Advanced Studies in Pure Mathematics 70, 2016
Minimal Models and Extremal Rays (Kyoto, 2011)
pp. 277–298

On symplectic hypersurfaces

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§1. Introduction

A symplectic variety is a normal complex variety X with a holomorphic symplectic form ω on the regular part X_{reg} and with rational Gorenstein singularities. Affine symplectic varieties arise in many different ways such as closures of nilpotent orbits of a complex simple Lie algebra, as Slodowy slices to such nilpotent orbits or as symplectic reductions of holomorphic symplectic manifolds with Hamiltonian actions. Many examples of affine symplectic varieties tend to require large embedding codimensions compared to their dimensions.

In this article we treat the rarest case, namely affine symplectic *hypersurfaces*. For technical reasons we also impose the condition that X admit a good $\mathbb{C}^*$-action, i.e. that its affine coordinate ring $A = \mathbb{C}[X]$ be positively graded, $A = \oplus_{i\geq 0} A_i$ with $A_0 = \mathbb{C}$, and that ω is also homogeneous of positive weight s. This condition is satisfied in all examples we know. Finally, such a homogeneous symplectic hypersurface X is called *indecomposable* if the unique fixed point of the $\mathbb{C}^*$-action is a Poisson subscheme of X. As the term indecomposable indicates, such singularities are essential factors of more general hypersurfaces in the sense that every homogeneous hypersurface (X, ω) equivariantly decomposes into a product $W_1 \times ... \times W_k \times X'$, where X' is an indecomposable homogeneous hypersurface and each W_i is isomorphic to $\mathbb{C}^2$ with a standard symplectic form of the same weight s as ω (Lemma 2.5).

Indecomposable homogeneous hypersurfaces $X = \{f = 0\} \subset \mathbb{C}^{2n+1}$ have the remarkable property that the Poisson structure $\{-,-\} : A \times A \to A$ defined on the coordinate ring A by the symplectic structure

Received December 20, 2011.
Revised November 19, 2012.

extends to the ambient space (Lemma 2.7). Consequently, the deformation $X_t = \{f = t\}$ is a Poisson deformation, from which it follows that X admits a crepant resolution (Theorem 2.8).

Since homogeneous symplectic hypersurfaces have no local moduli (cf. [9], Proposition (3.5)), they arise in a discrete way. As is well known, an indecomposable homogeneous hypersurface of dimension 2 is a Kleinian singularity of type A, D or E. In higher dimensions, the classification is an open problem. At this moment we know of a series X_n, $n \geq 2$, of 4-dimensional examples and of a single 6-dimensional example $\hat{X}$. We found them originally as the transversal slices to certain nilpotent orbits in complex simple Lie algebras [6]. In this article, we give several different descriptions of the same hypersurfaces.

Given that these constructions all lead to the same examples it might be natural to ask: *Is every indecomposable homogeneous symplectic hypersurface isomorphic to an ADE surface singularity, one of the 4-dimensional hypersurfaces X_n, or the 6-dimensional hypersurface $\hat{X}$?*

In the final section we look at X_n from the view point of contact geometry. Let $Y \subset \mathbb{P}(2n-1, 2n-1, 2, 2, 2)$ be the 3-dimensional projective variety defined by the same equation as X_n. The symplectic structure on X_n induces a contact structure on the regular part Y^0 of Y with the contact line bundle $\mathcal{O}(2) := \mathcal{O}_{\mathbb{P}}(2)|_{Y^0}$. We construct an explicit birational map between Y and the projectivised cotangent bundle $\mathbb{P}(T^*_{\mathbb{P}^1 \times \mathbb{P}^1})$ so that this contact structure is transformed to the canonical contact structure on $\mathbb{P}(T^*_{\mathbb{P}^1 \times \mathbb{P}^1})$. More exactly, we take a resolution $\mu : \tilde{Y} \to Y$ by blowing up the singular locus of Y and construct a birational contraction map $\nu : \tilde{Y} \to \mathbb{P}(T^*_{\mathbb{P}^1 \times \mathbb{P}^1})$. The pull-back of both contact structures by μ and ν then determine the same contact structure on $\tilde{Y}$ outside some divisor F with $F \subset \mathrm{Exc}(\mu) \cap \mathrm{Exc}(\nu)$. Now this construction tells us that if we start from $\mathbb{P}(T^*_{\mathbb{P}^1 \times \mathbb{P}^1})$, then after a suitable birational modification we can reach a singular contact Fano 3-fold Y. It would be interesting to know if such phenomena occur more generally.

§2. The Poisson matrix

A symplectic variety in the sense of Beauville [1] is a normal complex variety X with a symplectic form ω on the regular part X_{reg} and the property that for some proper resolution of singularities $\pi : X' \to X$ the form $\pi^*\omega$ extends to a regular form on X'. One can show that the same property then holds for any proper resolution. Equivalently, it is sufficient to require that X have rational Gorenstein singularities [7].

A $\mathbb{C}^*$-action on an affine variety $X = \mathrm{Spec}(A)$ is called good if the homogeneous components of the corresponding grading of the coordinate ring $A = \mathbb{C}[X]$ satisfy $A_0 = \mathbb{C}$ and $A_d = 0$ for $d < 0$. In this case we write $\mathfrak{m} := \bigoplus_{d>0} A_d$ for the maximal ideal corresponding to the unique fixed point $O \in X$. Then $\mathfrak{m}/\mathfrak{m}^2$ is a finite dimensional $\mathbb{C}^*$-representation. We may choose homogeneous elements $\bar{x}_1, \ldots, \bar{x}_m \in A$ whose residue classes form a basis of eigenvectors for the action and which therefore generate the ring A. This yields an equivariant embedding $X \to \mathbb{C}^m$ of minimal codimension, with $\mathbb{C}^*$ acting linearly and contracting on $\mathbb{C}^m$.

Definition 2.1. — *A $2n$-dimensional* homogeneous symplectic hypersurface *is a symplectic variety (X, ω) with a good $\mathbb{C}^*$-action $\lambda : \mathbb{C}^* \times X \to X$ such that*

(1) *ω is homogeneous of degree s, i.e. $\lambda(t)^*\omega = t^s\omega$, and*

(2) $\dim T_O X = 2n+1$, *where $O \in X$ is the unique fixed point of X.*

Lemma 2.2. — *Let (X, ω) be homogeneous symplectic hypersurface. Then the degree s of ω is positive.*

Proof. Let $\pi : X' \to X$ be a $\mathbb{C}^*$-equivariant resolution of the singularities of X. The fixed point locus for the induced $\mathbb{C}^*$-action on X' consists of a finite number of smooth projective varieties F_i lying above the origin $0 \in X$. We prove that there is a fixed point q such that the action of $\mathbb{C}^*$ on the cotangent space $T_q^* X'$ has only non-negative weights.

For each fixed point q, we define $T_q^*(X')^{\geq 0}$ to be the subspace of $T_q^* X'$ spanned by eigenvectors with non-negative weights. By Theorem 4.1 of [2], for each F_i there exists a locally closed, smooth and $\mathbb{C}^*$-invariant subvariety X_i' of X' containing F_i such that $T_q^*(X_i') = T_q^*(X')^{\geq 0}$ for all $q \in F_i$. Let $p \in X'$ be a point such that $\pi(p) \neq 0$. Then the closure of the $\mathbb{C}^*$-orbit passing through p is contained in some X_i' by Theorem 4.2 of [2]. This means that there is a locally closed $\mathbb{C}^*$-invariant decomposition of X', $X' = \cup X_i'$. In particular, $\dim X_{i_0}' = \dim X'$ for some i_0. Then $T_q^*(X')^{\geq 0} = T_q^* X'$ for $q \in F_{i_0}$.

Let us take such a fixed point q. Then at least one weight must be positive, as the action on X' is non-trivial. By assumption ω^n extends to a regular $2n$-form ψ of degree ns on X'. At q it can be expressed in terms of local coordinates as $\psi = g(z_1, \ldots, z_{2n}) dz_1 \wedge \ldots \wedge dz_{2n}$, so that $ns = \deg(\psi) \geq \sum_i \deg(z_i) > 0$. Q.E.D.

Every symplectic variety (X, ω) carries a canonical Poisson structure: On the open regular part X_{reg} there is an isomorphism $\omega^{-1} : \Omega_X \to T_X$, and the Poisson bracket is defined by $\{f, g\} := df(\omega^{-1}(dg))$ for $f, g \in \mathcal{O}_X(U)$, $U \subset X_{\mathrm{reg}}$. As X is normal, this bracket can be uniquely extended for any two regular functions on X. If X is affine with coordinate ring A, the Poisson bracket is completely determined by its values on a set $\bar{x}_1, \ldots, \bar{x}_m$ of generators of A. The matrix $\bar{\Theta} \in A^{m\times m}$ with entries

$$\bar{\Theta}_{ij} := \{\bar{x}_i, \bar{x}_j\} \tag{2.1}$$

is skew-symmetric and satisfies the Jacobi identity

$$\sum_m \left(\bar{\Theta}_{im} \frac{\partial \bar{\Theta}_{jk}}{\partial \bar{x}_m} + \bar{\Theta}_{jm} \frac{\partial \bar{\Theta}_{ki}}{\partial \bar{x}_m} + \bar{\Theta}_{km} \frac{\partial \bar{\Theta}_{ij}}{\partial \bar{x}_m} \right) = 0. \tag{2.2}$$

In the following, we will refer to $\bar{\Theta}$ as the Poisson matrix of X. Assume now that (X, ω) is a homogeneous symplectic hypersurface of dimension $2n$ with an equivariant embedding $X \to \mathbb{C}^{2n+1}$ such that the coordinates $x_1, \ldots, x_{2n+1}$ of the ambient space have degree $d_i := \deg(x_i) > 0$ and X is defined by a homogeneous polynomial $f \in \mathbb{C}[x_1, \ldots, x_{2n+1}]$ of degree $d := \deg(f) > 0$. As ω is homogeneous of degree $s > 0$, the Poisson structure is homogeneous of degree $-s$ and

$$\deg(\bar{\Theta}_{ij}) = \deg(x_i) + \deg(x_j) - s = d_i + d_j - s. \tag{2.3}$$

There exists a direct explicit relation between the Poisson matrix of X and its defining equation f, which we will explain next.

Recall that the pfaffian of a skew-symmetric $2n \times 2n$-matrix B is a homogeneous polynomial $\mathrm{pf}(B)$ of the entries of B of degree n such that $\mathrm{pf}(B)^2 = \det(B)$. Explicitly,

$$\mathrm{pf}(B) = \sum_\pi \mathrm{sgn}(\pi) B_{\pi(1)\pi(2)} \cdots B_{\pi(2n-1)\pi(2n)}, \tag{2.4}$$

where π runs through a subset of permutations in S_{2n} such that the collections $T_\pi := \{\{\pi(1), \pi(2)\}, \ldots, \{\pi(2n-1), \pi(2n)\}\}$ represent every decomposition of $\{1, \ldots, 2n\}$ into n unordered pairs exactly once.

If B is a skew-symmetric $(2n+1)\times(2n+1)$-matrix, let $\mathrm{pf}(B)$ denote the vector whose i-th entry is given as $\mathrm{pf}(B)_i = (-1)^{i-1}\,\mathrm{pf}(B^i)$, where B^i is obtained from B by deleting the i-th row and column. It is well-known that $B\,\mathrm{pf}(B) = 0$.

Lemma 2.3. — *Let $X \subset \mathbb{C}^{2n+1}$ be a homogeneous symplectic hypersurface defined by a homogeneous polynomial $f \in \mathbb{C}[x_1, \ldots, x_{2n+1}]$ and let $\bar{\Theta}$ denote its Poisson matrix. Then there is a non-zero constant c such that*

$$\mathrm{pf}(\bar{\Theta}) = c\,\mathrm{grad}(f) \tag{2.5}$$

as vectors with values in $\mathbb{C}[X]$.

Proof. As $f = 0$ in $\mathbb{C}[X]$, it follows that

$$0 = \{x_i, f\} = \sum_j \bar{\Theta}_{ij} \frac{\partial f}{\partial x_j} \in \mathbb{C}[X], \tag{2.6}$$

or briefly: $\bar{\Theta}\,\mathrm{grad}(f) = 0$. On the other hand, $\bar{\Theta}\,\mathrm{pf}(\bar{\Theta}) = 0$. Now over the regular part of X, the derivative $\mathrm{grad}(f)$ vanishes nowhere according to the Jacobian criterion for smoothness. Moreover, $\bar{\Theta}$ has rank $2n$ since X is symplectic so that the kernel of $\bar{\Theta}$ is one-dimensional and at least one of the pfaffians $\mathrm{pf}(\bar{\Theta}^i)$ is non-zero. So $\mathrm{pf}(\bar{\Theta})$ also vanishes nowhere on X_{reg}. As both $\mathrm{grad}(f)$ and $\mathrm{pf}(\bar{\Theta})$ span the kernel of $\bar{\Theta}$, there is an invertible regular function c on X_{reg} such that $\mathrm{pf}(\bar{\Theta}) = c\,\mathrm{grad}(f)$ on X_{reg}. Since X is normal, the function c extends to an invertible regular function on X, and $\mathrm{pf}(\bar{\Theta}) = c\,\mathrm{grad}(f)$ holds everywhere on X. As c is homogeneous of some weight, it must be constant. Q.E.D.

Replacing f by some scalar multiple, we can and will assume from now on that for every homogeneous symplectic hypersurface the following fundamental relation between the defining equation and the Poisson matrix holds:

$$\boxed{\mathrm{pf}(\bar{\Theta}) = \mathrm{grad}(f)} \tag{2.7}$$

Definition 2.4. — *A homogeneous symplectic hypersurface X is* indecomposable *if its unique fixed point is a Poisson subscheme of X.*

Using the previous notations this is equivalent to saying that the homogeneous maximal ideal $\mathfrak{m}$ satisfies $\{\mathfrak{m}, A\} \subset \mathfrak{m}$ which in turn is equivalent to the condition that $\bar{\Theta}_{ij} \in \mathfrak{m}$ for all i, j.

The following decomposition lemma is due to Weinstein [10] if the underlying variety is smooth. For singular Poisson varieties an analogous statement in the formal category has been proved by Kaledin. In the weighted homogeneous situation the argument of Weinstein extends easily. In fact, the proof is easier than both in Weinstein's and Kaledin's situation as the choice of new coordinates can be carried out in finitely many steps.

Lemma 2.5. — *Let (X, ω) be a homogeneous symplectic hypersurface. Then there is an equivariant symplectic isomorphism $X \cong W_1 \times \ldots \times W_k \times X'$, where X' is an indecomposable homogeneous symplectic hypersurface and each W_i is isomorphic to $\mathbb{C}^2$ with symplectic form $dz_1 \wedge dz_2$ and homogeneous coordinates with* $\deg(z_1) + \deg(z_2) = s$.

Proof. Let $X \to \mathbb{C}^{2n+1}$ be a homogeneous embedding with linear coordinates x_i of degree $d_i > 0$, and let $\bar{\Theta}$ denote the corresponding

Poisson matrix. If $\bar{\Theta}_{ij} \in \mathfrak{m}$ for all index pairs, X is indecomposable, and we are done. Otherwise there are indices i, j such that $\bar{\Theta}_{ij}$ is a non-zero constant, and after an appropriate linear coordinate change, we may assume that $\bar{\Theta}_{12} = 1$.

For every $i > 1$ we may expand $\bar{\Theta}_{1i} = \sum_m x_2^m u_m$ as a polynomial in x_2 and put $\tilde{x}_i := x_i - \sum_m x_2^m a_m$ with $a_0 = 0$. Here the coefficients a_m are polynomials in the coordinates $x_1, x_3, \ldots, x_{2n+1}$ that have to be chosen in such a way so as to give

$$(2.8) \qquad 0 \stackrel{!}{=} \{x_1, \tilde{x}_i\} = \sum_m x_2^m (u_m - \{x_1, a_m\} - (m+1)a_{m+1}).$$

Thus we may set recursively $a_{m+1} = \frac{1}{m+1}(u_m - \{x_1, a_m\})$. As $\deg(a_m)$ is strictly decreasing for $m = 1, 2, \ldots$, all sums are in fact finite.

Hence, after renaming our variables we may assume that $\{x_1, x_i\} = 0$ for all $i \neq 2$. In a similar way, we may now consider the expansion $\{x_2, x_i\} = \sum_m v_m x_1^m$ and new coordinates $\tilde{x}_i = x_i - \sum_m a_m x_1^m$ with recursively defined polynomials a_m. In order that the new coordinate change should not destroy the just achieved orthogonality property $\{x_1, x_i\} = 0$ for $i > 2$, it is important to note that the coefficients v_m do not contain positive powers of x_2. Indeed this is a consequence of the Jacobi identity:

$$(2.9) \qquad \{x_1, \{x_2, x_i\}\} = \{\{x_1, x_2\}, x_i\} + \{x_2, \{x_1, x_i\}\} = \{1, x_i\} + \{x_2, 0\} = 0,$$

so that

$$(2.10) \qquad 0 = \{x_1, \sum_m v_m x_1^m\} = \sum_m \{x_1, v_m\} x_1^m = \sum_m \frac{\partial v_m}{\partial x_2} x_1^m.$$

Hence repeating the argument of the first step and renaming the variables we arrive at a set of coordinates satisfying $\{x_1, x_2\} = 1$ and $\{x_i, x_j\} = 0$ for $i \leq 2 < j$.

Let $\bar{\Theta}_{ij}$ be the Poisson matrix with respect to this new set of homogeneous generators so that $\bar{\Theta}_{ij} = 0$ if $i \leq 1$ and $j \geq 2$ and $\bar{\Theta}_{12} = 1$. It follows from the Jacobi identity that

$$(2.11) \qquad \frac{\partial \bar{\Theta}_{ij}}{\partial \bar{x}_1} = -\{x_2, \bar{\Theta}_{ij}\} = 0$$

and analogously that $\frac{\partial \bar{\Theta}_{ij}}{\partial \bar{x}_2} = 0$. This implies that $\bar{\Theta}_{ij} \in \mathbb{C}[\bar{x}_3, \ldots, \bar{x}_{2n+1}]$. Similarly, $\bar{f} = 0$ implies $\frac{\partial f}{\partial x_1} = -\{x_2, f\} = 0$ and so on, so that $f \in \mathbb{C}[x_3, \ldots, x_{2n+1}]$. This shows that there is a graded Poisson isomorphism

$A \cong \mathbb{C}[x_1, x_2] \otimes A'$ with $A' = \mathbb{C}[x_3, \dots, x_{2n+1}]/(f)$ where the symplectic form on the first factor is $dx_1 \wedge dx_2$ and where $\deg(x_1) + \deg(x_2) = s$.

The assertion follows by induction on the dimension of X. Q.E.D.

Lemma 2.6. — *Let $X \subset \mathbb{C}^{2n+1}$ be an indecomposable homogeneous symplectic hypersurface defined by a polynomial $f \in \mathbb{C}[x_1, \dots, x_{2n+1}]$.*

(1) *$f \in \mathfrak{n}^{n+1}$ where $\mathfrak{n} = (x_1, \dots, x_{2n+1})$.*

(2) *All partial derivatives $\partial f/\partial x_i$ are non-zero polynomials.*

Proof. 1. As X is indecomposable, all entries of the Poisson matrix are contained in the maximal ideal $\mathfrak{m} = (x_1, \dots, x_{2n+1}) \subset \mathbb{C}[X]$. Hence all coefficients of its pfaffian are contained in $\mathfrak{n}^m$ as each summand of $\mathrm{pf}(\bar{\Theta})_i$ is the product of n entries of the Poisson matrix. The assertion now follows from identity (2.7).

2. Consider the stratification $X = X_0 \supset X_1 \supset X_2 \supset \dots$, where X_{m+1} is the singular part of X_m with its reduced subscheme structure. Kaledin has shown that each X_m is a Poisson subscheme of X, and that the canonically induced Poisson structure on its normalisation $\tilde{X}_m \to X_m$ turns $\tilde{X}_m$ into a symplectic variety. In particular, all X_m are even-dimensional (possibly reducible) varieties. Let X_k denote the last non-empty piece of the stratification. It is a smooth symplectic variety and contains the origin as a Poisson subscheme. According to Kaledin [4], Lemma 1.4 and Theorem 2.5, this is impossible unless $X_k = \{O\}$. Now if $\partial f/\partial x_i$ were identically zero for some index i, i.e. if f were independent of x_i, every stratum X_m, including X_k would contain the line given by $x_j = 0$ for all $j \neq i$, a contradiction. Q.E.D.

Lemma 2.7. — *Let $X \subset \mathbb{C}^{2n+1}$ be an indecomposable homogeneous symplectic hypersurface. Then the Poisson structure on X can be uniquely extended to a homogeneous Poisson structure on the ambient space $\mathbb{C}^{2n+1}$. In particular, if Θ denotes the matrix $\Theta_{ij} = \{x_i, x_j\}$, where $x_1, \dots, x_{2n+1}$ are linear homogeneous coordinates on $\mathbb{C}^{2n+1}$, then, possibly after rescaling it, the defining equation f of X satisfies* $\mathrm{grad}(f) = \mathrm{pf}(\Theta)$.

Proof. The natural epimorphism $\mathbb{C}[x_1, \dots, x_{2n+1}] \to \mathbb{C}[X]$ is an isomorphism in all degrees less than $d = \deg(f)$. Thus the Poisson matrix $\bar{\Theta}$ of X can be uniquely lifted to a skew-symmetric matrix Θ with values in the polynomial ring if the degree condition $\deg(\bar{\Theta}_{ij}) = d_i + d_j - s < d$ is satisfied. And the bracket defined by $\{g, h\} := \sum_{ij} \Theta_{ij} \frac{\partial g}{\partial x_i} \frac{\partial h}{\partial x_j}$ will automatically satisfy the Jacobi-identity provided that all summands in equation (2.2) have degree $< d$. Hence it suffices to show that

$$d_i + d_j - s < d \qquad \text{and} \qquad d_i + d_j + d_k - 2s < d \tag{2.12}$$

for all pairwise distinct indices i, j, k.

For any finite subset $I \subset K := \{1, \ldots, 2n+1\}$ with an odd number of elements, let $\bar{\Theta}^I$ denote the skew-symmetric matrix obtained from $\bar{\Theta}$ by elimination of the i-th row and column for all $i \in I$. Every monomial that appears in the pfaffian $\mathrm{pf}(\bar{\Theta}^I)$ is of the form $\pm\bar{\Theta}_{i_1 i_2}\bar{\Theta}_{i_3 i_4}\cdots\bar{\Theta}_{i_{\ell-1} i_\ell}$ where $\{i_1, \ldots, i_\ell\} = K \setminus I$. Thus if $\mathrm{pf}(\bar{\Theta}^I) \neq 0$, then

$$\deg(\mathrm{pf}(\bar{\Theta}^I)) = \sum_{i \notin I} d_i - \frac{1}{2}(2n + 1 - |I|)s. \tag{2.13}$$

We apply this observation to submatrices of the form $\bar{\Theta}^i$, $\bar{\Theta}^{ijk}$ and $\bar{\Theta}^{ijkpq}$. For brevity, let $\delta = \sum_i d_i$. From the connection between the derivatives of f and $\bar{\Theta}$ we conclude that

$$d - d_i = \deg\left(\frac{\partial f}{\partial x_i}\right) = \deg(\mathrm{pf}(\bar{\Theta}^i)) = \delta - d_i - ns. \tag{2.14}$$

Hence $\delta = d + ns$.

If $n = 1$, one has $d = d_1 + d_2 + d_3 - s$, and hence $d_i + d_j - s = d - d_k < d$ and $d_i + d_j + d_k - 2s = d - s < d$ for $\{i, j, k\} = \{1, 2, 3\}$. Assume $n \geq 2$ for the rest of the proof.

Let i, j be distinct indices and assume that $\mathrm{pf}(\bar{\Theta}^{ijk}) \neq 0$ for some $k \in K \setminus \{i, j\}$. Then

$$0 \leq \deg(\mathrm{pf}(\bar{\Theta}^{ijk})) = \delta - d_i - d_j - d_k - (n-1)s = (d - d_k) - (d_i + d_j - s),$$

so that $d_i + d_j - s \leq d - d_k < d$. If on the other hand we had $\mathrm{pf}(\bar{\Theta}^{ijk}) = 0$ for all k, then $\bar{\Theta}^{ij}$ would have rank $\leq 2n - 4$, and hence $\mathrm{rk}(\bar{\Theta}^i) \leq 2n - 2$, so that $\mathrm{pf}(\bar{\Theta}^i) = 0$ contradicting the fact that $\partial f/\partial x_i \neq 0$ by Lemma 2.6.

Let i, j, k be distinct indices. If $n = 2$ and $\{1, 2, 3, 4, 5\} \setminus \{i, j, k\} = \{p, q\}$, then $d_i + d_j + d_k - 2s = d - d_p - d_q < d$. Hence assume that $n \geq 3$. Suppose that $\mathrm{pf}(\bar{\Theta}^{ijkpq}) \neq 0$ for some pair of indices $p, q \in K \setminus \{i, j, k\}$. Then

$$\begin{aligned} 0 \ \leq \ \deg(\mathrm{pf}(\bar{\Theta}^{ijkpq})) \quad &= \quad \delta - d_i - d_j - d_k - d_p - d_q - (n-2)s \\ &= \quad (d - d_p - d_q) - (d_i + d_j + d_k - 2s), \end{aligned}$$

so that $d_i + d_j + d_k - 2s \leq d - d_p - d_q < d$. If on the other hand we had $\mathrm{pf}(\bar{\Theta}^{ijkpq}) = 0$ for all p, q, then $\bar{\Theta}^{ijk}$ would have rank $\leq 2n - 6$, and hence $\mathrm{rk}\,\Theta^i \leq 2n - 2$, leading to the same contradiction as before. Q.E.D.

Theorem 2.8. — *Let $X \subset \mathbb{C}^{2n+1}$ be an indecomposable homogeneous symplectic hypersurface. Then X admits a crepant resolution.*

Proof. The equation of X defines a flat deformation $f : \mathbb{C}^{2n+1} \to \mathbb{C}$. By Lemma 2.7, the Poisson structure on X uniquely extends to a homogeneous Poisson structure on the polynomial ring, and since $\{x_i, f\} = \sum_j \Theta_{ij} \partial f / \partial x_j = 0$, the deformation is in fact a Poisson deformation. For any $t \neq 0$, the fibre $f^{-1}(t)$ is smooth. Hence it follows from Corollary 5.6 in [8] that X admits a crepant resolution. Q.E.D.

§3. Examples

The following indecomposable symplectic hypersurfaces are known to us:

(1) ADE-surface singularities. These come in two series A_n and D_n and three exceptional examples E_6, E_7 and E_8.
(2) A series of four-dimensional hypersurfaces X_n, $n \geq 2$, with equations $f_n = a^2x + 2aby + b^2z + (xz - y^2)^n \in \mathbb{C}[a, b, x, y, z]$.
(3) A single six-dimensional example $\hat{X}$.

If we search for higher-dimensional symplectic hypersurfaces, relation (2.7) suggests to start from a skew-symmetric $(2n + 1) \times (2n + 1)$-matrix Θ with values in the polynomial ring $\mathbb{C}[x_1, \ldots, x_{2n+1}]$. It is then easy to reconstruct the polynomial f from the pfaffian minors of Θ. Of course, this puts quite strong differential conditions on Θ: It must satisfy the Jacobi identity (2.2), and its pfaffian minors must satisfy the Schwarz integrability conditions

$$(-1)^{i-1} \frac{\partial \operatorname{pf}(\Theta^i)}{\partial x_j} = (-1)^{j-1} \frac{\partial \operatorname{pf}(\Theta^j)}{\partial x_i}. \tag{3.1}$$

And finally one has to check that $X = \{f = 0\}$ is indeed symplectic.

Conversely, if $f \in A = \mathbb{C}[x_1, \ldots, x_{2n+1}]$ defines a symplectic hypersurface $X = \{f = 0\} \subset \mathbb{C}^{2n+1}$, the Poisson matrix is determined as the middle part of a skew-symmetric minimal resolution of the Jacobian ideal J:

$$0 \longrightarrow A \xrightarrow{df} A^{\oplus 2n+1} \xrightarrow{\Theta} A^{\oplus 2n+1} \xrightarrow{df} J \tag{3.2}$$

3.1. Two-dimensional examples

Two-dimensional symplectic surface singularities are classical and well studied mathematical objects ever since Klein discussed the invariants of finite subgroups $G \subset \mathrm{SL}_2(\mathbb{C})$ and computed the equation of the embedding $\mathbb{C}^2/G \subset \mathbb{C}^3$. For a two-dimensional symplectic hypersurface $X = \{f = 0\} \subset \mathbb{C}^3$, relation (2.7) is equivalent to saying that

$$\{x_i, x_j\} = \sum_k \varepsilon_{ijk} \frac{\partial f}{\partial x_k}. \tag{3.3}$$

Here ε_{ijk} denotes the totally skew-symmetric tensor that equals the sign of the permutation $(1,2,3) \mapsto (i,j,k)$ if i,j and k are pairwise distinct and 0 else. The corresponding symplectic form is obtained as the residue

$$\omega = \mathrm{res}_f \frac{dx_1 \wedge dx_2 \wedge dx_3}{f}. \tag{3.4}$$

Note that any choice of a homogeneous polynomial f defines a Poisson structure. But X will be symplectic if and only if it is isomorphic to one of the quotient singularities $\mathbb{C}^2/G$ in the following list:

group G	type	equation f
cyclic C_n	A_{n-1}	$x^2 + y^2 + z^n$
binary dihedral D_n^*	D_{n+2}	$x^2 + y^2 z + z^{n+1}$
binary tetrahedral $\mathbb{T}^*$	E_6	$x^2 + y^3 + z^4$
binary octahedral $\mathbb{O}^*$	E_7	$x^2 + y^3 + yz^3$
binary icosahedral $\mathbb{I}^*$	E_8	$x^2 + y^3 + z^5$

3.2. Four-dimensional examples

We know three constructions to obtain the hypersurfaces X_n:

The first construction establishes X_n as the transversal slice to the orbit of certain nilpotent elements x in a simple Lie algebra $\mathfrak{g}$. We only sketch the construction and refer to [6] for details. By the theorem of Jacobson-Morosov, one may choose elements h, y such that the map $\mathfrak{sl}_2 \to \mathfrak{g}$, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \mapsto x$, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \mapsto h$, $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \mapsto y$, defines a Lie algebra homomorphism. The so-called Slodowy slice $S := x + \ker(\mathrm{ad}\, y)$ intersects the orbit of x for the adjoint action transversely. Let $N \subset \mathfrak{g}$ denote the cone of nilpotent elements. Then $S_0 := S \cap N$ is a symplectic variety. If $\mathfrak{g} = \mathfrak{sp}_{2n}$ is the Lie algebra of type C_n and x is a nilpotent element of Jordan type $[2n-2, 1, 1]$, then S_0 is isomorphic to the hypersurface X_n defined by the vanishing of $f_n := a^2 x + 2aby + b^2 z + (xz - y^2)^n$.

The second construction is based on the following ansatz: Let V denote an even-dimensional representation of the Lie algebra $\mathfrak{sl}_2$. A Poisson bracket on the symmetric algebra $A = S^*(\mathfrak{sl}_2 \oplus V)$ is determined by its value on pairs of vectors in $\mathfrak{sl}_2 \oplus V$: it then extends uniquely to A by its biderivative properties. We put $\{x, x'\} := [x, x']$ and $\{x, v\} := x.v$ for $x, x' \in \mathfrak{sl}_2$ and $v \in V$ using the Lie bracket on $\mathfrak{sl}_2$ and the action of $\mathfrak{sl}_2$ on V. It remains to choose a skew-symmetric map $\varphi := \{-,-\}|_{\Lambda^2 V}$:

$\Lambda^2 V \to A$ which we assume to take values in the subring $S^*(\mathfrak{sl}_2)$. The Jacobi relation can be thought of as a homomorphism $J : \Lambda^3(\mathfrak{sl}_2 \oplus V) \to A$. Its restriction to $\Lambda^3(\mathfrak{sl}_2) \oplus \Lambda^2(\mathfrak{sl}_2) \otimes V$ vanishes since $[-,-]$ is a Lie bracket and V is a representation. The vanishing of $J|_{\mathfrak{sl}_2 \otimes \Lambda^2 V}$ forces φ to be equivariant. So it remains to verify that $J|_{\Lambda^3 V}$ vanishes.

Assume now that $V = \mathbb{C}^2$ is the two-dimensional standard representation. As $\Lambda^3 V = 0$, the Jacobi condition is automatically satisfied for any equivariant homomorphism $\varphi : \Lambda^2 V = \mathbb{C} \to S^*(\mathfrak{sl}_2)$. So φ has to be a homogeneous element in the invariant subring $S^*(\mathfrak{sl}_2)^{\mathfrak{sl}_2}$. As is well-known, this subring is freely generated by the Casismir element Δ. Explicitly, one obtains in terms of a standard basis x, h, y of $\mathfrak{sl}_2$ and a basis e_0, e_1 of $\mathbb{C}^2$ the following Poisson matrices

$$\Theta_n = \begin{pmatrix} 0 & -2x & h & 0 & e_0 \\ 2x & 0 & -2y & e_0 & -e_1 \\ -h & 2y & 0 & e_1 & 0 \\ 0 & -e_0 & -e_1 & 0 & 2n\Delta^{n-1} \\ -e_0 & e_1 & 0 & -2n\Delta^{n-1} & 0 \end{pmatrix}, \tag{3.5}$$

where $\Delta = h^2 + 4xy$. Integrating the pfaffian vector $df_n = c_n \operatorname{pf}(\Theta_n)$ yields the expression

$$f_n = -ye_0^2 + he_0e_1 + xe_1^2 + \Delta^n. \tag{3.6}$$

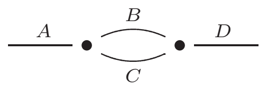
Up to a rescaling of the coordinates this is the same equation as in the first construction. The weights of the coordinates in this case are $\deg(x) = \deg(h) = \deg(y) = 2$ and $\deg(e_0) = \deg(e_1) = 2n - 1$.

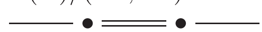
The third construction is due to Hanany and Mekareeya [3]. Let Γ denote a unitrivalent graph. This means that Γ is an undirected graph, possibly with loops, such that each vertex is the end point of exactly one or three edges. Here loops are counted twice. Attaching to each edge e a two-dimensional symplectic vector space V_e and to each inner vertex i the 8-dimensional $W_i = \bigotimes_{e\to i} V_e$, where the tensor product is taken over the three edges that end in i, we may form the symplectic vector space $W(\Gamma) := \bigoplus_i W_i$, where i runs through the set of inner vertices. The group $G(\Gamma) := \prod_e \mathrm{SL}(V_e)$, where e runs through the set of inner edges, acts on $W(\Gamma)$ preserving the symplectic form. Let $X(\Gamma) := W(\Gamma) /\!/\!/ G(\Gamma)$ denote the symplectic reduction. Based on physical considerations Hanany and Mekareeya argue that $X(\Gamma)$ is a symplectic variety that up to symplectic isomorphism depends only on the number $e(\Gamma)$ of exterior edges of Γ and its first Betti number $g(\Gamma)$. If Γ is read as the dual graph of a stable curve, $g(\Gamma)$ is the genus of that curve. Hanany and Mekareeya give the formula $\dim(X(\Gamma)) = 2(1+e(\Gamma))$, deduce from a calculation of the equivariant Hilbert series that $X(\Gamma)$ is a four-dimensional hypersurface if $e(\Gamma) = 1$, and state its defining equation.

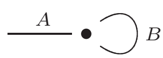
For completeness sake and in order to see that graphs with $e(\Gamma) = 1$ and $g(\Gamma) = n$ lead to our hypersurfaces X_n, we carry out the necessary invariant theoretic calculations in detail for the following graphs:

(3.7) 

For each loop of the form

(3.8) 

we need to consider the vector space $ABC \oplus BCD$, where we have dropped the tensor sign. We may consider $BC = \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$ as the fundamental representation of $\mathrm{SL}(B) \times \mathrm{SL}(C)/(-I, -I) = \mathrm{SO}_4$. This allows to simplify the diagram above to  where the double line indicates the fundamental representation of SO_4. Similarly, a loop

(3.9) 

leads to the vector space ABB. Again we consider $BB = \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C} \oplus \mathbb{C}^3$ as the sum of the trivial and the fundamental representation for the group $\mathrm{SL}(B)/(-I) = \mathrm{SO}_3$. We indicate this by a wriggled line . Thus we may replace the graph (3.7) by

(3.10) 

It follows from this reasoning that $W(\Gamma)$ is the space of representations for the following quiver:

$$\bullet \xrightarrow{x_1} \circ \xrightarrow{y_1} \bullet \xrightarrow{x_2} \circ \xrightarrow{y_2} \cdots \xrightarrow{y_{n-1}} \bullet \xrightarrow{x_n} \diamond\ , \tag{3.11}$$

where $\bullet$ correspond to copies of the fundamental representation $\mathbb{C}^2$ of SL_2, $\circ$ correspond to copies of the fundamental representation $\mathbb{C}^4$ of SO_4, and $\diamond$ corresponds to the representation $\mathbb{C} \oplus \mathbb{C}^3$ of SO_3. Using the symplectic and orthogonal forms on these representations we reinterpret tensor products as Hom spaces associated to the arrows. Then $W(\Gamma) = \{(x_1, y_1, \ldots, y_{n-1}, x_n)\}$, where $x_i \in \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^4)$ for $i = 1, \ldots, n$ and $y_i \in \mathrm{Hom}(\mathbb{C}^4, \mathbb{C}^2)$ for $i = 1, \ldots, n-1$. Let $x_i^* = J^{-1} x_i^t Q \in \mathrm{Hom}(\mathbb{C}^4, \mathbb{C}^2)$ and $y_i^* = Q^{-1} y_i^t J \in \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^4)$ denote the associated adjoint homomorphisms. Here J and Q denote the symplectic and quadratic form on

$\mathbb{C}^2$ and $\mathbb{C}^4$, respectively. Then the ideal I defined by momentum maps for the action of $G(\Gamma)$ is generated by the components of the elements $x_i x_i^* + y_i^* y_i$ and $y_i y_i^* + x_{i+1}^* x_{i+1}$ for $i = 1, \ldots, n-1$ and $\pi(x_n x_n^*)$, where $\pi : \mathfrak{so}_4 \to \mathfrak{so}_3$ denotes the projection dual to the inclusion $\mathfrak{so}_3 \to \mathfrak{so}_4$ associated to the representation of SO_3 on the vector space $\mathbb{C} \oplus \mathbb{C}^3$ at the end of the quiver. We calculate the symplectic reduction in three steps, taking invariants for the the groups SO_4 first, then for the groups SL_2, and finally for SO_3.

The invariant ring for the groups SO_4 is generated by x_n and the elements

$$a_i := x_i^* x_i,\ b_i := y_i x_i,\ c_i := y_i y_i^*,\ d_i := \det(x_i | y_i^*),\ \text{for } i = 1, \ldots, n-1.$$

The intersection with the momentum ideal is generated by

$$\pi(x_n x_n^*),\ c_i + a_{i+1},\ a_i^2 + b_i^* b_i,\ b_i a_i + c_i b_i,\ b_i b_i^* + c_i^2, d_i,\ \text{for } i = 1, \ldots, n-1,$$

where we have put $a_n := x_n^* x_n$ to simplify notations. This allows us to ignore the invariants d_i and c_i. We are left with the following set of generators

$$a_i \in \mathfrak{sl}_2,\ b_i \in \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^2),\ \text{for } i = 1, \ldots, n-1,\ \text{and } x_n \in \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^4),$$

with relations

$$\pi(x_n x_n^*),\quad b_i a_i - a_{i+1} b_i,\quad a_i^2 + b_i^* b_i,\quad b_i b_i^* + a_{i+1}^2.$$

Since a_i^2, $b_i b_i^*$, $b_i^* b_i$ are multiples of the identity, the corresponding relations can be rephrased as $a_1^2 = a_i^2 = -b_i b_i^* = -b_i^* b_i$. The generators a_i, b_i, x_n define a representation of the shortened quiver

$$\begin{tikzcd} \bullet \arrow[loop above, "a_1"] \arrow[r, "b_1"] & \bullet \arrow[loop above, "a_2"] \arrow[r, "b_2"] & \cdots \arrow[r, "b_{n-1}"] & \bullet \arrow[loop above, "a_n"] \arrow[r, "x_n"] & \diamond \end{tikzcd} \tag{3.12}$$

The invariant ring for the action of the groups SL_2 is generated by the components of all maps that are compositions of arrows forming a path from one end of the quiver to another or traces of compositions of arrows forming a closed loop. We can use the relations to move aside any of the loops a_i^2, $b_i b_i^*$ or $b_i^* b_i$. This reduces the number of generators to a_1, $u := x_n b_{n-1} \cdots b_1$ and $v := x_n x_n^*$,

$$\begin{tikzcd} \bullet \arrow[loop left, "a_1"] \arrow[r, "u"] & \diamond \arrow[loop right, "v"] \end{tikzcd} \tag{3.13}$$

satisfying the relations

$$u a_1 - v u,\ u u^* - \det(a_1)^{n-1} v,\ u^* u - \det(a_1)^{n-1} a_1,\ \mathrm{tr}(a_1^2) - \mathrm{tr}(v^2),\ \pi(v).$$

It remains to take invariants for the action of SO_3. The decomposition $\mathbb{C}^4 = \mathbb{C} \oplus \mathbb{C}^3$ yields corresponding decompositions

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad v = \begin{pmatrix} 0 & -w^* \\ w & z \end{pmatrix}. \tag{3.14}$$

As $z = \pi(v)$ is a relation, we may ignore z and continue to calculate with the SO_3-invariant generators a_1 and u_1 and the vector valued generators $u_2 \in \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^3)$ and $w \in \mathrm{Hom}(\mathbb{C}, \mathbb{C}^3)$. The remaining relations translate into

$$u_1 a_1 + w^* u_2, \quad u_2 a_1 - w u_1, \quad u_2 u_2^*, \quad \det(a_1) - w^* w. \tag{3.15}$$

$$u_2 u_1^* - \det(a_1)^{n-1} w, \quad u_1^* u_1^* + u_2^* u_2 - \det(a_1)^{n-1} a_1. \tag{3.16}$$

The invariants for the SO_3 action are generated by a_1, u_1: the further invariants

$$w^* w, \quad w^* u_2, \quad u_2^* u_2, \quad \det(w|u_2), \tag{3.17}$$

can be expressed in terms of a_1 and u_1 due to the given relations. So we end up with five generators x, y, z, a and b that are the components of

$$a_1 = \begin{pmatrix} y & x \\ -z & -y \end{pmatrix} \in \mathfrak{sl}_2 \quad \text{and} \quad u_1 = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}) \tag{3.18}$$

and satisfy the single equation

$$0 = \det(a_1)^n + u_1 a_1 u_1^* = a^2 x - 2aby + b^2 z + (xz - y^2)^n. \tag{3.19}$$

3.3. The six-dimensional example

At present we know only one six-dimensional indecomposable hypersurface, denoted $\hat{X}$. It looks rather special, and the following discussion might indicate that it is an exceptional example that is not contained in a series. We first encountered $\hat{X}$ as the slice to the six-dimensional nilpotent orbit in the nilpotent cone of the simple Lie algebra $\mathfrak{g}_2$ [6]. Its defining equation $\hat{f}$ is rather complicated, and it is easier to obtain it indirectly. $\hat{X}$ has the interesting property that it is completely determined by its singular locus $\Sigma \subset \mathbb{C}^7$, and we will explain how to recover $\hat{X}$ starting from Σ.

Let V denote the irreducible two-dimensional representation of the symmetric group S_3, realised as the kernel of the linear form $x_1 + x_2 + x_3$ in $\mathbb{C}^3$. The invariant ring $\mathbb{C}[V \oplus V^*]^{S_3}$ is generated by 3 polynomials a_1, a_2, a_3 of degree 2 that are obtained by the process of polarisation from the second elementary symmetric polynomial $x_1x_2 + x_2x_3 + x_3x_1$ and by 4 polynomials $b_1, \dots, b_4$ of degree 3 that are obtained by similarly polarising the third elementary symmetric polynomial $x_1x_2x_3$. This is a classical result treated for example by Weyl in [11, p. 36 ff.]. These invariants are explicitly given as follows: $a_1 = \sum_{i<j} x_ix_j$, $a_2 = \sum_{i,j} x_iy_j$, $a_3 = \sum_{i<j} y_iy_j$ and $b_1 = x_1x_2x_3$, $b_2 = x_1x_2y_3 + x_1y_2x_3 + y_1x_2x_3$, $b_3 = x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3$, $b_4 = y_1y_2y_3$ for the dual coordinates y_1, y_2, y_3 on V^*.

These polynomials define an embedding $\Sigma := (V \oplus V^*)/S_3 \to \mathbb{C}^7$. The relations among the invariants are generated by the following 2 polynomials of weighted degree 5 and 3 polynomials of degree 6:

$$\begin{aligned}
t_1 &= a_3b_2 - a_2b_3 + 3a_1b_4,\\
t_2 &= 3a_3b_1 - a_2b_2 + a_1b_3,\\
t_3 &= a_3(a_2^2 - 4a_1a_3) - 3b_3^2 + 9b_2b_4,\\
t_4 &= a_2(a_2^2 - 4a_1a_3) - 3b_2b_3 + 27b_1b_4,\\
t_5 &= a_1(a_2^2 - 4a_1a_3) - 3b_2^2 + 9b_1b_3;
\end{aligned} \tag{3.20}$$

We note in passing that the same quotient variety Σ can also be obtained as symplectic reduction for the action of SL_2 on $S^4\mathbb{C}^2 \oplus (S^4\mathbb{C}^2)^*$ or as the symplectic reduction for the action of SL_3 on $\mathfrak{sl}_3 \oplus \mathfrak{sl}_3^*$.

As a subring of $\mathbb{C}[V \oplus V^*]$ the graded coordinate ring $\mathbb{C}[\Sigma]$ inherits a canonical Poisson structure. The Poisson brackets $\{a_i, a_j\}$, $\{a_i, b_j\}$ and $\{b_i, b_j\}$ will have degrees 2, 3 and 4 respectively. But since the smallest degree of a relation among the a_i's and b_i's has degree 5, it follows that the Poisson structure uniquely extends to a homogeneous Poisson structure on the ambient space $\mathbb{C}^7$. Calculation gives the following Poisson matrix:

$$\begin{pmatrix}
0 & -2\,a_1 & -a_2 & 0 & -3\,b_1 & -2\,b_2 & -b_3\\
2\,a_1 & 0 & -2\,a_3 & 3\,b_1 & b_2 & -b_3 & -3\,b_4\\
a_2 & 2\,a_3 & 0 & b_2 & 2\,b_3 & 3\,b_4 & 0\\
0 & -3\,b_1 & -b_2 & 0 & \frac{2}{3}a_1^2 & \frac{2}{3}a_1a_2 & \frac{1}{3}a_2^2 - \frac{2}{3}a_1a_3\\
3\,b_1 & -b_2 & -2\,b_3 & -\frac{2}{3}a_1^2 & 0 & \frac{10}{3}a_1a_3 - \frac{1}{3}a_2^2 & \frac{2}{3}a_2a_3\\
2\,b_2 & b_3 & -3\,b_4 & -\frac{2}{3}a_1a_2 & \frac{1}{3}a_2^2 - \frac{10}{3}a_1a_3 & 0 & \frac{2}{3}a_3^2\\
b_3 & 3\,b_4 & 0 & \frac{2}{3}a_1a_3 - \frac{1}{3}a_2^2 & -\frac{2}{3}a_2a_3 & -\frac{2}{3}a_3^2 & 0
\end{pmatrix}$$

If we denote this matrix by $\hat{\Theta}$, its pfaffian $\mathrm{pf}(\hat{\Theta})$ allows to determine the hypersurface equation $\hat{f}$ via $d\hat{f} = c\,\mathrm{pf}(\hat{\Theta})$, up to some normalising constant c. The equation is rather complicated. One can express it using the relations between the invariants, i.e. the equations of Σ, as follows:

$$\hat{f} = a_1t_1^2 - a_2t_1t_2 + a_3t_2^2 + \frac{1}{12}(t_4^2 - 4t_3t_5). \tag{3.21}$$

Since $\hat{f} \in (t_1, \dots, t_5)^2$, the singular locus of $\hat{X} = \{\hat{f} = 0\}$ contains Σ, and an explicit calculation shows that Σ actually equals the *reduced* singular locus of $\hat{X}$.

One can also describe the Poisson matrix $\hat{\Theta}$ by the Poisson algebra approach described in the four-dimensional case: consider the four-dimensional irreducible representation $V = S^3\mathbb{C}^2$ of $\mathfrak{sl}_2$. The choice of an equivariant map $\varphi : \Lambda^2 V \to S^*(\mathfrak{sl}_2)$ gives rise to a Poisson structure on $A = S^*(\mathfrak{sl}_2 \oplus V)$ if certain conditions imposed by the Jacobi identity are satisfied: As there are equivariant decompositions $\Lambda^2 V = \mathbb{C} \oplus S^4\mathbb{C}^2$ and $S^*(\mathfrak{sl}_2) = \mathbb{C}[\Delta] \otimes \bigoplus_{m\geq 0} S^{2m}(\mathbb{C}^2)$, the space of homogeneous equivariant maps $\varphi : \Lambda^2 V \to S^*(\mathfrak{sl}_2)_N$ is two-dimensional for each even $N \geq 2$, generated by maps $\mathbb{C} \to \mathbb{C} \cdot \Delta^{N/2}$ and $S^4\mathbb{C}^2 \to S^4\mathbb{C}^2 \cdot \Delta^{N/2-1}$. However, and in contrast to the four-dimensional case, only for the degree $N = 2$ there is a map φ leading to a non-degenerate hypersurface: the one described above.

§4. Contact Fano 3-folds

Consider the 3-dimensional projective varieties $Y \subset \mathbb{P} := \mathbb{P}(2n-1, 2n-1, 2, 2, 2)$ defined by the weighted homogeneous polynomial $a^2x + 2aby + b^2z + (xz - y^2)^n = 0$ for each $n \geq 2$. Here the coordinates are given the degrees $|a| = |b| = 2n - 1$ and $|x| = |y| = |z| = 2$. As before, let X_n denote the symplectic hypersurface in $\mathbb{C}^5$ defined by the same equation. In this section we introduce a contact structure on Y and relate it with the projectivised cotangent bundle $\mathbb{P}(T^*_{\mathbb{P}^1\times\mathbb{P}^1})$ by explicit birational maps.

The singular locus of Y has two components: Y has quotient singularities of type $\frac{1}{2n-1}(1,1)$ along the smooth rational curve $C = \{x = y = z = 0\}$ and Du Val singularities of type D_{2n} along the smooth rational curve $D = \{a = b = 0,\ xz - y^2 = 0\}$. The projection map $p : \mathbb{C}^5 \setminus \{0\} \to \mathbb{P}(2n-1, 2n-1, 1^3)$ is a $\mathbb{C}^*$-bundle outside C and D. Define $Y^0 := Y \setminus (C \cup D)$ and $X_n^0 := p^{-1}(Y^0)$.

Recall that a contact structure on a complex manifold M of dimension $2d+1$ is an exact sequence of vector bundles

$$0 \longrightarrow D \longrightarrow T_M \xrightarrow{\theta} L \longrightarrow 0,$$

with $\mathrm{rk}(D) = 2d$ and $\mathrm{rk}(L) = 1$ so that $d\theta|_D$ induces a non-degenerate pairing on D. By using the formula for exterior derivation

$$d\theta(x,y) = x(\theta(y)) - y(\theta(x)) - \theta([x,y])$$

one can check that this is equivalent to saying that $[-,-] : D \times D \to L = T_M/D$ is non-degenerate. We call L the contact line bundle.

We shall introduce a contact structure on Y^0 with the contact line bundle $\mathcal{O}(2) := \mathcal{O}_{\mathbb{P}}(2)|_{Y^0}$. Let ω be a symplectic 2-form on X_n^0 of weight

2. By construction, the projection $p : X_n^0 \to Y^0$ is a $\mathbb{C}^*$-bundle, and X_n^0 is in fact isomorphic to the complement of the zero section of the line bundle $\mathcal{O}(-1)$ on Y^0. There is a canonical trivialisation $p^*\mathcal{O}(1) \cong \mathcal{O}_{X_n^0}$, and hence a trivialisation $p^*\mathcal{O}(i) \cong \mathcal{O}_{X_n^0}$ for any $i \in \mathbb{Z}$. Let ζ be the vector field which generates the $\mathbb{C}^*$-action. Since ω has weight 2, one can write $\omega(\zeta, \cdot) = p^*\theta$ for some appropriate element $\theta \in H^0(Y^0, \Omega^1_{Y^0} \otimes \mathcal{O}(2))$. This θ gives a contact structure on Y^0 with the contact line bundle $\mathcal{O}(2)$.

The rational map

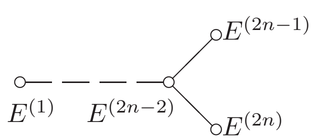
$$\mathbb{P}((2n-1)^2, 2^3) \dashrightarrow \mathbb{P}^2 = \mathbb{P}(2^3), (a : b : x : y : z) \to (x : y : z)$$

induces a rational map $Y \dashrightarrow \mathbb{P}^2$. To eliminate the indeterminancy of the rational map, we take the blow-up Y_1 of Y along C. Let $F_1 \subset Y_1$ be the exceptional divisor of the blowing-up. Notice that F_1 is a $\mathbb{P}^1$-bundle over C. Then the rational map actually becomes a morphism $f_1 : Y_1 \to \mathbb{P}^2$. Let us consider the fibres of f_1. For $(1 : \mu : \lambda) \in \mathbb{P}^2$, the fibre $f_1^{-1}(1 : \mu : \lambda)$ is isomorphic to the quasi-homogeneous hypersurface of $\mathbb{P}(2n-1, 2n-1, 2)$ defined by

$$a^2 + 2ab\mu + b^2\lambda + (\lambda - \mu^2)^n x^{2n-1} = 0.$$

If $(1 : \mu : \lambda) \in \{xz - y^2 = 0\}$, then it is a multiple fibre with multiplicity 2. If $(1 : \mu : \lambda) \notin \{xz - y^2 = 0\}$, then the fibre is a smooth rational curve. In other words, f_1 is a conic bundle whose discriminant locus D' is $\{xz - y^2 = 0\}$ and all singular fibres are non-reduced.

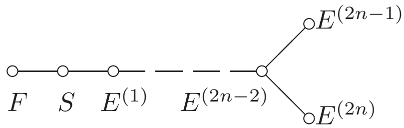
Set $S_1 := f_1^{-1}(D')_{red}$. Then S_1 is a $\mathbb{P}^1$-bundle over D'. Since the blowing-up $Y_1 \to Y$ does not change an open neighborhood of $D \subset Y$, its inverse image D_1 by the blowing-up is isomorphic to D. Moreover, D_1 is a section of the $\mathbb{P}^1$-bundle $S_1 \to D'$. The singular locus of Y_1 coincides with D_1. As Y_1 has Du Val singularities of type D_{2n} along D_1, one can take its minimal resolution $\tilde{Y} \to Y_1$. The exceptional locus of the minimal resolution consists of $2n$ divisors $E^{(1)}, \dots, E^{(2n)}$ intersecting each other according to the following D_{2n}-configuration:



Here the vertices correspond to the exceptional surfaces and the edges correspond to the intersection curves. Each surface $E^{(i)}$ is a $\mathbb{P}^1$-bundle over D and each intersection curve is a section of the $\mathbb{P}^1$-bundle map.

Let S and F be respectively the proper transforms of S_1 and F_1 by the map $\tilde{Y} \to Y_1$. Then S intersects with F along a section of the

$\mathbb{P}^1$-bundle structure. There are no intersetion of F with $E^{(i)}$'s. On the other hand, S intersects with only $E^{(1)}$. Notice that $E^{(1)} \cap S$ is a section of the ruled surface $E^{(1)}$, which is disjoint from $E^{(1)} \cap E^{(2)}$:



One can blow down successively these divisors along their rulings in the following order: S, $E^{(1)}$, $\ldots$, $E^{(2n-3)}$, and finally F. We call the resulting variety Z. The existence of such birational contraction maps are justified in the following way. Let us consider Y_1 and S_1. Let ℓ_1 be a fibre of $S_1 \to D'$. We prove that $(K_{Y_1}, \ell_1) = -1$. Let $\ell \subset Y$ be the image of ℓ_1 by the map $\pi_1 : Y_1 \to Y$. By an explicit calculation we see that $(\mathcal{O}(1), \ell) = \frac{1}{2} \cdot \frac{1}{2n-1}$. Since $K_Y = \mathcal{O}_Y(-4)$, one has $(K_Y, \ell) = -\frac{2}{2n-1}$. Since $K_{X_1} = (\pi_1)^* K_Y - \frac{2n-3}{2n-1} \cdot F_1$ and $(F_1, \ell_1) = 1$, we see that $(K_{Y_1}, \ell_1) = -1$. Denote by π_2 the minimal resolution $\tilde{Y} \to Y_1$. The proper transform S of S_1 by π_2 is isomorphic to S_1; hence there is a $\mathbb{P}^1$-bundle map $S \to D'$. Let $\tilde{\ell}$ be a fibre of this map. Then, since $K_{\tilde{Y}} = (\pi_2)^* K_{Y_1}$, we see that

$$(K_{\tilde{Y}}, \tilde{\ell}) = -1.$$

Let m_i be a fibre of the $\mathbb{P}^1$-bundle structure of $E^{(i)}$. Then we have

$$(K_{\tilde{Y}}, m_i) = 0.$$

By Nakano-Fujiki criterion one has a bimeromorphic map $\nu_1 : \tilde{Y} \to Z_1$ to a Moishezon manifold Z_1, where ν_1 contracts all rulings of S to points. As S intersects with $E^{(1)}$ along a section, we have

$$(K_{Z_1}, \nu_1(m_1)) = (K_{\tilde{Y}}, m_1) - 1 = -1.$$

Then we get a bimeromorphic map $\nu_2 : Z_1 \to Z_2$, where ν_2 contracts all rulings of $\nu_1(E^{(1)})$ to points. We can further continue the same procedures in the order of $E^{(2)}, \ldots, E^{(2n-3)}$ and finally F. As a consequence we have a sequence of birational contraction maps

$$\tilde{Y} \to Z_1 \to Z_2 \to \ldots \to Z.$$

In the remainder we denote by ν the map $\tilde{Y} \to Z$ and by μ the map $\tilde{Y} \to Y$.

Lemma 4.1. — *Z has a contact structure.*

Proof. The birational map $\pi : \tilde{Y} \to Y$ is a crepant resolution of Y around D. As remarked above, Y^0 has a contact form $\eta \in \Gamma(Y^0, \Omega^1_{Y^0} \otimes \mathcal{O}(2))$ with the contact line bundle $\mathcal{O}(2)$. Take a point $x \in D$. Since $\mathcal{O}(2)$ is a line bundle around D, one can trivialise $\mathcal{O}(2)$ on an open neighbourhood $x \in U \subset Y$. Then η is regarded as a 1-form on U_{reg} such that $\eta \wedge d\eta$ is a nowhere-vanishing 3-form on U_{reg}. This 3-form extends to a generator of the invertible dualising sheaf ω_U. Set $\tilde{U} := \pi^{-1}(U)$ and $\pi_U := \pi|_{\tilde{U}}$. Then $(\pi_U)^*(\eta \wedge d\eta)$ is a nowhere-vanishing 3-form on $\tilde{U}$ because π_U gives a crepant resolution of U. This shows that $(\pi_U)^*\eta$ is a contact 1-form on $\tilde{U}$ with the contact line bundle $(\pi_U)^*(\mathcal{O}(2)|_U)$. As a consequence, $\tilde{Y}$ has a contact structure outside $F = \pi^{-1}(C)$.

Let $\beta \subset Z$ be the image of F by the birational morphism $\nu : \tilde{Y} \to Z$. Note that $\dim \beta = 1$. Let us consider the birational morphism

$$\tilde{Y} - \nu^{-1}(\beta) \to Z - \beta.$$

There is an open subset Z^0 of $Z - \beta$ such that $\nu^{-1}(Z^0) \cong Z^0$ and such that the complement of Z^0 in $Z - \beta$ has at least codimension 2. The restriction of the contact structure on $\tilde{Y} - F$ to $\nu^{-1}(Z^0)$ gives a contact structure of Z^0. Since the complement of Z^0 in Z has at least codimension 2, the contact structure uniquely extends to a contact structure on Z. Q.E.D.

Lemma 4.2. — *Z is isomorphic to $\mathbb{P}(T^*_{\mathbb{P}^1 \times \mathbb{P}^1})$.*

Proof. We cover D by three orbifold charts $W_x \to Y$, $W_y \to Y$ and $W_z \to Y$, where $W_x := \{x = 1\} \subset \mathbb{C}^5$, $W_y := \{y = 1\}$ and $W_z := \{z = 1\}$. Note that each map is a $\mathbb{Z}_2$-cover onto its image. Let V be the union of these images. The blowing up of each chart along the singular locus is $\mathbb{Z}_2$-equivariant, and three pieces $\tilde{W}_x/\mathbb{Z}_2$, $\tilde{W}_y/\mathbb{Z}_2$ and $\tilde{W}_z/\mathbb{Z}_2$ are glued together to give a partial resolution $V' \to V$. Since it does not change anything outside D, it gives a partial resolution $Y' \to Y$. The exceptional locus E' of the partial resolution is a $\mathbb{P}^1$-bundle over D and Y' has A_{2n-1}-singularities along a section of this $\mathbb{P}^1$-bundle. Note that the partial resolution $Y' \to Y$ eliminate the indeterminancy of the rational map

$$Y - - \to \mathbb{P}^1, (a : b : x : y : z) \to (a : b)$$

and gives a morphism $Y' \to \mathbb{P}^1$. Each fibre of E' is isomorphically mapped onto $\mathbb{P}^1$ by the morphism. This, in particular, shows that E' has two $\mathbb{P}^1$-bundle structures. Hence we see that $E' \cong \mathbb{P}^1 \times \mathbb{P}^1$. By the definition, $\tilde{Y} \to Y$ factors through $Y' : \tilde{Y} \to Y' \to Y$. The proper transform of E' by the birational map $\tilde{Y} \to Y$ is nothing but $E^{(2n-1)}$. By the argument above, $E^{(2n-1)} \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Next look at $E^{(2n-2)}$. It has a $\mathbb{P}^1$-bundle structure whose fibres correspond to exceptional curves of the map $\tilde{Y} \to Y$. Since it has three disjoint sections (corresponding to the intersections with $E^{(2n-3)}$, $E^{(2n-1)}$ and $E^{(2n)}$), we also see that $E^{(2n-2)} \cong \mathbb{P}^1 \times \mathbb{P}^1$. Define $\sigma := E^{(2n-2)} \cap E^{(2n-3)}$.

We write $E^{(2n-2)}$, $E^{(2n-1)}$ and $E^{(2n)}$ for their images in Z by the map $\tilde{Y} \to Z$.

Pick a fibre α of $E^{(2n-2)} \subset Z$. Since $(E^{(2n-2)}, \alpha)_Z = 0$, the curve α can move aside $E^{(2n-2)}$ in a parameter space of dimension 2. We prove that there is a morphism $Z \to \mathbb{P}^1 \times \mathbb{P}^1$ whose fibres are all deformation equivalent to α. The linear system which gives the morphism is $|\mathcal{O}_Z(E^{(2n-2)})|$. To prove that this linear system is free from base points, it suffices to show that $|\mathcal{O}_{E^{(2n-2)}}(E^{(2n-2)})|$ is free from base points by the exact sequence

$$0 \to H^0(Z, \mathcal{O}_Z) \to H^0(Z, \mathcal{O}_Z(E^{(2n-2)})) \to H^0(E^{(2n-2)}, \mathcal{O}_{E^{(2n-2)}}(E^{(2n-2)})) \to 0.$$

Note that

$$K_{\tilde{Y}} = \mu^* K_Y - \frac{2n-3}{2n-1} F.$$

We can also write $K_{\tilde{Y}}$ by a linear combination of $\nu^* K_Z$, S, F and $E^{(1)}$, $\ldots$, $E^{(2n-3)}$. By using the two expression of $K_{\tilde{Y}}$, one can write

$$\nu^* K_Z = \mu^* K_Y - 2E^{(2n-3)} + \text{other terms}.$$

Restricting this to $E^{(2n-2)}$ we get $K_Z|_{E^{(2n-2)}} = -4\alpha - 2\sigma$ since $(K_Y, D) = -4$, which easily follows from the fact that $K_Y = \mathcal{O}(-4)$ and $(\mathcal{O}(1), D) = 1$.

Now, by the adjunction formula $K_{E^{(2n-2)}} = K_Z + E^{(2n-2)}|_{E^{(2n-2)}}$ we see that $E^{(2n-2)}|_{E^{(2n-2)}} \sim 2\alpha$. The corresponding linear system is free from base points.

Since $h^0(Z, \mathcal{O}_Z(E^{(2n-2)})) = 4$, we have a morphism $Z \to \mathbb{P}^3$. Since $(E^{(2n-2)})^3 = 0$ and $(E^{(2n-2)})^2 \sim 2\alpha$, the image has 2 dimension. Moreover, since $(E^{(2n-1)}, \alpha) = 1$ and $E^{(2n-1)} \cong \mathbb{P}^1 \times \mathbb{P}^1$, the morphism is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ with a section $E^{(2n-1)}$. As we have seen in Lemma 4.1, Z has a contact structure. Moreover the morphism defined here is a Legendre $\mathbb{P}^1$-bundle. By [5], it then follows that Z is a projectivised cotangent bundle $\mathbb{P}(T^*_{\mathbb{P}^1 \times \mathbb{P}^1})$. Q.E.D.

Remark 4.3. — *Let X'_n be a Slodowy slice to a niloptent orbit $\mathcal{O}_{[4n-3,1^3]}$ of $\mathfrak{so}_{4n}$ with $n \geq 2$. Then one can check that X'_n is isomorphic to the complete intersection of $\mathbb{C}^6(\alpha, \beta, \gamma, x, y, z)$ defined by two equations $f = g = 0$ with*

$$f = \alpha x + 2\beta y + \gamma z = 0 \qquad \text{and} \qquad g = \alpha\gamma - \beta^2 + 1/4(xz - y^2)^{2n-1} = 0.$$

With the new coordinates $A = \alpha - \frac{1}{2}z(xz-y^2)^{n-1}$, $B = \beta + \frac{1}{2}y(xz - y^2)^{n-1}$ and $C = \gamma - \frac{1}{2}x(xz-y^2)^{n-1}$ the equations $f = 0$ and $g = 0$ respectively become

$$Ax + 2By + Cz + (xz - y^2)^n = 0 \qquad \text{and} \qquad AC - B^2 = 0.$$

It follows from this description that $\tau(a, b, x, y, z) := (a^2, ab, b^2, x, y, z)$ defines a double covering $\tau : X_n \to X'_n$. Note that τ is ramified precisely over the singular locus of X'_n. Moreover, X'_n is equipped with the Kostant-Kirillov 2-form ω' on the regular locus. Then $\tau^*\omega'$ is equivalent to the Kostant-Kirillov 2-form ω on X_n by Theorem (3.1) in [9].

Let Y' be the 3-dimensional projective variety in $\mathbb{P}(2n-1, 2n-1, 2n-1, 1, 1, 1)$ defined by $f = g = 0$. The degrees of the coordinates are $|\alpha| = |\beta| = |\gamma| = 2n-1$ and $|x| = |y| = |z| = 1$. Then Y' admits a contact structure on its regular part. Moreover, by the observation above, we immediately see that $Y \cong Y'$ as contact varieties.

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