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On finiteness of B-representations and semi-log canonical abundance

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Dedicated to Professor Shigefumi Mori on his 60th Birthday

Abstract.

We give a new proof of the finiteness of **B**-representations. As a consequence of the finiteness of **B**-representations and Kollár's gluing theory on lc centers, we prove that the (relative) abundance conjecture for slc pairs is implied by the abundance conjecture for log canonical pairs.

§1. Introduction

Throughout this note, the ground field will be the field $\mathbb{C}$ of complex numbers. It is well known that even though the log minimal model program is focused on the study of log pairs (X, Δ) where X is a normal variety, for technical reasons it is often necessary to deal with log pairs (X, Δ) where X is a semi-normal variety. This naturally occurs in proofs by induction on the dimension where, for example, we restrict to the reduced part of the boundary of a dlt pair (X, Δ) (cf. e.g. [Kolláretal92, KMM94, Birkar12, HX13, FG11]) or when we study moduli of pairs, as normal varieties can degenerate to

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non-normal ones (cf. [Kollár13, HX13]). In [Fujino00], O. Fujino first used the **B**-representation to study the semi-log canonical abundance conjecture and proved the conjecture in the 3 dimensional case. Recently, J. Kollár has developed a useful technique for gluing log canonical centers (cf. [Kollár13, Kollár11]) that reduces many questions on semi-normal pairs to questions on their normalizations. An important result used in Kollár's theory and in Fujino's work (cf. [Fujino00]), is the finiteness of **B**-representations, which was first proved by Deligne-Nakamura-Ueno in the klt case, and then generalized to the log canonical case by Fujino-Gongyo (cf. [FG11]). Recall the following.

Definition 1.1. Let (X, Δ) be a projective dlt pair. We define the birational automorphism group $\mathrm{Bir}(X, \Delta)$ of (X, Δ) to be the group of all birational maps g of X such that if we take a common resolution

$$\begin{tikzcd} & Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "g"] & & X \end{tikzcd}$$

then $p^*(K_X + \Delta) = q^*(K_X + \Delta)$. We call the induced homomorphism

$$\rho_m : \mathrm{Bir}(X, \Delta) \to \mathrm{Aut}(H^0(X, \mathcal{O}_X(m(K_X + \Delta))))$$

the **B-representation** of the pair (X, Δ). As far as we know, **B**-birational maps and **B**-representations for general log pairs were first explicitly introduced in [Fujino00].

In this note we first aim to give a new proof of the following result.

Theorem 1. *Given a projective dlt pair (X, Δ) such that $K_X + \Delta$ is a semi-ample $\mathbb{Q}$-divisor. There exists $m \in \mathbb{N}$, such that the image of the* **B***-representation*

$$\rho_M : \mathrm{Bir}(X, \Delta) \to \mathrm{Aut}(H^0(X, \mathcal{O}_X(M(K_X + \Delta))))$$

is finite for any positive integer M divisible by m.

Remark 1.2. Theorem 1 was proven by different methods in [FG11, 1.1]. The argument in the current note was originally contained in [HMX14]. During the preparation of [HMX14] we were informed of [FG11] and we decided to include Theorem 1 in a separated paper. Our proof uses the case when the Kodaira dimension is 0, which was proved by Gongyo (cf. [Gongyo13]) using ideas from [Fujino00].

The second part of our paper is focused on the study of semi-log canonical abundance. One of the main applications of the finiteness of **B**-representations is to prove that abundance for semi-log canonical pairs, follows from abundance for log canonical pairs (cf. [Fujino00, FG11]). Using Kollár's gluing theory, as a consequence of Theorem 1, we prove the following.

Theorem 2. *Let (X, Δ) be a semi-log canonical pair, $f : X \to S$ a projective morphism, $n : \bar{X} \to X$ the normalization and write $n^*(K_X + \Delta) = K_{\bar{X}} + \bar{\Delta} + \bar{D}$, where $\bar{D}$ is the double locus. If $K_{\bar{X}} + \bar{\Delta} + \bar{D}$ is semi-ample over S, then $K_X + \Delta$ is semi-ample over S.*

As a corollary we recover the following result conjectured by C. Birkar (cf. [Birkar12, 1.2]), which is known to be a natural step of the log canonical minimal model program in the relative case (cf. [KMM94, Section 7]).

Corollary 1.3. *Let (X, Δ) be a $\mathbb{Q}$-factorial dlt pair which is projective over a variety S, and $T := \lfloor \Delta \rfloor$ where Δ is a $\mathbb{Q}$-divisor. Suppose that*

(1) *$K_X + \Delta$ is nef over S,*
(2) *$(K_X + \Delta)|_{T_i}$ is semi-ample over S for each component T_i of T,*
(3) *$K_X + \Delta - \epsilon P$ is semi-ample over S for some $\mathbb{Q}$-divisor $P \geq 0$ with* $\mathrm{Supp}(P) = T$ *and for any sufficiently small rational number $\epsilon > 0$.*

Then, $K_X + \Delta$ is semi-ample over S.

Another corollary is the following result, which answers a question raised by J. Kollár in the problem session in MSRI in March 2009.

Corollary 1.4. *Let (X, Δ) be a semi-log canonical pair and $f : X \to S$ a projective morphism, such that $K_X + \Delta \equiv_{\mathbb{Q},S} 0$. Then $K_X + \Delta \sim_{\mathbb{Q},S} 0$.*

Remark 1.5. A result due to O. Fujino and Y. Gongyo (cf. [FG11, 4.13]), implies Theorem 2 when the base S is projective. Recent work of J. Kollár on gluing lc centers provides us with a technique to prove the general case.

The absolute case of Corollary 1.4 or the case when S is projective is also known (cf. [Gongyo13]). However, the general relative case does not seem to be available anywhere in the literature (see the remark in [FG11, 4.16]).

§2. Hodge theoretic construction

Construction 2.1 (cf. [Kollár07a, 8.4.6]). Let (X, Δ) be a log canonical pair, and $f : X \to Y$ a proper surjective morphism of normal varieties with connected fibers such that $n = \dim X - \dim Y$ and $K_X + \Delta \sim_{\mathbb{Q},Y} 0$.

Let $p : W \to X$ be a log resolution of (X, Δ). Write

$$p^*(K_X + \Delta) = K_W + E + F - G,$$

where E and G are integral effective divisors with no common components and $F = \{F\}$. Let a be an integer such that aF is an integral divisor. Denote by $\phi = f \circ p : W \to Y$.

Let $Y^0 \subset Y$ be a smooth open subset such that ϕ is smooth over Y^0. We denote by $\bullet^0$ the base change over Y^0. Replacing Y^0 by an open subset, we may assume that $(K_X + \Delta)|_{X^0} \sim_{\mathbb{Q}} 0$. We define the line bundle $V^0 = \omega^{-1}_{W^0}(G^0 - E^0)$ so that $V^{0\otimes a} \cong \mathcal{O}_{W^0}(aF^0)$. This data defines a local system $\mathbb{V}^0$ on $W^0 \setminus \operatorname{Supp}(E^0 \cup F^0)$ (cf. [Kollár07a, 8.4.6], [EV92, 3.2] and its proof).

Consider the normalization of the corresponding μ_a-cover $\pi : W' \to W^0$, and denote by E' the reduced divisor supported on $\pi^* E^0$. The push-forward $\pi_*(\mathbb{C}|_{W' \setminus E'})$ has a μ_a-action. If we decompose

$$\pi_*(\mathbb{C}|_{W' \setminus E'}) = \bigoplus_i \pi_*(\mathbb{C}|_{W' \setminus E'})^{(i)}$$

into the corresponding eigenspaces, then $\mathbb{V}^0$ is isomorphic to the restriction of $\pi_*(\mathbb{C}|_{W' \setminus E'})^{(1)}$ to $W^0 \setminus (E^0 \cup F^0)$. We denote $\pi_*(\mathbb{C}|_{W' \setminus E'})^{(1)}$ by $\mathbb{V}$. So $\mathbb{V}$ is determined up to the choice of a unit in $\mathcal{O}_{Y^0}$. However, $(R^n \phi_* \mathbb{V})^{\otimes a}$ (we will sloppily denote $\phi|_{W^0 \setminus E^0}$ by ϕ) is a well defined local system on Y^0 (cf. [Kollár07a, 8.4.7]).

Denote by $\phi' : W' \to Y^0$ the composite morphism $\phi \circ \pi$ (and its restriction to open subsets). Then $R^n \phi'_* \mathbb{V}$ is a direct summand of $R^n {\phi'}_* \mathbb{C}|_{W' \setminus E'}$ which carries a variation of mixed Hodge structure. We remark that even though W' has quotient singularities, locally (W', E') is a finite quotient of a log smooth pair and hence Hodge theoretically it behaves as a smooth variety with a simple normal crossing divisor, at least for $\mathbb{Q}$-coefficients (cf. [Steenbrink77, Section 1]).

The local system $R^n \phi_* \mathbb{V}$ gives a variation of mixed Hodge structure on Y^0 (see [FF12, FFS14] and the references therein). It follows from [Steenbrink77, 1.18] that the bottom piece of the Hodge filtration gives an isomorphism

$$F^n R^n {\phi'}_*(\mathbb{C}|_{W' \setminus E'}) \cong {\phi'}_* \omega_{W'/Y}(E').$$

By taking the corresponding eigenspaces, we conclude that $F^n R^n \phi_*(\mathbb{V}|_{W^0 \setminus E^0}) \cong \phi_* \mathcal{O}_{W^0}(G^0)$ is a line bundle over Y^0 (cf. [Kollár07a, 8.4.5(7')]).

Denote this line bundle by L. Since L is of rank 1, there exists an integer i such that, $L \subset W_{n+i}(R^n\phi_*\mathbb{V})$ but $L \not\subset W_{n+i-1}(R^n\phi_*\mathbb{V})$. Let $\mathbb{H}$ be the smallest pure sub-$\mathbb{Q}$-VHS of $\mathrm{Gr}^W_{n+i}(R^n\phi_*\mathbb{V})$ which contains L.

Lemma 2.2. $\mathbb{H}$ *does not depend on the choice of the resolution of* W.

Remark 2.3. Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be two local systems defined over Zariski dense open subsets $U_1, U_2 \subset Y$ such that $\mathbb{H}_1|_{U_1\cap U_2} \cong \mathbb{H}_2|_{U_2 \cap U_2}$. Since Y is normal, then there is a unique local system $\mathbb{H}$ (up to a unique isomorphism) defined over $U_1 \cup U_2$, such that its restriction to U_i is isomorphic to $\mathbb{H}_i$.

Proof of Lemma 2.2. From the above remark, it suffices to verify the lemma for any nonempty open set $Y^0 \subset Y$. For two resolutions W_1, W_2, we can assume that there exists a morphism $\psi : W_2 \to W_1$. By shrinking Y^0, we can also assume that W_1 and W_2 are both smooth over Y^0 (cf. Remark 2.3). If $p_1^*(K_X + \Delta) = K_{W_1} + E_1 + F_1 - G_1$, and aF_1 is integral, then

$$p_2^*(K_X + \Delta) = \psi^*(K_{W_1} + E_1 + F_1 - G_1) = K_{W_2} + E_2 + F_2 - G_2,$$

and aF_2 is easily seen to be integral. Applying the construction 2.1 for both W_i (here we choose the same unit in $\mathcal{O}_{Y^0}$ for W_i), we have that W_2' is the normalization of $W_2^0 \times_{W_1^0} W_1'$ and we denote by $\psi' : W_2' \to W_1'$. Let $E^* = \mathrm{Supp}(\psi'^{-1}E_1')$, then we know that $E_2' \subset E^*$.

$$\begin{tikzcd}
E_2' \subset E^* \subset W_2' \arrow[r] \arrow[d, "\psi'"'] & E_2 \subset W_2 \arrow[d, "\psi"'] \\
E_1' \subset W_1' \arrow[r] & E_1 \subset W_1
\end{tikzcd}$$

Therefore, there exist morphisms between mixed Hodge structures on Y^0

$$i : R^n\phi'_{2*}(\mathbb{C}|_{W_2' \setminus E_2'}) \to R^n\phi'_{2*}(\mathbb{C}|_{W_2' \setminus E^*}) \qquad \text{and}$$

$$j : R^n\phi'_{1*}(\mathbb{C}|_{W_1' \setminus E_1'}) \to R^n\phi'_{2*}(\mathbb{C}|_{W_2' \setminus E^*}).$$

Applying the Hodge filtration $F^n(\cdot)$, it follows from [Steenbrink77, 1.18] that

$$F^n R^n\phi'_{2*}(\mathbb{C}|_{W_2' \setminus E_2'}) \cong \phi'_{2*}\omega_{W_2'/Y}(E_2'),$$

and similarly for the other pairs. Thus we have morphisms

$$F^n i : \phi'_{2*}\omega_{W'_2/Y}(E'_2) \to \phi'_{2*}\omega_{W'_2/Y}(E^*) \qquad \text{and}$$
$$F^n j : \phi'_{1*}\omega_{W'_1/Y}(E'_1) \to \phi'_{2*}\omega_{W'_2/Y}(E^*).$$

which are isomorphisms by Lemma 2.4.

Replacing Y^0 by a smaller open set, by the discussion in 2.1, there are morphisms between mixed Hodge structures,

$$R^n\phi_{1*}\mathbb{V}_1 \to R^n\phi_{2*}(\mathbb{C}|_{W'_2\setminus E^*}) \leftarrow R^n\phi_{2*}\mathbb{V}_2.$$

We conclude that if we take $F^n(\cdot)$ of each term above, then we obtain an induced isomorphism $F^n(R^n\phi_{1*}\mathbb{V}_1) \to F^n(R^n\phi_{2*}\mathbb{V}_2)$. Since polarized-VHSs form a semi-simple category, considering $\mathrm{Gr}^W_{n+i}R^n\phi_{j*}\mathbb{V}_j$, we see that the images of $\mathbb{H}_1$ and $\mathbb{H}_2$ are mapped to the same pure Hodge substructure of $\mathrm{Gr}^W_{n+i}R^n\phi_{2*}(\mathbb{C}|_{W'_2\setminus E^*})$. Q.E.D.

Lemma 2.4. *Let $f : Y_2 \to Y_1$ be a birational morphism between normal projective varieties. Let Δ_1 be reduced divisor on Y_1, such that (Y_1, Δ_1) is log canonical. Let $f_*^{-1}(\Delta_1) \le \Delta_2 \le f_*^{-1}(\Delta_1) + \mathrm{Ex}(f)$ be an effective Weil divisor on Y_2 whose support contains all divisors of discrepancy -1 with respect to the pair (Y_1, Δ_1). Then there is a natural morphism $f_*\mathcal{O}_{Y_2}(K_{Y_2}+\Delta_2) \to \mathcal{O}_{Y_1}(K_{Y_1}+\Delta_1)$ inducing an isomorphism*

$$H^0(Y_2, \mathcal{O}_{Y_2}(K_{Y_2} + \Delta_2)) \cong H^0(Y_1, \mathcal{O}_{Y_1}(K_{Y_1} + \Delta_1)).$$

Proof. By assumption, we can write

$$F + f^*(K_{Y_1} + \Delta_1) = K_{Y_2} + \Delta_2 + E,$$

where $F, E \ge 0$, F is exceptional and $\lfloor E \rfloor = 0$. Therefore,

$$\begin{aligned} H^0(Y_1, \mathcal{O}_{Y_1}(K_{Y_1} + \Delta_1)) &\cong H^0(Y_2, \mathcal{O}_{Y_2}(\lfloor F + f^*(K_{Y_1} + \Delta_1)\rfloor)) \\ &\cong H^0(Y_2, \mathcal{O}_{Y_2}(K_{Y_2} + \Delta_2 + \lfloor E\rfloor)) \cong H^0(Y_2, \mathcal{O}_{Y_2}(K_{Y_2} + \Delta_2)). \end{aligned}$$

Q.E.D.

§3. Finiteness of B-representations

In this section we prove Theorem 1. By assumption, there exists a positive integer m, such that $|m(K_X + \Delta)|$ is base point free and it induces an algebraic fibration structure $f : X \to Y$, so that

$$Y = \mathrm{Proj} \bigoplus_{d\ge 0} H^0(X, \mathcal{O}_X(dm(K_X + \Delta))).$$

It follows from Kawamata's canonical bundle formula (cf. [Kawamata98, Ambro99]) that we can write

$$K_X + \Delta \sim_{\mathbb{Q}} f^*(K_Y + B + J),$$

where B is the boundary part and J is the moduli part. Let $P \in Y$ be a codimension 1 point, then the coefficient of P in B is defined by

$$1 - \operatorname{lct}(X, \Delta; f^{-1}(P))$$

where the log canonical threshold is computed over a neighborhood of the generic point of P. In particular, B is an effective $\mathbb{Q}$-divisor. The moduli part $J = J(X, \Delta)$ is defined as an equivalence class of $\mathbb{Q}$-divisors, coming from Hodge theory.

Remark 3.1. Note that [Kawamata98] defines J only in the case when the restriction $K_F + \Delta_F = (K_X + \Delta)|_F$ to a general fiber F is klt. If $K_F + \Delta_F$ is only dlt, then consider S a minimal stratum of $\lfloor \Delta \rfloor$ that dominates Y. Let $(\pi \circ f_S) : S \to Y_S \to Y$ be the Stein factorization. Restricting to this stratum we obtain a pair $K_S + \Delta_S = (K_X + \Delta)|_S$ which is klt when restricted to a general fiber F_S of $S \to Y_S$. We may then define the boundary and moduli parts $B_S = B(S, \Delta_S)$ and $J_S = J(S, \Delta_S)$ on Z_S by applying [Kawamata98] to (S, Δ_S). We claim that B_S is the pullback of the divisor $B := B(X, \Delta)$ on Y. To see this, note that replacing Y by a big open subset and X by an appropriate birational model, we may assume that for any codimension 1 point $P \in Y$, $(X, \operatorname{Supp}(\Delta + f^{-1}(P)))$ is log smooth. Let b_P be the coefficient of $B(X, \Delta)$ at P, then $(X, \Delta + b_P f^*(P))$ is dlt and there is a divisor E on X dominating P of coefficient 1 in $\Delta + b_P f^*(P)$. By [Kollár13, 4.42], every irreducible component of $\pi^{-1}(P)$ is dominated by a divisor F on S of coefficient 1 in $\Delta_S + f^*(P)|_S = \Delta_S + f_S^*\pi^*(P)$. The claim now follows easily.

Alternatively, one can define $J(X, \Delta)$ directly by using VMHS as in [Kawamata11], [FF12] and [FFS14]. By what we have observed above, these two definitions coincide, that is $J(X, \Delta) = \frac{1}{\deg \pi}\pi_* J(S, \Delta_S)$.

For any $g \in \operatorname{Bir}(X, \Delta)$ and any $d \geq 0$ we have homomorphisms $\rho_{dm} : \operatorname{Bir}(X, \Delta) \to \operatorname{Aut}_{\mathbb{C}} H^0(X, \mathcal{O}_X(dm(K_X + \Delta)))$ where

$$\rho_{dm}(g) = g^* : H^0(X, \mathcal{O}_X(dm(K_X + \Delta))) \to H^0(X, \mathcal{O}_X(dm(K_X + \Delta))).$$

Consider the induced homomorphism

$$\chi : \operatorname{Bir}(X, \Delta) \to \operatorname{Aut}(Y).$$

For any element $g \in \mathrm{Bir}(X,\Delta)$, there is a commutative diagram

Let F be the geometric generic fiber of f and $n = \dim F = \dim X - \dim Y$ be the relative dimension. Then for any d, we have a short exact sequence

$$1 \to G \to \rho_{dm}(\mathrm{Bir}(X,\Delta)) \to \chi(\mathrm{Bir}(X,\Delta)) \to 1,$$

where $G \subset H^0(Y, \mathcal{O}_Y^*) = \mathbb{C}^*$. We know that G is finite (cf. [Gongyo13, 4.9]), so it suffices to show the following result.

Theorem 3. *The image* $\chi(\mathrm{Bir}(X,\Delta)) \subset \mathrm{Aut}(Y)$ *is finite.*

First it is easy to see the following result.

Lemma 3.2. *The image of* $\chi(\mathrm{Bir}(X,\Delta))$ *is contained in* $\mathrm{Aut}(Y,B)$.

Proof. Let $P \in Y$ be a codimension 1 point, as we noted, the coefficient of P in B is defined by $1 - \mathrm{lct}(X,\Delta; f^{-1}(P))$. This number is unchanged if we replace (X,Δ) by any log resolution. Let W be a log resolution such that the following diagram is commutative

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, "g"] & & X. \end{tikzcd}$$

Since $g \in \mathrm{Bir}(X,\Delta)$, we know that we can write $K_W + \Delta_W = p^*(K_X + \Delta) = q^*(K_X + \Delta)$. The coefficient of P in $\chi(g)^*B$ is the same as the coefficient of $\chi(g)_*P$ in B, which is

$$\begin{aligned} 1 - \mathrm{lct}(X,\Delta; f^{-1}(\chi(g)_*P)) &= 1 - \mathrm{lct}(W,\Delta_W; (f\circ q)^{-1}(\chi(g)_*P)) \\ &= 1 - \mathrm{lct}(W,\Delta_W; (\chi(g)\circ f\circ p)^{-1}(\chi(g)_*P)). \end{aligned}$$

Since $\chi(g) \in \mathrm{Aut}(Y)$, the right hand side is the same as $1-\mathrm{lct}(W,\Delta_W;(f\circ p)^{-1}(P))$, which is the coefficient of P in B. Thus $\chi(g) \in \mathrm{Aut}(Y,B)$. Q.E.D.

By the following result, we see that the birational maps also preserve $\mathbb{H}$, where $\mathbb{H}$ is the local system defined in 2.1,

Proposition 3.3. *(1) Let* $g \in \mathrm{Bir}(X,\Delta)$. *If we assume* $\mathbb{H}$ *is defined over an open set* $Y^0 \subset Y$, *then over* $Y^0 \cap \chi(g)^{-1}(Y^0)$, *there exists an isomorphism* $i_g : \chi(g)^*\mathbb{H} \cong \mathbb{H}$.
(2) Let $g_1, g_2 \in \mathrm{Bir}(X,\Delta)$, *then over* $Y^0 \cap \chi(g_1)^{-1}(Y^0) \cap \chi(g_2)^{-1}(Y^0)$ *we have* $i_{g_1} \circ i_{g_2} = i_{g_1\circ g_2}$.

Proof. (1) Let W be a common log resolution,

$$\begin{tikzcd} & W \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \arrow[rr, dashed, "g"] \arrow[d, "f"'] & & X \arrow[d, "f"] \\ Y \arrow[rr, "\chi(g)"] & & Y \end{tikzcd}$$

Since $p_1^*(K_X + \Delta) = p_2^*(K_X + \Delta)$, we obtain the same local system on some open subset $W^0 \subset W$, whose image is an open subset $Y^0 \subset Y$. We can shrink Y^0, so that $R^n\phi_{1*}(\mathbb{V})$ (resp. $R^n\phi_{2*}(\mathbb{V})$) is defined over $\chi(g)^{-1}(Y^0)$ (resp. Y^0), where we denote $f \circ p_i$ by ϕ_i.

Since $p_1^*(K_X+\Delta) = p_2^*(K_X+\Delta)$, we conclude $\chi(g)^*(L) \cong L$. Therefore, because of the semi-simplicity of VHS, the isomorphism

$$\chi(g)^* : R^n\phi_{1*}(\mathbb{V}) \to R^n\phi_{2*}(\mathbb{V})$$

which sends L to L, will send $\mathbb{H}$ to $\mathbb{H}$.

(2) This also follows by a similar argument since $(\chi(g_1 \circ g_2))^*L = L$. Q.E.D.

Since $\chi(\mathrm{Bir}(X, \Delta))$ preserves the polarization on Y, it is a subgroup of $\mathrm{PGL}(N)$ for some N. Let G be the Zariski closure of $\chi(\mathrm{Bir}(X, \Delta))$ in $\mathrm{PGL}(N)$, in particular G is an algebraic group. Thus, to show that $\chi(\mathrm{Bir}(X, \Delta))$ is finite, we only need to verify that G does not contain $\mathbb{G}_a$ or $\mathbb{G}_m$. We will show that G does not contain $\mathbb{G}_m$. (The argument for $\mathbb{G}_a$ is similar and we leave it to the reader.)

Choose $Y_0 \subset Y \setminus B$ to be an open set which the Hodge structure $\mathbb{H}$ does not degenerate, and define an open set

$$\widetilde{Y} = \bigcup_{g \in \chi(\mathrm{Bir}(X,\Delta))} g(Y_0).$$

Therefore, $\widetilde{Y}$ is invariant under the action of G as it is invariant under the Zariski dense subgroup $\chi(\mathrm{Bir}(X, \Delta))$ of G. By Proposition 3.3 and Remark 2.3, $\mathbb{H}$ is non-degenerate over $\widetilde{Y}$.

Then for a general point $z \in \widetilde{Y}$, we consider the closure of the orbit $o : \mathbb{P}^1 \to Y$ of $\mathbb{G}_m \cdot z$. Since $\widetilde{Y}$ is G-invariant, we know that the pull back $o^*\mathbb{H}$ is well defined over $\mathbb{G}_m \subset \mathbb{P}^1$. Let $\pi : Y' \to Y$ be a G-equivariant resolution of $(Y, Y \setminus \widetilde{Y})$, thus $\mathbb{H}' = \pi^*\mathbb{H}$ is defined outside a simple normal crossing divisor (see [Kollár07b]). We can write $\pi^*(K_Y + B + J) = K_{Y'} + B' + J'$ where $(Y', B' + J')$ is the decomposition which the Kawamata subadjunction formula yields on Y'. In particular, (Y', B') is sub log canonical.

Let $\phi : Z \to Y'$ be a branched covering such that $\phi^*\mathbb{H}'$ has unipotent monodromies and hence admits a canonical extension over Z (see [Kollár07a, 8.10.10]). Let $\phi_C : C \to \mathbb{P}^1$ be the normalization of $\mathbb{P}^1 \times_Y Z$. As $o(\mathbb{P}^1)$ is the compactification of a general orbit, we know that ϕ_C is of degree d. Then we know that $\phi_C^*(o^*\mathbb{H})$ has a canonical extension from the preimage of $\phi_C^{-1}(\mathbb{G}_m)$ to C which coincides with the restriction of the canonical extension of $\phi^*\mathbb{H}'$ over C (see [Deligne70], [Kollár07a, 8.10.8]). Therefore, if $J_Z = \bar{E}^{n+i,0}(\phi^*\mathbb{H}')$, then

$$J_C = \bar{E}^{n+i,0}(\phi^*\mathbb{H}'|_C) = \bar{E}^{n+i,0}(\phi^*\mathbb{H}')|_C = J_Z|_C.$$

As $\mathbb{G}_m \cong \mathbb{A}^1 \setminus \{0\} \subset \mathbb{P}^1$. There exists a ramified cover $d : \mathbb{P}^1 \to \mathbb{P}^1$, such that if we denote by $o_d = o \circ d$, then $o_d^*(\mathbb{H})$ is defined over $\mathbb{A}^1 \setminus \{0\}$, and it has unipotent monodromies near 0 and ∞. As $o_d^*(\mathbb{H})$ is non-degenerate on $\mathbb{P}^1 \setminus \{0, \infty\}$, $o_d^*\mathbb{H}$ is trivial by Rigidity Theorem (see [Deligne71, 4.1.2], [Schmid73, 7.24]). Since $\bar{E}^{n+i,0}(o_d^*(\mathbb{H})) = \mathcal{O}_{\mathbb{P}^1}(J_d)$, we have

$$\frac{1}{d} d_* J_d \sim_{\mathbb{Q}} \frac{1}{\deg \phi_C} \phi_{C*} J_C \sim_{\mathbb{Q}} J'|_{\mathbb{P}^1} = 0,$$

where the first equality follows from the fact that the moduli part is well defined for any choice of base change such that the monodromy is unipotent, and since ϕ is finite the second equality follows from restricting both sides of $\frac{1}{\deg \phi}(\phi_* J_Z) \sim_{\mathbb{Q}} J'$ under the lifting $\mathbb{P}^1 \to Y'$ of o.

On the other hand, by Lemma 3.4, we have

$$\deg(o^*(K_Y + B + J)) = \deg((K_{Y'} + B')|_{\mathbb{P}^1}) + \deg(J'|_{\mathbb{P}^1}) \leq 0.$$

Since $K_Y + B + J$ is ample, this is a contradiction and so this completes the proof of Theorem 3.

Lemma 3.4. *Let (Y, B) be a sub log canonical pair and $\psi : \mathbb{G}_m \times (Y, B) \to (Y, B)$ a faithful action. For general $t \in Y$, if we denote by $\psi_t : \mathbb{P}^1 \times \{t\} \to Y$ the closure of the orbit, then* $\deg \psi_t^*(K_Y + B) \leq 0$.

The following lemma is due to Iitaka (cf. [Iitaka81]). Since we could not find an explicit reference, we include a short argument for the reader's convenience.

Proof. Let $\pi : Y' \to Y$ be a $\mathbb{G}_m$-equivariant log resolution of (Y, B) and $B' = \mathrm{Supp}(\pi_*^{-1}(B) + \mathrm{Ex}(\pi))$. Thus, for a general point $t \in Y$, we have that

$$\mathbb{G}_m \cdot t \cap (\mathrm{Supp}(B) \cup \pi(\mathrm{Ex}(\pi))) = \emptyset.$$

We then have the lifting $\phi_s : \mathbb{P}^1 \to Y'$ of ψ_t. Since (Y, B) is log canonical, $\pi^*(K_Y + B) \leq K_{Y'} + B'$. We may assume that there is a quasi-projective smooth variety T with a point $s \in T$ and a generically finite dominating morphism, $\phi_T : \mathbb{P}^1 \times T \to Y'$ such that $\phi_T^{-1}(B') \subset \{0, \infty\} \times T$, and $\phi_T|_{\mathbb{P}^1 \times \{s\}} = \phi_s$. Then by the log pull back formula,we conclude that

$$\phi_T^*(K_{Y'} + B') \leq K_{\mathbb{P}^1 \times T} + \{0\} \times T + \{\infty\} \times T.$$

Therefore, we conclude that

$$\deg \psi_t^*(K_Y + B) \leq \deg \phi_s^*(K_{Y'} + B') \leq \mathbb{P}^1 \cdot (K_{\mathbb{P}^1 \times T} + \{0\} \times T + \{\infty\} \times T),$$

which is computed by $\deg(K_{\mathbb{P}^1} + \{0\} + \{\infty\}) = 0$. Q.E.D.

§4. Abundance for slc pairs

In this section, we will study semi-log canonical abundance. By [Fujino00], it is known that if S is a point, then Theorem 1 implies Theorem 2. By [FG11], Theorem 2 is also known when S is projective. The gluing theory for log canonical centers, developed by J. Kollár (cf. [Kollár12, Kollár13]), provides a very powerful tool to study semi-log canonical varieties and allows us to prove Theorem 2 in full generality.

More specifically, Kollár's recent theory of giving any lc center a source and a spring provides a useful tool for checking that the pro-finite equivalence relation is finite (cf. [Kollár11]). We note that this is not true for general pro-finite equivalence relations. The fact that we are gluing lc centers for relatively projective pairs plays an essential role here.

First, the main theorem for abstract gluing theory is the following.

Theorem 4 ([Kollár13, 9.21])**.** *Let (X, S_*) be an excellent scheme or algebraic space over a field of characteristic 0 with a stratification. Assume that (X, S_*) satisfies the conditions (HN) and (HSN). Let $R \rightrightarrows X$ be a finite, set theoretic, stratified equivalence relation. Then*

(1) *the geometric quotient X/R exists,*
(2) *$\pi : X \to X/R$ is stratifiable and*
(3) *$(X/R, \pi_* S_*)$ also satisfies the conditions (HN) and (HSN).*

We now recall the following notation in [Kollár13, 4.28], which was built on the earlier work of F. Ambro (cf. [Ambro03]) and O. Fujino (cf. [Fujino99]).

Definition 4.1. *We call $f : (X, \Delta) \to Y$ a crepant log structure if*
(1) (X, Δ) *is log canonical,*
(2) f *is projective, surjective, with connected fibers,*
(3) $K_X + \Delta \sim_{\mathbb{Q},f} 0$.
We note that, by taking a dlt modification of (X, Δ) *(cf.* [KK10, 3.1]*), we can assume that* (X, Δ) *is dlt.*

The proof of Theorem 2 needs the stratification we develop in [HX13, Section 3]. More precisely, we let $\bar{g} : \bar{X} \to \bar{Y}$ be the morphism over S induced by the relatively semiample $\mathbb{Q}$-divisor $K_{\bar{X}} + \bar{\Delta} + \bar{D}$ and we consider the pro-finite equivalence relation

$$(\sigma_1, \sigma_2) : \bar{T} \rightrightarrows \bar{Y}$$

given by the image of the pro-finite relation $D^n \rightrightarrows \bar{X}$. Recall that D^n denotes the normalization of $\bar{D}$ on $\bar{X}$. Note that $\bar{T}$ and $\bar{Y}$ are normal. We may assume that $\bar{g}$ is induced by $|m(K_{\bar{X}} + \bar{\Delta} + \bar{D})|$ for some sufficiently divisible positive integer m. If we write $K_{D^n} + \Theta = (K_{\bar{X}} + \bar{\Delta} + \bar{D})|_{D^n}$, then (D^n, Θ) is also log canonical.

As in [HX13, Section 3], we consider the minimal qlc stratifications $S_*\bar{T}$ and $S_*\bar{Y}$ given by the crepant log structures $(D^n, \Theta) \to \bar{T}$ and $(\bar{X}, \bar{\Delta} + \bar{D}) \to \bar{Y}$. We note that if (X^d, Δ^d) is a dlt modification of $(\bar{X}, \bar{\Delta} + \bar{D})$ (cf. [KK10, 3.1]), then

$$(X^d, \Delta^d) \to (\bar{X}, \bar{\Delta} + \bar{D}) \to \bar{Y}$$

gives the same minimal qlc stratification on $\bar{Y}$.

Proposition 4.1. *We have the following facts.*
(1) *The stratifications $S_*\bar{T}$ and $S_*\bar{Y}$ satisfy conditions (HN) and (HSN).*
(2) *The pro-finite relation $(\sigma_1, \sigma_2) : \bar{T} \rightrightarrows \bar{Y}$ is stratified.*

Proof. The first statement follows from [KK10, 5.7] and the second one follows from [HX13, 3.11] which is a consequence of [KK10, 1.7]. Q.E.D.

By Theorem 4, to prove that the quotient $\bar{Y}/\bar{T}$ exists, we only need to verify that $\bar{T} \rightrightarrows \bar{Y}$ generates a finite relation $R \to \bar{Y}$. By [Kollár13, 9.55], it suffices to check this over the generic point of each stratum V^0

of $S_i\bar{Y}$. We work over the generic point of $\bar{f}(V^0)$, where $\bar{f} : \bar{Y} \to S$ is the induced morphism. Let $\eta \in S$ be one such point and $\bar{\eta}$ its algebraic closure. We must verify that the equivalence relation $R \times_S \bar{\eta} \subset \bar{Y} \times \bar{Y} \times_S \bar{\eta}$ is finite over $V^0 \times \bar{\eta}$.

Let Z be one of the minimal lc centers of (X^d, Δ^d) which dominates the closure V of V^0. $(Z, \mathrm{Diff}^*_Z \Delta^d)$ is dlt and its restriction over V^0 is klt. We take the Stein factorization $Z \to \widetilde{V} \to V$. We will need the following results from [Kollár11].

Proposition 4.2 ([Kollár11, 1]). *For different choices of the minimal non-klt centers Z and Z', we have*

(1) $(Z, \mathrm{Diff}^*_Z \Delta^d)$ *is* **B**-*birational to* $(Z', \mathrm{Diff}^*_{Z'} \Delta^d)$ *over* V, *and*
(2) $\widetilde{V}$ *is isomorphic to* $\widetilde{V}'$ *over* V.

Definition 4.2 ([Kollár11, 18]). *We denote by* $\mathrm{Src}(V, X^d, \Delta^d) := (Z, \mathrm{Diff}^*_Z \Delta^d)$ *a source of the lc center* V *and* $\mathrm{Spr}(V, X^d, \Delta^d) := \widetilde{V}$ *its spring. As in Proposition 4.2,* $\mathrm{Src}(V, X^d, \Delta^d)$ *is determined up to* **B**-*equivalence.*

Let V^0_{ij} be an i-dimensional stratum, and $\widetilde{V}^0_{ij}$ the pre-image of V^0_{ij} under the morphism $\widetilde{V}_{ij} \to V_{ij}$. We define $\mathrm{Spr}_i(\bar{Y}, X^d, \Delta^d) = \amalg_j \widetilde{V}^0_{ij}$, where the disjoint union runs over all i-dimensional strata V^0_{ij}. Then the main structural result is the following.

Theorem 4.3 ([Kollár11, 28]). *Let $R \subset \bar{Y} \times \bar{Y}$ be the relation generated by $\bar{T} \rightrightarrows \bar{Y}$ as above. Let $p_i : \mathrm{Spr}_i(\bar{Y}, X^d, \Delta^d) \to S_i\bar{Y}$ be the induced finite morphisms. Let $\bar{\eta}_{ij}$ be the algebraic closure of the generic point of $\bar{f}(V_{ij})$. Then*

$$((p_i \times p_i)^{-1}(R \cap (S_i\bar{Y} \times S_i\bar{Y})) \times_S \bar{\eta}_{ij}) \cap (\widetilde{V}^0_{ij} \times \widetilde{V}^0_{ij} \times_S \bar{\eta}_{ij})$$

is a subset of the graph $\cup_g \Gamma(\chi(g))$ for all $g \in \mathrm{Bir}(Z_{\bar{\eta}_{ij}}, \mathrm{Diff}^*_{Z_{\bar{\eta}_{ij}}} \Delta^d)$.

Proof of Theorem 2. By Theorem 3, we have that $\cup_g \Gamma(\chi(g))$ is finite over $\widetilde{V}^0_{ij} \times \bar{\eta}_{ij}$, and so the hypotheses of Theorem 4 are satisfied. Thus there exists a quotient Y of $\bar{T} \rightrightarrows \bar{Y}$. Following the proof of [HX13, 3.1], we see that there exists a line bundle L on Y whose pull back to X is isomorphic to $\mathcal{O}_X(m(K_X + \Delta))$ for some integer $m > 0$. Then it is easy to see that L is relatively ample over S, which means that $K_X + \Delta$ is relatively semi-ample over S. Q.E.D.

Proof of Corollary 1.3. The argument is the same as in [KMM94, 7.4]. We include it here for the reader's convenience. It follows from Theorem 2 that $m(K_T + \Delta_T) := m(K_X + \Delta)|_T$ is base point free over

S for any sufficiently divisible integer $m > 0$. By assumption (3), the relative base locus of $m(K_X + \Delta)$ is contained in the support of T. We write

$$m(K_X+\Delta)-T=(m-1)(K_X+\Delta-\epsilon P)+K_X+\Delta-(T-(m-1)\epsilon P).$$

Since $K_X + \Delta - \epsilon P$ is semi-ample over S, and $(X, \Delta - (T - (m-1)\epsilon P))$ is klt for $0 < \epsilon \ll 1$, by Kollár's injectivity theorem (cf. [Kollár86]), we have that

$$R^1 f_*\mathcal{O}_X(m(K_X+\Delta)-T) \xrightarrow{T} R^1 f_*\mathcal{O}_X(m(K_X+\Delta))$$

is an injection and hence $f_*\mathcal{O}_X(m(K_X+\Delta)) \to f_*\mathcal{O}_T(m(K_T+\Delta_T))$ is surjective. We have a commutative diagram.

$$\begin{array}{ccc} f^*f_*\mathcal{O}_X(m(K_X+\Delta)) & \longrightarrow & f^*f_*\mathcal{O}_T(m(K_T+\Delta_T)) \\ \downarrow & & \downarrow \\ \mathcal{O}_X(m(K_X+\Delta)) & \longrightarrow & \mathcal{O}_T(m(K_T+\Delta_T)) \end{array}$$

Since the upper arrow and the right arrow are surjective, $m(K_X + \Delta)$ is relatively globally generated along T over S and so $m(K_X + \Delta)$ is relatively globally generated over S. Q.E.D.

Proof of Corollary 1.4. By Theorem 2, we may assume that X is irreducible, i.e., we only need to treat the case that (X, Δ) is log canonical. We may assume that S is normal and dominated by X. By induction, we may assume that $Corollary 1.4$ is known for lower dimensions. Replacing (X, Δ) by a dlt modification, we may assume that (X, Δ) is $\mathbb{Q}$-factorial and dlt. Let $T = \lfloor \Delta \rfloor$. If we write $(K_X + \Delta)|_T = K_T + \Delta_T$, then (T, Δ_T) is a dslt pair.

We run a $(K_X + \Delta - \epsilon T)$-MMP with scaling by an ample divisor H for a small positive integer ϵ. If T has a component which dominates S, then the MMP ends with a Fano contraction, and we can apply the same argument as in [Gongyo13]. Otherwise, all components of T are vertical over S. Since over the generic point of S, we have that $K_X + \Delta$ is klt, then we know that it is $\mathbb{Q}$-linearly equivalent to 0 (cf. [Nakayama04, V, 4.9]).

Therefore, $(X, \Delta - \epsilon T)$ is a klt pair, whose generic fiber has a good model. By, [HX13, 1.1 and 2.9], we conclude that the MMP with scaling terminates with a good model X' of $(X, \Delta - \epsilon T)$ over S. Since X' is indeed a minimal model for all $(X, \Delta - \epsilon' T)$ with $0 < \epsilon' < \epsilon$, which is good again by [HX13, 1.1].

The MMP sequence is $(K_X + \Delta)$-trivial, so (X', Δ') is a lc pair. We let $\mu : (X^d, \Delta^d) \to (X', \Delta')$ be a dlt model. Let T' be the strict transform of T on X', then $(X', \Delta' - \epsilon T')$ is klt and the non-klt locus of (X', Δ') is contained in $\mathrm{Supp}(T')$. Since each μ-exceptional divisor has coefficient 1 in Δ^d, it follows that $\lfloor \Delta^d \rfloor = \mathrm{Supp}(\mu^* T')$. Thus if $T^d = \mu^* T'$, then $K_{X^d} + \Delta^d - \epsilon T^d = \mu^*(K_{X'} + \Delta' - \epsilon T')$ is semiample over S. By induction on the dimension, $K_\Sigma + \Delta_\Sigma$ is semiample over S for all components Σ of $\lfloor \Delta^d \rfloor = \mathrm{Supp}(T^d)$. By Corollary 1.3, $K_{X^d} + \Delta^d$ is semi-ample over S and hence so is $K_{X'} + \Delta'$. Since each step of the MMP is $(K_X + \Delta)$-trivial, this implies that $K_X + \Delta$ is semi-ample over S, i.e., $K_X + \Delta \sim_{\mathbb{Q},S} 0$. Q.E.D.

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