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Spherical multiple flags

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Abstract.

For a reductive group G, the products of projective varieties homogeneous under G that are spherical for the diagonal action of G have been classified by Stembridge. We consider the B-orbit closures in these spherical varieties and prove that under some mild restrictions they are normal, Cohen-Macaulay and have a rational resolution.

§1. Introduction

A classical problem in geometric representation theory is to prove regularity properties of B-orbit closures inside a G-variety X. Here and henceforth G is a connected reductive group over an algebraically closed field k and B is a Borel subgroup of G. The most famous example of such a theorem is the result of Mehta and Ramanathan [12] that Schubert varieties (that is, B-orbit closures inside $X = G/P$ a projective rational homogeneous space) are normal, Cohen-Macaulay and have a rational resolution. For general *spherical varieties* (*i.e.*, normal G-varieties with finitely many B-orbits), this is more complicated and the B-orbit closures are not normal in general (for a survey of partial results in this direction, cf. [14, Section 4.4]). In this paper, we restrict our attention to products of homogeneous spaces. Our result is the following

Theorem 1. *Assume that G is simply laced (i.e., with simple factors of types A, D, E only). Let P_1, P_2 be two* cominuscule *(see Definition 2.6) parabolic subgroups of G and let $X = G/P_1 \times G/P_2$. Then the B-orbit closures inside X, for the diagonal action are normal, Cohen-Macaulay and have a rational resolution.*

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To prove these regularity properties, we need to study in more detail the B-orbit structure of X and the *weak order* (cf. Definition 3.1) among the B-orbits. We prove the following two facts, whose proof constitutes most of the paper, and which we hope might be of independent interest:

(a) the minimal B-orbits with respect to the weak order are $B \times B$-stable (see Theorem 3.13), hence their closures are products of Schubert varieties,

(b) the action maps $P \times^B \mathcal{O} \to P\mathcal{O}$, where $\mathcal{O}$ is a B-orbit in X, $P \supseteq B$ a minimal parabolic subgroup with $P\mathcal{O} \neq \mathcal{O}$, are birational (see Corollary 4.20).

With these results in hand, the structure of the proof of Theorem 1 is as follows. For a B-orbit closure $\bar{\mathcal{O}} \subseteq X$, we find a minimal (with respect to the weak order) B-orbit $\mathcal{O}' \preccurlyeq \mathcal{O}$. Since by (a) the orbit closure $\bar{\mathcal{O}}'$ is a product of two Schubert varieties, it admits a rational resolution $Z \to \bar{\mathcal{O}}'$ (for example, the product of two Bott-Samelson resolutions [10] of the two factors – recall that a rational resolution of a variety X is a proper birational morphism $p : Y \to X$ with Y non singular, with $p_*\mathcal{O}_Y = \mathcal{O}_X$ and $R^i p_* \mathcal{O}_Y = R^i p_* \omega_Y = 0$ for $i > 0$). Since $\mathcal{O}' \preccurlyeq \mathcal{O}$ there exists a sequence $P_{\gamma_1}, \cdots, P_{\gamma_i}$ of minimal parabolic subgroups of G containing B raising $\mathcal{O}'$ to $\mathcal{O}$ (see Definition 3.1). Then using (b) we prove that the action map

$$P_{\gamma_r} \times^B \ldots \times^B P_{\gamma_1} \times Z \to \bar{\mathcal{O}}$$

is a rational resolution of $\bar{\mathcal{O}}$. The main part consists in proving normality of $\bar{\mathcal{O}}$. We proceed by descending induction in the weak order: maximal B-orbits closures are G-stable and normal as locally trivial fibrations over homogeneous varieties with Schubert varieties as fibers. In the induction step, we use (a) and (b) again and follow ideas of Brion [4]. Finally, Cohen-Macaulayness follows from general arguments from Brion [4, Section 3, Remark 2] and the fact that this holds for the G-orbit closures.

In addition, we show that the "simply-lacedness" assumption in Theorem 1 is necessary. In Section 5 we consider the spherical variety $X = (\mathrm{Sp}_6/P)^2$ where P is the stabilizer in Sp_6 of a 3-dimensional isotropic subspace of k^6. We find a B-orbit in X whose closure is not normal (moreover the property (b) above fails). We do not know whether Theorem 1 holds without the assumption that P_1 and P_2 be cominuscule. Note that examples of such pairs with X spherical are quite restricted – see [11] and [19] for a complete list. The main reason for the assumption that P_1, P_2 are cominuscule is that in such case the G-orbits are induced from symmetric varieties (see Definition 3.4 and Corollary 2.9) in which case minimal orbits for the weak order are closed.

The case of Theorem 1 when X is a product of two Graßmann varieties was proved in [2] thanks to a detour into quiver representations. It was one of the motivations of this work to present a direct proof of this result. It was also inspired by a complete combinatorial description of the weak order in a product of two Graßmann varieties due to Smirnov [16], where the two phenomena (a) and (b) mentioned above have been observed.

The structure of the paper is as follows. In Section 2, we recall the notion of opposite pairs of parabolic subgroups and show how one can reduce the study of G-orbits inside $G/P_1 \times G/P_2$ to the case when (P_1, P_2) is an opposite pair. In that case $G/P_1 \times G/P_2$ is a symmetric variety which turns out to be very important. In Section 3, we recall the definition and basic properties of the weak order among the B-orbits in a spherical variety X and prove (a) (Theorem 3.13). In Section 4, we introduce a distance function between torus-fixed points in X, generalizing a previous notion introduced in [7] and used in [6, 8, 9, 5] to study quantum cohomology of cominuscule rational homogeneous spaces. We use it to prove (b) (Corollary 4.20). The proof of Theorem 1 occupies Section 5, and our counterexample with G non-simply laced can be found in Section 6.

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§2. Structure of G-orbits

Let G be a connected reductive algebraic group, T a maximal torus of G and B a Borel subgroup of G containing T. Let $W = N_G(T)/T$ be the Weyl group associated to T. Let P_1 and P_2 be two parabolic subgroups of G containing B and define

$$X = G/P_1 \times G/P_2.$$

The variety X has finitely many G-orbits. Any orbit is of the form: $G \cdot (P_1, wP_2)$ for some $w \in W$ and is isomorphic to G/H with

$$H = P_1 \cap P_2^w$$

where $P_2^w = wP_2w^{-1}$. The inclusion morphism $\iota : G/H \to G/P_1 \times G/P_2$ is induced by the morphism $G \to G \times G$ defined by $g \mapsto (g, gn_w)$ where n_w is any representative of w in $N_G(T)$.

In this section we prove a structure result on G-orbits which reduces the study to the case of an opposite pair (P_1, P_2) (see Definition 2.1). For this we fix a G-orbit $G \cdot (P_1, wP_2) \simeq G/H$ of X with $w \in W$ and $H = P_1 \cap P_2^w$.

Recall that if $\chi : \mathbb{G}_m \to T$ is a cocharacter of T, we may define a parabolic subgroup P_χ of G as follows:

$$P_\chi = \{g \in G \ / \ \lim_{t\to 0} \chi(t) g \chi(t)^{-1} \text{ exists}\}.$$

In the above definition, the limit exists if the map $\mathbb{G}_m \to G$, $t \mapsto \chi(t)g\chi(t)^{-1}$ extends to $\mathbb{A}^1 \supseteq \mathbb{G}_m$. Note that P_χ contains T. Any parabolic subgroup containing T can be defined this way. The possible cocharacters for a given parabolic P are of the form $\chi_P + \chi$ with χ dominant for a unique cocharacter χ_P. We have $P = P_{\chi_P}$. For example, the cocharacter $\chi_{P_2^w}$ of P_2^w is $w(\chi_{P_2})$.

Definition 2.1. A pair (P_1, P_2) is called *opposite* if $w_0(\chi_{P_1}) = -\chi_{P_2}$, where w_0 is the longest element of the Weyl group.

Note that this is a variant of the classical definition of opposite parabolic subgroups: a pair (P_1, P_2) is an opposite pair if and only if P_2 is opposite to $P_1^{w_0}$.

Definition 2.2. Define the parabolic subgroup R of G by its cocharacter

$$\chi_R = \chi_{P_1} + w(\chi_{P_2}).$$

Denote by L_R the Levi subgroup of R containing T and by U_R the unipotent radical of R. We have a semidirect product $R = L_R \ltimes U_R$.

Lemma 2.3. *Let $w_0^{L_R}$ be the longest element of the Weyl group of L_R.*

(i) The parabolic subgroup R contains the intersection $P_1 \cap P_2^w$.

(ii) The pair (Q_1, Q_2) with $Q_1 = L_R \cap P_1$ and $Q_2 = (P_2^w \cap L_R)^{w_0^{L_R}}$ is an opposite pair in L_R.

Proof. (i) This is obvious by definition.

(ii) We have the equality $\chi_{P_1}|_{L_R} + w(\chi_{P_2})|_{L_R} = 0$ proving the result. □

Definition 2.4. We set $K = L_R \cap H$.

Lemma 2.5. *The subgroup K is the Levi subgroup of Q_1 and $Q_2^{w_0^{L_R}}$.*

Proof. Let α be a root of L_R. Then $\langle \chi_{P_1} + w(\chi_{P_2}), \alpha\rangle = 0$. Note that we have $\chi_{Q_1} = \chi_{P_1}|_{L_R}$ and $\chi_{Q_2} = w_0^{L_R}(w(\chi_{P_2})|_{L_R})$. This implies $\chi_{Q_1} = -w_0^{L_R}(\chi_{Q_2})$ and that α is a root of K if and only if $\langle \chi_{Q_1}, \alpha\rangle = 0 = \langle w_0^{L_R}(\chi_{Q_2}), \alpha\rangle$. □

Since $H \subset R$, we have a G-equivariant morphism $p : G/H \to G/R$, which is a locally trivial fibration with fiber isomorphic to R/H. In other words we have an isomorphism $G/H \simeq G \times^R R/H$. Consider the quotient map $\mathrm{pr} : R \to R/U_R = L_R$ where U_R is the unipotent radical of R. This defines an R-action on L_R.

Recall the following definition (this is the dual definition of a minuscule weight, see [1, Chapter 6. Exercice 24]).

Definition 2.6. A parabolic subgroup is *cominuscule* if its associated cocharacter χ_P satisfies $|\langle \chi_P, \alpha\rangle| \leq 1$ for any root α.

Lemma 2.7. *If P_1 and P_2 are cominuscule, then H contains U_R, and the image $\mathrm{pr}(H)$ of H under the projection $\mathrm{pr} : R \to L_R = R/U_R$ is K.*

Proof. We write U_α for the 1-dimensional T-stable unipotent subgroup of G whose Lie algebra is the eigenspace with weight α.

The group U_R is the product of the groups U_α for $\langle \chi_R, \alpha\rangle > 0$. Note that P_1 (resp. P_2^w) contains the groups U_α for $\langle \chi_{P_1}, \alpha\rangle \geq 0$ (resp. $\langle w(\chi_{P_2}), \alpha\rangle \geq 0$). Now let α with $\langle \chi_R, \alpha\rangle > 0$ and $\langle \chi_{P_1}, \alpha\rangle < 0$. Then $\langle w(\chi_{P_2}), \alpha\rangle = \langle \chi_R, \alpha\rangle - \langle \chi_{P_1}, \alpha\rangle \geq 2$ contradicting the fact that P_2 is cominuscule. This implies $U_R \subset P_1$. The same argument gives $U_R \subset P_2^w$ proving the inclusion $U_R \subset H$.

The map pr is defined as follows. For any $r \in R$, there is a unique decomposition $r = lu$ with $l \in L_R$ and $u \in U_R$. We have $\mathrm{pr}(r) = l$. Since $H \subset R = L_R U_R$ and $U_R \subset H$, it follows that $H = (L_R \cap H)U_R = KU_R$ and hence $\mathrm{pr}(H) = K$. □

For P_1 and P_2 cominuscule, we have a R-equivariant morphism $R/H \to L_R/K$ (the action of R on L_R/K is given by the group morphism $\mathrm{pr} : R \to L_R$). We get a morphism

$$G/H \simeq G \times^R R/H \to G \times^R L_R/K.$$

Note that since $K = Q_1 \cap Q_2^{w_0^{L_R}}$, the diagonal embedding $L_R \to L_R \times L_R$ induces an embedding $L_R/K \to L_R/Q_1 \times L_R/Q_2^{w_0^{L_R}}$.

Lemma 2.8. *With the above notation.*

(i) The variety L_R/K is the dense L_R-orbit in $L_R/Q_1 \times L_R/Q_2^{w_0^{L_R}}$.
(ii) The fiber of $R/H \to L_R/\mathrm{pr}(H)$ is isomorphic to $U_R/U_R \cap H$.

(iii) If P_1 and P_2 are cominuscule, then $R/H \to L_R/\mathrm{pr}(H)$ is an isomorphism.

Proof. (i) Follows from the fact that (Q_1, Q_2) is an opposite pair.

(ii) The statement on the fiber is clear by construction.

(iii) This follows from Lemma 2.7: we have an inclusion $U_R \subset H$ proving the triviality of the fiber $U_R/U_R \cap H$ of the map. □

Corollary 2.9. *For P_1 and P_2 cominuscule, G/H is isomorphic to $G \times^R L_R/K$.*

Remark 2.10. If P_1 and P_2 are cominuscule, the G-orbit G/H is obtained by parabolic induction from L_R/K (see Definition 3.4) that is to say from a quotient $L_R/Q_1 \cap Q_2^{w_0^{L_R}}$ with (Q_1, Q_2) an opposite pair in L_R.

§3. Minimal orbits for the weak order

Recall that a G-spherical variety, or simply a spherical variety X is a normal G-variety with a dense B-orbit. This in particular implies that the set $B(X)$ of B-orbits is finite.

In this section we first recall general results on B-orbits in a spherical variety X. We then apply these results to the case where $X = G/P_1 \times G/P_2$ with P_1 and P_2 cominuscule parabolic subgroups.

3.1. Weak order

Let X be a spherical variety and let $\mathcal{O}$ be a B-orbit in X. There is a natural partial order, called the weak order on the set $B(X)$ of B-orbits in X defined as follows. Recall that a minimal parabolic subgroup is a parabolic subgroup with semisimple rank one. The following definition was introduced in [15].

Definition 3.1. Let $\mathcal{O}$ be a B-orbit in X.

(i) If P is a minimal parabolic subgroup containing B such that $\mathcal{O}$ is not P-stable, we say that P *raises* $\mathcal{O}$.

(ii) The *weak order* is the order generated by the following cover relations $\mathcal{O} < \mathcal{O}'$ where $\mathcal{O}$ is any B-orbit in X and where $\mathcal{O}'$ is the dense B-orbit in $P\mathcal{O}$ for P a minimal parabolic raising $\mathcal{O}$.

By results of [15] or [3] three cases can occur. Recall that there exists a morphism $P \times^B \mathcal{O} \to P\mathcal{O}$ induced by the action. Recall also that the rank $\mathrm{rk}(Z)$ of a B-variety Z is the minimal codimension of U-orbits with U the unipotent radical of B.

Lemma 3.2. *Let $\mathcal{O}$ be a B-orbit in X and let P be a minimal parabolic subgroup raising $\mathcal{O}$. Let $\mathcal{O}'$ be the dense B-orbit in $P\mathcal{O}$. Then* $\dim \mathcal{O}' = \dim \mathcal{O} + 1$ *and one of the following three cases occurs:*

(U) *The P-orbit $P\mathcal{O}$ contains two B-orbits $\mathcal{O}$ and $\mathcal{O}'$ and $P \times^B \mathcal{O} \to P\mathcal{O}$ is birational. We have* $\mathrm{rk}(\mathcal{O}') = \mathrm{rk}(\mathcal{O})$.

(N) *The P-orbit $P\mathcal{O}$ contains two B-orbits $\mathcal{O}$ and $\mathcal{O}'$ and $P \times^B \mathcal{O} \to P\mathcal{O}$ is of degree 2. We have* $\mathrm{rk}(\mathcal{O}') = \mathrm{rk}(\mathcal{O}) + 1$.

(T) *The P-orbit $P\mathcal{O}$ contains three B-orbits $\mathcal{O}$, $\mathcal{O}'$ and $\mathcal{O}''$ and $P \times^B \mathcal{O} \to P\mathcal{O}$ is birational. We have* $\dim \mathcal{O} = \dim \mathcal{O}''$ *and* $\mathrm{rk}(\mathcal{O}') = \mathrm{rk}(\mathcal{O}) + 1 = \mathrm{rk}(\mathcal{O}'') + 1$.

The following graph was introduced in [15] and [3].

Definition 3.3. We define a graph $\Gamma(X)$ whose vertices are the elements in $B(X)$ and whose edges are the pairs $(\mathcal{O}, \mathcal{O}')$ with $\mathcal{O}$ raised to $\mathcal{O}'$ by a minimal parabolic subgroup P. We say that an edge is of type U, N or T if we are in the corresponding U, N or T situation of the previous lemma.

Let R be a parabolic subgroup of G and let L_R be its Levi quotient. Let Y be a L_R-variety. We write B_{L_R} for the image of $B \cap R$ in L_R. Note that this is a Borel subgroup of L_R. The following definition was introduced in [3].

Definition 3.4. We say that a G-variety X is obtained from Y by *parabolic induction* if X is of the form $X = G \times^R Y$ where Y is a L_R-variety (the R-action on Y is defined by $r \cdot y = \bar{r} \cdot y$ for $y \in Y$ and $\bar{r} \in L_R = R/U_R$ the class of $r \in R$).

The following result is a direct application of [3, Lemma 6].

Lemma 3.5. *Let $X = G \times^R Y$ be obtained by parabolic induction from Y.*

(ı) The variety X is G-spherical if and only if Y is L_R-spherical.

Assume that X is spherical.

(ıı) The set $B(X)$ is in bijection with the product $B_{L_R}(Y) \times B(G/R)$. The bijection $B_{L_R}(Y) \times B(G/R) \to B(X)$ is given by $(\mathcal{O}, BgR/R) \mapsto BgR \times^R \mathcal{O}$. Furthermore, the edges are of two types:

- *of type $((\mathcal{O}, BgR/R), (\mathcal{O}, Bg'R/R))$ with $(BgR/R, Bg'R/R)$ an edge of $B(G/R)$. These edges are of type* U;
- *or of the form $((\mathcal{O}, BgR/R), (\mathcal{O}', BgR/R))$ with $(\mathcal{O}, \mathcal{O}')$ an edge of $B_{L_R}(Y)$. The edges $((\mathcal{O}, BgR/R), (\mathcal{O}', BgR/R))$ and $(\mathcal{O}, \mathcal{O}')$ have the same type.*

Let P_1 and P_2 be cominuscule parabolic subgroups and let $X = G/P_1 \times G/P_2$. The following result was proved in [11] (see also [19] for a complete classification of products of projective homogeneous G-varieties which are G-spherical).

Proposition 3.6. *The variety X is G-spherical.*

Consider a G-orbit $G \cdot (P_1, wP_2) \simeq G/H$ of X with $w \in W$ and $H = P_1 \cap P_2^w$ and recall the notation from Section 2. Corollary 2.9 gives the isomorphism

$$G/H \simeq G \times^R L_R/K.$$

In particular, by Lemma 3.5, to describe the weak order on G/H we only need to study the weak order on L_R/K. Thanks to Lemma 2.3, it is therefore enough to consider the case where (P_1, P_2) is an opposite pair and w is the longest element.

3.2. Minimal orbits: The case of opposite pairs

In this subsection, we consider the spherical variety $X = G/P_1 \times G/P_2$ with P_1 and P_2 two cominuscule parabolic subgroups of G such that (P_1, P_2) is an opposite pair. We pick the dense G-orbit in X *i.e.* the orbit $G \cdot (P_1, wP_2) \simeq G/H$ with $H = P_1 \cap P_2^w$ and $w = w_0$ the longest element of W.

We start with the following result (see [13, Proposition 4.5] for a proof).

Proposition 3.7. *The group H is a symmetric subgroup of G.*

We continue with results on minimal length representatives: for P a parabolic subgroup of G containing B, we write W_P for its Weyl group and W^P for the subset of W of minimal length representatives of the quotient W/W_P.

Lemma 3.8. *Let w_{P_1} and w_{P_2} be the longest elements in W^{P_1} and W^{P_2}, then $w_{P_2} = w_{P_1}^{-1}$.*

Proof. The lengths of w_{P_1} and w_{P_2} are equal to the dimensions of G/P_1 and G/P_2. Since (P_1, P_2) is an opposite pair, these dimensions are equal and $w_{P_1}(\chi_{P_1}) = -\chi_{P_2}$. Thus $l(w_{P_1}^{-1}) = l(w_{P_2})$ and we compute $w_{P_1}^{-1}(\chi_{P_2}) = -\chi_{P_1} = w_{P_2}(\chi_{P_2})$. Therefore $w_{P_1}^{-1}$ is in the same class as w_{P_2} in W/W_{P_2} proving the result. □

Lemma 3.9. *Let $u \in W^{P_1}$, there exists a unique $u^\vee \in W^{P_2}$ such that $(uP_1, u^\vee P_2)$ is in the dense G-orbit in $G/P_1 \times G/P_2^w$. We have the formulas*

$$u^{-1}u^\vee = w_{P_2} \text{ and } l(u) + l(u^\vee) = l(w_{P_2}).$$

where w_{P_2} is the longest element in W^{P_2}.

Proof. Recall that (P_1, w_0P_2) is the the dense G-orbit. In particular (uP_1, uw_0P_2) is in the dense G-orbit. Let $v \in W$ such that (uP_1, vP_2) is also in the dense G-orbit *i.e.* we have $u(\chi_{P_1}) = -v(\chi_{P_2})$. Because (P_1, P_2) is an opposite pair we have $w_{P_1}(\chi_{P_1}) = -\chi_{P_2}$ thus we get $w_{P_1}^{-1}(\chi_{P_2}) = u^{-1}v(\chi_{P_2})$ and the equality $w_{P_1}^{-1} = u^{-1}v$ in W/W_{P_2}. Let $v' \in W_{P_2}$ such that the equality $w_{P_1}^{-1} = u^{-1}vv'$ holds in W. By the previous lemma we get $w_{P_2} = u^{-1}vv'$. Write $w_{P_1} = u'u$ with $l(w_{P_1}) = l(u) + l(u')$ (this is possible since $u \in W^{P_1}$). Note that the have $u' = v'^{-1}v^{-1}$ and therefore $l(w_{P_2}) = l(u^{-1}) + l(u')$ and the expression $w_{P_2} = u^{-1}u'^{-1}$ is length additive. Since $w_{P_2} \in W^{P_2}$ this implies $u'^{-1} \in W^{P_2}$. The element $u^\vee = u'^{-1}$ satisfies the conclusions of the lemma. □

Lemma 3.10. *The B-orbit $B \cdot (uP_1, u^\vee P_2)$ is a $B \times B$-orbit.*

Proof. Recall that we have the following equalities

$$B{\cdot}uP_1 = \prod_{\alpha>0,\ U_{u^{-1}(\alpha)}\not\subset P_1} U_\alpha{\cdot}uP_1 \text{ and } B{\cdot}u^\vee P_2 = \prod_{\alpha>0,\ U_{u^{\vee -1}(\alpha)}\not\subset P_2} U_\alpha{\cdot}u^\vee P_2.$$

We are thus left to prove that there is no positive root α with $U_{u^{-1}(\alpha)} \not\subset P_1$ and $U_{u^{\vee -1}(\alpha)} \not\subset P_2$. Let α be such a root. We have the inequalities $\langle \chi_{P_1}, u^{-1}(\alpha)\rangle < 0$ and $\langle \chi_{P_2}, u^{\vee -1}(\alpha)\rangle < 0$. By Lemma 3.9, the second inequality is equivalent to $\langle w_{P_2}(\chi_{P_2}), u^{-1}(\alpha)\rangle < 0$. But since $w_{P_2'}(\chi_{P_2}) = -\chi_{P_1}$ this leads to a contradiction with the first inequality. □

Lemma 3.11. *The minimal orbits for the weak order in G/H are closed.*

Proof. This follows from the fact that the statement holds true for symmetric homogeneous spaces (see [18]) and the fact that H is a symmetric subgroup (Proposition 3.7). □

Proposition 3.12. *The minimal B-orbits in G/H are $B \times B$-orbits.*

Proof. Let $\mathcal{O}$ be a minimal B-orbit for the weak order. By Lemma 3.11, the orbit $\mathcal{O}$ is closed in G/H. Let $p : G \to G/H$ and $q : G \to G/B$ be the two projections. Then $qp^{-1}(\mathcal{O})$ is a closed H-orbit in G/B. Since H contains the maximal torus T, the closed orbit $qp^{-1}(\mathcal{O})$ contains a T-fixed point. This implies that $\mathcal{O}$ contains a T-fixed point. This T-fixed point is of the form $z = (uP_1, vP_2)$ for $u, v \in W$. By Lemma 3.9, we get $v = u^\vee$ and by Lemma 3.10 we get that $\mathcal{O}$ is a $B \times B$-orbit. □

3.3. Minimal orbits: General case

In this subsection, we consider the spherical variety $X = G/P_1 \times G/P_2$ with P_1 and P_2 two cominuscule parabolic subgroups. We pick a G-orbit $G \cdot (P_1, wP_2) \simeq G/H$ of X with $H = P_1 \cap P_2^w$ and $w \in W$.

Theorem 3.13. *The minimal B-orbits in G/H are $B \times B$-orbits.*

Proof. According to Lemma 3.5, a minimal B-orbit is of the form $BgR \times^R \mathcal{O}$ where BgR/R is a minimal B-orbit in $B(G/R)$ and $\mathcal{O}$ is a minimal B_{L_R}-orbit in L_R/K. Therefore BgR/R is a point and $\mathcal{O}$ is a $B_{L_R} \times B_{L_R}$-orbit. The result follows. □

§4. Distance and rank

In this section we consider $X = G/P_1 \times G/P_2$ with G simply laced and P_1, P_2 cominuscule. We prove that there is no edge of type N in the graph $B(X)$.

By definition of the weak order, we only need to consider $B(G/H)$ for G/H a G-orbit with $H = P_1 \cap P_2^w$ in X. Note that thanks to Lemma 3.5 and Corollary 2.9, we only need to prove this result for opposite pairs. We shall specify when we assume that the pair (P_1, P_2) is an opposite pair.

4.1. Distance

In this subsection we introduce a *distance* $d(x, y)$ between T-fixed points $xP_1 \in G/P_1$ and $yP_2 \in G/P_2$ and prove that it is closely related to the rank of the B-orbit of (xP_1, yP_2). Let ϖ_{P_i} be the fundamental weight corresponding to the cocharacter χ_{P_i}. Denote by $V_{\varpi_{P_i}}$ the irreducible representation of highest weight ϖ_{P_i} and by $\Pi_{\varpi_{P_i}}$ the set of weights of $V_{\varpi_{P_i}}$ for $i \in \{1, 2\}$ (note that this representation might not be a representation of the group G itself but only of a finite cover of G). Recall that $W \cdot \varpi_{P_i}$ the W-orbit of ϖ_{P_i} is equal to $\Pi_{\varpi_{P_i}}$ in our situation since G is simply laced and both weights are cominuscule therefore minuscule. Recall also that the map $W^{P_i} \to \Pi_{\varpi_i}, u \mapsto u(\varpi_{P_i})$ is bijective and that the Schubert cells in G/P_i are of the form $\Omega_u = BuP_i/P_i$ for a unique $u \in W^{P_i}$. Fix $(\ ,\ )$ a W-invariant scalar product on weights and write $|\cdot|$ for the associated norm. Note that, since G is assumed to be simply laced, all the roots have the same length. We choose $(\ ,\ )$ so that roots have length 2.

The following definition generalises a definition introduced in [7].

Definition 4.1. For $\lambda_i \in \Pi_{\varpi_{P_i}}$ define $d(\lambda_1, \lambda_2) = (\varpi_{P_1}, \varpi_{P_2}) - (\lambda_1, \lambda_2)$.

Remark 4.2. (i) The distance $d(\lambda_1,\lambda_2)$ is W-invariant.
(ii) If $\varpi_{P_1}=\varpi_{P_2}$, then we have $d(\lambda_1,\lambda_2)=\frac{1}{2}|\lambda_1-\lambda_2|^2$.

Lemma 4.3. *We have* $d(\lambda_1,\lambda_2)\in[0,(\varpi_{P_1},\varpi_{P_2}-w_{P_2}(\varpi_{P_2}))]$.

Proof. Since the distance is W-invariant, we have $d(\lambda_1,\lambda_2)=d(\varpi_{P_1},\mu)$ for some $\mu\in\Pi_{\varpi_{P_2}}$. We have $d(\varpi_{P_1},\mu)=(\varpi_{P_1},\varpi_{P_2}-\mu)$. Since ϖ_{P_2} is the highest weight of $V_{\varpi_{P_2}}$ and $w_{P_2}(\varpi_{P_2})$ the lowest weight, the result follows. □

Lemma 4.4. *We have* $d(\lambda_1,\lambda_2)=0$ *if and only if* λ_1 *and* λ_2 *belong to the same chamber.*

Proof. If λ_1 and λ_2 belong to the same chamber, then letting W act we may assume that this chamber is the dominant chamber. In particular $\lambda_i=\varpi_{P_i}$ and the distance vanishes. Conversely, we may assume by letting W act that $\lambda_1=\varpi_{P_1}$. We proceed by induction on $\varpi_{P_2}-\lambda_2$. If $\lambda_2=\varpi_{P_2}$, we are done. Otherwise $\lambda_2<\varpi_{P_2}$ and there exists a simple root α such that

$$\lambda_2<s_\alpha(\lambda_2)=\lambda_2+\alpha\leq\varpi_{P_2}.$$

Furthermore, since $d(\lambda_1,\lambda_2)=d(\varpi_{P_1},\lambda_2)=(\varpi_{P_1},\varpi_{P_2}-\lambda_2)=0$ we must have $(\varpi_{P_1},\alpha)=0$. Then we have $0=d(s_\alpha(\varpi_{P_1}),s_\alpha(\lambda_2))=d(\varpi_{P_1},s_\alpha(\lambda_2))$. By induction, ϖ_{P_1} and $s_\alpha(\lambda_2)$ are in the same chamber. The same is therefore true for $s_\alpha(\varpi_{P_1})=\varpi_{P_1}$ and λ_2. □

Corollary 4.5. *If* $d(\lambda_1,\lambda_2)>0$*, then there exists a root* α *with* $(\lambda_1,\alpha)(\lambda_2,\alpha)<0$.

Proof. If there is no root α with $(\lambda_1,\alpha)(\lambda_2,\alpha)<0$, then λ_1 and λ_2 are in the same chamber and $d(\lambda_1,\lambda_2)=0$ by the previous lemma. □

Lemma 4.6. *For* $(\lambda_1,\alpha)(\lambda_2,\alpha)<0$*, we have the following formula* $d(\lambda_1,s_\alpha(\lambda_2))=d(\lambda_1,\lambda_2)-1$.

Proof. For P_i cominuscule and G simply laced, we have $(\lambda_i,\alpha)\in\{-1,0,1\}$. The result follows from this by an easy computation. □

Corollary 4.7. *Let* $d=d(\lambda_1,\lambda_2)$.
(i) There exists a sequence $(\gamma_i)_{i\in[1,d]}$ *or roots such that if* $(\mu_i)_{i\in[0,d]}$ *is defined by* $\mu_d=\lambda_2$ *and* $\mu_{i-1}=s_{\gamma_i}(\mu_i)$*, then* $d(\lambda_1,\mu_i)=i$.
(ii) The roots $(\gamma_i)_{i\in[1,d]}$ *are pairwise orthogonal and satisfy the inequality* $(\lambda_1,\gamma_i)(\lambda_2,\gamma_i)<0$ *for all* $i\in[1,d]$.

Proof. (i) We proceed by induction on d. By the former corollary, if $d > 0$, there exists a root α with $(\lambda_1, \alpha)(\lambda_2, \alpha) < 0$. Set $\gamma_d = \alpha$ and $\mu_{d-1} = s_\alpha(\lambda_2)$, then $d(\lambda_1, \mu_{d-1}) = d - 1$. We conclude by induction.

(ii) Note that in the sequence $(\gamma_k)_{k\in[1,d]}$, we may replace γ_k by its opposite. Therefore we may assume that $(\lambda_1, \gamma_k) < 0$ (and thus $(\mu_k, \gamma_k) > 0$) for all $i \in [1, d]$. We first prove by induction on $j - i$ the vanishing $(\gamma_i, \gamma_j) = 0$ for all $i < j$. By induction assumption, we have

$$\mu_i = s_{\gamma_{i+1}} \cdots s_{\gamma_j}(\mu_j) = \mu_j - \sum_{k=i+1}^{j} (\gamma_k, \mu_k)\gamma_k = \mu_j - \sum_{k=i+1}^{j} \gamma_k.$$

We get, again using induction

$$1 \geq (\gamma_i, \mu_j) = (\gamma_i, \mu_i) + \sum_{k=i+1}^{j} (\gamma_k, \mu_k)(\gamma_i, \gamma_k) = 1 + (\gamma_i, \gamma_j).$$

In particular we get $(\gamma_i, \gamma_j) \leq 0$. If $(\gamma_i, \gamma_j) = -1$, then $\gamma_i + \gamma_j$ would be a root and we would have $(\lambda_1, \gamma_i + \gamma_j) \geq -1$. But $(\lambda_1, \gamma_i + \gamma_j) = -2$ a contradiction. The second condition easily follows. □

We can prove a converse of the above statement.

Lemma 4.8. *If $(\gamma_i)_{i\in[1,d]}$ is a sequence of pairwise orthogonal roots such that for all $i \in [1, d]$, we have $(\lambda_1, \gamma_i)(\lambda_2, \gamma_i) < 0$, then $d(\lambda_1, \lambda_2) \geq d$.*

Proof. Define the sequence $(\mu_i)_{i\in[0,d]}$ of weights as above: $\mu_d = \lambda_2$ and $\mu_{i-1} = s_{\gamma_i}(\mu_i)$. We have $d(\lambda_1, \mu_{i+1}) = d(\lambda_1, \mu_i) - 1$ for all i, the result follows. □

Corollary 4.9. *The distance $d(\lambda_1, \lambda_2)$ is the maximal length of sequences $(\gamma_i)_{i\in[1,d]}$ of pairwise orthogonal roots satisfying $(\lambda_1, \gamma_i)(\lambda_2, \gamma_i) < 0$ for all $i \in [1, d]$.*

4.2. Connection with the rank

We define a map $\Phi : B(X) \to W^{P_1} \times W^{P_2}$ as follows. Let $\mathcal{O}$ be a B-orbit in $G/P_1 \times G/P_2$. Then the images of $\mathcal{O}$ in G/P_1 and in G/P_2 are Schubert cells Ω_u and Ω_v with $(u, v) \in W^{P_1} \times W^{P_2}$. We put

$$\Phi(\mathcal{O}) = (u, v).$$

Remark 4.10. We defined the distance on the pairs of weights in $\Pi_1 \times \Pi_2$. We extend this definition to $W^{P_1} \times W^{P_2}$ by setting $d(u, v) = d(u(\varpi_{P_1}), v(\varpi_{P_2}))$.

Lemma 4.11. *Let $\mathcal{O}, \mathcal{O}' \in B(X)$ with $\mathcal{O} \leq \mathcal{O}'$ for the weak order. Let $(u, v) = \Phi(\mathcal{O})$ and $(u', v') = \Phi(\mathcal{O}')$. We have $d(u, v) - d(u', v') \leq \mathrm{rk}(\mathcal{O}') - \mathrm{rk}(\mathcal{O})$.*

Proof. Choose a sequence $(P_{\gamma_i})_{i \in [1,r]}$ of minimal parabolic subgroups raising $\mathcal{O}$ to $\mathcal{O}'$. Here γ_i for $i \in [1, r]$ denotes the simple root whose opposite is a root of P_{γ_i}. Let us write $\mathcal{O}_i$ for the dense B-orbit in $P_{\gamma_i} \cdots P_{\gamma_1} \mathcal{O}$ and write $\Phi(\mathcal{O}_i) = (u_i, v_i)$. We have the three possibilities:

- if $(\gamma_{i+1}, u_i(\varpi_{P_1})) = 1$, then we have $u_{i+1} = s_{\gamma_{i+1}} u_i$ and the equality $u_{i+1}(\varpi_{P_1}) = s_{\gamma_{i+1}} u_i(\varpi_{P_1}) = u_i(\varpi_{P_1}) - \gamma_{i+1}$,
- if $(\gamma_{i+1}, u_i(\varpi_{P_1})) = 0$, then we have $u_{i+1} = u_i$ and $u_{i+1}(\varpi_{P_1}) = u_i(\varpi_{P_1}) = s_{\gamma_{i+1}} u_i(\varpi_{P_1})$,
- if $(\gamma_{i+1}, u_i(\varpi_{P_1})) = -1$, then we have $u_{i+1} = u_i$ and the equality $u_{i+1}(\varpi_{P_1}) = u_i(\varpi_{P_1})$.

The same possibilities occur for v_i. There are only two cases for which we have $d(u_{i+1}, v_{i+1}) \neq d(u_i, v_i)$, namely for $(\gamma_{i+1}, u_i(\varpi_{P_1})) = 1$ and $(\gamma_{i+1}, v_i(\varpi_{P_2})) = -1$ and for $(\gamma_{i+1}, u_i(\varpi_{P_1})) = -1$ and $(\gamma_{i+1}, v_i(\varpi_{P_2})) = 1$. In both cases we have $d(u_{i+1}, v_{i+1}) = d(u_i, v_i) - 1$ by Lemma 4.6.

We claim that the following inequality holds

$$\mathrm{rk}(\mathcal{O}_{i+1}) - \mathrm{rk}(\mathcal{O}_i) \geq d(u_i, v_i) - d(u_{i+1}, v_{i+1}).$$

Since $\mathrm{rk}(\mathcal{O}_{i+1}) \geq \mathrm{rk}(\mathcal{O}_i)$ this is clear in all cases where $d(u_{i+1}, v_{i+1}) = d(u_i, v_i)$. The last two cases are symmetric, we only treat the first one. Remark that the orbit $\mathcal{O}_{i+1} = P_{\gamma_{i+1}} \mathcal{O}_i$ contains the orbit $\mathcal{O}_i$ and another orbit. Indeed, if y is the T-fixed element in Ω_{v_i}, then there exists an element of the form (x, y) in $\mathcal{O}_i$. The element $s_{\gamma_{i+1}}(x, y)$ is in $\mathcal{O}_{i+1}$ and $s_{\gamma_{i+1}}(y)$ is a T-fixed point different from y. Since the image by the second projection to G/P_2 of $\mathcal{O}_i$ and $\mathcal{O}_{i+1}$ is Ω_{v_i} which does not contain $s_{\gamma_{i+1}}(y)$ there is a third orbit $\mathcal{O}'_i$ contained in $\mathcal{O}_{i+1}$ and containing $s_{\gamma_{i+1}}(y)$. In particular $\mathrm{rk}(\mathcal{O}_{i+1}) = \mathrm{rk}(\mathcal{O}_i) + 1$. The claim is proved.

Summing up we get the desired inequality. □

Proposition 4.12. *Assume that (P_1, P_2) is an opposite pair and that w is the longest element of the Weyl group W of G. Then we have $d(\varpi_{P_1}, w(\varpi_{P_2})) \geq \mathrm{rk}(X)$.*

Proof. Consider the dense G-orbit G/H with $H = P_1 \cap P_2^{w_0}$ in X. This is the orbit of $([w_{P_1}(\varpi_{P_1})], [\varpi_{P_2}])$. We have a surjective morphism $p_1 : G/H \to G/P_1$ and we consider the fiber of $[w_{P_1}(\varpi_{P_1})]$ which is isomorphic to $P_1^{w_{P_1}} \cdot [\varpi_{P_2}] \simeq P_1^{w_{P_1}} / P_1^{w_{P_1}} \cap P_2 \simeq P_2^- / P_2^- \cap P_2 \simeq L_2 U_{P_2}^- / L_2$ where $U_{P_2}^-$ is the unipotent radical of P_2^- and L_2 is the Levi subgroup containing T. We have a trivialisation of the morphism $p_1 : G/H \to$

G/P_1 over the open subset $U_{P_1} \cdot [w_{P_1}(\varpi_{P_1})] \simeq U_{P_1}$ and therefore an open B-stable subset of X isomorphic to

$$U_{P_1} \times L_2 U_{P_2}^- / L_2.$$

The rank of X as a G-variety is therefore the rank of $L_2 U_{P_2}^- / L_2 \simeq U_{P_2}$ as L_2-variety. Note that L_2 acts on $U_{P_2}^-$ by conjugation. To compute the rank we want to compute the dimension of the quotient $U_{P_2}^-/U$ where U is a maximal unipotent subgroup of L_2 (and therefore acts on $U_{P_2}^-$ by conjugation).

Recall that, since P_2 is cominuscule, the root subgroups U_α contained in P_2^- commute pairwise and the multiplication $\prod U_\alpha \to U_{P_2}^-$ is a T-equivariant isomorphism of algebraic groups (where the action of T is given by conjugation and the product on the left hand-side is taken in any order and runs over the roots α with $\langle \chi_{P_2}, \alpha \rangle < 0$). We may therefore consider $U_{P_2}^-$ as an additive group. This structure extends to a vector space structure with a T-linear action. As T-vector space $U_{P_2^-}$ admits a T-equivariant decomposition as direct sum of the U_α for $(\alpha, \varpi_{P_2}) = -1$. The action of $U_\beta \subset U$ on $U_{P_2}^-$ induces a morphism $U_\beta \times U_\alpha \to U_\alpha \times U_{\alpha+\beta}$ defined by $(b, a) \mapsto (a, c_{\alpha,\beta} ab)$ for some constant $c_{\alpha,\beta}$ (non vanishing if $\alpha + \beta$ is a root, see [17, Proposition 8.2.3]).

We define a sequence $(R_i, \theta_i)_{i \in [1,s]}$ of pairs consisting of a root system R_i and a root $\theta_i \in R_i$ by induction. Let $R_1 = R$ be the root system of G and let θ_1 be the highest root of R_1. Assuming that (R_i, θ_i) is defined, consider the root system R_i' of roots in R_i orthogonal to θ_i. The root system R_i' might be reducible and is the product of its irreducible components. We define R_{i+1} as the irreducible component of R_i' containing the roots α satisfying $(\alpha, \varpi_{P_2}) \neq 0$. The root θ_{i+1} is then the highest root in R_{i+1}. Note that this definition is related to the classical construction of Kostant cascades (see [20, Section 40.5]) although we only consider a subset of the cascade of the full root system (the elements of the cascade not contained in $\varpi_{P_2}^\perp$).

We define the following sets of negative roots

$$\begin{aligned}
\mathcal{C}_i &= \{\gamma \in R^- \mid \text{ there exists } \gamma' \in R^- \text{ with } \gamma + \gamma' = -\theta_i\} \\
\mathcal{A}_i &= \{\alpha \in \mathcal{C}_i \mid (\alpha, \varpi_{P_2}) = -1\} \\
\mathcal{B}_i &= \{\beta \in \mathcal{C}_i \mid (\beta, \varpi_{P_2}) = 0\}.
\end{aligned}$$

Lemma 4.13. *We have $\mathcal{C}_i \subset R_i^-$.*

Proof. Let $\gamma \in \mathcal{C}_i$, then $\gamma \geq -\theta_i$. We prove that this condition implies $\gamma \in R_i$. By induction we only need to prove the statement for $\gamma \in R_{i-1}$. For $\gamma \in R_{i-1} \setminus R_i$, there exists a simple root of R_{i-1} non orthogonal to θ_{i-1} appearing with a negative coefficient in the expression of γ as linear combination of simple roots of R_{i-1}. This implies $\gamma \not\geq -\theta_i$. □

Lemma 4.14. *We have* $\mathcal{C}_i = \{\gamma \in R_i^- \mid (\gamma, -\theta_i) = 1\} \subset R_i^- \setminus (R_{i+1}^- \cup \{-\theta_i\})$.

Proof. Let $\gamma, \gamma' \in R^-$ with $\gamma + \gamma' = -\theta_i$. Then $2 = (\gamma', \gamma') = 4 + 2(\theta_i, \gamma)$ proving $(-\theta_i, \gamma) = 1$. Conversely, let $\gamma \in R_i^-$ with $(\gamma, -\theta_i) = 1$. Then $\gamma' = s_\gamma(-\theta_i) = -\theta_i - \gamma \in R_i^-$ and $\gamma, \gamma' \in \mathcal{C}_i$. For $\gamma \in \mathcal{C}_i$ we have $\gamma > -\theta_i$ and $(\gamma, -\theta_i) = 1$ thus $\gamma \notin R_{i+1}$. □

Lemma 4.15. *We have* $\mathcal{A}_i = \{\alpha \in R_i^- \setminus (R_{i+1}^- \cup \{-\theta_i\}) \mid (\alpha, \varpi_{P_2}) = -1\}$.

Proof. The inclusion of $\mathcal{A}_i$ in the right hand side follows from Lemma 4.14. Conversely let $\alpha \in R_i^- \setminus (R_{i+1}^- \cup \{-\theta_i\})$ with $(\alpha, \varpi_{P_2}) = -1$. If $(-\theta_i, \alpha) = 0$ then $\alpha \in R_{i+1}$. This is not the case therefore $(-\theta_i, \alpha) \neq 0$. Since θ_i is the highest root of R_i and $\alpha \neq -\theta_i$, we have $(\alpha, -\theta_i) = 1$. By Lemma 4.14 we have $\alpha \in \mathcal{C}_i$. □

Lemma 4.16. *Let* $\alpha \in \mathcal{A}_i$ *and* $\alpha' \in \mathcal{A}_j$ *and set* $\beta = -\theta_i - \alpha$ *and* $\beta' = -\theta_j - \alpha'$.

1. *Then* $\beta \in \mathcal{B}_i$ *and* $\beta' \in \mathcal{B}_j$.
2. *If* $i \neq j$, *then* $\alpha \neq \alpha'$ *and* $\beta \neq \beta'$.
3. *If* $i = j$, *then* $\beta + \beta'$ *is not a root.*

Proof. 1. The inclusion $\beta \in \mathcal{B}_i$ follows from the definition since we have the equality $(\theta_i, \varpi_{P_2}) = -1 = (\alpha, \varpi_{P_2})$. The same argument gives $\beta' \in \mathcal{B}_j$.

2. Since $\alpha, \beta \in \mathcal{C}_i$ and $\alpha', \beta' \in \mathcal{C}_j$, the assertion follows from the following claim: $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$. To prove the claim we may assume $i < j$. We have $R_j \subset R_{i+1} \subset R_i$ and the claim follows by Lemma 4.14.

3. Assume that $\beta + \beta'$ is a root. By Lemma 4.14, we have $(\beta + \beta', -\theta_i) = 2$. This implies $\beta + \beta' = -\theta_i$. Thus $\beta = \alpha'$ and $\beta' = \alpha$. Since P_2 is cominuscule and α, α' are roots of $U_{P_2}^-$, the sum $\alpha + \alpha'$ is not a root. A contradiction. □

We set $U(\theta_i) = \prod_{\alpha \in \mathcal{A}_i \cup \{\theta_i\}} U_\alpha \subset U_{P_2}^-$ for $i \in [1, s]$. These are subspaces of $U_{P_2}^-$. Note that for α, β roots of $U_{P_2}^-$ we have $U_\alpha U_\beta = U_\beta U_\alpha$ so that we do not have to take care of the order of the product. We also define $U_i = \prod_{\beta \in \mathcal{B}_i} U_{-\beta} \subset U$ for $i \in [1, s]$. Note that by Lemma 4.16, the above $U_{-\beta}$ commute so that we can take any order for this product.

Lemma 4.17. *Let* $U_i \subset U$ *act on* $U_{P_2}^-$ *by conjugation. Then* $U_i \cdot U_{-\theta_i} = U(\theta_i)$.

Proof. For $\beta \in \mathcal{B}_i$, the action maps $U_{-\beta} \times U_{-\theta_i}$ onto $U_{-\theta_i} \times U_{-\theta_i-\beta}$. Since the map $\mathcal{B}_i \to \mathcal{A}_i$ defined by $\beta \mapsto -\theta_i - \beta$ is bijective, an easy induction gives the equality $U_i \cdot U_{-\theta_i} = U(\theta_i)$. □

Corollary 4.18. *We have* $U \cdot (\prod_{i=1}^s U_{-\theta_i}) = U_{P_2}^-$.

Proof. Since $U_{P_2}^- = \prod_{i=1}^s U(\theta_i)$ and $\prod_{i=1}^s U_i \subset U$, we may apply inductively the previous Lemma. □

In particular, we see that $\mathrm{rk}(X) = \dim U_{P_2}^-/U \leq s$. But $(\theta_i)_{i\in[1,s]}$ is a sequence of pairwise orthogonal roots satisfying the following formulas $(\theta_i, \varpi_{P_2}) = 1$ and $(\theta_i, w_{P_1}(\varpi_{P_1})) = (\theta_i, -\varpi_{P_2}) = -1$ thus by Corollary 4.9 we have $d(\varpi_{P_1}, w_{P_2}(\varpi_{P_2})) = d(w_{P_1}(\varpi_{P_1}), \varpi_{P_2}) \geq s$ and the proposition is proved. □

Theorem 4.19. *Assume that* P_1 *and* P_2 *are opposite and* w *is the longest element in* W. *Let* $\mathcal{O} \in B(X)$ *and set* $\Phi(\mathcal{O}) = (u, v)$. *Then* $\mathrm{rk}(\mathcal{O}) + d(u, v) = \mathrm{rk}(X)$.

Proof. Let $(P_{\gamma_i})_{i\in[1,k]}$ be a sequence of minimal parabolic subgroups raising a minimal orbit $\mathcal{O}'$ to $\mathcal{O}$ and let $(P_{\gamma_i})_{i\in[k+1,r]}$ be a sequence of minimal parabolic subgroups raising $\mathcal{O}$ to the dense B-orbit in G. Write $\mathcal{O}_0 = \mathcal{O}'$ and $\mathcal{O}_i$ for the dense B-orbit in $P_{\gamma_i} \cdots P_{\gamma_1}\mathcal{O}'$. We have $\mathcal{O}_k = \mathcal{O}$. Set $\Phi(\mathcal{O}_i) = (u_i, v_i)$. According to the proof of Lemma 4.11, the equality $d(u_{i+1}, v_{i+1}) = d(u_i, v_i) - 1$ implies the equality $\mathrm{rk}(\mathcal{O}_{i+1}) = \mathrm{rk}(\mathcal{O}_i) + 1$. In particular, we get $d(u_0, v_0) = d(u_0, v_0) - d(u_r, v_r) \leq \mathrm{rk}(X) - \mathrm{rk}(\mathcal{O}') \leq \mathrm{rk}(X) \leq d(1, w_{P_2})$. But since $\mathcal{O}'$ is minimal for the weak order we have by Theorem 3.13 the equality $v_0 = u_0^\vee$ and by Lemma 3.9 we have $u_0^{-1}u_0^\vee = w_{P_2}$. Therefore $d(u_0, v_0) = d(1, u_0^{-1}v_0) = d(1, w_{P_2})$ and we have equality in all the inequalities. The result follows. □

Corollary 4.20. *There is no edge of type* N *in the graph* $\Gamma(X)$.

Proof. By Lemma 3.5 and Corollary 2.9, we may assume that P_1 and P_2 are opposite and w is the longest element.

Choose any minimal orbit $\mathcal{O}$ in X and any sequence $(P_{\gamma_i})_{i\in[1,r]}$ of minimal parabolic subgroups raising $\mathcal{O}$ to X. Write $\mathcal{O}_i$ for the dense B-orbit in $P_{\gamma_i} \cdots P_{\gamma_1}\mathcal{O}$ and set $\Phi(\mathcal{O}_i) = (u_i, v_i)$. According to the proof of Lemma 4.11 and to Lemma 4.6, the equality $d(u_{i+1}, v_{i+1}) = d(u_i, v_i) - 1$ implies the equality $\mathrm{rk}(\mathcal{O}_{i+1}) = \mathrm{rk}(\mathcal{O}_i) + 1$ and occurs only when $(u_i(\varpi_{P_1}), \gamma_{i+1})(v_i(\varpi_{P_2}), \gamma_{i+1}) < 0$. All the edges corresponding to such a raising by $P_{\gamma_{i+1}}$ are of type T by the above proof. But since $d(u_0, v_0) =$

$\mathrm{rk}(X) - \mathrm{rk}(\mathcal{O})$ there is no other edge of $\Gamma(X)$ raising the rank. Since edges of type N raise the rank there is no such edge. □

§5. Proof of Theorem 1

We want to use the technique developed by Brion in [3] and [4] to prove normality of the B-orbit closures. In particular Brion proves the following.

Proposition 5.1. *Let X be G-spherical variety such that the graph $\Gamma(X)$ has no edge of type N. Let Y be a B-stable subvariety such that for all minimal parabolic subgroups P raising Y the variety PY is normal, then the non normal locus in Y is G-invariant.*

Proof. We only sketch the proof and refer to [4, Proof of Theorem 1] for a complete proof. We shall need the following two results.

For $\mathcal{O}$ a B-orbit and P raising $\mathcal{O}$, let Y a B-stable subvariety containing $\mathcal{O}$ as an open subset. Consider the map $\pi : P \times^B Y \to PY$. For any sheaf $\mathcal{F}$ on $P \times^B Y$, we have $R^i\pi_*\mathcal{F} = 0$ for $i > 1$ and $R^1\pi_*\mathcal{O}_{P\times^B Y} = 0$ (see for example [3, Page 294]).

For $\mathcal{G}$ a B-linearized sheaf on Y, write $P \times^B \mathcal{G}$ for the corresponding P-linearised sheaf induced on $P \times^B Y$. If $\mathcal{G}$ is linearized under the stabiliser of Y in G and if $\pi_*(P \times^B \mathcal{G}) = 0$ for any P raising Y, then $\mathrm{Supp}(\mathcal{G})$ is G-invariant (see for example [3, Lemma 8]).

Assume that PY is normal for all P raising Y, let $\nu : Z \to Y$ be the normalization and let $\mathcal{F}$ be the cokernel of the map $\mathcal{O}_Y \to \nu_*\mathcal{O}_Z$. The sheaf $\mathcal{F}$ is linearised under the stabiliser of Y in G. The previous assertions imply that we have an exact sequence

$$0 \to \pi_*\mathcal{O}_{P\times^B Y} \to \pi_*(P \times^B \nu)_*\mathcal{O}_{P\times^B Z} \to \pi_*(P \times^B \mathcal{F}) \to 0$$

if P raises Y. But PY is normal and both morphisms π, $\pi \circ (P \times^B \nu)$ are proper and birational. By Zariski Main Theorem the first map is an isomorphism and $\pi_*(P \times^B \mathcal{F}) = 0$. We conclude that $\mathrm{Supp}(\mathcal{F})$ is G-invariant. Q.E.D.

Consider $X = G/P_1 \times G/P_2$ with P_1 and P_2 cominuscule. The variety X is G-spherical and has a unique closed G-orbit Z obtained as the image of the map $G/P_1 \cap P_2$ induced by the diagonal embedding $G \to G \times G$. To prove Theorem 1, we therefore only have to prove the normality of B-orbit closures containing Z.

Let Y' be a B-orbit closure containing Z. There exists a minimal orbit closure Y and a sequence of minimal parabolic subgroups $(P_{\gamma_i})_{i\in[1,r]}$ such that with $Y_0 = Y$ and $Y_i = P_{\gamma_i} \cdots P_{\gamma_1} Y$ for $i \geq 1$, the parabolic

$P_{\gamma_{i+1}}$ raises Y_i to Y_{i+1} for all i and $Y_r = Y'$. Consider the morphism $\pi : P_{\gamma_r} \times^B \cdots \times^B P_{\gamma_1} \times^B Y \to Y'$ which is birational by Lemma 3.2 and Corollary 4.20.

Proposition 5.2. *The inverse image* $\pi^{-1}(Z)$ *of* Z *is* $P_{\gamma_r} \times^B \cdots \times^B P_{\gamma_1} \times^B (Z \cap Y)$ *and the fibers of the map* $\pi^{-1}(Z) \to Z$ *are connected and generically reduced.*

Proof. Since Z is G-stable, the inverse image of Z by the action $G \times X \to X$ is $G \times Z$. This implies that the inverse image $\pi^{-1}(Z)$ has to be contained in $P_{\gamma_r} \times^B \cdots \times^B P_{\gamma_1} \times^B (Z \cap Y)$ and thus equal to $P_{\gamma_r} \times^B \cdots \times^B P_{\gamma_1} \times^B (Z \cap Y)$. But Y is a minimal B-orbit closure and as such (Theorem 3.13) is a product $X_u^{P_1} \times X_v^{P_2}$ of Schubert varieties with $u \in W^{P_1}$ and $v \in W^{P_2}$ (we write here X_u^P for the orbit closure of BuP/P in G/P). Recall that the closed G-orbit is $Z = G/P_1 \cap P_2$ embedded diagonally in $G/P_1 \times G/P_2$. Let $p_i : G/P_1 \cap P_2 \to G/P_i$ be the projection for $i \in \{1, 2\}$. The intersection of $Y = X_u^{P_1} \times X_v^{P_2}$ with Z is the intersection of Schubert varieties

$$p_1^{-1}(X_u^{P_1}) \cap p_2^{-1}(X_v^{P_2})$$

in Z. In particular it is reduced. This implies that the generic fiber of the map $\pi^{-1}(Z) \to Z$ is reduced.

To prove the connectedness of the fibers, we proceed by induction on r. We have a commutative diagram

$$\begin{array}{ccc} P_r \times^B \cdots \times^B P_1 \times^B (Z \cap Y) & \longrightarrow & Z \\ \big\downarrow & \nearrow_{\phi} & \\ P_r \times^B (Z \cap P_{r-1} \cdots P_1 Y) & & \end{array}$$

and it is enough to prove the connectedness of the fibers of ϕ. Let $Z' = Z \cap P_{r-1} \cdots P_1 Y$ and let $\mathcal{O}$ be a B-orbit in Z. The fiber over a point $z \in \mathcal{O}$ is given by $\{[p, z'] \in P_r \times^B Z' \mid pz' = z\}$. In particular $z' = p^{-1}z \in Z' \cap P_r\mathcal{O}$. Since $\mathcal{O}$ is a B-orbit in the projective homogeneous space Z, we have $P_r\mathcal{O} = \mathcal{O} \cup \mathcal{O}'$ with an edge of type U between $\mathcal{O}$ and $\mathcal{O}'$. If $P_r\mathcal{O} \subset Z'$, then the fiber is $\{[p, p^{-1}z] \mid p \in P_r\}$ and is isomorphic to $\mathbb{P}^1$. Otherwise, we must have $\dim \mathcal{O} < \dim \mathcal{O}'$ and $P_r\mathcal{O} \cap Z' = \mathcal{O}$. In that case $\phi^{-1}(P_r\mathcal{O}) = P_r \times^B \mathcal{O}$. The restriction of ϕ is the map $P_r \times^B \mathcal{O} \to P_r\mathcal{O}$ which is an isomorphism (it is birational since the edge is of type U and P_r-equivariant). $\square$

Corollary 5.3. *Let Y' be a B-orbit closure in X containing Z such that for any parabolic subgroup P raising Y', the variety PY' is normal, then Y' is normal.*

Proof. By Proposition 5.1, the non normal locus of Y' is G-invariant and therefore is either empty or contains Z. Let $\nu : \widetilde{Y}' \to Y'$ be the normalisation. Let Y be the closure of a minimal B-orbit and let $(P_{\gamma_i})_{i\in[1,r]}$ be a sequence of minimal parabolic subgroups as above. The variety Y is a product of Schubert varieties and therefore normal. It follows that the morphism $\pi : P_{\gamma_r} \times^B \cdots \times^B P_{\gamma_1} \times^B Y \to Y'$ factors through ν. Since the fibers of π over Z are connected, so are the fibers of ν over Z. Furthermore, since the general fiber of π over Z is reduced, so is the general fiber of ν. This implies that ν is an isomorphism on an open subset of Z and hence Z contains normal points of Y'. The non normal locus of Y' is therefore empty. □

Proof of Theorem 1. We prove the normality of B-orbit closures by descending induction with respect to the weak order.

A maximal B-orbit $\mathcal{O}$ is a G-orbit therefore of the form G/H with $H = P_1 \cap P_2^w$. We thus have $\mathcal{O} \simeq G \times^{P_1} P_1P_2^w/P_2^w$. The closure is then a locally trivial fibration over G/P_1 with fiber the Schubert variety $\overline{P_1P_2^w}/P_2^w$. It is normal since Schubert varieties are normal by [12].

Let Y be a B-orbit closure. Recall that the non normal locus is either empty or contains the closed orbit Z. If Y does not contain Z, then it must be normal. If Y contains Z, then by induction assumption, the hypothesis of Corollary 5.3 are satisfied and Y is normal.

The Cohen-Macaulay property follows from a general argument in [4, Section 3, Remark 2]. It will also follow from the existence of a rational resolution. For this, let Y' be a B-orbit closure and Y and $(P_{\gamma_i})_{i\in[1,r]}$ be the closure of a minimal B-orbit and a sequence of minimal parabolics raising Y to Y'. The variety Y is a product of Schubert varieties by Theorem 3.13. Let $\widetilde{Y}$ be the product of the Bott-Samelson resolutions of these varieties. Then by the same arguments as in [4, Section 3, end of Remark 2] the morphism $P_{\gamma_r} \times^B \cdots \times^B P_{\gamma_1} \times^B \widetilde{Y} \to Y'$ is a rational resolution. □

§6. Example of non normal closures

In this section we give an counterexample to Theorem 1 and Corollary 4.20 for G non simply laced.

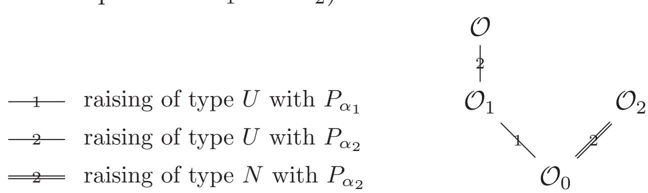
Let $(e_i)_{i\in[1,6]}$ be the canonical basis in k^6. Define the symplectic form ω on k^6 by $\omega(e_i, e_j) = \delta_{7,i+j}$ for all $i < j$. Let G be the symplectic group Sp_6 of linear automorphisms preserving ω. Let $P = P_1 = P_2$ be the stabiliser of the 3-dimensional isotropic subspace $\langle e_1, e_2, e_3\rangle$. Then $X = G/P \times G/P$ is the set of pairs of maximal (of dimension 3) subspaces in k^6 isotropic for ω. Consider the full flag

$$\langle e_1\rangle \subset \langle e_1, e_2\rangle \subset \langle e_1, e_2, e_3\rangle \subset \langle e_1, e_2, e_3, e_4\rangle \subset \langle e_1, e_2, e_3, e_4, e_5\rangle \subset k^6$$

and the Borel subgroup B of G stabilising this complete flag. We denote by T the maximal torus defined by the basis $(e_i)_{i\in[1,6]}$. We denote by α_1, α_2 and α_3 the simple roots of G with notation as in [1].

Proposition 6.1. *The closure of the B-orbit $\mathcal{O}$ of the element $x = (\langle e_3, e_1 + e_5, e_2 + e_6\rangle, \langle e_4, e_5, e_6\rangle)$ is not normal.*

Proof. To prove this result, we describe B-orbits $\mathcal{O}_0$, $\mathcal{O}_1$ and $\mathcal{O}_2$ in $\mathcal{O}$ such that the graph $B(X)$ contains the following subgraph (we denote by P_{α_1} and P_{α_2} the minimal parabolic subgroups containing B associated to the simple roots α_1 and α_2).



Graph 1. Subgraph of $\Gamma(X)$

If such a subgraph exists, we claim that the closure of $\mathcal{O}$ is not normal. This was proved in [14, Corollary 4.4.5], we reproduce the simple proof for the convenience of the reader: the morphism $P_{\alpha_2} \times^B \mathcal{O}_1 \to \mathcal{O}$ is birational while its restriction $P_{\alpha_2} \times^B \mathcal{O}_0 \to \mathcal{O}_2$ has non connected fibres. Zariski's Main Theorem gives the conclusion.

We are therefore left to prove that the above graph is indeed a subgraph of $\Gamma(X)$. We define the orbits $\mathcal{O}_0$, $\mathcal{O}_1$ and $\mathcal{O}_2$ as follows:

$\mathcal{O}_0$ is the B-orbit of $x_0 = (\langle e_1, e_2 + e_4, e_3 + e_5\rangle, \langle e_4, e_5, e_6\rangle)$

$\mathcal{O}_1$ is the B-orbit of $x_1 = (\langle e_2, e_1 + e_4, e_3 + e_6\rangle, \langle e_4, e_5, e_6\rangle)$

$\mathcal{O}_2$ is the B-orbit of $x_2 = (\langle e_1, e_3 + e_4, e_2 + e_5\rangle, \langle e_4, e_5, e_6\rangle)$.

We first prove the following equalities: $P_{\alpha_1}x_0 = P_{\alpha_1}x_1$, $P_{\alpha_2}x_0 = P_{\alpha_2}x_2$ and $P_{\alpha_2}x_1 = P_{\alpha_2}x$. For this is is enough to produce elements $p_1 \in P_{\alpha_1}$, $p_2 \in P_{\alpha_2}$ and $p \in P_{\alpha_2}$ such that $p_1x_0 = x_1$, $p_2x_0 = x_2$ and $px_1 = x$. It is enough to take p_1, p_2, p as follows:

$$p_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad p_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$p = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Computing the stabiliser of x_i for $i \in \{0,1,2,\emptyset\}$ in B, it is easy to compute the dimensions $\dim \mathcal{O}_0 = 8$, $\dim \mathcal{O}_1 = 9$, $\dim \mathcal{O}_2 = 9$ and $\dim \mathcal{O} = 10$. Note also that the orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ are distinct: write $x_i = (V_i, W_i)$ for $i \in \{1,2\}$, we have that V_1 is in the B-orbit of $\langle e_3, e_5, e_6\rangle$ while V_2 is in the B-orbit of $\langle e_1, e_4, e_5\rangle$. This proves that the above graph has the correct shape and we are left to proving that the types of the edges are as in Graph 1 above.

To decide if the edge is of type U, T or N we use the following criteria (see [15, Page 405] or [3, Page 268]): let P be a minimal parabolic subgroup raising a B-orbit $\mathcal{O}$ to a B-orbit $\mathcal{O}'$. Let $x \in \mathcal{O}'$ and P_x its stabiliser in P. Denote by S the image of P_x in $\mathrm{Aut}(P/B) = \mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$. Then we have:

- the edge is of type U if S contains a positive dimensional unipotent subgroup,
- the edge is of type T if S is a maximal torus in $\mathrm{Aut}(P/B)$,
- the edge is of type N if S is the normaliser of a maximal torus in $\mathrm{Aut}(P/B)$.

An easy computation of stabilisers proves that the edges are of the above type finishing the proof. Q.E.D.

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