Multiplier ideals via Mather discrepancy

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Dedicated to Professor Shigefumi Mori on the occasion of his 60th birthday

Abstract.

We define a version of multiplier ideals, the *Mather multiplier ideals*, on a variety with arbitrary singularities, using the Mather discrepancy and the Jacobian ideal. In this context we prove a relative vanishing theorem, thus obtaining restriction, subadditivity and summation theorems. The Mather multiplier ideals also satisfy a Skoda type result. As an application, we obtain a Briançon-Skoda type formula for the integral closures of ideals on a variety with arbitrary singularities.

§1. Introduction

The main goal of this paper is to introduce and prove the basic properties of *Mather multiplier ideals*, a variant of the usual multiplier ideals which have found wide applications recently in higher-dimensional algebraic geometry. Typically, the definition of multiplier ideals is given in terms of a suitable resolution of singularities, and an ingredient that plays a fundamental role is the discrepancy (or relative canonical divisor). In order for the discrepancy to be well-defined, one needs to make some assumptions on the singularities, namely to assume that the ambient variety is normal and $\mathbf{Q}$-Gorenstein. The main advantage of Mather multiplier ideals is that they can be defined on any variety (by which we mean an integral scheme of finite type over an algebraically closed field of characteristic zero). This agrees with the usual multiplier ideals on normal, locally complete intersection varieties.

Given an arbitrary variety X and a nonzero ideal $\mathfrak{a}$ on X, we consider a log resolution $f \colon Y \longrightarrow X$ for the product of $\mathfrak{a}$ with the Jacobian ideal

$\mathcal{J}ac_X$ of X (see §1 for the relevant definitions). Such a resolution has the property that the image of the canonical map $f^*(\Omega_X^n) \longrightarrow \Omega_Y^n$ (where $n = \dim(X)$) can be written as $\mathcal{O}_Y(-\widehat{K}_{Y/X}) \cdot \Omega_Y^n$, for some effective divisor $\widehat{K}_{Y/X}$ on Y. This is the *Mather discrepancy divisor*, and if we write $\mathcal{J}ac_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$, then the difference $\widehat{K}_{Y/X} - J_{Y/X}$ plays the role that the discrepancy plays in the usual definition of multiplier ideals. More precisely, if $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-Z_{Y/X})$ and if $t \in \mathbf{R}_{\geq 0}$, then the Mather multiplier ideal of $\mathfrak{a}$ with exponent t is given by

$$\widehat{\mathcal{J}}(X, \mathfrak{a}^t) := f_*\mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} - \lfloor tZ_{Y/X} \rfloor).$$

A priori this is only a fractional ideal, but we show that it is, in fact, contained in $\mathcal{O}_X$. If X is normal and locally complete intersection, then the difference $\widehat{K}_{Y/X} - J_{Y/X}$ is equal to the usual discrepancy $K_{Y/X}$, hence the Mather multiplier ideals on X agree with the usual multiplier ideals.

The importance of multiplier ideals comes from the role they play in connection with vanishing theorems (see [11, Chapter 9]). The basic result that accounts for this connection is a relative vanishing theorem. We prove a similar result in the context of Mather multiplier ideals. With the above notation, this says that we have the vanishing of all higher direct images:

$$R^i f_*\mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} - \lfloor tZ_{Y/X} \rfloor) = 0 \text{ for all } i > 0.$$

As in the "classical" case, this leads to some important properties of Mather multiplier ideals: restriction theorems (we prove three such results, in various settings), a Subadditivity Theorem and a Summation Theorem, as well as a Skoda-type Theorem (see [11] for the "classical" statements of these results). In proving these results, we make use of the approach developed by Eugene Eisenstein in [5] for proving a version of the Restriction Theorem, and a Subadditivity Theorem for multiplier ideals on singular varieties. We obtain as a corollary the following analogue of the Briançon-Skoda Theorem on arbitrary varieties: if $\mathfrak{a}$ is an ideal on X, and $\dim(X) = n$, then the integral closure of $\mathcal{J}ac_X \cdot \mathfrak{a}^m$ is contained in $\mathfrak{a}^{m-n+1}$ for every $m \geq n$.

The notion of Mather discrepancy is suggested by the approach to singularities via spaces of arcs (see [6]). This replacement for the usual discrepancy was first introduced in [3], where it was applied to relate divisorial valuations to contact loci in arc spaces for singular varieties. It was subsequently used in [8] to define new versions of some familiar invariants of singularities (such as the log canonical threshold and the

minimal log discrepancies) on arbitrary varieties. Around the same time that our present work was done, related ideas have been developed in the paper [4] of Tommaso de Fernex and Roi Docampo, in which the authors also undertake a study of invariants of singularities defined from the point of view of Mather discrepancies, and relate these to rational and du Bois singularities.

Acknowledgement We are grateful to Takehiko Yasuda and Orlando Villamayor for useful conversations with the second author on Nash blow-ups and factorizing resolutions, respectively.

§2. Definition of Mather multiplier ideal

Recall that all our varieties are assumed to be irreducible and reduced, over a fixed algebraically closed field k of characteristic zero. We begin by recalling the definition of Nash blow-up and Mather discrepancy, following [3].

Let X be an n-dimensional variety. The sheaf $\Omega_X^n = \wedge^n \Omega_X$ is invertible over the smooth locus X_{reg} of X, hence the morphism

$$\pi \colon \mathbf{P}(\Omega_X^n) = \mathcal{P}roj(\mathrm{Sym}(\Omega_X^n)) \longrightarrow X$$

is an isomorphism over X_{reg}. The *Nash blow-up* $\widehat{X}$ is the closure of $\pi^{-1}(X_{\mathrm{reg}})$ in $\mathbf{P}(\Omega_X^n)$ (with the reduced scheme structure). Note that by construction we have a projective birational morphism $\nu \colon \widehat{X} \longrightarrow X$ and a surjective morphism $\nu^*(\Omega_X^n) \longrightarrow \mathcal{O}_{\mathbf{P}(\Omega_X^n)}(1)|_{\widehat{X}}$. Furthermore, this is universal in the following sense: given a birational morphism of varieties $\phi \colon Y \longrightarrow X$, this factors through ν if and only if $\phi^*(\Omega_X^n)$ admits a morphism onto an invertible sheaf $\mathcal{L}$ on Y (which identifies $\mathcal{L}$ to the quotient of $\phi^*(\Omega_X^n)$ by its torsion).

Recall that a *resolution of singularities* of X is a projective birational morphism $f \colon Y \longrightarrow X$, with Y smooth. This is a *log resolution* of a nonzero ideal $\mathfrak{a}$ on X if $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D)$ for an effective divisor D, the exceptional locus of f is an effective divisor E, and $D \cup E$ has simple normal crossings. Given several ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$, a log resolution of the product $\mathfrak{a}_1 \cdot \ldots \cdot \mathfrak{a}_r$ gives a simultaneous log resolution of all $\mathfrak{a}_i$, such that the divisors corresponding to each of the $\mathfrak{a}_i$ and the exceptional divisor have simple normal crossings.

Definition 2.1 ([3])**.** *Let X be a variety of dimension n and $f \colon Y \longrightarrow X$ a resolution of singularities that factors through the Nash blow-up of X. In this case the image of the canonical homomorphism*

$$f^*(\Omega_X^n) \longrightarrow \Omega_Y^n$$

is an invertible sheaf of the form $J \cdot \Omega_Y^n$, where J is the invertible ideal sheaf on Y that defines an effective divisor supported on the exceptional locus of f. This divisor is called the Mather discrepancy divisor *and* we denote it by $\widehat{K}_{Y/X}$. We refer to [3] for further details about Nash blow-ups and Mather discrepancies.

Definition 2.2. *Recall that the* Jacobian ideal $\mathcal{J}ac_X$ *of a variety X is the nth Fitting ideal* $\mathrm{Fitt}_n(\Omega_X)$ *of Ω_X. This is an ideal whose support is the singular locus of X. If $f: Y \longrightarrow X$ is a log resolution of $\mathcal{J}ac_X$, we denote by $J_{Y/X}$ the effective divisor on Y such that $\mathcal{J}ac_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$.*

Remark 2.1. Every log-resolution of $\mathcal{J}ac_X$ factors through the Nash blow-up. Indeed, it follows from [12, Lemma 1] that given a proper birational morphism $\varphi: Y \longrightarrow X$ of varieties of dimension n, the following are equivalent:

(1)   The smallest Fitting ideal $\mathrm{Fitt}_n(\varphi^*\Omega_X)$ is invertible;
(2)   The projective dimension $\mathrm{pd}(\varphi^*\Omega_X) \leq 1$ and $\varphi^*\Omega_X/(\varphi^*\Omega_X)^{tor}$ is locally free of rank n, where $(\varphi^*\Omega_X)^{tor}$ is the torsion part of $\varphi^*\Omega_X$.

If φ is a log resolution of $\mathcal{J}ac_X$, then $\mathrm{Fitt}_n(\varphi^*\Omega_X) = \mathcal{J}ac_X \cdot \mathcal{O}_Y$ is invertible. Therefore by Lipman's result $\varphi^*\Omega_X/(\varphi^*\Omega_X)^{tor}$ is locally free of rank n, and we deduce that φ factors through $\widehat{X}$ by the universal property of the Nash blow-up.

Definition 2.3. *Let X be a variety, $\mathfrak{a} \subseteq \mathcal{O}_X$ a nonzero ideal on X, and $t \in \mathbf{R}_{\geq 0}$. Given a log resolution $f: Y \longrightarrow X$ of $\mathcal{J}ac_X \cdot \mathfrak{a}$, we denote by $Z_{Y/X}$ the effective divisor on Y such that $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-Z_{Y/X})$. The* Mather multiplier ideal *of $\mathfrak{a}$ of exponent t is defined by*

$$\widehat{\mathcal{J}}(X, \mathfrak{a}^t) = f_* \left(\mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} - \lfloor tZ_{Y/X} \rfloor) \right),$$

where for a real divisor D we denote by $\lfloor D \rfloor$ the largest integral divisor $\leq D$.

Remark 2.2. The Mather multiplier ideal does not depend on the choice of f. Indeed, arguing as in the proof of [11, Theorem 9.2.18], we see that it is enough to show that if $g: Y' \longrightarrow Y$ is such that $f' = f \circ g$ is another log resolution of $\mathcal{J}ac_X \cdot \mathfrak{a}$, then
(1)
$$\mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} - \lfloor tZ_{Y/X} \rfloor) = g_* \left(\mathcal{O}_{Y'}(\widehat{K}_{Y'/X} - J_{Y'/X} - \lfloor tZ_{Y'/X} \rfloor) \right).$$

Since $\widehat{K}_{Y'/X} = g^*(\widehat{K}_{Y/X}) + K_{Y'/Y}$, where $K_{Y'/Y}$ is the (usual) discrepancy of h, and $J_{Y'/X} = g^*(J_{Y/X})$ and $Z_{Y'/X} = g^*(Z_{Y/X})$, then (1)

follows from [11, Lemma 9.2.19], which is the main ingredient in showing that the usual definition of multiplier ideals is independent of log resolution.

Remark 2.3. For nonzero ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ on a variety X, one can similarly define a mixed Mather multiplier ideal $\widehat{\mathcal{J}}(X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r})$ for every $t_1, \ldots, t_r \in \mathbf{R}_{\geq 0}$. With the notation in Definition 2.3, if f is a log resolution of $\mathcal{J}ac_X \cdot \mathfrak{a}_1 \cdot \ldots \cdot \mathfrak{a}_r$, and if we put $\mathfrak{a}_i \cdot \mathcal{O}_Y = \mathcal{O}_Y(-Z_i)$, then

$$\widehat{\mathcal{J}}(X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r}) = f_* \left(\mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} - \lfloor t_1 Z_1 + \ldots + t_r Z_r \rfloor) \right).$$

For simplicity, we will mostly consider Mather multiplier ideals for one ideal, but all statements have obvious generalizations to the mixed case.

Remark 2.4. If X is normal and locally a complete intersection, then $\widehat{\mathcal{J}}(X, \mathfrak{a}^t) = \mathcal{J}(X, \mathfrak{a}^t)$, where the right hand side is the usual multiplier ideal (see [11, Chapter 9] for definition). Indeed, in this case the image of the canonical map $\Omega_X^n \longrightarrow \omega_X$ is $\mathcal{J}ac_X \cdot \omega_X$, hence $\widehat{K}_{Y/X} - J_{Y/X} = K_{Y/X}$. In particular, we see that $\widehat{\mathcal{J}}(X, \mathfrak{a}^t) = \mathcal{J}(X, \mathfrak{a}^t)$ if X is smooth.

Remark 2.5. For now, the Mather multiplier ideal is only a fractional ideal of $\mathcal{O}_X$, with $\widehat{\mathcal{J}}(X, \mathfrak{a}^t) \subset i_*(\mathcal{O}_{X_{reg}})$, where $i \colon X_{reg} \hookrightarrow X$ is the inclusion of the smooth locus. This inclusion implies that $\widehat{\mathcal{J}}(X, \mathfrak{a}^t)$ is an ideal of $\mathcal{O}_X$ if X is normal. We will see in Corollary 2.10 that, in fact, the same is true on every variety X.

The assertions in the next proposition are an immediate consequence of the definition.

Proposition 2.6. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be nonzero ideals on the variety X.*
(1) *If $\mathfrak{a} \subseteq \mathfrak{b}$, then $\widehat{\mathcal{J}}(X, \mathfrak{a}^t) \subseteq \widehat{\mathcal{J}}(X, \mathfrak{b}^t)$ for every $t \in \mathbf{R}_{\geq 0}$.*
(2) *If $s \leq t$ are in $\mathbf{R}_{\geq 0}$, then $\widehat{\mathcal{J}}(X, \mathfrak{a}^t) \subseteq \widehat{\mathcal{J}}(X, \mathfrak{a}^s)$.*
(3) *If $t \in \mathbf{R}_{\geq 0}$ and m is a positive integer, then $\widehat{\mathcal{J}}(X, (\mathfrak{a}^m)^t) = \widehat{\mathcal{J}}(X, \mathfrak{a}^{mt})$.*

Definition 2.4. *Let A be a variety and X a reduced closed subscheme of A. A morphism $\varphi_A \colon \overline{A} \longrightarrow A$ is a factorizing resolution of X inside A if the following hold:*
(1) *φ_A is an isomorphism at the generic point of every irreducible component of X. In particular, the strict transform $\overline{X}$ of X is defined.*
(2) *The morphisms φ_A and $\varphi_X := \varphi_A|_{\overline{X}}$ are resolutions of singularities of A and X, respectively, and the union of $\overline{X}$ with the exceptional locus $\mathrm{Exc}(\varphi_A)$ has simple normal crossings.*

(3)   *If I_X and $I_{\overline{X}}$ are the defining ideals of X and $\overline{X}$ in A and $\overline{A}$, respectively, then there exists an effective divisor $R_{X/A}$ on $\overline{A}$ such that*

$$I_X \cdot \mathcal{O}_{\overline{A}} = I_{\overline{X}} \cdot \mathcal{O}_{\overline{A}}(-R_{X/A}).$$

Note that the divisor $R_{X/A}$ in (3) is supported on $\mathrm{Exc}(\phi_A)$, hence by assumption has simple normal crossings with $\overline{X}$. Bravo and Villamayor proved in [1] that if A is smooth and X is reduced and equidimensional, then factorizing resolutions of X in A exist. The following strengthening is due to Eisenstein [5, Lemma 3.1].

Lemma 2.7. *Let $X \subseteq A$ be a reduced closed subscheme of a variety A. Let $\varphi_1 \colon A' \longrightarrow A$ be a birational morphism from a smooth variety A' that is an isomorphism at the generic points of the irreducible components of X. Let X' be the strict transform of X in A', and let E be a divisor on A' with simple normal crossing support, such that no component of X' is contained in E. Then there exists a morphism $\varphi_2 \colon \overline{A} \longrightarrow A'$ such that $\varphi_A := \varphi_1 \circ \varphi_2$ is a factorizing resolution of X inside A, and $\overline{X} \cup \mathrm{Exc}(\varphi_A) \cup \mathrm{Supp}(\varphi_2^* E)$ has simple normal crossings, where $\overline{X}$ is the strict transform of X in $\overline{A}$.*

The following lemma is essentially [5, Lemma 4.4].

Lemma 2.8. *Let $X \subseteq A$ be a reduced equidimensional closed subscheme of a smooth variety A, and let $\varphi_A \colon \overline{A} \longrightarrow A$ be a factorizing resolution of X inside A. Let $c = \mathrm{codim}(X, A)$ and $I_X \cdot \mathcal{O}_{\overline{A}} = I_{\overline{X}} \cdot \mathcal{O}_{\overline{A}}(-R_{X/A})$. If $\varphi_A|_{\overline{X}}$ factors through the blow-up along the ideal $\mathcal{J}ac_X$, then*

$$(K_{\overline{A}/A} - cR_{X/A})|_{\overline{X}} = \widehat{K}_{\overline{X}/X} - J_{\overline{X}/X}.$$

Proposition 2.9. *Let X be a closed subvariety of the smooth variety A, with $\mathrm{codim}(X, A) = c$. If $\tilde{\mathfrak{a}} \subseteq \mathcal{O}_A$ is an ideal on A whose restriction $\mathfrak{a} = \tilde{\mathfrak{a}} \cdot \mathcal{O}_X$ to X is nonzero, then for every $t \in \mathbf{R}_{\geq 0}$*

$$\widehat{\mathcal{J}}(X, \mathfrak{a}^t) \subseteq \mathcal{J}(A, \tilde{\mathfrak{a}}^t) \cdot \mathcal{O}_X.$$

Proof. Let $\mathfrak{b} \subseteq \mathcal{O}_A$ be the pull-back of $\mathcal{J}ac_X$ by the canonical surjection $\mathcal{O}_A \longrightarrow \mathcal{O}_X$. We begin by choosing a log resolution $\varphi_1 \colon A' \longrightarrow A$ of $\mathfrak{b} \cdot \tilde{\mathfrak{a}}$. Let E be the divisor on A' such that $\mathcal{O}_{A'}(-E) = \tilde{\mathfrak{a}}\mathfrak{b} \cdot \mathcal{O}_{A'}$. Applying Lemma 2.7, we obtain a factorizing resolution $\varphi_A \colon \overline{A} \longrightarrow A$ of X inside A, that is also a log resolution of $\mathfrak{b} \cdot \tilde{\mathfrak{a}}$. Let $R_{X/A}$ be the divisor on $\overline{A}$ such that

$$I_X \cdot \mathcal{O}_{\overline{A}} = I_{\overline{X}} \cdot \mathcal{O}_{\overline{A}}(-R_{X/A}),$$

where $\overline{X}$ is the strict transform of X in $\overline{A}$. As φ_A factors through φ_1, the morphism $\varphi_X := \varphi_A|_{\overline{X}}$ is a log resolution of $\mathcal{J}ac_X$. Let $\tilde{\mathfrak{a}} \cdot \mathcal{O}_{\overline{A}} = \mathcal{O}_{\overline{A}}(-Z_{\overline{A}/A})$ and $D = K_{\overline{A}/A} - \lfloor tZ_{\overline{A}/A} \rfloor - cR_{X/A}$. The exact sequence

$$0 \longrightarrow I_{\overline{X}} \cdot \mathcal{O}_{\overline{A}}(D) \longrightarrow \mathcal{O}_{\overline{A}}(D) \longrightarrow \mathcal{O}_{\overline{X}}(D|_{\overline{X}}) \longrightarrow 0$$

yields the exact sequence

$$(2) \quad 0 \longrightarrow \varphi_{A*}(I_{\overline{X}} \cdot \mathcal{O}_{\overline{A}}(D)) \longrightarrow \varphi_{A*}(\mathcal{O}_{\overline{A}}(D)) \longrightarrow \varphi_{X*}(\mathcal{O}_{\overline{X}}(D|_{\overline{X}}))$$
$$\longrightarrow R^1\varphi_{A*}(I_{\overline{X}} \cdot \mathcal{O}_{\overline{A}}(D)).$$

We claim that the last term in this sequence is zero. Indeed, let $\psi \colon \tilde{A} \longrightarrow \overline{A}$ be the blow-up along $\overline{X}$ with exceptional divisor T. If $\mathcal{F} = \mathcal{O}_{\tilde{A}}(-T + \psi^*D)$, then the projection formula gives $\psi_*\mathcal{F} = I_{\overline{X}} \cdot \mathcal{O}_{\overline{A}}(D)$. For $p > 0$, we have

$$R^p\psi_*(\mathcal{F}) = \mathcal{O}_{\overline{A}}(D) \otimes R^p\psi_*(\mathcal{O}_{\tilde{A}}(-T)) = 0, \quad R^p(\varphi_A \circ \psi)_*(\mathcal{F}) = 0.$$

The second vanishing follows from the Local Vanishing Theorem for multiplier ideals (see [11, Theorem 9.4.1]), using the fact that if $I_X \cdot \mathcal{O}_{\tilde{A}} = \mathcal{O}_{\tilde{A}}(-F)$, then we can write

$$-T + \psi^*(D) = K_{\tilde{A}/A} - \lfloor cF + t\psi^*(Z_{\overline{A}/A}) \rfloor.$$

Using the Leray spectral sequence

$$E_2^{p,q} = R^p\varphi_{A*}R^q\psi_*(\mathcal{F}) \Rightarrow R^{p+q}(\varphi_A \circ \psi)_*(\mathcal{F}),$$

we obtain $R^1\varphi_{A*}(\psi_*\mathcal{F}) = 0$, as claimed.

Since $R_{X/A}$ is effective, we have by definition $\varphi_{A*}(\mathcal{O}_{\overline{A}}(D)) \subseteq \mathcal{J}(A, \tilde{\mathfrak{a}}^t)$, hence the exact sequence (2) gives

$$\varphi_{X*}(\mathcal{O}_{\overline{X}}(D|_{\overline{X}})) \subseteq \mathcal{J}(A, \tilde{\mathfrak{a}}^t) \cdot \mathcal{O}_X.$$

By Lemma 2.8 we have

$$\varphi_{X*}(\mathcal{O}_{\overline{X}}(D|_{\overline{X}})) = \varphi_{X*}\left(\mathcal{O}_{\overline{X}}(\widehat{K}_{\overline{X}/X} - J_{\overline{X}/X} - \lfloor tZ_{\overline{X}/X} \rfloor)\right) = \widehat{\mathcal{J}}(X, \mathfrak{a}^t),$$

which completes the proof of the proposition. Q.E.D.

Corollary 2.10. *Given a variety X, a nonzero ideal $\mathfrak{a}$ on X, and $t \in \mathbf{R}_{\geq 0}$, the following hold:*

(1) *The Mather multiplier ideal $\widehat{\mathcal{J}}(X, \mathfrak{a}^t)$ is an ideal of $\mathcal{O}_X$.*
(2) *The ideal $\widehat{\mathcal{J}}(X, \mathfrak{a}^t)$ is integrally closed.*

(3) If $\nu\colon X^\nu \longrightarrow X$ is the normalization of X, then the integral closure $\overline{\mathcal{J}ac_X}$ of $\mathcal{J}ac_X$ satisfies

$$\overline{\mathcal{J}ac_X} \subseteq (\mathcal{O}_X : \nu_*\mathcal{O}_{X^\nu}).$$

Proof. (1) follows from Proposition 2.9, since X can be locally embedded in a smooth variety, and the result is known (and easy to prove) for usual multiplier ideals on smooth varieties. For (2), it is enough to note that if $f\colon Y \longrightarrow X$ is a birational morphism, with Y normal, and E is a Weil divisor on Y such that $\mathcal{I} := f_*\mathcal{O}_Y(E)$ is an ideal of $\mathcal{O}_X$, then $\mathcal{I}$ is integrally closed.

In order to show (3), let $\varphi\colon Y \longrightarrow X$ be a log resolution of $\mathcal{J}ac_X$. We have the inclusions

$$\overline{\mathcal{J}ac_X} \subseteq \varphi_*\mathcal{O}_Y(-J_{Y/X}) \subseteq \varphi_*\mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X}) = \widehat{\mathcal{J}}(X, \mathcal{O}_X) \subseteq \mathcal{O}_X.$$

Since φ factors through ν, it follows that $\varphi_*\mathcal{O}_Y(-J_{Y/X})$ is a $\nu_*\mathcal{O}_{X^\nu}$-module, and we deduce the inclusion in (3). Q.E.D.

Example 2.11. If $X = \operatorname{Spec} k[x^2, x^3]$, then

$$\widehat{\mathcal{J}}(X, \mathcal{O}_X) = (x^2) \subset k[x^2, x^3].$$

Note that X is a curve with a cusp at the origin and has the normalization $\nu\colon \overline{X} = \operatorname{Spec} k[x] \longrightarrow X$. The morphism ν is a resolution of singularities and factors through the Nash blow-up (since the normalization of a curve factors through every blow-up). As $\nu^*\Omega_X$ is generated by dx^2, dx^3, the image of the canonical map $\nu^*\Omega_X \longrightarrow \Omega_{\overline{X}}$ is generated by $x\,dx$ and $x^2\,dx$, hence $\mathcal{O}_{\overline{X}}(-\widehat{K}_{\overline{X}/X}) = (x)$. On the other hand, $\mathcal{J}ac_X = (x^3, x^4) \subset k[x^2, x^3]$ and $\mathcal{J}ac_X\mathcal{O}_{\overline{X}} = (x^3) \subset k[x]$. Therefore $\mathcal{O}_{\overline{X}}(\widehat{K}_{\overline{X}/X} - J_{\overline{X}/X}) = (x^2) \subset k[x]$, so that $\widehat{\mathcal{J}}(X, \mathcal{O}_X) = (x^2)$, as an ideal of $k[x^2, x^3]$.

§3. The local vanishing theorem and applications

We first recall that for every variety X of dimension n we have a canonical morphism $\eta_X\colon \Omega_X^n \longrightarrow \omega_X$, where ω_X is the canonical sheaf of X. This is clear when X is normal, since in that case $\omega_X \simeq i_*(\Omega_{X_{\mathrm{reg}}}^n)$, where $i\colon X_{\mathrm{reg}} \hookrightarrow X$ is the inclusion of the smooth locus. In the general case, let $\nu\colon X' \longrightarrow X$ be the normalization, so we have $\eta_{X'}\colon \Omega_{X'}^n \longrightarrow \omega_{X'}$. Since ν is finite and surjective, we have an isomorphism $\nu_*(\omega_{X'}) \simeq \mathcal{H}om_{\mathcal{O}_X}(\nu_*(\mathcal{O}_{X'}), \omega_X)$, and the inclusion $\mathcal{O}_X \hookrightarrow \nu_*(\mathcal{O}_{X'})$ induces a

canonical morphism $\nu_*(\omega_{X'}) \longrightarrow \omega_X$. We then obtain η_X as the composition

$$\Omega_X^n \longrightarrow \nu_*(\Omega_{X'}^n) \overset{\nu_*(\eta_{X'})}{\longrightarrow} \nu_*(\omega_{X'}) \longrightarrow \omega_X.$$

The following is a consequence of [6, Proposition 9.1].

Lemma 3.1. *Let $X \subseteq \mathbf{A}^N$ be a closed subvariety of dimension n, defined by the ideal $I_X = (f_1, \ldots, f_d)$, and let $c = N - n$. If the subscheme M defined by c functions $f_{i_1}, \ldots, f_{i_c}$ has pure codimension c, and $M = X$ at the generic point of X, then there exists an isomorphism of ω_X with an ideal of $\mathcal{O}_X$*

$$\omega_X \simeq ((I_M : I_X) + I_X)/I_X.$$

Under this isomorphism, the image of the canonical map $\eta_X \colon \Omega_X^n \longrightarrow \omega_X$ corresponds to $\mathcal{J}ac_M|_X$.

Remark 3.2. Suppose that $X \subseteq \mathbf{A}^N$ is as in Lemma 3.1. If $F_i := \sum_{j=1}^d a_{ij} f_j$, with a_{ij} general elements of k, then $I_X = (F_1, \ldots, F_d)$ and every set of distinct c elements $F_{i_1}, \ldots, F_{i_c}$ defines a subscheme M that satisfies the conditions in Lemma 3.1.

Lemma 3.3. *For every n-dimensional affine variety X, there exist a non-zero regular function h on X and ideals $I, G \subseteq \mathcal{O}_X$ such that there is a surjective morphism $\Omega_X^n \longrightarrow I$ and we have*

(3) $$h \cdot \mathcal{J}ac_X = I \cdot G.$$

Proof. We apply Lemma 3.1 using the notation in Remark 3.2. For every subset $J \subseteq \{1, \ldots, d\}$ with c elements, we obtain a subscheme M_J of $\mathbf{A}^N$ defined by $(F_i \mid i \in J)$, that satisfies the conditions in Lemma 3.1. In particular, we obtain an isomorphism $\alpha_J \colon \omega_X \longrightarrow \mathfrak{q}_J \subseteq \mathcal{O}_X$ that maps the image of $\Omega_X^n \longrightarrow \omega_X$ onto $\mathcal{J}ac_{M_J}|_X$.

Let us fix a subset J_0 as above, so α_{J_0} induces a surjective map from Ω_X^n onto $I = \mathcal{J}ac_{M_{J_0}}|_X$. Note that the isomorphism $\alpha_J \circ \alpha_{J_0}^{-1} \colon \mathfrak{q}_{J_0} \longrightarrow \mathfrak{q}_J$ is given by $\alpha_J \circ \alpha_{J_0}(u) = \frac{b_J}{a_J} u$ for some nonzero $a_J, b_J \in \mathcal{O}(X)$ (if $\mathfrak{q}$ is an ideal in a domain R, and $\varphi \colon \mathfrak{q} \longrightarrow R$ is an R-linear map, then for every nonzero $u, v \in \mathfrak{q}$ we have $\frac{\varphi(u)}{u} = \frac{\varphi(u)v}{uv} = \frac{\varphi(uv)}{uv} = \frac{u\phi(v)}{uv} = \frac{\varphi(v)}{v}$). Furthermore, after multiplying each a_J by a suitable factor, we may assume that all a_J are equal, and we denote the common value by h.

In particular, by considering the image of Ω_X^n, we deduce $h \cdot \mathcal{J}ac_{M_J}|_X = b_J \cdot I$ for every J. Since $\mathcal{J}ac_X = \sum_J \mathcal{J}ac_{M_J}|_X$, we obtain (3) by taking $G = \sum_J (b_J)$. Q.E.D.

Remark 3.4. This lemma gives another proof for Remark 2.1.

The following theorem gives an analogue for the Local Vanishing Theorem for multiplier ideals (see [11, Theorem 9.4.1]) in the context of Mather multiplier ideals.

Theorem 3.5. *Let $\mathfrak{a}$ be a nonzero ideal on the variety X, and let $t \in \mathbf{R}_{\geq 0}$. If $f \colon Y \longrightarrow X$ is a log-resolution of $\mathfrak{a} \cdot \mathcal{J}ac_X$, and we put $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-Z_{Y/X})$, then*

$$R^i f_* \left(\mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} - \lfloor tZ_{Y/X} \rfloor) \right) = 0 \ \ for \ all \ i > 0.$$

Proof. Since the assertion is local on X, we may assume that X is affine. It follows from Remark 2.1 that we can factor f through the Nash blow-up as $Y \xrightarrow{\psi} \widehat{X} \xrightarrow{\varphi} X$. Let $\mathcal{O}_{\widehat{X}}(1) = \mathcal{O}_{\mathbf{P}(\Omega_X^n)}(1)|_{\widehat{X}}$. With I as in Lemma 3.3, we have a surjection of graded algebras $\mathrm{Sym}(\Omega_X^n) \longrightarrow \oplus_{i=0}^\infty I^i$. This induces a closed immersion $\mathrm{Bl}_I(X) \hookrightarrow \mathbf{P}(\Omega_X^n)$, hence an isomorphism $\mathrm{Bl}_I(X) \simeq \widehat{X}$ over X, which yields $\mathcal{O}_{\widehat{X}}(1) \simeq I \cdot \mathcal{O}_{\widehat{X}}$.

It follows from [3, Proposition 1.7] that

$$\mathcal{O}_Y(\widehat{K}_{Y/X}) \simeq \mathcal{O}_Y(K_Y) \otimes \psi^*(\mathcal{O}_{\widehat{X}}(1))^{-1} \simeq \mathcal{O}_Y(K_Y) \otimes (I \cdot \mathcal{O}_Y)^{-1}.$$

Since f is a log resolution of $\mathcal{J}ac_X$, we deduce from Lemma 3.3 that we may write $G \cdot \mathcal{O}_Y = \mathcal{O}_Y(-B)$ for a divisor B on Y, and we have

$$\mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} - \lfloor tZ_{Y/X} \rfloor) \simeq \mathcal{O}_Y(K_Y) \otimes (I \cdot \mathcal{O}_Y)^{-1} \otimes IG \cdot \mathcal{O}_Y \otimes \mathcal{O}_Y(-\lfloor tZ_{Y/X} \rfloor)$$

$$\simeq \mathcal{O}_Y(K_Y - B - \lfloor tZ_{Y/X} \rfloor).$$

Since both $-B$ and $-Z_{Y/X}$ are relatively nef over X, the Kawamata-Viehweg Vanishing Theorem (see for example [10, Theorem 1-2-3]) implies the vanishing in the theorem.   Q.E.D.

Remark 3.6. The vanishing in Theorem 3.5 holds, with the same proof, for mixed Mather multiplier ideals. More precisely, if $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ are nonzero ideals on X, and $f \colon Y \longrightarrow X$ is a log resolution of $\mathcal{J}ac_X \cdot \mathfrak{a}_1 \cdot \ldots \cdot \mathfrak{a}_r$ such that $\mathfrak{a}_i \cdot \mathcal{O}_Y = \mathcal{O}_Y(-Z_i)$, then

$$R^i f_* \left(\mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} - \lfloor t_1 Z_1 + \ldots + t_r Z_r \rfloor) \right) = 0$$

for every $t_1, \ldots, t_r \in \mathbf{R}_{\geq 0}$ and every $i > 0$.

According to our principle to replace the usual discrepancy $K_{Y/X}$ by $\widehat{K}_{Y/X} - J_{Y/X}$ we can define "canonical singularities" for a not necessarily **Q**-Gorenstein variety. The paper [4] introduces the notion of J-canonical singularities. For an arbitrary variety X, we say that X has J-canonical

singularities if $\widehat{K}_{Y/X} - J_{Y/X} \geq 0$ for a log resolution $f : Y \longrightarrow X$ of $\mathcal{J}ac_X$ (and automatically for all log resolutions of $\mathcal{J}ac_X$). They prove the following statement ([4, Theorem 7.5]), which we also obtain as a corollary of our results.

Corollary 3.7. *If a variety X has J-canonical singularities, then X is normal and has rational singularities.*

For the proof of the corollary, we use the following generalization of Fujita's vanishing theorem (the original Fujita's vanishing theorem is in [7]).

Proposition 3.8 ([10], Theorem 1-3-1)**.** *Let $f : Y \longrightarrow X$ be a proper birational morphism from a non-singular variety Y onto a variety X with divisors $L, \tilde{L}$ on Y. Assume that there exist $\mathbf{Q}$-divisors $D, \tilde{D}$ and an effective divisor E on Y such that the following conditions are satisfied:*
  (1) *supp D and supp $\tilde{D}$ are divisors with only simple normal crossings and $\lfloor D \rfloor = \lfloor \tilde{D} \rfloor = 0$;*
  (2) *both $-L - D$ and $-\tilde{L} - \tilde{D}$ are f-nef;*
  (3) *$K_Y \sim L + \tilde{L} + E$; and*
  (4) *E is exceptional for f.*
Then, $R^i f_* \mathcal{O}_Y(L) = 0$ for $i > 0$.

Proof of Corollary 3.7. Let $\nu : X^\nu \longrightarrow X$ be the normalization and $f : Y \longrightarrow X$ be a log resolution of $\mathcal{J}ac_X$. By the assumption $\widehat{K}_{Y/X} - J_{Y/X} \geq 0$, we obtain

$$\widehat{\mathcal{J}}(X, \mathcal{O}_X) = f_* \mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X}) \supseteq f_* \mathcal{O}_Y = \nu_* \mathcal{O}_{X^\nu}.$$

Since $\widehat{\mathcal{J}}(X, \mathcal{O}_X)$ is an ideal of $\mathcal{O}_X$, we obtain $X^\nu = X$. Now, we will prove $R^i f_* \mathcal{O}_Y = 0$ for $i > 0$. Note that $\widehat{K}_{Y/X} - J_{Y/X} = K_Y - B$ with $-B$ being f-nef, which we obtained in the proof of Theorem 3.5. Let $D = \tilde{D} = L = 0$, $E = \widehat{K}_{Y/X} - J_{Y/X}$ and $\tilde{L} = B$, then $-L - D = 0$ and $-\tilde{L} - \tilde{D} = -B$ are both f-nef. The condition $K_Y \sim L + \tilde{L} + E$ is also satisfied. Then, we can apply Fujita's vanishing theorem and obtain $R^i f_* \mathcal{O}_Y = 0$ for $i > 0$. $\square$

We apply the above Local Vanishing Theorem (Theorem 3.5) to prove two more restriction results. We say that a proper subscheme X of a variety B is a hyperplane section if B can be embedded as a closed subvariety of a smooth variety A, and if there is a smooth hypersurface H in A such that $X = B \cap H$.

Theorem 3.9 (Restriction theorem for hyperplane sections). *Let X be a closed subvariety of a variety B which is a hyperplane section, such that $X \cap B_{\mathrm{reg}} \neq \emptyset$. If $\tilde{\mathfrak{a}} \subseteq \mathcal{O}_B$ is an ideal on B such that its restriction $\mathfrak{a} = \tilde{\mathfrak{a}} \cdot \mathcal{O}_X$ is non-zero, then for every $t \in \mathbf{R}_{\geq 0}$ we have*

$$\widehat{\mathcal{J}}(X, \mathfrak{a}^t) \subseteq \widehat{\mathcal{J}}(B, \tilde{\mathfrak{a}}^t) \cdot \mathcal{O}_X.$$

Proof. Let $B \hookrightarrow A$ be a closed embedding in a smooth variety, and H a smooth hypersurface in A such that $X = B \cap H$. We denote by $\tilde{\tilde{\mathfrak{a}}}$ and $\mathfrak{b}_B$ the pull back of $\tilde{\mathfrak{a}}$ and $\mathcal{J}ac_B$, respectively, by the canonical projection $\mathcal{O}_A \longrightarrow \mathcal{O}_B$, and by $\mathfrak{b}_X$ the pull back of $\mathcal{J}ac_X$ by the canonical projection $\mathcal{O}_A \longrightarrow \mathcal{O}_X$.

We claim that there is a projective, birational morphism $\varphi_A \colon \overline{A} \longrightarrow A$ that is an isomorphism at the generic points of B, H and X, such that if $\overline{B}$, $\overline{H}$ and $\overline{X}$ denote the proper transforms in $\overline{A}$ of B, H and X, respectively, then the following hold:

  (i) φ_A is a log resolution of $\tilde{\tilde{\mathfrak{a}}} \cdot \mathfrak{b}_B \cdot \mathfrak{b}_X$,
  (ii) φ_A is a factorizing resolution of B, and also of H, inside A,
  (iii) $\varphi_H := \varphi_A|_{\overline{H}} \colon \overline{H} \longrightarrow H$ is a factorizing resolution of X inside H and $\varphi_B := \varphi_A|_{\overline{B}} \colon \overline{B} \longrightarrow B$ is a factorizing resolution of X inside B,
  (iv) φ_B and $\varphi_X := \varphi_A|_{\overline{X}} \colon \overline{X} \longrightarrow X$ are log resolutions of $\tilde{\mathfrak{a}} \cdot \mathcal{J}ac_B$ and $\mathfrak{a} \cdot \mathcal{J}ac_X$, respectively.

We construct φ_A as follows. We first take a log resolution $\varphi' \colon A' \longrightarrow A$ for the product of $\tilde{\tilde{\mathfrak{a}}}$, $\mathfrak{b}_B$, $\mathfrak{b}_X$, and $I_{H/A}$, where $I_{H/A}$ is the defining ideal of H in A. Since $X \cap B_{\mathrm{reg}} \neq \emptyset$, we may and will assume that φ' is an isomorphism at the generic points of B, H and X. Let E be the simple normal crossings divisor on A' such that $\mathcal{O}_{A'}(-E) = \tilde{\tilde{\mathfrak{a}}} \cdot \mathfrak{b}_B \cdot \mathfrak{b}_X \cdot I_{H/A} \cdot \mathcal{O}_{A'}$. We now apply Lemma 2.7 to (A, A', E, B) to obtain a factorizing resolution $\varphi_A \colon \overline{A} \longrightarrow A' \longrightarrow A$ of B inside A. Note that φ_A is also a factorizing resolution of H inside A, since it is an embedded resolution of (A, H) and the factorizing condition (3) in Definition 2.4 is automatic, as H is a hypersurface. Therefore φ_A satisfies condition (ii) above, and conditions (i) and (iv) also follow by construction. As φ_A is a factorizing resolution of B inside A, we have

$$I_{B/A} \cdot \mathcal{O}_{\overline{A}} = I_{\overline{B}/\overline{A}} \cdot \mathcal{O}_{\overline{A}}(-R_{\overline{B}/\overline{A}})$$

for an effective divisor $R_{\overline{B}/\overline{A}}$ on $\overline{A}$, where $I_{B/A}$ and $I_{\overline{B}/\overline{A}}$ are the defining ideals of B in A and $\overline{B}$ in $\overline{A}$, respectively. It is clear that $\overline{X}$ is a connected component of $\overline{B} \cap \overline{H}$. After blowing-up the other components, we may

assume that $\overline{X} = \overline{B} \cap \overline{H}$. Since $I_{X/H} = I_{B/A} \cdot \mathcal{O}_H$ and $I_{\overline{X}/\overline{H}} = I_{\overline{B}/\overline{A}} \cdot \mathcal{O}_{\overline{H}}$, we obtain

$$I_{X/H} \cdot \mathcal{O}_{\overline{H}} = I_{\overline{X}/\overline{H}} \cdot \mathcal{O}_{\overline{H}}(-R_{B/A}|_{\overline{H}}),$$

which shows the first part of (iii), and that we may take $R_{X/H} = R_{B/A}|_{\overline{H}}$. Using similarly that $I_{X/B} = I_{H/A} \cdot \mathcal{O}_B$ and $I_{\overline{X}/\overline{B}} = I_{\overline{H}/\overline{A}} \cdot \mathcal{O}_{\overline{B}}$, we obtain

$$I_{X/B} \cdot \mathcal{O}_{\overline{B}} = I_{\overline{X}/\overline{B}} \cdot \mathcal{O}_{\overline{B}}(-R_{H/A}|_{\overline{B}}).$$

This gives the second part of (iii), and that we may take $R_{X/B} = R_{H/A}|_{\overline{B}}$. Therefore φ_A satisfies all our requirements.

Since φ_A is a factorizing resolution of B, Lemma 2.8 gives

$$(4) \qquad (K_{\overline{A}/A} - cR_{B/A})|_{\overline{B}} = \widehat{K}_{\overline{B}/B} - J_{\overline{B}/B},$$

where $c = \operatorname{codim}(B, A)$. As φ_A is a factorizing resolution of the smooth hypersurface H, Lemma 2.8 implies

$$(5) \qquad (K_{\overline{A}/A} - R_{H/A})|_{\overline{H}} = K_{\overline{H}/H}.$$

Finally, since φ_H is a factorizing resolution of X inside H, and we have $\operatorname{codim}(X, H) = c$, Lemma 2.8 gives

$$(6) \qquad (K_{\overline{H}/H} - cR_{X/H})|_{\overline{X}} = \widehat{K}_{\overline{X}/X} - J_{\overline{X}/X}.$$

Using (4), (5), (6), and the formulas for $R_{X/H}$ and $R_{X/B}$, we obtain

$$(7) \qquad (\widehat{K}_{\overline{B}/B} - J_{\overline{B}/B} - R_{X/B})|_{\overline{X}} = \widehat{K}_{\overline{X}/X} - J_{\overline{X}/X}.$$

Let $\tilde{\mathfrak{a}} \cdot \mathcal{O}_{\overline{B}} = \mathcal{O}_{\overline{B}}(-Z_{\overline{B}/B})$ and $D = \widehat{K}_{\overline{B}/B} - J_{\overline{B}/B} - \lfloor tZ_{\overline{B}/B} \rfloor - R_{X/B}$. We have an exact sequence:

$$(8) \quad 0 \longrightarrow \varphi_{B*}(I_{\overline{X}/\overline{B}} \cdot \mathcal{O}_{\overline{B}}(D)) \longrightarrow \varphi_{B*}(\mathcal{O}_{\overline{B}}(D)) \longrightarrow \varphi_{X*}(\mathcal{O}_{\overline{X}}(D|_{\overline{X}}))$$

$$\longrightarrow R^1 \varphi_{B*}(I_{\overline{X}/\overline{B}} \cdot \mathcal{O}_{\overline{B}}(D)).$$

Arguing as in the proof of Proposition 2.9, we see that the last term in this sequence is zero: instead of using the Local Vanishing Theorem for multiplier ideals, we use Theorem 3.5 (in the version mentioned in Remark 3.6). Since $R_{X/B}$ is effective, we have an inclusion $\varphi_{B*}(\mathcal{O}_{\overline{B}}(D)) \subseteq \widehat{\mathcal{J}}(B, \tilde{\mathfrak{a}}^t)$, hence the exact sequence (8) gives

$$\varphi_{X*}(\mathcal{O}_{\overline{X}}(D|_{\overline{X}})) \subseteq \widehat{\mathcal{J}}(B, \tilde{\mathfrak{a}}^t) \cdot \mathcal{O}_X.$$

On the other hand, it follows from (7) that

$$\varphi_{X*}(\mathcal{O}_{\overline{X}}(D|_{\overline{X}})) = \varphi_{X*}\left(\mathcal{O}_{\overline{X}}(\widehat{K}_{\overline{X}/X} - J_{\overline{X}/X} - \lfloor tZ_{\overline{X}/X}\rfloor)\right) = \widehat{\mathcal{J}}(X, \mathfrak{a}^t),$$

and this completes the proof of the theorem.   Q.E.D.

Corollary 3.10 (Restriction theorem for Cohen-Macaulay varieties with proper intersection). *Let B and H be Cohen-Macaulay subvarieties of a smooth ambient variety A, such that $X = B \cap H$ is irreducible and reduced, and the intersection is proper, that is, $\dim(X) = \dim(B) + \dim(H) - \dim(A)$. If $X \cap B_{\mathrm{reg}} \neq \emptyset$, and $\tilde{\mathfrak{a}} \subset \mathcal{O}_B$ is an ideal on B whose restriction $\mathfrak{a} = \tilde{\mathfrak{a}} \cdot \mathcal{O}_X$ to X is nonzero, then for every $t \in \mathbf{R}_{\geq 0}$ we have*

$$(9) \qquad\qquad \widehat{\mathcal{J}}(X, \mathfrak{a}^t) \subseteq \widehat{\mathcal{J}}(B, \tilde{\mathfrak{a}}^t) \cdot \mathcal{O}_X.$$

Proof. Consider the diagonal $\Delta \subset A \times A$. Since H and B intersect properly, it follows that $X \simeq \Delta \cap (H \times B)$ is of codimension $N = \dim(A)$ in $H \times B$. Both $A \times A$ and Δ are smooth, hence Δ is locally defined in $A \times A$ by N equations. Since the inclusion (9) is a local statement, we may replace A by a suitable open cover, and therefore we may assume that A is affine, and the ideal of Δ in $A \times A$ is generated by N equations $f_1, \ldots, f_N \in \mathcal{O}(A \times A)$.

We claim that there is a sequence of varieties

$$X = X_0 \subset X_1 \subset \cdots \subset X_N = H \times B$$

such that X_i is a hyperplane section of X_{i+1} for each i with $0 \leq i < N$. Let $H_1, \ldots, H_N$ be general elements of the linear system $\Lambda = \{\sum_i \lambda_i f_i \mid \lambda_i \in k\}$, and let $X_i = (H \times B) \cap H_1 \cap \ldots \cap H_{N-i}$. Note that $X_0 = X$, and we make the convention $X_N = H \times B$. By Bertini's Theorem, each $H_1 \cap \ldots \cap H_i$ is smooth, hence X_i is a hyperplane section in X_{i+1} for $i < N$. Furthermore, $H \times B$ is Cohen-Macaulay by assumption, hence each X_i is Cohen-Macaulay. In particular, no X_i has embedded components. We now prove by descending induction that X_i is irreducible and reduced for every i with $1 \leq i \leq N$. The case $i = N$ is clear, and let us assume that X_{i+1} is irreducible and reduced for some $i \geq 1$. The base locus of $\Lambda|_{H \times B}$ is equal to X, hence another application of Bertini's theorem implies that $X_i \smallsetminus X$ is irreducible and reduced. Since X is not an irreducible component of X_i by dimension considerations, it follows that X_i is irreducible and generically reduced, hence reduced (recall that X_i has no embedded components). This completes the proof of our claim.

Let $p_2: H \times B \longrightarrow B$ be the canonical projection. Applying successively Theorem 3.9 for the above sequence of hyperplane sections, we

obtain

(10) $$\widehat{\mathcal{J}}(X, \mathfrak{a}^t) \subseteq \widehat{\mathcal{J}}(H \times B, p_2^*(\tilde{\mathfrak{a}})^t) \cdot \mathcal{O}_X.$$

On the other hand, if we consider an appropriate resolution of $H \times B$ of the form $\overline{H} \times \overline{B} \longrightarrow H \times B$, and if we denote by $q_1 \colon \overline{H} \times \overline{B} \longrightarrow \overline{H}$ and $q_2 \colon \overline{H} \times \overline{B} \longrightarrow \overline{B}$ the two projections, then we have

$$\widehat{K}_{(\overline{H} \times \overline{B})/(H \times B)} = q_1^* \widehat{K}_{\overline{H}/H} + q_2^* \widehat{K}_{\overline{B}/B}, \ J_{(\overline{H} \times \overline{B})/(H \times B)} = q_1^* J_{\overline{H}/H} + q_2^* J_{\overline{B}/B}.$$

Using also the Künneth formula, these imply

$$\widehat{\mathcal{J}}(H \times B, q_2^*(\tilde{\mathfrak{a}})^t) = q_1^* \widehat{\mathcal{J}}(H, \mathcal{O}_H) \otimes q_2^* \widehat{\mathcal{J}}(B, \tilde{\mathfrak{a}}^t),$$

and we thus obtain

$$\widehat{\mathcal{J}}(H \times B, p_2^*(\tilde{\mathfrak{a}})^t) \cdot \mathcal{O}_X \subseteq (\widehat{\mathcal{J}}(H, \mathcal{O}_H) \cdot \mathcal{O}_X) \cdot (q_2^* \widehat{\mathcal{J}}(B, \tilde{\mathfrak{a}}^t) \cdot \mathcal{O}_X) \subseteq \widehat{\mathcal{J}}(B, \tilde{\mathfrak{a}}^t) \cdot \mathcal{O}_X.$$

Combining this with (10), we obtain (9). Q.E.D.

Question 3.11. *Does the assertion in Corollary 3.10 hold if we drop the assumption that H and B are Cohen-Macaulay?*

On the other hand, the following example shows that one does not have a restriction theorem when X is an arbitrary subvariety of a singular[1] variety B.

Example 3.12. Let B be the hypersurface defined by $x_1^d + \cdots + x_n^d = 0$ in $\mathbf{A}^n$, with $d \geq n \geq 3$. Note that $(B, 0)$ is a non-canonical normal isolated singularity. B is a complete intersection, hence

$$\widehat{\mathcal{J}}(B, \mathcal{O}_B) = \mathcal{J}(B, \mathcal{O}_B) = f_*(K_{Y/X}),$$

where $f \colon Y \longrightarrow X$ is a log resolution of $\mathcal{J}ac_B$. Since B is non-canonical, we see that $\widehat{\mathcal{J}}(B, \mathcal{O}_B)$ is contained in the maximal ideal defining 0. Suppose now that $X \subset B$ is a line through 0. Since X is smooth, we have $\widehat{\mathcal{J}}(X, \mathcal{O}_X) = \mathcal{J}(X, \mathcal{O}_X) = \mathcal{O}_X$. Therefore the inclusion $\widehat{\mathcal{J}}(X, \mathcal{O}_X) \subseteq \widehat{\mathcal{J}}(B, \mathcal{O}_B)_X$ does not hold.

We can now apply our vanishing result in Theorem 3.5 to deduce a version of the Subadditivity Theorem for Mather multiplier ideals. We refer to [11, §9.5.B] and [5, Theorem 6.5] for the corresponding result in the setting of usual multiplier ideals. In particular, our approach follows closely the one in [5], which treats the case of singular varieties. We start with the following adjunction-type formula, which is [5, Lemma 6.3].

---

[1]When the ambient variety B is smooth, the restriction theorem is provided by Proposition 2.9.

Lemma 3.13. *Let X be an n-dimensional variety and $Y \longrightarrow X$ a log resolution of $\mathcal{J}ac_X$. Consider $A = X \times X$ and $A' = Y \times Y$, and let $\varphi \colon \overline{A} \longrightarrow A' \longrightarrow A$ be a factorizing resolution of the diagonal Δ inside A, provided by Lemma 2.7. If $\overline{\Delta}$ is the strict transform of Δ in $\overline{A}$, then we have the following inequality of divisors on $\overline{A}$:*

$$\left(\widehat{K}_{\overline{A}/A} - J_{\overline{A}/A} - nR_{\Delta/A} \right) |_{\overline{\Delta}} \geq \widehat{K}_{\overline{\Delta}/\Delta} - 2J_{\overline{\Delta}/\Delta}.$$

Theorem 3.14. *If $\mathfrak{a}$ and $\mathfrak{b}$ are nonzero ideals on a variety X, then for every $s, t \in \mathbf{R}_{\geq 0}$ we have*

$$\widehat{\mathcal{J}}(X, \mathcal{J}ac_X \mathfrak{a}^s \mathfrak{b}^t) \subseteq \widehat{\mathcal{J}}(X, \mathfrak{a}^s) \cdot \widehat{\mathcal{J}}(X, \mathfrak{b}^t).$$

Proof. Let $g \colon Y \longrightarrow X$ be a log resolution of $\mathcal{J}ac_X \cdot \mathfrak{a} \cdot \mathfrak{b}$. We write $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-Z_{Y/X})$ and $\mathfrak{b} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-W_{Y/X})$. Let $g \times g \colon A' = Y \times Y \longrightarrow A := X \times X$ be the product map. We denote by $\tilde{p}$ and $\tilde{q}$ the first and the second projections from $Y \times Y$ to Y, and by p and q the first and the second projections from $X \times X$ to X. Let E be the divisor on A' such that $\mathcal{O}_{A'}(-E) = p^*(\mathfrak{a} \cdot \mathcal{J}ac_X) \cdot q^*(\mathfrak{b} \cdot \mathcal{J}ac_X)$. We consider a factorizing resolution $\varphi \colon \overline{A} \longrightarrow A' \longrightarrow A$ of the diagonal Δ in A, obtained by applying Lemma 2.7 with respect to E. Let $\overline{\Delta}$ be the strict transform of Δ in $\overline{A}$, and let $\varphi_\Delta = \varphi|_{\overline{\Delta}}$. If we write $\mathfrak{a} \cdot \mathcal{O}_{\overline{\Delta}} = \mathcal{O}_{\overline{\Delta}}(-Z_{\overline{\Delta}/\Delta})$ and $\mathfrak{b} \cdot \mathcal{O}_{\overline{\Delta}} = \mathcal{O}_{\overline{\Delta}}(-W_{\overline{\Delta}/\Delta})$, then after identifying X and Δ, we obtain

$$\widehat{\mathcal{J}}(X, \mathcal{J}ac_X \mathfrak{a}^s \mathfrak{b}^t) = \varphi_{\Delta *} \left(\widehat{K}_{\overline{\Delta}/\Delta} - 2J_{\overline{\Delta}/\Delta} - \lfloor sZ_{\overline{\Delta}/\Delta} \rfloor - \lfloor tW_{\overline{\Delta}/\Delta} \rfloor \right),$$

and by Lemma 3.13, the right hand side is contained in

$$(11) \quad \varphi_{\Delta *} \left(\widehat{K}_{\overline{A}/A} - J_{\overline{A}/A} - \lfloor s\tilde{p}^* Z_{\overline{Y}/X} \rfloor - \lfloor t\tilde{q}^* W_{\overline{Y}/X} \rfloor - nR_{\Delta/A}|_{\overline{\Delta}} \right),$$

where $n = \dim(X)$. Arguing as in the proof of Theorem 3.9, we see using Theorem 3.5 that the ideal in (11) is contained in

$$\widehat{\mathcal{J}}(X \times X, (p^* \mathfrak{a})^s \cdot (q^* \mathfrak{b})^t) \cdot \mathcal{O}_\Delta.$$

Considering the log resolution $g \times g \colon A' \longrightarrow A$ of $\mathcal{J}ac_A \cdot p^*(\mathfrak{a}) \cdot q^*(\mathfrak{b})$, and noting that $\widehat{K}_{A'/A} = \tilde{p}^* \widehat{K}_{Y/X} + \tilde{q}^* \widehat{K}_{Y/X}$ and $J_{A'/A} = \tilde{p}^* J_{Y/X} + \tilde{q}^* J_{Y/X}$, we obtain

$$\widehat{\mathcal{J}}(X \times X, (p^* \mathfrak{a})^s \cdot (q^* \mathfrak{b})^t) \cdot \mathcal{O}_\Delta = \widehat{\mathcal{J}}(X, \mathfrak{a}^s) \cdot \widehat{\mathcal{J}}(X, \mathfrak{b}^t),$$

and we obtain the inclusion in the statement of the theorem.   Q.E.D.

We now give a Skoda-type formula for Mather multiplier ideas (see [11, §11.1.A] for the statement in the context of usual multiplier ideals). The proof follows the one in *loc. cit.*, using our vanishing result in Theorem 3.5.

Theorem 3.15. *If X is a variety of dimension n, and $\mathfrak{a} \subseteq \mathcal{O}_X$ is a nonzero ideal on X, then for every $t \geq n$ we have*

$$\widehat{\mathcal{J}}(X, \mathfrak{a}^t) = \mathfrak{a} \cdot \widehat{\mathcal{J}}(X, \mathfrak{a}^{t-1}).$$

In particular, if t is an integer, then

$$\widehat{\mathcal{J}}(X, \mathfrak{a}^t) = \mathfrak{a}^{t-n+1} \cdot \widehat{\mathcal{J}}(X, \mathfrak{a}^{n-1})$$

Proof. The inclusion $\mathfrak{a} \cdot \widehat{\mathcal{J}}(X, \mathfrak{a}^{t-1}) \subseteq \widehat{\mathcal{J}}(X, \mathfrak{a}^t)$ holds for every $t \geq 1$, and it is an immediate consequence of the definition of the Mather multiplier ideal. Therefore from now on we focus on the reverse inclusion.

If X is covered by open subsets U_i, then it is clearly enough to prove the theorem for each U_i. Since X is n-dimensional, after taking a suitable open cover, we may assume that X is affine, and there is a reduction $\mathfrak{b}$ of $\mathfrak{a}$ generated by n elements $s_1, \ldots, s_n$ (see [2, Proposition 4.6.8]). This means that $\mathfrak{b} \subseteq \mathfrak{a}$, and there is $m \geq 1$ such that $\mathfrak{b}\mathfrak{a}^m = \mathfrak{a}^{m+1}$.

Let $\varphi \colon Y \longrightarrow X$ be a log-resolution of $\mathfrak{a} \cdot \mathfrak{b} \cdot \mathcal{J}ac_X$. We write $\mathfrak{a} \cdot \mathcal{O}_Y = \mathfrak{b} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-Z_{Y/X})$ and consider $\mathcal{L}_s = \mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} - \lfloor sZ_{Y/X} \rfloor)$ for every $s \in \mathbf{R}_{\geq 0}$. It follows from Theorem 3.5 that $R^p\varphi_*\mathcal{L}_s = 0$ for $p > 0$.

If V is the k-vector space generated by $s_1, \ldots, s_n$, then we have on Y an exact Koszul complex

$$0 \longrightarrow \wedge^n V \otimes_k \mathcal{O}_Y(nZ_{Y/X}) \longrightarrow \cdots \longrightarrow V \otimes_k \mathcal{O}_Y(Z_{Y/X}) \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

By tensoring with $\mathcal{L}_t$ we get the exact complex

$$0 \longrightarrow \wedge^n V \otimes_k \mathcal{L}_{t-n} \longrightarrow \cdots \longrightarrow V \otimes_k \mathcal{L}_{t-1} \longrightarrow \mathcal{L}_t \longrightarrow 0.$$

Since $R^p\varphi_*\mathcal{L}_{t-i} = 0$ for $0 \leq i \leq n$ and for all $p > 0$, we deduce that we have an exact complex on X

$$0 \longrightarrow \wedge^n V \otimes_k \varphi_*\mathcal{L}_{t-n} \longrightarrow \cdots \longrightarrow V \otimes_k \varphi_*\mathcal{L}_{t-1} \longrightarrow \varphi_*\mathcal{L}_t \longrightarrow 0.$$

In particular, the map $V \otimes_k \varphi_*\mathcal{L}_{t-1} \longrightarrow \varphi_*\mathcal{L}_t$ is surjective, hence

$$\widehat{\mathcal{J}}(X, \mathfrak{a}^t) = \mathfrak{b} \cdot \widehat{\mathcal{J}}(X, \mathfrak{a}^{t-1}) \subseteq \mathfrak{a} \cdot \widehat{\mathcal{J}}(X, \mathfrak{a}^{t-1}).$$

This completes the proof of the theorem. Q.E.D.

Corollary 3.16 (Briançon-Skoda type formula)**.** *If $\mathfrak{a}$ is an ideal on an n-dimensional variety X, then for every integer $m \geq n$ we have*

$$\overline{\mathcal{J}ac_X \cdot \mathfrak{a}^m} \subseteq \mathfrak{a}^{m-n+1}.$$

Proof. We may assume that $\mathfrak{a}$ is nonzero, since otherwise the assertion is trivial. We keep the notation in the proof of Theorem 3.15. Since the divisor $\widehat{K}_{Y/X}$ is effective, we have an inclusion

$$\mathcal{O}_Y(-J_{Y/X} - mZ_{Y/X}) \subseteq \mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} - mZ_{Y/X}).$$

We deduce

$$\overline{\mathcal{J}ac_X \cdot \mathfrak{a}^m} \subseteq \varphi_*(\mathcal{O}_Y(-J_{Y/X} - mZ_{Y/X}))$$

$$\subseteq \varphi_*(\mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} - mZ_{Y/X})) = \widehat{\mathcal{J}}(X, \mathfrak{a}^m).$$

Using Theorem 3.15, we get

$$\overline{\mathcal{J}ac_X \cdot \mathfrak{a}^m} \subseteq \widehat{\mathcal{J}}(X, \mathfrak{a}^m) = \mathfrak{a}^{m-n+1} \cdot \widehat{\mathcal{J}}(X, \mathfrak{a}^{n-1}) \subseteq \mathfrak{a}^{m-n+1}.$$

Q.E.D.

We also have a version of the Summation Theorem (see [9, Corollary 2]) for Mather multiplier ideals.

Theorem 3.17. *If $\mathfrak{a}_1, \ldots, \mathfrak{a}_r, \mathfrak{b}$ are nonzero ideals on the variety X, then for every $s, t \in \mathbf{R}_{\geq 0}$ we have*

$$(12) \qquad \widehat{\mathcal{J}}\left(X, (\mathfrak{a}_1 + \ldots + \mathfrak{a}_r)^s \mathfrak{b}^t\right) = \sum_{s_1 + \ldots + s_r = s} \widehat{\mathcal{J}}(X, \mathfrak{a}_1^{s_1} \cdots \mathfrak{a}_r^{s_r} \mathfrak{b}^t).$$

Proof. Let $\varphi \colon Y \longrightarrow X$ be a log resolution of the product of $\mathfrak{b}$, $\mathcal{J}ac_X$, and of all $\mathfrak{a}_i + \mathfrak{a}_j$, for $1 \leq i, j \leq r$. It follows from [9, Lemma 3.1] that φ is also a log resolution of $\mathfrak{b} \cdot (\mathfrak{a}_1 + \ldots + \mathfrak{a}_r) \cdot \mathcal{J}ac_X$. Furthermore, if we write $\mathfrak{a}_i \cdot \mathcal{O}_Y = \mathcal{O}_Y(-A_i)$, $\mathfrak{b} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-B)$, and $(\mathfrak{a}_1 + \ldots + \mathfrak{a}_r) \cdot \mathcal{O}_Y = \mathcal{O}_Y(-A)$, then A is the largest effective divisor on Y with $A \leq A_i$ for all i. For every $s_1, \ldots, s_r$ with $\sum_i s_i = s$, we have

$$\lfloor s_1 A_1 + \ldots + s_r A_r \rfloor \geq \sum_{i=1}^r \lfloor sA \rfloor,$$

and the inclusion "$\supseteq$" in (12) then follows from definition of Mather multiplier ideals.

We now consider the reverse inclusion. It is shown in [9, §3] that there is an exact complex on Y

$$\mathcal{E}_\bullet \colon \ 0 \longrightarrow \mathcal{E}_{r-1} \longrightarrow \ldots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow 0$$

with the following properties:

(1) $\mathcal{E}_0 = \mathcal{O}_Y(-A)$.

(2) $\mathcal{E}_1$ is a finite direct sum of sheaves of the form $\mathcal{O}_Y(-\lfloor s_1 A_1 + \ldots + s_r A_r + tB \rfloor)$, with $\sum_i s_i = s$.

(3) Every $\mathcal{E}_i$ with $2 \leq i \leq r-1$ is a finite direct sum of sheaves of the form $\mathcal{O}_Y(-\lfloor a_1 A_1 + \ldots + a_r A_r + bB \rfloor)$, for some $a_1, \ldots, a_r, b \in \mathbf{R}_{\geq 0}$.

It follows from the above properties and Theorem 3.5 that

$$R^i \varphi_*(\mathcal{E}_j \otimes \mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X})) = 0 \text{ for all } i > 0 \text{ and } j \geq 0.$$

This implies that the push-forward $\varphi_*(\mathcal{E}_\bullet \otimes \mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X}))$ is exact. The surjectivity of the last map in this complex, and properties (1) and (2) give the inclusion "$\subseteq$" in (12). Q.E.D.

We end with some considerations concerning an asymptotic version of Mather multiplier ideals. For basic facts about graded sequences of ideals and their multiplier ideals, we refer to [11, Chapter 11]. Let X be a variety, and let $\mathfrak{a}_\bullet = (\mathfrak{a}_m)_{m \geq 1}$ be a graded sequence of ideals on X, that is, $\mathfrak{a}_p \cdot \mathfrak{a}_q \subseteq \mathfrak{a}_{p+q}$ for all $p, q \geq 1$. We assume that some $\mathfrak{a}_p$ is nonzero. Given $t \in \mathbf{R}_{\geq 0}$ and $p, m \geq 1$ such that $\mathfrak{a}_p$ is nonzero, we obtain using Proposition 2.6

$$\widehat{\mathcal{J}}(X, \mathfrak{a}_p^{t/p}) = \widehat{\mathcal{J}}(X, (\mathfrak{a}_p^m)^{t/mp}) \subseteq \widehat{\mathcal{J}}(X, \mathfrak{a}_{pm}^{t/mp}).$$

It follows from the Noetherian property that there is an ideal $\widehat{\mathcal{J}}(X, \mathfrak{a}_\bullet^t)$ such that $\widehat{\mathcal{J}}(X, \mathfrak{a}_p^{t/p}) = \widehat{\mathcal{J}}(X, \mathfrak{a}_\bullet^t)$ whenever p is divisible enough. This is the *asymptotic Mather multiplier ideal* of $\mathfrak{a}_\bullet$. It is straightforward to extend Theorems 3.9, 3.10, 3.14, and 3.17 to the setting of graded sequences of ideals. We leave for the reader the task of formulating these extensions. We only note that if $\mathfrak{a}_\bullet^{(1)}, \ldots, \mathfrak{a}_\bullet^{(r)}$ are graded sequences of ideals, then their product and sum have as their p^{th} terms $\prod_{i=1}^r \mathfrak{a}_p^{(i)}$, and respectively, $\sum_{i_1 + \ldots + i_r = p} \mathfrak{a}_{i_1}^{(1)} \cdot \ldots \cdot \mathfrak{a}_{i_r}^{(r)}$.

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