x_n)$ does not depend on t, or in other words, $f(t,\overline{t},x_3,\dots,x_n)=f(0,0,x_3,\dots,x_n)$ holds.*

For example, the formal power series $x_1+_F x_2+_F\cdots+_F x_n$ is $\mathbb{L}$-supersymmetric. Formal power series with the $\mathbb{L}$-supersymmetric property form a subring of $\Lambda^{\mathbb{L}}(\mathbf{x}_n)$, and we denote it by $\Gamma^{\mathbb{L}}(\mathbf{x}_n)$. Hereafter we shall call $\Gamma^{\mathbb{L}}(\mathbf{x}_n)$ the *ring of $\mathbb{L}$-supersymmetric functions* in the n variables $x_1,\dots,x_n$. We also define $\Gamma^{\mathbb{L}}_+(\mathbf{x}_n)$ to be the subring of $\Gamma^{\mathbb{L}}(\mathbf{x}_n)$ consisting of $f(x_1,\dots,x_n)\in\Gamma^{\mathbb{L}}(\mathbf{x}_n)$ such that $f(t,x_2,\dots,x_n)-$

$f(0, x_2, \dots, x_n)$ is divisible by $t +_F t$. We also define

$$\Gamma^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b}) := \mathbb{L}[[\mathbf{b}]] \,\hat{\otimes}_{\mathbb{L}}\, \Gamma^{\mathbb{L}}(\mathbf{x}_n) \quad \text{and} \quad \Gamma^{\mathbb{L}}_{+}(\mathbf{x}_n|\mathbf{b}) := \mathbb{L}[[\mathbf{b}]] \,\hat{\otimes}_{\mathbb{L}}\, \Gamma^{\mathbb{L}}_{+}(\mathbf{x}_n).$$

Proposition 4.4.

(1) *For $\lambda \in \mathcal{SP}_n$, $P^{\mathbb{L}}_{\lambda}(x_1, \dots, x_n)$ and $Q^{\mathbb{L}}_{\lambda}(x_1, \dots, x_n)$ are in $\Gamma^{\mathbb{L}}(\mathbf{x}_n)$. Moreover $Q^{\mathbb{L}}_{\lambda}(x_1, \dots, x_n)$ is an element of $\Gamma^{\mathbb{L}}_{+}(\mathbf{x}_n)$.*

(2) *For $\lambda \in \mathcal{SP}_n$, $P^{\mathbb{L}}_{\lambda}(x_1, \dots, x_n|\mathbf{b})$ is in $\Gamma^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b})$, whereas $Q^{\mathbb{L}}_{\lambda}(x_1, \dots, x_n|\mathbf{b})$ is in $\Gamma^{\mathbb{L}}_{+}(\mathbf{x}_n|\mathbf{b})$.*

We can prove this proposition along the same lines as in Ikeda-Naruse [25, Propositions 3.1, 3.2]. However we shall give a proof for convenience of the reader. Before proving this proposition, we prepare two simple lemmas.

Lemma 4.5. *Let $f(x, y) \in \mathbb{L}[[x, y]]$ be a formal power series. Suppose that $f(\overline{y}, y) = 0$. Then $f(x, y)$ is divisible by $x +_F y$.*

Proof. By the assumption, $f(x, y)$ is divisible by $x - \overline{y}$. On the other hand, we see that $x +_F y = (x - \overline{y}) \times u(x, y)$ with $u(x, y) \in \mathbb{L}[[x, y]]$, a unit. From this the claim follows. Q.E.D.

Lemma 4.6. *Let $r \leq n$, and $f(x_1, \dots, x_r)$ be a formal power series in $\mathbb{L}[[x_1, \dots, x_r]]$. Then the following function is $\mathbb{L}$-supersymmetric.*

$$R_n(x_1, \dots, x_n) = \sum_{w \in S_n} w \left[f(x_1, \dots, x_r) \prod_{i=1}^{r} \prod_{j=i+1}^{n} \frac{x_i +_F x_j}{x_i +_F \overline{x}_j} \right]. \tag{35}$$

Furthermore if $f(x_1, \dots, x_r)$ is divisible by $\prod_{i=1}^{r} x_i = x_1 \cdots x_r$, then $R_n(x_1, \dots, x_n)$ has the stability in the following sense:

$$R_{n+1}(x_1, \dots, x_n, 0) = (n + 1 - r) R_n(x_1, \dots, x_n).$$

Proof. By the definition, $R_n(x_1, \dots x_n)$ is a symmetric formal power series. We shall show that this formal power series has $\mathbb{L}$-supersymmetric property. Let $x_p = t$, $x_q = \overline{t}$ for arbitrary integers p, q such that $1 \leq p < q \leq n$. We claim that each summand corresponding to $w \in S_n$ does not depend on t. For convenience we put $F_n(x_1, \dots, x_n) = \prod_{i=1}^{r} \prod_{j=i+1}^{n} \dfrac{x_i +_F x_j}{x_i +_F \overline{x}_j}$. If one of p, q is in $\{w(1), \dots, w(r)\}$, then we see directly that the term

$$F_n(x_{w(1)}, \dots, x_{w(n)}) = \prod_{i=1}^{r} \prod_{j=i+1}^{n} \frac{x_{w(i)} +_F x_{w(j)}}{x_{w(i)} +_F \overline{x}_{w(j)}}$$

vanishes because of $t +_F \overline{t} = 0$. If $p, q \in \{w(r+1), \dots, w(n)\}$, then the term

$$F_n(x_{w(1)}, \dots, x_{w(n)}) = \prod_{i=1}^{r} \frac{x_{w(i)} +_F x_{w(i+1)}}{x_{w(i)} +_F \overline{x}_{w(i+1)}} \times \cdots \times \frac{x_{w(i)} +_F x_{w(n)}}{x_{w(i)} +_F \overline{x}_{w(n)}}$$

does not depend on t because of the identity $\dfrac{x_{w(i)} +_F t}{x_{w(i)} +_F \overline{t}} \times \dfrac{x_{w(i)} +_F \overline{t}}{x_{w(i)} +_F t} = 1$. Note that in this case $f(x_{w(1)}, \dots, x_{w(r)})$ does not depend on t. Thus the first assertion is proved.

For the second assertion, we argue as follows: In the defining equation of R_{n+1}, we divide the summation into two parts: one is the summation for $w \in S_{n+1}$ such that $n+1 \in \{w(1), \dots, w(r)\}$, and the other is the summation for $w \in S_{n+1}$ such that $n+1 \in \{w(r+1), \dots, w(n+1)\}$. Let $R_{n+1}(x_1, \dots, x_{n+1}) = \sum_1 + \sum_2$ be such decomposition. If we set $x_{n+1} = 0$, then each summand in $\sum_1$ becomes zero because $f(x_{w(1)}, \dots, x_{w(r)}) = 0$ by the assumption. Therefore we have only to consider the entries in $\sum_2$. If $n+1 \in \{w(r+1), \dots, w(n+1)\}$, we have $n+1 = w(k)$ for some $r+1 \le k \le n+1$. If $k = n+1$, namely $w(n+1) = n+1$, the element $w \in S_{n+1}$ naturally defines an element $w' \in S_n$, i.e., $w'(i) = w(i)$ $(1 \le i \le n)$. Then we have $f(x_{w(1)}, \dots, x_{w(r)}) = f(x_{w'(1)}, \dots, x_{w'(r)})$. If $r+1 \le k \le n$, we consider the permutation $(w(n+1) \quad n+1)w$. Since this permutation fixes $n+1$, it defines naturally an element $w' \in S_n$, i.e., $w'(i) = w(i)$ $(1 \le i \le n,\ i \ne k)$ and $w'(k) = w(n+1)$. Then we also have $f(x_{w(1)}, \dots, x_{w(r)}) = f(x_{w'(1)}, \dots, x_{w'(r)})$. In either case, we see directly that

$$F_{n+1}(x_{w(1)}, \dots, x_{w(n+1)}) = F_n(x_{w'(1)}, \dots, x_{w'(n)}),$$

when we put $x_{n+1} = 0$. Therefore $\sum_2$ becomes

$$\begin{aligned}&(n-r+1) \sum_{w' \in S_n} f(x_{w'(1)}, \dots, x_{w'(r)}) F_n(x_{w'(1)}, \dots, x_{w'(n)}) \\ &= (n-r+1) R_n(x_1, \dots, x_n)\end{aligned}$$

when we put $x_{n+1} = 0$, and the second assertion is proved. Q.E.D.

Proof of Proposition 4.4. The assertion (2) follows immediately from (1). Let us prove (1). First assertion follows from Lemma 4.6 when we put $f(x_1, \dots, x_r) = [\mathbf{x}|0]^\lambda = \prod_{i=1}^r [x_i|0]^{\lambda_i} = \prod_{i=1}^r x_i^{\lambda_i}$, or $[[\mathbf{x}|0]]^\lambda = \prod_{i=1}^r [[x_i|0]]^{\lambda_i} = \prod_{i=1}^r (x_i +_F x_i) x_i^{\lambda_i - 1}$.

For the second statement, we shall show that $Q_\lambda^{\mathbb{L}}(t, x_2, \dots, x_n) - Q_\lambda^{\mathbb{L}}(0, x_2, \dots, x_n)$ is divisible by $t +_F t$. In the defining equation (33), we divide the summation into two parts: one is the summation for $w \in S_n$ such that $1 \in \{w(1), \dots, w(r)\}$, and the other is the summation for $w \in S_n$ such that $1 \in \{w(r+1), \dots, w(n)\}$. Let $Q^{\mathbb{L}}_\lambda(t, x_2, \dots, x_n) - Q^{\mathbb{L}}_\lambda(0, x_2, \dots, x_n) = \sum_1 + \sum_2$ be such decomposition. Since $[[x_1|\mathbf{b}]]^{\lambda_1} = (x_1 +_{\scriptscriptstyle F} x_1)[x_1|\mathbf{b}]^{\lambda_1 - 1}$, we see easily that each summand in $\sum_1$ is divisible by $t +_{\scriptscriptstyle F} t$. Each summand in $\sum_2$ is also divisible by $t +_{\scriptscriptstyle F} t$ because the numerator of the following term

$$\prod_{i=1}^{r} \frac{x_{w(i)} +_{\scriptscriptstyle F} t}{x_{w(i)} +_{\scriptscriptstyle F} \overline{t}} - 1 = \prod_{i=1}^{r} \frac{(x_{w(i)} +_{\scriptscriptstyle F} t) - (x_{w(i)} +_{\scriptscriptstyle F} \overline{t})}{x_{w(i)} +_{\scriptscriptstyle F} \overline{t}}$$

is divisible by $t +_{\scriptscriptstyle F} t$ from Lemma 4.5. Therefore we have the required result. Q.E.D.

4.4. Stability Property

In this subsection, we discuss the *stability property* of our functions $P^{\mathbb{L}}_\lambda$ and $Q^{\mathbb{L}}_\lambda$. First consider the formal power series $P^{\mathbb{L}}_\lambda(x_1, \dots, x_n)$ and $Q^{\mathbb{L}}_\lambda(x_1, \dots, x_n)$ ($\lambda \in \mathcal{SP}_n$). They have the following stability property:

Proposition 4.7. *Let λ be a strict partition of length $r \leq n$. Then*

(1) $P^{\mathbb{L}}_\lambda(x_1, \dots, x_{n-1}, 0) = P^{\mathbb{L}}_\lambda(x_1, \dots, x_{n-1})$,

(2) $Q^{\mathbb{L}}_\lambda(x_1, \dots, x_{n-1}, 0) = Q^{\mathbb{L}}_\lambda(x_1, \dots, x_{n-1})$,

where the right-hand sides are zero if $r = n$.

Proof. First note that both $[\mathbf{x}|0]^\lambda = \prod_{i=1}^r [x_i|0]^{\lambda_i} = \prod_{i=1}^r x_i^{\lambda_i}$ and $[[\mathbf{x}|0]]^\lambda = \prod_{i=1}^r [[x_i|0]]^{\lambda_i} = \prod_{i=1}^r (x_i +_{\scriptscriptstyle F} x_i) x_i^{\lambda_i - 1}$ are divisible by $\prod_{i=1}^r x_i = x_1 \cdots x_r$. Therefore the result follows from Lemma 4.6. Q.E.D.

For each positive integer n, let $\varphi_{n+1} : \Gamma^{\mathbb{L}}(\mathbf{x}_{n+1}) \longrightarrow \Gamma^{\mathbb{L}}(\mathbf{x}_n)$ be a homomorphism of graded $\mathbb{L}$-algebras given by the specialization $x_{n+1} = 0$. Then $\{\Gamma^{\mathbb{L}}(\mathbf{x}_n), \varphi_n\}_{n=1,2,\dots}$ form an inverse system of graded $\mathbb{L}$-algebras. Define the *ring of* $\mathbb{L}$*-supersymmetric fucntions* $\Gamma^{\mathbb{L}}(\mathbf{x})$ to be the inverse limit of this system. Namely, we define

$$\Gamma^{\mathbb{L}}(\mathbf{x}) := \varprojlim_n \Gamma^{\mathbb{L}}(\mathbf{x}_n).$$

For each strict partition $\lambda \in \mathcal{SP}$, by Proposition 4.7, the sequence $\{P^{\mathbb{L}}_\lambda(x_1, \dots, x_n)\}$ defines an element

$$P^{\mathbb{L}}_\lambda(\mathbf{x}) := \varprojlim_n P^{\mathbb{L}}_\lambda(x_1, \dots, x_n) \in \Gamma^{\mathbb{L}}(\mathbf{x}).$$

Similarly we define

$$\Gamma_+^{\mathbb{L}}(\mathbf{x}) := \varprojlim_n \Gamma_+^{\mathbb{L}}(\mathbf{x}_n),$$

which is a subring of $\Gamma^{\mathbb{L}}(\mathbf{x})$. Also for each strict partition $\lambda \in \mathcal{SP}$, the sequence $\{Q_\lambda^{\mathbb{L}}(x_1, \ldots, x_n)\}$ defines an element

$$Q_\lambda^{\mathbb{L}}(\mathbf{x}) := \varprojlim_n Q_\lambda^{\mathbb{L}}(x_1, \ldots, x_n) \in \Gamma_+^{\mathbb{L}}(\mathbf{x}).$$

Note that we have the following inclusion relations:

$$\Gamma_+^{\mathbb{L}}(\mathbf{x}_n) \subset \Gamma^{\mathbb{L}}(\mathbf{x}_n) \subset \Lambda^{\mathbb{L}}(\mathbf{x}_n) = \mathbb{L}[[e_1(\mathbf{x}_n), \ldots, e_n(\mathbf{x}_n)]].$$

This implies that

$$\Gamma_+^{\mathbb{L}}(\mathbf{x}) \subset \Gamma^{\mathbb{L}}(\mathbf{x}) \subset \Lambda^{\mathbb{L}}(\mathbf{x}) = \mathbb{L}[[e_1(\mathbf{x}), e_2(\mathbf{x}), \ldots]].$$

From this, $\Gamma^{\mathbb{L}}(\mathbf{x})$ and $\Gamma_+^{\mathbb{L}}(\mathbf{x})$ inherit the Hopf algebra structure over $\mathbb{L}$ from $\Lambda^{\mathbb{L}}(\mathbf{x}) = \mathbb{L}[[e_1(\mathbf{x}), e_2(\mathbf{x}), \ldots]]$.

Next consider the stability property of $P_\lambda^{\mathbb{L}}(x_1, \ldots, x_n|\mathbf{b})$ and $Q_\lambda^{\mathbb{L}}(x_1, \ldots, x_n|\mathbf{b})$ ($\lambda \in \mathcal{SP}_n$). The homomorphisms of graded $\mathbb{L}[[\mathbf{b}]]$-algebras $\psi_{n+1} : \Gamma^{\mathbb{L}}(\mathbf{x}_{n+1}|\mathbf{b}) \longrightarrow \Gamma^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b})$ given by the specialization $x_{n+1} = 0$ define an inverse system $\{\Gamma^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b}), \psi_n\}_{n=1,2,\ldots}$. Then we define

$$\Gamma^{\mathbb{L}}(\mathbf{x}|\mathbf{b}) := \varprojlim_n \Gamma^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b}) = \varprojlim_n \mathbb{L}[[\mathbf{b}]] \,\hat{\otimes}_{\mathbb{L}}\, \Gamma^{\mathbb{L}}(\mathbf{x}_n) \cong \mathbb{L}[[\mathbf{b}]] \,\hat{\otimes}_{\mathbb{L}}\, \Gamma^{\mathbb{L}}(\mathbf{x}).$$

Similarly, we define

$$\Gamma_+^{\mathbb{L}}(\mathbf{x}|\mathbf{b}) := \varprojlim_n \Gamma_+^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b}) = \varprojlim_n \mathbb{L}[[\mathbf{b}]] \,\hat{\otimes}_{\mathbb{L}}\, \Gamma_+^{\mathbb{L}}(\mathbf{x}_n) \cong \mathbb{L}[[\mathbf{b}]] \,\hat{\otimes}_{\mathbb{L}}\, \Gamma_+^{\mathbb{L}}(\mathbf{x}).$$

In contrast to Proposition 4.7, the functions $P_\lambda^{\mathbb{L}}(x_1, \ldots, x_n|\mathbf{b})$ and $Q_\lambda^{\mathbb{L}}(x_1, \ldots, x_n|\mathbf{b})$ have the following stability property:

Proposition 4.8. *Let λ be a strict partition of length $r \leq n$. Then*

(1) $P_\lambda^{\mathbb{L}}(x_1, \ldots, x_{n-2}, 0, 0|\mathbf{b}) = P_\lambda^{\mathbb{L}}(x_1, \ldots, x_{n-2}|\mathbf{b})$,

(2) $Q_\lambda^{\mathbb{L}}(x_1, \ldots, x_{n-1}, 0|\mathbf{b}) = Q_\lambda^{\mathbb{L}}(x_1, \ldots, x_{n-1}|\mathbf{b})$,

where the right-hand sides are zero if $r = n$.

Proof. The proof of (1) goes along the same lines as in Ikeda-Naruse [23, Proposition 8.2] (see also Ikeda-Naruse [25, Remark 3.1]). The assertion (2) follows from Lemma 4.6 because $[[\mathbf{x}|\mathbf{b}]]^\lambda = \prod_{i=1}^r [[x_i|\mathbf{b}]]^{\lambda_i} = \prod_{i=1}^r (x_i +_F x_i)[x_i|\mathbf{b}]^{\lambda_i - 1}$ is divisible by $\prod_{i=1}^r x_i = x_1 \cdots x_r$. Q.E.D.

Following Ikeda-Naruse [23, Proposition 8.2], we call the above stability property (1) the *stability mod* 2.

Remark 4.9. *Note that* (*see also Ikeda-Naruse* [25, Remark 3.1]) *the usual stability property* $P^{\mathbb{L}}_{\lambda}(x_1, \ldots, x_{n-1}, 0|\mathbf{b}) = P^{\mathbb{L}}_{\lambda}(x_1, \ldots, x_{n-1}|\mathbf{b})$ *does not hold in general. For from Example* 4.2*, we have* $P^{\mathbb{L}}_{(1)}(x_1|\mathbf{b}) = x_1 +_F b_1$*, whereas* $P^{\mathbb{L}}_{(1)}(x_1, 0|\mathbf{b}) = x_1 +_F b_1 + b_1 \dfrac{x_1}{\overline{x}_1}$.

Therefore in the case of $P^{\mathbb{L}}_{\lambda}(x_1, \ldots, x_n|\mathbf{b})$, there exist well-defined *even* and *odd* limit functions. In what follows, we shall use only the *even* limit functions. To be precise, for each strict partition $\lambda \in \mathcal{SP}$, we define the functions $P^{\mathbb{L}}_{\lambda}(\mathbf{x}_n|\mathbf{b})^+ \in \Gamma^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b})$ by

$$P^{\mathbb{L}}_{\lambda}(\mathbf{x}_n|\mathbf{b})^+ = P^{\mathbb{L}}_{\lambda}(x_1, \ldots, x_n|\mathbf{b})^+ := \begin{cases} P^{\mathbb{L}}_{\lambda}(x_1, \ldots, x_n|\mathbf{b}) & \text{if } n \text{ is even,} \\ P^{\mathbb{L}}_{\lambda}(x_1, \ldots, x_n, 0|\mathbf{b}) & \text{if } n \text{ is odd.} \end{cases}$$

By Proposition 4.8 (1), the sequence $\{P^{\mathbb{L}}_{\lambda}(x_1, \ldots, x_n|\mathbf{b})^+\}$ defines an element

$$P^{\mathbb{L}}_{\lambda}(\mathbf{x}|\mathbf{b})^+ := \varprojlim_{n} P^{\mathbb{L}}_{\lambda}(x_1, \ldots, x_n|\mathbf{b})^+ \in \Gamma^{\mathbb{L}}(\mathbf{x}|\mathbf{b}).$$

Similarly, for each strict partition $\lambda \in \mathcal{SP}$, define

$$Q^{\mathbb{L}}_{\lambda}(\mathbf{x}|\mathbf{b}) := \varprojlim_{n} Q^{\mathbb{L}}_{\lambda}(x_1, \ldots, x_n|\mathbf{b}) \in \Gamma^{\mathbb{L}}_{+}(\mathbf{x}|\mathbf{b}).$$

This is a well defined element of $\Gamma^{\mathbb{L}}_{+}(\mathbf{x}|\mathbf{b})$ because of Proposition 4.8 (2). Finally, note that we have the following inclusion relations:

$$\begin{aligned} \Gamma^{\mathbb{L}}_{+}(\mathbf{x}_n|\mathbf{b}) \subset \Gamma^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b}) \subset \Lambda^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b}) &= \mathbb{L}[[\mathbf{b}]] \,\hat{\otimes}_{\mathbb{L}}\, \Lambda^{\mathbb{L}}(\mathbf{x}_n) \\ &= \mathbb{L}[[\mathbf{b}]] \,\hat{\otimes}_{\mathbb{L}}\, \mathbb{L}[[e_1(\mathbf{x}_n), \ldots, e_n(\mathbf{x}_n)]]. \end{aligned}$$

This implies that

$$\Gamma^{\mathbb{L}}_{+}(\mathbf{x}|\mathbf{b}) \subset \Gamma^{\mathbb{L}}(\mathbf{x}|\mathbf{b}) \subset \Lambda^{\mathbb{L}}(\mathbf{x}|\mathbf{b}) \cong \mathbb{L}[[\mathbf{b}]] \,\hat{\otimes}_{\mathbb{L}}\, \mathbb{L}[[e_1(\mathbf{x}), e_2(\mathbf{x}), \ldots]].$$

4.5. Universal factorial Schur functions

4.5.1. *Definition of* $s^{\mathbb{L}}_{\lambda}(\mathbf{x}_n|\mathbf{b})$ Let $\mathcal{P}_n$ denote the set of all partitions of length $\leq n$. For a positive integer n, we set $\rho_n = (n, n-1, \ldots, 2, 1)$. The *factorial Schur function* $s_{\lambda}(x|a)$ (for its definition, see e.g., Ikeda-Naruse [23, §5.1], Macdonald [46, I, §3, Examples 20], Molev-Sagan [54, p.4431]) and the *factorial Grothendieck polynomial* $G_{\lambda}(x_1, \ldots, x_n|b)$ (for its definition, see Ikeda-Naruse [25, (2.13), (2.14)], McNamara [49, Definition 4.1]) can also be generalized in the universal setting. Here we assign the variables $x_1, x_2, \ldots, b_1, b_2, \ldots$ degree $\deg(x_i) = \deg(b_i) = 1$ $(i = 1, 2, \ldots)$, and use $\deg^h(a_{i,j}) = 1 - i - j$ $(i, j \geq 1)$. For partitions $\lambda, \mu \in \mathcal{P}_n$, $\lambda + \mu$ is a partition of length $\leq n$ defined by $(\lambda + \mu)_i := \lambda_i + \mu_i$ $(1 \leq i \leq n)$.

Definition 4.10 (Universal factorial Schur functions). *For a partition $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{P}_n$, we define the universal factorial Schur functions $s^{\mathbb{L}}_\lambda(x_1, \ldots, x_n|\mathbf{b})$ to be*

$$(36)\quad s^{\mathbb{L}}_\lambda(\mathbf{x}_n|\mathbf{b}) = s^{\mathbb{L}}_\lambda(x_1, \ldots, x_n|\mathbf{b}) := \sum_{w \in S_n} w \left[\frac{[\mathbf{x}|\mathbf{b}]^{\lambda + \rho_{n-1}}}{\prod_{1 \leq i < j \leq n} (x_i +_F \overline{x}_j)} \right].$$

Remark 4.11. *The non-equivariant version, i.e.,* $\mathbf{b} = 0$*, of our functions are already defined by Fel'dman* [17, Definition 4.2]*. These are called the generalized Schur polynomials there. We thank the referee who pointed out this fact to us.*

By the same reason for the universal factorial Schur P- and Q-functions, $s^{\mathbb{L}}_\lambda(x_1, \ldots, x_n|\mathbf{b})$ is a well-defined element of $\Lambda^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b})$, namely it is a symmetric formal power series with coefficients in $\mathbb{L}[[\mathbf{b}]]$ in the variables $x_1, \ldots, x_n$. It is also a homogeneous formal power series of total degree $|\lambda|$. In Definition 4.10, if we put $a_{i,j} = 0$ for all $i, j \geq 1$, the functions $s^{\mathbb{L}}_\lambda(\mathbf{x}_n|\mathbf{b})$ reduce to the usual factorial Schur polynomials $s_\lambda(\mathbf{x}_n|\mathbf{b})$. If we put $a_{1,1} = \beta$, $a_{i,j} = 0$ for all $(i, j) \neq (1, 1)$, then $s^{\mathbb{L}}_\lambda(\mathbf{x}_n|\mathbf{b})$ reduce to the factorial Grothendieck polynomials $G_\lambda(\mathbf{x}_n|\mathbf{b})$. Note that unlike the usual Schur polynomials, $s^{\mathbb{L}}_\emptyset(x_1, \ldots, x_n|\mathbf{b}) \neq 1$. For instance,

$$s^{\mathbb{L}}_\emptyset(x_1, x_2|\mathbf{b}) = \frac{x_1 +_F b_1}{x_1 +_F \overline{x}_2} + \frac{x_2 +_F b_1}{x_2 +_F \overline{x}_1} = 1 + a_{1,2}x_1x_2 + a_{1,1}a_{1,2}b_1x_1x_2 + \cdots.$$

We also define

$$s^{\mathbb{L}}_\lambda(\mathbf{x}_n) = s^{\mathbb{L}}_\lambda(x_1, \ldots, x_n) := s^{\mathbb{L}}_\lambda(x_1, \ldots, x_n|0).$$

It is a homogeneous symmetric formal power series in $\Lambda^{\mathbb{L}}(\mathbf{x}_n)$ of total degree $|\lambda|$. By definition, we have

$$(37)\qquad s^{\mathbb{L}}_\lambda(x_1, \ldots, x_n) = s_\lambda(x_1, \ldots, x_n) + g_\lambda(x_1, \ldots, x_n),$$

where $s_\lambda(x_1, \ldots, x_n)$ is the usual Schur polynomial and $g_\lambda(x_1, \ldots, x_n)$ is a symmetric formal power series with coefficients in $\mathbb{L}$ in the variables $x_1, \ldots, x_n$ whose degree with respect to $x_1, \ldots, x_n$ is strictly higher than $|\lambda|$. From this, one can show that $s^{\mathbb{L}}_\lambda(x_1, \ldots, x_n)$, $\lambda \in \mathcal{P}_n$, are linearly independent over $\mathbb{L}$ by the same argument as with the case of $P^{\mathbb{L}}_\lambda$'s (see §4.2).

4.5.2. *Basis Theorem for* $s^{\mathbb{L}}_{\lambda}(\mathbf{x}_n)$ It is well-known that the usual Schur functions (polynomials) $s_\lambda(x_1, \ldots, x_n)$, where $\ell(\lambda) \leq n$, form a $\mathbb{Z}$-basis of $\Lambda_n = \Lambda(\mathbf{x}_n) = \mathbb{Z}[x_1, \ldots, x_n]^{S_n}$ (see Macdonald [46, I, §3, (3.2)]). Also the Grothendieck polynomials $G_\lambda(x_1, \ldots, x_n)$ ($\lambda \in \mathcal{P}_n$) form a $\mathbb{Z}[\beta]$-basis of $\mathbb{Z}[\beta][x_1, \ldots, x_n]^{S_n}$ (see e.g., Ikeda-Naruse [25, Corollary 2.1], McNamara [49, Theorem 3.4]). Our functions $s^{\mathbb{L}}_{\lambda}(x_1, \ldots, x_n)$, $\lambda \in \mathcal{P}_n$, also have the similar property. Indeed one can prove the following Basis Theorem:

Proposition 4.12 (Basis Theorem)**.** $s^{\mathbb{L}}_{\lambda}(x_1, \ldots, x_n)$ $(\lambda \in \mathcal{P}_n)$ *form a formal* $\mathbb{L}$*-basis for* $\Lambda^{\mathbb{L}}(\mathbf{x}_n)$.

Proof. We already remarked the linear independence of $s^{\mathbb{L}}_{\lambda}(x_1, \ldots, x_n)$'s, $\lambda \in \mathcal{P}_n$, over $\mathbb{L}$. Therefore it only remains to prove that an arbitrary symmetric formal power series $f(x_1, \ldots, x_n) \in \Lambda^{\mathbb{L}}(\mathbf{x}_n)$ can be written as a formal $\mathbb{L}$-linear combination of $s^{\mathbb{L}}_{\lambda}(x_1, \ldots, x_n)$'s. This follows immediately from (37) and the fact that the usual Schur polynomials $s_\lambda(x_1, \ldots, x_n)$, $\lambda \in \mathcal{P}_n$, form a formal $\mathbb{L}$-basis for the ring $\Lambda^{\mathbb{L}}(\mathbf{x}_n) = \mathbb{L}[[e_1(\mathbf{x}_n), \ldots, e_n(\mathbf{x}_n)]]$. Q.E.D.

4.5.3. *Vanishing Property for* $s^{\mathbb{L}}_{\lambda}(\mathbf{x}_n|\mathbf{b})$ In order to prove the linear independence of $s^{\mathbb{L}}_{\lambda}(x_1, \ldots, x_n|\mathbf{b})$, $\lambda \in \mathcal{P}_n$, over $\mathbb{L}[[\mathbf{b}]]$, we shall make use of the following vanishing property for $s^{\mathbb{L}}_{\lambda}(\mathbf{x}_n|\mathbf{b})$ (for the vanishing property of the factorial Schur polynomials $s_\lambda(x|a)$, see Molev-Sagan [54, Theorem 2.1]. For the Grothendieck polynomials $G_\lambda(x_1, \ldots, x_n|b)$, see Ikeda-Naruse [25, Proposition 2.2], McNamara [49, Theorem 4.4]). Given a partition $\mu \in \mathcal{P}_n$, define the sequence

$$\overline{\mathbf{b}}_\mu := (\overline{b}_{\mu_1}, \overline{b}_{\mu_2}, \ldots, \overline{b}_{\mu_i}, \ldots, \overline{b}_{\mu_n}).$$

Then we have the following. Here we identify a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ with its Young diagram

$$D(\lambda) = \{(i,j) \in \mathbb{Z}^2 \mid i \geq 1,\ 1 \leq j \leq \lambda_i\}$$

(see Macdonald [46, I, p.2]).

Proposition 4.13 (Vanishing Property)**.** *Let* $\lambda, \mu \in \mathcal{P}_n$. *Then we have*

$$s^{\mathbb{L}}_{\lambda}(\overline{\mathbf{b}}_{\mu+\rho_n}|\mathbf{b}) = \begin{cases} 0 & \text{if} \quad \mu \not\supset \lambda, \\ \displaystyle\prod_{(i,j)\in\lambda} (\overline{b}_{\lambda_i+n-i+1} +_F b_{n+j-{}^t\lambda_j}) & \text{if} \quad \mu = \lambda, \end{cases}$$

where $\rho_n = (n, n-1, \ldots, 2, 1)$ *and* ${}^t\lambda$ *is the diagram conjugate to* λ.

Proof. The proof is essentially identical to that of Molev-Sagan [54, Theorem 2.1 (Vanishing Theorem)]. See also Ivanov [26, Theorem 1.5 (the zero property)]. We shall exhibit a proof for the sake of completeness. Let us prove the first assertion. We use the expression (36):

$$
\begin{aligned}
s^{\mathbb{L}}_{\lambda}(\mathbf{x}_n|\mathbf{b}) &= \sum_{w\in S_n} w\left[\frac{[\mathbf{x}|\mathbf{b}]^{\lambda+\rho_{n-1}}}{\prod_{1\le i<j\le n}(x_i +_F \overline{x}_j)}\right] \\
&= \sum_{w\in S_n}\left[\frac{\prod_{i=1}^n [x_{w(i)}|\mathbf{b}]^{\lambda_i+n-i}}{\prod_{1\le i<j\le n}(x_{w(i)} +_F \overline{x}_{w(j)})}\right].
\end{aligned}
\tag{38}
$$

We wish to show that this becomes zero when we make a substitution

$$\mathbf{x}_n = \overline{\mathbf{b}}_{\mu+\rho_n} = (\overline{b}_{\mu_1+n}, \overline{b}_{\mu_2+n-1}, \ldots, \overline{b}_{\mu_i+n-i+1}, \ldots, \overline{b}_{\mu_n+1}).$$

The condition $\lambda \not\subset \mu$ implies that there exists an index k such that $\mu_k < \lambda_k$ (and hence $\mu_k+1 \le \lambda_k$). For an arbitrary permutation $w \in S_n$, there exists a positive integer $1 \le j \le k$ such that $w(j) \ge k$. Thus we have $\mu_{w(j)} \le \mu_k < \lambda_k \le \lambda_j$, and hence the following inequalities hold:

$$\mu_{w(j)} + n - w(j) + 1 \le \mu_k + n - k + 1 \le \lambda_k + n - k \le \lambda_j + n - j.$$

Therefore the term $[x_{w(j)}|\mathbf{b}]^{\lambda_j+n-j}$ becomes zero when we specialize $x_{w(j)}$ to $\overline{b}_{\mu_{w(j)}+n-w(j)+1}$, and the first assertion follows.

Next consider the case $\mu = \lambda$. We shall show that in the expression (38), the terms other than the one corresponding to $w = e$ vanish when we set $\mathbf{x}_n = \overline{\mathbf{b}}_{\lambda+\rho_n}$. For $w \ne e$, there exists a positive integer $1 \le k \le n$ such that $w(k) > k$ (and hence $w(k) \ge k+1$). Thus we have an inequality

$$\lambda_{w(k)} + n - w(k) + 1 \le \lambda_k + n - k.$$

Therefore the term $[x_{w(k)}|\mathbf{b}]^{\lambda_k+n-k}$ becomes zero when we specialize $x_{w(k)}$ to $\overline{b}_{\lambda_{w(k)}+n-w(k)+1}$. The term corresponding to $w = e$ is

$$
\begin{aligned}
&\frac{[\overline{\mathbf{b}}_{\lambda+\rho_n}|\mathbf{b}]^{\lambda+\rho_{n-1}}}{\prod_{1\le i<j\le n}(\overline{b}_{\lambda_i+n-i+1} +_F b_{\lambda_j+n-j+1})} \\
= \; &\frac{\prod_{i=1}^n [\overline{b}_{\lambda_i+n-i+1}|\mathbf{b}]^{\lambda_i+n-i}}{\prod_{1\le i<j\le n}(\overline{b}_{\lambda_i+n-i+1} +_F b_{\lambda_j+n-j+1})} \\
= \; &\frac{\prod_{i=1}^n \prod_{j=1}^{\lambda_i+n-i}(\overline{b}_{\lambda_i+n-i+1} +_F b_j)}{\prod_{i=1}^n \prod_{j=i+1}^{n}(\overline{b}_{\lambda_i+n-i+1} +_F b_{\lambda_j+n-j+1})}.
\end{aligned}
$$

By cancellation, we obtain the required formula. Q.E.D.

4.5.4. *Algebraic localization map of type A_∞* In §4.5.2, we have proven the Basis Theorem for $s^{\mathbb{L}}_\lambda(\mathbf{x}_n)$'s. We shall prove the Basis Theorem for $s^{\mathbb{L}}_\lambda(\mathbf{x}_n|\mathbf{b})$'s in §4.5.5. In order to prove this, we shall exploit the "localization technique". This subsubsection will be devoted to a brief introduction of such technique. Here we shall introduce the following two devices: the *GKM ring* and the *algebraic localization map*. In order to define these two notions, we freely use the standard notations and conventions about the Weyl group, the root system, the Grassmannian elements of type A_∞. We collect them in the Appendix 6.2.1 for the reader's convenience.

First we introduce the GKM ring. Let L denote a free $\mathbb{Z}$-module with a basis $\{t_i\}_{i\geq 1}$. The positive roots $\Delta^+_A \subset L$ of type A_∞ are given by

$$\Delta^+_A = \{\alpha_{j,i} = t_j - t_i \mid j > i \geq 1\}.$$

The simple roots are given by $\alpha_i = \alpha_{i+1,i}$ $(i \geq 1)$. We define a map $e : L \longrightarrow \mathbb{L}[[\mathbf{b}]]$ by setting $e(t_i) := b_i$ $(i \geq 1)$ and by the rule $e(\alpha + \alpha') := e(\alpha) +_F e(\alpha')$ for $\alpha, \alpha' \in L$. Note that by definition, we have $e(-\alpha) = \overline{e(\alpha)}$ for $\alpha \in L$. For the simple root $\alpha_i = t_{i+1} - t_i$ $(i \geq 1)$, we have

$$e(\alpha_i) = e(t_{i+1} - t_i) = b_{i+1} +_F \overline{b}_i \quad (i \geq 1).$$

Let $\mathrm{Map}\,(\mathcal{P}_n, \mathbb{L}[[\mathbf{b}]])$ denote the set of all maps $\psi : \mathcal{P}_n \longrightarrow \mathbb{L}[[\mathbf{b}]],\ \lambda \longmapsto \psi_\lambda$. It has a natural $\mathbb{L}[[\mathbf{b}]]$-algebra structure under pointwise multiplication $(\psi\cdot\varphi)_\lambda := \psi_\lambda \cdot \varphi_\lambda$ for $\psi, \varphi \in \mathrm{Map}\,(\mathcal{P}_n, \mathbb{L}[[\mathbf{b}]])$ and scalar multiplication $(c\cdot\psi)_\lambda := c\cdot\psi_\lambda$ for $c \in \mathbb{L}[[\mathbf{b}]]$, $\psi \in \mathrm{Map}\,(\mathcal{P}_n, \mathbb{L}[[\mathbf{b}]])$. Hereafter we use the identification $\mathrm{Map}\,(\mathcal{P}_n, \mathbb{L}[[\mathbf{b}]]) \cong \prod_{\mu\in\mathcal{P}_n}(\mathbb{L}[[\mathbf{b}]])_\mu$. Thus an element $\psi \in \mathrm{Map}\,(\mathcal{P}_n, \mathbb{L}[[\mathbf{b}]])$ can be regarded as a collection $\psi = (\psi_\mu)_{\mu\in\mathcal{P}_n}$ with $\psi_\mu \in \mathbb{L}[[\mathbf{b}]]$. With these notations, we introduce the GKM ring.

Definition 4.14 (GKM ring of type A_∞)**.** *Let $\Psi^{(n)}_A$ be a subring of* $\mathrm{Map}\,(\mathcal{P}_n, \mathbb{L}[[\mathbf{b}]]) \cong \prod_{\mu\in\mathcal{P}_n}(\mathbb{L}[[\mathbf{b}]])_\mu$ *that consists of elements $\psi = (\psi_\mu)_{\mu\in\mathcal{P}_n}$ satisfying the following condition* (*GKM condition*) :

$$(39)\quad \psi_{s_\alpha\mu} - \psi_\mu \in e(-\alpha)\,\mathbb{L}[[\mathbf{b}]] \quad \textit{for all}\ \mu \in \mathcal{P}_n\ \textit{and all}\ \alpha \in \Delta^+ = \Delta^+_A.$$

It is easy to see that $\Psi^{(n)}_A$ is indeed a subring (more precisely, an $\mathbb{L}[[\mathbf{b}]]$-subalgebra) of $\prod_{\mu\in\mathcal{P}_n}(\mathbb{L}[[\mathbf{b}]])_\mu$. We call the ring $\Psi^{(n)}_A$ the *GKM ring of type A_∞*.

Next we shall introduce the algebraic localization map of type A_∞. For each partition $\mu \in \mathcal{P}_n$, define an $\mathbb{L}[[\mathbf{b}]]$-algebra homomorphism

$$\phi^{(n)}_{\mu,A} : \Lambda^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b}) = \mathbb{L}[[\mathbf{b}]]\,\hat{\otimes}_{\mathbb{L}}\Lambda^{\mathbb{L}}(\mathbf{x}_n) \longrightarrow \mathbb{L}[[\mathbf{b}]],\ F = F(\mathbf{x}_n|\mathbf{b}) \longmapsto \phi^{(n)}_{\mu,A}(F)$$

by $\phi^{(n)}_{\mu,A}(F) := F(\overline{\mathbf{b}}_{\mu+\rho_n}|\mathbf{b})$.

Definition 4.15 (Algebraic localization map of type A_∞)**.** *Define the homomorphism of* $\mathbb{L}[[\mathbf{b}]]$*-algebras to be*

(40)
$$\Phi^{(n)}_A : \Lambda^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b}) \longrightarrow \prod_{\mu\in\mathcal{P}_n} (\mathbb{L}[[\mathbf{b}]])_\mu, \ F \longmapsto \Phi^{(n)}_A(F) := (\phi^{(n)}_{\mu,A}(F))_{\mu\in\mathcal{P}_n}.$$

We call the homomorphism $\Phi^{(n)}_A$ the *algebraic localization map of type* A_∞.

Lemma 4.16.

(1) *The image of* $\Phi^{(n)}_A$ *is contained in the GKM ring* $\Psi^{(n)}_A$*, that is,* $\mathrm{Im}\,(\Phi^{(n)}_A) \subset \Psi^{(n)}_A$*.*

(2) *The homomorphism* $\Phi^{(n)}_A$ *is injective.*

Proof. (1) For an arbitrary $F = F(\mathbf{x}_n|\mathbf{b}) \in \Lambda^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b})$, we have to show that $F(\overline{\mathbf{b}}_{s_\alpha\mu+\rho_n}|\mathbf{b}) - F(\overline{\mathbf{b}}_{\mu+\rho_n}|\mathbf{b}) \in e(-\alpha)\mathbb{L}[[\mathbf{b}]]$ for all $\mu\in\mathcal{P}_n$ and for all $\alpha\in\Delta^+$. More accurately, we have to show that

$$F(\overline{\mathbf{b}}_{s_{t_j-t_i}\mu+\rho_n}|\mathbf{b}) - F(\overline{\mathbf{b}}_{\mu+\rho_n}|\mathbf{b}) \in \langle \overline{b}_j +_F b_i\rangle \quad \text{for } j > i \geq 1,$$

where $\langle \overline{b}_j +_F b_i\rangle$ denotes an ideal of $\mathbb{L}[[\mathbf{b}]]$ generated by $\overline{b}_j +_F b_i$. This is a direct consequence of the action of the reflection $s_{t_j-t_i}$ on $\mu\in\mathcal{P}_n$ (see Appendix 6.2.1) and Lemma 4.5.

(2) We shall prove the following statement:

> For $F\in\Lambda^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b})$, suppose that there exists a positive integer N (which may depend on n) such that $\phi^{(n)}_\mu(F) = 0$ for all partitions $\mu\in\mathcal{P}_n$ containing $(N^n) = (N, N, \ldots, N)$. Then we have $F = 0$.

From this, the conclusion of (2) immediately follows. Let us prove the above statement by induction on the number n of x-variables. For the case $n = 1$, the proof is easy. For if $F = F(x_1|\mathbf{b}) \in \mathbb{L}[[\mathbf{b}]][[x_1]]$ satisfies the assumption, we have $F(\overline{b}_{N+k}|\mathbf{b}) = 0$ for all $k \geq 1$. This implies that $F(x_1|\mathbf{b})$ is divisible by $x_1 - \overline{b}_{N+k}$ for all $k\geq 1$. From this, we conclude that $F = 0$ as a formal power series in x_1 with coefficients in $\mathbb{L}[[\mathbf{b}]]$.

Next we consider the case $n > 1$, and assume that the above assertion holds for the case of $n-1$ variables $\mathbf{x}_{n-1}$. Suppose that $F = F(\mathbf{x}_n|\mathbf{b}) \in \Lambda^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b})$ satisfies the assumption $\phi^{(n)}_\mu(F) = 0$ for all $\mu \supset (N^n)$. For each $k\geq 1$, we put $F_k = F_k(\mathbf{x}_{n-1}|\mathbf{b}) := F(\mathbf{x}_{n-1}, \overline{b}_{N+k}|\mathbf{b})$. Then we see easily that $F_k \in \Lambda^{\mathbb{L}}(\mathbf{x}_{n-1}|\mathbf{b})$ and $\phi^{(n-1)}_\nu(F_k) = 0$ for all

$\nu \in \mathcal{P}_{n-1}$ such that $\nu \supset ((N+k)^{n-1}))$. Therefore by the induction hypothesis, we have $F_k = 0$ as a formal power series in $x_1, \dots, x_{n-1}$ with coefficients in $\mathbb{L}[[\mathbf{b}]]$. Thus F depends only on the variable x_n. But F vanishes when we set $x_n = \overline{b}_{N+k}$ for all $k \geq 1$, and hence we deduce that $F = 0$ as required. Q.E.D.

4.5.5. *Basis Theorem for* $s^{\mathbb{L}}_{\lambda}(\mathbf{x}_n|\mathbf{b})$ The factorial Schur polynomials $s_{\lambda}(x_1, \dots, x_n|a)$, where $\ell(\lambda) \leq n$, form a $\mathbb{Z}[a]$-basis of $\mathbb{Z}[a] \otimes_{\mathbb{Z}} \Lambda_n$ (see Macdonald [46, I, §3, Examples 20]). For the factorial Grothendieck polynomials $G_{\lambda}(x_1, \dots, x_n|\mathbf{b})$, readers are referred to Ikeda-Naruse [25, Lemma 2.5], McNamara [49, Theorems 4.6, 4.9]. By the technique introduced in the previous subsubsection, we shall prove the Basis Theorem for the universal factorial Schur functions $s^{\mathbb{L}}_{\lambda}(x_1, \dots, x_n|\mathbf{b})$'s.

Theorem 4.17 (Basis Theorem). $s^{\mathbb{L}}_{\lambda}(\mathbf{x}_n|\mathbf{b})$ $(\lambda \in \mathcal{P}_n)$ *form a formal* $\mathbb{L}[[\mathbf{b}]]$*-basis for* $\Lambda^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b})$.

The proof of this theorem will be divided into two steps: the linear independence property and the generation (spanning) property.

Proof of Theorem 4.17 (*Linear independence property*). We show the linear independence of $s^{\mathbb{L}}_{\lambda}(\mathbf{x}_n|\mathbf{b})$'s over $\mathbb{L}[[\mathbf{b}]]$ by using the Vanishing Property (Proposition 4.13). Suppose that there exists a linear relation of the form

$$\sum_{\lambda} c_{\lambda}(\mathbf{b})\, s^{\mathbb{L}}_{\lambda}(\mathbf{x}_n|\mathbf{b}) = 0 \quad (c_{\lambda}(\mathbf{b}) \in \mathbb{L}[[\mathbf{b}]]). \tag{41}$$

Let μ be *minimal* (with respect to the containement) among all partitions in (41) such that $c_{\lambda}(\mathbf{b}) \neq 0$. We set $\mathbf{x}_n = \overline{\mathbf{b}}_{\mu+\rho_n}$ in (41). Then using the first part of Proposition 4.13 and the choice of μ, we obtain

$$c_{\mu}(\mathbf{b}) s^{\mathbb{L}}_{\mu}(\overline{\mathbf{b}}_{\mu+\rho_n}|\mathbf{b}) = 0.$$

By the second part of Proposition 4.13, we have $s^{\mathbb{L}}_{\mu}(\overline{\mathbf{b}}_{\mu+\rho_n}|\mathbf{b}) \neq 0$, and hence we have $c_{\mu}(\mathbf{b}) = 0$. We repeat this process, and we finally conclude that all the coefficients $c_{\lambda}(\mathbf{b})$ turn out to be zero. Q.E.D.

Proof of Theorem 4.17 (*generation* (*spanning*) *property*). Let $F = F(\mathbf{x}_n|\mathbf{b}) \in \Lambda^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b})$. We wish to express F as a formal $\mathbb{L}[[\mathbf{b}]]$-linear combination of $s^{\mathbb{L}}_{\lambda}$, $\lambda \in \mathcal{P}_n$. Define the *support* of F by

$$\mathrm{Supp}\,(F) := \{\mu \in \mathcal{P}_n \mid \phi^{(n)}_{\mu}(F) \neq 0\} \subset \mathcal{P}_n.$$

Let $\nu \in \mathrm{Supp}\,(F)$ be a *minimal* element (with respect to the containment). We know from Lemma 4.16 (1) that $\Phi^{(n)}(F) = (\phi^{(n)}_{\mu}(F))_{\mu \in \mathcal{P}_n} \in$

$\Psi^{(n)}$, therefore by the GKM condition (39) and the minimality of ν, we see that $\phi_\nu^{(n)}(F)$ is divisible by $e(-\alpha)$ for all $\alpha \in \mathrm{Inv}\,(\nu)$. Since the elements $\{e(-\alpha) \mid \alpha \in \mathrm{Inv}\,(\nu)\}$ are relatively prime, $\phi_\nu^{(n)}(F)$ is divisible by their product $\prod_{\alpha \in \mathrm{Inv}\,(\nu)} e(-\alpha)$. Here we know from the Vanishing Property (Proposition 4.13) and (52) (see Appendix 6.2.1) that

$$\phi_\nu^{(n)}(s_\nu^{\mathbb{L}}) = s_\nu^{\mathbb{L}}(\overline{\mathbf{b}}_{\nu+\rho_n}|\mathbf{b}) = \prod_{(i,j)\in\nu} (\overline{b}_{\nu_i+n-i+1} +_F b_{n+j-{}^t\nu_j}) = \prod_{\alpha \in \mathrm{Inv}\,(\nu)} e(-\alpha).$$

Thus we have $\phi_\nu^{(n)}(F) = c_\nu \cdot \phi_\nu^{(n)}(s_\nu^{\mathbb{L}})$ for some $c_\nu \in \mathbb{L}[[\mathbf{b}]]$. Let

$$F' := F - c_\nu \cdot s_\nu^{\mathbb{L}}.$$

Then we have $\phi_\nu^{(n)}(F') = \phi_\nu^{(n)}(F) - c_\nu \cdot \phi_\nu^{(n)}(s_\nu^{\mathbb{L}}) = 0$, and hence $\nu \notin \mathrm{Supp}\,(F')$. Moreover, for every $\mu \in \mathrm{Supp}\,(F') \setminus \mathrm{Supp}\,(F)$ (obviously, $\mu \neq \nu$), we have $0 \neq \phi_\mu^{(n)}(F') = \phi_\mu^{(n)}(F) - c_\nu \cdot \phi_\mu^{(n)}(s_\mu^{\mathbb{L}}) = -c_\nu \cdot \phi_\mu^{(n)}(s_\nu^{\mathbb{L}}) = -c_\nu \cdot s_\nu^{\mathbb{L}}(\overline{\mathbf{b}}_{\mu+\rho_n}|\mathbf{b})$, and hence $\nu < \mu$. Therefore $\mathrm{Supp}\,(F') \setminus \mathrm{Supp}\,(F)$ consists of elements strictly greater than ν. Then we apply to F' the above argument, and repeat this. Eventually, we will obtain the function $\tilde{F}$ of the form $\tilde{F} = F - \sum_{\lambda \in \mathcal{P}_n} c_\lambda \cdot s_\lambda^{\mathbb{L}}$ with $c_\lambda \in \mathbb{L}[[\mathbf{b}]]$ whose restriction $\phi_\mu^{(n)}(\tilde{F})$ to all $\mu \in \mathcal{P}_n$ vanish, i.e., $\Phi^{(n)}(\tilde{F}) = 0$. Since the homomorphism $\Phi^{(n)}$ is injective (Lemma 4.16 (2)), we have $\tilde{F} = 0$, and hence we obtain the required expression. Q.E.D.

Corollary 4.18 (Corollary to the proof of Theorem 4.17)**.** *The algebraic localization map $\Phi_A^{(n)}$ is onto the GKM ring $\Psi_A^{(n)}$.*

Proof. Let $\psi = (\psi_\mu)_{\mu \in \mathcal{P}_n} \in \Psi_A^{(n)}$, and put $\mathrm{Supp}\,(\psi) = \{\mu \in \mathcal{P}_n \mid \psi_\mu \neq 0\} \subset \mathcal{P}_n$. Applying the same argument as in the proof of the above theorem, one sees that ψ is the image of a certain function of the form $\sum_{\lambda \in \mathcal{P}_n} c_\lambda \cdot s_\lambda^{\mathbb{L}}$, whence the result follows. Q.E.D.

4.6. Factorization Formula

The following *factorization* property (cf. Pragacz [55, Proposition 2.2], Ikeda-Naruse [25, Proposition 2.3]) will be useful in the proof of the Basis Theorem below (Theorem 4.20).

Proposition 4.19 (Factorization Formula)**.** *For a positive integer $k \geq 1$, let ρ_k denote the partition $(k, k-1, \ldots, 2, 1)$ (and $\rho_0 = \emptyset$ by convention).*

(1) *For a positive integer n, we have*

$$\begin{aligned} P^{\mathbb{L}}_{\rho_{n-1}}(x_1,\dots,x_n|\mathbf{b}) &= \left(\prod_{1\le i<j\le n}(x_i +_F x_j)\right) s^{\mathbb{L}}_{\emptyset}(x_1,\dots,x_n|\mathbf{b}),\\ Q^{\mathbb{L}}_{\rho_{n}}(x_1,\dots,x_n|\mathbf{b}) &= \left(\prod_{1\le i\le j\le n}(x_i +_F x_j)\right) s^{\mathbb{L}}_{\emptyset}(x_1,\dots,x_n|\mathbf{b}). \end{aligned}$$

(2) *For a a positive integer n and a partition $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)\in\mathcal{P}_n$, we have*

$$\begin{aligned} P^{\mathbb{L}}_{\rho_{n-1}+\lambda}(x_1,\dots,x_n|\mathbf{b}) &= \left(\prod_{1\le i<j\le n}(x_i +_F x_j)\right) s^{\mathbb{L}}_{\lambda}(x_1,\dots,x_n|\mathbf{b}),\\ Q^{\mathbb{L}}_{\rho_{n}+\lambda}(x_1,\dots,x_n|\mathbf{b}) &= \left(\prod_{1\le i\le j\le n}(x_i +_F x_j)\right) s^{\mathbb{L}}_{\lambda}(x_1,\dots,x_n|\mathbf{b}). \end{aligned}$$

Proof. The first formulas (1) follow immediately from (2) once we put $\lambda=\emptyset$. We shall show the formulas (2). We first prove the case of $P^{\mathbb{L}}_{\lambda}$. Note that the length of $\rho_{n-1}+\lambda$ is $n-1$ or n. In any case, the product in the expression (33) becomes $[\mathbf{x}|\mathbf{b}]^{\lambda+\rho_{n-1}}\prod_{1\le i<j\le n}\dfrac{x_i +_F x_j}{x_i +_F \overline{x}_j}$. Therefore we modify it as follows:

$$[\mathbf{x}|\mathbf{b}]^{\lambda+\rho_{n-1}}\prod_{1\le i<j\le n}\frac{x_i +_F x_j}{x_i +_F \overline{x}_j} = \prod_{1\le i<j\le n}(x_i +_F x_j)\times\frac{[x|\mathbf{b}]^{\lambda+\rho_{n-1}}}{\prod_{1\le i<j\le n}(x_i +_F \overline{x}_j)}.$$

Since $\prod_{1\le i<j\le n}(x_i +_F x_j)$ is symmetric, we have from (33),

$$\begin{aligned} P^{\mathbb{L}}_{\rho_{n-1}+\lambda}(x_1,\dots,x_n|\mathbf{b}) &= \prod_{1\le i<j\le n}(x_i +_F x_j)\\ &\qquad\times\sum_{w\in S_n} w\left[\frac{[x|\mathbf{b}]^{\lambda+\rho_{n-1}}}{\prod_{1\le i<j\le n}(x_i +_F \overline{x}_j)}\right]\\ &= \prod_{1\le i<j\le n}(x_i +_F x_j)\times s^{\mathbb{L}}_{\lambda}(x_1,\dots,x_n|\mathbf{b}). \end{aligned}$$

For the case of $Q^{\mathbb{L}}_\lambda$, the product in the expression (33) becomes

$$\begin{aligned}
&[[\mathbf{x}|\mathbf{b}]]^{\lambda+\rho_n}\prod_{1\le i<j\le n}\frac{x_i +_{F} x_j}{x_i +_{F} \overline{x}_j}\\
=\;&\prod_{i=1}^{n}(x_i +_{F} x_i)[x_i|\mathbf{b}]^{\lambda_i+n-i}\times\prod_{1\le i<j\le n}\frac{x_i +_{F} x_j}{x_i +_{F} \overline{x}_j}\\
=\;&\prod_{1\le i\le j\le n}(x_i +_{F} x_j)\times\frac{[\mathbf{x}|\mathbf{b}]^{\lambda+\rho_{n-1}}}{\prod_{1\le i<j\le n}(x_i +_{F} \overline{x}_j)},
\end{aligned}$$

and the result follows. Q.E.D.

4.7. Basis Theorem for $P^{\mathbb{L}}_\lambda(\mathbf{x})$, $Q^{\mathbb{L}}_\lambda(\mathbf{x})$

In [55, Theorem 2.11], Pragacz showed that the usual Schur P-polynomials $P_\lambda(x_1,\dots,x_n)$ ($\lambda\in\mathcal{SP}_n$) form a $\mathbb{Z}$-basis for the ring $\Gamma_n=\Gamma(\mathbf{x}_n)$ of "supersymmetric polynomials" (cf. Macdonald [46, III, (8.9)]). Their K-theoretic analogues $GP_\lambda(x_1,\dots,x_n)$ ($\lambda\in\mathcal{SP}_n$) form a $\mathbb{Z}[\beta]$-basis for the ring $G\Gamma_n$ of "K-supersymmetric polynomials" (Ikeda-Naruse [25, Theorem 3.1]). Our functions $P^{\mathbb{L}}_\lambda(x_1,\dots,x_n)$ ($\lambda\in\mathcal{SP}_n$) also have the similar property. Namely we have the following theorem:

Theorem 4.20 (Basis Theorem)**.**

(1) *The formal power series $P^{\mathbb{L}}_\lambda(x_1,\dots,x_n)$ ($\lambda\in\mathcal{SP}_n$) form a formal $\mathbb{L}$-basis of $\Gamma^{\mathbb{L}}(\mathbf{x}_n)$.*

(2) *The formal power series $Q^{\mathbb{L}}_\lambda(x_1,\dots,x_n)$ ($\lambda\in\mathcal{SP}_n$) form a formal $\mathbb{L}$-basis of $\Gamma^{\mathbb{L}}_+(\mathbf{x}_n)$.*

Proof. For the proof of this theorem, we make use of the same strategy as in Ikeda-Naruse [25, Theorem 3.1], Pragacz [55, Theorem 2.11 (Q)], and thus we shall only prove the case (1) when n is even. First note that $P^{\mathbb{L}}_\lambda(x_1,\dots,x_n)$ ($\lambda\in\mathcal{SP}_n$) are linearly independent over $\mathbb{L}$ as we showed earlier (see §4.2). We use the induction on the number of the variables n. Let $n=2$ and $f(x_1,x_2)$ be an $\mathbb{L}$-supersymmetric function. We may assume that the constant term of f is zero, namely $f(0,0)=0$. Then $\mathbb{L}$-supersymmetricity implies that $f(t,\overline{t})=f(0,0)=0$. Therefore by Lemma 4.5, $f(x_1,x_2)$ is divisible by $x_1 +_{F} x_2$. Thus f can be written as $f(x_1,x_2)=(x_1 +_{F} x_2)g(x_1,x_2)$ for some symmetric function $g(x_1,x_2)$. By the Basis Theorem for $s^{\mathbb{L}}_\lambda(\mathbf{x}_n)$'s (Proposition 4.12), we can write $g(x_1,x_2)=\sum_{\lambda\in\mathcal{P}_2}c_\lambda s^{\mathbb{L}}_\lambda(x_1,x_2)$, $c_\lambda\in\mathbb{L}$. Then by the factorization property (Proposition 4.19), we have

$$f(x_1,x_2)=\sum_{\lambda\in\mathcal{P}_2}c_\lambda(x_1 +_{F} x_2)s^{\mathbb{L}}_\lambda(x_1,x_2)=\sum_{\lambda\in\mathcal{P}_2}c_\lambda P^{\mathbb{L}}_{\rho_1+\lambda}(x_1,x_2).$$

Thus $f(x_1, x_2)$ is an $\mathbb{L}$-linear combination of $P^{\mathbb{L}}_{\mu}(x_1, x_2)$'s, $\mu \in \mathcal{SP}_2$ (note that for $\lambda \in \mathcal{P}_2$, we have $\rho_1 + \lambda \in \mathcal{SP}_2$).

For $n \geq 4$, we proceed as follows. Let $f(x_1, \dots, x_n)$ be an $\mathbb{L}$-supersymmetric function. Notice that $f(x_1, \dots, x_{n-2}, t, \overline{t}) = f(x_1, \dots, x_{n-2}, 0, 0)$ holds. Put $f_1(x_1, \dots, x_{n-2}) := f(x_1, \dots, x_{n-2}, 0, 0)$. Since $f_1(x_1, \dots, x_{n-2})$ is also $\mathbb{L}$-supersymmetric, we can write f_1 as an $\mathbb{L}$-linear combination of $P^{\mathbb{L}}_{\lambda}(x_1, \dots, x_{n-2})$'s with $\lambda \in \mathcal{SP}_{n-2}$ by the induction hypothesis. Thus we have the following expression:

$$f_1(x_1, \dots, x_{n-2}) = \sum_{\lambda \in \mathcal{SP}_{n-2}} c_\lambda P^{\mathbb{L}}_{\lambda}(x_1, \dots, x_{n-2}), \quad c_\lambda \in \mathbb{L}.$$

Consider the function $g(x_1, \dots, x_n) := \sum_{\lambda \in \mathcal{SP}_{n-2}} c_\lambda P^{\mathbb{L}}_{\lambda}(x_1, \dots, x_n)$. Note that $g(x_1, \dots, x_{n-2}, 0, 0) = f_1(x_1, \dots, x_{n-2}) = f(x_1, \dots, x_{n-2}, 0, 0)$ holds because of the stability of $P^{\mathbb{L}}_{\lambda}$. Put $h(x_1, \dots, x_n) := f(x_1, \dots, x_n) - g(x_1, \dots, x_n)$. Then we have

$$\begin{aligned} h(x_1, \dots, x_{n-2}, t, \overline{t}) &= h(x_1, \dots, x_{n-2}, 0, 0) \\ &= f(x_1, \dots, x_{n-2}, 0, 0) - g(x_1, \dots, x_{n-2}, 0, 0) = 0. \end{aligned}$$

This implies that $x_{n-1} +_F x_n$ divides h. Since h is symmetric, we see that h is a multiple of $V := \prod_{1 \leq i < j \leq n} (x_i +_F x_j)$. Thus we have

$$f(x_1, \dots, x_n) = g(x_1, \dots, x_n) + V \cdot s(x_1, \dots, x_n)$$

for some symmetric function s. By the Basis Theorem for $s^{\mathbb{L}}_{\lambda}(\mathbf{x}_n)$'s (Proposition 4.12) again, we can write

$$s(x_1, \dots, x_n) = \sum_{\lambda \in \mathcal{P}_n} d_\lambda s^{\mathbb{L}}_{\lambda}(x_1, \dots, x_n), \quad d_\lambda \in \mathbb{L}.$$

Using the factorization theorem (Proposition 4.19) again, we have

$$\begin{aligned} f &= g + V \cdot s \\ &= \sum_{\lambda \in \mathcal{SP}_{n-2}} c_\lambda P^{\mathbb{L}}_{\lambda}(x_1, \dots, x_n) + \sum_{\lambda \in \mathcal{P}_n} d_\lambda \left(\prod_{1 \leq i < j \leq n} (x_i +_F x_j) \right) s^{\mathbb{L}}_{\lambda}(x_1, \dots, x_n) \\ &= \sum_{\lambda \in \mathcal{SP}_{n-2}} c_\lambda P^{\mathbb{L}}_{\lambda}(x_1, \dots, x_n) + \sum_{\lambda \in \mathcal{P}_n} d_\lambda P^{\mathbb{L}}_{\rho_{n-1}+\lambda}(x_1, \dots, x_n). \end{aligned}$$

Note that if $\lambda \in \mathcal{P}_n$, then $\rho_{n-1} + \lambda \in \mathcal{SP}_n$. Thus f can be written as an $\mathbb{L}$-linear combination of $P^{\mathbb{L}}_{\mu}(x_1, \dots, x_n)$'s, $\mu \in \mathcal{SP}_n$. Q.E.D.

Taking limit $n \to \infty$, we obtain the following:

Corollary 4.21 (Basis Theorem)**.**
(1) *The formal power series* $P^{\mathbb{L}}_{\lambda}(\mathbf{x})$ $(\lambda \in \mathcal{SP})$ *form a formal* $\mathbb{L}$*-basis of* $\Gamma^{\mathbb{L}}(\mathbf{x})$.
(2) *The formal power series* $Q^{\mathbb{L}}_{\lambda}(\mathbf{x})$ $(\lambda \in \mathcal{SP})$ *form a formal* $\mathbb{L}$*-basis of* $\Gamma^{\mathbb{L}}_{+}(\mathbf{x})$.

4.8. Vanishing Property

Various factorial analogues of Schur P- and Q- polynomials (functions) satisfy the vanishing property (see Ivanov [26, Theorem 1.5 (the zero property)], [27, Theorem 5.3 (Vanishing property)], Ikeda-Naruse [23, Proposition 8.3], Ikeda-Mihalcea-Naruse [24, Proposition 4.2], Ikeda-Naruse [25, Proposition 7.1]). In this subsection, we shall prove the vanishing property for the functions $P^{\mathbb{L}}_{\lambda}(\mathbf{x}_n|\mathbf{b})^+$'s (for its definition, see §4.4) and $Q^{\mathbb{L}}_{\lambda}(\mathbf{x}_n|\mathbf{b})$'s. For a strict partition $\mu = (\mu_1, \ldots, \mu_r)$ of length r, we set

$$\overline{\mathbf{b}}_{\mu} := (\overline{b}_{\mu_1}, \ldots, \overline{b}_{\mu_r}, 0, 0, \ldots).$$

We also set

$$\begin{aligned} \mathrm{sh}(\mu) &:= (\mu_1 + 1, \ldots, \mu_r + 1) && \text{if } r \text{ is even,} \\ \mathrm{sh}(\mu) &:= (\mu_1 + 1, \ldots, \mu_r + 1, 1) && \text{if } r \text{ is odd.} \end{aligned}$$

Here we consider only *even* variable case $P^{\mathbb{L}}_{\lambda}(x_1, \ldots, x_{2n}|\mathbf{b})$ because of the mod 2 stability (see Proposition 4.8).

Proposition 4.22 (Vanishing Property)**.** *Let* $\lambda, \mu \in \mathcal{SP}$. *Then we have*

(1)

$$P^{\mathbb{L}}_{\lambda}(\overline{\mathbf{b}}_{\mathrm{sh}(\mu)}|\mathbf{b})^+ = 0 \quad \textit{if} \quad \mu \not\supset \lambda,$$

$$P^{\mathbb{L}}_{\lambda}(\overline{\mathbf{b}}_{\mathrm{sh}(\lambda)}|\mathbf{b})^+ = \prod_{i=1}^{r} \left(\prod_{\substack{1 \le j \le \lambda_i, \\ j \neq \lambda_p + 1 \ \textit{for} \ i+1 \le p \le r}} (\overline{b}_{\lambda_i + 1} +_F b_j) \cdot \prod_{j=i+1}^{r} (\overline{b}_{\lambda_i+1} +_F \overline{b}_{\lambda_j+1}) \right).$$

(2)

$$Q^{\mathbb{L}}_{\lambda}(\overline{\mathbf{b}}_{\mu}|\mathbf{b}) = 0 \quad \textit{if} \quad \mu \not\supset \lambda,$$

$$Q^{\mathbb{L}}_{\lambda}(\overline{\mathbf{b}}_{\lambda}|\mathbf{b}) = \prod_{i=1}^{r} \left(\prod_{\substack{1 \le j \le \lambda_i - 1, \\ j \neq \lambda_p \ \textit{for} \ i+1 \le p \le r}} (\overline{b}_{\lambda_i} +_F b_j) \cdot \prod_{j=i}^{r} (\overline{b}_{\lambda_i} +_F \overline{b}_{\lambda_j}) \right).$$

Proof. We will only prove (1) when the length $\ell(\mu) = r$ is *even*. The proofs of the remaining cases are similar. Using the stability property (Proposition 4.8), $P^{\mathbb{L}}_{\lambda}(\overline{\mathbf{b}}_{\mathrm{sh}(\mu)}|\mathbf{b})^{+}$ can be evaluated as $P^{\mathbb{L}}_{\lambda}(\overline{b}_{\mu_1+1}, \overline{b}_{\mu_2+1}, \dots, \overline{b}_{\mu_r+1}|\mathbf{b})$.

If $\mu \not\supset \lambda$, there exists an index $1 \leq k \leq r$ such that $\mu_k < \lambda_k$ (and hence $\mu_k + 1 \leq \lambda_k$). For an arbitrary permutation $w \in S_r$, there exists a positive integer $1 \leq j \leq k$ such that $w(j) \geq k$. Then we have inequalities $\mu_{w(j)} \leq \mu_k < \lambda_k \leq \lambda_j$. Therefore the term

$$[x_{w(j)}|\mathbf{b}]^{\lambda_j} = (x_{w(j)} +_F b_1)(x_{w(j)} +_F b_2) \cdots (x_{w(j)} +_F b_{\mu_{w(j)}+1}) \\ \cdots (x_{w(j)} +_F b_{\lambda_j})$$

vanishes when we specialize $x_{w(j)}$ to $\overline{b}_{\mu_{w(j)}+1}$. This means that $P^{\mathbb{L}}_{\lambda}(\overline{\mathbf{b}}_{\mathrm{sh}(\mu)}|\mathbf{b}) = 0$.

For the case $\mu = \lambda$, we shall show that in the defining equation (33) of $P^{\mathbb{L}}_{\lambda}(x_1, \dots, x_r|\mathbf{b})$, each term corresponding to $w \in S_r$ other than the identity e becomes zero when we evaluate them at $(x_1, \dots, x_r) = \overline{\mathbf{b}}_{\mathrm{sh}(\lambda)} = (\overline{b}_{\lambda_1+1}, \dots, \overline{b}_{\lambda_r+1})$. For $w \neq e$, there exists an index $1 \leq k \leq r$ such that $w(k) > k$. Thus we have $\lambda_{w(k)} < \lambda_k$ (and hence $\lambda_{w(k)} + 1 \leq \lambda_k$). Then it is obvious that

$$[x_{w(k)}|\mathbf{b}]^{\lambda_k} = (x_{w(k)} +_F b_1)(x_{w(k)} +_F b_2) \cdots (x_{w(k)} +_F b_{\lambda_{w(k)}+1}) \\ \cdots (x_{w(k)} +_F b_{\lambda_k})$$

becomes zero when we specialize $x_{w(k)}$ to $\overline{b}_{\lambda_{w(k)}+1}$. The term corresponding to $w = e$ is

$$\begin{aligned} &\prod_{i=1}^{r} [\overline{b}_{\lambda_i+1}|\mathbf{b}]^{\lambda_i} \prod_{i=1}^{r} \prod_{j=i+1}^{r} \frac{\overline{b}_{\lambda_i+1} +_F \overline{b}_{\lambda_j+1}}{\overline{b}_{\lambda_i+1} +_F b_{\lambda_j+1}} \\ = &\prod_{i=1}^{r} \left((\overline{b}_{\lambda_i+1} +_F b_1) \cdots (\overline{b}_{\lambda_i+1} +_F b_{\lambda_i}) \times \prod_{j=i+1}^{r} \frac{\overline{b}_{\lambda_i+1} +_F \overline{b}_{\lambda_j+1}}{\overline{b}_{\lambda_i+1} +_F b_{\lambda_j+1}} \right). \end{aligned}$$

By cancellation, we obtain the desired value. Q.E.D.

4.9. Algebraic localization map of types B_∞, C_∞, or D_∞

In §4.7, we have proven the Basis Theorem for $P^{\mathbb{L}}_{\lambda}(\mathbf{x}_n)$'s and $Q^{\mathbb{L}}_{\lambda}(\mathbf{x}_n)$'s. We shall prove the Basis Theorem for $P^{\mathbb{L}}_{\lambda}(\mathbf{x}_n|\mathbf{b})^{+}$'s and $Q^{\mathbb{L}}_{\lambda}(\mathbf{x}_n|\mathbf{b})$'s in §4.10. For the proof, we apply the same technique as the case of $s^{\mathbb{L}}_{\lambda}(\mathbf{x}_n|\mathbf{b})$ (see §4.5.5), namely the Vanishing Property (Proposition 4.22) and the localization technique. Because the idea of the proof is the same as that of Theorem 4.17, we only exhibit the definitions and the results. We mainly follow the notation and convention as in Ikeda-Naruse [25, §4] and Ikeda-Mihalcea-Naruse [24, §3] (we collect the necessary data in the Appendix 6.2.2).

Let us introduce the GKM ring and the algebraic localization maps of type X_∞ for $X = B, C$, or D. As with the type A_∞ case (§4.5.4), denote by $\mathrm{Map}\,(\mathcal{SP}_n, \mathbb{L}[[\mathbf{b}]])$ the set of all maps from $\mathcal{SP}_n$ to $\mathbb{L}[[\mathbf{b}]]$. It is an $\mathbb{L}[[\mathbf{b}]]$-algebra under pointwise multiplication and scalar multiplication. We identify $\mathrm{Map}\,(\mathcal{SP}_n, \mathbb{L}[[\mathbf{b}]])$ with the product ring $\prod_{\mu \in \mathcal{SP}_n} (\mathbb{L}[[\mathbf{b}]])_\mu$.

Definition 4.23 (GKM ring of type X_∞)**.** *Let X be B, C, or D. Let $\Psi_X^{(n)}$ be a subring of $\mathrm{Map}\,(\mathcal{SP}_n, \mathbb{L}[[\mathbf{b}]])$ that consists of elements $\psi = (\psi_\mu)_{\mu \in \mathcal{SP}_n}$ satisfying the following GKM condition:*

$$\psi_{s_\alpha \mu} - \psi_\mu \in e(-\alpha)\,\mathbb{L}[[\mathbf{b}]] \quad \text{for all } \mu \in \mathcal{SP}_n \text{ and all } \alpha \in \Delta_X^+.$$

We call $\Psi_X^{(n)}$ the *GKM ring of type X_∞*.

Next we shall introduce the algebraic localization map of type X_∞ for $X = C, D$. For each strict partition $\mu \in \mathcal{SP}_n$, define the $\mathbb{L}[[\mathbf{b}]]$-algebra homomorphisms to be

$$\begin{aligned}
&\phi_{\mu,C}^{(n)} : \Gamma_+^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b}) = \mathbb{L}[[\mathbf{b}]] \,\hat{\otimes}_{\mathbb{L}}\, \Gamma_+^{\mathbb{L}}(\mathbf{x}_n) \longrightarrow \mathbb{L}[[\mathbf{b}]],\\
&F = F(\mathbf{x}_n|\mathbf{b}) \longmapsto \phi_{\mu,C}^{(n)}(F) := F(\overline{\mathbf{b}}_\mu|\mathbf{b}),\\
&\phi_{\mu,D}^{(n)} : \Gamma^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b}) = \mathbb{L}[[\mathbf{b}]] \,\hat{\otimes}_{\mathbb{L}}\, \Gamma^{\mathbb{L}}(\mathbf{x}_n) \longrightarrow \mathbb{L}[[\mathbf{b}]],\\
&F = F(\mathbf{x}_n|\mathbf{b}) \longmapsto \phi_{\mu,D}^{(n)}(F) := F(\overline{\mathbf{b}}_{\mathrm{sh}(\mu)}|\mathbf{b}).
\end{aligned}$$

Definition 4.24 (Algebraic localization map of type X_∞)**.** *Define the homomorphisms of $\mathbb{L}[[\mathbf{b}]]$-algebras to be*

$$\Phi_C^{(n)} : \Gamma_+^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b}) \longrightarrow \prod_{\mu \in \mathcal{SP}_n} (\mathbb{L}[[\mathbf{b}]])_\mu,\ F \longmapsto \Phi_C^{(n)}(F) := (\phi_{\mu,C}^{(n)}(F))_{\mu \in \mathcal{SP}_n},$$

$$\Phi_D^{(n)} : \Gamma^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b}) \longrightarrow \prod_{\mu \in \mathcal{SP}_n} (\mathbb{L}[[\mathbf{b}]])_\mu,\ F \longmapsto \Phi_D^{(n)}(F) := (\phi_{\mu,D}^{(n)}(F))_{\mu \in \mathcal{SP}_n}.$$

We call the homomorphism $\Phi_X^{(n)}$ the *algebraic localization map of type X_∞*. Concerning the above algebraic localization maps, the following lemma can be proved analogously (see the proof of Lemma 4.16):

Lemma 4.25. *Let X be C or D. Then we have*

(1) *The image of $\Phi_X^{(n)}$ agrees with the GKM ring $\Psi_X^{(n)}$: $\mathrm{Im}\,(\Phi_X^{(n)}) = \Psi_X^{(n)}$.*

(2) *The homomorphism $\Phi_X^{(n)}$ is injective.*

4.10. Basis Theorem for $P_\lambda^{\mathbb{L}}(\mathbf{x}|\mathbf{b})^+$, $Q_\lambda^{\mathbb{L}}(\mathbf{x}|\mathbf{b})$

The GKM ring and the algebraic localization map of type X_∞ for $X = B, C$, or D, and an analogous argument as in the Basis Theorem for $s_\lambda^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b})$'s (Theorem 4.17) enable us to prove the Basis Theorem for $P_\lambda^{\mathbb{L}}(\mathbf{x}|\mathbf{b})^+$'s and $Q_\lambda^{\mathbb{L}}(\mathbf{x}|\mathbf{b})$'s.

Theorem 4.26 (Basis Theorem)**.**

(1) *The formal power series $P_\lambda^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b})^+$ ($\lambda \in \mathcal{SP}_n$) form a formal $\mathbb{L}[[\mathbf{b}]]$-basis of $\Gamma^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b})$.*

(2) *The formal power series $Q_\lambda^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b})$ ($\lambda \in \mathcal{SP}_n$) form a formal $\mathbb{L}[[\mathbf{b}]]$-basis of $\Gamma_+^{\mathbb{L}}(\mathbf{x}_n|\mathbf{b})$.*

Taking limit $n \to \infty$, we obtain the following:

Corollary 4.27 (Basis Theorem)**.**

(1) *The formal power series $P_\lambda^{\mathbb{L}}(\mathbf{x}|\mathbf{b})^+$ ($\lambda \in \mathcal{SP}$) form a formal $\mathbb{L}[[\mathbf{b}]]$-basis of $\Gamma^{\mathbb{L}}(\mathbf{x}|\mathbf{b})$.*

(2) *The formal power series $Q_\lambda^{\mathbb{L}}(\mathbf{x}|\mathbf{b})$ ($\lambda \in \mathcal{SP}$) form a formal $\mathbb{L}[[\mathbf{b}]]$-basis of $\Gamma_+^{\mathbb{L}}(\mathbf{x}|\mathbf{b})$.*

§5. Dual universal (factorial) Schur P- and Q-functions

In the previous section, we have constructed "cohomology bases" $\{P_\lambda^{\mathbb{L}}(\mathbf{x}|\mathbf{b})^+\}$ and $\{Q_\lambda^{\mathbb{L}}(\mathbf{x}|\mathbf{b})\}$. Our next task is to construct the corresponding "homology bases" $\{\widehat{p}_\lambda^{\mathbb{L}}(\mathbf{y}|\mathbf{b})\}$ and $\{\widehat{q}_\lambda^{\mathbb{L}}(\mathbf{y}|\mathbf{b})\}$. In this section, we consider this problem only in the non-equivariant case, i.e., $\mathbf{b} = 0$. Namely, we shall construct certain functions $\widehat{p}_\lambda^{\mathbb{L}}(\mathbf{y})$ (resp. $\widehat{q}_\lambda^{\mathbb{L}}(\mathbf{y})$) for strict partitions $\lambda \in \mathcal{SP}$ dual to $Q_\lambda^{\mathbb{L}}(\mathbf{x})$ (resp. $P_\lambda^{\mathbb{L}}(\mathbf{x})$). Here we use the countably infinite set of variables $\mathbf{y} = (y_1, y_2, \ldots)$. Their degrees are given by $\deg(y_i) = 1$ and $\deg_h(a_{i,j}) = i + j - 1$ ($i, j \geq 1$). Let $\Lambda_{\mathbb{L}}(\mathbf{y}) = \mathbb{L}_* \otimes_{\mathbb{Z}} \Lambda(\mathbf{y})$ be the ring of symmetric functions in $\mathbf{y}$ with coefficients in $\mathbb{L}_*$ (see the beginning of §4.2).

5.1. One row case

First we shall construct the required functions corresponding to the "one rows", that is, strict partitions (k) ($k = 1, 2, \ldots$). We put

$$\Delta(t; \mathbf{y}) := \prod_{j=1}^{\infty} \frac{1 - \overline{t} y_j}{1 - t y_j} \in \Lambda_{\mathbb{L}}(\mathbf{y})[[t]].$$

Then we define $\widehat{q}_k^{\mathbb{L}}(\mathbf{y}) \in \Lambda_{\mathbb{L}}(\mathbf{y})$ $(k = 0, 1, 2, \ldots)$ as the coefficients of the following expansion:

$$\Delta(t; \mathbf{y}) = \sum_{k=0}^{\infty} \widehat{q}_k^{\mathbb{L}}(\mathbf{y}) t^k. \tag{42}$$

Next we shall define $\widehat{p}_k^{\mathbb{L}}(\mathbf{y}) \in \Lambda_{\mathbb{L}}(\mathbf{y})$ $(k = 1, 2, \ldots)$ as the coefficients of the following expansion:

$$\Delta(t; \mathbf{y}) = 1 + (t +_{F} t) \sum_{k=1}^{\infty} \widehat{p}_k^{\mathbb{L}}(\mathbf{y}) t^{k-1}. \tag{43}$$

We set $\widehat{p}_0^{\mathbb{L}}(\mathbf{y}) := 1$ by convention. In order to make the above definition valid, we have to verify that $\Delta(t; \mathbf{y}) - 1$ is divisible by $t +_{F} t$. This follows from Lemma 4.5. More concretely, if we write $t +_{F} t = 2t + \sum_{k=2}^{\infty} \alpha_k^{\mathbb{L}} t^k$ with $\alpha_k^{\mathbb{L}} \in \mathbb{L}$, we have

$$\widehat{q}_1^{\mathbb{L}}(\mathbf{y}) = 2\widehat{p}_1^{\mathbb{L}}(\mathbf{y}), \quad \widehat{q}_k^{\mathbb{L}}(\mathbf{y}) = 2\widehat{p}_k^{\mathbb{L}}(\mathbf{y}) + \sum_{j=1}^{k-1} \alpha_{k+1-j}^{\mathbb{L}} \widehat{p}_j^{\mathbb{L}}(\mathbf{y}) \ (k \geq 2). \tag{44}$$

Remark 5.1.

(1) *Notice that comparing* (42) (*resp.* (43)) *with the equation* (23) (*resp.* (25)), *we see immediately that* $\widehat{q}_k^{\mathbb{L}}(\mathbf{y})$ $(k = 1, 2, \ldots)$ (*resp.* $\widehat{p}_k^{\mathbb{L}}(\mathbf{y})$ $(k = 1, 2, \ldots)$) *coincide with* $\widehat{q}_k^{MU}(\mathbf{y})$ (*resp.* $\widehat{p}_k^{MU}(\mathbf{y})$) *in Definition* 3.1 (*resp. Definition* 3.5).

(2) *It follows from Definition* 4.1 *that* $P_{(k)}^{\mathbb{L}}(x_1) = x_1^k$ *and* $Q_{(k)}^{\mathbb{L}}(x_1) = (x_1 +_{F} x_1) x_1^{k-1}$ *for* $k = 1, 2, \ldots$. *Therefore we can write* (42) *and* (43) *as*

$$\Delta(x_1; \mathbf{y}) = \prod_{j=1}^{\infty} \frac{1 - \overline{x}_1 y_j}{1 - x_1 y_j} = \sum_{k=0}^{\infty} P_{(k)}^{\mathbb{L}}(x_1) \, \widehat{q}_k^{\mathbb{L}}(\mathbf{y}) = \sum_{k=0}^{\infty} Q_{(k)}^{\mathbb{L}}(x_1) \, \widehat{p}_k^{\mathbb{L}}(\mathbf{y}). \tag{45}$$

5.2. Definition of $\widehat{p}_\lambda^{\mathbb{L}}(\mathbf{y})$ and $\widehat{q}_\lambda^{\mathbb{L}}(\mathbf{y})$

In §5.1, we have constructed $\widehat{p}_k^{\mathbb{L}}(\mathbf{y})$ and $\widehat{q}_k^{\mathbb{L}}(\mathbf{y})$ corresponding to the one rows (k) $(k = 1, 2, \ldots)$. In this subsection, we extend these functions to arbitrary strict partitions $\lambda \in \mathcal{SP}$. For this end, we argue as follows. Define

$$\Gamma_{\mathbb{L}}(\mathbf{y}) := \text{ the } \mathbb{L}\text{-subalgebra of } \Lambda_{\mathbb{L}}(\mathbf{y}) \text{ generated by } \widehat{p}_k^{\mathbb{L}}(\mathbf{y}),\ k = 1, 2, \ldots,$$
$$\Gamma_{\mathbb{L}}^{+}(\mathbf{y}) := \text{ the } \mathbb{L}\text{-subalgebra of } \Lambda_{\mathbb{L}}(\mathbf{y}) \text{ generated by } \widehat{q}_k^{\mathbb{L}}(\mathbf{y}),\ k = 1, 2, \ldots.$$

As we remarked in the previous subsection, $\widehat{p}_k^{\mathbb{L}}(\mathbf{y}) = \widehat{p}_k^{MU}(\mathbf{y})$ and $\widehat{q}_k^{\mathbb{L}}(\mathbf{y}) = \widehat{q}_k^{MU}(\mathbf{y})$ for $k = 1, 2, \ldots$, and hence the above algebras $\Gamma_{\mathbb{L}}(\mathbf{y})$ and $\Gamma_{\mathbb{L}}^{+}(\mathbf{y})$ coincide with the algebras ${\Gamma'}_*^{MU}$ and Γ_*^{MU} respectively defined in Definitions 3.7 and 3.4. In particular, we have

$$\Gamma_{\mathbb{L}}(\mathbf{y}) = \mathbb{L}[\widehat{p}_1^{\mathbb{L}}(\mathbf{y}), \widehat{p}_3^{\mathbb{L}}(\mathbf{y}), \ldots, \widehat{p}_{2i-1}^{\mathbb{L}}(\mathbf{y}), \ldots] \ (\cong MU_*(\Omega Sp)),$$
$$\Gamma_{\mathbb{L}}^{+}(\mathbf{y}) = \mathbb{L}[\widehat{q}_1^{\mathbb{L}}(\mathbf{y}), \widehat{q}_2^{\mathbb{L}}(\mathbf{y}), \ldots, \widehat{q}_i^{\mathbb{L}}(\mathbf{y}), \ldots]/(\widehat{q}^{\mathbb{L}}(T)\widehat{q}^{\mathbb{L}}(\overline{T}) = 1) \ (\cong MU_*(\Omega_0 SO)),$$

where $\widehat{q}^{\mathbb{L}}(T) := \sum_{k\geq 0} \widehat{q}_k^{\mathbb{L}}(\mathbf{y})T^k$. Also $\Gamma_{\mathbb{L}}(\mathbf{y})$ and $\Gamma_{\mathbb{L}}^{+}(\mathbf{y})$ have the Hopf algebra structure over $\mathbb{L}$ as explained in §3.2.

For later discussion, we prepare the following: For a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, define the monomials

$$\begin{aligned} \widehat{p}_{[\lambda]}^{\mathbb{L}}(\mathbf{y}) &:= \prod_{i=1}^{r} \widehat{p}_{\lambda_i}^{\mathbb{L}}(\mathbf{y}) = \widehat{p}_{\lambda_1}^{\mathbb{L}}(\mathbf{y})\widehat{p}_{\lambda_2}^{\mathbb{L}}(\mathbf{y}) \cdots \widehat{p}_{\lambda_r}^{\mathbb{L}}(\mathbf{y}), \\ \widehat{q}_{[\lambda]}^{\mathbb{L}}(\mathbf{y}) &:= \prod_{i=1}^{r} \widehat{q}_{\lambda_i}^{\mathbb{L}}(\mathbf{y}) = \widehat{q}_{\lambda_1}^{\mathbb{L}}(\mathbf{y})\widehat{q}_{\lambda_2}^{\mathbb{L}}(\mathbf{y}) \cdots \widehat{q}_{\lambda_r}^{\mathbb{L}}(\mathbf{y}). \end{aligned}$$

Note that there exist some relations among $\widehat{p}_k^{\mathbb{L}}(\mathbf{y})$'s and $\widehat{q}_k^{\mathbb{L}}(\mathbf{y})$'s (see the relations (24) and (28)), and these monomials are not linearly independent over $\mathbb{L}$. Then we define the $\mathbb{L}$-submodule of $\Gamma_{\mathbb{L}}(\mathbf{y})$ (resp. $\Gamma_{\mathbb{L}}^{+}(\mathbf{y})$) spanned by these monomials $\widehat{p}_{[\lambda]}^{\mathbb{L}}(\mathbf{y})$'s (resp. $\widehat{q}_{[\lambda]}^{\mathbb{L}}(\mathbf{y})$'s) with $\lambda \in \mathcal{P}_n$:

$$\begin{aligned} \Gamma_{\mathbb{L}}^{(n)}(\mathbf{y}) &:= \sum_{\lambda \in \mathcal{P}_n} \mathbb{L}\, \widehat{p}_{[\lambda]}^{\mathbb{L}}(\mathbf{y}) \ \subset \ \Gamma_{\mathbb{L}}(\mathbf{y}), \\ \Gamma_{\mathbb{L}}^{(n),+}(\mathbf{y}) &:= \sum_{\lambda \in \mathcal{P}_n} \mathbb{L}\, \widehat{q}_{[\lambda]}^{\mathbb{L}}(\mathbf{y}) \ \subset \ \Gamma_{\mathbb{L}}^{+}(\mathbf{y}). \end{aligned}$$

Now consider the iterated product of $\Delta(x_i; \mathbf{y})$'s, and their limit.

$$\begin{aligned} \Delta(\mathbf{x}_n; \mathbf{y}) &= \Delta(x_1, \ldots, x_n; \mathbf{y}) := \prod_{i=1}^{n} \Delta(x_i; \mathbf{y}) = \prod_{i=1}^{n}\prod_{j=1}^{\infty} \frac{1 - \overline{x}_i y_j}{1 - x_i y_j}, \\ \Delta(\mathbf{x}; \mathbf{y}) &:= \varprojlim_n \Delta(\mathbf{x}_n; \mathbf{y}) = \prod_{i=1}^{\infty}\prod_{j=1}^{\infty} \frac{1 - \overline{x}_i y_j}{1 - x_i y_j}. \end{aligned}$$

We can think of $\Delta(\mathbf{x}_n; \mathbf{y})$ (resp. $\Delta(\mathbf{x}; \mathbf{y})$) as an element of $\Lambda^{\mathbb{L}}(\mathbf{x}_n) \hat{\otimes}_{\mathbb{L}} \Lambda_{\mathbb{L}}(\mathbf{y})$ (resp. $\Lambda^{\mathbb{L}}(\mathbf{x}) \hat{\otimes}_{\mathbb{L}} \Lambda_{\mathbb{L}}(\mathbf{y})$).

Proposition 5.2.

(1) *$\Delta(x_1, \dots, x_n; \mathbf{y})$ is $\mathbb{L}$-supersymmetric in the variables $x_1, \dots, x_n$. Therefore $\Delta(x_1, \dots, x_n; \mathbf{y})$ is in $\Gamma^{\mathbb{L}}(\mathbf{x}_n) \hat{\otimes}_{\mathbb{L}} \Lambda_{\mathbb{L}}(\mathbf{y})$. Moreover, $\Delta(x_1, \dots, x_n; \mathbf{y})$ is in $\Gamma^{\mathbb{L}}_{+}(\mathbf{x}_n) \hat{\otimes}_{\mathbb{L}} \Lambda_{\mathbb{L}}(\mathbf{y})$.*

(2) *Furthermore, we have $\Delta(x_1, \dots, x_n; \mathbf{y})$ is in $\Gamma^{\mathbb{L}}_{+}(\mathbf{x}_n) \hat{\otimes}_{\mathbb{L}} \Gamma^{(n)}_{\mathbb{L}}(\mathbf{y})$ and $\Delta(x_1, \dots, x_n; \mathbf{y})$ is in $\Gamma^{\mathbb{L}}(\mathbf{x}_n) \hat{\otimes}_{\mathbb{L}} \Gamma^{(n),+}_{\mathbb{L}}(\mathbf{y})$.*

Proof. (1) By the definition, it is obvious that $\Delta(t, \overline{t}, x_3, \dots, x_n) = \Delta(0, 0, x_3, \dots, x_n)$ holds. Moreover, using Lemma 4.5, we see easily that $\Delta(t, x_2, \dots, x_n; \mathbf{y}) - \Delta(0, x_2, \dots, x_n; \mathbf{y})$ is divisible by $t +_{F} t$, and hence the first assertion is proved.

(2) Using (42), we have

$$\prod_{i=1}^{n} \prod_{j=1}^{\infty} \frac{1 - \overline{x}_i y_j}{1 - x_i y_j} = \prod_{i=1}^{n} \left(\sum_{\lambda_i = 0}^{\infty} x_i^{\lambda_i} \widehat{q}^{\mathbb{L}}_{\lambda_i}(\mathbf{y}) \right) = \sum_{\lambda \in \mathcal{P}_n} m_{\lambda}(\mathbf{x}_n) \, \widehat{q}^{\mathbb{L}}_{[\lambda]}(\mathbf{y}),$$

where $m_{\lambda}(\mathbf{x}_n)$ denotes the *monomial symmetric polynomial* corresponding to a partition $\lambda \in \mathcal{P}_n$. Using (44), we see that $\widehat{q}^{\mathbb{L}}_{[\lambda]}(\mathbf{y})$ can be written as a certain $\mathbb{L}$-linear combination of the monomials of the form $\widehat{p}^{\mathbb{L}}_{[\mu]}(\mathbf{y})$ with $\mu \in \mathcal{P}_n$. From these, the result follows immediately. Q.E.D.

Taking limit $n \to \infty$ and using the Basis Theorem (Corollary 4.21), we can expand $\Delta(\mathbf{x}; \mathbf{y})$ in terms of a basis $\{Q^{\mathbb{L}}_{\lambda}(\mathbf{x})\}_{\lambda \in \mathcal{SP}}$ for $\Gamma^{\mathbb{L}}_{+}(\mathbf{x})$ (resp. a basis $\{P^{\mathbb{L}}_{\lambda}(\mathbf{x})\}_{\lambda \in \mathcal{SP}}$ for $\Gamma^{\mathbb{L}}(\mathbf{x})$). Then we will obtain the required functions as the coefficients of these expansions. Thus we make the following definition:

Definition 5.3 (Dual universal Schur P and Q-functions)**.** *We define $\widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y})$ and $\widehat{q}^{\mathbb{L}}_{\lambda}(\mathbf{y})$ for strict partitions $\lambda \in \mathcal{SP}$ by the following identities* (*Cauchy identities*) :

$$\begin{aligned} \Delta(\mathbf{x}; \mathbf{y}) &= \prod_{i,j \geq 1} \frac{1 - \overline{x}_i y_j}{1 - x_i y_j} = \sum_{\lambda \in \mathcal{SP}} Q^{\mathbb{L}}_{\lambda}(\mathbf{x}) \, \widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y}), \\ \Delta(\mathbf{x}; \mathbf{y}) &= \prod_{i,j \geq 1} \frac{1 - \overline{x}_i y_j}{1 - x_i y_j} = \sum_{\lambda \in \mathcal{SP}} P^{\mathbb{L}}_{\lambda}(\mathbf{x}) \, \widehat{q}^{\mathbb{L}}_{\lambda}(\mathbf{y}). \end{aligned} \tag{46}$$

By the Definition 5.3 and Proposition 5.2 (2), we see that for a strict partition λ of length r,

$$\widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y}) \in \Gamma^{(r)}_{\mathbb{L}}(\mathbf{y}) \subset \Gamma_{\mathbb{L}}(\mathbf{y}) \quad \text{and} \quad \widehat{q}^{\mathbb{L}}_{\lambda}(\mathbf{y}) \in \Gamma^{(r),+}_{\mathbb{L}}(\mathbf{y}) \subset \Gamma^{+}_{\mathbb{L}}(\mathbf{y}).$$

Remark 5.4.

(1) *By using the Basis Theorem (Theorem 4.20), we can also define $\widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y})$ and $\widehat{q}^{\mathbb{L}}_{\lambda}(\mathbf{y})$ for strict partitions $\lambda \in \mathcal{SP}_n$ by the following identities:*

$$\begin{aligned} \Delta(\mathbf{x}_n;\mathbf{y}) &= \prod_{i=1}^{n}\prod_{j=1}^{\infty}\frac{1-\overline{x}_i y_j}{1-x_i y_j} = \sum_{\lambda\in\mathcal{SP}_n} Q^{\mathbb{L}}_{\lambda}(\mathbf{x}_n)\,\widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y}),\\ \Delta(\mathbf{x}_n;\mathbf{y}) &= \prod_{i=1}^{n}\prod_{j=1}^{\infty}\frac{1-\overline{x}_i y_j}{1-x_i y_j} = \sum_{\lambda\in\mathcal{SP}_n} P^{\mathbb{L}}_{\lambda}(\mathbf{x}_n)\,\widehat{q}^{\mathbb{L}}_{\lambda}(\mathbf{y}). \end{aligned}$$

Then by the stability property of $P^{\mathbb{L}}_{\lambda}(\mathbf{x}_n)$ and $Q^{\mathbb{L}}_{\lambda}(\mathbf{x}_n)$ (see Proposition 4.7), we see that the definition of $\widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y})$ and $\widehat{q}^{\mathbb{L}}_{\lambda}(\mathbf{y})$ does not depend on n. In particular, in view of (45), we see that $\widehat{p}^{\mathbb{L}}_{(k)}(\mathbf{y})$ and $\widehat{q}^{\mathbb{L}}_{(k)}(\mathbf{y})$ corresponding to the one rows (k) ($k = 1, 2, \ldots$) agree with $\widehat{p}^{\mathbb{L}}_{k}(\mathbf{y})$ and $\widehat{q}^{\mathbb{L}}_{k}(\mathbf{y})$ defined in §5.1 respectively.

(2) *By using $\Delta(\mathbf{x};\mathbf{y})$ and the Basis Theorem (Corollary 4.27), one can formally define the dual universal factorial Schur P- and Q-functions $\widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y}|\mathbf{b})$, $\widehat{q}^{\mathbb{L}}_{\lambda}(\mathbf{y}|\mathbf{b})$. However, in order to make this definition valid, we have to remove some technical difficulties. We hope to return to this problem elsewhere.*

The functions $\widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y})$ (resp. $\widehat{q}^{\mathbb{L}}_{\lambda}(\mathbf{y})$) are elements of $\Gamma_{\mathbb{L}}(\mathbf{y}) \subset \Lambda_{\mathbb{L}}(\mathbf{y})$ (resp. $\Gamma^{+}_{\mathbb{L}}(\mathbf{y}) \subset \Lambda_{\mathbb{L}}(\mathbf{y})$), and hence symmetric functions of total degree $|\lambda|$. If we put $a_{i,j} = 0$ for all $i, j \geq 1$, $\widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y})$ (resp. $\widehat{q}^{\mathbb{L}}_{\lambda}(\mathbf{y})$) reduce to the usual Schur P-functions $P_{\lambda}(\mathbf{y})$ (resp. Q-functions $Q_{\lambda}(\mathbf{y})$). Therefore we have

$$(47)\qquad \begin{aligned} \widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y}) &= P_{\lambda}(\mathbf{y}) + \text{lower order terms in } \mathbf{y},\\ \widehat{q}^{\mathbb{L}}_{\lambda}(\mathbf{y}) &= Q_{\lambda}(\mathbf{y}) + \text{lower order terms in } \mathbf{y}. \end{aligned}$$

Here are some examples of these functions (see also Examples 3.2 and 3.6):

Example 5.5.

$$\begin{aligned} \widehat{p}^{\mathbb{L}}_{1}(\mathbf{y}) &= P_1(\mathbf{y}),\\ \widehat{p}^{\mathbb{L}}_{2}(\mathbf{y}) &= P_2(\mathbf{y}) - a_{1,1}h_1(\mathbf{y}),\\ \widehat{p}^{\mathbb{L}}_{3}(\mathbf{y}) &= P_3(\mathbf{y}) + a_{1,1}h_2(\mathbf{y}) - 2a_{1,1}h_1(\mathbf{y})^2 + (a_{1,1}^2 - a_{1,2})h_1(\mathbf{y}).\\ \widehat{q}^{\mathbb{L}}_{1}(\mathbf{y}) &= Q_1(\mathbf{y}),\\ \widehat{q}^{\mathbb{L}}_{2}(\mathbf{y}) &= Q_2(\mathbf{y}) - a_{1,1}h_1(\mathbf{y}),\\ \widehat{q}^{\mathbb{L}}_{3}(\mathbf{y}) &= Q_3(\mathbf{y}) + 2a_{1,1}h_2(\mathbf{y}) - 3a_{1,1}h_1(\mathbf{y})^2 + a_{1,1}^2 h_1(\mathbf{y}). \end{aligned}$$

5.3. Basis Theorem for $\widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y})$, $\widehat{q}^{\mathbb{L}}_{\lambda}(\mathbf{y})$

As expected, the functions $\{\widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y})\}_{\lambda\in\mathcal{SP}}$ (resp. $\{\widehat{q}^{\mathbb{L}}_{\lambda}(\mathbf{y})\}_{\lambda\in\mathcal{SP}}$) constitute an $\mathbb{L}$-basis for $\Gamma_{\mathbb{L}}(\mathbf{y})$ (resp. $\Gamma^{+}_{\mathbb{L}}(\mathbf{y})$).

Theorem 5.6 (Basis Theorem)**.**

(1) $\{\widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y})\}_{\lambda\in\mathcal{SP}}$ *are linearly independent over* $\mathbb{L}$ *and form an* $\mathbb{L}$-*basis of* $\Gamma_{\mathbb{L}}(\mathbf{y})$.

(2) $\{\widehat{q}^{\mathbb{L}}_{\lambda}(\mathbf{y})\}_{\lambda\in\mathcal{SP}}$ *are linearly independent over* $\mathbb{L}$ *and form an* $\mathbb{L}$-*basis of* $\Gamma^{+}_{\mathbb{L}}(\mathbf{y})$.

Proof. We only prove the assertion (1). (2) can be proved similarly. From (47) and the similar argument as in §4.2, we see that $\widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y})$'s, $\lambda\in\mathcal{SP}$, are linearly independent over $\mathbb{L}$. In particular, an $\mathbb{L}$-submodule $\tilde{\Gamma}_{\mathbb{L}}(\mathbf{y})$ of $\Gamma_{\mathbb{L}}(\mathbf{y})$ generated by $\widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y})$'s, $\lambda\in\mathcal{SP}$, is a direct sum: $\tilde{\Gamma}_{\mathbb{L}}(\mathbf{y}) = \bigoplus_{\lambda\in\mathcal{SP}}\mathbb{L}\,\widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y}) \subset \Gamma_{\mathbb{L}}(\mathbf{y})$. We wish to show that $\tilde{\Gamma}_{\mathbb{L}}(\mathbf{y})$ agrees with $\Gamma_{\mathbb{L}}(\mathbf{y})$, or equivalently, the set $\{\widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y})\}_{\lambda\in\mathcal{SP}}$ spans $\Gamma_{\mathbb{L}}(\mathbf{y})$ over $\mathbb{L}$. To this end, it is sufficient to show that monomials $\widehat{p}^{\mathbb{L}}_{k_1}(\mathbf{y})\widehat{p}^{\mathbb{L}}_{k_2}(\mathbf{y})\cdots\widehat{p}^{\mathbb{L}}_{k_r}(\mathbf{y})$ ($k_1\geq 1,\ldots,k_r\geq 1,\ r\geq 1$) are expressed as $\mathbb{L}$-linear combinations of $\widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y})$'s, $\lambda\in\mathcal{SP}$, since $\Gamma_{\mathbb{L}}(\mathbf{y})$ is generated by $\widehat{p}^{\mathbb{L}}_{k}(\mathbf{y})$ ($k=1,2,\ldots$) as an $\mathbb{L}$-algebra. We prove this by making use of the *Hopf algebra property* of $\Gamma^{\mathbb{L}}_{+}(\mathbf{x})$. Let $\mathbf{x}=(x_1,x_2,\ldots)$ and $\mathbf{x}'=(x'_1,x'_2,\ldots)$ be two countable sets of variables. Then the function $Q^{\mathbb{L}}_{\lambda}(\mathbf{x},\mathbf{x}')$ can be written in terms of the two sets separately, that is,

$$Q^{\mathbb{L}}_{\lambda}(\mathbf{x},\mathbf{x}') = \sum_{\mu,\,\nu\in\mathcal{SP}} \widehat{c}^{\lambda}_{\mu,\,\nu}\, Q^{\mathbb{L}}_{\mu}(\mathbf{x})\, Q^{\mathbb{L}}_{\nu}(\mathbf{x}') \quad (\widehat{c}^{\lambda}_{\mu,\,\nu}\in\mathbb{L}). \tag{48}$$

This gives the coproduct (comultiplication, diagonal map) $\phi:\Gamma^{\mathbb{L}}_{+}(\mathbf{x})\longrightarrow\Gamma^{\mathbb{L}}_{+}(\mathbf{x})\otimes_{\mathbb{L}}\Gamma^{\mathbb{L}}_{+}(\mathbf{x})$. Note that by the Basis Theorem for $Q^{\mathbb{L}}_{\lambda}(\mathbf{x})$'s (Corollary 4.21), the coefficients $\widehat{c}^{\lambda}_{\mu,\,\nu}\in\mathbb{L}$ are uniquely determined in the above expression. On the other hand, we have by Definition 5.3,

$$\begin{aligned}
\Delta(\mathbf{x};\mathbf{y}) &= \prod_{i,j\geq 1}\frac{1-\overline{x}_i y_j}{1-x_i y_j} = \sum_{\mu\in\mathcal{SP}} Q^{\mathbb{L}}_{\mu}(\mathbf{x})\,\widehat{p}^{\mathbb{L}}_{\mu}(\mathbf{y}),\\
\Delta(\mathbf{x}';\mathbf{y}) &= \prod_{i,j\geq 1}\frac{1-\overline{x'_i} y_j}{1-x'_i y_j} = \sum_{\nu\in\mathcal{SP}} Q^{\mathbb{L}}_{\nu}(\mathbf{x}')\,\widehat{p}^{\mathbb{L}}_{\nu}(\mathbf{y}).
\end{aligned}$$

Multiplying these two expressions, we have

$$\sum_{\lambda\in\mathcal{SP}} Q^{\mathbb{L}}_{\lambda}(\mathbf{x},\mathbf{x}')\,\widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y}) = \Delta(\mathbf{x},\mathbf{x}';\mathbf{y}) = \sum_{\mu,\,\nu\in\mathcal{SP}} \widehat{p}^{\mathbb{L}}_{\mu}(\mathbf{y})\widehat{p}^{\mathbb{L}}_{\nu}(\mathbf{y})Q^{\mathbb{L}}_{\mu}(\mathbf{x})Q^{\mathbb{L}}_{\nu}(\mathbf{x}').$$

By (48), the left-hand side turns into

$$\sum_{\mu,\,\nu\in\mathcal{SP}}\left(\sum_{\lambda\in\mathcal{SP}}\widehat{c}^{\lambda}_{\mu,\,\nu}\widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y})\right)Q^{\mathbb{L}}_{\mu}(\mathbf{x})Q^{\mathbb{L}}_{\nu}(\mathbf{x}'),$$

and therefore we have the following product formula (we used the Basis Theorem for $Q^{\mathbb{L}}_{\lambda}(\mathbf{x})$'s again):

$$\widehat{p}^{\mathbb{L}}_{\mu}(\mathbf{y})\widehat{p}^{\mathbb{L}}_{\nu}(\mathbf{y})=\sum_{\lambda\in\mathcal{SP}}\widehat{c}^{\lambda}_{\mu,\,\nu}\,\widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y}). \tag{49}$$

Notice that by our convention of the grading of $\mathbb{L}=\mathbb{L}_*$, the right-hand side is necessarily a finite sum, and hence is contained in an $\mathbb{L}$-submodule $\tilde{\Gamma}_{\mathbb{L}}(\mathbf{y})=\bigoplus_{\lambda\in\mathcal{SP}}\mathbb{L}\,\widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y})$. Thus $\tilde{\Gamma}_{\mathbb{L}}(\mathbf{y})$ is closed under multiplication. Especially, in the case $\mu=(k)$ and $\nu=(l)$, i.e., one rows, the product $\widehat{p}^{\mathbb{L}}_{k}(\mathbf{y})\widehat{p}^{\mathbb{L}}_{l}(\mathbf{y})$ is contained in $\tilde{\Gamma}_{\mathbb{L}}(\mathbf{y})$. Iterating use of the product formula (49) yields the required result. Q.E.D.

5.4. Hopf algebra structure

As mentioned in §3.1, the ring of symmetric functions Λ has a structure of a self-dual Hopf algebra over $\mathbb{Z}$. Also its subalgebras Γ and Γ' have Hopf algebra structures over $\mathbb{Z}$ which are dual to each other. In this subsection, we shall mention the Hopf algebra structure of our rings $\Gamma^{\mathbb{L}}(\mathbf{x})$, $\Gamma^{\mathbb{L}}_{+}(\mathbf{x})$ (see §4.4) and $\Gamma_{\mathbb{L}}(\mathbf{y})$, $\Gamma^{+}_{\mathbb{L}}(\mathbf{y})$ (see §5.2). As we saw in the proof of the Basis Theorem (Theorem 5.6), through the Cauchy identity (46), the coproduct formula (48) of $Q^{\mathbb{L}}_{\lambda}(\mathbf{x})$ determines the product formula (49) of $\widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y})$'s. By an analogous argument, one sees easily that the product formula

$$Q^{\mathbb{L}}_{\mu}(\mathbf{x})\,Q^{\mathbb{L}}_{\nu}(\mathbf{x})=\sum_{\lambda\in\mathcal{SP}}c^{\lambda}_{\mu,\,\nu}\,Q^{\mathbb{L}}_{\lambda}(\mathbf{x})\quad(c^{\lambda}_{\mu,\,\nu}\in\mathbb{L})$$

determines the coproduct formula

$$\widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y},\mathbf{y}')=\sum_{\mu,\,\nu\in\mathcal{SP}}c^{\lambda}_{\mu,\,\nu}\,\widehat{p}^{\mathbb{L}}_{\mu}(\mathbf{y})\,\widehat{p}^{\mathbb{L}}_{\nu}(\mathbf{y}'),$$

where $\mathbf{y}=(y_1,y_2,\ldots)$ and $\mathbf{y}'=(y'_1,y'_2,\ldots)$ are two countable sets of variables. Thus we obtain the following result as a formal consequence of the Cauchy identities (46). Here we write ϕ for the coproduct map.

Proposition 5.7 (Duality)**.**

(1) (*Duality between* $\Gamma_+^{\mathbb{L}}(\mathbf{x})$ *and* $\Gamma_{\mathbb{L}}(\mathbf{y})$)
If $Q_\lambda^{\mathbb{L}}(\mathbf{x})\, Q_\mu^{\mathbb{L}}(\mathbf{x}) = \sum_{\nu \in \mathcal{SP}} c_{\lambda,\mu}^{\nu}\, Q_\nu^{\mathbb{L}}(\mathbf{x})$, *then*

$$\phi(\widehat{p}_\nu^{\mathbb{L}}(\mathbf{y})) = \sum_{\lambda,\,\mu \in \mathcal{SP}} c_{\lambda,\mu}^{\nu}\, \widehat{p}_\lambda^{\mathbb{L}}(\mathbf{y}) \otimes \widehat{p}_\mu^{\mathbb{L}}(\mathbf{y}).$$

If $\phi(Q_\nu^{\mathbb{L}}(\mathbf{x})) = \sum_{\lambda,\,\mu \in \mathcal{SP}} \widehat{c}_{\lambda,\mu}^{\nu}\, Q_\lambda^{\mathbb{L}}(\mathbf{x}) \otimes Q_\mu^{\mathbb{L}}(\mathbf{x})$, *then*

$$\widehat{p}_\lambda^{\mathbb{L}}(\mathbf{y})\, \widehat{p}_\mu^{\mathbb{L}}(\mathbf{y}) = \sum_{\nu \in \mathcal{SP}} \widehat{c}_{\lambda,\mu}^{\nu}\, \widehat{p}_\nu^{\mathbb{L}}(\mathbf{y}).$$

(2) (*Duality between* $\Gamma^{\mathbb{L}}(\mathbf{x})$ *and* $\Gamma_{\mathbb{L}}^{+}(\mathbf{y})$)
If $P_\lambda^{\mathbb{L}}(\mathbf{x})\, P_\mu^{\mathbb{L}}(\mathbf{x}) = \sum_{\nu \in \mathcal{SP}} d_{\lambda,\mu}^{\nu}\, P_\nu^{\mathbb{L}}(\mathbf{x})$, *then*

$$\phi(\widehat{q}_\nu^{\mathbb{L}}(\mathbf{y})) = \sum_{\lambda,\,\mu \in \mathcal{SP}} d_{\lambda,\mu}^{\nu}\, \widehat{q}_\lambda^{\mathbb{L}}(\mathbf{y}) \otimes \widehat{q}_\mu^{\mathbb{L}}(\mathbf{y}).$$

If $\phi(P_\nu^{\mathbb{L}}(\mathbf{x})) = \sum_{\lambda,\,\mu \in \mathcal{SP}} \widehat{d}_{\lambda,\mu}^{\nu}\, P_\lambda^{\mathbb{L}}(\mathbf{x}) \otimes P_\mu^{\mathbb{L}}(\mathbf{x})$, *then*

$$\widehat{q}_\lambda^{\mathbb{L}}(\mathbf{y})\widehat{q}_\mu^{\mathbb{L}}(\mathbf{y}) = \sum_{\nu \in \mathcal{SP}} \widehat{d}_{\lambda,\mu}^{\nu}\, \widehat{q}_\nu^{\mathbb{L}}(\mathbf{y}).$$

By the Basis Theorems (Corollary 4.21 and Theorem 5.6), we can define the $\mathbb{L}$-bilinear pairing between the rings $\Gamma_+^{\mathbb{L}}(\mathbf{x})$ and $\Gamma_{\mathbb{L}}(\mathbf{y})$,

$$[-,-] : \Gamma_+^{\mathbb{L}}(\mathbf{x}) \times \Gamma_{\mathbb{L}}(\mathbf{y}) \longrightarrow \mathbb{L},$$

by setting $[Q_\lambda^{\mathbb{L}}(\mathbf{x}), \widehat{p}_\mu^{\mathbb{L}}(\mathbf{y})] = \delta_{\lambda,\mu}$. This pairing induces an $\mathbb{L}$-module homomorphism

$$\kappa : \Gamma_+^{\mathbb{L}}(\mathbf{x}) \longrightarrow \mathrm{Hom}_{\mathbb{L}}(\Gamma_{\mathbb{L}}(\mathbf{y}), \mathbb{L}), \quad F \longmapsto \kappa(F) = [F, -].$$

Using the Basis Theorems (Corollary 4.21 and Theorem 5.6) again, one can show that κ is an isomorphism of $\mathbb{L}$-modules. Furthermore, both $\Gamma_+^{\mathbb{L}}(\mathbf{x})$ and $\mathrm{Hom}_{\mathbb{L}}(\Gamma_{\mathbb{L}}(\mathbf{y}), \mathbb{L})$ have a Hopf algebra structure over $\mathbb{L}$. Using the above duality (Proposition 5.7), one can also show that κ is actually an isomorphism of Hopf algebras over $\mathbb{L}$. As we remarked earlier, we know that $\Gamma_{\mathbb{L}}(\mathbf{y}) \cong \Gamma_*^{MU} \cong MU_*(\Omega Sp)$, and therefore $\mathrm{Hom}_{\mathbb{L}}(\Gamma_{\mathbb{L}}(\mathbf{y}), \mathbb{L}) \cong$

$MU^*(\Omega Sp)$ as Hopf algebras over $\mathbb{L}$. Thus we have the following isomorphism of Hopf algebras over $\mathbb{L}$:

$$\Gamma^{\mathbb{L}}_{+}(\mathbf{x}) \cong \Gamma^*_{MU} \cong MU^*(\Omega Sp).$$

Similarly we define the pairing

$$[-,-] : \Gamma^{\mathbb{L}}(\mathbf{x}) \times \Gamma^{+}_{\mathbb{L}}(\mathbf{y}) \longrightarrow \mathbb{L},$$

by setting $[P^{\mathbb{L}}_{\lambda}(\mathbf{x}), \widehat{q}^{\mathbb{L}}_{\mu}(\mathbf{y})] = \delta_{\lambda,\,\mu}$. By means of this pairing and the same argument as above, we have the following isomorphism of Hopf algebras over $\mathbb{L}$:

$$\Gamma^{\mathbb{L}}(\mathbf{x}) \cong {\Gamma'}^*_{MU} \cong MU^*(\Omega_0 SO).$$

Summing up the results so far, we have the following:

Theorem 5.8. *There is a symmetric function realization as Hopf algebras over* $\mathbb{L}_*$:

$$\begin{aligned} MU_*(\Omega Sp) &\cong {\Gamma'}^{MU}_{*} \cong \Gamma_{\mathbb{L}}(\mathbf{y}) = \mathbb{L}[\widehat{p}^{\mathbb{L}}_1(\mathbf{y}), \widehat{p}^{\mathbb{L}}_3(\mathbf{y}), \ldots, \widehat{p}^{\mathbb{L}}_{2i-1}(\mathbf{y}), \ldots] \\ &= \bigoplus_{\lambda \in \mathcal{SP}} \mathbb{L}\, \widehat{p}^{\mathbb{L}}_{\lambda}(\mathbf{y}) \ \subset \ \Lambda_{\mathbb{L}}(\mathbf{y}), \\ MU_*(\Omega_0 SO) &\cong \Gamma^{MU}_{*} \cong \Gamma^{+}_{\mathbb{L}}(\mathbf{y}) \\ &= \mathbb{L}[\widehat{q}^{\mathbb{L}}_1(\mathbf{y}), \widehat{q}^{\mathbb{L}}_2(\mathbf{y}), \ldots, \widehat{q}^{\mathbb{L}}_i(\mathbf{y}), \ldots]/(\widehat{q}^{\mathbb{L}}(T)\widehat{q}^{\mathbb{L}}(\overline{T}) = 1) \\ &= \bigoplus_{\lambda \in \mathcal{SP}} \mathbb{L}\, \widehat{q}^{\mathbb{L}}_{\lambda}(\mathbf{y}) \ \subset \ \Lambda_{\mathbb{L}}(\mathbf{y}). \end{aligned}$$

Dually, there is a symmetric function realization as Hopf algebras over $\mathbb{L}^*$:

$$\begin{aligned} MU^*(\Omega Sp) &\cong \Gamma^*_{MU} \cong \Gamma^{\mathbb{L}}_{+}(\mathbf{x}) = \prod_{\lambda \in \mathcal{SP}} \mathbb{L}\, Q^{\mathbb{L}}_{\lambda}(\mathbf{x}) \ \subset \ \Lambda^{\mathbb{L}}(\mathbf{x}), \\ MU^*(\Omega_0 SO) &\cong {\Gamma'}^*_{MU} \cong \Gamma^{\mathbb{L}}(\mathbf{x}) = \prod_{\lambda \in \mathcal{SP}} \mathbb{L}\, P^{\mathbb{L}}_{\lambda}(\mathbf{x}) \ \subset \ \Lambda^{\mathbb{L}}(\mathbf{x}). \end{aligned}$$

5.5. Concluding remarks

(1) In §3.3, we introduced the functions $\tilde{q}^{MU}_k(\mathbf{x}) \in \Gamma^*_{MU}$ ($k = 1, 2, \ldots$), and described the ring Γ^*_{MU} in terms of these functions. At present, we have not been able to obtain a general formula relating the functions $\tilde{q}^{MU}_k(\mathbf{x})$ ($k = 1, 2, \ldots$) and $Q^{\mathbb{L}}_{\lambda}(\mathbf{x})$ ($\lambda \in \mathcal{SP}$) under the isomorphism $\Gamma^*_{MU} \cong \Gamma^{\mathbb{L}}_{+}(\mathbf{x})$.

(2) By the result of Buch [11], we observe that the *stable Grothendieck polynomials* $G_\lambda(\mathbf{x})$, $\lambda \in \mathcal{P}$, represent the Schubert classes of the K-theory (more precisely, K-cohomology) of the infinite Grassmannain $BU \simeq \Omega SU$. Dual functions $g_\lambda(\mathbf{y})$, $\lambda \in \mathcal{P}$, called the *dual stable Grothendieck polynomials*, are defined and studied by Lam-Pylyavskyy [38, §9.1]. By the construction, these dual functions $\{g_\lambda(\mathbf{y})\}$ represent the Schubert classes of the K-homology $K_*(\Omega SU)$. There is a "type C" analogue of the above story: If we specialize $a_{1,1} = \beta$, $a_{i,j} = 0$ for all $(i,j) \neq (1,1)$ in Definition 5.3, the resulting functions will be denoted by $gp_\lambda(\mathbf{y})$ and $gq_\lambda(\mathbf{y})$, $\lambda \in \mathcal{SP}$. By definition, these functions are dual to the K-theoretic Schur P- and Q-functions $GP_\lambda(\mathbf{x})$ and $GQ_\lambda(\mathbf{x})$ due to Ikeda-Naruse [25, Definition 2.1]. As shown in that paper, the functions $\{GP_\lambda(\mathbf{x})\}$ and $\{GQ_\lambda(\mathbf{x})\}$ represent the Schubert classes for the K-cohomology of infinite maximal isotropic Grassmannians $Sp/U \simeq \Omega Sp$ and $SO/U \simeq \Omega_0 SO$ (see Ikeda-Naruse [25, Corollary 8.1]). Therefore the functions $\{gp_\lambda(\mathbf{y})\}$ and $\{gq_\lambda(\mathbf{y})\}$ are expected to represent the Schubert classes for the (non-equivariant) K-homology $K_*(\Omega Sp)$ and $K_*(\Omega_0 SO)$ of the loop spaces ΩSp and $\Omega_0 SO$. We return to this problem elsewhere.

§6. Appendix

6.1. Universal factorial Schur functions

In §4.5, we defined the *universal factorial Schur functions* in the n-variables $s^{\mathbb{L}}_\lambda(\mathbf{x}_n|\mathbf{b})$, $\lambda \in \mathcal{P}_n$. In order to consider the limit function as $n \to \infty$, we need to modify its definition. In fact, we are able to generalize the *double Schur function* $s_\lambda(x||a)$ and the *dual Schur function* $\widehat{s}_\lambda(y||a)$ due to Molev [53, §2.1, §3.1] in the universal setting. In this appendix, we only exhibit the definition and basic property of our functions. Details will be discussed elsewhere. Here we use a doubly infinite sequence $\mathbf{b}_\mathbb{Z} = (\ldots, b_{-2}, b_{-1}, b_0, b_1, b_2, \ldots)$ in place of $\mathbf{b} = (b_1, b_2, \ldots)$. First we introduce another variant of the ordinary k-th power t^k (see §4.2 and Molev [53, p.7]). For a fixed positive integer n, we define

$$[t\,||\,\mathbf{b}_\mathbb{Z}]^k_n := \prod_{i=1}^{k}(t +_{\scriptscriptstyle F} b_{n+1-i}) = (t +_{\scriptscriptstyle F} b_n)(t +_{\scriptscriptstyle F} b_{n-1}) \cdots (t +_{\scriptscriptstyle F} b_{n+1-k}),$$

where we set $[t \,||\, \mathbf{b}_{\mathbb{Z}}]^0_n := 1$. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, we set

$$[\mathbf{x} \,||\, \mathbf{b}_{\mathbb{Z}}]^{\lambda}_n := \prod_{i=1}^{r} [x_i \,||\, \mathbf{b}_{\mathbb{Z}}]^{\lambda_i}_n.$$

Definition 6.1 (Universal factorial Schur functions)**.** *For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{P}_n$, we define*

$$s^{\mathbb{L}}_{\lambda}(\mathbf{x}_n \,||\, \mathbf{b}_{\mathbb{Z}}) = s^{\mathbb{L}}_{\lambda}(x_1, \ldots, x_n \,||\, \mathbf{b}_{\mathbb{Z}}) := \sum_{w \in S_n} w \left[\frac{[\mathbf{x} \,||\, \mathbf{b}_{\mathbb{Z}}]^{\lambda + \rho_{n-1}}_n}{\prod_{1 \le i < j \le n} (x_i +_F \overline{x}_j)} \right],$$

where $\rho_{n-1} = (n-1, n-2, \ldots, 1, 0)$.

This is a symmetric formal power series with coefficients in $\mathbb{L}$ in the variables $x_1, \ldots, x_n$ and $b_n, b_{n-1}, \ldots, b_{2-\lambda_1}$. One sees immediately from the Definitions 4.10 and 6.1 that $s^{\mathbb{L}}_{\lambda}(\mathbf{x}_n|\mathbf{b})$ turns into $s^{\mathbb{L}}_{\lambda}(\mathbf{x}_n \,||\, \mathbf{b}_{\mathbb{Z}})$ by changing the parameters $b_i \to b_{n-i+1}$ $(i = 1, 2, \ldots)$, and vice versa.

As with the case of $s^{\mathbb{L}}_{\lambda}(\mathbf{x}_n|\mathbf{b})$ (see Proposition 4.13), the functions $s^{\mathbb{L}}_{\lambda}(\mathbf{x}_n \,||\, \mathbf{b}_{\mathbb{Z}})$ also satisfy the Vanishing Property. Given a partition $\mu \in \mathcal{P}_n$, we introduce the sequence $\overline{\mathbf{b}}_{I-\mu}$ by

$$\overline{\mathbf{b}}_{I-\mu} := (\overline{b}_{1-\mu_1}, \overline{b}_{2-\mu_2}, \ldots, \overline{b}_{i-\mu_i}, \ldots).$$

Proposition 6.2 (Vanishing Property)**.** *Let $\lambda, \mu \in \mathcal{P}_n$. Then we have*

$$s^{\mathbb{L}}_{\lambda}(\overline{\mathbf{b}}_{I-\mu} \,||\, \mathbf{b}_{\mathbb{Z}}) = \begin{cases} 0 & \text{if} \quad \mu \not\supset \lambda, \\ \displaystyle\prod_{(i,j) \in \lambda} (\overline{b}_{i-\lambda_i} +_F b_{{}^t\lambda_j - j + 1}) & \text{if} \quad \mu = \lambda. \end{cases}$$

Since the functions $\{s^{\mathbb{L}}_{\lambda}(\mathbf{x}_n \,||\, \mathbf{b}_{\mathbb{Z}})\}_{n \ge 1}$ have the stability property under the evaluation map $x_n = \overline{b}_n$, we can take limit $n \to \infty$ to obtain the limit function $s^{\mathbb{L}}_{\lambda}(\mathbf{x}||\, \mathbf{b}_{\mathbb{Z}})$.

Using the Cauchy identity and an analogous argument that we did in §5, we can define the dual functions $\widehat{s}^{\mathbb{L}}_{\lambda}(\mathbf{y}||\mathbf{b}_{\mathbb{Z}})$ for $\lambda \in \mathcal{P}_n$. These functions are a generalization of Molev's dual Schur functions $\widehat{s}_{\lambda}(y||a)$ in the universal setting. Here we use one more set of variables $\mathbf{y} = (y_1, y_2, \ldots)$.

Definition 6.3 (Dual universal factorial Schur functions)**.** *For a partition $\lambda \in \mathcal{P}_n$, we define $\widehat{s}^{\mathbb{L}}_{\lambda}(\mathbf{y} \,||\, \mathbf{b}_{\mathbb{Z}})$ by the following identities* (*Cauchy identities*) :

$$\prod_{i=1}^{n} \prod_{j=1}^{\infty} \frac{1 - \overline{b}_i y_j}{1 - x_i y_j} = \sum_{\lambda \in \mathcal{P}_n} s^{\mathbb{L}}_{\lambda}(\mathbf{x}_n \,||\, \mathbf{b}_{\mathbb{Z}})\, \widehat{s}^{\mathbb{L}}_{\lambda}(\mathbf{y} \,||\, \mathbf{b}_{\mathbb{Z}}).$$

For a geometric meaning of Molev's double and dual Schur functions, readers are referred to Lam-Shimozono [41].

6.2. Root data

6.2.1. *Type A_∞*

Weyl group of type A_∞ Let $S_\infty := \varinjlim_n S_n$ be the *infinite* symmetric group, that is, the Weyl group of type A_∞: $W(A_\infty) \cong S_\infty$. An element w of S_∞ is identified with a permutation of the set of positive integers $\mathbb{N} := \{1, 2, \ldots\}$ such that $w(i) = i$ for all but a finite number of $i \in \mathbb{N}$. We shall use the *one-line notation* $w = w(1)\,w(2) \cdots$ to denote an element $w \in S_\infty$. The group S_∞ is also a Coxeter group generated by the *simple reflections* $\{s_i\}_{i \in I}$, where the index set $I = \{1, 2, \ldots\}$. More explicitly, the simple reflections are given by the *simple transpositions*, i.e., $s_i = (i\ i+1)$ $(i \geq 1)$. The defining relations of $\{s_i\}_{i \in I}$ are given by

$$\begin{array}{rcll} s_i^2 & = & 1 & (i \geq 1), \\ s_i s_{i+1} s_i & = & s_{i+1} s_i s_{i+1} & (i \geq 1), \\ s_i s_j & = & s_j s_i & (i, j \geq 1,\ |i-j| \geq 2). \end{array}$$

Root system of type A_∞ We fix notation about the root system of type A_∞. Let L denote a free $\mathbb{Z}$-module with a basis $\{t_i\}_{i \geq 1}$. For $j > i \geq 1$, we define the *positive root* $\alpha_{j,i}$ to be $\alpha_{j,i} := t_j - t_i$, and denote the set of all positive roots by Δ_A^+ [10], namely

$$\Delta_A^+ := \{\alpha_{j,i} = t_j - t_i \mid j > i \geq 1\} \subset L.$$

The set of *negative roots* is defined by $\Delta_A^- := -\Delta_A^+ = \{-\alpha \mid \alpha \in \Delta_A^+\}$. We also set $\Delta_A := \Delta_A^+ \coprod \Delta_A^-$ and call it the *root system of type A_∞*. The following elements of Δ_A^+ are called the *simple roots*:

$$\alpha_i := \alpha_{i+1,i} = t_{i+1} - t_i \quad (i \geq 1).$$

The Weyl group S_∞ acts on the lattice L by the usual permutation on t_i's, and hence on the root system Δ. The action of an element $w \in S_\infty$ on a root $\alpha \in \Delta$ will be denoted by $w \cdot \alpha$ or $w(\alpha)$ in the sequel. For a positive root $\alpha \in \Delta^+$, we have $\alpha = w(\alpha_i)$ for some $i \in I$ and some $w \in S_\infty$. Then we set $s_\alpha = s_{w(\alpha_i)} = w s_i w^{-1}$ (thus for a simple root α_i itself, we have $s_{\alpha_i} = s_i$ $(i \in I)$).

[10] In what follows, we often drop the subscript A for brevity when there is no fear of confusion.

Grassmannian elements

Definition 6.4. *An element $w \in S_\infty$ is called a Grassmannian element of descent n if the condition*

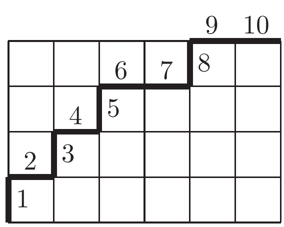
$$w(1) < w(2) < \cdots < w(n),\ w(n+1) < w(n+2) < \cdots$$

is satisfied for some fixed positive integer n.

The set of all Grassmannian elements of descent n will be denoted by $S_\infty^{(n)}$ in the sequel. Note that $S_\infty^{(n)}$ is equal to the set of elements $w \in S_\infty$ sucht that $\ell(ws_i) > \ell(w)$ for $i \neq n$, where $\ell(w)$ is the length of w. In other words, $S_\infty^{(n)}$ is the set of *minimal length coset representatives* of the quotient group $S_\infty/(S_n \times S_\infty)$, where $S_n \times S_\infty$ is a subgroup of S_∞ generated by the elements $\{s_i\}_{I\setminus\{n\}}$. It is well known that there is a bijection between the set $\mathcal{P}_n$ of partitions of length $\leq n$ and $S_\infty^{(n)}$. To be precise, the bijection is given as follows: For $w \in S_\infty^{(n)}$, we define a partition $\lambda_w = ((\lambda_w)_1, (\lambda_w)_2, \ldots, (\lambda_w)_n) \in \mathcal{P}_n$ by

$$(\lambda_w)_i := w(n+1-i) - (n+1-i)\ (1 \leq i \leq n).$$

Conversely, given a partition $\lambda \in \mathcal{P}_n$, we can construct a Grassmannian permutation w_λ of descent n by the following manner. First note that when considered as a Young diagram, λ is contained in the rectangle $n \times (N-n)$ for sufficiently large N, and we identify λ with a path starting from the south-west corner to the north-east corner of the rectangle. We assign numbers $1, 2, \ldots, N$ to each step. For example, for $\lambda = (4, 2, 1, 0)$ with $n = 4$, $N = 10$, we have the following picture:



If the assigned numbers of the vertical steps are $i_1 < i_2 < \cdots < i_n$, and those of the horizontal steps are $j_1 < j_2 < \cdots < j_{N-n}$, then the corresponding Grassmannian permutation is

$$w_\lambda = i_1\, i_2\, \ldots\, i_n\, j_1\, j_2\, \ldots\, j_{N-n}.$$

More explicity, we have

(50) $i_k = \lambda_{n-k+1} + k\ (1 \leq k \leq n), \quad j_k = n + k - {}^t\lambda_k\ (1 \leq k \leq N-n),$

where ${}^t\lambda$ is the conjugate of λ. In the above example, we have $w_\lambda = 1\,3\,5\,8\;\,2\,4\,6\,7\,9\,10$. Note that the set $\mathcal{P}_n$ (resp. $S_\infty^{(n)}$) is a partially ordered set given by the containment $\lambda \subset \mu$ of partitions (resp. the Bruhat-Chevalley ordering), and the above bijection preserves these partial orderings.

Furthermore a reduced expression of w_λ can be obtained by the following manner. For each box (cell) $\alpha = (i, j) \in \lambda$, the *content* of α is defined to be $c(\alpha) := j - i$. We fill in each box $\alpha = (i, j) \in \lambda$ with the number $n + c(\alpha) = n - i + j$. For the above example ($\lambda = (4, 2, 1)$, $n = 4$), the numbering is given by the following picture:

$$\begin{array}{|c|c|c|c|} \hline 4 & 5 & 6 & 7 \\ \hline 3 & 4 & \multicolumn{2}{c}{} \\ \cline{1-2} 2 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

We read the entries of the boxes of the Young diagram of λ from right to left starting from the bottom row to the top row and obtain the sequence $i_1\ i_2\ \cdots\ i_{|\lambda|}$. We then let $s_{i_1} s_{i_2} \cdots s_{i_{|\lambda|}}$ be the product of the corresponding simple reflections, which gives a reduced expression of w_λ. For the above example, we have

$$w_\lambda = 1\,3\,5\,8\,2\,4\,6\,7\,9\,10 = s_2 \cdot s_4 s_3 \cdot s_7 s_6 s_5 s_4.$$

<u>Action of S_∞ on $\mathcal{P}_n$</u> By means of the above bijection $\mathcal{P}_n \xrightarrow{\sim} S_\infty^{(n)}$, $\lambda \longmapsto w_\lambda$, the group S_∞ acts naturally on the set $\mathcal{P}_n$. We only describe the action of the simple reflections s_i $(i \in I)$ on partitions $\lambda \in \mathcal{P}_n$. Given a partition $\lambda \in \mathcal{P}_n$, a box $\alpha = (i, j) \in \lambda$ (resp. $\alpha = (i, j) \notin \lambda$ and $i \leq n$) is *removable* (resp. *addable*) if $\lambda \setminus \{\alpha\}$ (resp. $\lambda \cup \{\alpha\}$) is again a Young diagram of a partition in $\mathcal{P}_n$, i.e., $(i, j+1) \notin \lambda$ and $(i+1, j) \notin \lambda$ (resp. $(i, j-1) \in \lambda$ and $(i-1, j) \in \lambda$). Furthermore, we fill in each box $\alpha = (i, j) \in \lambda$ with the number $n + c(\alpha) = n + j - i$. Then λ is called *k-removable* (resp. *k-addable*) if there is a removable box $\alpha \in \lambda$ (resp. an addable box $\alpha \notin \lambda$) such that $c(\alpha) = k - n$. Under the above convention, the action of the simple reflections s_i $(i \in I)$ on partitions $\lambda \in \mathcal{P}_n$ is given by the following manner:

(1) $s_i\lambda < \lambda$ if and only if λ is i-removable.
(2) $s_i\lambda > \lambda$ if and only if λ is i-addable.

Otherwise s_i acts trivially on λ. For a general $w \in S_\infty$, write $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ as a product of simple reflections, and apply the above process successively to a partition λ to obtain the action $w\cdot\lambda$.

Let $\lambda \in \mathcal{P}_n$ be a partition, and $w_\lambda \in S_\infty^{(n)}$ the corresponding Grassmannian element. Define its *inversion set* by

$$\mathrm{Inv}\,(\lambda) := \{\alpha \in \Delta_A^+ \mid s_\alpha\lambda < \lambda\} = \{\alpha \in \Delta_A^+ \mid s_\alpha w_\lambda < w_\lambda\}.$$

In view of the above action of S_∞ on λ (or more directly by considering the condition $s_\alpha w_\lambda < w_\lambda$), we can describe the inversion set $\mathrm{Inv}\,(\lambda)$ explicitly for a given $\lambda \in \mathcal{P}_n$.

$$(51)\qquad \begin{aligned} \mathrm{Inv}\,(\lambda) &= \{t_{\lambda_i+n-i+1} - t_{n+j-{}^t\lambda_j} \mid (i,j)\in\lambda\} \\ &= \{t_{w_\lambda(n+1-i)} - t_{w_\lambda(n+j)} \mid (i,j)\in\lambda\}. \end{aligned}$$

Euler classes Lastly we define a map $e : L \longrightarrow \mathbb{L}[[\mathbf{b}]]$ by setting $e(t_i) := b_i$ $(i \geq 1)$ and by the rule $e(\alpha+\alpha') := e(\alpha) +_F e(\alpha')$ for $\alpha, \alpha' \in L$. Note that by definition, we have $e(-\alpha) = \overline{e(\alpha)}$ for $\alpha \in L$. For the simple root $\alpha_i = t_{i+1} - t_i$ $(i \geq 1)$, we have

$$e(\alpha_i) = e(t_{i+1} - t_i) = b_{i+1} +_F \overline{b}_i \quad (i \geq 1).$$

In particular, for a partition $\lambda \in \mathcal{P}_n$, it follows from (51) that

$$(52)\qquad \begin{aligned} \prod_{\alpha\in\mathrm{Inv}\,(\lambda)} e(-\alpha) &= \prod_{(i,j)\in\lambda} (\overline{b}_{\lambda_i+n-i+1} +_F b_{n+j-{}^t\lambda_j}) \\ &= \prod_{(i,j)\in\lambda} (\overline{b}_{w_\lambda(n+1-i)} +_F b_{w_\lambda(n+j)}). \end{aligned}$$

6.2.2. *Type B_∞, C_∞, D_∞*

Weyl groups of type X_∞ Let $X = B, C$, or D, and $W = W(X_\infty)$ be the Weyl group of type X_∞. This is a Coxeter group generated by the simple reflections $\{s_i\}_{i\in J}$, where the index set $J = I \sqcup \{0\} = \{0,1,2,\ldots\}$ for $X = B, C$, and $J = I \sqcup \{\hat{1}\} = \{\hat{1},1,2,\ldots\}$ for $X = D$. If W is of type C_∞ (or B_∞), the defining relations of $\{s_i\}_{i\in J}$ are give by

$$\begin{aligned} s_i^2 &= 1 && (i = 0,1,2,\ldots), \\ s_is_{i+1}s_i &= s_{i+1}s_is_{i+1} && (i\geq 1), \\ s_is_j &= s_js_i && (i,j\geq 1,\ |i-j|\geq 2), \\ s_0s_1s_0s_1 &= s_1s_0s_1s_0, && s_0s_i = s_is_0 \quad (i\geq 1). \end{aligned}$$

If W is of type D_∞, the defining relations are given by

$$\begin{array}{rcll} s_i^2 &=& 1 & (i = \hat{1}, 1, 2, \ldots), \\ s_i s_{i+1} s_i &=& s_{i+1} s_i s_{i+1} & (i \geq 1), \\ s_i s_j &=& s_j s_i & (i, j \geq 1,\ |i-j| \geq 2), \\ s_{\hat{1}} s_2 s_{\hat{1}} &=& s_2 s_{\hat{1}} s_2, & s_{\hat{1}} s_j = s_j s_{\hat{1}} \quad (j \neq 2). \end{array}$$

It is also well known that the Weyl group $W(X_\infty)$ of type X_∞ can be realized as a (sub) group of *signed* or *barred* permutations. As before, let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of positive integers. Denote by $\overline{\mathbb{N}} = \{\overline{1}, \overline{2}, \ldots\}$ a "negative" copy of $\mathbb{N}$ (thus $\overline{\overline{i}} = i$ for $i \in \mathbb{N}$). A *signed permutation* w of $\mathbb{N}$ is defined as a bijection on the set $\mathbb{N} \cup \overline{\mathbb{N}}$ such that $\overline{w(i)} = w(\overline{i})$ for all $i \in \mathbb{N}$ and $w(i) = i$ for all but a finite number of i. Denote by $\overline{S}_\infty$ the group of all signed permutations. We shall use the one-line notation $w = w(1)\, w(2) \cdots$ to denote an element $w \in \overline{S}_\infty$ (we have only to specify $w(i)$ for $i \in \mathbb{N}$ because of the condition $w(\overline{i}) = \overline{w(i)}$). Then the above-mentioned realization is given by the following identifications:

$$s_0 = (1\ \overline{1}), \quad s_i = (i\ i+1)(\overline{i+1}\ \overline{i}) \quad \text{for } i \geq 1.$$

Then $W(B_\infty)$ and $W(C_\infty)$ can be identified with $\overline{S}_\infty$. Further the simple reflection $s_{\hat{1}}$ is identified with $s_0 s_1 s_0$, in other words, $s_{\hat{1}} = (\overline{2}\ 1)(\overline{1}\ 2)$. Then $W(D_\infty)$ can be identified with the subgroup $\overline{S}_{\infty,+}$ of $\overline{S}_\infty$ generated by $s_{\hat{1}}$ and s_i $(i \geq 1)$.

Root systems of type X_∞ Let L denote a free $\mathbb{Z}$-module with a basis $\{t_i\}_{i \geq 1}$. The set of positive roots is defined respectively by

$$\begin{array}{lrcl} \text{Type } B_\infty: & \Delta_B^+ &=& \{t_i \mid i \geq 1\} \cup \{t_j \pm t_i \mid j > i \geq 1\}, \\ \text{Type } C_\infty: & \Delta_C^+ &=& \{2t_i \mid i \geq 1\} \cup \{t_j \pm t_i \mid j > i \geq 1\}, \\ \text{Type } D_\infty: & \Delta_D^+ &=& \{t_j \pm t_i \mid j > i \geq 1\}. \end{array}$$

The set of negative roots is defined by $\Delta_X^- := -\Delta_X^+ = \{-\alpha \mid \alpha \in \Delta_X^+\}$, and set $\Delta_X := \Delta_X^+ \coprod \Delta_X^-$. The following elements of Δ_X^+ are the simple roots:

$$\begin{array}{ll} \text{Type } B_\infty: & \alpha_0 = t_1, \quad \alpha_i = t_{i+1} - t_i \ (i \geq 1), \\ \text{Type } C_\infty: & \alpha_0 = 2t_1, \quad \alpha_i = t_{i+1} - t_i \ (i \geq 1), \\ \text{Type } D_\infty: & \alpha_{\hat{1}} = t_1 + t_2, \quad \alpha_i = t_{i+1} - t_i \ (i \geq 1). \end{array} \tag{53}$$

Grassmannian elements We introduce the Grassmannian elements in the case $X = B, C$, or D. For simplicity, we only deal with the case $X = B, C$. For $X = D$, we need an appropriate modification (see e.g., Ikeda-Mihalcea-Naruse [24, §3.4], Ikeda-Naruse [25, §4.3]).

Definition 6.5. *An element $w \in \overline{S}_\infty$ is called a Grassmannian element if the condition*

$$w(1) < w(2) < \cdots < w(i) < \cdots$$

is satisfied, where the ordering is given by

$$\cdots < \overline{m} < \cdots < \overline{2} < \overline{1} < 1 < 2 < \cdots < m < \cdots .$$

Let n be a fixed positive integer. We are concerned with a Grassmannian element $w \in \overline{S}_\infty$ satisfying the condition

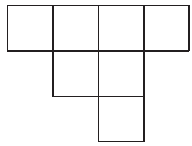
$$w(1) < w(2) < \cdots < w(r) < 1, \quad \overline{1} < w(r+1) < w(r+2) < \cdots$$

for some $1 \leq r \leq n$. In other words, it is a Grassmannian element in $\overline{S}_\infty$ whose first r values $w(1), w(2), \dots, w(r)$ $(r \leq n)$ are negative. The set of all such Grassmannian elements is denoted by $\overline{S}_\infty^{(n)}$ in the sequel. Then there is a bijection between the set $\mathcal{SP}_n$ of strict partitions of length $\leq n$ and $\overline{S}_\infty^{(n)}$. The bijection is given explicitly as follows: For $w \in \overline{S}_\infty^{(n)}$, let $1 \leq r \leq n$ be the number such that $w(1) < \cdots < w(r) < 1$ and $\overline{1} < w(r+1) < w(r+2) < \cdots$. Then we define an r-tuple of positive integers $\lambda_w = ((\lambda_w)_1, (\lambda_w)_2, \dots, (\lambda_w)_r)$ by $(\lambda_w)_i := \overline{w(i)}$ for $1 \leq i \leq r$. By the above ordering, we see immediately that λ_w is a strict partition of length $\leq n$, i.e., $\lambda_w \in \mathcal{SP}_n$. Conversely, for a strict partition $\lambda = (\lambda_1, \dots, \lambda_r)$ of length $r \leq n$, we define the element $w_\lambda \in \overline{S}_\infty^{(n)}$ by $w_\lambda(i) = \overline{\lambda}_i$ $(1 \leq i \leq r)$, and the remaining values $w_\lambda(j)$ $(j \geq r+1)$ are given by the increasing sequence of positive integers from $\mathbb{N} \setminus \{\lambda_r, \lambda_{r-1}, \dots, \lambda_1\}$. For example, the Grassmannian element $w = \overline{6}\,\overline{4}\,\overline{3}\,\overline{1}\,2\,5\,7 \cdots$ corresponds to the strict partition $\lambda = (6, 4, 3, 1)$. Note that the set $\mathcal{SP}_n$ (resp. $\overline{S}_\infty^{(n)}$) is a partially ordered set given by the containment $\lambda \subset \mu$ of strict partitions (resp. Bruhat-Chevalley ordering), and the above bijection preserves these partial orderings.

Furthermore a reduced expression of w_λ can be obtained by the following manner. First we associate to each strict partition $\lambda = (\lambda_1, \dots, \lambda_r)$, $\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0$ the *shifted Young diagram*

$$D'(\lambda) := \{(i, j) \in \mathbb{Z}^2 \mid 1 \leq i \leq r,\ i \leq j \leq \lambda_i + i - 1\}.$$

For example,



is the shifted Young diagram of a strict partition $\lambda = (4, 2, 1)$. We shall identify a strict partition λ with its shifted Young diagram if there is no fear of confusion. For each box $\alpha = (i, j) \in D'(\lambda)$, we define its content $c(\alpha) \in I \sqcup \{0\}$ by $c(\alpha) := j - i$ (for $X = B, C$). We fill in the number $c(\alpha)$ to each box $\alpha \in D'(\lambda)$. For example, for $\lambda = (4, 2, 1)$, the numbering is given by the following picture:

0	1	2	3
	0	1	
		0	

Type B, C

We read the entries of the boxes of the shifted Young diagram of λ from right to left starting from the bottom row to the top row and obtain the sequence $i_1\, i_2, \ldots, i_{|\lambda|}$. We let $s_{i_1} s_{i_2} \cdots s_{i_{|\lambda|}}$ be the product of the corresponding simple reflections, which gives a reduced expression of w_λ. For the above example, we have

$$w_\lambda = \overline{4}\,\overline{2}\,\overline{1}3 \cdots = s_0 \cdot s_1 s_0 \cdot s_3 s_2 s_1 s_0.$$

Action of the Weyl group $W(X_\infty)$ on $\mathcal{SP}_n$ By means of the above bijection $\mathcal{SP}_n \xrightarrow{\sim} \overline{S}_\infty^{(n)}$, $\lambda \longmapsto w_\lambda$, the group $\overline{S}_\infty$ acts on the set $\mathcal{SP}_n$. We shall describe the action of the simple reflections s_i ($i \in I \sqcup \{0\}$) on strict partitions $\lambda \in \mathcal{SP}_n$. Given a strict partition $\lambda \in \mathcal{SP}_n$, a box $\alpha = (i, j) \in \lambda$ (resp. $\alpha = (i, j) \notin \lambda$ and $i \le n$) is removable (resp. addable) if $\lambda \setminus \{\alpha\}$ (resp. $\lambda \cup \{\alpha\}$) is again a shifted Young diagram of a strict partition in $\mathcal{SP}_n$, i.e., $j = \lambda_i + i - 1$ and $\lambda_{i+1} \le \lambda_i - 2$ (resp. $j = \lambda_i + i$ and $\lambda_i \le \lambda_{i-1} - 2$). Furthemore, we fill in each box $\alpha = (i, j) \in \lambda$ with the number $c(\alpha) = j - i$. Then λ is called k-removable (resp. k-addable) if there is a removable box $\alpha \in \lambda$ (resp. an addable box $\alpha \notin \lambda$) such that $c(\alpha) = k$. Under the above covention, the action of the simple reflections s_i $(i \in I \sqcup \{0\})$ on strict partitions $\lambda \in \mathcal{SP}_n$ is given by the following manner:

(1) $s_i\lambda < \lambda$ if and only if λ is i-removable.
(2) $s_i\lambda > \lambda$ if and only if λ is i-addable.

Otherwise s_i acts trivially on λ. For a general $w \in \overline{S}_\infty$, write $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ as a product of simple reflections, and apply the above process successively to a strict partition λ to obtain the action $w \cdot \lambda$.

Let $\lambda \in \mathcal{SP}_n$ be a strict partition, and $w_\lambda \in \overline{S}_\infty^{(n)}$ the corresponding Grassmannian element. Define its inversion set by

$$\mathrm{Inv}_X(\lambda) = \{\alpha \in \Delta_X^+ \mid s_\alpha\lambda < \lambda\} = \{\alpha \in \Delta_X^+ \mid s_\alpha w_\lambda < w_\lambda\}.$$

In view of the above action of $\overline{S}_\infty$ on λ, we can describe the inversion set $\mathrm{Inv}_X(\lambda)$ explicitly for a given $\lambda \in \mathcal{SP}_n$. For instance, the set $\mathrm{Inv}_C(\lambda)$ for a strict partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathcal{SP}_n$, $\ell(\lambda) = r$, is described as follows: Let $w_\lambda \in \overline{S}_\infty^{(n)}$ be the corresponding Grassmannian element. Recall that $w_\lambda(i) = \overline{\lambda}_i$ $(1 \le i \le r)$ and $w_\lambda(j)$ $(j \ge r+1)$ are obtained by the increasing sequence of positive integers from $\mathbb{N} \setminus \{\lambda_r, \lambda_{r-1}, \ldots, \lambda_1\}$. From this, we have

(54)
$$\begin{aligned}\mathrm{Inv}_C(\lambda) &= \{\, t_{\overline{w_\lambda(i)}} + t_{\overline{w_\lambda(j)}} \mid (i,j) \in \lambda\} \\ &= \{t_{\lambda_i} + t_{\lambda_j}\ (1 \le i \le r,\ i \le j \le r)\} \\ &\quad \cup \{t_{\lambda_i} + t_{\overline{j}}\ (1 \le i \le r,\ 1 \le j \le \lambda_i - 1,\ j \ne \lambda_{i+1}, \ldots, \lambda_r)\}.\end{aligned}$$

Here λ is identified with its associated *shifted* diagram, and $t_{\overline{i}}$ for $i \in \mathbb{N}$ is understood to be $-t_i$.

Euler classes We define a map $e : L \longrightarrow \mathbb{L}[[\mathbf{b}]]$ by setting $e(t_i) := b_i$ $(i \ge 1)$ and by the rule $e(\alpha + \alpha') := e(\alpha) +_F e(\alpha')$ for $\alpha, \alpha' \in L$. For the simple roots α_i $(i \ge 0)$ and $\alpha_{\hat{1}}$ given in (53), we have

$$\begin{aligned}&\text{Type } B_\infty: \quad e(\alpha_0) = b_1, \quad e(\alpha_i) = b_{i+1} +_F \overline{b}_i\ (i \ge 1),\\ &\text{Type } C_\infty: \quad e(\alpha_0) = b_1 +_F b_1, \quad e(\alpha_i) = b_{i+1} +_F \overline{b}_i\ (i \ge 1),\\ &\text{Type } D_\infty: \quad e(\alpha_{\hat{1}}) = b_1 +_F b_2, \quad e(\alpha_i) = b_{i+1} +_F \overline{b}_i\ (i \ge 1).\end{aligned}$$

In particular, for a strict partition $\lambda \in \mathcal{SP}_n$, it follows from (54) that

(55)
$$\begin{aligned}\prod_{\alpha \in \mathrm{Inv}_C(\lambda)} e(-\alpha) &= \prod_{(i,j)\in\lambda} (b_{w_\lambda(i)} +_F b_{w_\lambda(j)}) \\ &= \prod_{i=1}^{r} \left(\prod_{j=i}^{r} (\overline{b}_{\lambda_i} +_F \overline{b}_{\lambda_j}) \cdot \prod_{\substack{1 \le j \le \lambda_i - 1 \\ j \neq \lambda_p \text{ for } i+1 \le p \le r}} (\overline{b}_{\lambda_i} +_F b_j) \right).\end{aligned}$$

Notice that this value is equal to the specialization $Q^{\mathbb{L}}_\lambda(\overline{\mathbf{b}}_\lambda|\mathbf{b})$ in Proposition 4.22. Similarly, an analogous result holds for $P^{\mathbb{L}}_\lambda(\mathbf{x}_n|\mathbf{b})^+$.

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