

Resolution of singularities of an idealistic filtration in dimension 3 after Benito-Villamayor

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Dedicated to Prof. Shigefumi Mori on the occasion of his 60-th birthday

Abstract.

We establish an algorithm for resolution of singularities of an idealistic filtration in dimension 3 (a local version) in positive characteristic, incorporating the method recently developed by Benito-Villamayor into our framework. Although (a global version of) our algorithm only implies embedded resolution of surfaces in a smooth ambient space of dimension 3, a classical result known before, we introduce some new invariant which effectively measures how much the singularities are improved in the process of our algorithm and which strictly decreases after each blow up. This is in contrast to the well-known Abhyankar-Moh pathology of the increase of the residual order under blow up and the phenomenon of the “Kangaroo” points observed by Hauser.

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§1. Outline of the paper

The goals of this paper are two-fold. The first goal is to present the general mechanism of resolution of singularities (a local version) in the framework of the Idealistic Filtration Program in positive characteristic. The classical algorithm in characteristic zero works by induction on dimension based upon the notion of a hypersurface of maximal contact. Our algorithm in positive characteristic works by induction on the invariant “ σ ” based upon the notion of a leading generator system (cf. [18]). Roughly speaking, the general mechanism splits into two parts; the first part is to reduce the problem in the general case to the one in the so-called “monomial case”, and the second part is to solve the problem in the monomial case. The first part of the general mechanism works in arbitrary dimension. The second part is quite subtle and difficult in positive characteristic (while it is easy in the classical setting in characteristic zero). The second goal is to establish the algorithm in dimension 3, by actually solving the problem of resolution of singularities in the monomial case. We incorporate the method recently developed by Benito-Villamayor [5] into our framework. We introduce some new invariant, which effectively measures how much the singularities are improved in the process of our algorithm and which strictly decreases after each blow up. This is in clear contrast to the well-known Abhyankar-Moh pathology of the increase of the residual order (cf. [21]) and the phenomenon of the “Kangaroo” points observed by Hauser (cf. [16]) and others. We note that the algorithm by Benito-Villamayor (cf. [3][4][5]), which works by induction on dimension based upon the notion of a generic projection, is different from our algorithm, and that even the setting of the monomial case is different from ours by definition. It is something of a surprise that we can share the “same” method when our and their approaches are different. Establishing the algorithm in dimension 4 or above remains as an open problem.

We remark that (a global version, which will be published elsewhere, of) our algorithm in dimension 3 only yields embedded resolution of surfaces in a nonsingular ambient 3-fold over an algebraically closed field of any characteristic. This is a classical result known for more than 50 years since the time of Abhyankar, Hironaka and others (cf. [1][12][15][17]). The front line of research, thanks to the works of Abhyankar, Cossart-Piltant, and Cutkosky (cf. [1][9][10][11][12]), goes way beyond, establishing resolution of singularities of a 3-fold over an algebraically closed field of positive characteristic. We believe, however, that the method of our paper should provide a first step toward the open problem of embedded resolution of singularities of 3-folds in a nonsingular ambient 4-fold in positive characteristic, and also in higher dimensional cases.

The outline of the paper goes as follows.

After §1, which describes the outline of the paper, we give a brief overview of the contents of the paper in §2. In §3, we present a quick review on the algorithm in characteristic zero. Our algorithm in positive characteristic is modeled upon the classical algorithm in characteristic zero. The review is given in such a way that the reader can see the similarities and differences between our algorithm and the classical algorithm through an easy and accessible comparison. In §4, we present the general mechanism of our algorithm for resolution of singularities in positive characteristic. In §5, we present a solution to the problem of resolution of singularities in the monomial case in dimension 3, thus completing our algorithm as a whole in dimension 3.

We assume, throughout the entire paper, that the base field is an algebraically closed field $k = \bar{k}$ of characteristic zero $\text{char}(k) = 0$ or positive characteristic $\text{char}(k) = p > 0$. Therefore, there is no danger in *not* distinguishing the two notions “smooth over k ” and “regular” (meaning that every local ring of a variety is a regular local ring), and in using the word “nonsingular” as a synonym. In the case where the base field k is perfect, our algorithm over its algebraic closure \bar{k} is invariant under the action of the Galois group $\text{Gal}(\bar{k}/k)$ and hence descends to the algorithm over the original base field k . The case where the base field is not perfect will be investigated elsewhere.

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§2. Overview

The problem of resolution of singularities in its simplest form is stated as follows:

Problem 1 (Resolution of singularities). *Given an algebraic variety X over k , find a proper birational map $X \xleftarrow{\pi} \tilde{X}$ from a nonsingular variety \tilde{X} .*

The above problem is reduced to the following problem of embedded resolution of singularities, if our solution to the latter is *functorial* in the sense that it is stable under the pull-back by smooth morphisms.

Problem 2 (Embedded resolution of singularities). *Given an algebraic variety X , embedded as a closed subvariety in a smooth ambient variety W over k , i.e., $X \subset W$, find a sequence starting from $(X_0 \subset W_0) = (X \subset W)$*

$$(X_0 \subset W_0) \leftarrow \cdots \leftarrow (X_i \subset W_i) \xrightarrow{\pi_{i+1}} (X_{i+1} \subset W_{i+1}) \leftarrow \cdots \leftarrow (X_l \subset W_l),$$

where $W_i \leftarrow W_{i+1}$ is a blow up with smooth center $C_i \subset W_i$ which does not contain the strict transform X_i of X_0 in year i , i.e., $C_i \not\supset X_i$, and which is transversal to the exceptional divisor D_i , i.e., $C_i \pitchfork D_i$ (maybe contained in D_i), such that the last strict transform X_l is nonsingular and transversal to D_l .

Note that in the above formulation we do *not* require that the center is contained in the singular locus of the strict transform $C_i \subset \text{Sing}(X_i)$ or even that the center is contained in the strict transform $C_i \subset X_i$.

Note also that we say C_i is transversal to D_i , denoted by $C_i \pitchfork D_i$, if at any closed point $P \in W_i$ there exist a regular system of parameters (x_1, \dots, x_d) at P , taken from $\mathfrak{m}_P \subset \mathcal{O}_{W,P}$, and subsets $A, B \subset \{1, \dots, d = \dim W_i\}$ such that $C_i = \bigcap_{\alpha \in A} \{x_\alpha = 0\}$ and $D_i = \bigcup_{\beta \in B} \{x_\beta = 0\}$ in a neighborhood of P .

The implication “Solution to Problem 2 (in a functorial way) \Rightarrow Solution to Problem 1” can be seen as follows: Given an algebraic variety X , decompose it into the union of affine open subvarieties $X = \bigcup_{\lambda \in \Lambda} X_\lambda$ with embeddings $X_\lambda \subset W_\lambda = \mathbb{A}^{n_\lambda}$. Take embedded resolutions of singularities $(X_\lambda \subset W_\lambda) \leftarrow (\widetilde{X}_\lambda \subset \widetilde{W}_\lambda)$. We have only to see that the \widetilde{X}_λ ’s patch together by the *functoriality* to obtain resolution of singularities $X = \bigcup_{\lambda \in \Lambda} X_\lambda \leftarrow \widetilde{X} = \bigcup_{\lambda \in \Lambda} \widetilde{X}_\lambda$. For the detail of the proof, we refer the reader to [23].

How can we solve Problem 2 (in a functorial way so that we can solve Problem 1 also)? We would like to use “induction” on dimension. However, the inductive scheme to approach Problem 2 is not clear as stated, at least not obvious. We will present in §3 the reformulation by Hironaka where in characteristic zero the inductive scheme on dimension is more transparent and used classically, and in §4 another reformulation in our framework where in positive characteristic the inductive scheme on the invariant “ σ ” emerges.

We give an overview of the contents of our paper in the following.

§3 is devoted to a quick review on the algorithm in characteristic zero.

In §3.1, we give the precise statement of the reformulation by Hironaka (following the language used by Villamayor). In short, the reformulation turns the problem of embedded resolution of singularities into a game of reducing the order of an ideal on a nonsingular ambient variety. We start from a triplet of data $(W, (\mathcal{I}, a), E)$, where W is a nonsingular variety over k , (\mathcal{I}, a) is a pair consisting of a *nonzero* coherent ideal sheaf \mathcal{I} on W and a fixed positive integer $a \in \mathbb{Z}_{>0}$, and where E is a simple normal crossing divisor on W , called a boundary (divisor). We define its singular locus to be $\text{Sing}(\mathcal{I}, a) := \{P \in W \mid \text{ord}_P(\mathcal{I}) \geq a\}$. Then we are required to find a sequence of transformations (see §3.1 for the precise definition of a transformation) starting from $(W_0, (\mathcal{I}_0, a), E_0) = (W, (\mathcal{I}, a), E)$

$$(W_0, (\mathcal{I}_0, a), E_0) \leftarrow \cdots \leftarrow (W_i, (\mathcal{I}_i, a), E_i) \xrightarrow{\pi_i^{\pm 1}} (W_{i+1}, (\mathcal{I}_{i+1}, a), E_{i+1}) \\ \leftarrow \cdots \leftarrow (W_l, (\mathcal{I}_l, a), E_l)$$

such that $\text{Sing}(\mathcal{I}_l, a) = \emptyset$. That is to say, the order of the last ideal \mathcal{I}_l is everywhere below a . We call such a sequence “resolution of singularities for $(W, (\mathcal{I}, a), E)$ ”.

In this reformulation, the inductive scheme of the problem can be stated, though naive, simply as follows: Given a triplet of data $(W, (\mathcal{I}, a), E)$, find another triplet of data $(H, (\mathcal{J}, b), F)$ with $H \underset{\text{closed}}{\subset} W$ and $\dim H = \dim W - 1$ such that constructing resolution of singularities for $(W, (\mathcal{I}, a), E)$ is equivalent to constructing one for $(H, (\mathcal{J}, b), F)$, a property symbolically denoted by

$$(W, (\mathcal{I}, a), E) \underset{\text{equivalent}}{\sim} (H, (\mathcal{J}, b), F).$$

In §3.2, we discuss the inductive scheme on dimension of the algorithm in characteristic zero. It starts with the following key inductive lemma: Given $(W, (\mathcal{I}, a), E)$ assumed to be under a certain condition (\star) , the lemma constructs $(H, (\mathcal{J}, b), F)$ with $H \underset{\text{closed}}{\subset} W$ and $\dim H = \dim W - 1$ (locally around a fixed point $P \in W$) which satisfies one of the following two.

(i) The ideal \mathcal{J} is a zero sheaf, i.e., $\mathcal{J} \equiv 0$: In this case, we take the transformation with center $H = \text{Sing}(\mathcal{I}, a)$

$$(W, (\mathcal{I}, a), E) \leftarrow (\widetilde{W}, (\widetilde{\mathcal{I}}, a), \widetilde{E}).$$

After the transformation, we have $\text{Sing}(\widetilde{\mathcal{I}}, a) = \emptyset$ and hence resolution of singularities is achieved.

(ii) The ideal \mathcal{J} is *not* a zero sheaf, i.e., $\mathcal{J} \not\equiv 0$: In this case, we have

$$(W, (\mathcal{I}, a), E) \underset{\text{equivalent}}{\sim} (H, (\mathcal{J}, b), F).$$

Therefore, by constructing resolution of singularities for $(H, (\mathcal{J}, b), F)$ by induction on dimension, we achieve resolution of singularities for $(W, (\mathcal{I}, a), E)$.

The hypersurface “ H ” in the key inductive lemma is called a *hypersurface of maximal contact*.

There are three shortcomings of the above key inductive lemma when we look at the goal of establishing the algorithm in characteristic zero:

- ① we have to impose condition (\star) on $(W, (\mathcal{I}, a), E)$,
- ② the construction of $(H, (\mathcal{J}, b), F)$ is only local, and
- ③ the invariant “ord” may increase after a transformation, even though our ultimate goal is to reduce its value to be below the fixed level a .

In order to overcome these shortcomings, we adopt the following mechanism.

(1) We introduce a pair of invariants (w-ord, s) and its associated triplet of data $(W, (\mathcal{K}, \kappa), G)$, called the modification of the original triplet $(W, (\mathcal{I}, a), E)$, having the following properties. (We remark that, when we want to emphasize the dimension of the ambient variety, we add “dim” to the pair of invariants as the first factor, making the pair into a triplet (dim, w-ord, s).)

- The maximum locus of the pair of invariants coincides with the singular locus of the modification, i.e.,

$$\text{MaxLocus}(\text{w-ord}, s) = \text{Sing}(\mathcal{K}, \kappa).$$

Moreover, after each transformation, the value of the pair (w-ord, s) never increases, and the locus where the value of the pair takes the same maximum value as the original one coincides with the singular locus of the transformation of the modification. (Note that the transformations of $(W, (\mathcal{K}, \kappa), G)$ induce those of $(W, (\mathcal{I}, a), E)$.) This means that resolution of singularities for $(W, (\mathcal{K}, \kappa), G)$ implies the strict decrease of the (maximum) value of the pair (w-ord, s).

- Even though the original triplet $(W, (\mathcal{I}, a), E)$ may not satisfy condition (\star) , the modification $(W, (\mathcal{K}, \kappa), G)$ does.

(2) We apply the key inductive lemma to $(W, (\mathcal{K}, \kappa), G)$. We find a triplet $(H, (\mathcal{J}, b), F)$ with $H \subset_{\text{closed}} W$ and $\dim H = \dim W - 1$ such that we are either in Case (i), or in Case (ii) where resolution of singularities for $(H, (\mathcal{J}, b), F)$ implies the one for $(W, (\mathcal{K}, \kappa), G)$.

(3) In Case (i), by a single blow up with center H , we achieve resolution of singularities for $(W, (\mathcal{K}, \kappa), G)$. In Case (ii), by induction on dimension, we achieve resolution of singularities for $(H, (\mathcal{J}, b), F)$, hence for $(W, (\mathcal{K}, \kappa), G)$. In both cases, we have the strict decrease of the (maximum) value of the pair (w-ord, s).

(4) Repeatedly decreasing the value of the pair (w-ord, s) this way, we reach the case where w-ord = 0. The condition w-ord = 0 is equivalent to saying that the ideal is generated by some monomial of the defining equations of the components in the boundary divisor E . Thus we call the case where w-ord = 0 *the monomial case*.

(5) Finally, we have only to construct resolution of singularities in the monomial case, which can be done easily in characteristic zero.

The description of the inductive scheme above is only local, and hence we have overcome shortcomings ① and ③ only so far. The way we overcome shortcoming ② is discussed in §3.5 via the strand of invariants woven in §3.3.

In §3.3, we present the inductive scheme explained in §3.2 in terms of weaving the strand of invariants “ $\text{inv}_{\text{classic}}$ ”. The strand “ $\text{inv}_{\text{classic}}$ ” consists of the units of the form $(\dim H^j, \text{w-ord}^j, s^j)$, the dimension of the hypersurface of maximal contact H^j followed by the pair as described in §3.1 computed from the triplet $(H^j, (\mathcal{J}^j, b^j), F^j)$ at the j -th stage, and ends either with $(\dim H^m, \infty)$ or with $(\dim H^m, 0, \Gamma)$ depending on whether the last triplet $(H^m, (\mathcal{J}^m, b^m), F^m)$ is in Case (i) of the key inductive lemma or it is in the monomial case. That is to say, “ $\text{inv}_{\text{classic}}$ ” takes the following form

$$\begin{aligned} \text{“inv}_{\text{classic}}\text{”} &= (\dim H^0 = \dim W, \text{w-ord}^0, s^0) \cdots (\dim H^j, \text{w-ord}^j, s^j) \\ &\quad \cdots (\dim H^{m-1}, \text{w-ord}^{m-1}, s^{m-1}) \begin{cases} (\dim H^m, \infty), \text{ or} \\ (\dim H^m, 0, \Gamma). \end{cases} \end{aligned}$$

Note that we do not include the invariant s in the last unit.

We choose the center of blow up to be H^m when $\text{w-ord}^m = \infty$ according to Case (i) of the key inductive lemma, or the maximum locus of the invariant Γ on H^m when $\text{w-ord}^m = 0$ according to the procedure in the monomial case (which is discussed in §3.4). This is equivalent to choosing the center of blow up to be the maximum locus of “ $\text{inv}_{\text{classic}}$ ”. This leads to the decrease of the (maximum) value of “ $\text{inv}_{\text{classic}}$ ”. Showing that the (maximum) value of “ $\text{inv}_{\text{classic}}$ ” cannot decrease infinitely many times, we achieve resolution of singularities for $(W, (\mathcal{I}, a), E)$.

In §3.4, we discuss the procedure of constructing resolution of singularities in the monomial case, the only remaining task to complete the classical algorithm. In characteristic zero, this can be done easily and purely from the combinatorial data obtained by looking at the monomial in consideration, manifested as the invariant Γ .

The strand “ $\text{inv}_{\text{classic}}$ ” a priori depends on the choice of the hypersurfaces of maximal contact we take in the process of weaving, and it is a priori only locally defined. However, the strand “ $\text{inv}_{\text{classic}}$ ” is actually independent of the choice, and hence it is globally well-defined. This can be shown classically by the so-called Hironaka’s trick, or more recently by inserting Włodarczyk’s “homogenization” or the first author’s “differential saturation” into the construction of the modification. Therefore,

the process of resolution of singularities, where we take the center of blow up to be the maximum locus of “ $\text{inv}_{\text{classic}}$ ”, is also globally well-defined. This is how we overcome shortcoming ② of the key inductive lemma, accomplishing the globalization of the algorithm in §3.5.

This completes the overview of §3.

§4 is devoted to describing the general mechanism of our algorithm in positive characteristic, which is closely modeled upon the algorithm in characteristic zero explained in §3. (We remark that our algorithm is also valid in characteristic zero.)

In §4.1, we give the statement of a further reformulation (of Problem 2), which allows us to present the inductive structure on the invariant “ σ ”. We start from a triplet of data (W, \mathcal{R}, E) , where we replace the pair (\mathcal{I}, a) in the classical triplet $(W, (\mathcal{I}, a), E)$ with \mathcal{R} . Here $\mathcal{R} = \bigoplus_{a \in \mathbb{Z}_{\geq 0}} (\mathcal{I}_a, a)$ represents an idealistic filtration of i.f.g. type (short for “integrally and finitely generated type”), i.e., a finitely generated and graded (by the nonnegative integer $a \in \mathbb{Z}_{\geq 0}$ called the level of the ideal \mathcal{I}_a in the first factor) \mathcal{O}_W -algebra satisfying the condition $\mathcal{O}_W = \mathcal{I}_0 \supset \mathcal{I}_1 \supset \mathcal{I}_2 \supset \cdots \supset \mathcal{I}_a \supset \cdots$. We define its singular locus to be $\text{Sing}(\mathcal{R}) := \{P \in W \mid \text{ord}_P(\mathcal{I}_a) \geq a \quad \forall a \in \mathbb{Z}_{\geq 0}\}$. Then we are required to find a sequence of transformations (see §4.1 for the precise definition of a transformation) starting from $(W_0, \mathcal{R}_0, E_0) = (W, \mathcal{R}, E)$

$$\begin{aligned} (W_0, \mathcal{R}_0, E_0) &\leftarrow \cdots \leftarrow (W_i, \mathcal{R}_i, E_i) \xleftarrow{\pi_{i+1}} (W_{i+1}, \mathcal{R}_{i+1}, E_{i+1}) \\ &\leftarrow \cdots \leftarrow (W_l, \mathcal{R}_l, E_l) \end{aligned}$$

such that $\text{Sing}(\mathcal{R}_l, a) = \emptyset$. We call such a sequence “resolution of singularities for (W, \mathcal{R}, E) ”. So far it is perfectly parallel to the story in characteristic zero, and there is nothing unique to the case in positive characteristic. In fact, resolution of singularities for $(W, (\mathcal{I}, a), E)$ is equivalent to resolution of singularities for (W, \mathcal{R}, E) where the idealistic filtration of i.f.g. type is given by the formula $\mathcal{R} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} (\mathcal{I}^{\lceil n/a \rceil}, n)$. However, the reformulation allows us to introduce the invariant σ , which plays the key role in our inductive scheme.

In §4.2, we discuss the inductive scheme of our algorithm on the invariant σ in positive characteristic. Here, unlike in characteristic zero, we have no key inductive lemma. In fact, the statement of the key inductive lemma fails to hold in positive characteristic as is demonstrated by an example due to R. Narasimhan. That is to say, there is *no* smooth Hypersurface of Maximal Contact (HMC for short) in general. However, we can still introduce the notion of a *Leading Generator System* (LGS for short), which should be considered as a collective substitute in positive

characteristic for the notion of an HMC in characteristic zero. Our basic strategy is to follow the construction of the algorithm in characteristic zero, replacing an HMC (leading to the induction on dimension) with an LGS (leading to the induction on the invariant σ). The description of the new inductive scheme goes as follows.

(1) We introduce a triplet of invariants $(\sigma, \tilde{\mu}, s)$.

If $(\sigma, \tilde{\mu}, s) = (\sigma, \infty, 0)$ or $(\sigma, 0, 0)$, then we do not construct the modification (W', \mathcal{R}', E') .

- In case $(\sigma, \tilde{\mu}, s) = (\sigma, \infty, 0)$, we blow up with center $C = \text{Sing}(\mathcal{R})$. The nonsingularity of C is guaranteed by the Nonsingularity Principle (cf. [18] [19]), while the transversality of C to the boundary E is guaranteed by the invariant $s = 0$. After the blow up, the singular locus becomes empty, and resolution of singularities for (W, \mathcal{R}, E) is achieved.

- In case $(\sigma, \tilde{\mu}, s) = (\sigma, 0, 0)$, we are in the monomial case by definition, and we go to Step (5).

If $(\sigma, \tilde{\mu}, s) \neq (\sigma, \infty, 0)$ or $(\sigma, 0, 0)$, then we construct its associated triplet of data (W', \mathcal{R}', E') , called the modification of the original triplet (W, \mathcal{R}, E) , having the following properties. (Actually the ambient space for the modification stays the same, i.e., $W' = W$.)

- Resolution of singularities for (W', \mathcal{R}', E') implies the strict decrease of the (maximum) value of the triplet $(\sigma, \tilde{\mu}, s)$. Note that the value of the triplet never increases after transformations.

- Either the value of σ strictly decreases, or the value of σ stays the same but the number of the components in the boundary strictly drops. In either case, we have $(\sigma, \#E) > (\sigma', \#E')$.

(2) There is *no* key inductive lemma in our new setting.

(3) When $(\sigma, \tilde{\mu}, s) \neq (\sigma, \infty, 0)$ or $(\sigma, 0, 0)$ in (1), we achieve resolution of singularities for (W', \mathcal{R}', E') by induction on $(\sigma, \#E)$, which implies the strict decrease of the (maximum) value of the triplet $(\sigma, \tilde{\mu}, s)$.

(4) Repeatedly decreasing the value of the triplet $(\sigma, \tilde{\mu}, s)$, we reach the case where $(\sigma, \tilde{\mu}, s) = (\sigma, \infty, 0)$ or $(\sigma, 0, 0)$, the monomial case.

(5) Finally we have only to construct resolution of singularities in the monomial case in order to achieve resolution of singularities of the original triplet (W, \mathcal{R}, E) . However, the problem of resolution of singularities in the monomial case in positive characteristic is quite subtle and very difficult. We only provide a solution in dimension 3 in §5.

In §4.3, we present the inductive scheme explained in §4.2 in terms of weaving the strand of invariants “ inv_{new} ”. The strand “ inv_{new} ” consists of the units of the form $(\sigma^j, \tilde{\mu}^j, s^j)$ computed from the triplet $(W^j, \mathcal{R}^j, E^j)$ at the j -th stage, and ends either with $(\sigma^m, \infty, 0)$ or with

$(\sigma^m, 0, 0)$. That is to say, “ inv_{new} ” takes the following form

$$\begin{aligned} \text{“inv}_{\text{new}}\text{”} &= (\sigma^0, \tilde{\mu}^0, s^0) \cdots (\sigma^j, \tilde{\mu}^j, s^j) \\ &\cdots (\sigma^{m-1}, \tilde{\mu}^{m-1}, s^{m-1}) \begin{cases} (\sigma^m, \infty, 0), \text{ or} \\ (\sigma^m, 0, 0). \end{cases} \end{aligned}$$

Note that we *do* compute and include the invariant s in the last unit, in contrast to the weaving of “ $\text{inv}_{\text{classic}}$ ”.

Resolution of singularities of the last m -th triplet $(W^m, \mathcal{R}^m, E^m)$ can be achieved by taking the transformation with center $\text{Sing}(\mathcal{R}^m)$ when $(\sigma^m, \tilde{\mu}^m, s^m) = (\sigma^m, \infty, 0)$ or by resolution of singularities in the monomial case when $(\sigma^m, \tilde{\mu}^m, s^m) = (\sigma^m, 0, 0)$. This leads to the decrease of the value of “ inv_{new} ”. Showing that the value of “ inv_{new} ” cannot decrease infinitely many times, we (should) achieve resolution of singularities for (W, \mathcal{R}, E) (assuming that the problem of resolution of singularities in the monomial case is solved).

There is one important remark to make. In §4.2 we do *not* claim that the maximum locus of the triplet $(\sigma, \tilde{\mu}, s)$ coincides with the singular locus $\text{Sing}(\mathcal{R}')$ of the modified idealistic filtration, and in §4.3 we do *not* claim that the strand “ inv_{new} ” is a global invariant whose maximum locus gives the globally well-defined center of blow up for the algorithm. The discussion of our new inductive scheme and the weaving of the new strand of invariants are, therefore, only local. (See Remark 3 for the precise meaning of “local”.) In fact, “ inv_{new} ” as presented is independent of the choice of the LGS’s we take in the process of weaving, just like “ $\text{inv}_{\text{classic}}$ ” is independent of the choice of the HMC’s we take in the process of weaving in characteristic zero. Nevertheless, the gap between $\text{MaxLocus}(\sigma, \tilde{\mu}, s)$ and $\text{Sing}(\mathcal{R}')$ may occur when $\tilde{\mu} = 1$, and hence that the maximum locus of “ inv_{new} ” does not provide the global center of blow up as it is. This calamity is unique to our setting, and never existed in the classical setting. We can fix this calamity by making certain adjustments to the strand, the description of which is rather technical and will be published elsewhere. Therefore, in this paper, we restrict ourselves to the local description, which, we believe, still captures the essence of the inductive scheme on the invariant σ .

In §4.4, we mention briefly the reason why resolution of singularities in the monomial case is difficult in positive characteristic, while it is easy in characteristic zero.

This completes the overview of §4.

§5 is devoted to the detailed discussion of resolution of singularities in the monomial case in dimension 3, incorporating the method recently

developed by Benito-Villamayor into our framework with some improvements. In fact, we introduce a new invariant, which strictly drops after each blow up and hence effectively shows the termination of the procedure. This is the most subtle and difficult part of this paper.

The invariant τ represents the number of the elements in the LGS, which takes the value 0, 1, 2, 3 in dimension 3 and which never decreases under transformations. When $\tau = 0, 2, 3$, resolution of singularities in the monomial case is rather easy. Therefore, we focus our attention to the case where $\tau = 1$ in §5.

We are thus in the situation where (analytically) we have a unique element (h, p^e) in the LGS and where h is of the following form via Weierstrass Preparation Theorem with respect to a regular system of parameters (x, y, z) at $P \in W$, taken from $\widehat{\mathfrak{m}}_P \subset \widehat{\mathcal{O}}_{W,P}$,

$$h = z^{p^e} + a_1 z^{p^e-1} + \cdots + a_{p^e} \quad \text{with} \quad a_i \in k[[x, y]]$$

satisfying $\text{ord}_P(a_i) > i$ for $i = 1, \dots, p^e$. We also have a monomial

$$(x^\alpha y^\beta, a) \in \mathcal{R}$$

of the defining equations x and y of the components of the boundary divisor E (actually E_{young}) such that every element $(f, \lambda) \in \mathcal{R}$ is divisible by $(x^\alpha y^\beta)^{\lambda/a}$ modulo h , a consequence of the condition $\tilde{\mu} = 0$.

A naive idea for resolution of singularities in the monomial case may be stated as follows: Carry out the algorithm for resolution of singularities of $(x^\alpha y^\beta, a)$ on the hypersurface $\{z = 0\}$.

A bad news is that the above naive idea does not work for the reason that $(z, 1) \notin \mathcal{R}$ in general, which has the following bad implications:

(1) Even though we see that the coefficients a_i for $i = 1, \dots, p^e - 1$ are under control (in the sense that a_i is divisible by $(x^\alpha y^\beta)^{i/a}$), the constant term a_{p^e} is not well controlled. (The idealistic filtration \mathcal{R} in the monomial case is *not* differentially saturated but only saturated for $\{\partial^n / \partial z^n \mid n \in \mathbb{Z}_{\geq 0}\}$ in general. However, this is good enough to control the coefficients a_i for $i = 1, \dots, p^e - 1$.) This leads to the calamity that a candidate for the center determined by the naive idea may not even be contained in the singular locus.

(2) The hypersurface $\{z = 0\}$ may not be of maximal contact. That is to say, after a blow up, its strict transform may no longer “contact” (contain) the singular locus at all.

In §5.1, we introduce the process of “cleaning” in order to eliminate the “mess” described in the bad news above. The idea of “cleaning” can be seen already in the work of Abhyankar and Hironaka, in the

definition of the “residual order”. Here we follow the process refined by Benito-Villamayor. After cleaning, the invariant

$$H(P) := \min \left\{ \frac{1}{p^e} \text{ord}_P(a_{p^e}), \mu(P) = \frac{\alpha + \beta}{a} \right\}$$

is independent of the choice of h or a regular system of parameters (x, y, z) . The invariant H is well-defined not only at a closed point $P \in W$ but also at the generic point $\xi_{\{x=0\}}$ (resp. $\xi_{\{y=0\}}$) of a component $\{x=0\}$ (resp. $\{y=0\}$) of the boundary divisor E .

In §5.2, we give the description of the procedure of resolution of singularities, depending on the description of the singular locus. Since we are in the monomial case, we have $\text{Sing}(\mathcal{R}) \subset \{x=0\} \cup \{y=0\}$. The description of the singular locus restricted to $\{x=0\}$, locally around P , is given as follows according to the value of the invariant H :

$$\text{Sing}(\mathcal{R}) \cap \{x=0\} = \begin{cases} \{z=x=0\} & \text{if } H(\xi_{\{x=0\}}) \geq 1 \\ P & \text{if } H(\xi_{\{x=0\}}) < 1. \end{cases}$$

That is to say, by looking at the invariant H , we can tell if the singular locus has dimension 1, where it is a nonsingular curve, or has dimension zero, where it is an isolated point. We have a similar description of the singular locus restricted to $\{y=0\}$.

The procedure goes as follows:

Step 1. Check if $\dim \text{Sing}(\mathcal{R}) = 1$. If yes, then blow up the 1-dimensional components one by one. Since the invariant H strictly decreases for the component of the boundary divisor involved in the blow up, this step comes to an end after finitely many times with the dimension of the singular locus dropping to 0.

Step 2. Once $\dim \text{Sing}(\mathcal{R}) = 0$, blow up the isolated points in the singular locus.

Step 3. Go back to Step 1.

Repeat these steps.

§5.3 is devoted to showing termination of the procedure described in §5.2. We closely follow the beautiful and delicate argument recently developed by Benito-Villamayor, which analyzes the behavior of the monomial, the invariant H , and the newton polygon of the constant term a_{p^e} under blow ups. Their argument is an extension of the classical ideas of Abhyankar and Hironaka, but highly refined taking into consideration the condition that we are in the monomial case. Benito-Villamayor also uses “a proof by contradiction” in some part of their argument for

termination of the procedure. That is to say, they derive a contradiction, assuming the existence of an infinite sequence of the procedure. Therefore, their argument is not effective. They also use some “stratification” of the configuration of the boundary divisors. We introduce a new and explicit invariant, which makes the termination argument effective and which allows us to eliminate the use of “stratification” from our argument.

This completes the overview of §5, and hence the overview of the entire paper.

§3. A quick review on the algorithm in characteristic zero

The goal of this section is to give a quick review on the algorithm in characteristic zero, upon which our algorithm in positive characteristic is closely modeled. For the overview of this section, we refer the reader to §2.

3.1. Reformulation of the problem by Hironaka

First, we present the reformulation of the problem by Hironaka. The form of the presentation we use here is due to Villamayor.

Problem 3 (Hironaka’s reformulation). *Suppose we are given the triplet of data $(W, (\mathcal{I}, a), E)$, where W is a nonsingular variety over k , (\mathcal{I}, a) is a pair consisting of a coherent ideal sheaf \mathcal{I} on W and a positive integer $a \in \mathbb{Z}_{>0}$, and E is a simple normal crossing divisor on W .*

We define its singular locus to be

$$\text{Sing}(\mathcal{I}, a) := \{P \in W \mid \text{ord}_P(\mathcal{I}) \geq a\}.$$

We only consider a nonzero ideal sheaf \mathcal{I} for the resolution problem.

Then construct a sequence of transformations starting from $(W_0, (\mathcal{I}_0, a), E_0) = (W, (\mathcal{I}, a), E)$

$$\begin{aligned} (W_0, (\mathcal{I}_0, a), E_0) &\leftarrow \cdots \leftarrow (W_i, (\mathcal{I}_i, a), E_i) \xleftarrow{\pi_{i+1}} (W_{i+1}, (\mathcal{I}_{i+1}, a), E_{i+1}) \\ &\leftarrow \cdots \leftarrow (W_l, (\mathcal{I}_l, a), E_l) \end{aligned}$$

such that $\text{Sing}(\mathcal{I}_l, a) = \emptyset$. We call such a sequence “resolution of singularities for $(W, (\mathcal{I}, a), E)$ ”.

We note that the transformation

$$(W_i, (\mathcal{I}_i, a), E_i) \xleftarrow{\pi_{i+1}} (W_{i+1}, (\mathcal{I}_{i+1}, a), E_{i+1})$$

is required to satisfy the following conditions:

- (1) $W_i \xrightarrow{\pi_{i+1}^{-1}} W_{i+1}$ is a blow up with smooth center $C_i \subset W_i$,
- (2) $C_i \subset \text{Sing}(\mathcal{I}_i, a)$, and C_i is transversal to E_i (maybe contained in E_i), i.e., $C_i \pitchfork E_i$,
- (3) $\mathcal{I}_{i+1} = \mathcal{I}(\pi_{i+1}^{-1}(C_i))^{-a} \cdot \pi_{i+1}^{-1}(\mathcal{I}_i)\mathcal{O}_{W_{i+1}}$,
- (4) $E_{i+1} = E_i \cup \pi_{i+1}^{-1}(C_i)$.

Lemma 1. *A solution to Hironaka’s reformulation provides a solution to the problem of embedded resolution of singularities.*

Proof. Given $X \subset W$, set $(W, (\mathcal{I}, a), E) = (W, (\mathcal{I}_X, 1), \emptyset)$. Then $\text{Sing}(\mathcal{I}_0, a) = \text{Sing}(\mathcal{I}_X, 1) = X = X_0$. Take resolution of singularities for $(W, (\mathcal{I}, a), E)$. Observe that, if X_i , the strict transform of X in year i , is an irreducible component of $\text{Sing}(\mathcal{I}_i, a)$ and if X_i is not an irreducible component of the center C_i in year i (and X_j has not been an irreducible component of C_j in year j for $0 \leq j < i$), then X_{i+1} is an irreducible component of $\text{Sing}(\mathcal{I}_{i+1}, a)$ in year $(i + 1)$. Since $\text{Sing}(\mathcal{I}_l, a) = \emptyset$, we conclude that X_m must be an irreducible component of C_m for some $m < l$. Since the center C_m is nonsingular by requirement, so is X_m . Therefore, the truncation of the sequence up to year m provides a sequence for embedded resolution. (The other requirements for embedded resolution as stated in Problem 2 follow automatically from the construction.) Q.E.D.

The inductive scheme for solving Hironaka’s reformulation can be simply stated in the following naive form.

Naive Inductive Scheme: *Given a triplet $(W, (\mathcal{I}, a), E)$, find another triplet $(H, (\mathcal{J}, b), F)$ with $H \underset{\text{closed}}{\subset} W$ and $\dim H = \dim W - 1$ such that the problem of constructing resolution of singularities for $(W, (\mathcal{I}, a), E)$ is equivalent to constructing one for $(H, (\mathcal{J}, b), F)$, i.e.,*

$$(W, (\mathcal{I}, a), E) \underset{\text{equivalent}}{\sim} (H, (\mathcal{J}, b), F).$$

As it is, the above inductive scheme is too naive to hold in general. In §3.2, we first state the key inductive lemma, which realizes the naive inductive scheme under a certain extra condition called (\star) , and then discuss how to turn it into the real inductive scheme, which works in general without the extra condition.

3.2. Inductive scheme on dimension

Lemma 2 (Key Inductive Lemma). *Given $(W, (\mathcal{I}, a), E)$ and a closed point $P \in \text{Sing}(\mathcal{I}, a)$, suppose that the following condition (\star) is satisfied:*

$$(\star) \quad \begin{cases} \text{ord}_P(\mathcal{I}) &= a \\ E &= \emptyset. \end{cases}$$

Then there exists $(H, (\mathcal{J}, b), F)$ with $H \overset{\text{closed}}{\subset} W$ and $\dim H = \dim W - 1$, in a neighborhood of P , which satisfies one of the following two.

(i) The ideal \mathcal{J} is a zero sheaf, i.e., $\mathcal{J} \equiv 0$: In this case, we take the transformation with center $H = \text{Sing}(\mathcal{I}, a)$

$$(W, (\mathcal{I}, a), E) \leftarrow (\widetilde{W}, (\widetilde{\mathcal{I}}, a), \widetilde{E}).$$

After the transformation, we have $\text{Sing}(\widetilde{\mathcal{I}}, a) = \emptyset$ and hence we achieve resolution of singularities for $(W, (\mathcal{I}, a), E)$ (in a neighborhood of P).

(ii) The ideal \mathcal{J} is not a zero sheaf, i.e., $\mathcal{J} \not\equiv 0$: In this case, we have

$$(W, (\mathcal{I}, a), E) \underset{\text{equivalent}}{\sim} (H, (\mathcal{J}, b), F).$$

Therefore, constructing resolution of singularities for $(H, (\mathcal{J}, b), F)$ by induction on dimension, we achieve resolution of singularities for $(W, (\mathcal{I}, a), E)$ (in a neighborhood of P).

Proof. Take $f \in \mathcal{I}_P$ such that $\text{ord}_P(f) = a$. Then there exists a differential operator δ of $\text{deg } \delta = a - 1$ such that $\text{ord}_P(\delta f) = 1$. We note that this is exactly the place where we use the “characteristic zero” condition.

We have only to set

$$\begin{cases} H &= \{h = 0\} \text{ where } h = \delta f, \\ (\mathcal{J}, b) &= (\text{Coeff}(\mathcal{I})|_H, a!), \\ F &= E|_H = \emptyset, \end{cases}$$

where the “coefficient ideal”, denoted by $\text{Coeff}(\mathcal{I})$, is defined by the formula

$$\text{Coeff}(\mathcal{I}) := \sum_{j=0}^{a-1} \{ \text{Diff}^j(\mathcal{I}) \}^{a!/(a-j)}$$

with $\text{Diff}^j(\mathcal{I})$ being the sheaf characterized at the stalk for a point $Q \in W$ by

$$\text{Diff}^j(\mathcal{I})_Q = \{ \theta(g) \mid g \in \mathcal{I}_Q, \theta : \text{a differential operator of } \text{deg}(\theta) \leq j \}.$$

We note that the condition $E = \emptyset$ is only used to guarantee that H intersects E transversally. The condition on the boundary divisor can be weakened (and E can be non-empty) to the following: we can find such $H = \{h = 0\}$, with $h \in \text{Diff}^{a-1}(\mathcal{I})_P$ and $\text{ord}_P(h) = 1$, that is transversal to E . We then call the condition $(\star)_{\text{weakened}}$. Q.E.D.

Remark 1. We remark that the statement of the key inductive lemma fails to hold in positive characteristic, as the following example by R. Narasimhan shows:

Consider $(W, (\mathcal{I}, a), E)$ defined by

$$\begin{cases} W &= \mathbb{A}^4 = \text{Spec } k[x, y, z, w] \text{ with } \text{char}(k) = 2, \\ (\mathcal{I}, a) &= ((f), 2) \text{ with } f = w^2 + x^3y + y^3z + z^7x, \\ E &= \emptyset. \end{cases}$$

Note that the triplet $(W, (\mathcal{I}, a), E)$ satisfies the condition (\star) at the origin \mathbb{O} .

Then we have a curve C , parametrized by t , sitting inside of the singular locus

$$C = \{x = t^{15}, y = t^{19}, z = t^7, w = t^{32}\} \subset \text{Sing}(\mathcal{I}, a).$$

Observe that the curve C has full embedding dimension at the origin \mathbb{O} , i.e.,

$$\text{embedding-dim}_{\mathbb{O}} C = 4 = \dim W.$$

Therefore, there exists *no* smooth hypersurface H which contains the singular locus $\text{Sing}(\mathcal{I}, a)$, and hence there exists *no* smooth hypersurface of maximal contact.

We list the shortcomings of the key inductive lemma toward establishing the algorithm for resolution of singularities for $(W, (\mathcal{I}, a), E)$ in general.

List of shortcomings of the key inductive lemma

- ① We have to impose condition (\star) on $(W, (\mathcal{I}, a), E)$.
- ② We construct $(H, (\mathcal{J}, b), F)$ only locally, and hence the resolution process is only locally constructed by induction.
- ③ The invariant “ord” may strictly increase under transformations, even though our ultimate goal is to reduce the invariant “ord” to be below the fixed level a .

Now we describe the mechanism which overcomes all of the shortcomings above in one stroke, turning the key inductive lemma into the real inductive structure in characteristic zero.

Mechanism to overcome the shortcomings in the list

Outline of the mechanism

- (1) Given $(W, (\mathcal{I}, a), E)$ (Precisely speaking, the triplet sits in the middle of the sequence, say in year “ i ”, for resolution of singularities. However, we omit the subscript “ $()_i$ ” indicating the year for simplicity

of the notation.), we introduce a pair of invariants $(\text{w-ord}, s)$ and its associated triplet of data $(W, (\mathcal{K}, \kappa), G)$, called the modification of the original triplet, having the following properties.

- The maximum locus of the pair $(\text{w-ord}, s)$, which is an upper semi-continuous function, coincides with the singular locus $\text{Sing}(\mathcal{K}, \kappa)$ of the modification, i.e.,

$$\text{MaxLocus}(\text{w-ord}, s) = \text{Sing}(\mathcal{K}, \kappa).$$

Moreover, after each transformation, the value of the pair $(\text{w-ord}, s)$ never increases, and the locus where the value of the pair takes the same maximum value as the original one coincides with the singular locus of the transformation of the modification. (Note that the transformations of $(W, (\mathcal{K}, \kappa), G)$ induce those of $(W, (\mathcal{I}, a), E)$.) This means that resolution of singularities for $(W, (\mathcal{K}, \kappa), G)$ implies the strict decrease of the (maximum) value of the pair $(\text{w-ord}, s)$.

- Even though the original triplet $(W, (\mathcal{I}, a), E)$ may not satisfy condition (\star) (or condition $(\star)_{\text{weakened}}$), the modification $(W, (\mathcal{K}, \kappa), G)$ satisfies $(\star)_{\text{weakened}}$.

(2) We apply the key inductive lemma to $(W, (\mathcal{K}, \kappa), G)$. We find a triplet $(H, (\mathcal{J}, b), F)$ with $H \subset_{\text{closed}} W$ and $\dim H = \dim W - 1$ such that we are either in Case (i), or in Case (ii) where resolution of singularities for $(H, (\mathcal{J}, b), F)$ implies the one for $(W, (\mathcal{K}, \kappa), G)$. (We note that $G = E_{\text{new}}$ may not be empty. However, we can still find a hypersurface of maximal contact H which is transversal to $G = E_{\text{new}}$. Therefore, the triplet $(H, (\mathcal{J}, b), F)$ with $F = G|_H$ works.)

(3) In Case (i), by a simple blow up with center H , we accomplish resolution of singularities for $(W, (\mathcal{K}, \kappa), G)$. In Case (ii), by induction on dimension, we achieve resolution of singularities for $(H, (\mathcal{J}, b), F)$, hence for $(W, (\mathcal{K}, \kappa), G)$. In both cases, we have the strict decrease of the (maximum) value of the pair $(\text{w-ord}, s)$.

(4) By repeating the above procedures (1), (2), (3) and decreasing the value of the pair $(\text{w-ord}, s)$, we reach the stage where $\text{w-ord} = 0$, i.e., $\overline{\mathcal{I}} = \mathcal{O}_W$. This means that we are in the monomial case, where the ideal \mathcal{I} is generated by some monomial of the defining equations of the components in $E_{\text{young}} \subset E$.

(5) Finally, we have only to construct resolution of singularities for the triplet in the monomial case, which can be done easily in characteristic zero.

We give the detailed and explicit description of the pair $(\text{w-ord}, s)$, the triplets $(W, (\mathcal{K}, \kappa), G)$ and $(H, (\mathcal{J}, b), F)$ in the following:

Description of the pair (w-ord, s)

◦ w-ord: It is the so-called (normalized) weak order. It is the order of $\overline{\mathcal{I}}$ (divided by the level a), where $\overline{\mathcal{I}}$ is obtained from \mathcal{I} by dividing it as much as possible by the defining equations of the components in E_{young} . The symbol E_{young} refers to the union of the exceptional divisors created after the process of resolution of singularities began. By abuse of notation, the symbol E_{young} (or E) sometimes refers to the set having its components as its elements.

◦ s : It is the number of the components in $E_{\text{old}} = E \setminus E_{\text{new}}$. The symbol E_{new} refers to the union of the exceptional divisors created after the time when the current value of w-ord first started.

Description of the triplet ($W, (\mathcal{K}, \kappa), G$)

$$W = W, \quad (\mathcal{K}, \kappa) = \text{Bdry}(\text{Comp}(\mathcal{I}, a)), \quad \text{and} \quad G = E \setminus E_{\text{old}} = E_{\text{new}},$$

where

• $\text{Comp}(\mathcal{I}, a)$ is either the transformation of the one in the previous year if w-ord stays the same, or

$$(\mathcal{I}^M + \overline{\mathcal{I}}^a, M \cdot a) \quad \text{with} \quad M = \text{w-ord} \cdot a$$

if w-ord strictly decreases, and where

• $\text{Bdry}(\text{Comp}(\mathcal{I}, a))$ is either the transformation of the one in the previous year if (w-ord, s) stays the same, or

$$(\mathcal{C} + (\sum_{D \in E_{\text{old}}} \mathcal{I}(D))^c, c) \quad \text{where} \quad (\mathcal{C}, c) = \text{Comp}(\mathcal{I}, a),$$

if (w-ord, s) strictly decreases.

We note that we have $E_{\text{new}} \subset E_{\text{young}}$ and that they may not be equal in general. We also note that, if the value of the pair (w-ord, s) stays the same as in the previous year, then (\mathcal{K}, κ) is the transformation of the one in the previous year. We remark that the symbols ‘‘Comp’’ and ‘‘Bdry’’ represent the ‘‘Companion’’ modification and the ‘‘Boundary’’ modification, respectively.

Description of the triplet ($H, (\mathcal{J}, b), F$)

Case: The value of the pair (w-ord, s) stays the same as in the previous year.

In this case, we simply take $(H, (\mathcal{J}, b), F)$ to be the transformation of the one in the previous year under blow up.

Case: The value of the pair (w-ord, s) strictly decreases.

In this case, we construct $(H, (\mathcal{J}, b), F)$ as follows.

$$\begin{cases} H & = \text{the strict transform of } H_{i_{\text{old}}}, \\ (\mathcal{J}, b) & = (\text{Coeff}(\mathcal{K})|_H, \kappa!), \\ F & = G|_H, \end{cases}$$

where $H_{i_{\text{old}}}$ is taken in the following way: We go back to the year $\iota := i_{\text{old}}$ when the current value of w-ord first started. Let $(\mathcal{C}_\iota, c_\iota) = \text{Comp}(\mathcal{I}_\iota, a)$ be the companion modification constructed in year ι . We take $f_\iota \in (\mathcal{C}_\iota)_{P_\iota}$ and a differential operator δ_ι of $\text{deg } \delta_\iota = c_\iota - 1$ such that $\text{ord}_{P_\iota}(f_\iota) = c_\iota$ and $\text{ord}_{P_\iota}(\delta_\iota f_\iota) = 1$. We set $H_{i_{\text{old}}} = H_\iota = \{\delta_\iota f_\iota = 0\}$.

This completes the discussion of the mechanism to achieve resolution of singularities for $(W, (\mathcal{I}, a), E)$ in general. We note, however, that we only overcome shortcomings ① and ③ on the list, since the description so far is only local. We discuss in §3.5 how to overcome shortcoming ② and how to globalize the procedure via the strand of invariants woven in the next section.

3.3. Weaving of the classical strand of invariants “ $\text{inv}_{\text{classic}}$ ”

In §3.3, we interpret the inductive scheme explained in §3.2 in terms of weaving the strand of invariants “ $\text{inv}_{\text{classic}}$ ”, whose maximum locus (with respect to the lexicographical order) determines the center of blow up for the algorithm for resolution of singularities in characteristic zero.

We weave the strand of invariants “ $\text{inv}_{\text{classic}}$ ” consisting of the units of the form $(\dim H^j, \text{w-ord}^j, s^j)$, computed from the modifications $(H^j, (\mathcal{J}^j, b^j), F^j)$ constructed simultaneously along the weaving process. We note that we are adding “ $\dim H$ ” to the pair $(\text{w-ord}, s)$ as the first factor of the unit, in order to emphasize the role of the dimension in the inductive scheme.

Weaving Process

We describe the weaving process inductively.

Note that constructing a sequence for resolution of singularities by blowing up is referred to as “proceeding in the vertical direction” passing from one year to the next, indicated by the subscript “ i ”, while weaving the strand and constructing the modifications passing from one stage to the next, indicated by the superscript “ j ”, is referred to as “proceeding in the horizontal direction” staying in a fixed year.

Suppose we have already woven the strands and constructed the modifications up to year $(i - 1)$.

Now we are in year i (looking at the neighborhood of a closed point $P_i \in W_i$).

We start with $(W_i, (\mathcal{I}_i, a), E_i) = (H_i^0, (\mathcal{J}_i^0, b_i^0), F_i^0)$, just renaming the transformation $(W_i, (\mathcal{I}_i, a), E_i)$ in year i of the resolution sequence as the 0-th stage modification $(H_i^0, (\mathcal{J}_i^0, b_i^0), F_i^0)$ in year i .

Suppose that we have already woven the strand up to the $(j-1)$ -th unit

$$(\text{inv}_{\text{classic}})_i^{\leq j-1} = (\dim H_i^0, \text{w-ord}_i^0, s_i^0)(\dim H_i^1, \text{w-ord}_i^1, s_i^1) \cdots (\dim H_i^{j-1}, \text{w-ord}_i^{j-1}, s_i^{j-1})$$

and that we have also constructed the modifications up to the j -th one

$$(H_i^0, (\mathcal{J}_i^0, b_i^0), F_i^0), \dots, (H_i^{j-1}, (\mathcal{J}_i^{j-1}, b_i^{j-1}), F_i^{j-1}), (H_i^j, (\mathcal{J}_i^j, b_i^j), F_i^j).$$

Our task is to compute the j -th unit $(\dim H_i^j, \text{w-ord}_i^j, s_i^j)$ and construct the $(j+1)$ -th modification $(H_i^{j+1}, (\mathcal{J}_i^{j+1}, b_i^{j+1}), F_i^{j+1})$ (unless the weaving process is over at the j -th stage).

Computation of the j -th unit $(\dim H_i^j, \text{w-ord}_i^j, s_i^j)$

◦ $\dim H_i^j$: We just remark that we insert this first factor in characteristic zero only to emphasize the role of the dimension, which corresponds to the role of the invariant σ in our algorithm in positive characteristic.

◦ w-ord_i^j : We compute the second factor as follows.

$$\text{w-ord}_i^j = \begin{cases} \infty & \text{if } \mathcal{J}_i^j \equiv 0, \\ \text{ord}(\overline{\mathcal{J}_i^j})/b_i^j & \text{if } \mathcal{J}_i^j \not\equiv 0, \end{cases}$$

where the ideal $\overline{\mathcal{J}_i^j}$ is obtained from \mathcal{J}_i^j by dividing the latter as much as possible by the defining ideals of the components in $(F_i^j)_{\text{young}}$, i.e.,

$$\overline{\mathcal{J}_i^j} = \left(\prod_{D \in (F_i^j)_{\text{young}}} \mathcal{I}(D)^{-\text{ord}_{\eta(D)}(\mathcal{J}_i^j)} \right) \cdot \mathcal{J}_i^j$$

where $\eta(D)$ is the generic point of D and where $(F_i^j)_{\text{young}} (\subset F_i^j)$ is the union of the exceptional divisors created after the year when the value $(\text{inv}_{\text{classic}})_i^{\leq j-1} (\dim H_i^j)$ first started.

We note that, if $\text{w-ord}_i^j = \infty$ or 0, we declare that the $(j = m)$ -th unit is the last one, and we stop the weaving process at the m -th stage in year i . When $\text{w-ord}_i^j = \infty$, the third factor is *not* included. When $\text{w-ord}_i^j = 0$, we are in the monomial case and we insert the invariant Γ as the third factor instead of the invariant s .

$\circ s_i^j$: It is the number of the components in $(F_i^j)_{\text{old}} = F_i^j \setminus (F_i^j)_{\text{new}}$, where $(F_i^j)_{\text{new}} (\subset F_i^j)$ is the union of the exceptional divisors created after the year when the value $(\text{inv}_{\text{classic}})_i^{\leq j-1} (\dim H_i^j, \text{w-ord}_i^j)$ first started. We note that the third factor s is *only* included if $\text{w-ord}_i^j \neq \infty$ or 0 .

At the end, the weaving process of the strand comes to an end in a fixed year i , with $(\text{inv}_{\text{classic}})_i$ taking the following form

$$(\text{inv}_{\text{classic}})_i = (\dim H_i^0, \text{w-ord}_i^0, s_i^0) \cdots (\dim H_i^j, \text{w-ord}_i^j, s_i^j) \cdots (\dim H_i^{m-1}, \text{w-ord}_i^{m-1}, s_i^{m-1}) \begin{cases} (\dim H_i^m, \text{w-ord}_i^m = \infty), \text{ or} \\ (\dim H_i^m, \text{w-ord}_i^m = 0, \Gamma). \end{cases}$$

Termination in the horizontal direction

We note that termination of the weaving process in the horizontal direction is a consequence of the fact that going from the j -th unit to the $(j + 1)$ -th unit we have $\dim H_i^j > \dim H_i^{j+1}$ and that the dimension obviously satisfies the descending chain condition.

Construction of the $(j+1)$ -th modification $(H_i^{j+1}, (\mathcal{J}_i^{j+1}, b_i^{j+1}), F_i^{j+1})$

We note that we construct the $(j + 1)$ -th modification only when $\text{w-ord}_i^j \neq \infty$ or 0 .

Case: $(\text{inv}_{\text{classic}})_i^{\leq j} = (\text{inv}_{\text{classic}})_{i-1}^j$.

In this case, we simply take $(H_i^{j+1}, (\mathcal{J}_i^{j+1}, b_i^{j+1}), F_i^{j+1})$ to be the transformation of $(H_{i-1}^{j+1}, (\mathcal{J}_{i-1}^{j+1}, b_{i-1}^{j+1}), F_{i-1}^{j+1})$ under the blow up.

Case: $(\text{inv}_{\text{classic}})_i^{\leq j} < (\text{inv}_{\text{classic}})_{i-1}^j$.

In this case, we follow the construction described in the mechanism discussed in §3.2.

Starting from $(H_i^j, (\mathcal{J}_i^j, b_i^j), F_i^j)$, we firstly construct $(H_i^j, (\mathcal{K}_i^j, \kappa_i^j), G_i^j)$, whose description is given below.

$$H_i^j = H_i^j, (\mathcal{K}_i^j, \kappa_i^j) = \text{Bdry}(\mathcal{C}_i^j, c_i^j), \text{ and } G_i^j = F_i^j \setminus (F_i^j)_{\text{old}} = (F_i^j)_{\text{new}},$$

where

$$(\mathcal{C}_i^j, c_i^j) = \text{Comp}(\mathcal{J}_i^j, b_i^j), \quad \text{Bdry}(\mathcal{C}_i^j, c_i^j) = (\mathcal{C}_i^j + (\sum_{D \subset (F_i^j)_{\text{old}}} \mathcal{I}(D))^{c_i^j}, c_i^j),$$

and, denoting $(\text{inv}_{\text{classic}})_i^{\leq j-1} (\dim H_i^j, \text{w-ord}_i^j)$ by α_i^j , we set $\text{Comp}(\mathcal{J}_i^j, b_i^j)$ to be either the transformation of $\text{Comp}(\mathcal{J}_{i-1}^j, b_{i-1}^j)$ if $\alpha_i^j = \alpha_{i-1}^j$, or

$$(\mathcal{J}_i^j M_i^j + \overline{\mathcal{J}_i^j}^{b_i^j}, M_i^j \cdot b_i^j) \quad \text{with} \quad M_i^j = \text{w-ord}_i^j \cdot b_i^j$$

if $\alpha_i^j < \alpha_{i-1}^j$.

Then we secondly construct $(H_i^{j+1}, (\mathcal{J}_i^{j+1}, b_i^{j+1}), F_i^{j+1})$ as follows.

$$\left\{ \begin{array}{l} H_i^{j+1} = \text{the strict transform of } H_{i_{\text{old}}^j}^{j+1}, \\ (\mathcal{J}_i^{j+1}, b_i^{j+1}) = (\text{Coeff}(\mathcal{K}_i^j)|_{H_i^{j+1}}, (\kappa_i^j)!) \\ F_i^{j+1} = G_i^j|_{H_i^{j+1}} = (F_i^j)_{\text{new}}|_{H_i^{j+1}}, \end{array} \right.$$

where $H_{i_{\text{old}}^j}^{j+1}$ is taken in the following way: We go back to the year $\iota := i_{\text{old}}^j$ when the current value of $\text{inv}_i^{\leq j}(\dim H_i^j, \text{w-ord}_i^j)$ first started. Let $(\mathcal{C}_\iota^j, c_\iota^j) = \text{Comp}(\mathcal{I}_\iota^j, b_\iota^j)$ be the companion modification constructed in year ι . We take $f_\iota^j \in (\mathcal{C}_\iota^j)_{P_\iota}$ and a differential operator δ_ι^j of $\text{deg } \delta_\iota^j = c_\iota^j - 1$ such that $\text{ord}_{P_\iota}(f_\iota^j) = c_\iota^j$ and $\text{ord}_{P_\iota}(\delta_\iota^j f_\iota^j) = 1$. We set $H_{i_{\text{old}}^j}^{j+1} = H_\iota^{j+1} = \{\delta_\iota^j f_\iota^j = 0\}$.

Summary of the algorithm in $\text{char}(k) = 0$ in terms of “ $\text{inv}_{\text{classic}}$ ”

We start with $(W, (\mathcal{I}, a), E) = (W_0, (\mathcal{I}_0, a), E_0)$.

Suppose we have already constructed the resolution sequence up to year i

$$(W, (\mathcal{I}, a), E) = (W_0, (\mathcal{I}_0, a), E_0) \leftarrow \cdots \leftarrow (W_i, (\mathcal{I}_i, a), E_i).$$

We weave the strand of invariants in year i described as above

$$\begin{aligned} (\text{inv}_{\text{classic}})_i &= (\dim H_i^0, \text{w-ord}_i^0, s_i^0) \cdots (\dim H_i^j, \text{w-ord}_i^j, s_i^j) \\ &\cdots (\dim H_i^{m-1}, \text{w-ord}_i^{m-1}, s_i^{m-1}) \begin{cases} (\dim H_i^m, \text{w-ord}_i^m = \infty), \text{ or} \\ (\dim H_i^m, \text{w-ord}_i^m = 0, \Gamma). \end{cases} \end{aligned}$$

There are two cases according to the form of the last unit:

Case: $\text{w-ord}_i^m = \infty$.

In this case, we take the center of blow up in year i for the transformation to be the last hypersurface of maximal contact H_i^m .

Case: $\text{w-ord}_i^m = 0$.

In this case, we follow the procedure specified for resolution of singularities in the monomial case in §3.4, and take the center of blow up in year i for the transformation to be the maximum locus of the invariant Γ on H_i^m .

In both cases, the center of blow up coincides with the maximum locus of the strand $(\text{inv}_{\text{classic}})_i$.

Termination in the vertical direction

The value of the strand “ $\text{inv}_{\text{classic}}$ ” never increases after the blow up described as above. By construction, the maximum locus of $(\text{inv}_{\text{classic}})^{\leq j}$ coincides with the singular locus of $(H^{j+1}, (\mathcal{J}^{j+1}, b^{j+1}), F^{j+1})$. Therefore, resolution of singularities for $(H^{j+1}, (\mathcal{J}^{j+1}, b^{j+1}), F^{j+1})$ implies the strict decrease of $(\text{inv}_{\text{classic}})^{\leq j}$. In particular, in the first case, $(\text{inv}_{\text{classic}})^{\leq m-1}$ strictly decreases. In the second case, either $(\text{inv}_{\text{classic}})^{\leq m-1}$ strictly decreases, or while $(\text{inv}_{\text{classic}})^{\leq m-1}$ may remain the same (and hence so does $(\text{inv}_{\text{classic}})^{\leq m-1}(\dim H_i^m, \text{w-ord}_i^m = 0)$), the invariant Γ strictly decreases. After all, we conclude that the value of the strand strictly decreases after each blow up, i.e., we have $(\text{inv}_{\text{classic}})_i > (\text{inv}_{\text{classic}})_{i+1}$. Now we claim that the value of the strand “ $\text{inv}_{\text{classic}}$ ” cannot decrease infinitely many times. In fact, suppose by induction we have shown that the value of $(\text{inv}_{\text{classic}})^{\leq t-1}$ cannot decrease infinitely many times. Then after some year, the value of $(\text{inv}_{\text{classic}})^{\leq t-1}$ stabilizes. This in turn implies that the value b^t , the second factor of the pair in the t -th modification, stays the same, say b . Now the value of w-ord^t , having the fixed denominator b , cannot decrease infinitely many times, and neither can the value s^t being the nonnegative integer. Therefore, we conclude that the value of $(\text{inv}_{\text{classic}})^{\leq t}$ cannot decrease infinitely many times. As the value of t increases, the value of the dimension decreases by one. Since obviously the value of the dimension satisfies the descending chain condition, the increase of the value of t stops after finitely many times. Finally, therefore, we conclude that the value of the strand “ $\text{inv}_{\text{classic}}$ ” cannot decrease infinitely many times.

Therefore, the algorithm terminates after finitely many years, achieving resolution of singularities for $(W, (\mathcal{I}, a), E)$.

3.4. The monomial case in characteristic zero

The purpose of §3.4 is to discuss how to construct resolution of singularities for $(W, (\mathcal{I}, a), E)$ which is in the monomial case. (Precisely speaking, the triplet sits in the middle of the sequence, say in year “ i ”, for resolution of singularities. However, we omit the subscript “ $()_i$ ” indicating the year for simplicity of the notation.)

Recall that, in the monomial case, \mathcal{I} is a monomial of the ideals defining the components of $E_{\text{young}} = \bigcup_{t=1}^e D_t \subset E$ (See §3.3 for the definition of E_{young}), i.e.,

$$\mathcal{I} = \prod_{t=1}^e \mathcal{I}(D_t)^{c_t} \quad \text{with} \quad c_t \in \mathbb{Z}_{\geq 0}.$$

Invariant “ Γ ”

Definition 1 (Invariant “ Γ ”). Let the situation be as above. We define the invariant $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$ at $P \in \text{Sing}(\mathcal{I}, a)$ in the following way:

$$\begin{aligned} \Gamma_1 &= -\min\{n \mid \exists t_1, \dots, t_n \text{ s.t. } c_{t_1} + \dots + c_{t_n} \geq a, P \in D_{t_1} \cap \dots \cap D_{t_n}\}, \\ \Gamma_2 &= \max\{(c_{t_1} + \dots + c_{t_n})/a \mid n = -\Gamma_1, P \in D_{t_1} \cap \dots \cap D_{t_n}\}, \\ \Gamma_3 &= \max\left\{(t_1, \dots, t_n) \mid \begin{array}{l} n = -\Gamma_1, (c_{t_1} + \dots + c_{t_n})/a = \Gamma_2, \\ P \in D_{t_1} \cap \dots \cap D_{t_n}, t_1 < \dots < t_n \end{array} \right\}. \end{aligned}$$

It is immediate to see the following properties of the invariant “ Γ ”.

- (1) The invariant Γ is an upper semi-continuous function.
- (2) The maximum locus $\text{MaxLocus}(\Gamma)$ is nonsingular, since it is the intersection of some components in $E_{\text{young}} \subset E$, a simple normal crossing divisor.

Procedure and its termination

Now take the transformation with center $C = \text{MaxLocus}(\Gamma)$

$$(W, (\mathcal{I}, a), E) \xrightarrow{\pi} (W', (\mathcal{I}', a), E')$$

where $E' = E \cup \pi^{-1}(C)$ and $E'_{\text{young}} = E_{\text{young}} \cup \pi^{-1}(C) = \bigcup_{t=1}^{e+1} D_t$ with $D_{e+1} = \pi^{-1}(C)$. Then it is easy to see that $(W', (\mathcal{I}', a), E')$ is again in the monomial case and that the invariant Γ strictly decreases, i.e.,

$$\Gamma > \Gamma'.$$

Since the value of Γ cannot decrease infinitely many times, this procedure must terminate after finitely many years, achieving resolution of singularities for $(W, (\mathcal{I}, a), E)$ in the monomial case.

This completes the discussion on how to construct resolution of singularities for $(W, (\mathcal{I}, a), E)$ in the monomial case.

3.5. Globalization

The strand “ $\text{inv}_{\text{classic}}$ ” a priori depends on the choice of the hyper-surfaces of maximal contact we take in the process of weaving, and it is a priori only locally defined. However, the strand “ $\text{inv}_{\text{classic}}$ ” is actually independent of the choice, and hence it is globally well-defined. This can be shown classically by the so-called Hironaka’s trick, or more recently by incorporating Włodarczyk’s “homogenization” (cf.[23]) or the first author’s “differential saturation” (cf.[18]) in the construction of the modification. Therefore, the process of resolution of singularities, where

we take the center of blow up to be the maximum locus of “ $\text{inv}_{\text{classic}}$ ”, is also globally well-defined. This is how we overcome shortcoming (2) of the key inductive lemma.

This finishes the quick review on the classical algorithm in characteristic zero.

§4. Our algorithm in positive characteristic

The goal of this section is to discuss the general mechanism of our algorithm in positive characteristic, which is modeled closely upon the classical algorithm in characteristic zero reviewed in §3.

4.1. Reformulation of the problem in our setting

First, we present the reformulation of the problem in our setting.

Problem 4 (Reformulation in terms of an idealistic filtration (cf. [18] [19])). *Suppose we are given the triplet of data (W, \mathcal{R}, E) such that W is a nonsingular variety over k , $\mathcal{R} = \bigoplus_{a \in \mathbb{Z}_{\geq 0}} (\mathcal{I}_a, a)$ is an idealistic filtration of i.f.g. type (resp. of i.g. type), i.e., a finitely generated graded \mathcal{O}_W -algebra (resp. a graded \mathcal{O}_W -algebra) satisfying the condition $\mathcal{O}_W = \mathcal{I}_0 \supset \mathcal{I}_1 \supset \mathcal{I}_2 \cdots \supset \mathcal{I}_a \supset \cdots$, where “ a ” in the second factor specifies the “level” of the ideal \mathcal{I}_a in the first factor, and E is a simple normal crossing divisor on W .*

We define its singular locus to be

$$\text{Sing}(\mathcal{R}) := \{P \in W \mid \text{ord}_P(\mathcal{I}_a) \geq a \quad \forall a \in \mathbb{Z}_{\geq 0}\}.$$

(We note that we only consider \mathcal{R} with $\mathcal{I}_1 \neq 0$ for the resolution problem.)

Then construct a sequence of transformations starting from $(W_0, \mathcal{R}_0, E_0) = (W, \mathcal{R}, E)$

$$\begin{aligned} (W_0, \mathcal{R}_0, E_0) &\leftarrow \cdots \leftarrow (W_i, \mathcal{R}_i, E_i) \xrightarrow{\pi_{i+1}^{i+1}} (W_{i+1}, \mathcal{R}_{i+1}, E_{i+1}) \\ &\leftarrow \cdots \leftarrow (W_l, \mathcal{R}_l, E_l) \end{aligned}$$

such that $\text{Sing}(\mathcal{R}_l) = \emptyset$.

We call such a sequence “resolution of singularities for (W, \mathcal{R}, E) ”.

We note that the transformation

$$(W_i, \mathcal{R}_i, E_i) \xrightarrow{\pi_{i+1}^{i+1}} (W_{i+1}, \mathcal{R}_{i+1}, E_{i+1})$$

is required to satisfy the following conditions:

- (1) $W_i \xrightarrow{\pi_{i+1}^{i+1}} W_{i+1}$ is a blow up with smooth center $C_i \subset W_i$,

- (2) $C_i \subset \text{Sing}(\mathcal{R}_i)$, and C_i is transversal to E_i (maybe contained in E_i), i.e., $C \pitchfork E_i$;
 (3) $\mathcal{R}_{i+1} = \mathcal{G}(\bigcup_{a \in \mathbb{Z}_{>0}} (\mathcal{J}_{a,i+1}, a))$, where

$$\mathcal{J}_{a,i+1} = \mathcal{I}(\pi_{i+1}^{-1}(C_i))^{-a} \cdot \pi_{i+1}^{-1}(\mathcal{I}_{a,i}) \mathcal{O}_{W_{i+1}} \quad \text{for } a \in \mathbb{Z}_{\geq 0},$$

i.e., \mathcal{R}_{i+1} is the smallest idealistic filtration of i.f.g. type containing $(\mathcal{J}_{a,i+1}, a)$ for all $a \in \mathbb{Z}_{\geq 0}$ (We note that $\mathcal{I}_{a,i+1} \supset \mathcal{J}_{a,i+1}$ but they may not be equal in general.),

(4) $E_{i+1} = E_i \cup \pi_{i+1}^{-1}(C_i)$.

Remark 2 (Local version). Problem 4 is the “global” version of the problem of resolution of singularities for the triplet of data (W, \mathcal{R}, E) . In the following, we formulate its local version: Starting from a closed point $P \in \text{Sing}(\mathcal{R}) \subset W$ and its neighborhood, we have a sequence of closed points and their neighborhoods

$$P_0 \in \text{Sing}(\mathcal{R}_0) \subset W_0 \leftarrow P_1 \in \text{Sing}(\mathcal{R}_1) \subset W_1 \leftarrow \cdots \leftarrow P_i \in \text{Sing}(\mathcal{R}_i) \subset W_i$$

in the resolution sequence, where $W = W_0$, $\mathcal{R} = \mathcal{R}_0$ and $P = P_0$. After we choose the center $P_i \in C_i \subset \text{Sing}(\mathcal{R}_i) \subset W_i$ and take the transformation $W_i \xrightarrow{\pi_{i+1}^{-1}} W_{i+1}$ to extend the resolution sequence, the “devil” tries to choose a closed point $P_{i+1} \in \pi_i^{-1}(P_i) \cap \text{Sing}(\mathcal{R}_{i+1}) \subset W_i$. If $\pi_i^{-1}(P_i) \cap \text{Sing}(\mathcal{R}_{i+1}) = \emptyset$, then the devil loses. Our task is to provide a prescription on how to choose the center so that, no matter how the devil makes his choice, he will end up losing. That is to say, the prescription should guarantee that we ultimately reach year $i = l-1$ so that, with the choice of center C_{l-1} , after the blow up we have $\pi_l^{-1}(P_{l-1}) \cap \text{Sing}(\mathcal{R}_l) = \emptyset$.

Our algorithm discussed in this paper is exclusively for this local version of the problem of resolution of singularities for the triplet of data (W, \mathcal{R}, E) . The adjustments we have to make to our algorithm in order to solve the global version of the problem will be published elsewhere.

The notion of “the differential saturation” \mathcal{DR} (of an idealistic filtration \mathcal{R} of i.f.g. type) plays an important role in our algorithm.

Definition 2. Let \mathcal{R} be an idealistic filtration of i.f.g. type. We define its differential saturation \mathcal{DR} (at the level of the stalk for a point $P \in W$) as follows:

$$\mathcal{DR}_P := \mathcal{G}(\{(\delta f, \max\{a - \deg \delta, 0\}) \mid (f, a) \in \mathcal{R}_P, \delta: \text{a diff. op.}\}),$$

where the symbol $\mathcal{G}(S)$ denotes “an idealistic filtration of i.g. type generated by the set S ”, i.e., the smallest idealistic filtration of i.g. type containing S .

Remark 3.

(1) It follows immediately from the generalized product rule (cf. [18]) that \mathcal{DR} is again an idealistic filtration of i.f.g. type, $\mathcal{R} \subset \mathcal{DR}$, and that it is differentially saturated, i.e., $\mathcal{D}(\mathcal{DR}) = \mathcal{DR}$.

(2) The problem of constructing resolution of singularities for (W, \mathcal{R}, E) is equivalent to the one for (W, \mathcal{DR}, E) , i.e.,

$$(W, \mathcal{R}, E) \underset{\text{equivalent to}}{\sim} (W, \mathcal{DR}, E).$$

4.2. Inductive scheme on the invariant “ σ ”

The classical algorithm in characteristic zero works by induction on dimension, based upon the notion of a *smooth* hypersurface of maximal contact, as reviewed in §3. Narasimhan’s example (cf. Remark 1 in §3) tells us, however, that there is no hope of finding a *smooth* hypersurface of maximal contact in positive characteristic. The following proposition gives rise to the notion of “a leading generator system” (called an LGS for short), which we consider as a collective substitute in positive characteristic for the notion of a hypersurface of maximal contact (called an HMC for short) in characteristic zero. Our algorithm in positive characteristic works by induction on the invariant “ σ ”, based upon the notion of an LGS. Roughly speaking, introducing the notion of an LGS corresponds to considering *singular* hypersurfaces of maximal contact.

Definition of the invariant “ σ ”

Proposition 1 (cf. [18]). *Let $\mathcal{R} = \bigoplus_{a \in \mathbb{Z}_{\geq 0}} (\mathcal{I}_a, a)$ be an idealistic filtration of i.f.g. type. Assume that \mathcal{R} is differentially saturated, i.e., $\mathcal{R} = \mathcal{DR}$. Fix a closed point $P \in \text{Sing}(\mathcal{R}) \subset W$.*

Consider the leading algebra $L_P(\mathcal{R})$

$$L_P(\mathcal{R}) := \bigoplus_{a \in \mathbb{Z}_{\geq 0}} \{f \bmod \mathfrak{m}_P^{a+1} \mid (f, a) \in \mathcal{R}_P, f \in \mathfrak{m}_P^a\} \subset \bigoplus_{a \in \mathbb{Z}_{\geq 0}} \mathfrak{m}_P^a / \mathfrak{m}_P^{a+1}.$$

Then there exists a regular system of parameters $(x_1, \dots, x_t, x_{t+1}, \dots, x_d)$ at P such that the leading algebra takes the following form:

Case: $\text{char}(k) = 0$

$$L_P(\mathcal{R}) = k[x_1, \dots, x_t] \subset k[x_1, \dots, x_t, x_{t+1}, \dots, x_d] = \bigoplus_{a \in \mathbb{Z}_{\geq 0}} \mathfrak{m}_P^a / \mathfrak{m}_P^{a+1}.$$

Moreover, we observe the following: if we take an element $(h_i, 1) \in \mathcal{R}_P$ with $h_i \equiv x_i \pmod{\mathfrak{m}_P^2}$ ($i = 1, \dots, t$), then the hypersurface $\{h_i = 0\}$ is a hypersurface of maximal contact in the classical sense.

Case: $\text{char}(k) = p > 0$

$$L_P(\mathcal{R}) = k[x_1^{p^{e_1}}, \dots, x_t^{p^{e_t}}] \subset k[x_1, \dots, x_t, x_{t+1}, \dots, x_d] = \bigoplus_{a \in \mathbb{Z}_{\geq 0}} \mathfrak{m}_P^a / \mathfrak{m}_P^{a+1}$$

for some $0 \leq e_1 \leq \dots \leq e_t$.

Remark 4. The former *Case: char(k) = 0* in the above proposition can be regarded as a special case of the latter *Case: char(k) = p > 0*, by formally setting $p = \infty$, $0 = e_1 = \dots = e_t$ and $\infty^0 = 1$, where all the x^{p^e} -terms with $e > 0$ become “invisible” as p goes to ∞ .

Definition 3 (Leading Generator System (cf. [18] [19])). Let the situation be as in the proposition above. Take a subset $\mathbb{H} = \{(h_\alpha, p^{e_\alpha})\}_{\alpha=1}^t \subset \mathcal{R}_P$ with

$$h_\alpha \equiv x_\alpha^{p^{e_\alpha}} \pmod{\mathfrak{m}_P^{p^{e_\alpha}+1}} \quad \text{for } \alpha = 1, \dots, t.$$

We say that \mathbb{H} is a leading generator system for \mathcal{R}_P (called an LGS for short). A leading generator system for $\widehat{\mathcal{R}}_P$ is defined similarly.

Definition 4 (Invariant “ σ ” and “ τ ” (cf. [18] [19])). Let the situation be as in the proposition above. Then the invariants σ and τ are defined by the following formulas

$$\sigma(P) := (a_n)_{n \in \mathbb{Z}_{\geq 0}} \quad \text{where } a_n = d - \#\{e_\alpha \mid e_\alpha \leq n\}$$

where the value set of the invariant σ is given the lexicographical order, and

$$\tau(P) := \# \text{ of the elements in an LGS} = \#\mathbb{H} = t.$$

(We note that the invariants σ and τ are independent of the choice of a regular system of parameters or an LGS, and that it does not matter whether we compute them at the algebraic level or at the analytic level.) The moral here is that the more e_α ’s we have at the lower level, the smaller the value of the invariant σ is and hence we consider the better the LGS \mathbb{H} is.

Basic strategy to establish our algorithm in positive characteristic: Follow the construction of the algorithm in $\text{char}(k) = 0$, replacing the notion of an HMC to use the induction on dimension by the notion of an LGS to use the induction on the invariant σ .

Mechanism of the inductive scheme on the invariant σ

Given (W, \mathcal{R}, E) (Precisely speaking, the triplet sits in the middle of the sequence, say in year “ i ”, for resolution of singularities. However, we omit the subscript “ $()_i$ ” indicating the year for simplicity of the notation), we introduce a triplet of invariants $(\sigma, \tilde{\mu}, s)$ and its associated triplet of data (W', \mathcal{R}', E') . Together they form the following mechanism to realize the inductive scheme on the invariant σ (We note that there is *no* Inductive Lemma in our setting):

Outline of the mechanism

(1) We introduce a triplet of invariants $(\sigma, \tilde{\mu}, s)$.

If $(\sigma, \tilde{\mu}, s) = (\sigma, \infty, 0)$ or $(\sigma, 0, 0)$, then we do not construct the modification (W', \mathcal{R}', E') .

- In case $(\sigma, \tilde{\mu}, s) = (\sigma, \infty, 0)$, we blow up with center $C = \text{Sing}(\mathcal{R})$. The nonsingularity of C is guaranteed by the Nonsingularity Principle (cf. [18] [19]), while the transversality of C to the boundary E is guaranteed by the invariant $s = 0$. After the blow up, the singular locus becomes empty, and resolution of singularities for (W, \mathcal{R}, E) is achieved.

- In case $(\sigma, \tilde{\mu}, s) = (\sigma, 0, 0)$, we are in the monomial case by definition, and we go to Step (5).

If $(\sigma, \tilde{\mu}, s) \neq (\sigma, \infty, 0)$ or $(\sigma, 0, 0)$, then we construct its associated triplet of data (W', \mathcal{R}', E') , called the modification of the original triplet (W, \mathcal{R}, E) , having the following properties. (Actually the ambient space for the modification stays the same, i.e., $W' = W$.)

- Resolution of singularities for (W', \mathcal{R}', E') implies the strict decrease of the (maximum) value of the triplet $(\sigma, \tilde{\mu}, s)$. Note that the value of the triplet never increases after transformations.

- Either the value of σ strictly decrease, or the value of σ stays the same but the number of the components in the boundary strictly drops. In either case, we have $(\sigma, \#E) > (\sigma', \#E')$.

(2) There is *no* key inductive lemma in our new setting.

(3) When $(\sigma, \tilde{\mu}, s) \neq (\sigma, \infty, 0)$ or $(\sigma, 0, 0)$ in (1), we achieve resolution of singularities for (W', \mathcal{R}', E') by induction on $(\sigma, \#E)$, which implies the strict decrease of the (maximum) value of the triplet $(\sigma, \tilde{\mu}, s)$.

(4) Repeatedly decreasing the value of the triplet $(\sigma, \tilde{\mu}, s)$, we reach the case where $(\sigma, \tilde{\mu}, s) = (\sigma, \infty, 0)$ or $(\sigma, 0, 0)$, the monomial case.

(5) Finally we have only to construct resolution of singularities in the monomial case in order to achieve resolution of singularities of the original triplet (W, \mathcal{R}, E) . However, the problem of resolution of singularities in the monomial case in positive characteristic is quite subtle and very difficult. We only provide a solution in dimension 3 in §5.

Warning: Even though the invariant σ never increases after blow ups chosen in our algorithm, the number of exceptional divisors $\#E$ and hence the pair $(\sigma, \#E)$ may increase. Therefore, the description of “by induction on $(\sigma, \#E)$ ” in (3) above is slightly imprecise. The precise mechanism of the induction is manifested as the weaving of the new strand of invariants “ inv_{new} ”. See §4.3 for details.

Description of the triplet of invariants $(\sigma, \tilde{\mu}, s)$

◦ σ : It is the invariant σ associated to the differential saturation \mathcal{DR} of the idealistic filtration of i.f.g. type \mathcal{R} (cf. Definitions 2, 3, and 4).

◦ $\tilde{\mu}$: It is the (normalized) weak order modulo an LGS of the idealistic filtration \mathcal{R} , with respect to E_{young} .

◦ s : It is the number of the components in $E_{\text{aged}} = E \setminus E_{\text{young}}$.

We explain how to compute the invariant $\tilde{\mu}$ more in detail: Let \mathbb{H} be the LGS chosen. Given $f \in \widehat{\mathcal{O}_{W,P}}$, let $f = \sum c_{f,B} H^B$ be its power series expansion with respect to the LGS (and its associated regular system of parameters) (cf. [19]). Then we define

$$\begin{aligned} \mu_P(\mathcal{R}) &= \inf \left\{ \frac{1}{a} \text{ord}_P(c_{f,\mathbb{O}}) \mid (f, a) \in \mathcal{R}_P, a > 0 \right\} \\ &= \inf \left\{ \frac{1}{a} \text{ord}_P(c_{f,\mathbb{O}}) \mid (f, a) \in \widehat{\mathcal{R}}_P, a > 0 \right\}, \\ \mu_{P,D}(\mathcal{R}) &= \inf \left\{ \frac{1}{a} \text{ord}_{\xi_D}(c_{f,\mathbb{O}}) \mid (f, a) \in \mathcal{R}_P, a > 0 \right\} \\ &= \inf \left\{ \frac{1}{a} \text{ord}_{\xi_D}(c_{f,\mathbb{O}}) \mid (f, a) \in \widehat{\mathcal{R}}_P, a > 0 \right\}, \end{aligned}$$

where ξ_D is the generic point of a component D in E_{young} . (For the definition of E_{young} , see (iii) of the remark below.) Now we define the invariant $\tilde{\mu}$ by the following formula

$$\tilde{\mu} = \mu_P(\mathcal{R}) - \sum_{D \in E_{\text{young}}} \mu_{P,D}(\mathcal{R}).$$

It is straightforward to see via the formal coefficient lemma that $\tilde{\mu}$ is independent of the choice of the LGS (and its associated regular system of parameters) and that $\tilde{\mu}$ is a nonnegative rational number, since our idealistic filtration is of i.f.g. type (cf. [18][19]).

We make the following remarks on the technical but important points about the LGS (and its associated regular system of parameters), the idealistic filtration of i.f.g. type \mathcal{R} , and $E_{\text{young}} \subset E$ used in the computation above:

(i) The idealistic filtration of i.f.g.type \mathcal{R} used in the computation depends on the history of the behavior of the invariant σ .

Case: The value of σ remains the same as the one in the previous year.

In this case, we keep \mathcal{R} as it is, which is the transformation of the one in the previous year, even though we compute the invariant σ using the differential saturation \mathcal{DR} . We take our LGS (a priori only in \mathcal{DR}) to be the transformation of the one in the previous year, which hence sits inside of \mathcal{R} .

Case: The value of σ is strictly less than the one in the previous year.

In this case, we replace the original \mathcal{R} with its differential saturation. We take our LGS from this replaced $\mathcal{R} = \mathcal{DR}$, which is differentially saturated, and compute $\tilde{\mu}$ accordingly. We remark that, in this case, $E_{\text{young}} = \emptyset$ and hence that $\tilde{\mu} = \mu_P(\mathcal{R})$.

We note that, in year 0, we also replace the original \mathcal{R} with its differential saturation (cf. Remark 3 (2)).

(ii) The LGS $\mathbb{H} = \{(h_\alpha, p^{e_\alpha})\}_{\alpha=1}^t \subset \widehat{\mathcal{R}}_P$ and its associated regular system of parameters $X = (x_1, \dots, x_t, x_{t+1}, \dots, x_d) \subset \widehat{\mathcal{O}}_{W,P}$ are taken in such a way that they satisfy the condition (\heartsuit) consisting of the three requirements described below:

Condition (\heartsuit)

- $h_\alpha \equiv x_\alpha^{p^{e_\alpha}} \pmod{\widehat{\mathfrak{m}}^{p^{e_\alpha}+1}}$ for $\alpha = 1, \dots, t$,
- the idealistic filtration of i.f.g. type $\widehat{\mathcal{R}}_P$ is saturated for $\{\partial^n / \partial x_\alpha^n \mid n \in \mathbb{Z}_{\geq 0}, \alpha = 1, \dots, t\}$, and
- the defining equations for the components of E_{young} , which are transversal to the LGS, form a part of the regular system of parameters, i.e., $\{x_D \mid D \in E_{\text{young}}\} \subset \{x_{t+1}, \dots, x_d\}$.

(We remark that it is easy to find such $H \subset \mathcal{R}$ and $X \subset \mathcal{O}_{W,P}$ that satisfy all the requirements but the second one in Condition (\heartsuit) .)

We recall the following formal coefficient lemma (cf. [19]), where the assumption is slightly weaker than the original one in the sense that it does not require the idealistic filtration is \mathcal{D} -saturated. However, the same conclusion is valid with the same proof.

Formal Coefficient Lemma

Let \mathcal{R} be an idealistic filtration of i.f.g. type. Take a subset $\mathbb{H} = \{(h_\alpha, p^{e_\alpha})\}_{\alpha=1}^t \subset \widehat{\mathcal{R}}_P$ (not necessarily an LGS) and a regular system of

parameters $X = (x_1, \dots, x_t, x_{t+1}, \dots, x_d) \subset \widehat{\mathcal{O}_{W,P}}$, satisfying the first two requirements in condition (\heartsuit) . Then, for $(f, a) \in \widehat{\mathcal{R}}_P$, we have

$$(c_{f,B}, \max\{a - |[B]|, 0\}) \in \widehat{\mathcal{R}}_P$$

for any B , and in particular,

$$(c_{f,0}, a) \in \widehat{\mathcal{R}}_P,$$

where $f = \sum c_{f,B} H^B$ is the power series expansion of f with respect to \mathbb{H} and X (cf. [19]).

(iii) The symbol E_{young} refers to the union of the exceptional divisors created after the time when the current value of σ first started. Therefore, by construction, E_{young} is transversal to the LGS. We only use E_{young} in our algorithm, in contrast to the classical algorithm where we have to use both E_{young} and E_{new} (cf. §3).

Description of the triplet (W', \mathcal{R}', E')

$$W' = W, \quad \mathcal{R}' = \text{Bdry}(\text{Comp}(\mathcal{R})), \quad \text{and} \quad E' = E \setminus E_{\text{aged}} = E_{\text{young}},$$

where

- $\text{Comp}(\mathcal{R})$ is either the transformation of the one in the previous year if $(\sigma, \tilde{\mu})$ stays the same, or the one constructed below if $(\sigma, \tilde{\mu})$ strictly decreases, and
- $\text{Bdry}(\text{Comp}(\mathcal{R}))$ is either the transformation of the one in the previous year if $(\sigma, \tilde{\mu}, s)$ stays the same, or

$$\mathcal{G}(\text{Comp}(\mathcal{R}) \cup \{(x_D, 1) \mid D \in E_{\text{aged}}\})$$

if $(\sigma, \tilde{\mu}, s)$ strictly decreases, where $\mathcal{G}(S)$ is the idealistic filtration of i.g. type generated by the set S , i.e., the smallest idealistic filtration of i.g. type containing S , and x_D is the defining equation of a component $D \in E_{\text{aged}}$.

We note that, if the value of the triplet $(\sigma, \tilde{\mu}, s)$ stays the same as in the previous year, then (W', \mathcal{R}', E') is the transformation of the one in the previous year. We remark that the symbols “Comp” and “Bdry” represent the “Companion” modification and the “Boundary” modification, respectively.

Construction of $\text{Comp}(\mathcal{R})$

We describe the construction of the companion modification $\text{Comp}(\mathcal{R})$, first at the analytic level, following closely the construction in the

classical setting, and then at the algebraic level, showing that the companion modification at the analytic level “descends” to the one at the algebraic level via the argument of “étale descent”.

Construction at the analytic level

First, we take an LGS $\mathbb{H} \subset \widehat{\mathcal{R}}_P$ and its associated regular system of parameters $X \subset \widehat{\mathcal{O}}_{W,P}$ satisfying the condition (♥).

We set

$$\mathbb{M}_X = \prod_{D \in E_{\text{young}}} x_D^{\mu_{P,D}(\mathcal{R})}.$$

Recall that $\{x_D \mid D \in E_{\text{young}}\} \subset X$.

Fix a common multiple $L \in \mathbb{Z}_{>0}$ of the denominators of $\tilde{\mu}, \mu_P(\mathcal{R})$, and $\{\mu_{P,D} \mid D \in E_{\text{young}}\}$. Set

$$\Xi = \widehat{\mathcal{O}}_{W,P} \otimes_k k[x_{t+1}^{\pm \frac{1}{L}}, \dots, x_d^{\pm \frac{1}{L}}].$$

We consider the following notion of an idealistic filtration \mathcal{Q} in the generalized sense:

$$\mathcal{Q} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \left(\mathcal{Q}_{\frac{n}{L}}, \frac{n}{L} \right) \subset \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \left(\Xi, \frac{n}{L} \right)$$

is a graded $\widehat{\mathcal{O}}_{W,P}$ -algebra, where

- the grading is given by $\{n/L \mid n \in \mathbb{Z}_{\geq 0}\}$, and it is specified as the level in the second factor,
- $\mathcal{Q}_{n/L} \subset \Xi$ is an $\widehat{\mathcal{O}}_{W,P}$ -submodule with the $\widehat{\mathcal{O}}_{W,P}$ -module structure induced by the left multiplication on Ξ (We emphasize that the tensor “ \otimes ” is over k .) satisfying the condition $\mathcal{Q}_{0/L} \supset \mathcal{Q}_{1/L} \supset \mathcal{Q}_{2/L} \supset \dots$,
- the algebra structure is given by the addition and multiplication on Ξ , while the $\widehat{\mathcal{O}}_{W,P}$ -algebra structure is given by the left multiplication on the first factor of Ξ ,
- the differential operators act on the first factor, i.e., for a differential operator δ of $\deg \delta$ and

$$q = \left(\sum f \otimes g, n/L \right) \in \left(\Xi, n/L \right)$$

with $f \in \widehat{\mathcal{O}}_{W,P}$ and $g \in k[x_{t+1}^{\pm \frac{1}{L}}, \dots, x_d^{\pm \frac{1}{L}}]$, we set

$$\delta q = \left(\sum \delta f \otimes g, \max \{n/L - \deg \delta, 0\} \right).$$

We construct $\widehat{\text{Comp}}(\mathcal{R})_{\mathbb{H},X}$ in the following manner:

Step 1. We take the idealistic filtration \mathcal{Q}_1 in the generalized sense generated by $\{(f \otimes 1, a) \mid (f, a) \in \mathcal{R}_P\}$ and $\{(c_{f,0} \otimes (\mathbb{M}_X^{-1})^a, \tilde{\mu} \cdot a) \mid (f, a) \in \mathcal{R}_P\}$, i.e.,

$$\mathcal{Q}_1 = \mathcal{G}(\{(f \otimes 1, a), (c_{f,0} \otimes (\mathbb{M}_X^{-1})^a, \tilde{\mu} \cdot a) \mid (f, a) \in \mathcal{R}_P\}),$$

where $c_{f,0}$ is the constant term of the power series expansion $f = \sum c_{f,B} H^B$ with respect to \mathbb{H} and X .

Step 2. We take the idealistic filtration \mathcal{Q}_2 in the generalized sense to be the $\mathcal{D}_{E_{\text{young}}}$ -saturation of the idealistic filtration \mathcal{Q}_1 in the generalized sense, i.e.,

$$\mathcal{Q}_2 = \mathcal{D}_{E_{\text{young}}}(\mathcal{Q}_1),$$

where $\mathcal{D}_{E_{\text{young}}}$ represents the logarithmic differentials with respect to the simple normal crossing divisor E_{young} (cf. [18]).

Step 3. We take the integral level part $\mathcal{P} = \text{ILP}(\mathcal{Q}_2)$ of the idealistic filtration \mathcal{Q}_2 in the generalized sense. That is to say, $\mathcal{P} = \bigoplus_{a \in \mathbb{Z}_{\geq 0}} (\mathcal{P}_a, a)$ is a graded $\widehat{\mathcal{O}_{W,P}}$ -algebra where, for $a \in \mathbb{Z}_{\geq 0}$, we set $\mathcal{P}_a = (\mathcal{Q}_2)_{n/L}$ with $a = n/L$.

Step 4. By taking the “round up” and contraction of \mathcal{P} , we obtain the usual idealistic filtration of i.f.g. type $\mathcal{C} = \bigoplus_{a \in \mathbb{Z}_{\geq 0}} (\mathcal{C}_a, a)$ where we set

$$\mathcal{C}_a = \text{RUC}(\mathcal{P}_a) \quad \text{for } a \in \mathbb{Z}_{\geq 0}.$$

Note that the “round up” and contraction map $\text{RUC} : \mathcal{P}_a \rightarrow \widehat{\mathcal{O}_{W,P}}$ is given by $\text{RUC}(\sum f \otimes (\mathbb{M}_X^{-1})^l) = \sum f \cdot \lceil (\mathbb{M}_X^{-1})^l \rceil$ for $l \in \mathbb{Z}_{\geq 0}$, where

$$\lceil (\mathbb{M}_X^{-1})^l \rceil = \prod_{D \in E_{\text{young}}} x_D^{\lceil -\mu_{P,D}(\mathcal{R}) \cdot l \rceil}.$$

Since we have $\text{ord}_{\xi_D}(c_{f,0}) \geq \mu_{P,D}(\mathcal{R}) \cdot a$ for $(f, a) \in \mathcal{R}_P$ and $D \in E_{\text{young}}$ by definition, and since a logarithmic differential operator with respect to E_{young} does not decrease the power of x_D for $D \in E_{\text{young}}$, we conclude that, though the image of the RUC map is only in $\widehat{\mathcal{O}_{W,P}}[x_{t+1}^{\pm 1}, \dots, x_d^{\pm 1}]$ a priori, it actually lies within $\widehat{\mathcal{O}_{W,P}}[x_{t+1}, \dots, x_d] = \widehat{\mathcal{O}_{W,P}}$, i.e.,

$$\mathcal{C}_a = \text{RUC}(\mathcal{P}_a) \subset \widehat{\mathcal{O}_{W,P}} \quad \text{for } a \in \mathbb{Z}_{\geq 0}.$$

Step 5. We set $\mathcal{C} = \widehat{\text{Comp}(\mathcal{R})}_{\mathbb{H}, X}$.

Remark 5.

(1) We remark that \mathcal{C} is finitely generated. In fact, if $\{(f_\lambda, a_\lambda) \mid \lambda \in \Lambda\}$ with $\#\Lambda < \infty$ is a set of (local) generators for \mathcal{R} , then by setting

$$\begin{aligned} \mathcal{Q}_{1,\Lambda} = \mathcal{G} & \left(\{(f_\lambda \otimes 1, a_\lambda) \mid (f_\lambda, a_\lambda) \in \mathcal{R}_P, \lambda \in \Lambda\} \right. \\ & \left. \cup \{(c_{f_\lambda, \mathbb{O}} \otimes (\mathbb{M}_X^{-1})^{a_\lambda}, \tilde{\mu} \cdot a_\lambda) \mid (f, a_\lambda) \in \mathcal{R}_P, \lambda \in \Lambda\} \right), \end{aligned}$$

we see by the formal coefficient lemma applied to the case of an idealistic filtration in the generalized sense (the same proof works as in the usual case (cf. [19])) that $\mathcal{Q}_1 \subset \mathcal{D}_{E_{\text{young}}}(\mathcal{Q}_{1,\Lambda})$ while obviously we have $\mathcal{Q}_{1,\Lambda} \subset \mathcal{Q}_1$, and hence that

$$\mathcal{D}_{E_{\text{young}}}(\mathcal{Q}_1) = \mathcal{D}_{E_{\text{young}}}(\mathcal{Q}_{1,\Lambda}).$$

It follows easily from this that \mathcal{C} is finitely generated.

(2) In Step 1, we take the set $\{(c_{f,\mathbb{O}} \otimes (\mathbb{M}_X^{-1})^a, \tilde{\mu} \cdot a) \mid (f, a) \in \mathcal{R}_P\}$ as a part of the generators, where the elements (f, a) belong to the idealistic filtration of i.f.g. type \mathcal{R}_P at the algebraic level. Even if we replace this with $\{(c_{f,\mathbb{O}} \otimes (\mathbb{M}_X^{-1})^a, \tilde{\mu} \cdot a) \mid (f, a) \in \widehat{\mathcal{R}}_P\}$, where the elements (f, a) belong to the idealistic filtration of i.f.g. type $\widehat{\mathcal{R}}_P$ at the analytic (completion) level, the resulting \mathcal{C} will not change.

In fact, this can be seen as follows: Let \mathcal{C}_{an} be the one obtained by using the analytic one. (Note that $\mathcal{Q}_{1,\text{an}}$ and $\mathcal{Q}_{2,\text{an}} = \mathcal{D}_{E_{\text{young}}}(\mathcal{Q}_{1,\text{an}})$ are defined similarly, to be used in the proof of Lemma 3.) Since

$$\begin{aligned} & \{(c_{f,\mathbb{O}} \otimes (\mathbb{M}_X^{-1})^a, \tilde{\mu} \cdot a) \mid (f, a) \in \mathcal{R}_P\} \\ & \subset \{(c_{f,\mathbb{O}} \otimes (\mathbb{M}_X^{-1})^a, \tilde{\mu} \cdot a) \mid (f, a) \in \widehat{\mathcal{R}}_P\}, \end{aligned}$$

it is obvious that we have $\mathcal{C} \subset \mathcal{C}_{\text{an}}$. On the other hand, take $(f, a) \in \widehat{\mathcal{R}}_P$ with the power series expansion $f = \sum c_{f,B} H^B$ with respect to \mathbb{H} and X , and $c_{f,\mathbb{O}}$ its constant term. Then by the formal coefficient lemma, we conclude $(c_{f,\mathbb{O}}, a) \in \widehat{\mathcal{R}}_P = \mathcal{R}_P \otimes_{\mathcal{O}_{W,P}} \widehat{\mathcal{O}}_{W,P}$. This means that there exists a finite set $\{(f_\lambda, a) \in \mathcal{R}_P \mid \lambda \in \Lambda\}$ such that $c_{f,\mathbb{O}} = \sum r_\lambda f_\lambda$ for some $r_\lambda \in \widehat{\mathcal{O}}_{W,P}$.

Set $g = \sum r_\lambda c_{f_\lambda, \mathbb{O}}$ and $h = \sum r_\lambda (f_\lambda - c_{f_\lambda, \mathbb{O}})$ with $c_{f,\mathbb{O}} = g + h$. Then since $c_{h,\mathbb{O}} = 0$, we conclude $c_{f,\mathbb{O}} = c_{g,\mathbb{O}} + c_{h,\mathbb{O}} = c_{g,\mathbb{O}}$. Now since $\{(c_{f_\lambda, \mathbb{O}} \otimes (\mathbb{M}_X^{-1})^a, \tilde{\mu} \cdot a)\} \subset \mathcal{Q}_1$ and hence $(g \otimes (\mathbb{M}_X^{-1})^a, \tilde{\mu} \cdot a) \in \mathcal{Q}_1$, we conclude, by the formal coefficient lemma applied to the case of an idealistic filtration in the generalized sense, that

$$(c_{f,\mathbb{O}} \otimes (\mathbb{M}_X^{-1})^a, \tilde{\mu} \cdot a) = (c_{g,\mathbb{O}} \otimes (\mathbb{M}_X^{-1})^a, \tilde{\mu} \cdot a) \in \mathcal{D}_{E_{\text{young}}}(\mathcal{Q}_1) = \mathcal{Q}_2,$$

which implies $\mathcal{C}_{\text{an}} \subset \mathcal{C}$.

Therefore, we conclude $\mathcal{C} = \mathcal{C}_{\text{an}}$.

Lemma 3. *The companion modification $\widehat{\text{Comp}}(\mathcal{R})_{\mathbb{H}, X}$ constructed at the analytic level as above is independent of the choice of an LGS \mathbb{H} and its associated regular system of parameters X satisfying the condition (\heartsuit) . That is to say, if \mathbb{H}' and X' are another LGS and its associated regular system of parameters satisfying the condition (\heartsuit) , then we have*

$$\widehat{\text{Comp}}(\mathcal{R})_{\mathbb{H}, X} = \widehat{\text{Comp}}(\mathcal{R})_{\mathbb{H}', X'}.$$

We write, therefore,

$$\widehat{\text{Comp}}(\mathcal{R}) = \widehat{\text{Comp}}(\mathcal{R})_{\mathbb{H}, X}$$

omitting the reference to the LGS and its associated regular system of parameters used in the construction.

We call $\widehat{\text{Comp}}(\mathcal{R})$ the companion modification at the analytic level.

Proof. We consider the following two cases.

Case: For any $e \in \mathbb{Z}_{\geq 0}$ and $(h_\alpha, p^{e_\alpha}) \in \mathbb{H}$ with $e_\alpha = e$, the element h_α is a linear combination of

$$\{(h'_\beta)^p e^{-e'_\beta} \mid e'_\beta \leq e\}.$$

There is no condition on X or X' other than their being associated to \mathbb{H} and \mathbb{H}' , respectively.

In this case, we claim $\widehat{\text{Comp}}(\mathcal{R})_{\mathbb{H}', X'} \subset \widehat{\text{Comp}}(\mathcal{R})_{\mathbb{H}, X}$.

Take $(f, a) \in \mathcal{R}_P$. Let $f = \sum c_{f, B} H$ be the power series expansion of f with respect to \mathbb{H} and X , with its constant term $c_{f, \mathbb{O}}$. We have $(c_{f, \mathbb{O}} \otimes (\mathbb{M}_X^{-1})^a, \tilde{\mu} \cdot a) \in \mathcal{Q}_1$. Set $g = c_{f, \mathbb{O}}$ and $h = f - c_{f, \mathbb{O}}$. Let $g = \sum c'_{g, B} H'^B$ and $h = \sum c'_{h, B} H'^B$ be the power series expansions of g and h , respectively, with respect to \mathbb{H}' and X' . Now we conclude, by the formal coefficient lemma applied to the case of an idealistic filtration in the generalized sense, that

$$(c'_{g, \mathbb{O}} \otimes (\mathbb{M}_X^{-1})^a, \tilde{\mu} \cdot a) \in \mathcal{D}_{E_{\text{young}}}(\mathcal{Q}_{1, \text{an}}) = \mathcal{Q}_{2, \text{an}}.$$

On the other hand, the assumption of this case implies that $c'_{h, \mathbb{O}} = 0$ and hence that

$$c'_{g, \mathbb{O}} = c'_{g, \mathbb{O}} + c'_{h, \mathbb{O}} = c'_{g+h, \mathbb{O}} = c'_{f, \mathbb{O}},$$

the constant term of the power series expansion $f = \sum c'_{f,B} H'^B$ with respect to \mathbb{H}' and X' . Thus, we have

$$(c'_{f,\mathbb{O}} \otimes (\mathbb{M}_{X'}^{-1})^a, \tilde{\mu} \cdot a) \in \mathcal{D}_{E_{\text{young}}}(\mathcal{Q}_{1,\text{an}}) = \mathcal{Q}_{2,\text{an}}.$$

This implies by Remark 5 (2)

$$\widehat{\text{Comp}}(\mathcal{R})_{\mathbb{H}',X'} \subset \mathcal{C}_{\text{an}} = \widehat{\text{Comp}}(\mathcal{R})_{\mathbb{H},X}.$$

(We note that the generators we choose in Step 1 of the construction for $\widehat{\text{Comp}}(\mathcal{R})_{\mathbb{H}',X'}$ are of the form $(c'_{f,\mathbb{O}} \otimes (\mathbb{M}_{X'}^{-1})^a, \tilde{\mu} \cdot a)$ (not $\otimes (\mathbb{M}_{X'}^{-1})^a$), which are sitting inside of

$$\widehat{\mathcal{O}}_{W,P} \otimes_k k[x'_{t+1}{}^{\pm \frac{1}{L}}, \dots, x'_d{}^{\pm \frac{1}{L}}] \quad (\text{not of } \Xi = \widehat{\mathcal{O}}_{W,P} \otimes_k k[x_{t+1}{}^{\pm \frac{1}{L}}, \dots, x_d{}^{\pm \frac{1}{L}}]).$$

However, these differences only contribute to the multiplication of units after Step 3 and Step 4, and hence do not matter for us to conclude the inclusion above.)

Case: $X = X'$.

In this case, we also claim $\widehat{\text{Comp}}(\mathcal{R})_{\mathbb{H}',X'} \subset \widehat{\text{Comp}}(\mathcal{R})_{\mathbb{H},X}$.

Take $(f, a) \in \mathcal{R}_P$. Let $f = \sum c'_{f,B} H'$ be the power series expansion of f with respect to \mathbb{H}' and X' , with its constant term $c'_{f,\mathbb{O}}$. By the formal coefficient lemma, we have $(c'_{f,\mathbb{O}}, a) \in \widehat{\mathcal{R}}_P$. Set $g' = c'_{f,\mathbb{O}} \in \widehat{\mathcal{O}}_{W,P}$. Let $g' = \sum c_{g',B} H^B$ be the power series expansion of g' with respect to H and X with its constant term $c_{g',\mathbb{O}}$. Then the assumption of $X = X'$ implies $g' = c_{g',\mathbb{O}}$. Therefore, we conclude that

$$(c'_{f,\mathbb{O}} \otimes (\mathbb{M}_{X'}^{-1})^a, \tilde{\mu} \cdot a) \in \left\{ (c_{h,\mathbb{O}} \otimes (\mathbb{M}_{X'}^{-1})^a, \tilde{\mu} \cdot a) \mid (h, a) \in \widehat{\mathcal{R}}_P \right\}$$

and hence by Remark 5 (2) (cf. the note at the end of the previous case) that

$$\widehat{\text{Comp}}(\mathcal{R})_{\mathbb{H}',X'} \subset \mathcal{C}_{\text{an}} = \mathcal{C} = \widehat{\text{Comp}}(\mathcal{R})_{\mathbb{H},X}.$$

Observe that, given an LGS \mathbb{H} and its associated regular system of parameters X satisfying the condition (\heartsuit) , we can reach another LGS \mathbb{H}' and its associated regular system of parameters X' satisfying the condition (\heartsuit) by a transformation described in the former case followed by another transformation described in the latter case. Therefore, by the above analysis, we have $\widehat{\text{Comp}}(\mathcal{R})_{\mathbb{H}',X'} \subset \widehat{\text{Comp}}(\mathcal{R})_{\mathbb{H},X}$. Reversing the role of \mathbb{H} and X with that of \mathbb{H}' and X' , we then have $\widehat{\text{Comp}}(\mathcal{R})_{\mathbb{H},X} \subset \widehat{\text{Comp}}(\mathcal{R})_{\mathbb{H}',X'}$.

Finally we conclude $\widehat{\text{Comp}}(\mathcal{R})_{\mathbb{H},X} = \widehat{\text{Comp}}(\mathcal{R})_{\mathbb{H}',X'}$. Q.E.D.

Construction at the algebraic level

Proposition 2. *There exists an idealistic filtration of i.f.g. type $\text{Comp}(\mathcal{R})$ at the algebraic level (i.e., over $\mathcal{O}_{W,P}$) such that its completion coincides with the companion modification $\widehat{\text{Comp}}(\mathcal{R})$ at the analytic level, i.e.,*

$$\{\text{Comp}(\mathcal{R})\}^\wedge := \text{Comp}(\mathcal{R}) \otimes_{\mathcal{O}_{W,P}} \widehat{\mathcal{O}_{W,P}} = \widehat{\text{Comp}}(\mathcal{R}).$$

We call $\text{Comp}(\mathcal{R})$ the companion modification at the algebraic level.

Proof. Step 1. Descent to the Henselization level.

We first note that the ingredients that we used to construct the companion modification at the analytic level;

- (i) the LGS \mathbb{H} and its associated regular system of parameters X satisfying the condition (\heartsuit) ,
- (ii) the constant term $c_{f,\mathbb{O}}$ of the power series expansion $f = \sum c_{f,B} H^B$ for $(f, a) \in \mathcal{R}_P$ with respect to \mathbb{H} and X (which shows up in the form of $(c_{f,\mathbb{O}} \otimes (\mathbb{M}_X^{-1})^a, \tilde{\mu} \cdot a)$ in the construction of the companion modification),

actually can be taken at the Henselization level (i.e., they can be taken from the Henselization $(\mathcal{O}_{W,P})^h$ of $\mathcal{O}_{W,P}$).

In fact, (i) at the Henselization level is a consequence of the classical Weierstrass Preparation Theorem and Weierstrass Division Theorem (cf. the proof of Proposition 4 (1)). (Alternatively, (i) at the Henselization level can be seen using the same argument as the one used to see (ii) at the Henselization level.)

We see (ii) at the Henselization level as follows: Set $R = \mathcal{O}_{W,P}$, R^h its Henselization, and \widehat{R} its completion. By replacing R with some local ring of an étale cover of $\text{Spec } R$, we may assume that the LGS \mathbb{H} and the regular system of parameters X satisfying the condition (\heartsuit) are taken from R . Set $A = k[x_{t+1}, \dots, x_d]_{(x_{t+1}, \dots, x_d)}$, A^h its Henselization, and \widehat{A} its completion. By looking at the power series expansion with respect to \mathbb{H} and its associated regular system of parameters X (cf. [19]), we have

$$\phi : \widehat{R}/(h_1, \dots, h_t) \xrightarrow{\sim} \sum_K \widehat{A} X^K$$

where on the right hand side the subscript $K = (k_1, \dots, k_t, k_{t+1}, \dots, k_d) \in \mathbb{Z}_{\geq 0}^d$ for the summation varies in the finite range

$$\begin{cases} 0 \leq k_\alpha \leq p^{e_\alpha} - 1 & \text{for } \alpha = 1, \dots, t \\ k_\alpha = 0 & \text{for } \alpha = t + 1, \dots, d. \end{cases}$$

Take an element $f \in R \subset \widehat{R}$. What we want to show is $c_{f,0} \in R^h$.

It suffices to show

$$(\star) \quad \phi(f \bmod (h_1, \dots, h_t)) = c_{f,0} \in \sum_K A^h X^K.$$

We take the coordinate ring S of an affine open neighborhood $P \in \text{Spec } S \subset W$ such that $f \in S$ and $\{h_\alpha\}_{\alpha=1}^t, X \subset S$. We denote by $\mathfrak{m}_{S,P}$ the maximal ideal of S corresponding to the point P . Note that the ideal $(h_1, \dots, h_t, x_{t+1}, \dots, x_d)R$ is $\mathfrak{m}_{S,P}R = \mathfrak{m}_P = (x_1, \dots, x_t, x_{t+1}, \dots, x_d)$ -primary (in R). By shrinking $\text{Spec } S$ if necessary, we may assume that the only prime ideal containing the ideal $(h_1, \dots, h_t, x_{t+1}, \dots, x_d)$ is $\mathfrak{m}_{S,P}$.

We regard R, R^h, S, A, A^h as subrings of \widehat{R} , i.e., $R, R^h, S, A, A^h \subset \widehat{R}$, and we consider the subring $SA^h \subset \widehat{R}$ generated by $S, A^h \subset \widehat{R}$. By abuse of notation, we denote the image of the natural projection $SA^h \subset \widehat{R} \rightarrow \widehat{R}/(h_1, \dots, h_t)$ by $SA^h/(h_1, \dots, h_t)$.

We claim

$$(\star\star) \quad \phi(SA^h/(h_1, \dots, h_t)) = \sum_K A^h X^K,$$

which clearly implies the assertion (\star) .

In the following, we omit the isomorphism ϕ from the left hand side in order to ease the notation. Therefore, the claim $(\star\star)$ is expressed as an equality

$$SA^h/(h_1, \dots, h_t) = \sum_K A^h X^K.$$

Obviously, we have

$$SA^h/(h_1, \dots, h_t) \supset \sum_K A^h X^K.$$

Our goal is to show the equality after taking $\otimes_{A^h} \widehat{A}$, i.e.,

$$\text{L.H.S.} = SA^h/(h_1, \dots, h_t) \otimes_{A^h} \widehat{A} = \sum_K A^h X^K \otimes_{A^h} \widehat{A} = \text{R.H.S.},$$

which, since \widehat{A} is faithfully flat over A^h , implies the original equality above before taking $\otimes_{A^h} \widehat{A}$, i.e., $(\star\star)$.

- Analysis of R.H.S.

We have

$$\text{R.H.S.} = \sum_K A^h X^K \otimes_{A^h} \widehat{A} = \sum_K \widehat{A} X^K.$$

- Analysis of L.H.S.

In order to analyze L.H.S., we look at the morphism $\theta: \text{Spec}(SA^h/(h_1, \dots, h_t)) \rightarrow \text{Spec}(A^h)$. Observe that θ is quasi-finite, i.e.,

- (a) it is of finite type, and
- (b) it has finite fibers.

The assertion (a) is immediate, since S is finitely generated over k . In order to see the assertion (b), first look at the morphism $\text{Spec}(SA/(h_1, \dots, h_t)) \rightarrow \text{Spec}(A)$. The fiber of this morphism over the origin downstairs is the origin upstairs, since the ideal $(h_1, \dots, h_t, x_{t+1}, \dots, x_d)$ is $(x_1, \dots, x_t, x_{t+1}, \dots, x_d)$ -primary. Now by what is called ‘‘Zariski’s Main Theorem by Grothendieck’’ (cf. [22]), we conclude that the morphism has finite fibers. From this it follows easily that the morphism $\theta: \text{Spec}(SA^h/(h_1, \dots, h_t)) \rightarrow \text{Spec}(A^h)$ has finite fibers.

Therefore, since A^h is Henselian, we conclude that the morphism θ is finite, i.e., $SA^h/(h_1, \dots, h_t)$ is a finite A^h -module. This implies that $SA^h/(h_1, \dots, h_t) \otimes_{A^h} \widehat{A}$ is the (x_{t+1}, \dots, x_d) -adic completion of the A^h -module $SA^h/(h_1, \dots, h_t)$. The latter coincides with the $(h_1, \dots, h_t, x_{t+1}, \dots, x_d)$ -adic completion of $SA^h/(h_1, \dots, h_t)$ viewed as an SA^h -module. Since the ideal $(h_1, \dots, h_t, x_{t+1}, \dots, x_d)$ is $(x_1, \dots, x_t, x_{t+1}, \dots, x_d)$ -primary, the $(h_1, \dots, h_t, x_{t+1}, \dots, x_d)$ -adic completion coincides with the $(x_1, \dots, x_t, x_{t+1}, \dots, x_d)$ -adic completion. Observing that the $(x_1, \dots, x_t, x_{t+1}, \dots, x_d)$ -adic completion of SA^h is \widehat{R} , we see that the $(x_1, \dots, x_t, x_{t+1}, \dots, x_d)$ -adic completion of $SA^h/(h_1, \dots, h_t)$ is $\widehat{R}/(h_1, \dots, h_t)$. Summarizing and remembering the convention of expressing the isomorphism ϕ as an equality, we conclude

$$\text{L.H.S.} = SA^h/(h_1, \dots, h_t) \otimes_{A^h} \widehat{A} = \widehat{R}/(h_1, \dots, h_t) = \sum_K \widehat{A}X^K.$$

Therefore, we have

$$\text{L.H.S.} = \sum_K \widehat{A}X^K = \text{R.H.S.}$$

This completes the argument for Step 1.

Therefore, we conclude that, for each LGS \mathbb{H} and its associated regular system of parameters X satisfying the condition (\heartsuit) , taken at the Henselization level, we have an idealistic filtration of i.f.g. type $\text{Comp}(\mathcal{R})_{\mathbb{H}, X}$ at the Henselization level (i.e., all the ideals are those of $(\mathcal{O}_{W,P})^h$) such that $\widehat{\text{Comp}}(\mathcal{R}) = \widehat{\text{Comp}}(\mathcal{R})_{\mathbb{H}, X} = \{\text{Comp}(\mathcal{R})_{\mathbb{H}, X}\}^\wedge$.

Step 2. Descent to the algebraic level.

Let $U = \text{Spec } \mathcal{O}_{W,P}$. Noting that the Henselization is the direct limit of the local rings at the closed points over P on the étale covers of U , we can take a collection of étale covers $\pi_\lambda : U_\lambda \rightarrow U$ and idealistic filtrations of i.f.g. type $\text{Comp}(\mathcal{R})_\lambda$ over U_λ such that, for $Q \in \pi_\lambda^{-1}(P)$, we have $\{\text{Comp}(\mathcal{R})_{\lambda,Q}\}^h = \text{Comp}(\mathcal{R})_{\mathbb{H},X}$ for some \mathbb{H} and X described as in Step 1. This implies

$$\begin{aligned} \{\text{Comp}(\mathcal{R})_{\lambda,Q}\}^\wedge &= \{\{\text{Comp}(\mathcal{R})_{\lambda,Q}\}^h\}^\wedge = \{\text{Comp}(\mathcal{R})_{\mathbb{H},X}\}^\wedge \\ &= \widehat{\text{Comp}(\mathcal{R})_{\mathbb{H},X}} = \widehat{\text{Comp}(\mathcal{R})}. \end{aligned}$$

That is to say, the completion of $\text{Comp}(\mathcal{R})_{\lambda,Q}$ canonically coincides with the companion modification at the analytic level $\widehat{\text{Comp}(\mathcal{R})}$. This in turn implies that, over $U_\lambda \cap U_\mu = U_\lambda \times_U U_\mu$, we have

$$\text{Comp}(\mathcal{R})_\lambda|_{U_\lambda \cap U_\mu} \stackrel{\phi_{\lambda\mu}}{=} \text{Comp}(\mathcal{R})_\mu|_{U_\lambda \cap U_\mu},$$

and this identification $\phi_{\lambda\mu}$ is canonical (and hence the collection of these identifications automatically satisfies the cocycle condition $\phi_{\lambda\mu} \circ \phi_{\mu\nu} = \phi_{\lambda\nu}$). Now it is a consequence of the general étale descent argument (cf. [14][20]) that there exists an idealistic filtration of i.f.g. type $\text{Comp}(\mathcal{R})$ such that $\pi_\lambda^*(\text{Comp}(\mathcal{R})) = \text{Comp}(\mathcal{R})_\lambda$, and hence that $\{\text{Comp}(\mathcal{R})\}^\wedge = \widehat{\text{Comp}(\mathcal{R})}$.

This completes the proof of Proposition 2.

Q.E.D.

Detailed discussion of the mechanism

Note that constructing the resolution sequence by blowing up is referred to as “proceeding in the vertical direction”, where the process is numbered by “year”, while constructing the triplet of invariants and its associated modification in a fixed year, is referred to as “proceeding in the horizontal direction”, where the process is numbered by “stage”. See §4.3 for more details.

Proceeding in the horizontal direction

Proposition 3. *The value of the pair $(\sigma, \#E)$ strictly decreases as we proceed in the horizontal direction from (W, \mathcal{R}, E) to (W', \mathcal{R}', E') , i.e., we have*

$$(\sigma, \#E) > (\sigma', \#E').$$

Proof. We analyze the assertion in the following two cases. Note that, since $\mathcal{R} \subset \mathcal{R}'$, we have $\sigma \geq \sigma'$ in both cases.

Case: $\tilde{\mu} \neq 0$ or ∞ .

In this case, we claim $\sigma > \sigma'$. We take the LGS $\mathbb{H} = \{(h_\alpha, p^{e_\alpha})\}_{\alpha=1}^t$ and its associated regular system of parameters $X = (x_1, \dots, x_t, x_{t+1}, \dots, x_d)$ satisfying the condition (\heartsuit) as given in “Description of the triplet of invariants $(\sigma, \tilde{\mu}, s)$ ”. Since \mathcal{R} is an idealistic filtration of i.f.g. type, there exists $(f, a) \in \mathcal{R}_P$ such that $\mu_P(\mathcal{R}) = \text{ord}_P(c_{f,0})/a$, where $f = \sum c_{f,B} H^B$ is the power series expansion of f with respect to \mathbb{H} and X . Set

$$\mathbb{M} = \prod_{D \in E_{\text{young}}} x_D^{\mu_{P,D}(\mathcal{R})}.$$

Subcase: $\mu_{P,D}(\mathcal{R}) \cdot a \in \mathbb{Z}_{\geq 0} \quad \forall D \in E_{\text{young}}$.

In this subcase, we have $\mathbb{M}^a \in \widehat{\mathcal{O}_{W,P}}$ and $c_{f,0} \cdot (\mathbb{M}^{-1})^a = c_{f,0} \cdot (\mathbb{M}^a)^{-1} \in \widehat{\mathcal{O}_{W,P}}$ by definition. Moreover, by construction of the companion modification, we have $(c_{f,0} \cdot (\mathbb{M}^a)^{-1}, \tilde{\mu} \cdot a) \in \widehat{\text{Comp}}(\mathcal{R}) \subset \widehat{\mathcal{R}'}$. Note that

$$\tilde{\mu} \cdot a = \left(\mu_P(\mathcal{R}) - \sum_{D \subset E_{\text{young}}} \mu_{P,D}(\mathcal{R}) \right) \cdot a = \text{ord}_P(c_{f,0} \cdot (\mathbb{M}^{-1})^a).$$

Note also that, by the characterization of the power series expansion with respect to \mathbb{H} and X , we have

$$c_{f,0} = \sum_{K \in (\mathbb{Z}_{\geq 0})^d} b_{f,0,K} X^K$$

with $b_{f,0,K} \in k[[x_{t+1}, \dots, x_d]]$ and with $K = (k_1, \dots, k_t, k_{t+1}, \dots, k_d)$ varying in the range satisfying the condition

$$\begin{cases} 0 \leq k_\alpha \leq p^{e_\alpha} - 1 & \text{for } \alpha = 1, \dots, t \\ k_\alpha = 0 & \text{for } \alpha = t+1, \dots, d. \end{cases}$$

Therefore, we conclude

$$\text{In}(c_{f,0} \cdot (\mathbb{M}^{-1})^a) = c_{f,0} \cdot (\mathbb{M}^{-1})^a \bmod \widehat{\mathfrak{m}}^{\tilde{\mu} \cdot a + 1} = \sum_{K \in (\mathbb{Z}_{\geq 0})^d} a_{f,0,K} X^K$$

with $a_{f,0,K} \in k[x_{t+1}, \dots, x_d]$. Since $L_P(\mathcal{R}) = k[x_1^{p^{e_1}}, \dots, x_t^{p^{e_t}}]$, we conclude

$$\text{In}(c_{f,0} \cdot (\mathbb{M}^{-1})^a) \notin L_P(\mathcal{R}),$$

while by definition

$$\text{In}(c_{f,0} \cdot (\mathbb{M}^{-1})^a) \in L_P(\widehat{\mathcal{R}'}) = L_P(\mathcal{R}').$$

Therefore, we conclude $L_P(\mathcal{R}) \subsetneq L_P(\widehat{\mathcal{R}'})$ and hence $\sigma > \sigma'$.

Subcase: $\mu_{P,D}(\mathcal{R}) \cdot a \notin \mathbb{Z}_{\geq 0}$ for some $D \in E_{\text{young}}$.

In this case, take $l \in \mathbb{Z}_{>0}$ such that

$$l \cdot \mu_{P,D}(\mathcal{R}) \cdot a \in \mathbb{Z}_{\geq 0} \quad \forall D \in E_{\text{young}}.$$

Then we have $(\mathbb{M}^a)^l \in \mathcal{O}_{W,P}$ and $\{c_{f,\mathbb{O}} \cdot (\mathbb{M}^{-1})^a\}^l \in \widehat{\mathcal{O}_{W,P}}$ by definition. Moreover, by construction of the companion modification, we have $(\{c_{f,\mathbb{O}} \cdot (\mathbb{M}^{-1})^a\}^l, l \cdot \tilde{\mu} \cdot a) \in \widehat{\text{Comp}}(\mathcal{R}) \subset \widehat{\mathcal{R}'}$. Note that

$$l \cdot \tilde{\mu} \cdot a = l \cdot (\mu_P(\mathcal{R}) - \sum_{D \subset E_{\text{young}}} \mu_{P,D}(\mathcal{R})) \cdot a = \text{ord}_P \left(\{c_{f,\mathbb{O}} \cdot (\mathbb{M}^{-1})^a\}^l \right).$$

Note also that, since $\text{ord}_{\xi_D}(c_{f,\mathbb{O}}) \geq \mu_{P,D}(\mathcal{R}) \cdot a$ by definition, since $\text{ord}_{\xi_D}(c_{f,\mathbb{O}}) \in \mathbb{Z}_{\geq 0}$ and since $\mu_{P,D}(\mathcal{R}) \cdot a \notin \mathbb{Z}_{\geq 0}$ by the subcase assumption, we conclude

$$\text{ord}_{\xi_D}(c_{f,\mathbb{O}}) > \mu_{P,D}(\mathcal{R}) \cdot a.$$

This implies, $\{c_{f,\mathbb{O}} \cdot (\mathbb{M}^{-1})^a\}^l$ is divisible by x_D , and hence so is

$$\text{In} \left(\{c_{f,\mathbb{O}} \cdot (\mathbb{M}^{-1})^a\}^l \right) = \{c_{f,\mathbb{O}} \cdot (\mathbb{M}^{-1})^a\}^l \pmod{\widehat{\mathfrak{m}}^{l \cdot \tilde{\mu} \cdot a + 1}}.$$

Since $L_P(\mathcal{R}) = k[x_1^{p^{e_1}}, \dots, x_t^{p^{e_t}}]$ and since $x_D \in \{x_{t+1}, \dots, x_d\}$, we conclude

$$\text{In} \left(\{c_{f,\mathbb{O}} \cdot (\mathbb{M}^{-1})^a\}^l \right) \notin L_P(\mathcal{R}),$$

while by definition

$$\text{In} \left(\{c_{f,\mathbb{O}} \cdot (\mathbb{M}^{-1})^a\}^l \right) \in L_P(\widehat{\mathcal{R}'}) = L_P(\mathcal{R}').$$

Therefore, we conclude $L_P(\mathcal{R}) \subsetneq L_P(\widehat{\mathcal{R}'})$ and hence $\sigma > \sigma'$.

Case: $\tilde{\mu} = 0$ or ∞ .

In this case, we claim $(\sigma, \#E) > (\sigma', \#E')$. Observe $s \neq 0$. (If $s = 0$, then $(\sigma, \tilde{\mu}, s) = (\sigma, 0, 0)$ or $(\sigma, \infty, 0)$, and hence we do not construct the modification (W', \mathcal{R}', E') .) Therefore, by definition, there exists a divisor $D \in E_{\text{aged}}$ containing P . Thus, we have $\#E > \#(E \setminus E_{\text{aged}}) = \#E'$. (Note that “#” represents the number of the components passing through P .) Q.E.D.

Proceeding in the vertical direction

Proposition 4. *Let $(W, \mathcal{R}, E) \xleftarrow{\pi} (\widetilde{W}, \widetilde{\mathcal{R}}, \widetilde{E})$ be a transformation in the resolution sequence (i.e., $(W_i, \mathcal{R}_i, E_i) \xleftarrow{\pi_{i+1}} (W_{i+1}, \mathcal{R}_{i+1}, E_{i+1})$ in the sequence described in Problem 4 (cf. Remark 2)). Take a point $\widetilde{P} \in \pi^{-1}(P) \cap \text{Sing}(\widetilde{\mathcal{R}})$. Then the value of the triplet $(\sigma, \widetilde{\mu}, s)$ does not increase as we proceed in the vertical direction, i.e., we have*

$$(\sigma, \widetilde{\mu}, s) \geq (\widetilde{\sigma}, \widetilde{\mu}, \widetilde{s}) \quad (\text{i.e., } (\sigma_i, \widetilde{\mu}_i, s_i) \geq (\sigma_{i+1}, \widetilde{\mu}_{i+1}, s_{i+1})).$$

More precisely, we have the following:

(1) *The invariant σ does not increase, i.e., $\sigma \geq \widetilde{\sigma}$. When $\sigma = \widetilde{\sigma}$, the transformation of the LGS for $\widehat{\mathcal{R}}_P$ is an LGS for $\widehat{\widetilde{\mathcal{R}}}_P$. Moreover, the following property is preserved going from (W, \mathcal{R}, E) to $(\widetilde{W}, \widetilde{\mathcal{R}}, \widetilde{E})$: We can choose an LGS $\mathbb{H} = \{(h_\alpha, p^{e_\alpha})\}_{\alpha=1}^t \subset \widehat{\mathcal{R}}_P$ and a regular system of parameters $X = (x_1, \dots, x_t, x_{t+1}, \dots, x_d)$, taken from $\widehat{\mathcal{O}}_{W,P}$, satisfying the condition (♥) below;*

- $h_\alpha \equiv x_\alpha^{p^{e_\alpha}} \pmod{\widehat{\mathfrak{m}}_P^{e_\alpha+1}}$ for $\alpha = 1, \dots, t$,
- the idealistic filtration of i.f.g. type $\widehat{\mathcal{R}}_P$ is saturated for $\{\partial^n / \partial x_\alpha^n \mid n \in \mathbb{Z}_{\geq 0}, \alpha = 1, \dots, t\}$, and
- the defining equations for the components of E_{young} , which are transversal to the LGS, form a part of the regular system of parameters, i.e., $\{x_D \mid D \in E_{\text{young}}\} \subset \{x_{t+1}, \dots, x_d\}$.

(2) *When $\sigma = \widetilde{\sigma}$, the value of the invariant $\widetilde{\mu}$ does not increase, i.e., $(\sigma, \widetilde{\mu}) \geq (\sigma, \widetilde{\mu}) = (\widetilde{\sigma}, \widetilde{\mu})$.*

(3) *When $\sigma = \widetilde{\sigma}$, the value of the invariant s does not increase, i.e., $s \geq \widetilde{s}$, and hence combined with (2) we have*

$$(\sigma, \widetilde{\mu}, s) \geq (\sigma, \widetilde{\mu}, \widetilde{s}) = (\widetilde{\sigma}, \widetilde{\mu}, \widetilde{s}).$$

Proof. (1) Since $C \subset \text{Sing}(\mathcal{R})$ and since C is nonsingular, we conclude that there exists a regular system of parameters $(y_1, \dots, y_r, y_{r+1}, \dots, y_d) \subset \mathcal{O}_{W,P}$ such that

- $C = \{y_1 = \dots = y_r = 0\}$, and
- we have for $\alpha = 1, \dots, t$

$$\begin{cases} h_\alpha & \equiv x_\alpha^{p^{e_\alpha}} \pmod{\widehat{\mathfrak{m}}_P^{e_\alpha+1}}, \text{ and} \\ x_\alpha & \equiv \sum_{\beta=1}^r c_{\alpha,\beta} y_\beta \pmod{\widehat{\mathfrak{m}}_P^2} \text{ for some } c_{\alpha,\beta} \in k. \end{cases}$$

By replacing $(y_1, \dots, y_r, y_{r+1}, \dots, y_d)$ with some linear transformation and then by replacing $(x_1, \dots, x_t, x_{t+1}, \dots, x_d)$ with $(y_1, \dots, y_r, y_{r+1}, \dots, y_d)$, we may assume that we have a regular system of parameters $(x_1, \dots, x_t, x_{t+1}, \dots, x_d) \subset \widehat{\mathcal{O}_{W,P}}$ such that

- $C = \{x_1 = \dots = x_r = 0\}$ ($t \leq r$), and
- $h_\alpha \equiv x_\alpha^{p^{e_\alpha}} \pmod{\widehat{\mathfrak{m}}_P^{p^{e_\alpha}+1}}$ for $\alpha = 1, \dots, t$.

Observe that, if $t = r$, then after the blow up, for any point \tilde{P} over each x_α -chart ($\alpha = 1, \dots, t$) we have $\text{ord}_{\tilde{P}}(\tilde{h}_\alpha) = 0 < p^{e_\alpha}$ where $\tilde{h}_\alpha = h_\alpha/x_\alpha^{p^{e_\alpha}}$, and hence $\pi^{-1}(P) \cap \text{Sing}(\mathcal{R}) = \emptyset$. Therefore, we may assume $t < r$ and that our point $\tilde{P} \in \pi^{-1}(P) \cap \text{Sing}(\tilde{\mathcal{R}})$ is in the x_β -chart for some ($t < \beta \leq r$) with the regular system of parameters $(\tilde{x}_1, \dots, \tilde{x}_t, \tilde{x}_{t+1}, \dots, \tilde{x}_r, \tilde{x}_{r+1}, \dots, \tilde{x}_d)$ where

$$\tilde{x}_\alpha = \begin{cases} x_\alpha/x_\beta & \text{for } 1 \leq \alpha \leq r, \alpha \neq \beta \\ x_\beta & \text{for } \alpha = \beta, \\ x_\alpha & \text{for } r+1 \leq \alpha \leq d. \end{cases}$$

(For the indices $t+1 \leq \alpha \leq r, \alpha \neq \beta$, we may have to replace x_α with $x_\alpha - c_\alpha x_\beta$ for some $c_\alpha \in k$ if necessary.)

Now we look at the transformation $\tilde{h}_\alpha = h_\alpha/x_\beta^{p^{e_\alpha}}$ of

$$h_\alpha = \sum_{K=(k_1, \dots, k_d) \in (\mathbb{Z}_{\geq 0})^d} c_{\alpha,K} X^K \quad \text{with } c_{\alpha,K} \in k$$

for $\alpha = 1, \dots, t$. We compute

$$\tilde{h}_\alpha = \frac{h_\alpha}{x_\beta^{p^{e_\alpha}}} = \sum_{K=(k_1, \dots, k_d) \in (\mathbb{Z}_{\geq 0})^d} c_{\alpha,K} \frac{X^K}{x_\beta^{p^{e_\alpha}}} = \sum_{\tilde{K}=(\tilde{k}_1, \dots, \tilde{k}_d) \in (\mathbb{Z}_{\geq 0})^d} c_{\alpha,K} \tilde{X}^{\tilde{K}}$$

where $\tilde{k}_u = k_u$ for $u \neq \beta$ and $\tilde{k}_\beta = \sum_{u=1}^r k_u - p^{e_\alpha}$. Therefore, we have

$$\sum_{1 \leq u \leq r, u \neq \beta} k_u \geq p^{e_\alpha} \Rightarrow \text{deg}_{\tilde{X}}(\tilde{X}^{\tilde{K}}) \geq \text{deg}_X(X^K).$$

Therefore, the only terms $\tilde{X}^{\tilde{K}}$ with $\text{deg}_{\tilde{X}}(\tilde{X}^{\tilde{K}}) = p^{e_\alpha} < \text{deg}_X(X^K)$ that can possibly appear in \tilde{h}_α have to contain one of $\tilde{x}_\beta, \tilde{x}_{r+1}, \dots, \tilde{x}_d$.

Looking at $\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_t$ in the ascending order, we conclude that

- either $\sigma > \tilde{\sigma}$,
- or $\sigma = \tilde{\sigma}$, and we have

$$\widetilde{h}_\alpha \equiv \widetilde{x}_\alpha^{p^{e_\alpha}} + c_{\alpha,\beta} \widetilde{x}_\beta^{p^{e_\alpha}} + \sum_{u=r+1}^d c_{\alpha,u} \widetilde{x}_u^{p^{e_\alpha}} \pmod{\widehat{\mathfrak{m}}_P^{p^{e_\alpha}+1}}$$

with $\{c_{\alpha,\beta}, c_{\alpha,r+1}, \dots, c_{\alpha,d}\} \subset k$ for $\alpha = 1, \dots, t$, and hence $\widetilde{\mathbb{H}} = \{(\widetilde{h}_\alpha, p^{e_\alpha})\}_{\alpha=1}^t$ is an LGS for $\widetilde{\mathcal{R}}$.

This completes the proof of the first part of (1).

Next we look at the ‘‘Moreover’’ part of (1).

We show the existence of such an LGS $\mathbb{H} = \{(h_\alpha, p^{e_\alpha})\}_{\alpha=1}^t$ and its associated regular system of parameters $X = (x_1, \dots, x_t, x_{t+1}, \dots, x_d)$ that satisfy the condition (\heartsuit) by induction on the year .

When we are in the year when the value of σ first started (i.e., the value of σ is strictly less than the one in the previous year), we replace the original \mathcal{R} with its differential saturation (cf. the technical but important points in the description of the triplet of invariants $(\sigma, \tilde{\mu}, s)$ (i)). Thus we may assume \mathcal{R} is \mathcal{D} -saturated. Moreover, we have $E_{\text{young}} = \emptyset$. Therefore, we have only to take an LGS \mathbb{H} and X such that $h_\alpha \equiv x_\alpha^{p^{e_\alpha}} \pmod{\widehat{\mathfrak{m}}_P^{p^{e_\alpha}+1}}$ for $\alpha = 1, \dots, t$. The remaining requirements in the condition (\heartsuit) are then automatically satisfied.

Now we assume that in the current year we have such an LGS \mathbb{H} and its associated regular system of parameters X satisfying the condition (\heartsuit) . We show that, assuming $\sigma = \tilde{\sigma}$, even after the transformation we have such an LGS $\widetilde{\mathbb{H}}$ and \widetilde{X} that satisfy the condition (\heartsuit) .

Step 1. We modify our X so that the LGS \mathbb{H} and X are still associated, satisfy the condition (\heartsuit) as before, and now satisfy the extra condition that the center is defined by $C = \{x_1 = \dots = x_r = 0\}$ ($t \leq r$).

(a) We take another regular system of parameters $Y = (y_1, \dots, y_r, y_{r+1}, \dots, y_d)$ such that the center is defined by $C = \{y_1 = \dots = y_r = 0\}$. Note that, if $C \subset D$ for any component $D \in E_{\text{young}}$, then we include x_D in Y . Note that such x_D is included in X .

Then since $C \subset \text{Sing}(\mathcal{R})$, we conclude that

$$x_\alpha \equiv \sum_{\beta=1}^r c_{\alpha,\beta} y_\beta \pmod{\widehat{\mathfrak{m}}_P^2} \quad \text{for some } c_{\alpha,\beta} \in k$$

for $\alpha = 1, \dots, t$. Therefore, by taking a suitable linear transformation among $\{y_1, \dots, y_r\}$, we may assume

$$x_\alpha \equiv y_\alpha \pmod{\widehat{\mathfrak{m}}_P^2} \quad \text{for } \alpha = 1, \dots, t.$$

It is straightforward to see that, since \mathbb{H} and $X = (x_1, \dots, x_t, x_{t+1}, \dots, x_d)$ satisfy the condition (\heartsuit) , so does \mathbb{H} and $(y_1, \dots, y_t, x_{t+1}, \dots, x_d)$. By replacing X with $(y_1, \dots, y_t, x_{t+1}, \dots, x_d)$, we may assume that \mathbb{H} and X satisfy the condition (\heartsuit) , and $(x_1, \dots, x_t, y_{t+1}, \dots, y_r, y_{r+1}, \dots, y_d)$ is a regular system of parameters with $C = \{x_1 = \dots = x_t = y_{t+1} = \dots = y_r = 0\}$.

(b) We look at $\{y_\beta \mid t + 1 \leq \beta \leq r\}$. By applying the Weierstrass Division Theorem consecutively (and replacing the original y_β after multiplying some unit if necessary) or by simply looking at the power series expansion with respect to X , we write

$$y_\beta = \sum_{\alpha=1}^t q_{\beta,\alpha} x_\alpha + g(x_{t+1}, \dots, x_d) \quad \text{for } \beta = t + 1, \dots, r$$

with $q_{\beta,\alpha} \in \widehat{\mathcal{O}_{W,P}}$ and $g(x_{t+1}, \dots, x_d) \in k[[x_{t+1}, \dots, x_d]]$. Set

$$y'_\beta = y_\beta - \sum_{\alpha=1}^t q_{\beta,\alpha} x_\alpha \quad \text{for } \beta = t + 1, \dots, r.$$

Choose $\{x_{r+1}, \dots, x_d\} \subset \{x_{t+1}, \dots, x_d\}$ (after renumbering of x_{t+1}, \dots, x_d if necessary and keeping the condition $\{x_D \mid D \in E_{\text{young}}\} \subset \{x_{t+1}, \dots, x_d\}$) such that $(x_1, \dots, x_t, y'_{t+1}, \dots, y'_r, x_{r+1}, \dots, x_d)$ is a regular system of parameters. Now replace X with $(x_1, \dots, x_t, y'_{t+1}, \dots, y'_r, x_{r+1}, \dots, x_d)$. Then it is straightforward to see that, since \mathbb{H} and the previous X satisfy the condition (\heartsuit) , so do \mathbb{H} and the new X . Now by construction, the center C is defined by $C = \{x_1 = \dots = x_r = 0\}$ ($t \leq r$).

Step 2. Analysis after blow up.

As in the proof of the first part, we may assume $t < r$ and that our point $\tilde{P} \in \pi^{-1}(P) \cap \text{Sing}(\tilde{\mathcal{R}})$ is in the x_β -chart for some ($t < \beta \leq r$) with the regular system of coordinates $\tilde{X} = (\tilde{x}_1, \dots, \tilde{x}_t, \tilde{x}_{t+1}, \dots, \tilde{x}_r, \tilde{x}_{r+1}, \dots, \tilde{x}_d)$ where

$$\tilde{x}_\alpha = \begin{cases} x_\alpha/x_\beta & \text{for } 1 \leq \alpha \leq r, \alpha \neq \beta, \\ x_\beta & \text{for } \alpha = \beta, \\ x_\alpha & \text{for } r + 1 \leq \alpha \leq d. \end{cases}$$

(For the indices $t + 1 \leq \alpha \leq r, \alpha \neq \beta$, we may have to replace x_α with $x_\alpha - c_\alpha x_\beta$ for some $c_\alpha \in k$ if necessary.)

It is straightforward to see that the above \tilde{X} satisfies the following two conditions;

- the idealistic filtration of i.f.g. type $\widehat{\mathcal{R}}_{\widetilde{P}}$, which is (the completion of) the transformation of $\widehat{\mathcal{R}}_P$, is saturated for $\{\partial^n/\partial \widetilde{x}_\alpha^n \mid n \in \mathbb{Z}_{\geq 0}, \alpha = 1, \dots, t\}$, and

- the defining equations for the $\widetilde{E}_{\text{young}}$ form a part of the regular system of parameters, i.e., $\{\widetilde{x}_{\widetilde{D}} \mid \widetilde{D} \in \widetilde{E}_{\text{young}}\} \subset \{\widetilde{x}_{t+1}, \dots, \widetilde{x}_d\}$.

We have only to replace \widetilde{X} with $(\widetilde{x}'_1, \dots, \widetilde{x}'_t, \widetilde{x}_{t+1}, \dots, \widetilde{x}_d)$ in order to satisfy the remaining condition

- $\widetilde{h}_\alpha \equiv \widetilde{x}_\alpha^{p^{e_i}} \pmod{\widehat{\mathfrak{m}}_{\widetilde{P}}^{p^{e_\alpha}+1}}$ for $\alpha = 1, \dots, t$,

while keeping the other two requirements as above, where

$$\widetilde{x}'_\alpha = \widetilde{x}_\alpha + c_{\alpha,\beta}^{1/p^{e_\alpha}} \widetilde{x}_\beta + \sum_{u=r+1}^d c_{\alpha,u}^{1/p^{e_\alpha}} \widetilde{x}_u \quad \text{for } \alpha = 1, \dots, t$$

(using the same notation as in the proof of the first part).

Then the LGS $\widetilde{\mathbb{H}}$ and \widetilde{X} satisfy the condition (\heartsuit) .

This finishes the proof of “Moreover” part of (1).

(2) We use the same notation used in the proof of (1). We take an LGS \mathbb{H} and its associated regular system of parameters X , satisfying the condition (\heartsuit) and the extra condition that the center is defined by $C = \{x_1 = \dots = x_r = 0\}$ (see Step 1 in the proof of “Moreover” part of (1)). When $\sigma = \widetilde{\sigma}$, we make the following two observations.

- The transformation $\widetilde{\mathbb{H}}$ is an LGS of $\widetilde{\mathcal{R}}$.
- For $(f, a) \in \widehat{\mathcal{R}}_P$, let $\sum c_{f,B} H^B$ be the power series expansion of f with respect to \mathbb{H} and its associated regular system of parameters $(x_1, \dots, x_t, x_{t+1}, \dots, x_d)$, with the “constant term” being $c_{f,0}$. Look at its transformation (\widetilde{f}, a) with $\widetilde{f} = f/x_\beta^a$. The constant term $c_{\widetilde{f},0}$ of the transformation \widetilde{f} with respect to $\widetilde{\mathbb{H}}$ and its associated regular system of parameters $(\widetilde{x}'_1, \dots, \widetilde{x}'_t, \widetilde{x}_{t+1}, \dots, \widetilde{x}_d)$ is the transformation $\widetilde{c}_{f,0} = c_{f,0}/x_\beta^a$, where

$$\widetilde{x}'_\alpha = \widetilde{x}_\alpha + c_{\alpha,\beta}^{1/p^{e_\alpha}} \widetilde{x}_\beta + \sum_{u=r+1}^d c_{\alpha,u}^{1/p^{e_\alpha}} \widetilde{x}_u \quad \text{for } \alpha = 1, \dots, t.$$

Now the inequality $\widetilde{\mu} \geq \widetilde{\mu}$ follows from these two observations and the condition $C \subset \text{Sing}(\text{Comp}(\mathcal{R}))$ in our new setting by the same argument as the one used to show that the invariant w-ord does not increase under transformation in the classical setting.

(3) This follows immediately from the fact that the center C of blow up for the transformation π is nonsingular and transversal to the

boundary E (hence to E_{aged}), and from the fact that, since $\sigma = \tilde{\sigma}$, the aged part \tilde{E}_{aged} of \tilde{E} is the strict transform of E_{aged} by definition. Q.E.D.

4.3. Weaving of the new strand of invariants “inv_{new}”

In §4.3, we interpret the inductive scheme explained in §4.2 in terms of weaving the new strand of invariants “inv_{new}”.

We weave the strand of invariants “inv_{new}” consisting of the units of the form $(\sigma^j, \tilde{\mu}^j, s^j)$, computed from the modifications $(W^j, \mathcal{R}^j, E^j)$ constructed simultaneously along the weaving process.

Weaving Process

We describe the weaving process inductively.

Note that constructing a sequence for resolution of singularities by blowing up is referred to as “proceeding in the vertical direction” passing from one year to the next, indicated by the subscript “ i ”, while weaving the strand and constructing the modifications passing from one stage to the next, indicated by the superscript “ j ”, is referred to as “proceeding in the horizontal direction” staying in a fixed year.

Suppose we have already woven the strands and constructed the modifications up to year $(i - 1)$.

Now we are in year i (looking at the neighborhood of a point $P_i \in \text{Sing}(\mathcal{R}_i) \subset W_i$ (cf. Remark 2)).

We start with $(W_i, \mathcal{R}_i, E_i) = (W_i^0, \mathcal{R}_i^0, E_i^0)$, just renaming the transformation $(W_i, \mathcal{R}_i, E_i)$ in year i of the resolution sequence as the 0-th stage modification $(W_i^0, \mathcal{R}_i^0, E_i^0)$ in year i .

Suppose that we have already woven the strand up to the $(j - 1)$ -th unit

$$(\text{inv}_{\text{new}})_i^{\leq j-1} = (\sigma_i^0, \tilde{\mu}_i^0, s_i^0)(\sigma_i^1, \tilde{\mu}_i^1, s_i^1) \cdots (\sigma_i^{j-1}, \tilde{\mu}_i^{j-1}, s_i^{j-1})$$

and that we have also constructed the modifications up to the j -th one

$$(W_i^0, \mathcal{R}_i^0, E_i^0), (W_i^1, \mathcal{R}_i^1, E_i^1), \dots, (W_i^{j-1}, \mathcal{R}_i^{j-1}, E_i^{j-1}), (W_i^j, \mathcal{R}_i^j, E_i^j).$$

Our task is to compute the j -th unit $(\sigma_i^j, \tilde{\mu}_i^j, s_i^j)$ and construct the $(j + 1)$ -th modification $(W_i^{j+1}, \mathcal{R}_i^{j+1}, E_i^{j+1})$ (unless the weaving process is over at the j -th stage).

Computation of the j -th unit $(\sigma_i^j, \tilde{\mu}_i^j, s_i^j)$

◦ σ_i^j : It is the invariant σ associated to the differential saturation \mathcal{DR}_i^j of the idealistic filtration of i.f.g. type \mathcal{R}_i^j (cf. Proposition 1).

◦ $\tilde{\mu}_i^j$: It is the (normalized) weak order modulo LGS of the idealistic filtration of i.f.g. type \mathcal{R}_i^j with respect to $(E_i^j)_{\text{young}}$. (For the definition of $(E_i^j)_{\text{young}}(\subset E_i^j)$, see the description of the invariant s_i^j below.

We note that, in order to compute $\tilde{\mu}_i^j$,

- we keep \mathcal{R}_i^j as it is, which is the transformation of \mathcal{R}_{i-1}^j ,
- we take the LGS to be the transformation of the one in the previous year

if $(\text{inv}_{\text{new}})_i^{\leq j-1}(\sigma_i^j) = (\text{inv}_{\text{new}})_{i-1}^{\leq j-1}(\sigma_{i-1}^j)$,

- we replace the original \mathcal{R}_i^j with its differential saturation \mathcal{DR}_i^j ,
- we take the LGS from the replaced $\mathcal{R}_i^j = \mathcal{DR}_i^j$

if $(\text{inv}_{\text{new}})_i^{\leq j-1}(\sigma_i^j) < (\text{inv}_{\text{new}})_{i-1}^{\leq j-1}(\sigma_{i-1}^j)$.

◦ s_i^j : It is the number of the components in $(E_i^j)_{\text{aged}} = E_i^j \setminus (E_i^j)_{\text{young}}$, where $(E_i^j)_{\text{young}}(\subset E_i^j)$ is the union of the exceptional divisors created after the year when the value $(\text{inv}_{\text{new}})_i^{\leq j-1}(\sigma_i^j)$ first started. We note that this third factor is included even if $\tilde{\mu}_i^j = \infty$ or 0.

We note that, if $(\sigma_i^j, \tilde{\mu}_i^j, s_i^j) = (\sigma_i^j, \infty, 0)$ or $(\sigma_i^j, 0, 0)$, then we declare that the $(j = m)$ -th unit is the last one, and we stop the weaving process at the m -th stage in year i .

Thus the weaving process of the strand comes to an end in a fixed year i , with the strand $(\text{inv}_{\text{new}})_i$ taking the following form

$$(\text{inv}_{\text{new}})_i = (\sigma_i^0, \tilde{\mu}_i^0, s_i^0)(\sigma_i^1, \tilde{\mu}_i^1, s_i^1) \cdots (\sigma_i^j, \tilde{\mu}_i^j, s_i^j) \cdots (\sigma_i^m, \tilde{\mu}_i^m, s_i^m),$$

where $(\sigma_i^m, \tilde{\mu}_i^m, s_i^m) = (\sigma_i^m, \infty, 0)$ or $(\sigma_i^m, 0, 0)$.

Termination in the horizontal direction

We note that termination of the weaving process in the horizontal direction is a consequence of the fact that going from the j -th unit to the $(j + 1)$ -th unit we have $(\sigma_i^j, \#E_i^j) > (\sigma_i^{j+1}, \#E_i^{j+1})$ (cf. Proposition 3) and that the value set of $(\sigma, \#E)$ satisfies the descending chain condition.

Construction of the $(j + 1)$ -th modification $(W_i^{j+1}, \mathcal{R}_i^{j+1}, E_i^{j+1})$

We note that we construct the $(j + 1)$ -th modification only when $(\sigma_i^j, \tilde{\mu}_i^j, s_i^j) \neq (\sigma_i^j, \infty, 0)$ or $(\sigma_i^j, 0, 0)$.

We follow the construction described in the mechanism discussed in §4.2. Starting from $(W_i^j, \mathcal{R}_i^j, E_i^j)$, we construct $(W_i^{j+1}, \mathcal{R}_i^{j+1}, E_i^{j+1})$ as

below:

$$W_i^{j+1} = W_i^j, \quad \mathcal{R}_i^{j+1} = \text{Bdry} \left(\text{Comp}(\mathcal{R}_i^j) \right), \quad \text{and}$$

$$E_i^{j+1} = E_i^j \setminus (E_i^j)_{\text{aged}} = (E_i^j)_{\text{young}},$$

where

- denoting $(\text{inv}_{\text{new}})_i^{\leq j-1}(\sigma_i^j, \tilde{\mu}_i^j)$ by β_i^j , we set $\text{Comp}(\mathcal{R}_i^j)$ to be either the transformation of $\text{Comp}(\mathcal{R}_{i-1}^j)$ if $\beta_i = \beta_{i-1}$, or the one obtained by applying the construction in §4.2 if $\beta_i^j < \beta_{i-1}^j$, and
- denoting $(\text{inv}_{\text{new}})_i^{\leq j}$ by γ_i^j , we set $\text{Bdry}(\text{Comp}(\mathcal{R}_i^j))$ to be either the transformation of $\text{Bdry}(\text{Comp}(\mathcal{R}_{i-1}^j))$ if $\gamma_i^j = \gamma_{i-1}^j$, or

$$\mathcal{G}(\text{Comp}(\mathcal{R}_i^j) \cup \{(x_D, 1) \mid D \subset (E_i^j)_{\text{aged}}\})$$

if $\gamma_i^j < \gamma_{i-1}^j$.

We note that, if $\gamma_i^j = \gamma_{i-1}^j$, then $(W_i^{j+1}, \mathcal{R}_i^{j+1}, E_i^{j+1})$ is the transformation of $(W_{i-1}^{j+1}, \mathcal{R}_{i-1}^{j+1}, E_{i-1}^{j+1})$.

We also observe the following.

- * The ambient space remains the same throughout a fixed year i , i.e.,

$$W_i = W_i^0 = W_i^1 = \dots = W_i^j = W_i^{j+1} = \dots = W_i^{m-1} = W_i^m.$$

This is in clear contrast to the classical setting, where we take a consecutive sequence of the hypersurfaces of maximal contact (cf. §3.3)

$$W_i = H_i^0 \supset H_i^1 \supset \dots \supset H_i^j \supset H_i^{j+1} \supset \dots \supset H_i^{m-1} \supset H_i^m.$$

- * The idealistic filtration of i.f.g. type gets enlarged (not necessarily strictly) under modification, i.e.,

$$\mathcal{R}_i = \mathcal{R}_i^0 \subset \mathcal{R}_i^1 \subset \dots \subset \mathcal{R}_i^j \subset \mathcal{R}_i^{j+1} \dots \subset \mathcal{R}_i^{m-1} \subset \mathcal{R}_i^m.$$

- * The boundary divisor decreases (not necessarily strictly) under modification, i.e.,

$$E_i = E_i^0 \supset E_i^1 \supset \dots \supset E_i^j \supset E_i^{j+1} \supset \dots \supset E_i^{m-1} \supset E_i^m.$$

Summary of our algorithm in $\text{char}(k) = p > 0$ in terms of “ inv_{new} ”

We start with $(W, \mathcal{R}, E) = (W_0, \mathcal{R}_0, E_0)$.

Suppose we have already constructed the resolution sequence up to year i

$$(W, \mathcal{R}, E) = (W_0, \mathcal{R}_0, E_0) \leftarrow \cdots \leftarrow (W_i, \mathcal{R}_i, E_i).$$

We weave the strand of invariants in year i described as above

$$(\text{inv}_{\text{new}})_i = (\sigma_i^0, \tilde{\mu}_i^0, s_i^0)(\sigma_i^1, \tilde{\mu}_i^1, s_i^1) \cdots (\sigma_i^j, \tilde{\mu}_i^j, s_i^j) \cdots (\sigma_i^m, \tilde{\mu}_i^m, s_i^m),$$

where $(\sigma_i^m, \tilde{\mu}_i^m, s_i^m) = (\sigma_i^m, \infty, 0)$ or $(\sigma_i^m, 0, 0)$.

Case: $(\sigma_i^m, \tilde{\mu}_i^m, s_i^m) = (\sigma_i^m, \infty, 0)$.

In this case, we take the center of blow up in year i for the transformation to be the singular locus of the last modification $(W_i^m, \mathcal{R}_i^m, E_i^m)$, i.e., $\text{Sing}(\mathcal{R}_i^m)$, which is easily seen to be nonsingular as follows. We go back to the year $\iota := i_{\text{aged}}^m$ when the value of $(\text{inv}_{\text{new}})_i^{\leq m-1}(\sigma_i^m)$ first started. Observe that $\tilde{\mu}_i^m = \infty$ implies $\tilde{\mu}_\iota^m = \infty$. The Nonsingularity Principle (cf. [18] and [19]) applied to $\mathcal{R}_\iota^m = \mathcal{DR}_\iota^m$ tells us that there exists a regular system of parameters $(x_1, \dots, x_t, x_{t+1}, \dots, x_d)$ at P_ι such that $\mathcal{R}_\iota^m = \mathcal{G}((x_1^{p^{e_1}}, p^{e_1}), \dots, (x_t^{p^{e_t}}, p^{e_t}))$. Note that the center C_ι of blow up in year ι is contained in $\text{Sing}(\mathcal{R}_\iota^m)$. From this it follows inductively that there exists a regular system of parameters $(x_{1,i'}, \dots, x_{t,i'}, x_{t+1,i'}, \dots, x_{d,i'})$ at $P_{i'} \in \text{Sing}(\mathcal{R}_{i'}^m) \subset W_{i'}$ such that $\text{Sing}(\mathcal{R}_{i'}^j) = \mathcal{G}((x_{1,i'}^{p^{e_1}}, p^{e_1}), \dots, (x_{t,i'}^{p^{e_t}}, p^{e_t}))$ for $\iota \leq i' \leq i$. In particular, $\text{Sing}(\mathcal{R}_i^m)$ is nonsingular. We note that $\text{Sing}(\mathcal{R}_i^m) \subset \bigcap_{D \in E_i \setminus (E_i^m)_{\text{young}}} D$ by construction of the boundary modifications, and that $\text{Sing}(\mathcal{R}_i^m)$ is transversal to $(E_i^m)_{\text{young}}$. This implies that the center $\text{Sing}(\mathcal{R}_i^m)$ is transversal to E_i . After blow up, the singular locus of $(W_i^m, \mathcal{R}_i^m, E_i^m)$ disappears. Since resolution of singularities for $(W_i^m, \mathcal{R}_i^m, E_i^m)$ implies the strict decrease of the value of $(\text{inv}_{\text{new}})^{\leq m-1}$, we have $(\text{inv}_{\text{new}})_i^{\leq m-1} > (\text{inv}_{\text{new}})_{i+1}^{\leq m-1}$.

Case: $(\sigma_i^m, \tilde{\mu}_i^m, s_i^m) = (\sigma_i^m, 0, 0)$.

In this case, by following the procedure specified for resolution of singularities in the monomial case, we achieve resolution of singularities of the m -th modification, which implies the strict decrease of the value of $(\text{inv}_{\text{new}})^{\leq m-1}$. (It is possible that in the middle of the procedure the value of the $(\text{inv}_{\text{new}})^{\leq m-1}$ (or $(\text{inv}_{\text{new}})^{\leq m-1}(\sigma^m)$) strictly decreases).

Note: In both cases above, we are using the fact that the value of “ inv_{new} ” never increases after each transformation in the resolution sequence, in order to derive the strict decrease of $(\text{inv}_{\text{new}})^{\leq m-1}$ (or $(\text{inv}_{\text{new}})^{\leq m-1}(\sigma^m)$).

Termination in the vertical direction

By looking at the conclusions of the two cases above at the end of the weaving process in the horizontal direction, we conclude that in some year i' , the value of the strand strictly decreases, i.e., $(\text{inv}_{\text{new}})_i > (\text{inv}_{\text{new}})_{i'}$.

Now we claim that the value of the strand “ inv_{new} ” cannot decrease infinitely many times. In fact, suppose by induction we have shown that the value of $(\text{inv}_{\text{new}})^{\leq t-1}$ cannot decrease infinitely many times. Then after some year the value of $(\text{inv}_{\text{new}})^{\leq t-1}$ stabilizes. Since the value of the invariant σ satisfies the descending chain condition, after some year (say after year i'_{t-1}), the value of $(\text{inv}_{\text{new}})^{\leq t-1}(\sigma^t)$ stabilizes. Therefore, after year i'_{t-1} , we use the transformation of $\mathcal{R}_{i'_{t-1}}^t$ in order to compute the invariant $\tilde{\mu}$. (See (i) in the technical but important points in the computation of the triplet $(\sigma, \tilde{\mu}, s)$.) This implies that the denominator of the invariant $\tilde{\mu}$ is bounded, and hence that the invariant $\tilde{\mu}$ cannot decrease infinitely many times. Since the invariant s , being a nonnegative integer, cannot decrease infinitely many times, we conclude that $(\text{inv}_{\text{new}})^{\leq t-1}(\sigma^t, \tilde{\mu}^t, s^t) = (\text{inv}_{\text{new}})^{\leq t}$ cannot decrease infinitely many times.

For each t , let year i_t be the time when the stabilization of $(\text{inv}_{\text{new}})^{\leq t}$ first starts, i.e.,

$$(\text{inv}_{\text{new}})_{i_{t-1}}^{\leq t} > (\text{inv}_{\text{new}})_{i_t}^{\leq t} = (\text{inv}_{\text{new}})_i^{\leq t} \quad \text{for } i \geq i_t.$$

Note that $\{i_t\}$ is a (not necessarily strictly) increasing sequence, i.e., $i_t \leq i_{t'}$ if $t \leq t'$. Let $\sigma_t = \sigma_{i_t}^t$ be the first factor of the t -th unit of $(\text{inv}_{\text{new}})_{i_t}^{\leq t}$. Note that $\{\sigma_t\}$ is a (not necessarily strictly) decreasing sequence. That is to say, we have $\sigma_t \geq \sigma_{t+1}$, which follows easily if we look at year i_{t+1} and see $\sigma_t = \sigma_{i_t}^t = \sigma_{i_{t+1}}^t \geq \sigma_{i_{t+1}}^{t+1} = \sigma_{t+1}$.

We claim that we have either $\sigma_t > \sigma_{t+1}$ or $\sigma_t = \sigma_{t+1} > \sigma_{t+2}$. This can be seen by the following reasoning.

- First look at the t -th unit in year i_t , and observe $(\sigma_{i_t}^t, \tilde{\mu}_{i_t}^t, s_{i_t}^t) \neq (\sigma_{i_t}^t, \infty, 0)$ or $(\sigma_{i_t}, 0, 0)$. In fact, if $(\sigma_{i_t}^t, \tilde{\mu}_{i_t}^t, s_{i_t}^t) = (\sigma_{i_t}^t, \infty, 0)$ or $(\sigma_{i_t}, 0, 0)$, then the weaving process is over at the t -th stage in year i_t . When $(\sigma_{i_t}^t, \tilde{\mu}_{i_t}^t, s_{i_t}^t) = (\sigma_{i_t}^t, \infty, 0)$, by the single blow up with center $\text{Sing}(\mathcal{R}_{i_t}^t)$, we accomplish resolution of singularities for $(W_{i_t}^t, \mathcal{R}_{i_t}^t, E_{i_t}^t)$. This, however, implies the strict decrease of $(\text{inv}_{\text{new}})^{\leq t-1}$, contradicting its stability after year $i_{t-1} (\leq i_t)$. If $(\sigma_{i_t}^t, \tilde{\mu}_{i_t}^t, s_{i_t}^t) = (\sigma_{i_t}, 0, 0)$, then by the procedure of resolution of singularities in the monomial case, we accomplish resolution of singularities for $(W_{i_t}^t, \mathcal{R}_{i_t}^t, E_{i_t}^t)$, which implies the strict decrease of the value of $(\text{inv}_{\text{new}})^{\leq t-1}$. This also contradicts the stability

of $(\text{inv}_{\text{new}})^{\leq t-1}$ after year $i_{t-1} (\leq i_t)$. (It is possible that in the middle of the procedure the value of the $(\text{inv}_{\text{new}})^{\leq t-1}$ (or $(\text{inv}_{\text{new}})^{\leq t-1}(\sigma^t)$) strictly decreases. In the former case, it would contradict the stability of $(\text{inv}_{\text{new}})^{\leq t-1}$ after year $i_{t-1} (\leq i_t)$. In the latter case, it would contradict the stability of $(\text{inv}_{\text{new}})^{\leq t}$ after year i_t .)

- If $\tilde{\mu}_{i_t}^t \neq 0$ or ∞ , then $\sigma_{i_t}^t > \sigma_{i_t}^{t+1}$ (cf. the proof of Proposition 3). Since $\sigma_{i_t}^{t+1} \geq \sigma_{i_{t+1}}^{t+1}$, we conclude $\sigma_t = \sigma_{i_t}^t > \sigma_{i_t}^{t+1} \geq \sigma_{i_{t+1}}^{t+1} = \sigma_{t+1}$.

- We consider the case where $\tilde{\mu}_{i_t}^t = 0$. By the first observation, we have $s_{i_t}^t \neq 0$ and the weaving process continues onto the $(t+1)$ -th unit in year i_t . We have $\sigma_{i_t}^t \geq \sigma_{i_t}^{t+1}$.

Case: $\sigma_{i_t}^t > \sigma_{i_t}^{t+1}$.

We have $\sigma_t = \sigma_{i_t}^t > \sigma_{i_t}^{t+1} \geq \sigma_{i_{t+1}}^{t+1} = \sigma_{t+1}$.

Case: $\sigma_{i_t}^t = \sigma_{i_t}^{t+1}$.

Since year i_t is the time when the value of $(\text{inv}_{\text{new}})_{i_t}^{\leq t}$ first started, we have $(E_{i_t}^{t+1})_{\text{young}} = \emptyset$. The idealistic filtration of i.f.g. type $\mathcal{R}_{i_t}^{t+1} \supset \mathcal{R}_{i_t}^t$ contains a monomial of the defining equations of $(E_{i_t}^t)_{\text{young}}$. This implies $\tilde{\mu}_{i_t}^{t+1} \neq 0$ or ∞ . We have $\sigma_{i_t}^{t+1} \geq \sigma_{i_{t+1}}^{t+1}$.

Subcase: $\sigma_{i_t}^{t+1} > \sigma_{i_{t+1}}^{t+1}$.

We have $\sigma_t = \sigma_{i_t}^t = \sigma_{i_t}^{t+1} > \sigma_{i_{t+1}}^{t+1} = \sigma_{t+1}$.

Subcase: $\sigma_{i_t}^{t+1} = \sigma_{i_{t+1}}^{t+1}$.

Since $\tilde{\mu}_{i_t}^{t+1} \neq \infty$, we have $\tilde{\mu}_{i_{t+1}}^{t+1} \neq \infty$.

Subsubcase: $\tilde{\mu}_{i_{t+1}}^{t+1} \neq 0$.

We have $\sigma_{i_{t+1}}^{t+1} > \sigma_{i_{t+1}}^{t+2}$. This implies $\sigma_t = \sigma_{i_t}^t = \sigma_{i_t}^{t+1} = \sigma_{i_{t+1}}^{t+1} > \sigma_{i_{t+1}}^{t+2} \geq \sigma_{i_{t+2}}^{t+2} = \sigma_{t+2}$.

Subsubcase: $\tilde{\mu}_{i_{t+1}}^{t+1} = 0$.

By the first observation, we have $s_{i_{t+1}}^{t+1} \neq 0$. That is to say, there is a component D of $(E_{i_{t+1}}^{t+1})_{\text{aged}} \subset E_{i_{t+1}}^{t+1} = E_{i_{t+1}}^t \setminus (E_{i_{t+1}}^t)_{\text{aged}} = (E_{i_{t+1}}^t)_{\text{young}}$ passing through that point. $(E_{i_{t+1}}^t)_{\text{young}}$ is the union of the exceptional divisors created after the year when the value of $(\text{inv}_{\text{new}})_{i_{t+1}}^{\leq t-1}(\sigma_{i_{t+1}}^{t-1}) = (\text{inv}_{\text{new}})_{i_t}^{\leq t-1}(\sigma_{i_t}^t)$ first started. Therefore, D is transversal to the LGS of $\mathcal{R}_{i_{t+1}}^{t+1}$, which is the transformation of the LGS of $\mathcal{R}_{i_t}^t$ since $\sigma_{i_t}^t = \sigma_{i_t}^{t+1} = \sigma_{i_{t+1}}^{t+1}$. Since $\mathcal{R}_{i_{t+1}}^{t+2}$ contains $(x_D, 1)$, where x_D is the defining equation of D , we conclude $\sigma_{i_{t+1}}^{t+1} > \sigma_{i_{t+1}}^{t+2}$. Therefore, we have $\sigma_t = \sigma_{i_t}^t = \sigma_{i_t}^{t+1} = \sigma_{i_{t+1}}^{t+1} > \sigma_{i_{t+1}}^{t+2} \geq \sigma_{i_{t+2}}^{t+2} = \sigma_{t+2}$.

• We consider the case where $\tilde{\mu}_{i_t}^t = \infty$. By the first observation, we have $s_{i_t}^t \neq 0$ and the weaving process continues onto the $(t+1)$ -th unit in year i_t . We have $\sigma_{i_t}^t \geq \sigma_{i_t}^{t+1}$.

Case: $\sigma_{i_t}^t > \sigma_{i_t}^{t+1}$.

We have $\sigma_t = \sigma_{i_t}^t > \sigma_{i_t}^{t+1} \geq \sigma_{i_{t+1}}^{t+1} = \sigma_{t+1}$.

Case: $\sigma_{i_t}^t = \sigma_{i_t}^{t+1}$.

Since year i_t is the time when the value of $(\text{inv}_{\text{new}})_{i_t}^{\leq t}$ first started, we have $(E_{i_t}^{t+1})_{\text{young}} = \emptyset$. This implies $\tilde{\mu}_{i_t}^{t+1} \neq 0$. If $\tilde{\mu}_{i_t}^{t+1} \neq \infty$, then we can carry the same argument as above ($\tilde{\mu}_{i_t}^t = 0$ and *Case:* $\sigma_{i_t}^t = \sigma_{i_t}^{t+1}$) to conclude that either $\sigma_t > \sigma_{t+1}$ or $\sigma_t = \sigma_{t+1} > \sigma_{t+2}$. Therefore, we have only to consider the case where $\tilde{\mu}_{i_t}^{t+1} = \infty$. We have $\sigma_{i_t}^{t+1} \geq \sigma_{i_{t+1}}^{t+1}$.

Subcase: $\sigma_{i_t}^{t+1} > \sigma_{i_{t+1}}^{t+1}$.

We have $\sigma_t = \sigma_{i_t}^{t+1} = \sigma_{i_{t+1}}^{t+1} > \sigma_{t+1}$.

Subcase: $\sigma_{i_t}^{t+1} = \sigma_{i_{t+1}}^{t+1}$.

Since $\tilde{\mu}_{i_t}^{t+1} = \infty$, we have $\tilde{\mu}_{i_{t+1}}^{t+1} = \infty$. By the first observation, we have $s_{i_{t+1}}^{t+1} \neq 0$. That is to say, there is a component D of $(E_{i_{t+1}}^{t+1})_{\text{aged}} \subset E_{i_{t+1}}^{t+1} = E_{i_{t+1}}^t \setminus (E_{i_{t+1}}^t)_{\text{aged}} = (E_{i_{t+1}}^t)_{\text{young}}$ passing through that point. $(E_{i_{t+1}}^t)_{\text{young}}$ is the union of the exceptional divisors created after the year when the value of $(\text{inv}_{\text{new}})_{i_{t+1}}^{\leq t-1}(\sigma_{i_{t+1}}^t) = (\text{inv}_{\text{new}})_{i_t}^{\leq t-1}(\sigma_{i_t}^t)$ first started. Therefore, D is transversal to the LGS of $\mathcal{R}_{i_{t+1}}^{t+1}$, which is the transformation of the LGS of $\mathcal{R}_{i_t}^t$ since $\sigma_{i_t}^t = \sigma_{i_t}^{t+1} = \sigma_{i_{t+1}}^{t+1}$. Since $\mathcal{R}_{i_{t+1}}^{t+2}$ contains $(x_D, 1)$, where x_D is the defining equation of D , we conclude $\sigma_{i_{t+1}}^{t+1} > \sigma_{i_{t+1}}^{t+2}$. Therefore, we have $\sigma_t = \sigma_{i_t}^t = \sigma_{i_t}^{t+1} = \sigma_{i_{t+1}}^{t+1} > \sigma_{i_{t+1}}^{t+2} \geq \sigma_{i_{t+2}}^{t+2} = \sigma_{t+2}$.

This completes the reasoning for the claim that we have either $\sigma_t > \sigma_{t+1}$ or $\sigma_t = \sigma_{t+1} > \sigma_{t+2}$.

Now we finish the argument for termination in the vertical direction as follows.

Since the value of the invariant σ satisfies the descending chain condition, the increase of the value of t stops after finitely many times. Finally, therefore, we conclude that the value of the strand “ inv_{new} ” cannot decrease infinitely many times.

Therefore, the algorithm terminates after finitely many years, achieving resolution of singularities for (W, \mathcal{R}, E) .

4.4. Brief discussion on the monomial case in positive characteristic

Here in §4.4, we briefly discuss why the problem of resolution of singularities in the monomial case in positive characteristic is much more subtle and difficult than the one in characteristic zero.

Recall we say that the triplet (W, \mathcal{R}, E) is in the monomial case (at $P \in \text{Sing}(\mathcal{R}) \subset W$) in our setting if (and only if) the triplet of invariants takes the value $(\sigma, \tilde{\mu}, s) = (\sigma, 0, 0)$. (Precisely speaking, the triplet (W, \mathcal{R}, E) sits in the middle of constructing the sequence, say in year “ i ” and at stage “ j ”, for resolution of singularities. However, we omit the subscript and superscript “ $(\)_i^j$ ”, indicating the year and the stage, for simplicity of the notation.)

The description of the monomial case at the analytic level is given below.

SITUATION

- The condition $\tilde{\mu} = 0$ is interpreted as follows:

We can choose a regular system of parameters $(x_1, \dots, x_t, x_{t+1}, \dots, x_d)$, taken from $\widehat{\mathcal{O}_{W,P}}$, such that

- (1) the elements in the LGS $\mathbb{H} = \{(h_\alpha, p^{e_\alpha})\}_{\alpha=1}^t$ are of the form

$$h_\alpha = x_\alpha^{p^{e_\alpha}} + \text{higher terms} \quad \text{for } \alpha = 1, \dots, t,$$

(We sometimes call the higher terms in the above expression “the tail part”.)

- (2) there is a monomial $\mathbb{M} = \prod_{D \in E_{\text{young}}} x_D^{r_D}$ of the defining equations x_D of the components D in E_{young} with

$$(\mathbb{M}, a) \in \widehat{\mathcal{R}}_P \text{ for some } a \in \mathbb{Z}_{>0},$$

where $\{x_D \mid D \in E_{\text{young}}\} \subset \{x_\alpha \mid \alpha = t+1, \dots, d\}$ and $\sum r_D > a$,

- (3) the idealistic filtration of i.f.g. type $\widehat{\mathcal{R}}_P$ is saturated for $\{\partial^n / \partial x_\alpha^n \mid n \in \mathbb{Z}_{\geq 0}, \alpha = 1, \dots, t\}$,

satisfying the following condition: for an arbitrary $(f, \lambda) \in \widehat{\mathcal{R}}_P$ with $f = \sum c_{f,B} H^B$ being the power series expansion with respect to the LGS \mathbb{H} and the associated to the regular system of parameters $(x_1, \dots, x_t, x_{t+1}, \dots, x_d)$, we have $\mathbb{M}^{\lambda/a}$ dividing the constant term $c_{f, \mathbb{0}}$, i.e., $\mathbb{M}^{\lambda/a} \mid c_{f, \mathbb{0}}$. Note that, using the formal coefficient lemma, we see $(c_{f, \mathbb{0}}, \lambda) \in \widehat{\mathcal{R}}_P$.

- The condition $s = 0$ is of course equivalent to saying that there is no component of E_{aged} passing through P .

$$\boxed{\text{char}(k) = 0}$$

In fact, the above description $\boxed{\text{SITUATION}}$ of the monomial case is also valid when $\text{char}(k) = 0$, with all the elements of the LGS concentrated at level 1, i.e., $p^{e_\alpha} = 1$ for $\alpha = 1, \dots, t$. Moreover, we can replace x_α with h_α so that we have $h_\alpha = x_\alpha$ for $\alpha = 1, \dots, t$. Then there is *no* tail part. In order to construct resolution of singularities for (W, \mathcal{R}, E) , we have only to carry out the resolution process for $(V, ((\mathbb{M}), a)|_V, E|_V)$, which is the triplet in the monomial case in the classical setting, where $V = \{x_1 = \dots = x_t = 0\}$ is a nonsingular subvariety inside of W . Note that, since $E_{\text{aged}} = \emptyset$ (in a neighborhood of P , because $s = 0$), the third factor $E|_V$ is a simple normal crossing divisor on V .

$$\boxed{\text{char}(k) = p > 0}$$

In contrast to the case in $\text{char}(k) = 0$, the elements in the LGS may not be concentrated at level 1 in general. Therefore, we usually have the tail parts for those h_α 's at higher levels in $\text{char}(k) = p > 0$. The hypersurfaces defined by $\{h_\alpha = 0\}$ by those elements are singular hypersurfaces. Therefore, one is forced to analyze the monomial restricted to a singular subvariety (defined as the intersection of the singular hypersurfaces), or alternatively to analyze the combination of the monomial and the elements of the LGS with the tail parts included, while sticking to the original nonsingular ambient space. The latter is what we do in §5 of this paper in dimension 3.

In the simplest terms, NO or YES tail part is what makes the difference between the monomial case in $\text{char}(k) = 0$ and the one in $\text{char}(k) = p > 0$.

§5. Detailed discussion on the monomial case in dimension 3

The purpose of §5 is to discuss how to construct resolution of singularities (at the local level (cf. Remark 2)) in the monomial case. We refer the reader to §4.4 $\boxed{\text{SITUATION}}$ for the precise description of the monomial case.

5.1. Case analysis according to the invariant “ τ ”

In §5.1, we analyze the situation according to the value of the invariant τ .

Recall that the invariant τ is just the number of the elements in an LGS, and hence that, in dimension 3, it takes the value $\tau = 0, 1, 2, 3$.

It turns out that the analysis of the case $\tau = 0, 2$ or 3 is rather easy. We devote §5.2, §5.3, §5.4 to the analysis of the most subtle and difficult case $\tau = 1$.

Case: $\tau = 0$

In this case, there is no element in an LGS. We conclude that for an arbitrary $(f, \lambda) \in \widehat{\mathcal{R}}_P$ the monomial $\mathbb{M}^{\lambda/a}$ divides $c_{f, \mathbb{O}} = f$. Therefore, we can carry out the same algorithm for resolution of singularities of the triplet $(W, (\mathcal{I}, a), E) = (W, ((\mathbb{M}), a), E)$ in the monomial case in the classical setting in characteristic zero, using the invariant Γ discussed in §3.4. (Note that in the middle of the procedure the invariant σ may drop. If that happens, then we are no longer in the monomial case. In that case, we go through the mechanism described in §4.2 with the reduced new value of σ to reach the new monomial case.)

Case: $\tau = 1$

This case will be thoroughly discussed in §5.2, §5.3, §5.4.

Case: $\tau = 2$

In this case, we can choose a regular system of parameters (x, y, z) , taken from $\widehat{\mathcal{O}}_{W, P}$, such that

(1) the two elements in the LGS $\mathbb{H} = \{(h_1, p^{e_1}), (h_2, p^{e_2})\}$ are of the form

$$\begin{cases} h_1 &= z^{p^{e_1}} + \text{higher terms} \\ h_2 &= y^{p^{e_2}} + \text{higher terms,} \end{cases}$$

(2) there is a monomial $\mathbb{M} = x^r$ of the defining equation x of the component $\{x = 0\} \in E_{\text{young}}$ with

$$(\mathbb{M}, a) = (x^r, a) \in \widehat{\mathcal{R}}_P \text{ for some } a \in \mathbb{Z}_{>0},$$

(3) the idealistic filtration $\widehat{\mathcal{R}}_P$ is saturated for $\{\partial^n / \partial z^n, \partial^m / \partial y^m \mid n, m \in \mathbb{Z}_{\geq 0}\}$.

Then it is easy to see that $\text{Sing}(\mathcal{R}) = P$ (in a neighborhood of P) and hence that the only possible transformation is the blow up with center P . After blow up, we see that the (possibly) non-empty singular locus lies only over the x -chart. We also see that the singular locus, if non-empty, consists of a single point $\widetilde{P} \in \widetilde{W}$ (in a neighborhood of the inverse image of P) with a regular system of parameters $(\widetilde{x}, \widetilde{y}, \widetilde{z}) = (x, y/x, z/x)$. The new LGS

$$\widetilde{\mathbb{H}} = \{(\widetilde{h}_1, p^{e_1}), (\widetilde{h}_2, p^{e_2})\} = \{(h_1/x^{p^{e_1}}, p^{e_1}), (h_2/x^{p^{e_2}}, p^{e_2})\}$$

are of the form

$$\begin{cases} \widetilde{h}_1 &= (\widetilde{z}')^{p^{e_1}} + \text{higher terms} \\ \widetilde{h}_2 &= (\widetilde{y}')^{p^{e_2}} + \text{higher terms} \end{cases}$$

where (cf. the proof of Proposition 4)

$$\begin{cases} \widetilde{z}' &= \widetilde{z} + (c_z)^{1/p^{e_1}} \widetilde{x} \\ \widetilde{y}' &= \widetilde{y} + (c_y)^{1/p^{e_2}} \widetilde{x} \end{cases} \quad \text{for some } c_z, c_y \in k$$

(in case the invariant σ does not decrease). We calculate the new monomial to be

$$(\widetilde{x}^{\tau-a}, a).$$

Since the power of x in the above monomial cannot decrease infinitely many times, we achieve resolution of singularities after finitely many repetitions of this procedure. (Note that in the middle of the procedure the invariant σ may drop. If that happens, then we are no longer in the monomial case. In that case, we go through the mechanism described in §4.2 with the reduced new value of σ to reach the new monomial case.)

Case: $\tau = 3$

This case does not happen. In fact, suppose this case did happen. Then go back to the year $\iota := i_{\text{aged}}$ when the current value of σ first started. Since the current value of τ is equal to 3, so is the value of τ back in year ι . Take the blow up with center P_ι , which is the image of P in year ι . Then it is immediate to see that there is no singular locus (in a neighborhood of the inverse image of P_ι) after blow up. This implies in turn that $\text{Sing}(\mathcal{R}_\iota) = P_\iota$ (in a neighborhood of P_ι). The only possible transformation in the resolution sequence, therefore, is the blow up with center P_ι . After blow up, however, we already saw $\text{Sing}(\mathcal{R}_{\iota+1}) = \emptyset$ over P_ι . This is a contradiction, since $P_{\iota+1}$, which is the image of P in year $\iota + 1$, should be included in $\text{Sing}(\mathcal{R}_{\iota+1})$. (Note that we have $\iota < i$, since the value of $\widetilde{\mu}$ is never zero when the new value of σ starts in year ι and since the current value of $\widetilde{\mu}$ is zero being in the monomial case in year i .)

Focus on the case $\tau = 1$

In the following §5.2, §5.3, §5.4, we focus on, and restrict ourselves to, the case $\tau = 1$. We carry out the computation of the invariants at the analytic (completion) level, even though the centers of blow ups are chosen at the algebraic level, and hence all the procedures in the algorithm are carried out at the algebraic level.

We restate the SITUATION described in §4.4 in a slightly refined form in our particular case $\tau = 1$, for the sole purpose of fixing the notation for §5.2, §5.3, §5.4.

SITUATION

We can choose a regular system of parameters (x, y, z) , taken from $\widehat{\mathcal{O}_{W,P}}$, such that

(1) via the Weierstrass Preparation Theorem the unique element in the LGS $\mathbb{H} = \{(h, p^e)\}$ is of the form

$$h = z^{p^e} + a_1 z^{p^e-1} + a_2 z^{p^e-2} + \cdots + a_{p^e-1} z + a_{p^e}$$

with

$$a_i \in k[[x, y]] \quad \text{and} \quad \text{ord}_P(a_i) > i \quad \text{for} \quad i = 1, \dots, p^e,$$

(2) there is a monomial $\mathbb{M} = x^\alpha y^\beta$ of the defining equation(s) of the component(s) $H_x = \{x = 0\}$ (and possibly $H_y = \{y = 0\}$) in E_{young} with

$$(\mathbb{M}, a) = (x^\alpha y^\beta, a) \in \mathcal{R}_P \quad \text{and} \quad \alpha + \beta > a,$$

(We write $\mathbb{M}_u := x^{\alpha/a} y^{\beta/a}$ and call \mathbb{M}_u the *usual* monomial.)

(3) the idealistic filtration $\widehat{\mathcal{R}}_P$ is saturated for $\{\partial^n / \partial z^n \mid n \in \mathbb{Z}_{\geq 0}\}$, satisfying the following condition: for an arbitrary $(f, \lambda) \in \widehat{\mathcal{R}}_P$ with $f = \sum c_{f,B} H^B = \sum c_{f,b} h^b$ being the power series expansion with respect to the LGS $\mathbb{H} = \{(h, p^e)\}$ and its associated regular system of parameters (x, y, z) , we have $\mathbb{M}^{\lambda/a}$ dividing the constant term $c_{f,\mathbb{O}} = c_{f,0}$, i.e., $\mathbb{M}^{\lambda/a} \mid c_{f,\mathbb{O}} = c_{f,0}$.

In particular, by looking at $\partial^n h / \partial z^n$ for $n = 1, \dots, p^e - 1$, we see that the coefficient a_i is divisible by $(\mathbb{M}_u)^i$ for $i = 1, \dots, p^e - 1$ (but maybe not for $i = p^e$). That is to say,

$$(\mathbb{M}_u)^i \mid a_i, \quad \text{i.e.,} \quad x^{\lceil i\alpha/a \rceil} y^{\lceil i\beta/a \rceil} \mid a_i \quad \text{for} \quad i = 1, \dots, p^e - 1.$$

Throughout §5.2, §5.3, §5.4, we are under SITUATION described as above.

5.2. “Cleaning” and the invariant “H” (in the case $\tau = 1$)

The purpose of this section is to introduce the invariant “H” through the process of “cleaning”. We follow closely the argument developed by Benito-Villamayor [5], making some modifications to fit it into our own setting in the framework of the Idealistic Filtration Program.

Definition 5. We define the slope of h at P with respect to (x, y, z) by the formula

$$\text{Slope}_{h,(x,y,z)}(P) = \min \left\{ \frac{1}{p^e} \text{ord}_P(a_{p^e}), \mu(P) \right\}.$$

Remark 6.

(i) Since we are in the monomial case and hence $\tilde{\mu} = 0$ and since the monomial is $(\mathbb{M}, a) = (x^\alpha y^\beta, a)$, we compute

$$\mu(P) = \frac{1}{a} \text{ord}_P(x^\alpha y^\beta) = \frac{\alpha + \beta}{a} = \text{ord}_P(\mathbb{M}_u),$$

while

$$\mu(\xi_{H_x}) = \frac{\alpha}{a} \quad \text{and} \quad \mu(\xi_{H_y}) = \frac{\beta}{a}.$$

(ii) We have a “very good” control over the coefficients a_i except for the constant term a_{p^e} , in the sense that

$$(\mathbb{M}_u)^i = (x^{\alpha/a} y^{\beta/a})^i \mid a_i \quad \text{for} \quad i = 1, \dots, p^e - 1,$$

which implies

$$\frac{1}{i} \text{ord}_P(a_i) \geq \mu(P) \quad \text{for} \quad i = 1, \dots, p^e - 1.$$

Therefore, this control leads to the following observation

$$\begin{aligned} \text{Slope}_{h,(x,y,z)}(P) &= \min \left\{ \mu(P), \frac{1}{p^e} \text{ord}_P(a_{p^e}) \right\} \\ &= \min \left\{ \mu(P), \frac{1}{i} \text{ord}_P(a_i) ; i = 1, \dots, p^e \right\}. \end{aligned}$$

Definition 6 (Well-adaptedness (cf. [5])). We say h is well-adapted at P with respect to (x, y, z) if one of the following two conditions holds:

- A. $\text{Slope}_{h,(x,y,z)}(P) = \mu(P)$.
- B. $\text{Slope}_{h,(x,y,z)}(P) = \text{ord}_P(a_{p^e})/p^e < \mu(P)$ and the initial form $\text{In}_P(a_{p^e})$ is *not* a p^e -th power.

Similarly, we say h is well-adapted at ξ_{H_x} , where ξ_{H_x} is the generic point of the hypersurface $H_x = \{x = 0\}$ in E_{young} , if one of the following two conditions holds:

- A. $\text{Slope}_{h,(x,y,z)}(\xi_{H_x}) = \mu(\xi_{H_x}) = \alpha/a$.
- B. $\text{Slope}_{h,(x,y,z)}(\xi_{H_x}) = \text{ord}_{\xi_{H_x}}(a_{p^e})/p^e < \mu(\xi_{H_x})$ and the initial form $\text{In}_{\xi_{H_x}}(a_{p^e})$ is *not* a p^e -th power.

The notion of h being well-adapted at ξ_{H_y} , where ξ_{H_y} is the generic point of the hypersurface $H_y = \{y = 0\}$ in E_{young} , is defined in an identical manner.

Note that if

$$a_{p^e} = \sum_{k+l \geq d} c_{kl} x^k y^l = x^r \{g(y) + x \cdot \omega(x, y)\}$$

where $c_{kl} \in k$, $0 \neq g(y) \in k[[y]]$, $\omega(x, y) \in k[[x, y]]$, $d = \text{ord}_P(a_{p^e})$ and $r = \text{ord}_{\xi_{H_x}}(a_{p^e})$, then

$$\text{In}_P(a_{p^e}) = \sum_{k+l=d} c_{kl} x^k y^l \quad \text{and} \quad \text{In}_{\xi_{H_x}}(a_{p^e}) = x^r g(y).$$

Proposition 5.

(1) *There exist an LGS $\mathbb{H} = \{(h, p^e)\}$ and a regular system of parameters (x, y, z) , as described in SITUATION in §5.1, such that h is well-adapted at P , ξ_{H_x} and ξ_{H_y} simultaneously with respect to (x, y, z) . Note that we require the property that the idealistic filtration \mathcal{R}_P is $\{\frac{\partial^n}{\partial z^n} \mid n \in \mathbb{Z}_{\geq 0}\}$ -saturated with respect to the regular system of parameters (x, y, z) (cf. SITUATION (3)).*

(2) *If h is well-adapted at $* = P, \xi_{H_x}$ or ξ_{H_y} with respect to (x, y, z) , then $\text{Slope}_{h, (x, y, z)}(*)$ is independent of the choice of h and (x, y, z) .*

Proof.

(1) We start with h and (x, y, z) as given in SITUATION in §5.1.

Step 1. Modify h and (x, y, z) to be well-adapted at P .

Suppose we are in Case A or Case B as described in Definition 6. Then h is already well-adapted at P with respect to (x, y, z) and there is no modification needed.

Therefore, we may assume that we are not in either Case A or Case B. That is to say, we have

$$\text{Slope}_{h, (x, y, z)}(P) = \frac{1}{p^e} \text{ord}_P(a_{p^e}) < \mu(P),$$

and

$$\text{In}_P(a_{p^e}) = \sum_{k+l=d} c_{kl} x^k y^l, \quad \text{with } d = \text{ord}_P(a_{p^e}), \text{ is a } p^e\text{-th power.}$$

Take

$$\{\text{In}_P(a_{p^e})\}^{1/p^e} \in k[x, y],$$

and set

$$z' = z + \{\text{In}_P(a_{p^e})\}^{1/p^e}, \quad \text{i.e., } z = z' - \{\text{In}_P(a_{p^e})\}^{1/p^e}.$$

Plug this into

$$h = z^{p^e} + a_1 z^{p^e-1} + a_2 z^{p^e-2} + \cdots + a_{p^e-1} z + a_{p^e}$$

to obtain

$$h = z'^{p^e} + a'_1 z'^{p^e-1} + a'_2 z'^{p^e-2} + \cdots + a'_{p^e-1} z' + a'_{p^e}$$

with

$$a'_i \in k[[x, y]] \quad \text{for } i = 1, \dots, p^e - 1, p^e.$$

Since for $i = 1, \dots, p^e - 1$, we have (cf. Remark 6 (ii))

$$\frac{1}{i} \text{ord}_P(a_i) \geq \mu(P) > \frac{1}{p^e} \text{ord}_P(a_{p^e}) = \text{ord}_P\left(\{\text{In}_P(a_{p^e})\}^{1/p^e}\right),$$

we conclude that a'_{p^e} is of the form

$$a'_{p^e} = a_{p^e} - \text{In}_P(a_{p^e}) + \text{higher terms}$$

and hence that

$$\frac{1}{p^e} \text{ord}_P(a'_{p^e}) > \frac{1}{p^e} \text{ord}_P(a_{p^e}).$$

We go back to the starting point, replacing the original h and (x, y, z) by h' and $(x', y', z') = (x, y, z')$, with strictly increased $\text{ord}_P(a'_{p^e})/p^e$. Since $\mu(P) < \infty$, we conclude that, after finitely many repetitions of this process, we have to come to the situation where we are in Case A or Case B, i.e., where h is well-adapted at P with respect to (x, y, z) .

Step 2. Modify h and (x, y, z) further to be well-adapted at ξ_{H_x} without destroying the well-adaptedness at P .

Take h which is well-adapted at P with respect to (x, y, z) , as obtained through Step 1.

Suppose we are in Case A or Case B as described in the second half of Definition 6. Then h is already well-adapted at ξ_{H_x} with respect to (x, y, z) and there is no modification needed.

Therefore, we may assume that we are not in either Case A or Case B. That is to say, we have

$$\text{Slope}_{h, (x, y, z)}(\xi_{H_x}) = \frac{1}{p^e} \text{ord}_{\xi_{H_x}}(a_{p^e}) < \mu(\xi_{H_x}),$$

and

$$\text{In}_{\xi_{H_x}}(a_{p^e}) = x^r g(y) \text{ is a } p^e\text{-th power,}$$

where $a_{p^e} = x^r \{g(y) + x \cdot \omega(x, y)\}$ and $r = \text{ord}_{\xi_{H_x}}(a_{p^e})$. Take

$$\{\text{In}_{\xi_{H_x}}(a_{p^e})\}^{1/p^e} \in k[[y]][x],$$

and set

$$z' = z + \{\text{In}_{\xi_{H_x}}(a_{p^e})\}^{1/p^e}, \quad \text{i.e., } z = z' - \{\text{In}_{\xi_{H_x}}(a_{p^e})\}^{1/p^e}.$$

Then as in Step 1, we see

$$\frac{1}{p^e} \text{ord}_{\xi_{H_x}}(a'_{p^e}) > \frac{1}{p^e} \text{ord}_{\xi_{H_x}}(a_{p^e}).$$

We go back to the starting point, replacing the original h and (x, y, z) by h' and $(x', y', z') = (x, y, z')$, with strictly increased $\text{ord}_{\xi_{H_x}}(a'_{p^e})/p^e$. Since $\mu(\xi_{H_x}) < \infty$, we conclude that, after finitely many repetitions of this process, we have to come to the situation where we are in Case A or Case B, i.e., where h is well-adapted at ξ_{H_x} with respect to (x, y, z) .

The only issue here is to check, in the process, the property that h is well-adapted at P is preserved.

Case: $\text{ord}_P(a_{p^e})/p^e \geq \mu(P)$.

In this case, we have

$$\text{ord}_P \left(\{\text{In}_{\xi_{H_x}}(a_{p^e})\}^{1/p^e} \right) \geq \frac{1}{p^e} \text{ord}_P(a_{p^e}) \geq \mu(P)$$

and, for $i = 1, \dots, p^e - 1$, we have (cf. Remark 6 (ii))

$$\frac{1}{i} \text{ord}_P(a_i) \geq \mu(P).$$

Therefore, we conclude that

$$\frac{1}{p^e} \text{ord}_P(a'_{p^e}) \geq \mu(P),$$

and hence that h' stays well-adapted at P with respect to (x', y', z') .

Case: $\text{ord}_P(a_{p^e})/p^e < \mu(P)$.

In this case, we have

$$\text{ord}_P \left(\{\text{In}_{\xi_{H_x}}(a_{p^e})\}^{1/p^e} \right) \geq \frac{1}{p^e} \text{ord}_P(a_{p^e})$$

and, for $i = 1, \dots, p^e - 1$, we have (cf. Remark 6 (ii))

$$\frac{1}{i} \text{ord}_P(a_i) \geq \mu(P) > \frac{1}{p^e} \text{ord}_P(a_{p^e}).$$

Hence, since $\text{In}_P(a_{p^e})$ is not a p^e -th power and since the degree $d = \text{ord}_p(a_{p^e})$ -part of $\text{In}_{\xi_{H_x}}(a_{p^e})$ is a p^e -th power, we have

$$\text{In}_P(a'_{p^e}) = \text{In}_P(a_{p^e}) - \left\{ \text{the degree } d = \text{ord}_p(a_{p^e})\text{-part of } \text{In}_{\xi_{H_x}}(a_{p^e}) \right\}.$$

Therefore, we conclude that

$$\text{ord}_P(a'_{p^e}) = \text{ord}_P(\text{In}_P(a'_{p^e})) = \text{ord}_P(\text{In}_P(a_{p^e})) = \text{ord}_P(a_{p^e}) < \mu(P)$$

and $\text{In}_P(a'_{p^e})$ is not a p^e -th power, and hence that h' stays well-adapted at P with respect to (x', y', z') .

Step 3. Modify h and (x, y, z) still further to be well-adapted at ξ_{H_y} without destroying the well-adaptedness at P and ξ_{H_x} .

The process of this step is almost identical to that of Step 2, and hence is left to the reader as an exercise.

We note that the requirement as described in SITUATION (3) is met, since the original (x, y, z) satisfies the property and since we only modify z by adding the elements in $k[[x, y]]$ throughout the process.

This finishes the proof of (1).

(2) We only give a proof for the case where $* = P$, since the proof for the case where $* = \xi_{H_x}$ is identical.

Take h which is well-adapted at P with respect to (x, y, z) , with the property that the idealistic filtration \mathcal{R}_P is saturated for $\{\partial^n/\partial z^n \mid n \in \mathbb{Z}_{\geq 0}\}$.

We set

$$H(P) = \min \left\{ \mu(P), \max \left\{ \frac{1}{p^e} \text{ord}_P(h'|_{Z'}) \mid \begin{array}{l} h', (x', y', z'), \\ Z' = \{z' = 0\} \end{array} \right\} \right\}$$

where, computing the above “max”, we let h' and (x', y', z') vary among all such pairs consisting of the unique element in an LGS and a regular system of parameters that satisfy the condition

$$h' \equiv z'^{p^e} \pmod{\widehat{\mathfrak{m}}_P^{p^e+1}}.$$

It suffices to show

$$\text{Slope}_{h, (x, y, z)}(P) = H(P),$$

since the number $H(P)$ is obviously independent of the choice of h and (x, y, z) .

Observe

$$\text{Slope}_{h,(x,y,z)}(P) = \min \left\{ \frac{1}{p^e} \text{ord}_P(a_{p^e}), \mu(P) \right\} \leq H(P),$$

since

$$\frac{1}{p^e} \text{ord}_P(h|_Z) = \frac{1}{p^e} \text{ord}_P(a_{p^e})$$

and since h and (x, y, z) form such a pair consisting of the unique element in an LGS and a regular system of parameters that satisfy the condition

$$h \equiv z^{p^e} \pmod{\widehat{\mathfrak{m}}_P^{p^e+1}}.$$

Now we prove the inequality in the opposite direction

$$\text{Slope}_{h,(x,y,z)}(P) = \min \left\{ \frac{1}{p^e} \text{ord}_P(a_{p^e}), \mu(P) \right\} \geq H(P).$$

If $\text{Slope}_{h,(x,y,z)}(P) = \mu(P)$, then the above inequality obviously holds. Therefore, we may assume that $\text{Slope}_{h,(x,y,z)}(P) = \text{ord}_P(a_{p^e})/p^e < \mu(P)$ and that $\text{In}_P(a_{p^e})$ is not a p^e -th power.

Take an arbitrary pair h' and (x', y', z') as described in the definition of $H(P)$ above.

We claim that

$$\text{ord}_P(h'|_{Z'}) = \text{ord}_P(h|_{Z'}) \leq \text{ord}_P(h|_Z) = \text{ord}_P(a_{p^e}) < p^e \mu(P),$$

which implies the required inequality.

Let

$$h' = \sum c_B H^B = \sum_{b \in \mathbb{Z}_{\geq 0}} c_b h^b$$

be the power series expansion of h' with respect to the LGS $\mathbb{H} = \{(h, p^e)\}$ and its associated regular system of parameters (x, y, z) .

Since

$$h' \equiv z'^{p^e} \equiv c \cdot z^{p^e} \pmod{\widehat{\mathfrak{m}}_P^{p^e+1}} \quad \text{for some } c \in k^\times,$$

we conclude

$$h' = \sum_{b>0} c_b h^b + c_0 = u \cdot h + c_0$$

for some unit u in $\widehat{\mathcal{O}}_{W,P}$. Moreover, by the formal coefficient lemma, we have $(c_0, p^e) \in \widehat{\mathcal{R}}_P$. This implies that $(\mathbb{M}_u)^{p^e} \mid c_0$ and hence that $\text{ord}_P(c_0|_{Z'}) \geq \text{ord}_P(c_0) \geq p^e \mu(P)$.

Therefore, it suffices to prove

$$\text{ord}_P(h|_{Z'}) \leq \text{ord}_P(h|_Z) (= \text{ord}_P(a_{p^e}) < p^e \mu(P)).$$

(Then $\text{ord}_P(h|_{Z'}) = \text{ord}_P((u \cdot h)|_{Z'}) = \text{ord}_P((u \cdot h + c_0)|_{Z'}) = \text{ord}_P(h'|_{Z'})$.)

Now by the Weierstrass Preparation Theorem, we have

$$z' = v \cdot (z + w)$$

for some unit v in $\widehat{\mathcal{O}_{W,P}}$ and $w \in k[[x, y]]$. Since $Z' = \{z' = 0\} = \{z + w = 0\}$, by replacing z' with $z + w$, we may assume that z' is of the form

$$z' = z + w, \quad \text{i.e., } z = z' - w \quad \text{with } w \in k[[x, y]].$$

Plug this into

$$h = z^{p^e} + a_1 z^{p^e-1} + a_2 z^{p^e-2} + \cdots + a_{p^e-1} z + a_{p^e}$$

to obtain

$$h = z'^{p^e} + a'_1 z'^{p^e-1} + a'_2 z'^{p^e-2} + \cdots + a'_{p^e-1} z' + a'_{p^e}$$

with

$$a'_i \in k[[x, y]] \quad \text{for } i = 1, \dots, p^e - 1, p^e,$$

where

$$a'_{p^e} = (-w)^{p^e} + a_1(-w)^{p^e-1} + a_2(-w)^{p^e-2} + \cdots + a_{p^e-1}(-w) + a_{p^e}.$$

Observe that, since (cf. SITUATION in §5.1)

$$(\mathbb{M}_u)^i \mid a_i \quad \text{for } i = 1, \dots, p^e - 1,$$

we have

$$\frac{1}{i} \text{ord}_P(a_i) \geq \text{ord}_P(\mathbb{M}_u) = \mu(P) > \frac{1}{p^e} \text{ord}_P(a_{p^e}) \quad \text{for } 1 \leq i < p^e.$$

Case: $\text{ord}_P(w) > \text{ord}_P(a_{p^e})/p^e$.

In this case, we have

$$\text{ord}_P(h|_{Z'}) = \text{ord}_P(a'_{p^e}) = \text{ord}_P(a_{p^e}) = \text{ord}_P(h|_Z).$$

Case: $\text{ord}_P(w) = \text{ord}_P(a_{p^e})/p^e$.

In this case, since $\text{In}_P(a_{p^e})$ is not a p^e -th power, we see via the observation above that

$$\text{In}_P(a'_{p^e}) = \text{In}_P((-w)^{p^e}) + \text{In}_P(a_{p^e}) \neq 0$$

and

$$\text{ord}_P(h|_{Z'}) = \text{ord}_P(a'_{p^e}) = \text{ord}_P(a_{p^e}) = \text{ord}_P(h|_Z).$$

Case: $\text{ord}_P(w) < \text{ord}_P(a_{p^e})/p^e$.

In this case, we have

$$\begin{aligned} \text{ord}_P(h|_{Z'}) &= \text{ord}_P(a'_{p^e}) = \text{ord}_P((-w)^{p^e}) \\ &= p^e \text{ord}_P(w) < \text{ord}_P(a_{p^e}) = \text{ord}_P(h|_Z). \end{aligned}$$

Therefore, in all the cases above, we have

$$\text{ord}_P(h|_{Z'}) \leq \text{ord}_P(h|_Z).$$

This completes the proof of Proposition 5.

Q.E.D.

Definition 7 (Invariant “H”). We define the invariant H by the following formula

$$H(*) := \text{Slope}_{h,(x,y,z)}(*)$$

where h is well-adapted at $* = P, \xi_{H_x}$, or ξ_{H_y} with respect to (x, y, z) . By Proposition 5, the invariant H is independent of the choice of h and (x, y, z) .

Definition 8 (the *tight* monomial). We define the *tight* monomial \mathbb{M}_t by the formula

$$\mathbb{M}_t = x^{h_x} y^{h_y} \quad \text{where} \quad h_x = H(\xi_{H_x}), h_y = H(\xi_{H_y}).$$

Recall that the *usual* monomial \mathbb{M}_u is defined by the formula

$$\mathbb{M}_u = x^{\alpha/a} y^{\beta/a} \quad \text{where} \quad \frac{\alpha}{a} = \mu(\xi_{H_x}), \frac{\beta}{a} = \mu(\xi_{H_y}).$$

Note that we have $\mathbb{M}_t \mid \mathbb{M}_u$, that is to say, we have $0 \leq h_x \leq \mu(\xi_{H_x})$ and $0 \leq h_y \leq \mu(\xi_{H_y})$, which follow easily from the definition.

5.3. Description of the procedure (in the case $\tau = 1$)

Analysis of the singular locus $\text{Sing}(\mathcal{R})$ at P

First we analyze the singular locus $\text{Sing}(\mathcal{R})$ of the idealistic filtration \mathcal{R} of i.f.g. type at P .

Proposition 6. *We have the following description of the singular locus $\text{Sing}(\mathcal{R})$ at P , denoted by $\text{Sing}(\mathcal{R})_P$, according to the values of $h_x = H(\xi_{H_x})$ and $h_y = H(\xi_{H_y})$:*

$$\text{Sing}(\mathcal{R})_P = \begin{cases} V(z, x) \cup V(z, y) & \text{if } h_x \geq 1 \quad \text{and} \quad h_y \geq 1 \\ V(z, x) & \text{if } h_x \geq 1 \quad \text{and} \quad h_y < 1 \\ V(z, y) & \text{if } h_x < 1 \quad \text{and} \quad h_y \geq 1 \\ V(z, x, y) = P & \text{if } h_x < 1 \quad \text{and} \quad h_y < 1, \end{cases}$$

where “ V ” denotes the vanishing locus and where (x, y, z) is a regular system of parameters at P with respect to which h is well-adapted simultaneously at P , ξ_{H_x} , and ξ_{H_y} .

Proof. Note first that, since $(\mathbb{M}, a) = (x^\alpha y^\beta, a) \in \widehat{\mathcal{R}}_P$ with $a \in \mathbb{Z}_{>0}$, we have

$$\text{Sing}(\mathcal{R})_P \subset \{x = 0\} \cup \{y = 0\} = H_x \cup H_y.$$

Then the asserted description is a consequence of the following analysis of $\text{Sing}(\mathcal{R})_P \cap H_x$ (and that of $\text{Sing}(\mathcal{R})_P \cap H_y$, which is identical and hence omitted).

Case: $h_x \geq 1$.

In this case, we have

- $(h, p^e) \in \widehat{\mathcal{R}}_P$ with $h = z^{p^e} + a_1 z^{p^e-1} + \dots + a_{p^e-1} z + a_{p^e}$ being well-adapted both at P and ξ_{H_x} ,
- $x \mid a_i$ for $i = 0, \dots, p^e - 1$, since $\alpha/a \geq h_x \geq 1$ and since $\mathbb{M}_u \mid a_i$ for $i = 0, \dots, p^e - 1$,
- $x \mid a_{p^e}$, since $h_x \geq 1$,
- $h = 0$ on $\text{Sing}(\mathcal{R})_P$,

which imply

- $z = 0$ on $\text{Sing}(\mathcal{R})_P \cap H_x$.

Therefore, we conclude

$$\text{Sing}(\mathcal{R})_P \cap H_x \subset V(z, x).$$

On the other hand, for $Q \in V(z, x)$, we have

- $\text{ord}_Q(a_i) \geq \text{ord}_Q((\mathbb{M}_u)^i) \geq \alpha/a \cdot i \geq i$ for $i = 0, \dots, p^e - 1$, since $\mathbb{M}_u \mid a_i$ for $i = 0, \dots, p^e - 1$ and since $\alpha/a \geq h_x \geq 1$,
- $\text{ord}_Q(a_{p^e}) \geq h_x \cdot p^e \geq p^e$, since $h_x \geq 1$,

which imply

- $\text{ord}_Q(h) \geq p^e$.

Therefore, for any $(f, \lambda) \in \widehat{\mathcal{R}}_P$ with $f = \sum c_b h^b$ being the power series expansion with respect to the LGS $\mathbb{H} = \{(h, p^e)\}$ and its associated

regular system of parameters (x, y, z) , we have

$$\text{ord}_Q(f) \geq \lambda$$

since

• $\text{ord}_Q(c_b) \geq \text{ord}_Q((\mathbb{M}_u)^{\lambda - p^e \cdot b}) \geq \alpha/a \cdot (\lambda - p^e \cdot b) \geq \lambda - p^e \cdot b$ for b with $\lambda - p^e \cdot b \geq 0$, since $(c_b, \lambda - p^e \cdot b) \in \widehat{\mathcal{R}}_P$, which follows from the formal coefficient lemma.

Therefore, we conclude

$$\text{Sing}(\mathcal{R})_P \cap H_x \supset V(z, x),$$

and hence

$$\text{Sing}(\mathcal{R})_P \cap H_x = V(z, x).$$

Case: $1 > h_x > 0$.

In this case, we have

• $(h, p^e) \in \widehat{\mathcal{R}}_P$ with $h = z^{p^e} + a_1 z^{p^e - 1} + \dots + a_{p^e - 1} z + a_{p^e}$ being well-adapted both at P and ξ_{H_x} ,

• $x \mid a_i$ for $i = 0, \dots, p^e - 1$, since $\alpha/a \geq h_x > 0$ and since $\mathbb{M}_u \mid a_i$ for $i = 0, \dots, p^e - 1$,

• $x \mid a_{p^e}$, since $h_x > 0$,

• $h = 0$ on $\text{Sing}(\mathcal{R})_P$,

which imply

• $z = 0$ on $\text{Sing}(\mathcal{R})_P \cap H_x$.

Subcase: $h_x = \mu(\xi_x)$. In this subcase, the inequality $1 > h_x$ and the inclusion $(M, a) \in \widehat{\mathcal{R}}_P$ imply

$$\text{Sing}(\mathcal{R})_P \cap H_x \subset V(z, x, y).$$

Subcase: $h_x = \text{ord}_{\xi_x}(a_{p^e})/p^e$. In this subcase, we have $1 > h_x = r/p^e$ where $r = \text{ord}_{\xi_{H_x}}(a_{p^e})$, and hence $a_{p^e} = x^r \cdot \gamma(x, y)$ where $\gamma(x, y)$ is not divisible by x . Therefore, we conclude

$$\text{Sing}(\mathcal{R})_P \cap H_x \subset V(z, x) \cap \{\gamma(x, y) = 0\} = V(z, x, y).$$

Therefore, in both subcases, we have

$$\text{Sing}(\mathcal{R})_P \cap H_x = V(z, x, y) = P.$$

Case: $h_x = 0$.

Subcase: $\alpha/a = \mu(\xi_{H_x}) = 0$. In this subcase, we have $\beta/a = \mu(\xi_{H_y}) \geq 1$. Therefore, we conclude

$\text{Sing}(\mathcal{R})_P \cap H_x \subset V(z, x, y)$ and hence $\text{Sing}(\mathcal{R})_P \cap H_x = V(z, x, y) = P$.

Subcase: $\alpha/a = \mu(\xi_{H_x}) > 0$. In this subcase, we have

- $x \mid a_i$ for $i = 0, \dots, p^e - 1$, since $\alpha/a > 0$ and since $\mathbb{M}_u \mid a_i$ for $i = 0, \dots, p^e - 1$.

Set $a_{p^e} = g(y) + x \cdot \omega(x, y)$. Then $g(y) \neq 0$ and $g(y) = \text{In}_{\xi_{H_x}}(a_{p^e})$ is not a p^e -th power, since h is well-adapted at ξ_{H_x} . We also observe that, for $Q \in \text{Sing}(\mathcal{R})_P \cap H_x$, we have

$$p^e \leq \text{ord}_Q(h) \leq \text{ord}_Q(h|_{H_x}) = \text{ord}_Q(z^{p^e} + g(y)).$$

Therefore, there exists $e' < e$ such that

$$\frac{\partial^{p^{e'}}}{\partial y^{p^{e'}}}(z^{p^e} + g(y)) = \frac{\partial^{p^{e'}}}{\partial y^{p^{e'}}}g(y) \neq 0$$

and that

$$\text{ord}_Q \left(\frac{\partial^{p^{e'}}}{\partial y^{p^{e'}}}g(y) \right) = \text{ord}_Q \left(\frac{\partial^{p^{e'}}}{\partial y^{p^{e'}}}(z^{p^e} + g(y)) \right) \geq p^e - p^{e'} > 0.$$

This implies $y = 0$ at Q . Therefore, we conclude

$\text{Sing}(\mathcal{R})_P \cap H_x \subset V(z, x, y)$, and hence $\text{Sing}(\mathcal{R})_P \cap H_x = V(z, x, y) = P$.

This completes the proof of Proposition 6.

Q.E.D.

Description of the procedure for resolution of singularities

Now based upon the analysis of the support $\text{Sing}(\mathcal{R})_P$, we give the following (local) description of the procedure (around the point P) for resolution of singularities in the monomial case with $\tau = 1$:

Step 1. Check if $\dim \text{Sing}(\mathcal{R})_P = 1$.

If the answer is yes, then blow up the 1-dimensional components one by one. If there are two 1-dimensional components meeting at P , then we blow up the one associated to the boundary divisor with bigger H first. If the boundary divisors associated to the two 1-dimensional components have the same H , then we blow up first the one associated to the boundary divisor created later in the history. Since the invariant H strictly decreases under this procedure, this step comes to an end after finitely many times with the dimension of the singular locus dropping to 0.

If the answer is no, i.e., $\dim \text{Sing}(\mathcal{R})_P = 0$, then go to Step 2.

Step 2. Once $\dim \text{Sing}(\mathcal{R})_P = 0$, blow up the isolated point in the singular locus. Then go back to Step 1.

Repeat these steps.

(We note that, as long as the value of the invariant σ remains the same, we stay in the monomial case during the procedure above. We also note that in the middle of the procedure the invariant σ may drop. If the latter happens, we are no longer in the monomial case. In this case, we go through the mechanism described in §4.2 with the reduced new value of σ to reach the new monomial case.)

What remains to be shown is that the above procedure terminates after finitely many repetitions. This termination of the procedure is the main subject of §5.4.

5.4. Termination of the procedure (in the case $\tau = 1$)

Notion of a good/bad point (resp. hypersurface)

In order to analyze the termination of the procedure, we introduce the following notion of the point P being “good or bad” and the boundary divisor H_x (or H_y) being “good or bad”.

Definition 9 (“good/bad” point (cf. [5])). We say P is a *good* (resp. a *bad*) point if $\mu(P) - H(P) = 0$ (resp. > 0). Similarly, we say H_x is a *good* (resp. *bad*) hypersurface if $\mu(\xi_{H_x}) - H(\xi_{H_x}) = 0$ (resp. > 0), where ξ_{H_x} is the generic point of the hypersurface H_x .

The notion of H_y being a good or bad hypersurface is defined in an identical manner.

Lemma 4. *Let $W \xleftarrow{\pi} \widetilde{W}$ be the blow up with center P , E_P the exceptional divisor, $\widetilde{\mathcal{R}}$ the transformation of the idealistic filtration of i.f.g. type \mathcal{R} . Then P is a good (resp. bad) point if and only if E_P is a good (resp. bad) hypersurface.*

Proof. Take h and a regular system of parameters as described in **SITUATION** in §5.1. We may further assume that h is well-adapted at P with respect to the regular system of parameters (x, y, z) (cf. Proposition 5). Take a point $\widetilde{P} \in \pi^{-1}(P) \cap \text{Supp}(\widetilde{\mathcal{R}}) \subset \widetilde{W}$. Since the singular locus is empty over the z -chart, \widetilde{P} should be either in the x -chart or in the y -chart. Say, \widetilde{P} is in the x -chart with a regular system of parameters $(\widetilde{x}, \widetilde{y}, \widetilde{z}) = (x, y/x - c, z/x)$ for some $c \in k$. We also assume that the invariant σ stays the same, and hence (cf. Proposition 4) that (\widetilde{h}, p^e) is the unique element in the LGS at \widetilde{P} where

$$\widetilde{h} = \frac{\pi^*(h)}{x^{p^e}} = \widetilde{z}^{p^e} + \widetilde{a}_1 \widetilde{z}^{p^e-1} + \cdots + \widetilde{a}_{p^e} \quad \text{with} \quad \widetilde{a}_i = \frac{\pi^*(a_i)}{x^i} \in k[[\widetilde{x}, \widetilde{y}]].$$

We compute

$$\begin{aligned} \text{Slope}_{\tilde{h},(\tilde{x},\tilde{y},\tilde{z})}(\xi_{E_P}) &= \min \left\{ \frac{1}{p^e} \text{ord}_{\xi_{E_P}}(\tilde{a}_{p^e}), \mu(\xi_{E_P}) \right\} \\ &= \min \left\{ \frac{1}{p^e} \text{ord}_P(a_{p^e}) - 1, \mu(P) - 1 \right\} = \min \left\{ \frac{1}{p^e} \text{ord}_P(a_{p^e}), \mu(P) \right\} - 1 \\ &= H(P) - 1. \end{aligned}$$

Case: P is a good point, i.e., $H(P) = \mu(P)$.

In this case, we have

$$\text{Slope}_{\tilde{h},(\tilde{x},\tilde{y},\tilde{z})}(\xi_{E_P}) = H(P) - 1 = \mu(P) - 1 = \mu(\xi_{E_P}).$$

Therefore, we conclude that E_P is a good hypersurface (with \tilde{h} well-adapted at ξ_{E_P} with respect to $(\tilde{x}, \tilde{y}, \tilde{z})$ and $H(\xi_{E_P}) = \mu(\xi_{E_P})$).

Case: P is a bad point, i.e., $H(P) < \mu(P)$.

In this case, we have

$$\text{Slope}_{\tilde{h},(\tilde{x},\tilde{y},\tilde{z})}(\xi_{E_P}) = H(P) - 1 < \mu(P) - 1 = \mu(\xi_{E_P}).$$

Therefore, in order to see that E_P is a bad hypersurface, i.e., $H(\xi_{E_P}) < \mu(\xi_{E_P})$, we have only to show that \tilde{h} is well-adapted at ξ_{E_P} with respect to $(\tilde{x}, \tilde{y}, \tilde{z})$, i.e., $\text{In}_{\xi_{E_P}}(\tilde{a}_{p^e})$ is not a p^e -th power, and hence that $\text{Slope}_{\tilde{h},(\tilde{x},\tilde{y},\tilde{z})}(\xi_{E_P}) = H(\xi_{E_P})$.

Set $a_{p^e} = \sum_{k+l \geq d} c_{kl} x^k y^l$ where $d = \text{ord}_P(a_{p^e})$. Then

$$\begin{aligned} \pi^*(a_{p^e}) &= \sum_{k+l \geq d} c_{kl} x^{k+l} \left(\frac{y}{x}\right)^l = x^d \left\{ \sum_{k+l=d} c_{kl} \left(\frac{y}{x}\right)^l + x \cdot \Omega\left(x, \frac{y}{x}\right) \right\} \\ &= x^d \left\{ \phi\left(\frac{y}{x}\right) + x \cdot \Omega\left(x, \frac{y}{x}\right) \right\}, \end{aligned}$$

where $\phi(T) = \sum_{k+l=d} c_{kl} T^l$. Hence, we have

$$\begin{aligned} \tilde{a}_{p^e} &= \frac{\pi^*(a_{p^e})}{x^{p^e}} = x^{d-p^e} \left\{ \phi\left(\frac{y}{x} - c + c\right) + x \cdot \Omega\left(x, \frac{y}{x} - c + c\right) \right\} \\ &= \tilde{x}^{d-p^e} \left\{ \phi(\tilde{y} + c) + \tilde{x} \cdot \Omega(\tilde{x}, \tilde{y} + c) \right\}. \end{aligned}$$

Therefore, we conclude

$$\begin{aligned} \text{In}_P(a_{p^e}) &= \sum_{k+l=d} c_{kl} x^k y^l = x^d \phi\left(\frac{y}{x}\right) \text{ is a } p^e\text{-th power} \\ \iff p^e \mid d \text{ and } \phi(T) \text{ is a } p^e\text{-th power} \\ \iff \text{In}_{\xi_{E_P}}(\widetilde{a}_{p^e}) &= \widetilde{x}^{d-p^e} \phi(\widetilde{y} + c) \text{ is a } p^e\text{-th power.} \end{aligned}$$

Now since P is a bad point in this case, $\text{In}_P(a_{p^e})$ is not a p^e -th power. Therefore, by the above equivalence, $\text{In}_{\xi_{E_P}}(\widetilde{a}_{p^e})$ is not a p^e -th power, either.

This completes the proof of Lemma 4. Q.E.D.

Remark 7.

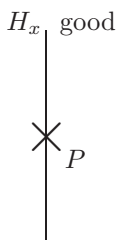
(1) The above proof is slightly sloppy in the sense that, after blow up, we may end up having $\text{ord}_{\widetilde{P}}(\widetilde{a}_{p^e}) = p^e$ and $\text{In}_{\widetilde{P}}(\widetilde{a}_{p^e}) = c\widetilde{x}^{p^e}$ for some $c \in k \setminus \{0\}$, even under the condition $\widetilde{P} \in \pi^{-1}(P) \cap \text{Supp}(\widetilde{\mathcal{R}}) \subset \widetilde{W}$ and the assumption that the invariant σ stays the same (cf. the proof of Proposition 4 (1)). Then we would have to replace $(\widetilde{x}, \widetilde{y}, \widetilde{z})$ with $(\widetilde{x}, \widetilde{y}, \widetilde{z}' = \widetilde{z} + c^{1/p^e} \widetilde{x})$ to guarantee condition (1) in SITUATION in §5.1. Accordingly, we have to analyze \widetilde{a}_{p^e}' . It is straightforward, however, to see that the same statement holds for \widetilde{a}_{p^e}' . The details are left to the reader as an exercise.

(2) As will be clear in the presentation that follows, especially in the way we classify the configurations and we define the new invariant “ inv_{MON} ”, the focus of our proof centers around the analysis looking at whether the hypersurface of our concern is good/bad. The notion of a point being good/bad, though related to our analysis via Lemma 4, is somewhat auxiliary.

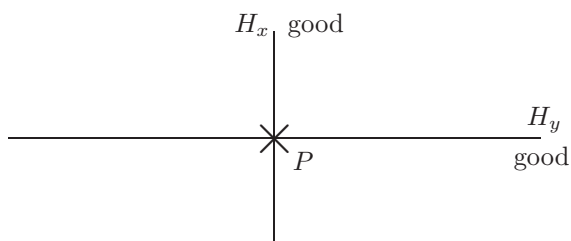
Configurations

Looking at the boundary divisor(s) in E_{young} at the point $P \in \text{Sing}(\mathcal{R})$ and seeing whether they are good or bad, we come up with the following classification of the “configurations”. Note that the pictures depict the configurations in a 2-dimensional manner, taking the intersection with the hypersurface $Z = \{z = 0\}$.

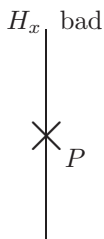
① The point P is only on one boundary divisor (in E_{young}), say H_x , which is good.



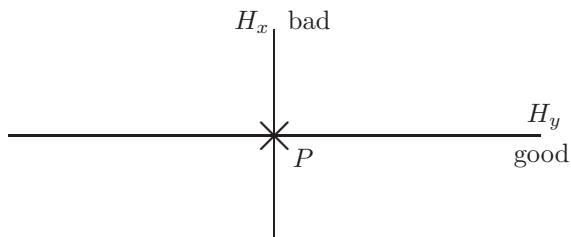
② The point P is at the intersection of two boundary divisors (in E_{young}), both of which are good.



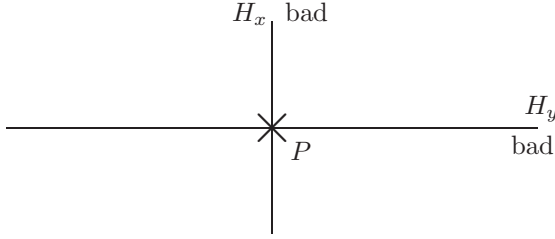
③ The point P is only on one boundary divisor (in E_{young}), say H_x , which is bad.



④ The point P is at the intersection of two boundary divisors (in E_{young}), one of which, say, H_x , is bad, while the other, say H_y , is good.



⑤ The point P is at the intersection of two boundary divisors (in E_{young}), say H_x and H_y , both of which are bad.



Basic strategy to show termination of the procedure

After the blow up $W \xrightarrow{\pi} \widetilde{W}$ specified by the procedure described in §5.3, we show that, at $\tilde{P} \in \pi^{-1}(P) \in \widetilde{W}$, one of the following holds:

- $\tilde{P} \notin \text{Sing}(\tilde{R})$, i.e., $\text{Sing}(\tilde{R}) = \emptyset$ in a neighborhood of \tilde{P} ,
- $\tilde{P} \in \text{Sing}(\tilde{R})$ and the invariant σ drops, or
- $\tilde{P} \in \text{Sing}(\tilde{R})$ and the invariant “ $\text{inv}_{\text{MON}}(P)$ ” strictly decreases,

i.e., $\text{inv}_{\text{MON}}(P) > \text{inv}_{\text{MON}}(\tilde{P})$. We note that “ $\text{inv}_{\text{MON}}(P)$ ” is a new invariant, which we introduce below, attached to the point $P \in \text{Sing}(\mathcal{R})$ in any one of the configurations ① through ⑤.

Since the invariant “ inv_{MON} ” cannot decrease infinitely many times, the procedure described in §5.3 terminates either with $\tilde{P} \notin \text{Supp}(\tilde{R})$ or with the drop of the invariant σ after finitely many repetitions.

This completes the description of the basic strategy.

Definition 10 (Invariant “ inv_{MON} ”). We define the invariant “ $\text{inv}_{\text{MON}}(P)$ ” associated to a point $P \in \text{Sing}(\tilde{R})$ in each of the configurations ① through ⑤ (cf. **Configurations**) as follows (Note that “MON” is short for “MONOMIAL”.):

$$\text{inv}_{\text{MON}}(P) = \begin{cases} (0, 0, \mu_x) & \text{in configuration ①,} \\ (0, 0, \min\{\mu_x, \mu_y\}, \max\{\mu_x, \mu_y\}) & \text{in configuration ②,} \\ (\rho_x, 0, \mu_x) & \text{in configuration ③,} \\ (\min\{\rho_x, \mu_x\}, \max\{\rho_x, \mu_x\}) & \text{in configuration ④,} \\ (\min\{\rho_x, \rho_y\}, \max\{\rho_x, \rho_y\}) & \text{in configuration ⑤,} \end{cases}$$

where $\mu_x = \mu(\xi_{H_x})$, $\mu_y = \mu(\xi_{H_y})$ and the invariant ρ is defined as below to determine ρ_x, ρ_y .

Proposition 7. *Let P be the point in configuration ③, ④, or ⑤, and let H_x be a bad boundary divisor (in E_{young}). Suppose h is well-adapted at ξ_{H_x} with respect to (x, y, z) (satisfying the conditions as described in SITUATION in §5.1 at the same time). Write*

$$a_{p^e} = x^r \{g(y) + x \cdot \omega(x, y)\} \quad \text{where } r = \text{ord}_{\xi_{H_x}}(a_{p^e}).$$

Set

$$\begin{aligned} \rho_{h,(x,y,z),H_x}(P) &= \frac{1}{p^e} \{ \text{res-ord}_P^{(p^e)}(\text{In}_{\xi_{H_x}}(a_{p^e})) - \text{ord}_{\xi_{H_x}}(a_{p^e}) \} \\ &= \frac{1}{p^e} \{ \text{res-ord}_P^{(p^e)}(x^r g(y)) - r \}, \end{aligned}$$

i.e.,

$$\rho_{h,(x,y,z),H_x}(P) = \begin{cases} \text{ord}_P(g(y)) / p^e & \text{in case } r \not\equiv 0 \pmod{p^e} \\ \text{res-ord}_P^{(p^e)}(g(y)) / p^e & \text{in case } r \equiv 0 \pmod{p^e}, \end{cases}$$

where $\text{res-ord}_P^{(p^e)}$ is the smallest degree of the term which appears with a nonzero coefficient and which is not a p^e -th power.

Then $\rho_{h,(x,y,z),H_x}(P)$ is independent of the choice of h and (x, y, z) .

Proof. Note first that $r = H(\xi_{H_x}) \cdot p^e$ is independent of the choice of h and (x, y, z) by Proposition 5 (2).

We set

$$\rho_x = \max \left\{ \frac{1}{p^e} \text{ord}_P \left(\left\{ (h'|_{Z'}) \cdot x'^{-r} \right\} \Big|_{Z' \cap H_{x'}} \right) \left| \begin{array}{l} h', (x', y', z'), \\ Z' = \{z' = 0\}, \\ H_{x'} = \{x' = 0\} \\ = H_x \end{array} \right. \right\},$$

where, computing the above “max”, we let h' and (x', y', z') vary among all such pairs consisting of the unique element in an LGS and a regular system of parameters that satisfy the condition

$$\begin{cases} h' & \equiv z'^{p^e} \pmod{\widehat{\mathfrak{m}}_P^{p^e+1}}, \text{ and} \\ \text{ord}_{\xi_{H_{x'}}}(h'|_{Z'}) & = \text{ord}_{\xi_{H_x}}(h'|_{Z'}) = r. \end{cases}$$

It suffices to show that

$$\rho_{h,(x,y,z),H_x}(P) = \rho_x,$$

as the number ρ_x is obviously independent of the choice of h and (x, y, z) .

Firstly we claim that the inequality

$$\rho_{h,(x,y,z),H_x}(P) \leq \rho_x$$

holds. Note that h and (x, y, z) form such a pair satisfying the above condition described in the definition of ρ_x computing the “max”, since h is well-adapted at ξ_{H_x} with respect to (x, y, z) .

Case: $r \not\equiv 0 \pmod{p^e}$.

In this case, we have

$$\rho_{h,(x,y,z),H_x}(P) = \frac{1}{p^e} \text{ord}_P(g(y)) = \frac{1}{p^e} \text{ord}_P(\{(h|z) \cdot x^{-r}\} |_{Z \cap H_x}) \leq \rho_x$$

by the definition of ρ_x above taking the “max” among all such pairs.

Case: $r \equiv 0 \pmod{p^e}$.

In this case, we modify z in the following way.

Set $g(y) = \sum_{n \in \mathbb{Z}_{\geq 0}} b_n y^n$ with $b_n \in k$. We have

$$x^r \left\{ \sum_{n < \text{res-ord}_P^{(p^e)}(g(y))} b_n y^n \right\} = w^{p^e} \quad \text{for some } w \in k[x, y],$$

since L.H.S. is a p^e -th power by the case assumption $r \equiv 0 \pmod{p^e}$ and by the definition of $\text{res-ord}_P^{(p^e)}$. Then set $z' = z + w$, i.e., $z = z' - w$.

Plug this into

$$h = z^{p^e} + a_1 z^{p^e-1} + a_2 z^{p^e-2} + \cdots + a_{p^e-1} z + a_{p^e}$$

to obtain

$$h = z'^{p^e} + a'_1 z'^{p^e-1} + a'_2 z'^{p^e-2} + \cdots + a'_{p^e-1} z' + a'_{p^e}$$

with $a'_i \in k[[x, y]]$ for $i = 1, \dots, p^e - 1, p^e$, where

$$a'_{p^e} = (-w)^{p^e} + a_1 (-w)^{p^e-1} + a_2 (-w)^{p^e-2} + \cdots + a_{p^e}.$$

Observe that, since (cf. SITUATION in §5.1)

$$(\mathbb{M}_u)^i \mid a_i \quad \text{for } i = 1, \dots, p^e - 1,$$

we have

$$\frac{1}{i} \text{ord}_{\xi_{H_x}}(a_i) \geq \text{ord}_{\xi_{H_x}}(\mathbb{M}_u) = \mu(\xi_{H_x}) > \frac{1}{p^e} \text{ord}_{\xi_{H_x}}(a_{p^e}) = \frac{r}{p^e}$$

for $i = 1, \dots, p^e - 1$, where the second strict inequality follows from the assumption that H_x is a bad boundary divisor.

Hence, we observe that a'_{p^e} is of the following form

$$a'_{p^e} = x^r \{g'(y) + x \cdot \omega'(x, y)\}$$

where

$$g'(y) = g(y) - w^{p^e} \cdot x^{-r} = \sum_{n \geq \text{res-ord}_P^{(p^e)}(g(y))} b_n y^n.$$

Therefore, we conclude that

$$\begin{aligned} \rho_{h,(x,y,z),H_x}(P) &= \frac{1}{p^e} \text{res-ord}_P^{(p^e)}(g(y)) = \frac{1}{p^e} \text{ord}_P(g'(y)) \\ &= \frac{1}{p^e} \text{ord}_P(\{(h|_{Z'}) \cdot x^{-r}\} |_{Z' \cap H_x}) \leq \rho_x. \end{aligned}$$

Secondly we prove the inequality in the opposite direction holds, i.e.,

$$\rho_{h,(x,y,z),H_x}(P) \geq \rho_x.$$

Take an arbitrary pair of h' and (x', y', z') satisfying the conditions described in the definition of ρ_x computing the “max”.

We claim that

$$\rho_{h,(x,y,z),H_x}(P) \geq \frac{1}{p^e} \text{ord}_P(\{(h'|_{Z'}) \cdot x'^{-r}\} |_{Z' \cap H_x}),$$

which implies the required inequality.

Let

$$h' = \sum c_B H^B = \sum_{b \in \mathbb{Z}_{\geq 0}} c_b h^b$$

be the power series expansion of h' with respect to the LGS $\mathbb{H} = \{(h, p^e)\}$ and its associated regular system of parameters (x, y, z) .

Since

$$h' \equiv z'^{p^e} \equiv c \cdot z^{p^e} \pmod{\widehat{\mathfrak{m}}_P^{p^e+1}} \quad \text{for some } c \in k^\times,$$

we conclude

$$h' = \sum_{b > 0} c_b h^b + c_0 = u \cdot h + c_0$$

for some unit u in $\widehat{\mathcal{O}}_{W,P}$.

Moreover, by the formal coefficient lemma, we have $(c_0, p^e) \in \widehat{\mathcal{R}}_P$. This implies that $(\mathbb{M}_u)^{p^e} | c_0$, and hence that

$$\text{ord}_{\xi_{H_x}}(c_0|_{Z'}) \geq \text{ord}_{\xi_{H_x}}(c_0) \geq p^e \mu(\xi_{H_x}) > r.$$

Hence we have

$$\left\{ (c_0|_{Z'}) \cdot x'^{-r} \right\} |_{Z' \cap H_{x'}} = \left\{ (c_0|_{Z'}) \cdot x^{-r} \right\} |_{Z' \cap H_x} = 0,$$

which implies

$$\begin{aligned} \text{ord}_P \left(\left\{ (h'|_{Z'}) \cdot x'^{-r} \right\} |_{Z' \cap H_{x'}} \right) &= \text{ord}_P \left(\left\{ ((u \cdot h + c_0)|_{Z'}) \cdot x'^{-r} \right\} |_{Z' \cap H_{x'}} \right) \\ &= \text{ord}_P \left(\left\{ (u \cdot h|_{Z'}) \cdot x'^{-r} \right\} |_{Z' \cap H_{x'}} \right) = \text{ord}_P \left(\left\{ (h|_{Z'}) \cdot x'^{-r} \right\} |_{Z' \cap H_{x'}} \right) \\ &= \text{ord}_P \left(\left\{ (h|_{Z'}) \cdot x^{-r} \right\} |_{Z' \cap H_x} \right). \end{aligned}$$

Therefore, it suffices to prove

$$\rho_{h, (x, y, z), H_x}(P) \geq \frac{1}{p^e} \text{ord}_P \left(\left\{ (h|_{Z'}) \cdot x^{-r} \right\} |_{Z' \cap H_x} \right).$$

Now by the Weierstrass Preparation Theorem, we have

$$z' = v \cdot (z + w)$$

for some unit v in $\widehat{\mathcal{O}_{W,P}}$ and $w \in k[[x, y]]$. Since $Z' = \{z' = 0\} = \{z + w = 0\}$, by replacing z' with $z + w$, we may assume that z' is of the form

$$z' = z + w, \quad \text{i.e., } z = z' - w \quad \text{with } w \in k[[x, y]].$$

Plug this into

$$h = z^{p^e} + a_1 z^{p^e-1} + a_2 z^{p^e-2} + \cdots + a_{p^e-1} z + a_{p^e}$$

to obtain

$$h = z'^{p^e} + a'_1 z'^{p^e-1} + a'_2 z'^{p^e-2} + \cdots + a'_{p^e-1} z' + a'_{p^e}$$

with $a'_i \in k[[x, y]]$ for $i = 1, \dots, p^e - 1, p^e$, where

$$a'_{p^e} = (-w)^{p^e} + a_1 (-w)^{p^e-1} + a_2 (-w)^{p^e-2} + \cdots + a_{p^e}.$$

Observe that, since (cf. SITUATION in §5.1)

$$(\mathbb{M}_u)^i \mid a_i \quad \text{for } i = 1, \dots, p^e - 1,$$

we have

$$\frac{1}{i} \text{ord}_{\xi_{H_x}}(a_i) \geq \text{ord}_{\xi_{H_x}}(\mathbb{M}_u) = \mu(\xi_{H_x}) > \frac{1}{p^e} \text{ord}_{\xi_{H_x}}(a_{p^e}) = \frac{r}{p^e}$$

for $i = 1, \dots, p^e - 1$. We claim that

$$\text{ord}_{\xi_{H_x}}(w) \geq \frac{r}{p^e}.$$

In fact, suppose $\text{ord}_{\xi_{H_x}}(w) < r/p^e$. Then using the above observation we would have

$$\text{ord}_{\xi_{H_x}}(h|_{Z'}) = \text{ord}_{\xi_{H_x}}(a'_{p^e}) = \text{ord}_{\xi_{H_x}}((-w)^{p^e}) < r.$$

By the equation $h' = u \cdot h + c_0$ and by the inequality $\text{ord}_{\xi_{H_x}}(c_0|_{Z'}) > r$, this would also imply

$$\text{ord}_{\xi_{H_x}}(h'|_{Z'}) = \text{ord}_{\xi_{H_x}}(h|_{Z'}) < r,$$

which is against the choice of h' and (x', y', z') that we started with, satisfying the conditions described in the definition of ρ_x computing the “max”.

Now we are at the stage to finish the argument to prove the inequality

$$\rho_{h,(x,y,z),H_x}(P) \geq \frac{1}{p^e} \text{ord}_P(\{(h|_{Z'}) \cdot x^{-r}\} |_{Z' \cap H_x}).$$

Case: $r \not\equiv 0 \pmod{p^e}$.

In this case, the claimed inequality $\text{ord}_{\xi_{H_x}}(w) \geq r/p^e$ implies the strict inequality $\text{ord}_{\xi_{H_x}}(w) > r/p^e$, since $\text{ord}_{\xi_{H_x}}(w)$ is an integer. Together with the observation, we conclude that

$$\text{In}_{\xi_{H_x}}(a'_{p^e}) = \text{In}_{\xi_{H_x}}(a_{p^e}) = x^r g(y),$$

and hence that

$$\rho_{h,(x,y,z),H_x}(P) = \frac{1}{p^e} \text{ord}_P(g(y)) = \frac{1}{p^e} \text{ord}_P(\{(h|_{Z'}) \cdot x^{-r}\} |_{Z' \cap H_x}).$$

Case: $r \equiv 0 \pmod{p^e}$.

In this case, set

$$w = x^{r/p^e} \{h(y) + x \cdot \theta(x, y)\} \quad \text{with} \quad h(y) \in k[[y]], \theta(x, y) \in k[[x, y]].$$

Together with the observation, we conclude that

$$\text{In}_{\xi_{H_x}}(a'_{p^e}) = \text{In}_{\xi_{H_x}}(a_{p^e}) - x^r h(y)^{p^e} = x^r \{g(y) - h(y)^{p^e}\},$$

and hence that

$$\begin{aligned} \rho_{h,(x,y,z),H_x}(P) &= \frac{1}{p^e} \text{res-ord}_P^{(p^e)}(g(y)) \geq \frac{1}{p^e} \text{ord}_P(g(y) - h(y)^{p^e}) \\ &= \frac{1}{p^e} \text{ord}_P(\{(h|_{Z'}) \cdot x^{-r}\} |_{Z' \cap H_x}). \end{aligned}$$

This completes the proof of Proposition 7.

Q.E.D.

Definition 11 (Invariant “ ρ ”). Let the situation be as described in Proposition 7. We define the invariant ρ of H_x at P , denoted by ρ_x , by the formula

$$\rho_x = \rho_{h,(x,y,z),H_x}(P).$$

Invariant ρ_y is defined in an identical manner, in case H_y is a bad boundary divisor (in E_{young}) passing through P in configuration ⑤.

Study of the behavior of the invariants under blow up

Case: blow up with 1-dimensional center

Claim 1. Let $W \xleftarrow{\pi} \widetilde{W}$ be the blow up with center being a 1-dimensional component $C = V(z, x)$ of $\text{Sing}(\mathcal{R})$ (cf. Proposition 6), and $\widetilde{\mathcal{R}}$ the transformation of the idealistic filtration \mathcal{R} of i.f.g. type. Then there is possibly only one point $\widetilde{P} \in \text{Sing}(\widetilde{\mathcal{R}}) \cap \pi^{-1}(P) \subset \widetilde{W}$, lying in the x -chart, with the regular system of parameters $(\widetilde{x}, \widetilde{y}, \widetilde{z}) = (x, y, z/x)$. The behavior of the invariants under blow up is described as follows (in case the invariant σ does not drop):

$$\begin{aligned} \mu_{\widetilde{x}} &= \mu_x - 1, & h_{\widetilde{x}} &= h_x - 1, & \text{and, in case } H_x \text{ is bad, } & \rho_{\widetilde{x}} &= \rho_x. \\ \mu_{\widetilde{y}} &= \mu_y, & h_{\widetilde{y}} &= h_y, & \text{and, in case } H_y \text{ is bad, } & \rho_{\widetilde{y}} &= \rho_y - 1. \end{aligned}$$

Proof. Write down

$$h = z^{p^e} + a_1 z^{p^e-1} + a_2 z^{p^e-2} + \cdots + a_{p^e-1} z + a_{p^e}$$

with

$$a_i \in k[[x, y]] \quad \text{and} \quad \text{ord}_P(a_i) > i \quad \text{for} \quad i = 1, \dots, p^e,$$

as described in SITUATION in §5.1. We may assume further that h is well-adapted at P , ξ_{H_x} and ξ_{H_y} simultaneously with respect to (x, y, z) (cf. Proposition 5).

It is straightforward to see that, if a point $\widetilde{P} \in \pi^{-1}(P) \subset \widetilde{W}$ lies in the z -chart, then $\text{ord}_P(\widetilde{h}) < p^e$, where $\widetilde{h} = \pi^*(h)/z^{p^e}$, and hence $\widetilde{P} \notin \text{Sing}(\widetilde{\mathcal{R}})$. Therefore, there is possibly only one point $\widetilde{P} \in \text{Sing}(\widetilde{\mathcal{R}}) \cap \pi^{-1}(P) \subset \widetilde{W}$, lying in the x -chart, with the regular system of parameters $(\widetilde{x}, \widetilde{y}, \widetilde{z}) = (x, y, z/x)$.

With respect to this regular system of parameters, we compute the transform of the monomial $(x^\alpha y^\beta, a)$ to be $(\widetilde{x}^{\alpha-a} \widetilde{y}^\beta, a)$. Therefore, we conclude

$$\mu_{\widetilde{x}} = \mu_x - 1, \quad \mu_{\widetilde{y}} = \mu_y.$$

Set

$$a_{p^e} = x^r \{g(y) + x \cdot \omega(x, y)\}$$

with

$$r = \text{ord}_{\xi_{H_x}}(a_{p^e}), \quad 0 \neq g(y) \in k[[y]], \quad \text{and} \quad \omega(x, y) \in k[[x, y]].$$

After blow up, we compute

$$\tilde{h} = \frac{\pi^*(h)}{x^{p^e}} = \tilde{z}^{p^e} + \tilde{a}_1 \tilde{z}^{p^e-1} + \tilde{a}_2 \tilde{z}^{p^e-2} + \cdots + \widetilde{a_{p^e-1}} \tilde{z} + \widetilde{a_{p^e}}$$

with

$$\tilde{a}_i = \frac{\pi^*(a_i)}{x^i} \quad \text{for} \quad i = 1, \dots, p^e.$$

In particular, we have

$$\widetilde{a_{p^e}} = \frac{\pi^*(a_{p^e})}{x^{p^e}} = x^{r-p^e} \{g(y) + x \cdot \omega(x, y)\} = \tilde{x}^{r-p^e} \{g(\tilde{y}) + \tilde{x} \cdot \omega(\tilde{x}, \tilde{y})\}.$$

(Note that we have $r \geq p^e$, since $h_x = r/p^e \geq 1$ (cf. Proposition 6).) Therefore, we compute

$$\begin{aligned} \text{Slope}_{\tilde{h},(\tilde{x},\tilde{y},\tilde{z})}(\xi_{H_{\tilde{x}}}) &= \min \left\{ \frac{1}{p^e} \text{ord}_{\xi_{H_{\tilde{x}}}}(\widetilde{a_{p^e}}), \mu(\xi_{H_{\tilde{x}}}) \right\} \\ &= \min \left\{ \frac{r-p^e}{p^e}, \mu_{\tilde{x}} \right\} = \min \left\{ \frac{1}{p^e} \text{ord}_{\xi_{H_x}}(a_{p^e}) - 1, \mu_x - 1 \right\} \\ &= \min \left\{ \frac{1}{p^e} \text{ord}_{\xi_{H_x}}(a_{p^e}), \mu_x \right\} - 1 = \text{Slope}_{h,(x,y,z)}(\xi_{H_x}) - 1. \end{aligned}$$

Hence, if $\text{Slope}_{h,(x,y,z)}(\xi_{H_x}) = \mu(\xi_{H_x})$, then

$$\text{Slope}_{\tilde{h},(\tilde{x},\tilde{y},\tilde{z})}(\xi_{H_{\tilde{x}}}) = \mu(\xi_{H_x}) - 1 = \mu(\xi_{H_{\tilde{x}}}).$$

If $\text{Slope}_{h,(x,y,z)}(\xi_{H_x}) < \mu(\xi_{H_x})$, then

$$\text{Slope}_{\tilde{h},(\tilde{x},\tilde{y},\tilde{z})}(\xi_{H_{\tilde{x}}}) = \text{Slope}_{h,(x,y,z)}(\xi_{H_x}) - 1 < \mu(\xi_{H_x}) - 1 = \mu(\xi_{H_{\tilde{x}}})$$

and

$$\text{In}_{\xi_{H_{\tilde{x}}}}(\widetilde{a_{p^e}}) = \tilde{x}^{r-p^e} g(\tilde{y}) \text{ is not a } p^e\text{-th power,}$$

since $\text{In}_{\xi_{H_x}}(a_{p^e}) = x^r g(y)$ is not a p^e -th power. Therefore, \tilde{h} is well-adapted at $\xi_{H_{\tilde{x}}}$ with respect to $(\tilde{x}, \tilde{y}, \tilde{z})$.

Therefore, we conclude

$$h_{\tilde{x}} = \text{Slope}_{\tilde{h},(\tilde{x},\tilde{y},\tilde{z})}(\xi_{H_{\tilde{x}}}) = \text{Slope}_{h,(x,y,z)}(\xi_{H_x}) - 1 = h_x - 1.$$

In case H_x is bad, i.e., $h_x < \mu_x$, so is $H_{\tilde{x}}$ since $h_{\tilde{x}} = h_x - 1 < \mu_x - 1 = \mu_{\tilde{x}}$. Moreover, we compute

$$\begin{aligned} \rho_{\tilde{x}} &= \rho_{\tilde{h},(\tilde{x},\tilde{y},\tilde{z}),H_{\tilde{x}}}(\tilde{P}) = \begin{cases} \text{ord}_{\tilde{P}}(g(\tilde{y}))/p^e & \text{if } r - p^e \not\equiv 0 \pmod{p^e}, \\ \text{res-ord}_{\tilde{P}}^{(p^e)}(g(\tilde{y}))/p^e & \text{if } r - p^e \equiv 0 \pmod{p^e} \end{cases} \\ &= \begin{cases} \text{ord}_P(g(y))/p^e & \text{if } r \not\equiv 0 \pmod{p^e}, \\ \text{res-ord}_P^{(p^e)}(g(y))/p^e & \text{if } r \equiv 0 \pmod{p^e} \end{cases} \\ &= \rho_{h,(x,y,z),H_x}(P) = \rho_x. \end{aligned}$$

In summary, we have

$$h_{\tilde{x}} = h_x - 1, \quad \rho_{\tilde{x}} = \rho_x.$$

The proof for the formulas

$$h_{\tilde{y}} = h_y, \quad \rho_{\tilde{y}} = \rho_y - 1$$

is similar, and left to the reader as an exercise.

Q.E.D.

Remark 8. The above proof is slightly sloppy in the sense that, after blow up, we may end up having $\text{ord}_{\tilde{P}}(\tilde{a}_{p^e}) = p^e$ and $\text{In}_{\tilde{P}}(\tilde{a}_{p^e}) = c\tilde{x}^{p^e} + c'\tilde{y}^{p^e}$ for some $(c, c') \in k^2 \setminus \{(0, 0)\}$, even under the condition $\tilde{P} \in \pi^{-1}(P) \cap \text{Sing}(\tilde{\mathcal{R}}) \subset \tilde{W}$ and the assumption that the invariant σ stays the same (cf. the proof of Proposition 4 (1)). Then we have to replace $(\tilde{x}, \tilde{y}, \tilde{z})$ with

$$(\tilde{x}, \tilde{y}, \tilde{z}' = \tilde{z} + c^{1/p^e}\tilde{x} + c'^{1/p^e}\tilde{y})$$

to guarantee condition (1) in SITUATION in §5.1. Accordingly, we have to analyze \tilde{a}_{p^e}' . It is straightforward, however, to see that the same calculations hold with \tilde{a}_{p^e}' . The details are left to the reader as an exercise.

Case: blow up with 0-dimensional center

Claim 2. Let $W \xrightarrow{\pi} \tilde{W}$ be the blow up with a 0-dimensional center $C = P = V(z, x, y) \in \text{Sing}(\mathcal{R})$ (cf. Proposition 6 and the description of the procedure in §5.3), and $\tilde{\mathcal{R}}$ the transformation of the idealistic filtration \mathcal{R} of i.f.g. type. Set $Z = \{z = 0\}$. Then a point $\tilde{P} \in \text{Sing}(\tilde{\mathcal{R}}) \cap \pi^{-1}(P) \subset \tilde{W}$ must be on the strict transform Z' of Z , lying either in the x -chart or in the y -chart. Assume that the invariant σ stays the same, i.e., $\sigma(P) = \sigma(\tilde{P})$. We make the following three observations regarding the behavior of the invariants under blow up. (We denote the strict

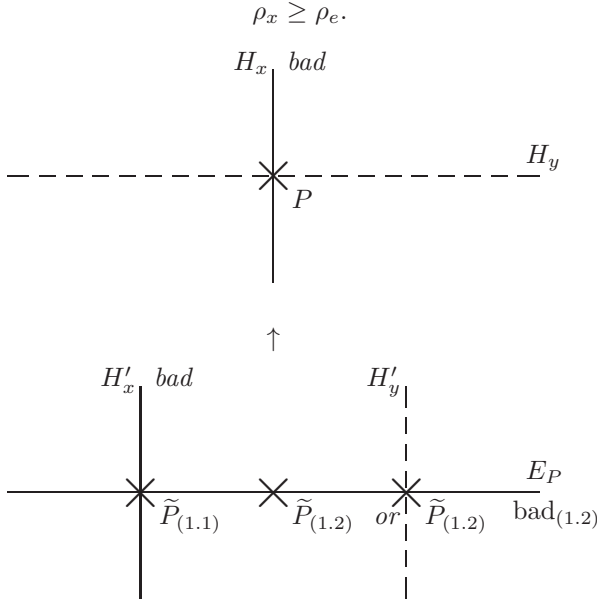
transforms of H_x and H_y by H'_x and H'_y , and the exceptional divisor by E_P . Note that the pictures depict the configurations in a 2-dimensional manner, taking the intersection with the hypersurface Z before blow up and with its strict transform Z' after blow up.):

(1) The point P is in case ③, ④ or ⑤.

(1.1) Suppose $h_x < 1$. Look at the point $\tilde{P} = E_P \cap H'_x \cap Z'$ in the y -chart with a regular system of parameters $(\tilde{x}, \tilde{y}, \tilde{z}) = (x/y, y, z/y)$. Then the hypersurface $H'_x = H_{\tilde{x}}$ is bad, and we have

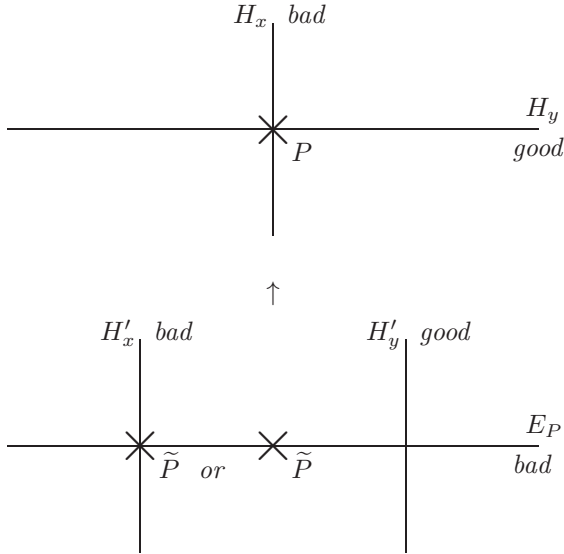
$$\rho_x > \rho_{\tilde{x}}.$$

(1.2) Suppose P is bad, and hence E_P is also bad (cf. Lemma 4). Look at a point $\tilde{P} \in (E_P \setminus H'_x) \cap Z'$ in the x -chart with a regular system of parameters $(\tilde{x} = e, \tilde{y}, \tilde{z}) = (x, y/x - c, z/x)$ for some $c \in k$. Then we have



(2) The point P is in case ④. Suppose P is bad, and hence E_P is also bad (cf. Lemma 4). Look at a point $\tilde{P} \in (E_P \setminus H'_y) \cap Z'$ with a regular system of parameters $(\tilde{x}, \tilde{y} = e, \tilde{z}) = (x/y - c, y, z/y)$ for some $c \in k$. Then we have

$$\mu_x > \rho_e.$$



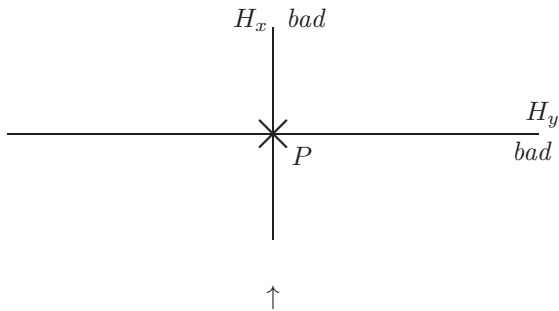
(3) The point P is in case ⑤. Suppose P is good. Then we have

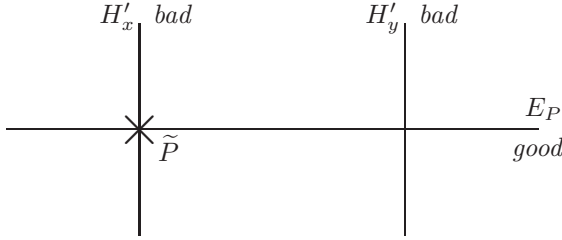
$$\rho_x > \mu_y \quad \text{and} \quad \rho_y > \mu_x.$$

Look at the point $\tilde{P} = E_P \cap H'_x \cap Z'$ in the y -chart with a regular system of parameters $(\tilde{x}, \tilde{y}, \tilde{z}) = (x/y, y, z/y)$. Since $\mu_{\tilde{x}} = \mu_x$, we have as a consequence

$$\rho_y > \mu_{\tilde{x}}.$$

We draw a similar conclusion looking at the point $E_P \cap H'_y \cap Z'$ in the x -chart.





Proof. (1) (1.1) Take (as described in SITUATION in §5.1)

$$h = z^{p^e} + a_1 z^{p^e-1} + a_2 z^{p^e-2} + \cdots + a_{p^e-1} z + a_{p^e}$$

which is well-adapted at ξ_{H_x} with respect to (x, y, z) (cf. Proposition 5). Set

$$a_{p^e} = x^r \{g(y) + x \cdot \omega(x, y)\} \text{ with } 0 \neq g(y) \in k[[y]], \omega(x, y) \in k[[x, y]].$$

Then we have

$$h_x = H(\xi_{H_x}) = \text{Slope}_{h, (x, y, z)}(\xi_{H_x}) = \frac{1}{p^e} \text{ord}_{\xi_{H_x}}(a_{p^e}) = \frac{r}{p^e} < \mu(\xi_{H_x}).$$

Since $h_x < 1$ by assumption, we have $r < p^e$. We compute

$$\widetilde{a}_{p^e} = \frac{a_{p^e}}{y^{p^e}} = \frac{(\widetilde{x}\widetilde{y})^r}{\widetilde{y}^{p^e}} \{g(\widetilde{y}) + \widetilde{x}\widetilde{y} \cdot \omega(\widetilde{x}\widetilde{y}, \widetilde{y})\} = \widetilde{x}^r \{\widetilde{g}(\widetilde{y}) + \widetilde{x} \cdot \widetilde{\omega}(\widetilde{x}, \widetilde{y})\}$$

where

$$\widetilde{g}(\widetilde{y}) = \widetilde{y}^{r-p^e} g(\widetilde{y}) \quad \text{and} \quad \widetilde{\omega}(\widetilde{x}, \widetilde{y}) = \widetilde{y}^{r-p^e+1} \omega(\widetilde{x}\widetilde{y}, \widetilde{y}).$$

We observe that $\widetilde{h} = h/y^{p^e}$ is well-adapted at $\xi_{H_{\widetilde{x}}}$ with respect to $(\widetilde{x}, \widetilde{y}, \widetilde{z})$, since

$$\text{Slope}_{\widetilde{h}, (\widetilde{x}, \widetilde{y}, \widetilde{z})}(\xi_{H_{\widetilde{x}}}) = \frac{r}{p^e} < \mu(\xi_{H_x}) = \mu(\xi_{H_{\widetilde{x}}})$$

and since

$$\text{In}_{\xi_{H_{\widetilde{x}}}}(\widetilde{a}_{p^e}) = \widetilde{x}^r \widetilde{g}(\widetilde{y}) = \widetilde{x}^r \widetilde{y}^{r-p^e} g(\widetilde{y})$$

is not a p^e -th power, a fact which follows easily from the fact that $\text{In}_{\xi_{H_x}}(a_{p^e}) = x^r g(y)$ is not a p^e -th power. Therefore, we conclude that

$$\begin{aligned} \rho_x &= \rho_{h,(x,y,z),H_x}(P) \\ &= \begin{cases} \text{ord}_P(g(y))/p^e & \text{in case } r \not\equiv 0 \pmod{p^e}, \\ \text{res-ord}_P^{(p^e)}(g(y))/p^e & \text{in case } r \equiv 0 \pmod{p^e}, \quad \text{i.e., } r = 0 \end{cases} \\ &> \begin{cases} \text{ord}_{\tilde{y}}(\tilde{y}^{r-p^e}g(\tilde{y}))/p^e & \text{in case } r \not\equiv 0 \pmod{p^e}, \\ \text{res-ord}_{\tilde{y}}^{(p^e)}(\tilde{y}^{-p^e}g(\tilde{y}))/p^e & \text{in case } r \equiv 0 \pmod{p^e}, \quad \text{i.e., } r = 0 \end{cases} \\ &= \begin{cases} \text{ord}_{\tilde{y}}(\tilde{g}(\tilde{y}))/p^e & \text{in case } r \not\equiv 0 \pmod{p^e}, \\ \text{res-ord}_{\tilde{y}}^{(p^e)}(\tilde{g}(\tilde{y}))/p^e & \text{in case } r \equiv 0 \pmod{p^e}, \quad \text{i.e. } r = 0 \end{cases} \\ &= \rho_{\tilde{h},(\tilde{x},\tilde{y},\tilde{z}),H_{\tilde{x}}}(\tilde{P}) = \rho_{\tilde{x}}. \end{aligned}$$

(Note that, even under the condition $\tilde{P} \in \pi^{-1}(P) \cap \text{Sing}(\tilde{\mathcal{R}}) \subset \tilde{W}$ and the assumption that the invariant σ stays the same, there is a possibility that we may end up having $\text{ord}_{\tilde{P}}(\tilde{a}_{p^e}) = p^e$ and $\text{In}_{\tilde{P}}(\tilde{a}_{p^e}) = c\tilde{y}^{p^e}$ for some $c \in k \setminus \{0\}$. (In this case, we necessarily have $r = 0$.) Then we have to replace $(\tilde{x}, \tilde{y}, \tilde{z})$ with $(\tilde{x}, \tilde{y}, \tilde{z}' = \tilde{z} + c^{1/p^e}\tilde{y})$ to guarantee condition (1) in SITUATION. Accordingly, we have to analyze \tilde{a}_{p^e}' . It is straightforward, however, to see that the same calculations hold with \tilde{a}_{p^e}' .)

(1) (1.2) Take (as described in SITUATION in §5.1)

$$h = z^{p^e} + a_1 z^{p^e-1} + a_2 z^{p^e-2} + \dots + a_{p^e-1} z + a_{p^e}$$

which is well-adapted at P and ξ_{H_x} simultaneously with respect to (x, y, z) (cf. Proposition 5).

Set

$$\begin{aligned} a_{p^e} &= \sum_{k+l \geq d} c_{kl} x^k y^l && \text{with } d = \text{ord}_P(a_{p^e}) > p^e \\ &= x^r \{g(y) + x \cdot \omega(x, y)\} && \text{with } r = \text{ord}_{\xi_{H_x}}(a_{p^e}). \end{aligned}$$

Then we compute

$$\begin{aligned} \pi^*(a_{p^e}) &= \sum_{k+l \geq d} c_{kl} x^{k+l} \left(\frac{y}{x}\right)^l = x^d \left\{ \sum_{k+l=d} c_{kl} \left(\frac{y}{x}\right)^l + x \cdot \Omega\left(x, \frac{y}{x}\right) \right\} \\ &= x^d \left\{ \phi\left(\frac{y}{x}\right) + x \cdot \Omega\left(x, \frac{y}{x}\right) \right\} \quad \text{where } \phi(T) = \sum_{k+l=d} c_{kl} T^l, \end{aligned}$$

and hence

$$\begin{aligned} \widetilde{a}_{p^e} &= \frac{\pi^*(a_{p^e})}{x^{p^e}} = x^{d-p^e} \left\{ \phi \left(\frac{y}{x} - c + c \right) + x \cdot \Omega \left(x, \frac{y}{x} - c + c \right) \right\} \\ &= \widetilde{x}^{d-p^e} \{ \phi(\widetilde{y} + c) + \widetilde{x} \cdot \Omega(\widetilde{x}, \widetilde{y} + c) \}. \end{aligned}$$

Moreover, just as in the analysis of the “*Case: P is a bad point*” in Lemma 4, we see that \widetilde{h} is well-adapted at ξ_{E_P} with respect to $(\widetilde{x}, \widetilde{y}, \widetilde{z})$, since

$$\begin{aligned} \text{Slope}_{\widetilde{h}, (\widetilde{x}, \widetilde{y}, \widetilde{z})}(\xi_{H_{\widetilde{x}}}) &= \frac{1}{p^e} \text{ord}_{\xi_{H_{\widetilde{x}}}}(\widetilde{a}_{p^e}) = \frac{d - p^e}{p^e} \\ &= \frac{1}{p^e} \text{ord}_P(a_{p^e}) - 1 \underset{\text{since } P \text{ is bad}}{<} \mu(P) - 1 = \mu(\xi_{H_{\widetilde{x}}}), \end{aligned}$$

and since $\text{In}_{\xi_{H_{\widetilde{x}}}}(\widetilde{a}_{p^e}) = \widetilde{x}^{d-p^e} \phi(\widetilde{y} + c)$ is not a p^e -th power, which is obvious if $d \not\equiv 0 \pmod{p^e}$ and which follows from the fact $\text{In}_P(a_{p^e}) = \sum_{k+l=d} c_{kl} x^k y^l$ is not a p^e -th power and hence $\phi(T) = \sum_{k+l=d} c_{kl} T^l$ is not a p^e -th power if $d \equiv 0 \pmod{p^e}$.

Therefore, we conclude

$$\begin{aligned} \rho_{\widetilde{x}=e} &= \rho_{\widetilde{h}, (\widetilde{x}, \widetilde{y}, \widetilde{z}), H_{\widetilde{x}}} \\ &= \begin{cases} \text{ord}_{\widetilde{y}}(\phi(\widetilde{y} + c)) / p^e & \text{in case } r \not\equiv 0 \pmod{p^e} \\ \text{res-ord}_{\widetilde{y}}^{(p^e)}(\phi(\widetilde{y} + c)) / p^e & \text{in case } r \equiv 0 \pmod{p^e} \end{cases} \\ &\leq \frac{1}{p^e} \text{deg } \phi(\widetilde{y} + c) = \frac{1}{p^e} \text{deg } \phi(y). \end{aligned}$$

That is to say, we have

$$(\star) \quad \rho_e \leq \frac{1}{p^e} \text{deg } \phi(y).$$

On the other hand, set

$$M = \begin{cases} \text{ord}_y(g(y)) & \text{in case } r \not\equiv 0 \pmod{p^e} \\ \text{res-ord}_y^{(p^e)}(g(y)) & \text{in case } r \equiv 0 \pmod{p^e} \end{cases}$$

Then we conclude, for all those (k, l) with $k + l = d$ and $c_{kl} \neq 0$, that we have

$$k + l = d \leq r + M, \quad \text{and} \quad k \geq r$$

and hence that

$$l = d - k \leq d - r \leq (r + M) - r = M.$$

This implies

$$(\star\star) \quad \deg \phi(y) = \deg \left(\sum_{k+l=d} c_{kl} y^l \right) \leq M.$$

From the inequalities (\star) and $(\star\star)$, we finally conclude

$$\rho_x = \frac{M}{p^e} \geq \frac{1}{p^e} \deg(\phi(y)) \geq \rho_e.$$

(Note that, even under the condition $\tilde{P} \in \pi^{-1}(P) \cap \text{Sing}(\tilde{\mathcal{R}}) \subset \tilde{W}$ and the assumption that the invariant σ stays the same, there is a possibility that we may end up having $\text{ord}_{\tilde{P}}(\tilde{a}_{p^e}) = p^e$ and $\text{In}_{\tilde{P}}(\tilde{a}_{p^e}) = c\tilde{x}^{p^e}$ for some $c \in k \setminus \{0\}$. Then we have to replace $(\tilde{x}, \tilde{y}, \tilde{z})$ with $(\tilde{x}, \tilde{y}, \tilde{z}' = \tilde{z} + c^{1/p^e} \tilde{x})$ to guarantee condition (1) in SITUATION in §5.1. Accordingly, we have to analyze \tilde{a}_{p^e}' . It is straightforward, however, to see that the same calculations hold with \tilde{a}_{p^e}' .)

(2) Take (as described in SITUATION in §5.1)

$$h = z^{p^e} + a_1 z^{p^e-1} + a_2 z^{p^e-2} + \cdots + a_{p^e-1} z + a_{p^e}$$

which is well-adapted at P and ξ_{H_y} simultaneously with respect to (x, y, z) (cf. Proposition 5). Set

$$a_{p^e} = \sum_{k+l \geq d} c_{kl} x^k y^l \quad \text{with} \quad d = \text{ord}_P(a_{p^e}).$$

Then we see

$$\frac{d}{p^e} = \frac{1}{p^e} \text{ord}_P(a_{p^e}) < \mu(P) \quad \text{and} \quad \text{In}_P(a_{p^e}) \text{ is not a } p^e\text{-th power,}$$

since P is bad and since h is well-adapted at P with respect to (x, y, z) . Then we conclude, for all those (k, l) with $k + l = d$ and $c_{kl} \neq 0$, that we have

$$\frac{\alpha + \beta}{a} = \mu(P) > \frac{d}{p^e} = \frac{k + l}{p^e} = \frac{k}{p^e} + \frac{l}{p^e} \geq \frac{k}{p^e} + \frac{\beta}{a},$$

since

$$\frac{l}{p^e} \geq \frac{1}{p^e} \text{ord}_{\xi_{H_y}}(a_{p^e}) \geq \mu(\xi_{H_y}) = \frac{\beta}{a},$$

where the second inequality follows from the assumption that H_y is good and that h is well-adapted at ξ_{H_y} with respect to (x, y, z) . Therefore, we conclude

$$\mu_x = \mu(\xi_{H_x}) = \frac{\alpha}{a} > \frac{k}{p^e}.$$

On the other hand, we compute

$$\begin{aligned} \pi^*(a_{p^e}) &= \sum_{k+l \geq d} c_{kl} \left(\frac{x}{y}\right)^k y^{k+l} = y^d \left\{ \sum_{k+l=d} c_{kl} \left(\frac{x}{y}\right)^k + y \cdot \Omega\left(\frac{x}{y}, y\right) \right\} \\ &= y^d \left\{ \varphi\left(\frac{x}{y}\right) + y \cdot \Omega\left(\frac{x}{y}, y\right) \right\}, \end{aligned}$$

where $\varphi(T) = \sum_{k+l=d} c_{kl} T^k$, and hence

$$\begin{aligned} \widetilde{a}_{p^e} &= \frac{\pi^*(a_{p^e})}{y^{p^e}} = y^{d-p^e} \left\{ \varphi\left(\frac{x}{y} - c + c\right) + y \cdot \Omega\left(\frac{x}{y} - c + c, y\right) \right\} \\ &= \widetilde{y}^{d-p^e} \left\{ \varphi(\widetilde{x} + c) + \widetilde{y} \cdot \Omega(\widetilde{x} + c, \widetilde{y}) \right\}. \end{aligned}$$

Moreover, just as in the analysis of the “*Case: P is a bad point*” in Lemma 4, we see that \widetilde{h} is well-adapted at ξ_{E_P} with respect to $(\widetilde{x}, \widetilde{y}, \widetilde{z})$, since

$$\begin{aligned} \text{Slope}_{\widetilde{h}, (\widetilde{x}, \widetilde{y}, \widetilde{z})}(\xi_{H_{\widetilde{y}}}) &= \frac{1}{p^e} \text{ord}_{\xi_{H_{\widetilde{y}}}}(\widetilde{a}_{p^e}) = \frac{d - p^e}{p^e} \\ &= \frac{1}{p^e} \text{ord}_P(a_{p^e}) - 1 \underset{\text{since } P \text{ is bad}}{<} \mu(P) - 1 = \mu(\xi_{H_{\widetilde{y}}}), \end{aligned}$$

and since $\text{In}_{\xi_{H_{\widetilde{y}}}}(\widetilde{a}_{p^e}) = \widetilde{y}^{d-p^e} \varphi(\widetilde{x} + c)$ is not a p^e -th power, which is obvious if $d \not\equiv 0 \pmod{p^e}$ and which follows from the fact $\text{In}_P(a_{p^e}) = \sum_{k+l=d} c_{kl} x^k y^l$ is not a p^e -th power and hence $\varphi(T) = \sum_{k+l=d} c_{kl} T^k$ is not a p^e -th power if $d \equiv 0 \pmod{p^e}$.

Therefore, we conclude

$$\begin{aligned} \rho_{\widetilde{y}=e} &= \rho_{\widetilde{h}, (\widetilde{x}, \widetilde{y}, \widetilde{z}), H_{\widetilde{y}}} = \begin{cases} \text{ord}_{\widetilde{x}}(\varphi(\widetilde{x} + c)) / p^e & \text{if } d \not\equiv 0 \pmod{p^e} \\ \text{res-ord}_{\widetilde{x}}^{(p^e)}(\varphi(\widetilde{x} + c)) / p^e & \text{if } d \equiv 0 \pmod{p^e} \end{cases} \\ &\leq \frac{1}{p^e} \text{deg } \varphi(\widetilde{y} + c) = \frac{1}{p^e} \text{deg } \varphi(y) = \frac{1}{p^e} \text{deg} \left(\sum_{k+l=d} c_{kl} T^k \right) < \frac{\alpha}{a} = \mu_x. \end{aligned}$$

(Note that, even under the condition $\widetilde{P} \in \pi^{-1}(P) \cap \text{Sing}(\widetilde{\mathcal{R}}) \subset \widetilde{W}$ and the assumption that the invariant σ stays the same, there is a possibility that we may end up having $\text{ord}_{\widetilde{P}}(\widetilde{a}_{p^e}) = p^e$ and $\text{In}_{\widetilde{P}}(\widetilde{a}_{p^e}) = c\widetilde{y}^{p^e}$ for some

$c \in k \setminus \{0\}$. (In this case, we necessarily have $r = 0$.) Then we have to replace $(\tilde{x}, \tilde{y}, \tilde{z})$ with $(\tilde{x}, \tilde{y}, \tilde{z}' = \tilde{z} + c^{1/p^e} \tilde{y})$ to guarantee condition (1) in SITUATION in §5.1. Accordingly, we have to analyze \tilde{a}_{p^e}' . It is straightforward, however, to see that the same calculations hold with \tilde{a}_{p^e}' .)

(3) Take (as described in SITUATION in §5.1)

$$h = z^{p^e} + a_1 z^{p^e-1} + a_2 z^{p^e-2} + \dots + a_{p^e-1} z + a_{p^e}$$

which is well-adapted at P and ξ_{H_x} simultaneously with respect to (x, y, z) (cf. Proposition 5).

Set

$$a_{p^e} = x^r \{g(y) + x \cdot \omega(x, y)\} \text{ with } 0 \neq g(y) \in k[[y]], \omega(x, y) \in k[[x, y]].$$

Set

$$M = \begin{cases} \text{ord}_P(g(y)) & \text{in case } r \not\equiv 0 \pmod{p^e}, \\ \text{res-ord}_P^{(p^e)}(g(y)) & \text{in case } r \equiv 0 \pmod{p^e} \end{cases}$$

so that

$$\rho_x = \rho_{h, (x, y, z), H_x}(P) = \frac{M}{p^e}.$$

Now we compute

$$\begin{aligned} \frac{r + M}{p^e} &\geq \frac{1}{p^e} \text{ord}_P(a_{p^e}) \underset{\text{since } P \text{ is good}}{\geq} \mu(P) = \frac{\alpha + \beta}{a} = \mu(\xi_{H_x}) + \frac{\beta}{a} \\ &\underset{\text{since } H_x \text{ is bad}}{>} \text{H}(\xi_{H_x}) + \frac{\beta}{a} = \frac{r}{p^e} + \frac{\beta}{a}. \end{aligned}$$

Therefore, we conclude

$$\rho_x = \frac{M}{p^e} > \frac{\beta}{a} = \mu(\xi_{H_y}) = \mu_y.$$

The proof for the inequality $\rho_y > \mu_x$ is identical.

This completes the proof of Claim 2.

Q.E.D.

Theorem 1. *Let $P \in \text{Sing}(\mathcal{R}) \subset W$ be a point in the monomial case as described in SITUATION in §5.1. Let $W \xleftarrow{\pi} \widetilde{W}$ be the blow up with center C specified by the procedure described in §5.3, and $\widetilde{\mathcal{R}}$ the transformation of the idealistic filtration \mathcal{R} of i.f.g. type.*

Then at $\tilde{P} \in \pi^{-1}(P) \in \widetilde{W}$, one of the following holds:

- $\tilde{P} \notin \text{Sing}(\widetilde{\mathcal{R}})$, i.e., $\text{Sing}(\widetilde{\mathcal{R}}) = \emptyset$ in a neighborhood of \tilde{P} ,

- $\tilde{P} \in \text{Sing}(\tilde{R})$ and the invariant σ drops, or
- $\tilde{P} \in \text{Sing}(\tilde{R})$ and the invariant “ $\text{inv}_{\text{MON}}(P)$ ” strictly decreases, i.e., $\text{inv}_{\text{MON}}(P) > \text{inv}_{\text{MON}}(\tilde{P})$.

Since the invariant “ inv_{MON} ” cannot decrease infinitely many times, the procedure described in §5.3 terminates either with $\tilde{P} \notin \text{Sing}(\tilde{R})$ or with the drop of the invariant σ after finitely many repetitions.

That is to say, the basic strategy to show termination of the procedure in the monomial case with $\tau = 1$ is established.

Proof. We have only to show (cf. Proposition 4) that, at $\tilde{P} \in \pi^{-1}(P) \in \tilde{W}$, assuming $\tilde{P} \in \text{Sing}(\tilde{R})$ and $\sigma(P) = \sigma(\tilde{P})$, we have

$$\text{inv}_{\text{MON}}(P) > \text{inv}_{\text{MON}}(\tilde{P}).$$

Case: $\dim C = 1$.

In this case, by Claim 1, it is easy to see that P and \tilde{P} are in the same configuration and

$$\text{inv}_{\text{MON}}(P) > \text{inv}_{\text{MON}}(\tilde{P}).$$

Case: $\dim C = 0$, i.e., $C = P$.

1. P is in configuration ① or ②

Since $P \in \text{Sing}(\mathcal{R})$, we have $h_x = \mu_x \geq 1$ in configuration ①, and we do not choose a point center $C = P$ in this case (cf. Proposition 6). So we only have to consider configuration ②, where we have $\mu_x = h_x < 1$ and $\mu_y = h_y < 1$. This implies $\mu_e < \min\{\mu_x, \mu_y\}$. Moreover, it is easy to see that P is necessarily a good point, hence E_P is also good, and that \tilde{P} is in configuration ②, since we assume $\tilde{P} \in \text{Sing}(\tilde{R})$. Now it is straightforward to see

$$\text{inv}_{\text{MON}}(P) > \text{inv}_{\text{MON}}(\tilde{P}).$$

2. P is in configuration ③

(i) $\tilde{P} = (E_P \setminus H'_x) \cap Z'$ with E_P being bad, and hence \tilde{P} is in configuration ③. In this subcase, we have $\rho_x \geq \rho_e$ by Claim 2 (1.2), while $\mu_x > \mu_x - 1 = \mu_e$. Therefore, we conclude

$$\text{inv}_{\text{MON}}(P) = (\rho_x, 0, \mu_x) > (\rho_e, 0, \mu_e) = \text{inv}_{\text{MON}}(\tilde{P}).$$

(ii) $\tilde{P} = E_P \cap H'_x \cap Z'$ with E_P being good, and hence \tilde{P} is in configuration ④. In this subcase, we have $\rho_x > \rho_{\tilde{x}}$ by Claim 2 (1.1). Therefore, we conclude

$$\text{inv}_{\text{MON}}(P) = (\rho_x, 0, \mu_x) > (\min\{\rho_{\tilde{x}}, \mu_{\tilde{x}}\}, \max\{\rho_{\tilde{x}}, \mu_{\tilde{x}}\}) = \text{inv}_{\text{MON}}(\tilde{P}).$$

(iii) $\tilde{P} = E_P \cap H'_x \cap Z'$ with E_P being bad, and hence \tilde{P} is in configuration ⑤. In this case, we have $\rho_x > \rho_{\tilde{x}}$ by Claim 2 (1.1). Therefore, we conclude

$$\text{inv}_{\text{MON}}(P) = (\rho_x, 0, \mu_x) > (\min\{\rho_{\tilde{x}}, \rho_{\tilde{y}}\}, \max\{\rho_{\tilde{x}}, \rho_{\tilde{y}}\}) = \text{inv}_{\text{MON}}(\tilde{P}).$$

3. P is in configuration ④

(i) $\tilde{P} = (E_P \setminus (H'_x \cup H'_y)) \cap Z'$ with E_P being bad, and hence \tilde{P} is in configuration ③. In this case, we have $\rho_x \geq \rho_e$ and $\mu_x > \rho_e (\geq 0)$ by Claim 2 (1.2) and (2). Therefore, we conclude

$$\text{inv}_{\text{MON}}(P) = (\min\{\rho_x, \mu_x\}, \max\{\rho_x, \mu_x\}) > (\rho_e, 0, \mu_e) = \text{inv}_{\text{MON}}(\tilde{P}).$$

(ii) $\tilde{P} = E_P \cap H'_x \cap Z'$ with E_P being good, and hence \tilde{P} is in configuration ④. In this case, we have $\rho_x > \rho_{\tilde{x}}$ by Claim 2 (1.1), while $\mu_x = \mu_{\tilde{x}}$. Therefore, we conclude

$$\begin{aligned} \text{inv}_{\text{MON}}(P) &= (\min\{\rho_x, \mu_x\}, \max\{\rho_x, \mu_x\}) \\ &> (\min\{\rho_{\tilde{x}}, \mu_{\tilde{x}}\}, \max\{\rho_{\tilde{x}}, \mu_{\tilde{x}}\}) = \text{inv}_{\text{MON}}(\tilde{P}). \end{aligned}$$

(iii) $\tilde{P} = E_P \cap H'_y \cap Z'$ with E_P being bad, and hence \tilde{P} is in configuration ④. In this case, we have $\rho_x \geq \rho_e$ by Claim 2 (1.2), while $\mu_x = \alpha/a > (\alpha + \beta)/a - 1 = \mu_e$ since $\beta/a = \mu_y = h_y < 1$. Therefore, we conclude

$$\begin{aligned} \text{inv}_{\text{MON}}(P) &= (\min\{\rho_x, \mu_x\}, \max\{\rho_x, \mu_x\}) \\ &> (\min\{\rho_e, \mu_e\}, \max\{\rho_e, \mu_e\}) = \text{inv}_{\text{MON}}(\tilde{P}). \end{aligned}$$

(iv) $\tilde{P} = E_P \cap H'_x \cap Z'$ with E_P being bad, and hence \tilde{P} is in configuration ⑤. In this case, we have $\rho_x > \rho_{\tilde{x}}$ and $\mu_x > \rho_{\tilde{y}=e}$ by Claim 2 (1.1) and (2). Therefore, we conclude

$$\begin{aligned} \text{inv}_{\text{MON}}(P) &= (\min\{\rho_x, \mu_x\}, \max\{\rho_x, \mu_x\}) \\ &> (\min\{\rho_{\tilde{x}}, \rho_{\tilde{y}}\}, \max\{\rho_{\tilde{x}}, \rho_{\tilde{y}}\}) = \text{inv}_{\text{MON}}(\tilde{P}). \end{aligned}$$

4. P is in configuration ⑤

(i) $\tilde{P} = (E_P \setminus (H'_x \cup H'_y)) \cap Z'$ with E_P being bad, and hence \tilde{P} is in configuration ③. In this case, we have $\rho_x \geq \rho_e$ and $\rho_y \geq \rho_e$ by Claim 2 (1.2). We also have $\rho_x, \rho_y > 0$ since $h_y, h_x < 1$ with $P \in \text{Sing}(\mathcal{R})$. Therefore, we conclude

$$\text{inv}_{\text{MON}}(P) = (\min\{\rho_x, \rho_y\}, \max\{\rho_x, \rho_y\}) > (\rho_e, 0, \mu_e) = \text{inv}_{\text{MON}}(\tilde{P}).$$

(ii) $\tilde{P} = E_P \cap H'_x \cap Z'$ with E_P being good, and hence \tilde{P} is in configuration ④. In this case, we have $\rho_x > \rho_{\tilde{x}}$ and $\rho_y > \mu_{\tilde{x}}$ by Claim 2 (1.1) and (3). Therefore, we conclude

$$\begin{aligned} \text{inv}_{\text{MON}}(P) &= (\min\{\rho_x, \rho_y\}, \max\{\rho_x, \rho_y\}) \\ &> (\min\{\rho_{\tilde{x}}, \mu_{\tilde{x}}\}, \max\{\rho_{\tilde{x}}, \mu_{\tilde{x}}\}) = \text{inv}_{\text{MON}}(\tilde{P}). \end{aligned}$$

(iii) $\tilde{P} = E_P \cap H'_x \cap Z'$ with E_P being bad, and hence \tilde{P} is in configuration ⑤. In this case, we have $\rho_x > \rho_{\tilde{x}}$ and $\rho_y \geq \rho_{\tilde{y}=e}$ by Claim 2 (1.1) and (1.2). Therefore, we conclude

$$\begin{aligned} \text{inv}_{\text{MON}}(P) &= (\min\{\rho_x, \rho_y\}, \max\{\rho_x, \rho_y\}) \\ &> (\min\{\rho_{\tilde{x}}, \rho_{\tilde{y}}\}, \max\{\rho_{\tilde{x}}, \rho_{\tilde{y}}\}) = \text{inv}_{\text{MON}}(\tilde{P}). \end{aligned}$$

This completes the proof of Theorem 1.

Q.E.D.

This completes the detailed discussion of the monomial case in dimension 3, and hence completes the presentation of (the local version of) our algorithm in dimension 3.

We finish this paper by making a couple of remarks.

Remark 9 (Invariant whose maximum locus determines the center of blow up in the monomial case?). The invariant “ inv_{MON} ” is only used to show effectively the termination of the procedure (in the monomial case), while the choice of the center is dictated by the study of the dimension of the singular locus (cf. Proposition 6 and the description of the procedure in 5.3). Actually all the existing algorithms, including the one in [5], use the analysis of the dimension of the singular locus for the choice of the center.

It would be desirable to have an invariant (manifested as the invariant Γ in characteristic zero), which satisfies the following properties:

- (1) it is upper semi-continuous,
- (2) its maximum locus determines the nonsingular center of blow up for constructing the sequence of transformations for resolution of singularities, and
- (3) it strictly drops after each blow up (over the center), and can not strictly decrease infinitely many times.

Such an invariant would not only show the termination effectively but also dictate the choice of the center.

Remark 10.

(1)(**Global version**) The presentation of our algorithm in this paper is restricted to the local version (cf. Remark 2). However, it is not so difficult, though technical and rather lengthy, to make some adjustments in order for us to turn the local version into the global version. The detailed discussion of these adjustments will be published elsewhere.

(2)(**Embedded resolution of singularities of a surface $X \subset W$ with $\dim W = 3$**) Consider a surface $X \subset W$ embedded in a nonsingular ambient space W of $\dim W = 3$. Then we can establish embedded resolution of singularities for $X \subset W$ (cf. Problem 2) in the following manner:

(i) The problem of embedded resolution of singularities for $X \subset W$ is reduced to the problem of resolution of singularities for the triplet $(W, (\mathcal{I}_X, 1), \emptyset)$ (cf. Lemma 1).

(ii) The problem of resolution of singularities for the triplet $(W, (\mathcal{I}_X, 1), \emptyset)$ is equivalent to the problem of resolution of singularities for $(W, \mathcal{R}, \emptyset)$ where the idealistic filtration of i.f.g.type is given by $\mathcal{R} = \mathcal{G}((\mathcal{I}_X, 1))$ (cf. §2. Overview).

(iii) The problem of resolution of singularities for $(W, \mathcal{R}, \emptyset)$ is solved by the global version of our algorithm in $\dim W = 3$.

(3)(**Embedded resolution of singularities of a surface $X \subset W$ with $\dim W$ arbitrary**) We can establish the global version of our algorithm for resolution of singularities of the triplet (W, \mathcal{R}, E) with $\dim W$ arbitrary, as long as it satisfies the condition $\tau(P) \geq \dim W - 2$ for all $P \in W$. As an application, we can give an alternative proof for embedded resolution of a surface $X \subset W$ with $\dim W$ arbitrary, a theorem established by [4] and [11]. We only describe the outline of our alternative proof below (while the details will be published elsewhere):

(i) We consider the Hilbert-Samuel function HS over X , and its maximum value $M = \max_{x \in X} \{\text{HS}(x)\}$.

(ii) Construct a covering $X = \bigcup_{\lambda \in \Lambda} U_\lambda$, and a pair $(\mathcal{I}_\lambda, a_\lambda)$ of an ideal \mathcal{I}_λ over U_λ with level $a_\lambda \in \mathbb{Z}_{>0}$ for each $\lambda \in \Lambda$ (in étale topology using the result due to Aroca (cf. [13]) or in Zariski topology using the result by [2]) satisfying the following property:

(♠) the singular locus of the pair coincides with the maximum locus of the Hilbert-Samuel function, i.e.,

$$\text{Sing}(\mathcal{I}_\lambda, a_\lambda) = \{P \in U_\lambda \mid \text{HS}(P) = M\},$$

and this relation persists through *any* sequence of transformations (and smooth morphisms).

(iii) For each $\lambda \in \Lambda$, consider the idealistic filtration of i.f.g.type $\mathcal{R}'_\lambda = \mathcal{G}(\mathcal{I}_\lambda, a_\lambda)$. We take its \mathcal{R} adical- $\&$ - \mathcal{D} ifferential saturation $\mathcal{R}_\lambda := \mathcal{RD}(\mathcal{R}'_\lambda)$ (cf. [18]). Then by the result of [6] we see that the \mathcal{R}_λ 's patch together, i.e., $\mathcal{R}_\lambda|_{U_\lambda \cap U_\mu} = \mathcal{R}_\mu|_{U_\mu \cap U_\lambda}$, to give rise to the idealistic filtration \mathcal{R} over W such that $\mathcal{R}|_{U_\lambda} = \mathcal{R}_\lambda$ for all $\lambda \in \Lambda$, that $\text{Sing}(\mathcal{R}) = \{P \in W \mid \text{HS}(P) = M\}$, and that this relation persists through any sequence of transformations (and smooth morphisms). Note that the \mathcal{R} - $\&$ - \mathcal{D} saturation gives the “biggest” idealistic filtration of i.f.g.type with the property (\spadesuit).

(iv) Observe that, for this idealistic filtration \mathcal{R} , we have $\tau(P) \geq \dim W - 2$ for all $P \in W$ by the result in [7][8].

(v) Apply the global version of our algorithm to obtain resolution of singularities for $(W, \mathcal{R}, E_M = \emptyset)$, which implies the strict decrease of (the maximum value of) the Hilbert-Samuel function.

(vi) Repeat the procedure. (Note that in the middle of the repetition of the procedure, the boundary E_M , which is the union of all the exceptional divisors created so far, may not be empty.)

(vii) Since the value of the Hilbert-Samuel function can not strictly decrease infinitely many times, the procedure must come to an end after finitely many repetitions, providing embedded resolution of singularities for $X \subset W$.

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