Advanced Studies in Pure Mathematics 70, 2016
Minimal Models and Extremal Rays (Kyoto, 2011)
pp. 397–420

On $\mathbb{Q}$-Fano 3-folds of Fano index 2

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To Shigefumi Mori in friendship and admiration

§1. Introduction

1.1. $\mathbb{Q}$-Fano 3-folds

A $\mathbb{Q}$-*Fano* 3-*fold* is a projective 3-fold X with at worst terminal singularities and ample anticanonical divisor $-K_X$. Here, bearing in mind Mori's fundamental notion of extremal ray, we assume also that X is $\mathbb{Q}$-factorial and has rank 1, that is, $\operatorname{Pic} X \simeq \mathbb{Z}$ or equivalently, $\operatorname{Cl} X \otimes \mathbb{Q} \simeq \mathbb{Q}$. We define the *Fano* and $\mathbb{Q}$-*Fano index* of X by:

$$\mathrm{q_F}(X) := \max\{q \in \mathbb{Z} \mid -K_X \sim qA \text{ with } A \text{ a Weil divisor}\},$$
$$\mathrm{q_{\mathbb{Q}}}(X) := \max\{q \in \mathbb{Z} \mid -K_X \sim_{\mathbb{Q}} qA \text{ with } A \text{ a Weil divisor}\},$$

where $\sim$ is linear equivalence and $\sim_{\mathbb{Q}}$ is $\mathbb{Q}$-linear equivalence. Clearly, $\mathrm{q_F}(X)$ divides $\mathrm{q_{\mathbb{Q}}}(X)$, and the two coincide unless $K_X + qA \in \operatorname{Cl} X$ is a nontrivial torsion element. An important invariant of a $\mathbb{Q}$-Fano 3-fold is its *genus* $\mathrm{g}(X) := \dim|-K_X| - 1$.

1.2. Background facts

Kaori Suzuki [Suz04] restricts the $\mathbb{Q}$-Fano index of X to one of

$$\mathrm{q_{\mathbb{Q}}}(X) \in \{1, \dots, 11, 13, 17, 19\}. \tag{1.2.1}$$

See also [Pro10b, Lemma 3.3]. Moreover, the following results are due to the first author.

Received March 5, 2012.
Revised August 6, 2014.
Y.P. acknowledges partial support from RFBR grants No. 11-01-92613-KO_a, 11-01-00336-a, the grant of Leading Scientific Schools No. 2998.2014.1, and Russian Academic Excellence Project '5–100'.
M.R. is partly supported as Scholar of KIAS.

1.2.2. Theorem ([Pro10b]). *Let X be a $\mathbb{Q}$-Fano 3-fold of $\mathbb{Q}$-Fano index $q := \mathrm{q}_{\mathbb{Q}}(X) \geq 9$. Then $\operatorname{Cl} X \simeq \mathbb{Z}$.*

(i) *If $q = 19$ then $X \simeq \mathbb{P}(3,4,5,7)$.*
(ii) *If $q = 17$ then $X \simeq \mathbb{P}(2,3,5,7)$.*
(iii) *If $q = 13$ and $\mathrm{g}(X) > 4$ then $X \simeq \mathbb{P}(1,3,4,5)$.*
(iv) *If $q = 11$ and $\mathrm{g}(X) > 10$ then $X \simeq \mathbb{P}(1,2,3,5)$.*
(v) *$q \neq 10$.*

1.2.3. Theorem ([Pro10c]). *Let X be a $\mathbb{Q}$-Fano 3-fold of $\mathbb{Q}$-Fano index q.*

(vi) *If $q = 9$ and $\mathrm{g}(X) > 4$ then $X \simeq X_6 \subset \mathbb{P}(1,2,3,4,5)$.*
(vii) *If $q = 8$ and $\mathrm{g}(X) > 10$ then $X \simeq X_6 \subset \mathbb{P}(1,2,3^2,5)$ or $X_{10} \subset \mathbb{P}(1,2,3,5,7)$.*
(viii) *If $q = 7$ and $\mathrm{g}(X) > 17$ then $X \simeq \mathbb{P}(1^2,2,3)$.*
(ix) *If $q = 6$ and $\mathrm{g}(X) > 15$ then $X \simeq X_6 \subset \mathbb{P}(1^2,2,3,5)$.*
(x) *If $q = 5$ and $\mathrm{g}(X) > 18$ then $X \simeq \mathbb{P}(1^3,2)$ or $X_4 \subset \mathbb{P}(1^2,2^2,3)$.*
(xi) *If $q = 4$ and $\mathrm{g}(X) > 21$ then $X \simeq \mathbb{P}^3$ or $X_4 \subset \mathbb{P}(1^3,2,3)$.*
(xii) *If $q = 3$ and $\mathrm{g}(X) > 20$ then $X \simeq X_2 \subset \mathbb{P}^4$ or $X_3 \subset \mathbb{P}(1^4,2)$.*

Here we study the case $\mathrm{q}_{\mathbb{Q}}(X) = 2$.

1.2.4. Theorem ([BS07b]). *The Hilbert series of $\mathbb{Q}$-Fano 3-folds with $q = \mathrm{q}_{\mathbb{Q}}(X) = \mathrm{q}_{\mathrm{F}}(X) = 2$ belong to at most 1492 cases.*

The online database [GRDB] lists the numerical type of candidates (the data going into the Hilbert series of their graded rings).

1.3. Main results

1.3.1. Main Theorem. *Let X be a $\mathbb{Q}$-Fano 3-fold of rank 1 with $\mathrm{q}_{\mathbb{Q}}(X) = \mathrm{q}_{\mathrm{F}}(X) = 2$ and K_X not Cartier. Let A be a Weil divisor on X such that $-K_X = 2A$.*

Then $\dim|A| \leq 4$. Moreover, if $\dim|A| = 4$, then X belongs to the single irreducible family constructed in 6.3.7 (see also 1.5).

1.3.2. Corollary. *A $\mathbb{Q}$-Fano 3-fold with $\mathrm{q}_{\mathbb{Q}}(X) = \mathrm{q}_{\mathrm{F}}(X) = 2$ and K_X not Cartier has $\mathrm{g}(X) \leq 16$.*

Remark 1.3.3. If K_X is Cartier and $\mathrm{q}_{\mathrm{F}}(X) = 2$, then X is a *del Pezzo variety* [Fuj90]. Two cases with $\operatorname{Cl} X \simeq \mathbb{Z}$ have $\dim|A| > 4$:

(a) the complete intersection of two quadrics $X = X_{2\cdot 2} \subset \mathbb{P}^5$, with $\dim|A| = 5$ and $\mathrm{g}(X) = 19$; and
(b) $X = X_5 \subset \mathbb{P}^6$ a section of the Grassmannian $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ by a subspace of codimension 3, with $\dim|A| = 6$ and $\mathrm{g}(X) = 23$. In this case X must be smooth by [Pro10a, Cor. 5.3].

1.4. Strategy of proof

Sections 4–5 contain the proof of Main Theorem 1.3.1. The Kawamata blowup of a $\frac{1}{r}(2,a,r-a)$ point initiates a Sarkisov link ending in a fibre space or a $\mathbb{Q}$-Fano 3-fold with $q \geq 3$; the assumption $\dim|A| \geq 4$ leads to a manageable case division. The auxiliary Section 3 treats the cases with $q \geq 3$, most of which lead to a contradiction, with just one surviving in Section 5 to characterize our Main Example.

1.5. The Main Example

Section 6 gives several constructions of the exceptional family of Main Theorem 1.3.1, $\mathbb{Q}$-Fanos X with $\operatorname{Cl} X = \mathbb{Z}\cdot A$, $K_X = -2A$ and $\dim|A| = 4$. They arise from the simplest type of Sarkisov link:

(1.5.1)
$$\begin{tikzcd} & E \subset X_1 \supset F \arrow[dl] \arrow[dr] & \\ P \in X & & Q \supset \Gamma_5 \ni P_0 \end{tikzcd}$$

starting from the nonsingular quadric hypersurface $Q \subset \mathbb{P}^4$, a point $P_0 \in Q$ and an irreducible curve $\Gamma_5 \subset E_0$ of degree 5 contained in the tangent hyperplane section $E_0 = Q \cap T_{P_0}Q$, with $\operatorname{mult}_{P_0}\Gamma_5 = 3$. We make the symbolic blowup $X_1 \to Q$ of Γ_5, then contract the birational transform $E_1 \simeq E_0 \simeq \mathbb{P}(2,1,1) \subset X_1$ of E_0 to a $\frac{1}{3}(1,2,2)$ orbifold point.

The symbolic blowup of $\Gamma \subset E_0 \subset Q$ is the relative Proj of the symbolic algebra $\mathcal{A} = \bigoplus \mathcal{I}_\Gamma^{[n]}$. For a singular curve Γ contained in a nodal surface, this is a local graded ring construction with a universal description, studied in much more detail and generality in Tom Ducat's thesis [Du15]. Compare [Du14].

1.6. Discussion

The study of $\mathbb{Q}$-Fanos divides into birational and biregular considerations. Biregular methods study projective embedding by multiples of A, or more precisely, generators and relations for the Gorenstein graded ring $R(X,A)$. This is effective when $R(X,A)$ has small codimension, especially if it is a hypersurface or codimension 2 complete intersection, etc. In contrast, birational methods are powerful when the linear system $|A|$ is large, implying a low canonical threshold, and allowing us to impose noncanonical singularities on $|A|$ and study X via the resulting Sarkisov link, aiming for a birational construction or a nonexistence result. The interest of this paper is as a meeting point of the two methods.

1.7. The fabulous half-elephant; more cases with $q = 2$

A surface section $F \in |A|$ of a $\mathbb{Q}$-Fano 3-fold X of index 2 is a del Pezzo surface (sometimes very singular). In a few cases where F has the simplest orbifold points such as $\frac{1}{3}(2,2)$ or $\frac{1}{5}(2,4)$, Reid and Suzuki [RS03] study such surfaces in terms of cascades of projections from nonsingular points. This foreshadows one construction of our Main Example in Section 6, and hints at other cases that might make interesting challenges, especially the X with $\dim|A| = 3$ or 2. Del Pezzo surfaces with only $\frac{1}{3}(2,2)$ orbifold points are classified in current work of Alessio Corti and Liana Heuberger [CH15]. Kuzma Khrabrov [Kh14] has partial results on $\mathbb{Q}$-Fano 3-folds X of index 2 with $\dim|A| \geq 2$.

§2. The method

2.1. Construction of a Sarkisov link [Ale94]

Let $\mathcal{M}$ be a linear system on X with no fixed part, and canonical threshold $c := \operatorname{ct}(X, \mathcal{M})$. Assume $-(K_X + c\mathcal{M})$ is ample. Then $(X, c\mathcal{M})$ is canonical but not terminal, so we can pull out an irreducible divisor $E \subset \widetilde{X}$ by an extremal divisorial extraction $f\colon \widetilde{X} \to X$, such that $\widetilde{X}$ has only terminal $\mathbb{Q}$-factorial singularities, $\rho(\widetilde{X}/X) = 1$, and f is $(K + c\mathcal{M})$-crepant:

$$K_{\widetilde{X}} + c\widetilde{\mathcal{M}} = f^*(K_X + c\mathcal{M}). \tag{2.1.1}$$

As in [Ale94], running a $(K + c\mathcal{M})$-MMP on $\widetilde{X}$ gives a Sarkisov link of type I or II:

$$\begin{tikzcd} & \widetilde{X} \arrow[r, dashed] \arrow[dl, "f"'] & \overline{X} \arrow[dr, "\overline{f}"] & \\ X & & & \widehat{X} \end{tikzcd} \tag{2.1.2}$$

where $\widetilde{X}$ and $\overline{X}$ have only $\mathbb{Q}$-factorial terminal singularities, $\rho(\widetilde{X}) = \rho(\overline{X}) = 2$, $\widetilde{X} \dashrightarrow \overline{X}$ is a chain of log flips, and $\overline{f}$ is a Mori extremal contraction, either a divisorial contraction to a $\mathbb{Q}$-Fano 3-fold $\widehat{X}$, or a Mori fibre space over a curve or surface $\widehat{X}$. In either case, $\rho(\widehat{X}) = 1$. We write $\widetilde{D}$ and $\overline{D}$ for the birational transform on $\widetilde{X}$ and $\overline{X}$ of a divisor or linear system D on X.

Assume that $K_X + \lambda\mathcal{M} + \Xi \sim_{\mathbb{Q}} 0$ for some $\lambda > c$ and an effective $\mathbb{Q}$-divisor Ξ. We can write

$$K_{\widetilde{X}} + \lambda\widetilde{\mathcal{M}} + \widetilde{\Xi} + aE \sim_{\mathbb{Q}} f^*(K_X + \lambda\mathcal{M} + \Xi) \sim_{\mathbb{Q}} 0, \tag{2.1.3}$$

where $a > 0$ is the log discrepancy of f. Note that if $K_X + \lambda\mathcal{M} + \Xi \sim 0$ then it is a Cartier divisor; we can assume that the mobile system $\widetilde{\mathcal{M}}$ has no common divisor with $\widetilde{\Xi}$. Then $K_{\widetilde{X}}$, $\lambda\widetilde{\mathcal{M}}$ and $\widetilde{\Xi}$ are all integral Weil divisors, and therefore a is an integer.

Remark 2.1.4. We use the extremal extraction $\widetilde{X} \to X$ with ray R and exceptional surface E to initiate a Sarkisov link. By (2.1.1), $K_{\widetilde{X}} + c\widetilde{\mathcal{M}}$ is nef and big, with

$$(2.1.5) \qquad K_{\widetilde{X}} \cdot R < 0, \quad (K_{\widetilde{X}} + c\widetilde{\mathcal{M}}) \cdot R = 0, \quad \text{so that } \widetilde{\mathcal{M}} \cdot R > 0.$$

The MMP that constructs the Sarkisov link proceeds by increasing λ in $K + \lambda\mathcal{M}$. Each step makes $K + \lambda\mathcal{M}$ bigger on the ray R, so on the exceptional surface E and its birational transforms. Thus the MMP can never contract the birational transform of E.

2.2. Case $\overline{f}$ not birational

Assume that $\overline{f}$ is not birational. Then $\widehat{X}$ is either a smooth rational curve or a del Pezzo surface with at worst Du Val singularities and $\rho(\widehat{X}) = 1$ [MP08]. We also have $\overline{f}(\overline{E}) = \widehat{X}$ by Remark 2.1.4, or because no multiple $n\overline{E}$ of the exceptional divisor $\overline{E}$ of f moves on $\overline{X}$. In this case we write $\overline{F}$ for a general fiber of $\overline{f}$. Let Θ be an ample Weil divisor on $\widehat{X}$ whose class generates $\operatorname{Cl}\widehat{X}$ modulo torsion. If $\widehat{X}$ is a surface with $K_{\widehat{X}}^2 = 1$, we take $\Theta = -K_{\widehat{X}}$.

2.2.1. For $\widehat{X}$ a surface, one of the following holds:

(i) $-K_{\widehat{X}} \cdot \Theta = 3$, $-K_{\widehat{X}} \sim 3\Theta$, $\widehat{X} \simeq \mathbb{P}^2$ and $\dim|\Theta| = 2$;

(ii) $-K_{\widehat{X}} \cdot \Theta = 2$, $-K_{\widehat{X}} \sim 4\Theta$, $\widehat{X} \simeq \mathbb{P}(1,1,2)$ and $\dim|\Theta| = 1$;

(iii) $-K_{\widehat{X}} \cdot \Theta = 1$, $-K_{\widehat{X}} \sim d\Theta$, where $d := K_{\widehat{X}}^2 \le 6$, and the minimal resolution of $\widehat{X}$ is a blowup of $\mathbb{P}^2$ at $9-d$ points in *almost general position*. In this case, $\dim|\Theta| = 0$ or 1. Moreover, by Kawamata–Viehweg vanishing and orbifold Riemann–Roch [YPG], for an ample Weil divisor $B \sim_{\mathbb{Q}} t\Theta$ we have

$$(2.2.2) \qquad \dim|B| \le \frac{t(t+d)}{2d}.$$

2.3. Case $\overline{f}$ birational

Assume that the contraction $\overline{f}$ is birational. In this case, $\widehat{X}$ is a $\mathbb{Q}$-Fano 3-fold and $\overline{f}$ contracts a unique exceptional divisor $\overline{F}$. Remark 2.1.4 implies that $\overline{E} \ne \overline{F}$ (or argue that $\overline{E} = \overline{F}$ would imply that $X \dashrightarrow \widehat{X}$ is an isomorphism in codimension, leading to a contradiction). Write $\widetilde{F} \subset \widetilde{X}$ and $F := f(\widetilde{F})$ for its birational transform. Set $\widehat{q} := \mathrm{q}_{\mathbb{Q}}(\widehat{X})$. For a divisor $\overline{D}$ on $\overline{X}$, we put $\widehat{D} := \overline{f}_*\overline{D}$.

2.4. Computer search for $\mathbb{Q}$-Fano 3-folds

All $\mathbb{Q}$-Fano 3-folds belong to a finite number of algebraic families [Kaw92]. In fact, Kawamata's proof implies that the possible "candidate" $\mathbb{Q}$-Fano 3-folds can be listed, although the volume of computation makes computer searches inevitable. This method was used in [Suz04], [BS07a], [BS07b], [Pro07], [Pro10b], [Pro10c]. See [GRDB] for explicit lists.

We outline the algorithm, starting with a useful remark.

Remark 2.4.1. The local analytic Weil divisor class group of a 3-fold $\mathbb{Q}$-factorial terminal point $P \in X$ is cyclic $\mathrm{Cl}(X, P) \simeq \mathbb{Z}/r$, and is generated by the canonical divisor K_X [Kaw88, Lemma 5.1]. In particular, if X is a $\mathbb{Q}$-Fano 3-fold, its local Gorenstein index r at every terminal point is coprime to the $\mathbb{Q}$-Fano index $q = \mathrm{q_F}(X)$.

2.4.2. Let X be a $\mathbb{Q}$-Fano 3-fold. For simplicity we assume that $q := \mathrm{q}_{\mathbb{Q}}(X) = \mathrm{q_F}(X) \geq 3$ (the only case we need in this section). Let A be a Weil divisor such that $-K_X \sim qA$ and $\mathcal{B}(X) = \{(r_P, b_P)\}$ the basket of orbifold points of X [YPG].

Step 1. We have the equality

$$-K_X \cdot c_2(X) + \sum_{P \in \mathcal{B}} \frac{r_P - 1}{r_P} = 24, \tag{2.4.3}$$

where $-K_X \cdot c_2(X) > 0$ [Kaw92]. Hence there is only a finite (but huge) number of possibilities for the basket $\mathcal{B}(X)$ and $-K_X \cdot c_2(X)$. Let $r := \mathrm{lcm}(\{r_P\})$ be the Gorenstein index of X.

Step 2. (1.2.1) says that $q \in \{3, \dots, 11, 13, 17, 19\}$. Remark 2.4.1 implies that $\gcd(q, r) = 1$, which eliminates some possibilities.

Step 3. In each case we compute A^3 by the formula

$$A^3 = \frac{12}{(q-1)(q-2)} \Big(1 - \frac{A \cdot c_2}{12} + \sum_{P \in B} c_P(-A) \Big).$$

(see [Suz04]), where c_P is the correction term in the orbifold Riemann–Roch formula [YPG]. The number rA^3 must be an integer [Suz04, Lemma 1.2].

Step 4. Next, the Bogomolov–Miyaoka inequality (see [Kaw92]) implies that

$$\left(4q^2 - 3q\right) A^3 \leq -4K_X \cdot c_2(X) \tag{2.4.4}$$

[Suz04, Prop. 2.2].

Step 5. Finally, the Kawamata–Viehweg vanishing theorem gives $\chi(tA) = h^0(tA) = 0$ for $-q < t < 0$. We check this condition using orbifold Riemann–Roch [YPG], [BRZ].

§3. On $\mathbb{Q}$-Fano 3-folds of Fano index ≥ 3

3.1. A result of Fujita

A *polarized variety* is a pair (X, S) consisting of a projective variety X and an ample Cartier divisor S on X. Its Δ-*genus* is defined as follows [Fuj90]:

$$\Delta(X, S) = \dim X + S^{\dim X} - h^0(X, \mathcal{O}_X(S)). \tag{3.1.1}$$

It is known that $\Delta(X, S) \geq 0$ and Fujita [Fuj90] classifies polarized varieties of small Δ-genera. We use the following easy consequence of Fujita's classification.

3.1.2. Lemma. *Let X be a $\mathbb{Q}$-Fano 3-fold and S an ample Weil divisor on X such that $\dim|S| > 0$, $|S|$ has no fixed components, and $-K_X \sim_{\mathbb{Q}} \lambda S$ with $\lambda \geq 2$. Assume that the pair $(X, |S|)$ is terminal. Then one of the following holds:*

(i) $X \simeq \mathbb{P}^3$, $\lambda = 4$, $\dim|S| = 3$;
(ii) $X \simeq \mathbb{P}^3$, $\lambda = 2$, $\dim|S| = 9$;
(iii) $X \simeq X_2 \subset \mathbb{P}^4$ *is a smooth quadric*, $\lambda = 3$, $\dim|S| = 4$;
(iv) X *is a del Pezzo 3-fold of degree* $1 \leq d \leq 5$, $\lambda = 2$, $\dim|S| = d + 1$;
(v) $X \simeq \mathbb{P}(1^3, 2)$, $\lambda = 5/2$, $\dim|S| = 6$.

Proof. Replace S with a general member of $|S|$. Since $(X, |S|)$ is terminal, the surface S is smooth and contained in the smooth locus of X [Ale94, 1.22]. By the adjunction formula we have $-K_S \sim (\lambda - 1)S_{|S}$. Hence S is a (smooth) del Pezzo surface and $(\lambda - 1)^2 S^3 = K_S^2$. Since $H^i(X, \mathcal{O}_X) = 0$ and $H^i(S, \mathcal{O}_S(S)) = 0$ for $i > 0$, by Riemann–Roch we have

$$h^0(X, \mathcal{O}_X(S)) = h^0(S, \mathcal{O}_S(S)) + 1 = \frac{\lambda}{2} S^3 + 2. \tag{3.1.3}$$

Therefore,

$$\Delta(X, S) = 3 + S^3 - \frac{\lambda}{2} S^3 - 2 = 1 + \frac{(2 - \lambda)S^3}{2} = 1 + \frac{(2 - \lambda)K_S^2}{2(\lambda - 1)^2}. \tag{3.1.4}$$

If $S \simeq \mathbb{P}^2$, then $\mathcal{O}_S(S) = \mathcal{O}_{\mathbb{P}^2}(l)$, where $3 = (\lambda - 1)l \geq l$. Then $\Delta(X, S) = 0$ and [Fuj90, Th. 5.10 and 5.15] gives cases (i) and (v). If $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$, then $\mathcal{O}_S(S) = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(k,k)$, where $k(\lambda-1)=2$. So, $\lambda = 2$ or 3, $\Delta(X,S)=0$, and [Fuj90, Th. 5.10 and 5.15] gives cases (ii) or (iii). Finally, if $S \not\simeq \mathbb{P}^2$, $\mathbb{P}^1\times\mathbb{P}^1$, then K_S is a primitive element of $\operatorname{Pic} S$. Hence $\lambda = 2$ and $\Delta(X,S)=1$. Then we have case (iv) [Fuj90, Ch. 1, §9]. Q.E.D.

Lemma 3.2 ([Pro10b, Th. 1.4 (vii)]). *Let X be a $\mathbb{Q}$-Fano 3-fold with terminal singularities and with $q := \mathrm{q}_{\mathbb{Q}}(X) \ge 5$. Let A be a Weil divisor such that $-K_X \sim_{\mathbb{Q}} qA$. If $\dim|A| \ge 2$, then $X \simeq \mathbb{P}(1^3,2)$.*

Proof. We first consider the case $\operatorname{rk}\operatorname{Cl}X = 1$ and $\mathrm{q}_{\mathbb{Q}}(X) = \mathrm{q}_{\mathrm{F}}(X)$ (in particular, X is a $\mathbb{Q}$-Fano 3-fold and $-K_X \sim qA$). Running a computer search as in 2.4 gives $-K_X^3 \ge 125/2$. Then by [Pro07] we have $X \simeq \mathbb{P}(1^3,2)$.

Next consider the case $\operatorname{rk}\operatorname{Cl}X > 1$ and $\mathrm{q}_{\mathbb{Q}}(X) = \mathrm{q}_{\mathrm{F}}(X)$. We get a contradiction in this case. Run the MMP. The property $-K_X \sim qA$ is preserved. At the end we get a $\mathbb{Q}$-Fano 3-fold $\overline{X}$ with $K_{\overline{X}} \sim q\overline{A}$ and $\dim|\overline{A}| \ge 2$. By (1.2.1) we have $\mathrm{q}_{\mathbb{Q}}(\overline{X}) = \mathrm{q}_{\mathrm{F}}(\overline{X}) = q$. By the above $\overline{X} \simeq \mathbb{P}(1^3,2)$ and $\dim|A| = \dim|\overline{A}| = 2$. Let $\overline{P} \in \overline{X}$ be the point of type $\frac{1}{2}(1,1,1)$. Consider the final step $g\colon \widetilde{X} \to \overline{X}$ of the MMP, a divisorial contraction, and let $\widetilde{E} \subset \widetilde{X}$ be its exceptional divisor. There are the following possibilities:

(a) $g(\widetilde{E}) = \overline{P}$. Then $K_{\widetilde{X}} \sim_{\mathbb{Q}} g^*K_{\overline{X}} + \frac{1}{2}\widetilde{E}$, $\widetilde{E} \simeq \mathbb{P}^2$, and $\mathcal{O}_{\widetilde{E}}(\widetilde{E}) \simeq \mathcal{O}_{\mathbb{P}^2}(-2)$ [Kaw96]. Hence, $\mathcal{O}_{\tilde{E}}(-K_{\widetilde{X}}) \simeq \mathcal{O}_{\mathbb{P}^2}(1)$. We get a contradiction because $-K_{\widetilde{X}}$ is divisible by $q \ge 5$.

(b) $g(\widetilde{E})$ is either a smooth point or a curve. In this case $g(\widetilde{E}) \not\subset \operatorname{Bs}|\overline{A}| = \{\overline{P}\}$. On the other hand, g is a $K_{\widetilde{X}}$-negative contraction, a contradiction.

Finally assume that the torsion part of $\operatorname{Cl}X$ is nontrivial. Every torsion element $\xi_1 \in \operatorname{Cl}X$ of order $n_1 > 1$ defines a $\boldsymbol{\mu}_{n_1}$-cover $\pi_1\colon X_1 \to X$ that is étale in codimension 2. Repeating the procedure we get a sequence

$$X_m \xrightarrow{\pi_m} X_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X. \tag{3.2.1}$$

with each π_k a $\boldsymbol{\mu}_{n_k}$-cover that is étale in codimension 2 and $\operatorname{Cl}X_m$ torsion free. By the above, $X_m \simeq \mathbb{P}(1^3,2)$. Since

$$h^0(X_m, \pi^*A) = h^0(X, A) = 3, \tag{3.2.2}$$

$\boldsymbol{\mu}_{n_m}$ acts trivially on $H^0(X_m,\pi^*A) = H^0(\mathcal{O}_{\mathbb{P}(1^3,2)}(1))$. On the other hand, we can take independent sections $x_1, x_2, x_3 \in H^0(\mathcal{O}_{\mathbb{P}(1^3,2)}(1))$ as orbinates at the $\frac{1}{2}(1,1,1)$-point $P_m \in X_m$. This contradicts that the point $(X_m, P_m)/\boldsymbol{\mu}_{n_m}$ is terminal. Q.E.D.

In a similar way to Lemma 3.2, one can prove the following.

Lemma 3.3 ([Pro10b, Th. 1.4 (vi)])**.** *Let X be a $\mathbb{Q}$-Fano 3-fold with terminal singularities and with $q := \mathrm{q}_{\mathbb{Q}}(X) \ge 7$. Let A be a Weil divisor such that $-K_X \sim_{\mathbb{Q}} qA$. If $\dim |A| \ge 1$, then $X \simeq \mathbb{P}(1^2, 2, 3)$.*

Proposition 3.4. *Let X be a $\mathbb{Q}$-Fano 3-fold and let $q := \mathrm{q}_{\mathbb{Q}}(X)$. Let $\mathcal{M}$ be a linear system on X such that $\dim \mathcal{M} \ge 4$ and $-K_X \sim 2\mathcal{M} + \Xi$, where Ξ is a nonzero effective Weil divisor. Then $\mathrm{Cl}\, X \simeq \mathbb{Z}\cdot\Xi$, $q = 2n+1$ is odd, and $\mathcal{M} \sim n\Xi$. Moreover, one of the following holds:*

(i) $q = 13$, $X \simeq \mathbb{P}(1,3,4,5)$;
(ii) $q = 11$, $X \simeq \mathbb{P}(1,2,3,5)$;
(iii) $q = 9$, $X \simeq X_6 \subset \mathbb{P}(1,2,3,4,5)$;
(iv) $q = 7$, $X \simeq \mathbb{P}(1^2,2,3)$;
(v) $q = 5$, $X \simeq X_4 \subset \mathbb{P}(1^2,2^2,3)$;
(vi) $q = 5$, $X \simeq \mathbb{P}(1^3,2)$;
(vii) $q = 3$, $X \simeq X_2 \subset \mathbb{P}^4$.

Proof. By assumption $q \ge 3$. If $q \ge 9$, then the assertion follows by [Pro10b, Prop. 3.6] and Theorem 1.2.3 (vi). So assume that $3 \le q \le 8$.

Let A be a Weil divisor with $-K_X \sim_{\mathbb{Q}} qA$ and n the integer such that $\mathcal{M} \sim_{\mathbb{Q}} nA$. If $\mathrm{Cl}\, X$ is torsion free, we can run the computer search 2.4. We get $q \ne 4$ and $\mathrm{g}(\widehat{X}) \ge 21$. Then by Theorem 1.2.3 we get one of cases (iv)–(vii). Thus from now on we assume that $\mathrm{Cl}\, X$ contains a nontrivial torsion element.

We may assume that $\mathcal{M}$ has no fixed part. If the pair $(X, \mathcal{M})$ is terminal, then X is in (vi) or (vii) by Lemma 3.1.2. Assume that $(X, \mathcal{M})$ is not terminal. Apply Construction 2.1 to $(X, \mathcal{M})$. We can write

$$K_{\widetilde{X}} + 2\widetilde{\mathcal{M}} + \widetilde{\Xi} + a\widetilde{E} \sim f^*(K_X + 2\mathcal{M} + \Xi) \sim 0,$$

where $a \in \mathbb{Z}_{>0}$. Hence,

$$K_{\overline{X}} + 2\overline{\mathcal{M}} + \overline{\Xi} + a\overline{E} \sim 0. \tag{3.4.1}$$

First consider the case of 2.2 where $\overline{f}$ is not birational. In particular, $\widehat{X}$ is either $\mathbb{P}^1$ or a del Pezzo surface as in 2.2.1.

Assume that $\overline{\mathcal{M}}$ is $\overline{f}$-horizontal. Restricting the relation (3.4.1) to a general fiber $\overline{F}$ of $\overline{f}$ we get

$$-K_{\overline{F}} \sim_{\mathbb{Q}} 2\overline{\mathcal{M}}_{|\overline{F}} + \overline{\Xi}_{|\overline{F}} + a\overline{E}_{|\overline{F}}, \tag{3.4.2}$$

where the divisors $\overline{\mathcal{M}}_{|\overline{F}}$ and $\overline{E}_{|\overline{F}}$ are ample. This is possible only if $\overline{F} \simeq \mathbb{P}^2$, $\widehat{X} \simeq \mathbb{P}^1$, $\mathcal{O}_{\overline{F}}(\overline{\mathcal{M}}) \simeq \mathcal{O}_{\overline{F}}(\overline{E}) \simeq \mathcal{O}_{\mathbb{P}^2}(1)$, and $a = 1$. From the exact sequence

$$(3.4.3) \qquad 0 \to \mathcal{O}_{\overline{X}}(\overline{\mathcal{M}} - \overline{F}) \to \mathcal{O}_{\overline{X}}(\overline{\mathcal{M}}) \to \mathcal{O}_{\overline{F}}(\overline{\mathcal{M}}) \to 0$$

we get

$$(3.4.4) \qquad h^0(\mathcal{O}_{\overline{X}}(\overline{\mathcal{M}} - \overline{F})) \ge h^0(\mathcal{O}_{\overline{X}}(\overline{\mathcal{M}})) - h^0(\mathcal{O}_{\overline{F}}(\overline{\mathcal{M}})) \ge 2.$$

Thus $\overline{\mathcal{M}} \ni \overline{F} + \overline{L}$, where $\overline{L} \in |\mathcal{M} - \overline{F}|$ is a mobile divisor. Hence there is a decomposition $-K_X \sim 2F + 2L + \Xi$. In particular, $q \ge 5$ and $F \sim_{\mathbb{Q}} L \sim_{\mathbb{Q}} A$. This implies that $\overline{f}$ has no multiple fibers. So, the group $\operatorname{Cl} \overline{X}$ is torsion free. Since $\mathcal{O}_{\overline{F}}(\overline{E}) \simeq \mathcal{O}_{\mathbb{P}^2}(1)$, the class of $\overline{E}$ is not divisible in $\operatorname{Cl} \overline{X}$. Hence $\operatorname{Cl} X$ is also torsion free, a contradiction.

Therefore, $\overline{\mathcal{M}}$ is $\overline{f}$-vertical. Then $\overline{\mathcal{M}} = \overline{f}^{*}\mathcal{B}$, where $\mathcal{B}$ is a linear system of Weil divisors on $\widehat{X}$ with $\dim \mathcal{B} \ge 4$. We use the notation of 2.2. Let $\overline{G} = \overline{f}^{*}\Theta$. We can write $\mathcal{B} \sim_{\mathbb{Q}} t\Theta$ for some $t \in \mathbb{Z}_{>0}$. Then

$$(3.4.5) \qquad -K_{\overline{X}} \sim_{\mathbb{Q}} 2t\overline{G} + \overline{\Xi} + a\overline{E},$$

so $8 \ge q \ge 2t + 1$ and $t \le 3$. If $\widehat{X} \simeq \mathbb{P}^1$, we obviously have $\dim \mathcal{B} \le 2$. Therefore, $\widehat{X}$ is a surface. Now we use 2.2.1.

If $t = 1$, then $\dim \mathcal{B} \le 2$, a contradiction. Consider the case $t = 2$. Then $\dim \mathcal{B} \ge 4$ only in the case $\widehat{X} \simeq \mathbb{P}^2$. Then $q \ge 5$, $G \sim_{\mathbb{Q}} A$, and $m = 2$. Since $\dim |G| \ge 2$, by Lemma 3.2 we have $X \simeq \mathbb{P}(1^3, 2)$. Consider the case $t = 3$. Then $q \ge 7$ and $G \sim_{\mathbb{Q}} A$. Since $\dim \mathcal{B} \ge 4$, we have either $\widehat{X} \simeq \mathbb{P}^2$, $\widehat{X} \simeq \mathbb{P}(1,1,2)$, or $K_{\widehat{X}}^2 = 1$. In either case $\dim |G| \ge 1$ (recall that if $K_{\widehat{X}}^2 = 1$, we take $\Theta = -K_{\widehat{X}}$). By Lemma 3.3 we get $X \simeq \mathbb{P}(1^2, 2, 3)$.

Now assume that $\overline{f}$ is birational. We have

$$(3.4.6) \qquad -K_{\widehat{X}} \sim 2\widehat{\mathcal{M}} + \widehat{\Xi} + a\widehat{E},$$

where, as usual, we write $\widehat{\Lambda} = \overline{f}_{*}\overline{\Lambda}$ for the birational transform of $\widehat{X}$ of a divisor (or a linear system) $\overline{\Lambda}$ on $\overline{X}$.

Clearly, $\dim \widehat{\mathcal{M}} \ge \dim \mathcal{M}$. If $(\widehat{X}, \widehat{\mathcal{M}})$ is not terminal, we can repeat the procedure 2.1 and continue. Thus we may assume that $(\widehat{X}, \widehat{\mathcal{M}})$ is as in (i)–(vii). In particular, $\operatorname{Cl} \widehat{X}$ is torsion free and $\widehat{\Xi} + a\widehat{E} \sim \widehat{\Theta}$, where $\widehat{\Theta}$ is the ample generator of $\operatorname{Cl} \widehat{X}$. So, $\widehat{\Xi} = 0$, $a = 1$, and $\widehat{E} \sim \widehat{\Theta}$. In particular, the class of $\widetilde{E}$ is a primitive element of $\operatorname{Cl} \widetilde{X} \simeq \mathbb{Z} \oplus \mathbb{Z}$. In this case, $\operatorname{Cl} X$ is also torsion free. Q.E.D.

§4. Proof of Main Theorem 1.3.1

Let X be a $\mathbb{Q}$-Fano 3-fold such that $-K_X \sim 2A$ for a primitive element $A \in \operatorname{Cl} X$. Assume that $\dim |A| \geq 4$ and K_X is not Cartier. We apply Construction 2.1 with $\mathcal{M} := |A|$, $\lambda = 2$ and $\Xi = 0$. By Lemma 3.1.2 the pair $(X, \mathcal{M})$ is not terminal. Hence in the notation of (2.1.3), the discrepancy $a > 0$. On the other hand, a is an integer. Therefore, $a \geq 1$.

Lemma 4.1. *The map* $\overline{f}$ *in* (2.1.2) *is birational.*

Proof. Suppose that $\overline{f}$ is not birational. Let $\overline{F}$ be a general fiber of $\overline{f}$. If $\overline{\mathcal{M}}$ is $\overline{f}$-vertical, then $\overline{\mathcal{M}} = \overline{f}^* \widehat{\mathcal{B}}$, where $\widehat{\mathcal{B}}$ is a linear system on $\widehat{X}$ whose class generates $\operatorname{Cl} \widehat{X} / \operatorname{Tors}$. But then $\dim \overline{\mathcal{M}} = \dim \widehat{\mathcal{B}} \leq 2$ by 2.2.1, contradicting our assumption.

Thus $\overline{\mathcal{M}}$ is $\overline{f}$-horizontal. Then $-K_{\overline{F}} = 2\overline{\mathcal{M}}_{|\overline{F}} + a\overline{E}_{|\overline{F}}$. This implies that $\overline{F} \simeq \mathbb{P}^2$, that is, $\overline{f}$ is generically a $\mathbb{P}^2$-bundle and $\mathcal{O}_{\overline{F}}(\overline{\mathcal{M}}) \simeq \mathcal{O}_{\mathbb{P}^2}(1)$. From the exact sequence

$$0 \to \mathcal{O}_{\overline{X}}(\overline{\mathcal{M}} - \overline{F}) \to \mathcal{O}_{\overline{X}}(\overline{\mathcal{M}}) \to \mathcal{O}_{\overline{F}}(\overline{\mathcal{M}}) \to 0 \tag{4.1.1}$$

we get

$$h^0(\mathcal{O}_{\overline{X}}(\overline{\mathcal{M}} - \overline{F})) \geq 2. \tag{4.1.2}$$

Therefore, $\overline{\mathcal{M}} \ni \overline{F} + \overline{L}$, where $\overline{F}$ and $\overline{L}$ are mobile divisors. This contradicts $\mathrm{q}_{\mathbb{Q}}(X) = 2$. Q.E.D.

4.2. Notation

When $\overline{f}$ is birational, $\widehat{X}$ is a $\mathbb{Q}$-Fano. Recall that we write $\widehat{\Lambda} = \overline{f}_* \overline{\Lambda}$ for the birational transform on $\widehat{X}$ of a divisor (or a linear system) $\overline{\Lambda}$ on $\overline{X}$. We have

$$-K_{\widehat{X}} \sim 2\widehat{\mathcal{M}} + a\widehat{E} \quad \text{with} \quad a > 0, \quad \dim \widehat{\mathcal{M}} \geq 4. \tag{4.2.1}$$

By Proposition 3.4 the class of $\widehat{E}$ is the ample generator of $\operatorname{Cl} \widehat{X} \simeq \mathbb{Z}$, $\widehat{q} = 2n + 1$, and $\widehat{\mathcal{M}} \subset |n\widehat{E}|$. Moreover, $\widehat{X}$ belongs to one of the possibilities listed in Proposition 3.4.

Assume first that $\widehat{q} > 3$. We consider the case $\widehat{q} = 3$ in the next section. We make frequent use of the following easy observation.

Remark 4.2.2. In the notation of 4.2, assume that there is a member $\widehat{M} \in \widehat{\mathcal{M}}$ such that $\widehat{M} = \widehat{L}_1 + \widehat{L}_2$, where $\widehat{L}_1$ and $\widehat{L}_2$ are effective ample Weil divisors. Then either $\operatorname{Supp} \widehat{L}_1 = \widehat{E}$ or $\operatorname{Supp} \widehat{L}_2 = \widehat{E}$.

Indeed, we can write

$$\overline{\mathcal{M}} \sim_{\mathbb{Q}} \overline{L}_1 + \overline{L}_2 + \gamma\overline{F}, \tag{4.2.3}$$

where $\overline{L}_i$ is the birational transform of $\widehat{L}_i$ and $\gamma \geq 0$. Therefore,

$$\mathcal{M} \sim f_*\chi_*^{-1}\overline{\mathcal{M}} \sim_{\mathbb{Q}} f_*\chi_*^{-1}\overline{L}_1 + f_*\chi_*^{-1}\overline{L}_2 + \gamma F. \tag{4.2.4}$$

Since the class of A is a primitive element of $\operatorname{Cl} X$, we have either $f_*\chi_*^{-1}\overline{L}_1 = 0$ or $f_*\chi_*^{-1}\overline{L}_2 = 0$ (and $\gamma = 0$).

4.2.5. Corollary. *Assume that we have* $\dim|n\widehat{E}| = 4$ *in the notation of 4.2. Then for any partition* $n = n_1 + n_2$, $n_i \in \mathbb{Z}$ *either* $\dim|n_1\widehat{E}| \leq 0$ *or* $\dim|n_2\widehat{E}| \leq 0$.

Proof. In this case $\widehat{\mathcal{M}} = |n\widehat{E}|$ is a complete linear system. Hence, one can take $\widehat{L}_i \in |n_i\widehat{E}|$. Q.E.D.

We consider the cases of Proposition 3.4 separately.

4.2.6. Cases (i), (iii) and (v). Then $\dim|n\widehat{E}| = 4$ and n is even. Apply Corollary 4.2.5 with $n_1 = n_2 = n/2$. We get a contradiction because $\dim|n_i\widehat{E}| > 0$.

4.2.7. Case (ii), $\widehat{X} \simeq \mathbb{P}(1,2,3,5)$. Then $n = 5$ and $\dim|n\widehat{E}| = 5$. Thus $\widehat{\mathcal{M}} \subset |5\widehat{E}|$ is a subsystem of codimension ≤ 1. Since $\dim 2\widehat{E}| = 1$, we can take $\widehat{L}_1 \in |2\widehat{E}|$ so that $\widehat{L}_1 \neq 2\widehat{E}$. Since $\dim|3\widehat{E}| = 2$, there exists a one dimensional family of divisors $\widehat{L}_2 \in |3\widehat{E}|$ such that $\widehat{L}_1 + \widehat{L}_2 \in \widehat{\mathcal{M}}$. So we may assume that $\widehat{L}_2 \neq 3\widehat{E}$. By Remark 4.2.2 we get a contradiction.

4.2.8. Case (iv), that is, $\widehat{X} \simeq \mathbb{P}(1^2,2,3)$. Then $n = 3$ and $\dim|n\widehat{E}| = 6$. Thus $\widehat{\mathcal{M}} \subset |3\widehat{E}|$ is a subsystem of codimension ≤ 2. Since $\dim|\widehat{E}| = 1$, —we can take $\widehat{L}_1 \in |\widehat{E}|$ so that $\widehat{L}_1 \neq \widehat{E}$. Since $\dim|2\widehat{E}| = 3$, there exists a one dimensional family of divisors $\widehat{L}_2 \in |2\widehat{E}|$ such that $\widehat{L}_1 + \widehat{L}_2 \in \widehat{\mathcal{M}}$. So we may assume that $\widehat{L}_2 \neq 2\widehat{E}$. By Remark 4.2.2 we get a contradiction.

4.2.9. Case (vi), $\widehat{X} \simeq \mathbb{P}(1^3,2)$. Then $n = 2$ and $\dim|n\widehat{E}| = 6$. Thus $\widehat{\mathcal{M}} \subset |2\widehat{E}|$ is a subsystem of codimension ≤ 2.

Assume that $\overline{f}(\overline{F})$ is a curve. Then

$$K_{\overline{X}} = \overline{f}^* K_{\widehat{X}} + \overline{F} \quad \text{and} \quad \overline{E} = \overline{f}^* \widehat{E} - \gamma\overline{F}.$$

Since any member of $|\widehat{E}|$ is smooth in codimension one, $\gamma \leq 1$. Moreover, since $n\overline{E}$ is not mobile for any n, we have $\gamma > 0$. Hence, $\gamma = 1$. So,

$$K_{\overline{X}} + 5\overline{E} + 4\overline{F} = \overline{f}^*(K_{\widehat{X}} + 5\widehat{E}) \sim 0.$$

This implies that $-K_X$ is divisible by 4, a contradiction.

Hence $\overline{f}(\overline{F}) \in \widehat{X}$ is a point, say $\widehat{P}$. If $\widehat{P} \in \widehat{X}$ is the point of index 2, then $\overline{f}$ is the blowup of the maximal ideal [Kaw96]. In this case $\overline{X}$ has exactly two extremal contractions: $\overline{f}$ and the $\mathbb{P}^1$-bundle induced by the projection $\mathbb{P}(1^3, 2) \dashrightarrow \mathbb{P}^2$. On the other hand, the second contraction must be birational, a contradiction. Hence $\overline{P} \in \widehat{X}$ is a smooth point.

Let $\widehat{\mathcal{L}} := |\widehat{E}|$. Take a general member $\widehat{L}_1 \in \widehat{\mathcal{L}}$. Dimension counting shows that there exists $\widehat{L}_2 \in \widehat{\mathcal{L}}$ such that $\widehat{L}_1 + \widehat{L}_2 \in \widehat{\mathcal{M}}$. If $\widehat{L}_2 \neq \widehat{E}$, we get a contradiction by Remark 4.2.2. Thus $\widehat{L}_2 = \widehat{E}$ for any choice of $\widehat{L}_1 \in \widehat{\mathcal{L}}$. Therefore, $\widehat{E} + \widehat{\mathcal{L}} \subset \widehat{\mathcal{M}}$ and we can write $\overline{\mathcal{M}} \sim_{\mathbb{Q}} \overline{\mathcal{L}} + \overline{E} + \gamma \overline{F}$, where $\gamma \geq 0$. Then

$$0 \sim K_{\overline{X}} + 2\overline{\mathcal{M}} + \overline{E} \sim_{\mathbb{Q}} K_{\overline{X}} + 2\overline{\mathcal{L}} + 3\overline{E} + 2\gamma\overline{F}. \tag{4.2.10}$$

Note that the only base point of $\widehat{\mathcal{L}}$ is the point of index 2. Hence, $\overline{\mathcal{L}} \sim_{\mathbb{Q}} \overline{f}^* \widehat{\mathcal{L}}$. Let $\widehat{\mathcal{L}}' \subset \widehat{\mathcal{L}}$ be the subsystem consisting of elements passing through $\widehat{P}$. Then we can write

$$\overline{\mathcal{L}}' \sim_{\mathbb{Q}} \overline{f}^* \widehat{\mathcal{L}}' - \delta\overline{F} \sim_{\mathbb{Q}} \overline{\mathcal{L}} - \delta\overline{F}, \quad \text{with } \delta > 0. \tag{4.2.11}$$

Therefore,

$$0 \sim_{\mathbb{Q}} K_{\overline{X}} + 2\overline{\mathcal{L}} + 3\overline{E} + 2\gamma\overline{F} \sim_{\mathbb{Q}} K_{\overline{X}} + 2\overline{\mathcal{L}}' + 3\overline{E} + 2(\delta+\gamma)\overline{F}. \tag{4.2.12}$$

This gives us $-K_X \sim_{\mathbb{Q}} 2\mathcal{L}' + 2(\delta+\gamma)F$ which contradicts $\mathrm{q}_{\mathbb{Q}}(X) = 2$.

§5. Conclusion of the proof of Main Theorem 1.3.1

This section considers Case (vii), when $\widehat{X} = Q \subset \mathbb{P}^4$ is a smooth quadric. Then $\widehat{\mathcal{M}} = |\mathcal{O}_Q(1)|$ is a complete linear system, and in particular is base point free. Thus $\overline{\mathcal{M}} \sim_{\mathbb{Q}} \overline{f}^* \widehat{\mathcal{M}}$. We also have $\widehat{E} \in |\mathcal{O}_Q(1)|$ and $\overline{f}(\overline{F}) \subset \widehat{E}$.

Lemma 5.1. $\Gamma := \overline{f}(\overline{F})$ *is a curve.*

Proof. Assume that $\overline{f}(\overline{F})$ is a point. Let $\widehat{\mathcal{M}}' \subset \widehat{\mathcal{M}}$ be the subsystem consisting of elements passing through $\overline{f}(\overline{F})$. Then we can write

$$\overline{\mathcal{M}}' \sim_{\mathbb{Q}} \overline{f}^* \widehat{\mathcal{M}}' - \delta\overline{F} \sim_{\mathbb{Q}} \overline{\mathcal{M}} - \delta\overline{F}, \quad \text{with } \delta > 0. \tag{5.1.1}$$

Therefore,

$$\begin{aligned} 0 &\sim_{\mathbb{Q}} \overline{f}^* (K_{\widehat{X}} + 2\widehat{\mathcal{M}}' + \widehat{E}) \sim_{\mathbb{Q}} \overline{f}^* (K_{\widehat{X}} + 2\widehat{\mathcal{M}} + \widehat{E}) \\ &\sim_{\mathbb{Q}} K_{\overline{X}} + 2\overline{\mathcal{M}} + \overline{E} \sim_{\mathbb{Q}} K_{\overline{X}} + 2\overline{\mathcal{M}}' + \overline{E} + 2\delta\overline{F}. \end{aligned} \tag{5.1.2}$$

This gives us $-K_X \sim_{\mathbb{Q}} 2\mathcal{M}' + 2\delta F$ which contradicts $\mathrm{q}_{\mathbb{Q}}(X) = 2$. Q.E.D.

Lemma 5.2. *$\overline{E} \simeq \mathbb{P}(1,1,2)$, $\mathbb{F}_2$, or $\mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. Clearly, $\widehat{E} \simeq \mathbb{P}(1,1,2)$ or $\mathbb{P}^1 \times \mathbb{P}^1$. In particular, the pair $(\widehat{X}, \widehat{E})$ is plt. Since $K_{\overline{X}} \sim_{\mathbb{Q}} \overline{f}^* K_{\widehat{X}} + \overline{F}$ and $\widehat{E}$ is smooth at the generic point of Γ, we have

$$K_{\overline{X}} + \overline{E} \sim \overline{f}^*(K_{\widehat{X}} + \widehat{E}). \tag{5.2.1}$$

Hence the pair $(\overline{X}, \overline{E})$ is plt and the divisor $K_{\overline{X}} + \overline{E}$ is Cartier. By adjunction, the surface $\overline{E}$ has at worst Du Val singularities. Moreover, $K_{\overline{E}} = \overline{f}^*_{|\overline{E}} K_{\widehat{E}}$, that is, the restriction $\overline{f}_{\overline{E}}$ is either an isomorphism or the minimal resolution of $\widehat{E}$. Q.E.D.

Lemma 5.3. *$-K_{\overline{X}}$ is nef.*

Proof. Recall that by our construction $\overline{X}$ has exactly two extremal rays. Denote them by R_1 and R_2. One of them, say R_1, is generated by nontrivial fibers of $\overline{f}$. Let C be an extremal curve on $\overline{X}$ that generates R_2. Assume that $-K_{\overline{X}}$ is not nef. Then $K_{\overline{X}} \cdot C > 0$ and C must be a flipped curve (because a divisorial contraction must be K-negative in our situation). Since $-K_{\overline{X}} \sim_{\mathbb{Q}} \overline{E} + 2\overline{f}^* \widehat{E}$, we have $\overline{E} \cdot C < 0$. In particular, $C \subset \overline{E}$. Since C is a flipped curve, it cannot be mobile on $\overline{E}$, that is, $\dim |C| = 0$. By Lemma 5.2, the only possibility is that $\overline{E} \simeq \mathbb{F}_2$ and C is the negative section of $\mathbb{F}_2$. But in this case C is contracted by $\overline{f}$ to a point, that is, the class of C lies in R_1, a contradiction. Q.E.D.

Lemma 5.4. *$K_{\overline{X}}$ is not Cartier at some point of $\overline{E}$.*

Proof. By (5.2.1) the divisor $K_{\overline{X}}$ is Cartier outside $\overline{E}$. Assume that $K_{\overline{X}}$ is Cartier near $\overline{E}$. Since $-K_{\overline{X}}$ is nef, the map $\overline{X} \dashrightarrow \widetilde{X}$ is either an isomorphism or a flop. In either case $\widetilde{X}$ has the same type of singularities as $\overline{X}$, that is, $K_{\widetilde{X}}$ is Cartier. By the classification of extremal contractions of Gorenstein terminal 3-folds [Cut88] the divisor $2K_X$ is Cartier. This contradicts the following remark. Q.E.D.

5.4.1. Corollary. *The curve Γ has a singular point that is not a local complete intersection.*

Proof. Indeed otherwise by [KM92, Prop. 4.10.1] the map $\overline{f}$ is the blowup of Γ and $K_{\overline{X}}$ is Cartier. Q.E.D.

5.4.2. Corollary. *$\widehat{E} \simeq \mathbb{P}(1,1,2)$, the curve Γ is not a Cartier divisor on $\widehat{E}$, and Γ is singular at the vertex of $\mathbb{P}(1,1,2)$.*

Lemma 5.5. $\deg\Gamma = 5$.

Proof. Let $\widehat{C} \subset \widehat{E}$ be a general hyperplane section. Since $-K_{\overline{X}}$ is nef,

$$0 \le -K_{\overline{X}} \cdot \overline{C} = -K_{\widehat{X}} \cdot \widehat{C} - (\Gamma \cdot \widehat{C})_{\widehat{E}} = 6 - \deg\Gamma. \tag{5.5.1}$$

Since Γ is not a Cartier divisor on $\widehat{E}$, its degree should be odd. If $\deg\Gamma \neq 5$, then Γ is either a line or a twisted cubic. In particular, it is smooth, a contradiction. Q.E.D.

5.6. Thus $\deg\Gamma = 5$ and Γ is singular. Then Γ can be given, in coordinates u_1, u_2, v for $E \simeq \mathbb{P}(1,1,2)$, by an equation $\gamma = v\alpha_3 + \beta_5$, where $\alpha_3(u_1,u_2)$ and $\beta_5(u_1,u_2)$ are homogeneous polynomial of the indicated degree. Thus P is a triple point of Γ and is its only singularity. Thus Γ is as in Main Example 1.5 and the conclusion of Theorem 1.3.1. By [KM92, Th. 4.9] the extraction $\overline{f}\colon \overline{X} \to Q = \widehat{X}$ of Γ is unique up to isomorphism over Q. Since $\rho(\overline{X}/Q) = 1$, the Sarkisov link (1.5.1) is uniquely determined. This completes the proof of Theorem 1.3.1.

§6. Examples

6.1. Symbolic blowup

This section is closely related to parts of Tom Ducat's thesis [Du15], and we acknowledge his help with our treatment.

Let $\Gamma \subset M$ be a reduced singular curve in a nonsingular 3-fold. The *symbolic blowup* of Γ in M is the relative Proj of the *symbolic algebra*, the graded algebra $\bigoplus_{n\ge 0} \mathcal{I}_\Gamma^{[n]}$, where $\mathcal{I}_\Gamma^{[n]}$ is the nth symbolic power, that is, the ideal in $\mathcal{O}_Q$ of functions vanishing n times at the generic point of Γ. In other words, in the primary decomposition of $\mathcal{I}_\Gamma^n$, ignore the embedded component at singular points P of Γ. (Primary decomposition is built into the computer algebra packages.)

In our case, Γ is a curve contained in a $\frac{1}{2}(1,1)$ orbifold point $P \in E_0 \subset M$ as a Weil divisor, whose class generates the local class group $\mathrm{Cl}_P E_0 \simeq \mathbb{Z}/2$. For simplicity, we treat $\Gamma \subset E_0 \subset M$ as germs around a singular point $P \in \Gamma$ in local analytic coordinates (but see 6.4). Write $\mathbb{C}^2_{\langle u_1,u_2\rangle}$ for the orbifold double cover of $P \in E_0$, and

$$(x_1, x_2, x_3) = (u_1^2, u_1u_2, u_2^2) \tag{6.1.1}$$

for the invariant monomials. Then M has local coordinates x_1, x_2, x_3, with $g = x_1x_3 - x_2^2$ the local equation of E_0, and $\Gamma \subset E_0$ corresponds to an invariant curve $\Gamma : (\gamma = 0) \subset \mathbb{C}^2$, with equation $\gamma = \gamma(u_1,u_2)$ an odd function of the orbinates.

To see $\Gamma \subset M$ in equations, first render into x_i the invariant multiples $u_1\gamma$ and $u_2\gamma$ of γ, say as

$$u_1\gamma = bx_1 - ax_2 = -f_2 \quad \text{and} \quad u_2\gamma = bx_2 - ax_3 = f_1, \tag{6.1.2}$$

where a, b are functions of x_1, x_2, x_3. Taking into account that a and b are in the maximal ideal (x_1, x_2, x_3) (because the curve Γ is singular at P, and not locally planar), and with a little massaging, we can put the generators of $\mathcal{I}_\Gamma$ and the syzygies between them in the determinantal form $\bigwedge^2 M = (f_1, f_2, g)$, where

$$M = \begin{pmatrix} x_1 & x_2 & a_2x_2 + a_3x_3 \\ x_2 & x_3 & b_1x_1 + b_2x_2 \end{pmatrix} \quad \text{and} \quad M \begin{pmatrix} f_1 \\ f_2 \\ g \end{pmatrix} \equiv 0. \tag{6.1.3}$$

In this case, the symbolic algebra needs just one further generator in degree 2, whose restriction to E_0 is the local equation

$$b^2x_1 - 2abx_2 + a^2x_3 \tag{6.1.4}$$

of the Cartier divisor $2\Gamma \subset E_0$. Rather than primary decomposition, we derive this final generator and its relations by unprojection.

Replacing f_1, f_2, g by ξ_1, ξ_2, η in (6.1.3) gives

$$\begin{pmatrix} x_1 & x_2 & a_2x_2 + a_3x_3 \\ x_2 & x_3 & b_1x_1 + b_2x_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \eta \end{pmatrix} = 0. \tag{6.1.5}$$

Equations (6.1.5) define the blowup of the ideal $\mathcal{I}_\Gamma = (f_1, f_2, g)$ as a codimension 2 complete intersection in $M \times \mathbb{P}^2_{\langle \xi_1, \xi_2, \eta \rangle}$, containing the "irrelevant" codimension 3 complete intersection $V(\xi_1, \xi_2, \eta)$. However, M has entries in (x_1, x_2, x_3), so it also contains the codimension 2 complete intersection $V(x_1, x_2, x_3)$ – the blowup of Γ must contain $\mathbb{P}^2$ over the origin (because Γ is not a local complete intersection). We rearrange (6.1.5) as

$$\begin{pmatrix} \xi_1 & \xi_2 + a_2\eta & a_3\eta \\ b_1\eta & \xi_1 + b_2\eta & \xi_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \tag{6.1.6}$$

The unprojection of $V(x_1, x_2, x_3)$ is given by the 4×4 Pfaffians of

$$\begin{pmatrix} \zeta & \xi_1 & \xi_2 + a_2\eta & a_3\eta \\ & b_1\eta & \xi_1 + b_2\eta & \xi_2 \\ & & x_3 & -x_2 \\ & & & x_1 \end{pmatrix}. \tag{6.1.7}$$

Geometrically, this is the blowup of $\mathcal{I}_\Gamma$ followed by the unprojection contracting $\mathbb{P}^2$ over the origin to a point. We can also view it simply as a practical means of writing down the generator h in degree 2 satisfying

$$(x_1, x_2, x_3)h = \bigwedge^2 \begin{pmatrix} f_1 & f_2 + a_2 g & a_3 g \\ b_1 g & f_1 + b_2 g & f_2 \end{pmatrix}, \tag{6.1.8}$$

so that clearly $h \in \mathcal{I}_\Gamma^{[2]}$, without computer algebra. In computational terms, this means that we can modify $f_1^2, f_1 f_2, f_2^2$ modulo $g \cdot \mathcal{I}_\Gamma$ to make them identically divisible by x_3, x_2, x_1 respectively, with (say)

$$f_1 f_2 - b_1 a_3 g^2 = -x_2 h, \tag{6.1.9}$$

where $h_{|E_0}$ is the equation (6.1.4) defining $2\Gamma \subset E_0$.

Proposition 6.2. *The symbolic algebra of $\mathcal{I}_\Gamma$ is generated by ξ_1, ξ_2, η in degree 1 (corresponding to f_1, f_2, g), and ζ in degree 2 (corresponding to h). The ideal of relations is generated by the maximal Pfaffians of* (6.1.7)*.*

Thus the symbolic blowup $M_1 \to M$ of Γ is the codimension 3 Gorenstein subvariety

$$M_1 \subset M \times \mathbb{P}(1, 1, 1, 2)_{\langle \xi_1, \xi_2, \eta, \zeta \rangle} \tag{6.2.1}$$

defined by the Pfaffians of (6.1.7)*. It has the following properties. If Γ is nonsingular at P it is not applicable. If Γ is singular it defines a morphism $M_1 \to M$ which is the ordinary blowup of $\mathcal{I}_\Gamma$ outside the origin. The birational transform $E_1 \subset M_1$ is isomorphic to E_0.*

The fibre of M_1 over the origin is $\mathbb{P}(2, 1)_{\langle \zeta, \eta \rangle}$ passing through P_ζ, which is a $\frac{1}{2}(1, 1, 1)$ orbifold point, and at most one more singular point. If $\operatorname{mult}_P \Gamma \geq 5$ then $P_\eta \in M_1$ has embedding dimension 3, so is not terminal. If $\operatorname{mult}_P \Gamma = 3$ then M_1 is terminal, and in fact:

(1) *M_1 is quasismooth if Γ has 3 distinct tangent branches.*
(2) *M_1 has a cA_1 point if Γ has a double tangent branch.*
(3) *M_1 has a cA_2 point if Γ has a triple tangent branch.*

In the local description, the cA_1 and cA_2 points of M_1 are arbitrary.

Proof. Although the precise statement is somewhat involved, the proof is easy. The generators and relations follow from the hyperplane section principle: indeed, the symbolic algebra restricted to E_0 is just an algebra of $\mathbb{Z}/2$ invariants, generated by $u_1\gamma, u_2\gamma$ and γ^2, and the restriction map is surjective by our choice of generators ξ_1, ξ_2 and ζ.

The birational transform $E_1 \to E_0$ is an isomorphism because the symbolic algebra of the singular (nonplanar) curve $\Gamma \subset M$ maps onto that of the $\mathbb{Q}$-Cartier divisor on $\Gamma \subset E_0$. The analysis of the singularities is straightforward. The cases correspond to the different possibilities for the cubic leading terms in γ coming from

$$b_1 u_1^3 + b_2 u_1^2 u_2 - a_2 u_1 u_2^2 - a_3 u_2^3. \qquad \text{Q.E.D.} \tag{6.2.2}$$

Remark 6.2.3. In the case that Γ has distinct tangent branches, its symbolic blowup can be done as an explicit construction in the nonsingular category that is folklore in the subject: blowing up P gives an exceptional $\Pi = \mathbb{P}^2$ with normal bundle $\mathcal{O}(-1)$. The 3 branches of Γ meet Π at noncollinear points, and blowing up Γ produces a dP_6 with a hexagon formed of the 3 blown up lines on Π together with the 3 lines joining them in Π, that are $(-1,-1)$ curves. Flopping these takes Π into $\Pi' = \mathbb{P}^2$ with normal bundle $\mathcal{O}(-2)$ by a standard quadratic transformation $\Pi \dashrightarrow \Pi'$, and Π' contracts to a $\frac{1}{2}(1,1,1)$ point.

6.3. Two examples

We apply this to contruct two families of $\mathbb{Q}$-Fano 3-folds X and Y of index 2 with $\mathrm{Cl} = \mathbb{Z} \cdot A$, each with a single $\frac{1}{3}(1,2,2)$ orbifold point and invariants

$$\begin{aligned} -K_X &= 2A_X \quad \text{with} \quad A_X^3 = \tfrac{10}{3}, \quad \dim |A_X| = 4, \\ -K_Y &= 2A_Y \quad \text{with} \quad A_Y^3 = \tfrac{7}{3}, \quad \dim |A_Y| = 3. \end{aligned} \tag{6.3.1}$$

Their Hilbert series come from this by the Ice Cream formula of [BRZ]:

$$\begin{aligned} P_{X,A_X}(t) &= \tfrac{1+t+t^2}{(1-t)^4} + \tfrac{t^2}{(1-t)^3(1-t^3)} = \tfrac{1+2t+4t^2+2t^3+t^4}{(1-t)^3(1-t^3)} \\ P_{Y,A_X}(t) &= \tfrac{1+t^2}{(1-t)^4} + \tfrac{t^2}{(1-t)^3(1-t^3)} = \tfrac{1+t+3t^2+t^3+t^4}{(1-t)^3(1-t^3)}. \end{aligned} \tag{6.3.2}$$

Example 6.3.3. Let $E_0 \subset \mathbb{P}^3$ be the ordinary quadratic cone and $\Gamma_7 \subset E_0 \subset \mathbb{P}^3$ a curve of degree 7 that is singular at the node $P \in E_0$, with $\mathrm{mult}_P \Gamma_7 = 3$. The symbolic blowup of Γ_7 defines an extremal extraction $Y_1 \to \mathbb{P}^3$, with the birational transform $E_1 \subset Y_1$ isomorphic to E_0 and to $\mathbb{P}(2,1,1)$.

Write B for the polarizing $\mathcal{O}(1)$ of $\mathbb{P}^3$ and its pullback to Y_1, so that $E_0 \sim 2B$, and let $F \subset Y_1$ be the scroll over Γ_7. Note that $2\Gamma_7 \sim 7B$ in $\mathrm{Pic}\, E_0$, so $2F \sim 7B$ in $\mathrm{Pic}\, E_1$. In $\mathrm{Cl}\, Y_1$ we have

$$2B = E_1 + F \quad \text{and} \quad K_{Y_1} = -4B + F = -2B - E_1. \tag{6.3.4}$$

We give Y_1 the polarizing divisor $A_1 = B + \frac{2}{3}E_1$. Then

$$3A_1 = 3B + 2E_1 = 7B - 2F \tag{6.3.5}$$

so that $3A_1$ is a Cartier divisor restricting to a linearly trivial divisor on E_1, that is, $\mathcal{O}_{E_1}(3A_1) \simeq \mathcal{O}_{E_1}$. Standard use of vanishing gives that $H^0(Y_1, \mathcal{O}_{Y_1}(3A_1)) \twoheadrightarrow H^0(\mathcal{O}_{E_1})$ is surjective, so that $|3A_1|$ is a free linear system, ample outside E_1, so contracts E_1 to a $\frac{1}{3}(2,1,1)$ orbifold point on a 3-fold Y.

Also, (6.3.4) gives

$$K_{Y_1} - \tfrac{1}{3}E_1 = -2B - \tfrac{4}{3}E_1 = -2A_1. \tag{6.3.6}$$

Hence $-K_Y = 2A$, with A an ample Weil divisor on Y. The contraction $Y_1 \to Y$ is the Kawamata blowup of $\frac{1}{3}(2,1,1)$, with discrepancy $\frac{1}{3}E_1$.

Example 6.3.7. The Main Example of Theorem 1.3.1 is almost the same. We start from the nonsingular quadric $Q \subset \mathbb{P}^4$ and the ordinary quadratic cone obtained as the intersection $E_0 = T_{P,Q} \cap Q$ with its tangent hyperplane at a point $P \in Q$. Let $\Gamma_5 \subset E_0$ be a irreducible quintic curve, assumed singular at P (it follows that $\operatorname{mult}_P \Gamma = 3$).

The symbolic blowup $X_1 \to Q$ of Γ_5 has exceptional scroll F, and birational transform $E_1 \simeq E_0$. As before, write $B = \mathcal{O}(1)$ for the polarizing divisor of Q, so that $E_0 \sim B$, and also for its pullback to X_1. Thus in $\operatorname{Cl} X_1$ we have

$$B = E_1 + F \quad \text{and} \quad K_{X_1} = -3B + F = -2B - E_1. \tag{6.3.8}$$

We give X_1 the polarising divisor $A_1 = B + \frac{2}{3}E_1$. Then $3A_1 = 3B + 2E_1 = 5B - 2F$ is a Cartier divisor with $\mathcal{O}_{E_1}(3A_1) \simeq \mathcal{O}_{E_1}$ with surjective restriction $H^0(\mathcal{O}_{X_1}(3A_1)) \twoheadrightarrow H^0(\mathcal{O}_{E_1})$. Thus $|3A_1|$ is a free linear system contracting E_1 to a $\frac{1}{3}(1,2,2)$ point. Now $K_{X_1} = -2B - G1 = -2A_1 + \frac{1}{3}E_1$ so that $-K_X = 2A$ with A an ample Weil divisor, and $X_1 \to X$ is the Kawamata blowup, with discrepancy $\frac{1}{3}E_1$.

6.4. Alternative graded ring constructions

We can treat the examples of 6.3 in graded ring terms. This is how we originally discovered them. Moreover, the algebra is interesting in its own right, and displays features that are possibly typical for index 2 Fano constructions.

The construction of Y is immediate. Its Hilbert series (6.3.2) is

$$\frac{1 - 2t^2 - 3t^3 + 3t^4 + 2t^5 - t^7}{(1-t)^4(1-t^2)^2(1-t^3)}, \tag{6.4.1}$$

indicating the codimension 3 subvariety $Y \subset \mathbb{P}(1^4, 2^2, 3)_{\langle x_{0\dots 3}, y_1, y_2, z\rangle}$ defined by the maximal Pfaffians of a 5×5 matrix of degrees

$$\begin{pmatrix} 3 & 2 & 2 & 2 \\ & 2 & 2 & 2 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}, \quad \text{typically} \quad \begin{pmatrix} z & y_1 & y_2 + a_2 & a_3 \\ & b_1 & y_1 + b_2 & y_2 \\ & & x_3 & -x_2 \\ & & & x_1 \end{pmatrix}, \tag{6.4.2}$$

with a_2, a_3, b_1, b_2 general quadratic forms in x_0, x_1, x_2, x_3. Every Y in this family is given in this way.

One can follow the argument back to see that in this case $\Gamma_7 \subset \mathbb{P}^3$ is defined by $\bigwedge^2 M = 0$ where

$$M = \begin{pmatrix} x_1 & x_2 & a_2 x_2 + a_3 x_3 \\ x_2 & x_3 & b_1 x_1 + b_2 x_2 \end{pmatrix}, \tag{6.4.3}$$

and the y_i in degree 2 are the rational forms solving

$$M \begin{pmatrix} y_1 \\ y_2 \\ 1 \end{pmatrix} = 0. \tag{6.4.4}$$

Plausible though it may seem at first sight, it is a mistake to confuse the codimension 2 variety $\overline{Y}_{4,4} \subset \mathbb{P}(1^4, 2^2)$ defined by these equations with the symbolic blowup Y_1 of Γ_7. The latter is a relative construction over $\mathbb{P}^3$, and is contained in $\mathbb{P}^3 \times \mathbb{P}^2$, so it has the ratios $f_1 : f_2 : g$ as regular functions, where $g = x_1 x_3 - x_2^2$ is the equation of E_0. It is not simply polarized or projectively Gorenstein. As we have seen, Y_1 has just one orbifold point of type $\frac{1}{2}(1,1,1)$.

In contrast, $\overline{Y}_{4,4}$ contains $\mathbb{P}(1,2,2)_{\langle x_0, y_1, y_2\rangle}$ with ideal (x_1, x_2, x_3), and has $\mathbb{P}^1_{\langle y_1, y_2\rangle}$ as $\frac{1}{2}(1,1)$ orbifold locus, so is not terminal. It is clearly obtained from $Y \subset \mathbb{P}(1^4, 2^2, 3)$ by eliminating z. Putting back z is a Type I or Kustin–Miller unprojection, with the Pfaffians of (6.4.2) giving the linear relations for z, so is perfectly valid as a construction of Y. However, the birational relation between Y and $\overline{Y}_{4,4}$ involves first the weighted blowup of the $\frac{1}{3}(1,2,2)$ point with the given weights $(1,2,2)$, not the Kawamata blowup with weights $(2,1,1)$, and this takes us outside the Mori category. A similar thing happens in many other constructions or attempted constructions of index 2 $\mathbb{Q}$-Fanos.

There is a similar narrative for the Main Example X, starting from $\Gamma_5 \subset E_0 \subset Q$. The Hilbert series P_X has the form $\frac{N(t)}{(1-t)^5(1-t^2)^2(1-t^3)}$

with numerator

$$
\begin{aligned}
N(t) = 1 - t^2 - 4t^3 - 4t^4 & \\
+ 4t^4 + 8t^5 + 4t^6 & \\
- 4t^6 - 4t^7 - t^8 + t^{10}. &
\end{aligned}
\tag{6.4.5}
$$

We keep the masked terms $-4t^4 + 4t^4$ to indicates that $R(X, A)$ needs 4 relations in degree 4. In fact, in order to have $\frac{1}{3}(1, 2, 2)_{x_4,y_1,y_2}$ at P_z, there must be 4 relations $zx_i = c_i$ to eliminate $x_0, \dots, x_3$ there.

Now eliminating z projects X to $\overline{X} \subset \mathbb{P}(1^5, 2^2)$ in codimension 3, with the Hilbert series

$$
\frac{1 - t^2 - 4t^3 + 4t^4 + t^5 - t^7}{(1-t)^5(1-t^2)^2},
\tag{6.4.6}
$$

which corresponds to the Pfaffians of a skew 5×5 matrix of degrees

$$
\begin{pmatrix} 1 & 1 & 1 & 2 \\ & 1 & 1 & 2 \\ & & 1 & 2 \\ & & & 2 \end{pmatrix}, \quad \text{typically} \quad \begin{pmatrix} x_0 & x_1 & x_2 & a \\ & x_2 & x_3 & b \\ & & x_4 & -y_1 \\ & & & y_2 \end{pmatrix}.
\tag{6.4.7}
$$

Here we choose coordinates on $\mathbb{P}^4$ with $Q : x_0x_4 - x_1x_3 + x_2^2 = 0$ and $P = (1, 0, \dots, 0)$, making $T_{P,Q} : x_4 = 0$ and $E_0 : x_1x_3 = x_2^2$. Let $\Gamma_5 \subset E_0 \subset \mathbb{P}^3$ be an irreducible curve of degree 5, assumed to be singular at P.

In (6.4.7), a and b are quadratic forms in $x_{0\dots4}$. The conditions that $\overline{X}$ defined by the Pfaffians of (6.4.7) contains $\mathbb{P}(1, 2, 2)_{\langle x_4,y_1,y_2\rangle}$ defined by the ideal $(x_0, \dots, x_3)$ is that a, b do not contain x_4^2. Then the 7 entries in the first two rows of (6.4.7) are in the ideal (x_0, x_1, x_2, x_3), so it is a Jerry$_{12}$. At the same time, the equations of $\Gamma_5 \subset E_0$ take the determinantal format $\bigwedge^2 M$ with M as in (6.1.3).

As before, unprojecting $\mathbb{P}(1, 2, 2)_{\langle x_4,y_1,y_2\rangle} \subset \overline{X}$ is a contruction of X as a Type I unprojection from $\overline{X} \supset \mathbb{P}(1, 2, 2)$, but $\overline{X}$ itself again has a line of $\frac{1}{2}$ points, so is not Mori category.

The "double Jerry" calculations of [T&J, 9.2] gives the unprojection variable z and most of its unprojection equations $zx_i = c_i$. Eliminating the pivot $m_{12} = x_0$ from the Pfaffians of (6.4.7), gives two equations

$$
\begin{pmatrix} x_1 & x_2 & a \\ x_2 & x_3 & b \end{pmatrix} \begin{pmatrix} y_2 \\ y_1 \\ x_4 \end{pmatrix} = 0 \quad \text{with} \quad \begin{aligned} a &= a_2x_2 + a_3x_3, \\ b &= b_1x_1 + b_2x_2, \end{aligned}
\tag{6.4.8}
$$

that we rearrange

$$\text{(6.4.9)} \qquad \begin{pmatrix} b_1x_4 & y_2+b_2x_4 & y_1 \\ y_2 & y_1+a_2x_4 & a_3x_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

From this we assemble a second Jerry_{12} matrix

$$\text{(6.4.10)} \qquad \begin{pmatrix} -z & b_1x_4 & y_2+b_2x_4 & y_1 \\ & y_2 & y_1+a_2x_4 & a_3x_4 \\ & & x_3 & -x_2 \\ & & & x_1 \end{pmatrix},$$

whose maximal Pfaffians provide the equations for x_1z, x_2z, x_3z. The final unprojection equation for x_0z

$$\text{(6.4.11)} \quad -x_0z = (b_1x_1+b_2x_2-a_2x_3)y_1 + a_3(x_3y_2+b_1x_2x_4+b_2x_3x_4).$$

exists by the theory of Kustin–Miller unprojection, but we don't know any smart way of deducing it. It has to be calculated by a laborious primary decomposition or colon ideal calculation, or by writing out the Kustin–Miller complexes.

The Jerry_{12} matrix (6.4.10) defines a codimension 3 subvariety in the family of our second example $Y \subset \mathbb{P}(1^4, 2^2, 3)$ (compare (6.4.2)), but specialized to contain $\mathbb{P}^2_{\langle x_1,x_2,x_3\rangle}$ defined by the codimension 4 ideal (x_4, y_1, y_2, z). This is a third construction of our Main Example X.

To do this from scratch: in Example 6.3.3, suppose that the curve $\Gamma_7 \subset E_0 \subset \mathbb{P}^3$ break up as the plane conic section $(x_0 = 0)$ plus a quintic Γ_5. Blowing up Γ_7 transforms the plane $(x_0 = 0)$ into a copy of $\mathbb{P}^2$ with normal bundle $\mathcal{O}(-1)$, that contracts to a point of the quadric $Q \subset \mathbb{P}^4$, taking E_0 and $\Gamma_5 \subset E_0$ isomorphically into the data for 6.3.7.

6.5. Summary: Three constructions of X

Our Main Example X can be obtained in three different ways

1. (1) The symbolic blowup of $\Gamma_5 \subset E_0 \subset Q$ followed by the contraction of E_1. Viewed from X, this is the Sarkisov link from its $\frac{1}{3}(1,2,2)$ point P of Section 1.5; it is initiated by the Kawamata blowup, that is the $(2,1,1)$ weighted blowup of P.
2. (2) Construct the codimension 3 variety $\overline{X} \subset \mathbb{P}(1^5, 2^2)$ given in the Pfaffian form (6.4.7), containing $\mathbb{P}(1,2,2)$, then unproject this plane. Viewed from X, this starts from the $(1,2,2)$ weighted blowup of P, which introduces a line of $\frac{1}{2}(1,1)$ orbifold points, so takes us out of the Mori category.

(3) Construct the codimension 3 variety $Y' \subset \mathbb{P}(1^4, 2^2, 3)$ as in Example 6.3.3, but specialized to contain $\mathbb{P}^2_{\langle x_1, x_2, x_3 \rangle}$. Its equations are the maximal Pfaffians of the Jerry_{12} matrix (6.4.10). One checks that Y' has 4 ordinary nodes on $\mathbb{P}^2$ as its only singularities for general choices of (a_2, a_3, b_1, b_2), so that it unprojects to a quasismooth X. Viewed from X, this starts from the ordinary blowup of a general point.

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