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Kählerian K3 surfaces and Niemeier lattices, II

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Abstract.

This is a continuation of our paper [13] where we considered markings of Kählerian K3 surfaces by Niemeier lattices and studied in details these markings and their applications to finite symplectic automorphism groups and non-singular rational curves of K3 surfaces for the first Niemeier lattices N_k, $k = 1, 2, \ldots, 21$.

Here we do the same for the remaining Niemeier lattices N_{22} and N_{23} with root systems $12A_2$ and $24A_1$ respectively.

Dedicated to Professor Shigeru Mukai
on occasion of his 60th Birthday

§1. Introduction

Studying of finite symplectic automorphism groups of Kählerian K3 surfaces started in our papers [10] (announcement, 1976) and [11] (1979). Some general theory of such groups was developed, and Abelian such groups were classified (14 non-trivial Abelian groups). Further, in [12, Remark 1.14.7] (1979) we showed, in particular, that all finite symplectic automorphism groups of K3 surfaces can be obtained from negative definite even unimodular lattices and their automorphism groups using primitive embeddings of even negative definite lattices into such unimodular lattices. By our results about existence of primitive embeddings of even lattices into even unimodular lattices in [12], for K3 surfaces it is enough to use negative definite even unimodular lattices of the rank 24. They are Niemeier lattices.

Later, all finite symplectic automorphism groups of Kählerian K3 surface were classified as abstract groups by Mukai [8] (1988), (see

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also Xiao [18], 1996). Kondō in [6] (1998) showed that this classification can be obtained by using primitive embeddings of lattices into Niemeier lattices (see also the important appendix to this paper by Mukai). This is similar to our considerations in [12, Remark 1.14.7]. Recently, Hashimoto [5] applied similar ideas of using Niemeier lattices to classify finite symplectic automorphism groups of Kälerian K3 surfaces similarly to our results about Abelian such groups in [11].

Thus, now it is clear that methods of using negative definite even unimodular lattices and Niemeier lattices are very powerful.

In [13], we used ideas and results of [12, Remark 1.14.7] to show that all twenty four Niemeier lattices are important for Kählerian K3 surfaces, their geometry and their symplectic automorphism groups. (Usually, the Niemeier lattice with the root system $24A_1$ related to Mathieu group M_{24} is used.) From our point of view, all twenty four Niemeier lattices are important for K3 surfaces.

In [13], we introduced and used *markings* of a Kählerian K3 surface X by Niemeier lattices. Using these markings, one can study, in particular, finite symplectic automorphism groups and non-singular rational curves on X. In [11], we demonstrated that to study finite symplectic automorphism groups of K3 surfaces, it is important to work not with algebraic K3 surfaces but with arbitrary Kählerian K3 surfaces. General Kählerian K3 surfaces X have negative definite Picard lattices S_X of the rank $\operatorname{rk} S_X \le 19$. By our results in [12], there exists a primitive embedding of S_X into one of 24 Niemeier lattices. One can study some arithmetic and geometry of S_X and of X using such primitive embedding. It is called a *marking of X.* All 24 Niemeier lattices are important for that. For Kählerian K3 surfaces with semi-negative definite and hyperbolic Picard lattice S_X we used some modifications of these markings.

In [13], we studied in details markings of Kählerian K3 surfaces by the first 21 Niemeier lattices N_k, $k = 1, 2, \ldots, 21$, for describing their non-singular rational curves and finite symplectic automorphism groups.

Here, we similarly study in details markings of Kählerian K3 surfaces by the remaining Niemeier lattices N_{22} and N_{23} with root systems $12A_2$ and $24A_1$ respectively. It follows from results by Kondō in [6], that any Kählerian K3 surface can be marked by one of Niemeier lattices N_k, $k = 1, 2, \ldots, 23$. One can avoid the difficult Leech lattice.

Unlike cases of N_k, $k = 1, 2, \ldots, 21$, considered in [13], for N_{22} and N_{23} we use classification of abstract finite symplectic automorphism groups of Kählerian K3 surfaces by Mukai [8] and its amplification by Xiao [18], and we follow to Hashimoto [5] in using GAP Program [4] to find so called KahK3 conjugacy classes of automorphism groups of N_{22} and N_{23}.

A preliminary variant of [13] and this paper was published as preprint [14].

§2. Existence of a primitive embedding of an even lattice into even unimodular lattices, according to [12]. Its application to Picard lattices of Kählerian K3 surfaces.

In this paper, we use notations, definitions and results of [12] about lattices (that is non-degenerate integral (over $\mathbb{Z}$) symmetric bilinear forms). In particular, $\oplus$ denotes the orthogonal sum of lattices, quadratic forms. For a prime p, we denote by $\mathbb{Z}_p$ the ring of p-adic integers, and by $\mathbb{Z}_p^*$ its group of invertible elements.

Let S be a lattice. Let $A_S = S^*/S$ be its discriminant group, and q_S its discriminant quadratic form on A_S where we assume that the lattice S is even: that is x^2 is even for any $x \in S$. We denote by $l(A_S)$ the minimal number of generators of the finite Abelian group A_S, and by $|A_S|$ its order. For a prime p, we denote by $q_{S_p} = q_{S\otimes\mathbb{Z}_p}$ the p-component of q_S (equivalently, the discriminant quadratic form of the p-adic lattice $S\otimes\mathbb{Z}_p$). A quadratic form on a group of order 2 is denoted by $q_\theta^{(2)}(2)$. A p-adic lattice $K(q_{S_p})$ of the rank $l(A_{S_p})$ with the discriminant quadratic form q_{S_p} is denoted by $K(q_{S_p})$. It is unique, up to isomorphisms, for $p \neq 2$, and for $p = 2$, if $q_{S_2} \not\cong q_\theta^{(2)}(2) \oplus q'$. We have the following result where an embedding $S \subset L$ of lattices is called *primitive* if L/S has no torsion.

Theorem 1. *(Theorem 1.12.2 in* [12]*).*

Let S be an even lattice of the signature $(t_{(+)}, t_{(-)})$, and $l_{(+)}$, $l_{(-)}$ are integers.

Then, there exists a primitive embedding of S into one of even unimodular lattices of the signature $(l_{(+)}, l_{(-)})$ if and only if the following conditions satisfy:

(1) $l_{(+)} - l_{(-)} \equiv 0 \mod 8$*;*

(2) $l_{(+)} - t_{(+)} \geq 0$, $l_{(-)} - t_{(-)} \geq 0$, $l_{(+)} + l_{(-)} - t_{(+)} - t_{(-)} \geq l(A_S)$*;*

(3) $(-1)^{l_{(+)}-t_{(+)}}|A_S| \equiv \det K(q_{S_p}) \mod (\mathbb{Z}_p{}^*)^2$ *for each odd prime p such that* $l_{(+)} + l_{(-)} - t_{(+)} - t_{(-)} = l(A_{S_p})$*;*

(4) $|A_S| \equiv \pm \det K(q_{S_2}) \mod (\mathbb{Z}_2{}^*)^2$, *if* $l_{(+)} + l_{(-)} - t_{(+)} - t_{(-)} = l(A_{S_2})$ *and* $q_{S_2} \not\cong q_\theta^{(2)}(2) \oplus q'$.

Remark that if the last inequality in (2) is strict then one does not need the conditions (3) and (4). If $q_{S2} \cong q_\theta^{(2)}(2) \oplus q'$, then one does not need the condition (4).

Father, a special case of Theorem 1 when $l_{(+)} = 3$ and $l_{(-)} = 19$, will be important for us. An even unimodular lattice of the signature $(3, 19)$ is unique up to isomorphisms, and it is isomorphic to the second cohomology lattice $H^2(X, \mathbb{Z})$ with the intersection pairing of Kählerian K3 surfaces X. We denote such lattice as L_{K3}. By Global Torelli Theorem [15], [2] and epimorphicity of Torelli map [7], [16], [17], its primitive negative definite sublattices $S \subset L_{K3}$ are isomorphic to Picard lattices of general (that is with negative definite Picard lattices) Kählerian K3 surfaces. See details in [13]. Thus, from Theorem 1, we obtain the following statement.

Theorem 2. *(Corollary of Theorem 1.12.2 in* [12]*). Let S be an even negative definite lattice.*

Then S is isomorphic to a Picard lattice of a Kählerin K3 surface (equivalently, S has a primitive embedding into even unimodular lattice L_{K3} of the signature $(3, 19)$) if and only if

(2) $\operatorname{rk} S \le 19$ *and* $\operatorname{rk} S + l(A_S) \le 22$*;*

(3) $-|A_S| \equiv \det K(q_{S_p}) \mod (\mathbb{Z}_p{}^*)^2$ *for each odd prime p such that* $\operatorname{rk} S + l(A_{S_p}) = 22$*;*

(4) $|A_S| \equiv \pm \det K(q_{S_2}) \mod (\mathbb{Z}_2{}^*)^2$*, if* $\operatorname{rk} S + l(A_{S_2}) = 22$ *and* $q_{S2} \not\cong q_\theta^{(2)}(2) \oplus q'$*.*

Remark that if the last inequality in (2) is strict then one does not need the conditions (3) and (4). If $q_{S2} \cong q_\theta^{(2)}(2) \oplus q'$, then one does not need the condition (4).

§3. Niemeier lattices and their KahK3 subgroups

In this Section, we consider only Niemeier lattices, but the same definitions and results are valid for all even negative definite unimodular (actually for arbitrary even negative definite) lattices. See Remark 1.14.7 in [12].

Further, negative definite even unimodular lattices N of the rank 24 are called *Niemeier lattices.* They were classified by Niemeier [9]. See also [3, Ch. 16, 18]. All elements with square (-2) of these lattices define root systems $\Delta(N)$ and generate root sublattices $[\Delta(N)] = N^{(2)} \subset N$ which are orthogonal sums of the root lattices A_n, $n \ge 1$, D_m, $m \ge 4$, or E_k, $k = 6, 7, 8$ with the corresponding root systems $\mathbb{A}_n$, $\mathbb{D}_m$, $\mathbb{E}_k$. We denote by the same letters their Dynkin diagrams. We fix standard bases by simple roots of these root systems and lattices according to [1] (see Figure 1). If there are several components, we put an additional second index which numerates components.

By Niemeier [9], there are 24 Niemeier lattices N, up to isomorphisms, and they are characterized by there root sublattices. Here is the list where $\oplus$ denotes the orthogonal sum of lattices:

$$N^{(2)} = [\Delta(N)] =$$

$$(1)\ D_{24},\ (2)\ D_{16}\oplus E_8,\ (3)\ 3E_8,\ (4)\ A_{24},\ (5)\ 2D_{12},\ (6)\ A_{17}\oplus E_7,$$

$$(7)\ D_{10}\oplus 2E_7,\ (8)\ A_{15}\oplus D_9,\ (9)\ 3D_8,\ (10)\ 2A_{12},\ (11)\ A_{11}\oplus D_7\oplus E_6,$$

$$(12)\ 4E_6,\ (13)\ 2A_9\oplus D_6,\ (14)\ 4D_6,\ (15)\ 3A_8,\ (16)\ 2A_7\oplus 2D_5,\ (17)\ 4A_6,$$

$$(18)\ 4A_5\oplus D_4,\ (19)\ 6D_4,\ (20)\ 6A_4,\ (21)\ 8A_3,\ (22)\ 12A_2,\ (23)\ 24A_1$$

give 23 Niemeier lattices N_k, where the number k is shown in brackets above. The last one, the *Leech lattice* (24) with $N^{(2)} = \{0\}$ has no roots. Further, we also denote by $N(R)$ the Niemeier lattice with the root system R.

We recall that a basis $P(N)$ of the root lattice $[\Delta(N)]$ by simple roots is defined up to the reflection group $W(N)$ which is generated by reflections $s_\delta : x \to x + (x\cdot\delta)\delta$, $x \in N$, in roots $\delta \in \Delta(N)$. We denote by $\Gamma(P(N))$ its Dynkin diagram. Let $A(N) \subset O(N)$ be the group of symmetries of the fixed basis $P(N)$. Thus, $g \in A(N)$ if and only if $g(P(N)) = P(N)$. Then we have the semi-direct product

$$O(N) = A(N) \ltimes W(N).$$

for the automorphism group of N. Equivalently, $P(N)$ is equivalent to a choice of the fundamental chamber for $W(N)$, and $A(N)$ is the group of symmetries of the fundamental chamber.

We use the basis of a root lattice A_n, D_n or E_k, $k = 6, 7, 8$, which is shown on Figure 1.

For A_n, $n \ge 1$, we denote $\epsilon_1 = (\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n)/(n+1)$. It gives the generator of the discriminant group $A_n^*/A_n \cong \mathbb{Z}/(n+1)\mathbb{Z}$.

For D_n, $n \ge 4$ and $n \equiv 0 \mod 2$, we denote $\epsilon_1 = (\alpha_1 + \alpha_3 + \cdots + \alpha_{n-3} + \alpha_{n-1})/2$, $\epsilon_2 = (\alpha_{n-1} + \alpha_n)/2$, $\epsilon_3 = (\alpha_1 + \alpha_3 + \cdots + \alpha_{n-3} + \alpha_n)/2$. They give all non-zero elements of the discriminant group $D_n^*/D_n \cong (\mathbb{Z}/2\mathbb{Z})^2$.

For D_n, $n \ge 4$ and $n \equiv 1 \mod 2$, we denote $\epsilon_1 = (\alpha_1 + \alpha_3 + \cdots + \alpha_{n-2})/2 + \alpha_{n-1}/4 - \alpha_n/4$, $\epsilon_2 = (\alpha_{n-1} + \alpha_n)/2$, $\epsilon_3 = (\alpha_1 + \alpha_3 + \cdots + \alpha_{n-2})/2 - \alpha_{n-1}/4 + \alpha_n/4$. They give all non-zero elements of $D_n^*/D_n \cong \mathbb{Z}/4\mathbb{Z}$.

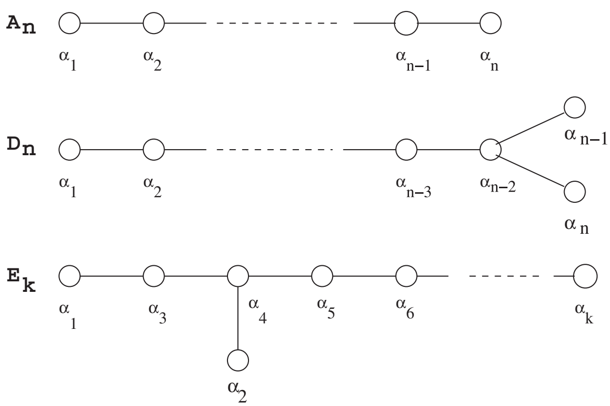

Fig. 1. Bases of Dynkin diagrams $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_k$.

For E_6, we denote $\epsilon_1 = (\alpha_1 - \alpha_3 + \alpha_5 - \alpha_6)/3$, $\epsilon_2 = (-\alpha_1 + \alpha_3 - \alpha_5 + \alpha_6)/3$. They give all non-zero elements of $E_6^*/E_6 \cong \mathbb{Z}/3\mathbb{Z}$.

For E_7, we denote $\epsilon_1 = (\alpha_2 + \alpha_5 + \alpha_7)/2$. It gives the non-zero element of $E_7^*/E_7 \cong \mathbb{Z}/2\mathbb{Z}$.

If the Dynkin diagram of a root lattice has several connected components, the second index of a basis numerates the corresponding connected component.

Definition 3.1. *Let N be a Niemeier lattice and $H \subset A(N)$ a subgroup of $A(N)$.*

The sublattice

$$N^H = \{x \in N \mid h(x) = x \ \forall h \in H\} \subset N$$

is called the invariant sublattice of H.

The orthogonal complement $N_H = (N^H)^\perp_N \subset N$ to N^H in N is called the coinvariant sublattice of H.

A subgroup $H \subset A(N)$ is called a KahK3 subgroup (that is a subgroup of Kählerian K3 surfaces) of $A(N)$ if its coinvariant sublattice N_H is isomorphic to a Picard lattice of a Kählerian K3 surface. Equivalently, there exists a primitive embedding $N_H \subset L_{K3}$. Equivalently, N_H satisfies to Theorem 2. See Sect. 2.

For $H \subset A(N)$, we denote by $Clos(H)$ the maximal subgroup of $A(N)$ with the same coinvariant sublattice N_H as for H. Thus,

$$Clos(H) = \{h \in A(N) \mid h|N^H \text{ is identity on } N^H\}.$$

Obviously, if H is a KahK3 subgroup of $A(N)$, then all subgroups of $Clos(H)$ are also KahK3 subgroups of $A(N)$.

For a subgroup $H \subset A(N)$ and $g \in A(N)$, the conjugate subgroup $H^g = gHg^{-1}$ has the coinvariant sublattice $N_{H^g} = g(N_H)$. Thus, *H and H^g are KahK3 subgroups simultaneously,* and we can speak about *KahK3 conjugacy classes of subgroups of $A(N)$.* To describe KahK3 subgroups of $A(N)$, it is sufficient to describe their KahK3 conjugacy classes. Equivalently, it is sufficient to describe *representatives of all KahK3 conjugacy classes of $A(N)$.*

In [13], we described KahK3 subgroups of $A(N_k)$ for the first 21 Niemeier lattices N_k, $k = 1, 2, \ldots, 21$.

In Section 4, we shall describe KahK3 conjugacy classes of $A(N_k)$ for remaining Niemeier lattices N_{22} and N_{23} and their applications to marking of Kählerian K3 surfaces (see [13] for details).

§4. Cases N_{23} and N_{22}, their KahK3 sugbroups and applications to Kählerian K3 surfaces

By [11] and [12], KahK3 subgroups $H \subset A(N_k)$ of Niemeier lattices (and all negative definite even unimodular lattices with the same definitions) are directly related to finite symplectic automorphism groups of Kählerian K3 surfaces. Let X be a Kählerian K3 surface and G its finite symplectic automorphism group. Let S_G be the orthogonal complement to the invariant sublattice $H^2(X, \mathbb{Z})^G$ in $H^2(X, \mathbb{Z})$ with respect to the intersection pairing. Then S_G with the action of G is isomorphic to one of N_H with the action of H on N_H for one of KahK3 subgroups H of one of Niemeier lattices N. The sets of such pairs (S_G, G) and (N_H, H) are the same up to isomorphisms.

We use Mukai's classification [8] of abstract finite symplectic automorphism groups of K3 surfaces and its amplification by Xiao [18]. We follow Xiao [18] and Hashimoto [5] in numbering them by $n = 1, 2, \ldots 81$ (see cases 23 and 22 below). For a fixed n and the corresponding abstract group H, we follow to Hashimoto [5] in using GAP Program [4] for finding conjugacy classes of H in $A23 = A(N_{23})$ and $A22 = A(N_{22})$. In particular, the GAP invariant i of this group (found by Hashimoto [5]) is very useful.Then the abstract finite group H is given by the GAP command H:=SmallGroup($|H|$,i). Its conjugacy classes in $A23$ and $A22$ are given by the GAP command IsomorphicSubgroups(A23,H) and IsomorphicSubgroups(A22,H) respectively when we identify $A23$ and $A22$ with the corresponding subgroups of the permutation group $\mathfrak{S}_{24}$ acting on bases of root sublattices of N_{23} and N_{22} respectively. For each of the conjugacy classes of H, we calculate (using algorithms and programs in [13]) invariants of Theorem 2 to find out if N_H has a primitive embedding into L_{K3} and it is a KahK3 conjugacy class. We give $\operatorname{rk} N_H$, the isomorphism class of $A_{N_H} = (N_H)^*/N_H$, and some additional invariants, if necessary, which show that the conjugacy class is a KahK3 conjugacy class according to Theorem 2. All other conjugacy classes of H do not satisfy Theorem 2, and they don't give KahK3 conjugacy classes. Surprisingly, these invariants are the same for all KahK3 conjugacy classses for the fixed n as our calculations show (we must say that our calculations agree to calculations by Hashimoto in [5]). We give them for each n. We numerate these conjugacy classes as $H_{n,m}$, where $m \in \mathbb{N}$ (and by $H_{41,1,1}$ and $H_{41,1,2}$ for $n = 41$ and $A(N_{23})$). For each of $H_{n,m}$ we give its orbits in the root basis which permit to find invariant and coinvariant sublattices of $H_{n,m}$ (see [13]). We must say that some of our calculations for $A(N_{23})$ repeat calculations by Mukai in Appendix to [6].

First, better to put before the case N_{23} because all $n = 1, 2, \ldots, 81$ give some KahK3 conjugacy classes in $A(N_{23})$. For N_{22}, more than half of $n = 1, 2, \ldots, 81$ give no conjugacy classes in $A(N_{22})$.

Case 23. For the Niemeier lattice N_{23}, we have

$$N = N_{23} = N(24A_1) =$$

$$= [24A_1,\ [1(00000101001100110101111)]] = [24A_1,$$

$$\epsilon_1+\epsilon_7+\epsilon_9+\epsilon_{12}+\epsilon_{13}+\epsilon_{16}+\epsilon_{17}+\epsilon_{19}+\epsilon_{21}+\epsilon_{22}+\epsilon_{23}+\epsilon_{24},$$

$$\epsilon_1+\epsilon_2+\epsilon_8+\epsilon_{10}+\epsilon_{13}+\epsilon_{14}+\epsilon_{17}+\epsilon_{18}+\epsilon_{20}+\epsilon_{22}+\epsilon_{23}+\epsilon_{24},$$

$$\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_9+\epsilon_{11}+\epsilon_{14}+\epsilon_{15}+\epsilon_{18}+\epsilon_{19}+\epsilon_{21}+\epsilon_{23}+\epsilon_{24},$$

$$\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_{10}+\epsilon_{12}+\epsilon_{15}+\epsilon_{16}+\epsilon_{19}+\epsilon_{20}+\epsilon_{22}+\epsilon_{24},$$

$$\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5+\epsilon_{11}+\epsilon_{13}+\epsilon_{16}+\epsilon_{17}+\epsilon_{20}+\epsilon_{21}+\epsilon_{23},$$

$$\epsilon_1+\epsilon_3+\epsilon_4+\epsilon_5+\epsilon_6+\epsilon_{12}+\epsilon_{14}+\epsilon_{17}+\epsilon_{18}+\epsilon_{21}+\epsilon_{22}+\epsilon_{24},$$

$$\epsilon_1+\epsilon_2+\epsilon_4+\epsilon_5+\epsilon_6+\epsilon_7+\epsilon_{13}+\epsilon_{15}+\epsilon_{18}+\epsilon_{19}+\epsilon_{22}+\epsilon_{23},$$

$$\epsilon_1+\epsilon_3+\epsilon_5+\epsilon_6+\epsilon_7+\epsilon_8+\epsilon_{14}+\epsilon_{16}+\epsilon_{19}+\epsilon_{20}+\epsilon_{23}+\epsilon_{24},$$

$$\epsilon_1+\epsilon_2+\epsilon_4+\epsilon_6+\epsilon_7+\epsilon_8+\epsilon_9+\epsilon_{15}+\epsilon_{17}+\epsilon_{20}+\epsilon_{21}+\epsilon_{24},$$

$$\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_5+\epsilon_7+\epsilon_8+\epsilon_9+\epsilon_{10}+\epsilon_{16}+\epsilon_{18}+\epsilon_{21}+\epsilon_{22},$$

$$\epsilon_1+\epsilon_3+\epsilon_4+\epsilon_6+\epsilon_8+\epsilon_9+\epsilon_{10}+\epsilon_{11}+\epsilon_{17}+\epsilon_{19}+\epsilon_{22}+\epsilon_{23},$$

$$\epsilon_1+\epsilon_4+\epsilon_5+\epsilon_7+\epsilon_9+\epsilon_{10}+\epsilon_{11}+\epsilon_{12}+\epsilon_{18}+\epsilon_{20}+\epsilon_{23}+\epsilon_{24},$$

$$\epsilon_1+\epsilon_2+\epsilon_5+\epsilon_6+\epsilon_8+\epsilon_{10}+\epsilon_{11}+\epsilon_{12}+\epsilon_{13}+\epsilon_{19}+\epsilon_{21}+\epsilon_{24},$$

$$\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_6+\epsilon_7+\epsilon_9+\epsilon_{11}+\epsilon_{12}+\epsilon_{13}+\epsilon_{14}+\epsilon_{20}+\epsilon_{22},$$

$$\epsilon_1+\epsilon_3+\epsilon_4+\epsilon_7+\epsilon_8+\epsilon_{10}+\epsilon_{12}+\epsilon_{13}+\epsilon_{14}+\epsilon_{15}+\epsilon_{21}+\epsilon_{23},$$

$$\epsilon_1+\epsilon_4+\epsilon_5+\epsilon_8+\epsilon_9+\epsilon_{11}+\epsilon_{13}+\epsilon_{14}+\epsilon_{15}+\epsilon_{16}+\epsilon_{22}+\epsilon_{24},$$

$$\epsilon_1+\epsilon_2+\epsilon_5+\epsilon_6+\epsilon_9+\epsilon_{10}+\epsilon_{12}+\epsilon_{14}+\epsilon_{15}+\epsilon_{16}+\epsilon_{17}+\epsilon_{23},$$

$$\epsilon_1+\epsilon_3+\epsilon_6+\epsilon_7+\epsilon_{10}+\epsilon_{11}+\epsilon_{13}+\epsilon_{15}+\epsilon_{16}+\epsilon_{17}+\epsilon_{18}+\epsilon_{24},$$

$$\epsilon_1+\epsilon_2+\epsilon_4+\epsilon_7+\epsilon_8+\epsilon_{11}+\epsilon_{12}+\epsilon_{14}+\epsilon_{16}+\epsilon_{17}+\epsilon_{18}+\epsilon_{19},$$

$$\epsilon_1+\epsilon_3+\epsilon_5+\epsilon_8+\epsilon_9+\epsilon_{12}+\epsilon_{13}+\epsilon_{15}+\epsilon_{17}+\epsilon_{18}+\epsilon_{19}+\epsilon_{20},$$

$$\epsilon_1+\epsilon_4+\epsilon_6+\epsilon_9+\epsilon_{10}+\epsilon_{13}+\epsilon_{14}+\epsilon_{16}+\epsilon_{18}+\epsilon_{19}+\epsilon_{20}+\epsilon_{21},$$

$$\epsilon_1+\epsilon_5+\epsilon_7+\epsilon_{10}+\epsilon_{11}+\epsilon_{14}+\epsilon_{15}+\epsilon_{17}+\epsilon_{19}+\epsilon_{20}+\epsilon_{21}+\epsilon_{22},$$

$$\epsilon_1+\epsilon_6+\epsilon_8+\epsilon_{11}+\epsilon_{12}+\epsilon_{15}+\epsilon_{16}+\epsilon_{18}+\epsilon_{20}+\epsilon_{21}+\epsilon_{22}+\epsilon_{23},$$

$$\epsilon_1+\epsilon_7+\epsilon_9+\epsilon_{12}+\epsilon_{13}+\epsilon_{16}+\epsilon_{17}+\epsilon_{19}+\epsilon_{21}+\epsilon_{22}+\epsilon_{23}+\epsilon_{24}]$$

where we denote $\alpha_i = \alpha_{1i}$ and $\epsilon_i = \epsilon_{1i}$ for $i = 1, 2, \ldots, 24$. The group $A(N_{23})$ is the Mathieu group M_{24} of the order 244823040 generated by

$$\varphi_1 = (\alpha_1)(\alpha_2\alpha_3 \ldots \alpha_{23}\alpha_{24}),\ \varphi_2 = (\alpha_4\alpha_{18}\alpha_{11}\alpha_8\alpha_{10})(\alpha_5\alpha_{14}\alpha_{15}\alpha_{20}\alpha_6)$$
$$(\alpha_9\alpha_{19}\alpha_{12}\alpha_{13}\alpha_{24})(\alpha_{16}\alpha_{21}\alpha_{23}\alpha_{22}\alpha_{17}),\ \varphi_3 = (\alpha_1\alpha_2)(\alpha_3\alpha_{24})(\alpha_4\alpha_{13})$$
$$(\alpha_5\alpha_{17})(\alpha_6\alpha_{19})(\alpha_7\alpha_{11})(\alpha_8\alpha_{21})(\alpha_9\alpha_{15})(\alpha_{10}\alpha_{22})(\alpha_{12}\alpha_{18})(\alpha_{14}\alpha_{23})$$
$$(\alpha_{16}\alpha_{20}).$$

(see [3, Ch. 16]).

Using results described at the beginning of this Sec. 4, and GAP Program [4], we obtain the following classification.

Classification of KahK3 conjugacy classes for $A(N_{23})$:

n=81, $H \cong M_{20}$ ($|H| = 960$, $i = 11357$): $\mathrm{rk}\, N_H = 19$, $(N_H)^*/N_H \cong \mathbb{Z}/40\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2$ and $\det(K((q_{N_H})_2)) \equiv \pm 2^5 \cdot 5 \mod (\mathbb{Z}_2^*)^2$.

$$H_{81,1} =$$
$$[(\alpha_1\alpha_{19}\alpha_6)(\alpha_3\alpha_{24}\alpha_{16})(\alpha_4\alpha_{20}\alpha_9)(\alpha_7\alpha_{10}\alpha_8)(\alpha_{11}\alpha_{23}\alpha_{13})(\alpha_{14}\alpha_{21}\alpha_{15}),$$
$$(\alpha_1\alpha_{11}\alpha_{23}\alpha_8)(\alpha_4\alpha_6\alpha_{13}\alpha_{20})(\alpha_5\alpha_{16})(\alpha_7\alpha_{17}\alpha_9\alpha_{14})$$
$$(\alpha_{10}\alpha_{21}\alpha_{19}\alpha_{15})(\alpha_{12}\alpha_{24})]$$

with orbits (here and in what follows we show orbits with more than one elements only) $\{\alpha_1, \alpha_{19}, \alpha_{11}, \alpha_6, \alpha_{15}, \alpha_{23}, \alpha_{13}, \alpha_{14}, \alpha_{10}, \alpha_8, \alpha_{20}, \alpha_{21}, \alpha_7, \alpha_9, \alpha_4, \alpha_{17}\}$, $\{\alpha_3, \alpha_{24}, \alpha_{16}, \alpha_{12}, \alpha_5\}$;

$$H_{81,2} =$$
$$[(\alpha_1\alpha_{16}\alpha_9)(\alpha_4\alpha_{10}\alpha_{12})(\alpha_5\alpha_{13}\alpha_{19})(\alpha_6\alpha_{20}\alpha_{17})(\alpha_7\alpha_{23}\alpha_{24})(\alpha_8\alpha_{21}\alpha_{11}),$$
$$(\alpha_1\alpha_{11}\alpha_{23}\alpha_8)(\alpha_4\alpha_6\alpha_{13}\alpha_{20})(\alpha_5\alpha_{16})(\alpha_7\alpha_{17}\alpha_9\alpha_{14})$$
$$(\alpha_{10}\alpha_{21}\alpha_{19}\alpha_{15})(\alpha_{12}\alpha_{24})]$$

with orbits $\{\alpha_1, \alpha_{16}, \alpha_{11}, \alpha_9, \alpha_5, \alpha_8, \alpha_{23}, \alpha_{14}, \alpha_{13}, \alpha_{21}, \alpha_{24}, \alpha_7, \alpha_{19}, \alpha_{20}, \alpha_{12}, \alpha_{17}, \alpha_{15}, \alpha_4, \alpha_6, \alpha_{10}\}$.

n=80, $H \cong F_{384}$ ($|H| = 384$, $i = 18135$): $\mathrm{rk}\, N_H = 19$, $(N_H)^*/N_H \cong (\mathbb{Z}/8\mathbb{Z})^2 \times \mathbb{Z}/4\mathbb{Z}$ and $\det(K((q_{N_H})_2)) \equiv \pm 2^8 \mod (\mathbb{Z}_2^*)^2$.

$$H_{80,1} =$$

$$[(\alpha_1\alpha_8\alpha_{15})(\alpha_4\alpha_{20}\alpha_{11})(\alpha_6\alpha_7\alpha_{21})(\alpha_9\alpha_{13}\alpha_{23})(\alpha_{14}\alpha_{17}\alpha_{19})(\alpha_{18}\alpha_{24}\alpha_{22}),$$
$$(\alpha_1\alpha_{11}\alpha_{23}\alpha_8)(\alpha_4\alpha_6\alpha_{13}\alpha_{20})(\alpha_5\alpha_{16})(\alpha_7\alpha_{17}\alpha_9\alpha_{14})$$
$$(\alpha_{10}\alpha_{21}\alpha_{19}\alpha_{15})(\alpha_{12}\alpha_{24})]$$

with orbits $\{\alpha_1,\alpha_8,\alpha_{11},\alpha_{15},\alpha_4,\alpha_{23},\alpha_{10},\alpha_{20},\alpha_6,\alpha_9,\alpha_{21},\alpha_7,\alpha_{13},\alpha_{14},$ $\alpha_{19},\alpha_{17}\}$, $\{\alpha_5,\alpha_{16}\}$, $\{\alpha_{12},\alpha_{24},\alpha_{22},\alpha_{18}\}$.

n=79, $H\cong\mathfrak{A}_6$ ($|H|=360$, $i=118$): $\operatorname{rk}N_H=19$ and $(N_H)^*/N_H\cong\mathbb{Z}/60\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$.

$$H_{79,1}=$$
$$[(\alpha_2\alpha_{22}\alpha_{15}\alpha_8\alpha_{23})(\alpha_3\alpha_{18}\alpha_{12}\alpha_7\alpha_4)(\alpha_5\alpha_{21}\alpha_{19}\alpha_{20}\alpha_{11})(\alpha_9\alpha_{10}\alpha_{13}\alpha_{14}\alpha_{24}),$$
$$(\alpha_2\alpha_{17}\alpha_{15}\alpha_8\alpha_{23})(\alpha_3\alpha_{14}\alpha_{10}\alpha_{13}\alpha_9)(\alpha_4\alpha_{24}\alpha_{18}\alpha_7\alpha_{12})(\alpha_5\alpha_{11}\alpha_{20}\alpha_{16}\alpha_{21})]$$

with orbits $\{\alpha_2,\alpha_{22},\alpha_{17},\alpha_{15},\alpha_8,\alpha_{23}\}$, $\{\alpha_3,\alpha_{18},\alpha_{14},\alpha_{12},\alpha_7,\alpha_{24},\alpha_{10},$ $\alpha_4,\alpha_9,\alpha_{13}\}$, $\{\alpha_5,\alpha_{21},\alpha_{11},\alpha_{19},\alpha_{20},\alpha_{16}\}$;

$$H_{79,2}=$$
$$[(\alpha_2\alpha_8\alpha_{18}\alpha_{20}\alpha_{10})(\alpha_3\alpha_{12}\alpha_{11}\alpha_{22}\alpha_{14})(\alpha_4\alpha_{15}\alpha_{24}\alpha_{19}\alpha_9)(\alpha_5\alpha_{23}\alpha_{17}\alpha_7\alpha_{13}),$$
$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_2,\alpha_8,\alpha_{13},\alpha_{18},\alpha_{14},\alpha_5,\alpha_{20},\alpha_7,\alpha_3,\alpha_{23},\alpha_{10},\alpha_{11},\alpha_{12},\alpha_{22},$ $\alpha_{17}\}$, $\{\alpha_4,\alpha_{15},\alpha_9,\alpha_{24},\alpha_{19},\alpha_{16}\}$.

n=78, $H\cong\mathfrak{A}_{4,4}$ ($|H|=288$, $i=1026$): $\operatorname{rk}N_H=19$, $(N_H)^*/N_H\cong\mathbb{Z}/24\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ and $\det(K((q_{N_H})_2)\equiv\pm2^5\cdot3^2\mod(\mathbb{Z}_2^*)^2$.

$$H_{78,1}=$$
$$[(\alpha_1\alpha_{19}\alpha_8)(\alpha_3\alpha_{22}\alpha_{15})(\alpha_5\alpha_{23}\alpha_{20})(\alpha_6\alpha_{21}\alpha_9)(\alpha_7\alpha_{11}\alpha_{16})(\alpha_{12}\alpha_{18}\alpha_{14}),$$
$$(\alpha_1\alpha_6\alpha_{23}\alpha_8)(\alpha_4\alpha_{11}\alpha_{16}\alpha_7)(\alpha_5\alpha_{21}\alpha_{14}\alpha_{12})$$
$$(\alpha_9\alpha_{19}\alpha_{20}\alpha_{18})(\alpha_{10}\alpha_{13})(\alpha_{15}\alpha_{22}),$$
$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_1,\alpha_{19},\alpha_6,\alpha_8,\alpha_{20},\alpha_{16},\alpha_{21},\alpha_{23},\alpha_{14},\alpha_5,\alpha_{18},\alpha_{11},\alpha_7,\alpha_9,$ $\alpha_{12},\alpha_4\}$, $\{\alpha_2,\alpha_{13},\alpha_{10}\}$, $\{\alpha_3,\alpha_{22},\alpha_{15}\}$;

$$H_{78,2}=$$
$$[(\alpha_4\alpha_{11}\alpha_{12})(\alpha_5\alpha_{18}\alpha_7)(\alpha_6\alpha_{23}\alpha_8)(\alpha_9\alpha_{14}\alpha_{16})(\alpha_{10}\alpha_{24}\alpha_{13})(\alpha_{19}\alpha_{20}\alpha_{21}),$$
$$(\alpha_1\alpha_{18}\alpha_7\alpha_5)(\alpha_4\alpha_{20}\alpha_{21}\alpha_8)(\alpha_6\alpha_{16}\alpha_9\alpha_{12})$$
$$(\alpha_{10}\alpha_{24})(\alpha_{11}\alpha_{14}\alpha_{23}\alpha_{19})(\alpha_{17}\alpha_{22}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_1,\alpha_{18},\alpha_7,\alpha_5\}$, $\{\alpha_2,\alpha_{13},\alpha_{10},\alpha_{24}\}$, $\{\alpha_3,\alpha_{22},\alpha_{17}\}$, $\{\alpha_4, \alpha_{11},\alpha_{20},\alpha_9,\alpha_{12},\alpha_{14},\alpha_{21},\alpha_6,\alpha_{23},\alpha_{16},\alpha_8,\alpha_{19}\}$.

n=77, $H \cong T_{192}$ ($|H| = 192$, $i = 1493$): $\mathrm{rk}\, N_H = 19$, $(N_H)^*/N_H \cong \mathbb{Z}/12\mathbb{Z} \times (\mathbb{Z}/4\mathbb{Z})^2$ and $\det(K((q_{N_H})_2)) \equiv \pm 2^6 \cdot 3 \mod (\mathbb{Z}_2^*)^2$.

$$H_{77,1} =$$

$$[(\alpha_1\alpha_{15}\alpha_8)(\alpha_3\alpha_{24}\alpha_{17})(\alpha_4\alpha_{20}\alpha_{16})(\alpha_5\alpha_{13}\alpha_6)(\alpha_{10}\alpha_{21}\alpha_{23})(\alpha_{12}\alpha_{14}\alpha_{22}),$$

$$(\alpha_1\alpha_{11}\alpha_{23}\alpha_8)(\alpha_4\alpha_6\alpha_{13}\alpha_{20})(\alpha_5\alpha_{16})$$

$$(\alpha_7\alpha_{17}\alpha_9\alpha_{14})(\alpha_{10}\alpha_{21}\alpha_{19}\alpha_{15})(\alpha_{12}\alpha_{24})]$$

with orbits $\{\alpha_1,\alpha_{15},\alpha_{11},\alpha_8,\alpha_{10},\alpha_{23},\alpha_{21},\alpha_{19}\}$, $\{\alpha_3,\alpha_{24},\alpha_{17},\alpha_{12},\alpha_9, \alpha_{14},\alpha_{22},\alpha_7\}$, $\{\alpha_4,\alpha_{20},\alpha_6,\alpha_{16},\alpha_5,\alpha_{13}\}$.

n=76, $H \cong H_{192}$ ($|H| = 192$, $i = 955$): $\mathrm{rk}\, N_H = 19$, $(N_H)^*/N_H \cong \mathbb{Z}/24\mathbb{Z} \times (\mathbb{Z}/4\mathbb{Z})^2$ and $\det(K((q_{N_H})_2) \equiv \pm 2^7 \cdot 3 \mod (\mathbb{Z}_2^*)^2$.

$$H_{76,1} =$$

$$[(\alpha_1\alpha_{17}\alpha_{19}\alpha_5)(\alpha_4\alpha_{24}\alpha_{21}\alpha_9)(\alpha_7\alpha_{23}\alpha_{20}\alpha_{22})$$

$$(\alpha_8\alpha_{16}\alpha_{14}\alpha_{15})(\alpha_{10}\alpha_{13})(\alpha_{12}\alpha_{18}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_1,\alpha_{17},\alpha_{19},\alpha_5,\alpha_{16},\alpha_{14},\alpha_{15},\alpha_8\}$, $\{\alpha_2,\alpha_{13},\alpha_{10}\}$, $\{\alpha_3,\alpha_{22}, \alpha_7,\alpha_{23},\alpha_{18},\alpha_{20},\alpha_{12},\alpha_{11}\}$, $\{\alpha_4,\alpha_{24},\alpha_9,\alpha_{21}\}$;

$$H_{76,2} =$$

$$[(\alpha_1\alpha_{18}\alpha_9\alpha_{21})(\alpha_4\alpha_{12}\alpha_5\alpha_{19})(\alpha_6\alpha_{23}\alpha_7\alpha_{20})$$

$$(\alpha_8\alpha_{14}\alpha_{11}\alpha_{16})(\alpha_{10}\alpha_{13})(\alpha_{17}\alpha_{24}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_1,\alpha_{18},\alpha_9,\alpha_7,\alpha_{21},\alpha_4,\alpha_{20},\alpha_{12},\alpha_6,\alpha_{11},\alpha_5,\alpha_{23},\alpha_{16},\alpha_{19}, \alpha_8,\alpha_{14}\}$, $\{\alpha_2,\alpha_{13},\alpha_{10}\}$, $\{\alpha_3,\alpha_{22}\}$, $\{\alpha_{17},\alpha_{24}\}$;

$$H_{76,3} =$$

$$[(\alpha_1\alpha_{19}\alpha_6\alpha_5)(\alpha_3\alpha_8\alpha_{15}\alpha_7)(\alpha_4\alpha_{14})$$

$$(\alpha_9\alpha_{22}\alpha_{23}\alpha_{10})(\alpha_{11}\alpha_{21}\alpha_{20}\alpha_{16})(\alpha_{12}\alpha_{18}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_1, \alpha_{19}, \alpha_6, \alpha_{16}, \alpha_5, \alpha_{11}, \alpha_{21}, \alpha_{20}\}$, $\{\alpha_2, \alpha_{13}\}$, $\{\alpha_3, \alpha_8, \alpha_{22}, \alpha_{15}, \alpha_{14}, \alpha_{23}, \alpha_7, \alpha_4, \alpha_{10}, \alpha_{12}, \alpha_{18}, \alpha_9\}$.

n=75, $H \cong 4^2\mathfrak{A}_4$ ($|H| = 192$, $i = 1023$): $\mathrm{rk}\, N_H = 18$, $(N_H)^*/N_H \cong (\mathbb{Z}/8\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2$ and $\det(K((q_{N_H})_2) \equiv \pm 2^8 \mod (\mathbb{Z}_2^*)^2$.

$$H_{75,1} =$$

$$[(\alpha_3\alpha_{16})(\alpha_4\alpha_5)(\alpha_6\alpha_{21})(\alpha_{10}\alpha_{20})(\alpha_{11}\alpha_{12})(\alpha_{13}\alpha_{17})(\alpha_{14}\alpha_{22})(\alpha_{23}\alpha_{24}),$$
$$(\alpha_1\alpha_9\alpha_7)(\alpha_3\alpha_{21}\alpha_{22})(\alpha_4\alpha_{14}\alpha_{24})(\alpha_5\alpha_{10}\alpha_{16})(\alpha_6\alpha_{20}\alpha_{23})(\alpha_{12}\alpha_{17}\alpha_{13}),$$
$$(\alpha_1\alpha_6\alpha_{22})(\alpha_3\alpha_{24}\alpha_{16})(\alpha_4\alpha_{20}\alpha_9)(\alpha_5\alpha_{10}\alpha_7)(\alpha_{11}\alpha_{13}\alpha_{12})(\alpha_{14}\alpha_{18}\alpha_{21})]$$

with orbits $\{\alpha_1, \alpha_9, \alpha_6, \alpha_7, \alpha_4, \alpha_{21}, \alpha_{20}, \alpha_{22}, \alpha_5, \alpha_{14}, \alpha_{10}, \alpha_{23}, \alpha_3, \alpha_{24}, \alpha_{18}, \alpha_{16}\}$, $\{\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{17}\}$.

n=74, $H \cong L_2(7)$ ($|H| = 168$, $i = 42$): $\mathrm{rk}\, N_H = 19$ and $(N_H)^*/N_H \cong \mathbb{Z}/28\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$.

$$H_{74,1} =$$

$$[(\alpha_6\alpha_8)(\alpha_9\alpha_{24})(\alpha_{10}\alpha_{22})(\alpha_{11}\alpha_{14})(\alpha_{12}\alpha_{13})(\alpha_{15}\alpha_{21})(\alpha_{17}\alpha_{20})(\alpha_{18}\alpha_{23}),$$
$$(\alpha_1\alpha_6\alpha_{22})(\alpha_3\alpha_{24}\alpha_{16})(\alpha_4\alpha_{20}\alpha_9)(\alpha_5\alpha_{10}\alpha_7)(\alpha_{11}\alpha_{13}\alpha_{12})(\alpha_{14}\alpha_{18}\alpha_{21})]$$

with orbits $\{\alpha_1, \alpha_6, \alpha_8, \alpha_{22}, \alpha_{10}, \alpha_7, \alpha_5\}$, $\{\alpha_3, \alpha_{24}, \alpha_9, \alpha_{16}, \alpha_4, \alpha_{20}, \alpha_{17}\}$, $\{\alpha_{11}, \alpha_{14}, \alpha_{13}, \alpha_{18}, \alpha_{12}, \alpha_{23}, \alpha_{21}, \alpha_{15}\}$;

$$H_{74,2} =$$

$$[(\alpha_5\alpha_{11})(\alpha_6\alpha_{10})(\alpha_7\alpha_{17})(\alpha_{12}\alpha_{19})(\alpha_{13}\alpha_{24})(\alpha_{14}\alpha_{15})(\alpha_{16}\alpha_{22})(\alpha_{18}\alpha_{20}),$$
$$(\alpha_1\alpha_6\alpha_{22})(\alpha_3\alpha_{24}\alpha_{16})(\alpha_4\alpha_{20}\alpha_9)(\alpha_5\alpha_{10}\alpha_7)(\alpha_{11}\alpha_{13}\alpha_{12})(\alpha_{14}\alpha_{18}\alpha_{21})]$$

with orbits $\{\alpha_1, \alpha_6, \alpha_{10}, \alpha_{22}, \alpha_7, \alpha_{16}, \alpha_{17}, \alpha_5, \alpha_3, \alpha_{11}, \alpha_{24}, \alpha_{13}, \alpha_{12}, \alpha_{19}\}$, $\{\alpha_4, \alpha_{20}, \alpha_{18}, \alpha_9, \alpha_{21}, \alpha_{14}, \alpha_{15}\}$.

n=73, $H \cong 2^4 D_{10}$ ($|H| = 160$, $i = 234$):

$$H_{73,1} =$$

$$[(\alpha_1\alpha_4\alpha_{20}\alpha_9\alpha_{19})(\alpha_3\alpha_5\alpha_{24}\alpha_{16}\alpha_{12})(\alpha_7\alpha_{21}\alpha_{17}\alpha_{15}\alpha_{13})(\alpha_8\alpha_{11}\alpha_{23}\alpha_{14}\alpha_{10}),$$
$$(\alpha_4\alpha_{13})(\alpha_5\alpha_{12})(\alpha_6\alpha_{17})(\alpha_7\alpha_9)(\alpha_8\alpha_{15})(\alpha_{11}\alpha_{21})(\alpha_{14}\alpha_{20})(\alpha_{16}\alpha_{24})]$$

with $Clos(H_{73,1}) = H_{81,1}$ above;

$$H_{73,2} =$$

$$[(\alpha_1\alpha_6\alpha_4\alpha_{24}\alpha_{15})(\alpha_5\alpha_8\alpha_{19}\alpha_{17}\alpha_9)(\alpha_7\alpha_{12}\alpha_{11}\alpha_{23}\alpha_{20})(\alpha_{10}\alpha_{14}\alpha_{13}\alpha_{16}\alpha_{21}),$$
$$(\alpha_1\alpha_9\alpha_{23}\alpha_{13})(\alpha_4\alpha_{10}\alpha_7\alpha_{19})(\alpha_5\alpha_{15}\alpha_{24}\alpha_8)$$

$$(\alpha_6\alpha_{14})(\alpha_{11}\alpha_{12}\alpha_{21}\alpha_{16})(\alpha_{17}\alpha_{20})]$$

with $Clos(H_{73,2}) = H_{81,2}$ above.

n=72, $H \cong \mathfrak{A}_4^2$ ($|H| = 144$, $i = 184$):

$$H_{72,1} = [(\alpha_1\alpha_4\alpha_{23}\alpha_7\alpha_{21}\alpha_{11})(\alpha_2\alpha_{10}\alpha_{13})(\alpha_3\alpha_{15}\alpha_{22})$$

$$(\alpha_5\alpha_9\alpha_{14}\alpha_{18}\alpha_6\alpha_{19})(\alpha_8\alpha_{20})(\alpha_{12}\alpha_{16}),$$

$$(\alpha_1\alpha_4\alpha_5\alpha_{21}\alpha_7\alpha_6)(\alpha_2\alpha_{10}\alpha_{13})(\alpha_3\alpha_{22}\alpha_{15})$$

$$(\alpha_8\alpha_{23}\alpha_{16}\alpha_{14}\alpha_{12}\alpha_{11})(\alpha_9\alpha_{18})(\alpha_{19}\alpha_{20})]$$

with $Clos(H_{72,1}) = H_{78,1}$ above;

$$H_{72,2} = [(\alpha_2\alpha_{10})(\alpha_3\alpha_{17}\alpha_{22})(\alpha_4\alpha_8\alpha_{11}\alpha_{21}\alpha_{12}\alpha_{14})$$

$$(\alpha_5\alpha_{18}\alpha_7)(\alpha_6\alpha_{20}\alpha_{19}\alpha_9\alpha_{16}\alpha_{23})(\alpha_{13}\alpha_{24}),$$

$$(\alpha_1\alpha_5)(\alpha_3\alpha_{17}\alpha_{22})(\alpha_4\alpha_{19}\alpha_{12}\alpha_9\alpha_{11}\alpha_8)$$

$$(\alpha_6\alpha_{23}\alpha_{20}\alpha_{21}\alpha_{14}\alpha_{16})(\alpha_7\alpha_{18})(\alpha_{10}\alpha_{13}\alpha_{24})]$$

with $Clos(H_{72,2}) = H_{78,2}$ above.

n=71, $H \cong F_{128}$ ($|H| = 128$, $i = 931$):

$$H_{71,1} =$$

$$[(\alpha_6\alpha_{15})(\alpha_8\alpha_{21})(\alpha_9\alpha_{23})(\alpha_{10}\alpha_{13})(\alpha_{11}\alpha_{17})(\alpha_{12}\alpha_{22})(\alpha_{14}\alpha_{20})(\alpha_{18}\alpha_{24}),$$

$$(\alpha_4\alpha_7)(\alpha_5\alpha_{16})(\alpha_6\alpha_{14})(\alpha_8\alpha_{11})(\alpha_9\alpha_{13})(\alpha_{12}\alpha_{24})(\alpha_{15}\alpha_{21})(\alpha_{17}\alpha_{20}),$$

$$(\alpha_1\alpha_6\alpha_{13}\alpha_{14}\alpha_{19}\alpha_{17}\alpha_9\alpha_{20})(\alpha_4\alpha_{11}\alpha_{23}\alpha_{21}\alpha_7\alpha_{15}\alpha_{10}\alpha_8)$$

$$(\alpha_5\alpha_{16})(\alpha_{12}\alpha_{22}\alpha_{24}\alpha_{18})]$$

with $Clos(H_{71,1}) = H_{80,1}$ above.

n=70, $H \cong \mathfrak{S}_5$ ($|H| = 120$, $i = 34$): $\mathrm{rk}\, N_H = 19$ and $(N_H)^*/N_H \cong \mathbb{Z}/60\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$.

$$H_{70,1} =$$

$$[(\alpha_5\alpha_{11})(\alpha_7\alpha_{12})(\alpha_8\alpha_{13})(\alpha_9\alpha_{20})(\alpha_{10}\alpha_{23})(\alpha_{14}\alpha_{21})(\alpha_{16}\alpha_{17})(\alpha_{19}\alpha_{22}),$$

$$(\alpha_2\alpha_{17}\alpha_{15}\alpha_8\alpha_{23})(\alpha_3\alpha_{14}\alpha_{10}\alpha_{13}\alpha_9)(\alpha_4\alpha_{24}\alpha_{18}\alpha_7\alpha_{12})(\alpha_5\alpha_{11}\alpha_{20}\alpha_{16}\alpha_{21})]$$

with orbits $\{\alpha_2, \alpha_{17}, \alpha_{16}, \alpha_{15}, \alpha_{21}, \alpha_8, \alpha_{14}, \alpha_5, \alpha_{13}, \alpha_{23}, \alpha_{10}, \alpha_{11}, \alpha_9, \alpha_{20}, \alpha_3\}$, $\{\alpha_4, \alpha_{24}, \alpha_{18}, \alpha_7, \alpha_{12}\}$, $\{\alpha_{19}, \alpha_{22}\}$.

$$H_{70,2} =$$

$$[(\alpha_3\alpha_9)(\alpha_4\alpha_8)(\alpha_6\alpha_{19})(\alpha_7\alpha_{24})(\alpha_{10}\alpha_{13})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{17})(\alpha_{14}\alpha_{22}),$$

$(\alpha_2\alpha_{17}\alpha_{15}\alpha_8\alpha_{23})(\alpha_3\alpha_{14}\alpha_{10}\alpha_{13}\alpha_9)(\alpha_4\alpha_{24}\alpha_{18}\alpha_7\alpha_{12})(\alpha_5\alpha_{11}\alpha_{20}\alpha_{16}\alpha_{21})]$

with orbits $\{\alpha_2,\alpha_{17},\alpha_{12},\alpha_{15},\alpha_4,\alpha_8,\alpha_{24},\alpha_{23},\alpha_7,\alpha_{18}\}$, $\{\alpha_3,\alpha_9,\alpha_{14},\alpha_{22},\alpha_{10},\alpha_{13}\}$, $\{\alpha_5,\alpha_{11},\alpha_{20},\alpha_{16},\alpha_{21}\}$, $\{\alpha_6,\alpha_{19}\}$.

n=69, $H\cong (Q_8*Q_8)\rtimes C_3$ ($|H|=96$, $i=204$):

$$H_{69,1}=$$

$$[(\alpha_3\alpha_{14}\alpha_7)(\alpha_4\alpha_{20}\alpha_5)(\alpha_6\alpha_{16}\alpha_{13})(\alpha_8\alpha_{23}\alpha_{11})(\alpha_9\alpha_{22}\alpha_{17})(\alpha_{10}\alpha_{15}\alpha_{19}),$$
$$(\alpha_1\alpha_{11}\alpha_{15}\alpha_{21}\alpha_{19}\alpha_{23})(\alpha_3\alpha_{12}\alpha_{14}\alpha_{22}\alpha_{24}\alpha_{17})(\alpha_4\alpha_{20}\alpha_5)$$
$$(\alpha_6\alpha_{16}\alpha_{13})(\alpha_7\alpha_9)(\alpha_8\alpha_{10}),$$
$$(\alpha_1\alpha_8\alpha_{11}\alpha_{21}\alpha_{10}\alpha_{19})(\alpha_3\alpha_7\alpha_{12}\alpha_{22}\alpha_9\alpha_{24})$$
$$(\alpha_4\alpha_6\alpha_5)(\alpha_{13}\alpha_{20}\alpha_{16})(\alpha_{14}\alpha_{17})(\alpha_{15}\alpha_{23})]$$

with $Clos(H_{69,1})=H_{77,1}$ above.

n=68, $H\cong 2^3D_{12}$, ($|H|=96$, $i=195$):

$$H_{68,1}=[(\alpha_1\alpha_5\alpha_7\alpha_{18})(\alpha_3\alpha_{22})(\alpha_4\alpha_{19}\alpha_{21}\alpha_{14})$$
$$(\alpha_6\alpha_{11}\alpha_9\alpha_{23})(\alpha_8\alpha_{16}\alpha_{20}\alpha_{12})(\alpha_{10}\alpha_{13}),$$
$$(\alpha_1\alpha_4\alpha_5\alpha_{21}\alpha_7\alpha_6)(\alpha_2\alpha_{10}\alpha_{13})(\alpha_3\alpha_{22}\alpha_{15})$$
$$(\alpha_8\alpha_{23}\alpha_{16}\alpha_{14}\alpha_{12}\alpha_{11})(\alpha_9\alpha_{18})(\alpha_{19}\alpha_{20})]$$

with $Clos(H_{68,1})=H_{78,1}$ above;

$$H_{68,2}=[(\alpha_2\alpha_{10}\alpha_{24}\alpha_{13})(\alpha_4\alpha_{21}\alpha_6\alpha_9)(\alpha_5\alpha_7)$$
$$(\alpha_8\alpha_{19}\alpha_{20}\alpha_{11})(\alpha_{12}\alpha_{14}\alpha_{16}\alpha_{23})(\alpha_{17}\alpha_{22}),$$
$$(\alpha_1\alpha_5)(\alpha_3\alpha_{17}\alpha_{22})(\alpha_4\alpha_{19}\alpha_{12}\alpha_9\alpha_{11}\alpha_8)$$
$$(\alpha_6\alpha_{23}\alpha_{20}\alpha_{21}\alpha_{14}\alpha_{16})(\alpha_7\alpha_{18})(\alpha_{10}\alpha_{13}\alpha_{24})]$$

with $Clos(H_{68,2})=H_{78,2}$ above.

n=67, $H\cong 4^2D_6$, ($|H|=96$, $i=64$):

$$H_{67,1}=[(\alpha_4\alpha_7\alpha_{19})(\alpha_6\alpha_8\alpha_9)(\alpha_{10}\alpha_{11}\alpha_{21})$$
$$(\alpha_{12}\alpha_{18}\alpha_{22})(\alpha_{13}\alpha_{15}\alpha_{20})(\alpha_{14}\alpha_{23}\alpha_{17}),$$
$$(\alpha_1\alpha_6\alpha_{13}\alpha_{14}\alpha_{19}\alpha_{17}\alpha_9\alpha_{20})(\alpha_4\alpha_{11}\alpha_{23}\alpha_{21}\alpha_7\alpha_{15}\alpha_{10}\alpha_8)$$
$$(\alpha_5\alpha_{16})(\alpha_{12}\alpha_{22}\alpha_{24}\alpha_{18})]$$

with $Clos(H_{67,1})=H_{80,1}$ above.

n=66, $H \cong 2^4C_6$ ($|H| = 96$, $i = 70$):

$$H_{66,1} = [(\alpha_1\alpha_5\alpha_{15}\alpha_8\alpha_{16}\alpha_{14})(\alpha_2\alpha_{13}\alpha_{10})$$

$$(\alpha_3\alpha_7\alpha_{11}\alpha_{23}\alpha_{12}\alpha_{22})(\alpha_4\alpha_{24}\alpha_{21})(\alpha_{17}\alpha_{19})(\alpha_{18}\alpha_{20}),$$

$$(\alpha_2\alpha_{10}\alpha_{13})(\alpha_5\alpha_{17}\alpha_8)(\alpha_7\alpha_{23}\alpha_{22})(\alpha_9\alpha_{24}\alpha_{21})(\alpha_{11}\alpha_{18}\alpha_{12})(\alpha_{15}\alpha_{19}\alpha_{16})]$$

with $Clos(H_{66,1}) = H_{76,1}$ above;

$$H_{66,2} = [(\alpha_1\alpha_4\alpha_5\alpha_{16}\alpha_{18}\alpha_{11})(\alpha_2\alpha_{10}\alpha_{13})(\alpha_3\alpha_{22})$$

$$(\alpha_6\alpha_8\alpha_{20}\alpha_{19}\alpha_{23}\alpha_{21})(\alpha_9\alpha_{14}\alpha_{12})(\alpha_{17}\alpha_{24}),$$

$$(\alpha_2\alpha_{10}\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9\alpha_{20}\alpha_8\alpha_{14}\alpha_{11})$$

$$(\alpha_5\alpha_{23}\alpha_7\alpha_{21}\alpha_{18}\alpha_{12})(\alpha_6\alpha_{19}\alpha_{16})(\alpha_{17}\alpha_{24})]$$

with $Clos(H_{66,2}) = H_{76,2}$ above;

$$H_{66,3} = [(\alpha_3\alpha_8\alpha_{18})(\alpha_4\alpha_{22}\alpha_{23})(\alpha_5\alpha_{19}\alpha_{21})$$

$$(\alpha_6\alpha_{20}\alpha_{11})(\alpha_7\alpha_{12}\alpha_{15})(\alpha_9\alpha_{14}\alpha_{10}),$$

$$(\alpha_1\alpha_5\alpha_{20}\alpha_{21}\alpha_6\alpha_{16})(\alpha_2\alpha_{13})(\alpha_3\alpha_4\alpha_7\alpha_{22}\alpha_{18}\alpha_9)$$

$$(\alpha_8\alpha_{15}\alpha_{12})(\alpha_{10}\alpha_{14}\alpha_{23})(\alpha_{11}\alpha_{19})]$$

with $Clos(H_{66,3}) = H_{76,3}$ above.

n=65, $H \cong 2^4D_6$ ($|H| = 96$, $i = 227$): $\mathrm{rk}\, N_H = 18$, $(N_H)^*/N_H \cong \mathbb{Z}/24\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2$ and $\det(K((q_{N_H})_2) \equiv \pm 2^7 \cdot 3 \mod (\mathbb{Z}_2^*)^2$.

$$H_{65,1} =$$

$$[(\alpha_1\alpha_6\alpha_{21})(\alpha_4\alpha_{17}\alpha_{14})(\alpha_7\alpha_{15}\alpha_8)(\alpha_9\alpha_{10}\alpha_{23})(\alpha_{11}\alpha_{20}\alpha_{19})(\alpha_{12}\alpha_{18}\alpha_{24}),$$

$$(\alpha_1\alpha_{11}\alpha_{23}\alpha_8)(\alpha_4\alpha_6\alpha_{13}\alpha_{20})(\alpha_5\alpha_{16})$$

$$(\alpha_7\alpha_{17}\alpha_9\alpha_{14})(\alpha_{10}\alpha_{21}\alpha_{19}\alpha_{15})(\alpha_{12}\alpha_{24})]$$

with orbits $\{\alpha_1, \alpha_{19}, \alpha_6, \alpha_7, \alpha_4, \alpha_{17}, \alpha_{20}, \alpha_{11}, \alpha_{14}, \alpha_{21}, \alpha_{15}, \alpha_{23}, \alpha_8, \alpha_{10}, \alpha_9, \alpha_{13}\}$, $\{\alpha_5, \alpha_{16}\}$, $\{\alpha_{12}, \alpha_{24}, \alpha_{18}\}$;

$$H_{65,2} =$$

$$[(\alpha_1\alpha_{10}\alpha_{19})(\alpha_4\alpha_{17}\alpha_{24})(\alpha_5\alpha_7\alpha_6)(\alpha_9\alpha_{20}\alpha_{16}),(\alpha_{11}\alpha_{15}\alpha_{21})(\alpha_{12}\alpha_{13}\alpha_{14}),$$

$$(\alpha_1\alpha_{11}\alpha_{23}\alpha_8)(\alpha_4\alpha_6\alpha_{13}\alpha_{20})(\alpha_5\alpha_{16})$$

$$(\alpha_7\alpha_{17}\alpha_9\alpha_{14})(\alpha_{10}\alpha_{21}\alpha_{19}\alpha_{15})(\alpha_{12}\alpha_{24})]$$

with orbits $\{\alpha_1, \alpha_{11}, \alpha_{10}, \alpha_{23}, \alpha_{19}, \alpha_{15}, \alpha_8, \alpha_{21}\}$, $\{\alpha_4, \alpha_{17}, \alpha_7, \alpha_9, \alpha_{24}, \alpha_6, \alpha_{20}, \alpha_{13}, \alpha_5, \alpha_{16}, \alpha_{14}, \alpha_{12}\}$;

$$H_{65,3} =$$

$$[(\alpha_1\alpha_7\alpha_{17})(\alpha_3\alpha_{22}\alpha_5)(\alpha_4\alpha_{23}\alpha_{20})(\alpha_6\alpha_{14}\alpha_8)(\alpha_9\alpha_{13}\alpha_{11})(\alpha_{10}\alpha_{19}\alpha_{21}),$$

$$(\alpha_1\alpha_{11}\alpha_{23}\alpha_8)(\alpha_4\alpha_6\alpha_{13}\alpha_{20})(\alpha_5\alpha_{16})$$

$$(\alpha_7\alpha_{17}\alpha_9\alpha_{14})(\alpha_{10}\alpha_{21}\alpha_{19}\alpha_{15})(\alpha_{12}\alpha_{24})]$$

with orbits $\{\alpha_1, \alpha_7, \alpha_{23}, \alpha_8, \alpha_{11}, \alpha_{17}, \alpha_{20}, \alpha_9, \alpha_6, \alpha_4, \alpha_{14}, \alpha_{13}\}$, $\{\alpha_3, \alpha_{22}, \alpha_{16}, \alpha_5\}$, $\{\alpha_{10}, \alpha_{19}, \alpha_{15}, \alpha_{21}\}$, $\{\alpha_{12}, \alpha_{24}\}$;

$$H_{65,4} = [(\alpha_1\alpha_{21}\alpha_{23})(\alpha_3\alpha_{22}\alpha_{12})(\alpha_4\alpha_{20}\alpha_{13})$$

$$(\alpha_5\alpha_{18}\alpha_{16})(\alpha_8\alpha_{11}\alpha_{10})(\alpha_9\alpha_{14}\alpha_{17}),$$

$$(\alpha_1\alpha_{11}\alpha_{23}\alpha_8)(\alpha_4\alpha_6\alpha_{13}\alpha_{20})(\alpha_5\alpha_{16})$$

$$(\alpha_7\alpha_{17}\alpha_9\alpha_{14})(\alpha_{10}\alpha_{21}\alpha_{19}\alpha_{15})(\alpha_{12}\alpha_{24})]$$

with orbits $\{\alpha_1, \alpha_8, \alpha_{21}, \alpha_{23}, \alpha_{15}, \alpha_{11}, \alpha_{10}, \alpha_{19}\}$, $\{\alpha_3, \alpha_{22}, \alpha_{24}, \alpha_{12}\}$, $\{\alpha_4, \alpha_{13}, \alpha_{20}, \alpha_6\}$, $\{\alpha_5, \alpha_{16}, \alpha_{18}\}$, $\{\alpha_7, \alpha_{17}, \alpha_{14}, \alpha_9, \}$.

n=64, $H \cong 2^4C_5$ ($|H| = 80$, $i = 49$):

$$H_{64,1} =$$

$$[(\alpha_3\alpha_5\alpha_{16}\alpha_{12}\alpha_{24})(\alpha_4\alpha_{11}\alpha_{15}\alpha_9\alpha_{23})(\alpha_6\alpha_{20}\alpha_{21}\alpha_{10}\alpha_8)(\alpha_7\alpha_{13}\alpha_{17}\alpha_{19}\alpha_{14}),$$

$$(\alpha_1\alpha_{13})(\alpha_4\alpha_{23})(\alpha_6\alpha_{21})(\alpha_7\alpha_{10})(\alpha_8\alpha_{17})(\alpha_9\alpha_{19})(\alpha_{11}\alpha_{14})(\alpha_{15}\alpha_{20})]$$

with $Clos(H_{64,1}) = H_{81,1}$ above;

$$H_{64,2} =$$

$$[(\alpha_1\alpha_4\alpha_5\alpha_{14}\alpha_8)(\alpha_6\alpha_{21}\alpha_{19}\alpha_9\alpha_{16})(\alpha_7\alpha_{12}\alpha_{20}\alpha_{11}\alpha_{23})(\alpha_{10}\alpha_{13}\alpha_{24}\alpha_{17}\alpha_{15}),$$

$$(\alpha_4\alpha_{13})(\alpha_5\alpha_{12})(\alpha_6\alpha_{17})(\alpha_7\alpha_9)(\alpha_8\alpha_{15})(\alpha_{11}\alpha_{21})(\alpha_{14}\alpha_{20})(\alpha_{16}\alpha_{24})]$$

with $Clos(H_{64,2}) = H_{81,2}$ above.

n=63, $H \cong M_9$ ($|H| = 72$, $i = 41$): $\mathrm{rk}\, N_H = 19$ and $(N_H)^*/N_H \cong \mathbb{Z}/18\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

$$H_{63,1} = [(\alpha_1\alpha_9\alpha_{17}\alpha_{14})(\alpha_4\alpha_{19})(\alpha_5\alpha_{15}\alpha_6\alpha_{21})$$

$$(\alpha_7\alpha_{11}\alpha_{23}\alpha_{22})(\alpha_{10}\alpha_{13}\alpha_{24}\alpha_{16})(\alpha_{12}\alpha_{20}),$$

$$(\alpha_1\alpha_{11}\alpha_{23}\alpha_8)(\alpha_4\alpha_6\alpha_{13}\alpha_{20})(\alpha_5\alpha_{16})$$

$$(\alpha_7\alpha_{17}\alpha_9\alpha_{14})(\alpha_{10}\alpha_{21}\alpha_{19}\alpha_{15})(\alpha_{12}\alpha_{24})]$$

with orbits $\{\alpha_1,\alpha_{23},\alpha_9,\alpha_{17},\alpha_{22},\alpha_7,\alpha_{11},\alpha_{14},\alpha_8\}$, $\{\alpha_4,\alpha_{20},\alpha_{19},\alpha_{16},\alpha_{12},\alpha_6,\alpha_{24},\alpha_{10},\alpha_{13},\alpha_{15},\alpha_{21},\alpha_5\}$.

n=62, $H\cong N_{72}$ ($|H|=72$, $i=40$): $\operatorname{rk} N_H=19$, $(N_H)^*/N_H\cong \mathbb{Z}/36\mathbb{Z}\times(\mathbb{Z}/3\mathbb{Z})^2$ and $\det(K((q_{N_H})_3))\equiv -2^2\cdot 3^4 \mod (\mathbb{Z}_3^*)^2$.

$$H_{62,1}=$$

$$[(\alpha_1\alpha_4)(\alpha_2\alpha_{15})(\alpha_7\alpha_{21})(\alpha_8\alpha_{20})(\alpha_{10}\alpha_{22})(\alpha_{11}\alpha_{16})(\alpha_{12}\alpha_{23})(\alpha_{14}\alpha_{19}),$$

$$(\alpha_1\alpha_{19})(\alpha_2\alpha_{18}\alpha_3\alpha_{23}\alpha_{10}\alpha_{22})(\alpha_4\alpha_{20}\alpha_{14})$$

$$(\alpha_5\alpha_{12}\alpha_{15})(\alpha_6\alpha_{16}\alpha_9\alpha_{24}\alpha_{17}\alpha_{11})(\alpha_7\alpha_{21})]$$

with orbits $\{\alpha_1,\alpha_4,\alpha_{19},\alpha_8,\alpha_{20},\alpha_{14}\}$, $\{\alpha_2,\alpha_{15},\alpha_{10},\alpha_{18},\alpha_{12},\alpha_{22},\alpha_{23},\alpha_5,\alpha_3\}$, $\{\alpha_6,\alpha_{11},\alpha_9,\alpha_{16},\alpha_{17},\alpha_{24}\}$, $\{\alpha_7,\alpha_{21}\}$.

n=61, $H\cong\mathfrak{A}_{4,3}$ ($|H|=72$, $i=43$): $\operatorname{rk} N_H=18$ and $(N_H)^*/N_H\cong(\mathbb{Z}/12\mathbb{Z})^2\times\mathbb{Z}/3\mathbb{Z}$.

$$H_{61,1}=$$

$$[(\alpha_1\alpha_{10}\alpha_{23})(\alpha_2\alpha_4\alpha_{11})(\alpha_3\alpha_{21}\alpha_{20})(\alpha_8\alpha_9\alpha_{17})(\alpha_{13}\alpha_{22}\alpha_{14})(\alpha_{16}\alpha_{19}\alpha_{24}),$$

$$(\alpha_2\alpha_{14})(\alpha_3\alpha_{17})(\alpha_4\alpha_8)(\alpha_7\alpha_{15})(\alpha_9\alpha_{13})(\alpha_{10}\alpha_{12})(\alpha_{11}\alpha_{21})(\alpha_{16}\alpha_{19}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_1,\alpha_{10},\alpha_{23},\alpha_{12}\}$, $\{\alpha_2,\alpha_4,\alpha_{21},\alpha_{13},\alpha_{11}\alpha_8,\alpha_{20},\alpha_{17},\alpha_{22}\alpha_{14},\alpha_9,\alpha_3\}$, $\{\alpha_7,\alpha_{18},\alpha_{15}\}$, $\{\alpha_{16},\alpha_{19},\alpha_{24}\}$.

n=60, $H\cong\Gamma_{26}a_2$ ($|H|=64$, $i=136$):

$$H_{60,1}=$$

$$[(\alpha_4\alpha_7)(\alpha_5\alpha_{16})(\alpha_6\alpha_8)(\alpha_{10}\alpha_{23})(\alpha_{11}\alpha_{14})(\alpha_{15}\alpha_{20})(\alpha_{17}\alpha_{21})(\alpha_{18}\alpha_{22}),$$

$$(\alpha_1\alpha_6\alpha_{23}\alpha_8\alpha_{19}\alpha_{17}\alpha_{10}\alpha_{21})(\alpha_4\alpha_{11}\alpha_{13}\alpha_{20}\alpha_7\alpha_{15}\alpha_9\alpha_{14})$$

$$(\alpha_5\alpha_{16})(\alpha_{12}\alpha_{18}\alpha_{24}\alpha_{22}),$$

$$(\alpha_1\alpha_6\alpha_{13}\alpha_{14}\alpha_{19}\alpha_{17}\alpha_9\alpha_{20})(\alpha_4\alpha_{11}\alpha_{23}\alpha_{21}\alpha_7\alpha_{15}\alpha_{10}\alpha_8)$$

$$(\alpha_5\alpha_{16})(\alpha_{12}\alpha_{22}\alpha_{24}\alpha_{18})]$$

with $Clos(H_{60,1})=H_{80,1}$ above.

n=59, $H\cong\Gamma_{23}a_2$ ($|H|=64$, $i=35$):

$$H_{59,1}=[(\alpha_4\alpha_{19})(\alpha_5\alpha_{16})(\alpha_6\alpha_{13}\alpha_{17}\alpha_{23})$$

$$(\alpha_8\alpha_{14}\alpha_{20}\alpha_{21})(\alpha_9\alpha_{11}\alpha_{10}\alpha_{15})(\alpha_{12}\alpha_{22}\alpha_{18}\alpha_{24}),$$

$$(\alpha_1\alpha_6\alpha_7\alpha_{11})(\alpha_4\alpha_{15}\alpha_{19}\alpha_{17})(\alpha_8\alpha_{10}\alpha_{20}\alpha_{13})$$

$$(\alpha_9\alpha_{21}\alpha_{23}\alpha_{14})(\alpha_{12}\alpha_{24})(\alpha_{18}\alpha_{22})]$$

with $Clos(H_{59,1}) = H_{80,1}$ above.

n=58, $H \cong \Gamma_{22}a_1$ ($|H| = 64$, $i = 32$):

$$H_{58,1,1} = [(\alpha_1\alpha_6\alpha_7\alpha_{11})(\alpha_4\alpha_{15}\alpha_{19}\alpha_{17})$$

$$(\alpha_8\alpha_{10}\alpha_{20}\alpha_{13})(\alpha_9\alpha_{21}\alpha_{23}\alpha_{14})(\alpha_{12}\alpha_{24})(\alpha_{18}\alpha_{22}),$$

$$(\alpha_1\alpha_6\alpha_{13}\alpha_{14}\alpha_{19}\alpha_{17}\alpha_9\alpha_{20})(\alpha_4\alpha_{11}\alpha_{23}\alpha_{21}\alpha_7\alpha_{15}\alpha_{10}\alpha_8)$$

$$(\alpha_5\alpha_{16})(\alpha_{12}\alpha_{22}\alpha_{24}\alpha_{18})]$$

with $Clos(H_{58,1,1}) = H_{80,1}$ above;

$$H_{58,1,2} = [(\alpha_1\alpha_9\alpha_7\alpha_{23})(\alpha_4\alpha_{10}\alpha_{19}\alpha_{13})$$

$$(\alpha_6\alpha_{14}\alpha_{11}\alpha_{21})(\alpha_8\alpha_{17}\alpha_{20}\alpha_{15})(\alpha_{12}\alpha_{22})(\alpha_{18}\alpha_{24}),$$

$$(\alpha_1\alpha_6\alpha_{13}\alpha_{14}\alpha_{19}\alpha_{17}\alpha_9\alpha_{20})(\alpha_4\alpha_{11}\alpha_{23}\alpha_{21}\alpha_7\alpha_{15}\alpha_{10}\alpha_8)$$

$$(\alpha_5\alpha_{16})(\alpha_{12}\alpha_{22}\alpha_{24}\alpha_{18})]$$

with $Clos(H_{58,1,2}) = H_{80,1}$ above.

n=57, $H \cong \Gamma_{13}a_1$ ($|H| = 64$, $i = 242$):

$$H_{57,1} = [(\alpha_1\alpha_3\alpha_7\alpha_{23})(\alpha_4\alpha_{20}\alpha_6\alpha_{10})$$

$$(\alpha_5\alpha_{14}\alpha_{21}\alpha_{22})(\alpha_9\alpha_{24}\alpha_{18}\alpha_{16})(\alpha_{11}\alpha_{13})(\alpha_{12}\alpha_{17}),$$

$$(\alpha_1\alpha_{10}\alpha_7\alpha_{20})(\alpha_3\alpha_6\alpha_{23}\alpha_4)(\alpha_5\alpha_{24}\alpha_{21}\alpha_{16})$$

$$(\alpha_9\alpha_{22}\alpha_{18}\alpha_{14})(\alpha_{11}\alpha_{12})(\alpha_{13}\alpha_{17}),$$

$$(\alpha_3\alpha_{24})(\alpha_4\alpha_6)(\alpha_5\alpha_{21})(\alpha_{10}\alpha_{14})(\alpha_{11}\alpha_{17})(\alpha_{12}\alpha_{13})(\alpha_{16}\alpha_{23})(\alpha_{20}\alpha_{22}),$$

$$(\alpha_3\alpha_{16})(\alpha_4\alpha_5)(\alpha_6\alpha_{21})(\alpha_{10}\alpha_{20})(\alpha_{11}\alpha_{12})(\alpha_{13}\alpha_{17})(\alpha_{14}\alpha_{22})(\alpha_{23}\alpha_{24})]$$

with $Clos(H_{57,1}) = H_{75,1}$ above.

n=56, $H \cong \Gamma_{25}a_1$ ($|H| = 64$, $i = 138$): $\mathrm{rk}\, N_H = 18$, $(N_H)^*/N_H \cong \mathbb{Z}/8\mathbb{Z} \times (\mathbb{Z}/4\mathbb{Z})^3$ and $\det(K((q_{N_H})_2)) \equiv \pm 2^9 \mod (\mathbb{Z}_2^*)^2$.

$$H_{56,1} =$$

$$[(\alpha_5\alpha_{23})(\alpha_6\alpha_{10})(\alpha_7\alpha_9)(\alpha_8\alpha_{16})(\alpha_{13}\alpha_{18})(\alpha_{14}\alpha_{24})(\alpha_{15}\alpha_{20})(\alpha_{19}\alpha_{21}),$$

$$(\alpha_1\alpha_{14})(\alpha_2\alpha_{23})(\alpha_3\alpha_5)(\alpha_6\alpha_{18})(\alpha_7,\alpha_8)(\alpha_9\alpha_{13})(\alpha_{10}\alpha_{16})(\alpha_{17}\alpha_{24}),$$

$$(\alpha_1\alpha_{14})(\alpha_2\alpha_{24})(\alpha_3\alpha_5)(\alpha_6\alpha_{10})(\alpha_{12}\alpha_{22})(\alpha_{16}\alpha_{18})(\alpha_{17}\alpha_{23})(\alpha_{19}\alpha_{20})]$$

with orbits $\{\alpha_1, \alpha_{14}, \alpha_{24}, \alpha_{17}, \alpha_2, \alpha_{23}, \alpha_5, \alpha_3\}$, $\{\alpha_6, \alpha_{10}, \alpha_{18}, \alpha_{16}, \alpha_{13}, \alpha_8, \alpha_9, \alpha_7\}$, $\{\alpha_{12}, \alpha_{22}\}$, $\{\alpha_{15}, \alpha_{20}, \alpha_{19}, \alpha_{21}\}$;

$$H_{56,2} =$$

$$[(\alpha_2\alpha_3)(\alpha_5\alpha_6)(\alpha_7\alpha_{18})(\alpha_8\alpha_{23})(\alpha_{10}\alpha_{20})(\alpha_{11}\alpha_{17})(\alpha_{15}\alpha_{16})(\alpha_{19}\alpha_{24}),$$

$$(\alpha_1\alpha_{14})(\alpha_2\alpha_{23})(\alpha_3\alpha_5)(\alpha_6\alpha_{18})(\alpha_7\alpha_8)(\alpha_9\alpha_{13})(\alpha_{10}\alpha_{16})(\alpha_{17}\alpha_{24}),$$

$$(\alpha_1\alpha_{14})(\alpha_2\alpha_{24})(\alpha_3\alpha_5)(\alpha_6\alpha_{10})(\alpha_{12}\alpha_{22})(\alpha_{16}\alpha_{18})(\alpha_{17}\alpha_{23})(\alpha_{19}\alpha_{20})]$$

with orbits $\{\alpha_1, \alpha_{14}\}$, $\{\alpha_2, \alpha_3, \alpha_{23}, \alpha_{24}, \alpha_5, \alpha_8, \alpha_{17}, \alpha_{19}, \alpha_6, \alpha_7, \alpha_{11}, \alpha_{20}, \alpha_{18}, \alpha_{10}, \alpha_{16}, \alpha_{15}\}$, $\{\alpha_9, \alpha_{13}\}$, $\{\alpha_{12}, \alpha_{22}\}$.

n=55, $H \cong \mathfrak{A}_5$ ($|H| = 60$, $i = 5$): $\operatorname{rk} N_H = 18$ and $(N_H)^*/N_H \cong \mathbb{Z}/30\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$.

$$H_{55,1} =$$

$$[(\alpha_2\alpha_{13}\alpha_{18}\alpha_7\alpha_5)(\alpha_3\alpha_{22}\alpha_{12}\alpha_{11}\alpha_{19})(\alpha_4\alpha_6\alpha_{15}\alpha_9\alpha_{14})(\alpha_{16}\alpha_{21}\alpha_{20}\alpha_{17}\alpha_{23}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_2, \alpha_{13}, \alpha_{18}, \alpha_7, \alpha_5\}$, $\{\alpha_3, \alpha_{22}, \alpha_{12}, \alpha_{11}, \alpha_{23}, \alpha_{19}, \alpha_{20}, \alpha_{16}, \alpha_{17}, \alpha_{21}\}$, $\{\alpha_4, \alpha_6, \alpha_9, \alpha_{15}, \alpha_{14}, \alpha_8\}$;

$$H_{55,2} =$$

$$[(\alpha_2\alpha_8\alpha_7\alpha_{17}\alpha_{11})(\alpha_3\alpha_{14}\alpha_{10}\alpha_{13}\alpha_{12})(\alpha_4\alpha_{19}\alpha_{16}\alpha_{15}\alpha_9)(\alpha_6\alpha_{23}\alpha_{20}\alpha_{18}\alpha_{22}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_2, \alpha_8, \alpha_{13}, \alpha_7, \alpha_{14}, \alpha_{12}, \alpha_{17}, \alpha_{18}, \alpha_{10}, \alpha_3, \alpha_{23}, \alpha_{11}, \alpha_{22}, \alpha_{20}, \alpha_6\}$, $\{\alpha_4, \alpha_{19}, \alpha_9, \alpha_{16}, \alpha_{15}\}$.

n=54, $H \cong T_{48}$ ($|H| = 48$, $i = 29$): $\operatorname{rk} N_H = 19$ and $(N_H)^*/N_H \cong \mathbb{Z}/24\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

$$H_{54,1} =$$

$$[(\alpha_1\alpha_{13}\alpha_{22})(\alpha_3\alpha_{10}\alpha_7)(\alpha_4\alpha_{23}\alpha_{24})(\alpha_5\alpha_9\alpha_{19})(\alpha_6\alpha_8\alpha_{14})(\alpha_{11}\alpha_{17}\alpha_{21}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_1, \alpha_2, \alpha_7, \alpha_{22}, \alpha_{10}, \alpha_3, \alpha_{13}, \alpha_{18}\}$, $\{\alpha_4, \alpha_{12}, \alpha_{16}, \alpha_{19}, \alpha_{24}, \alpha_9, \alpha_5, \alpha_{23}\}$, $\{\alpha_6, \alpha_8, \alpha_{14}\}$, $\{\alpha_{11}, \alpha_{21}, \alpha_{20}, \alpha_{17}\}$.

n=53, $H \cong 2^2Q_{12}$ ($|H| = 48$, $i = 30$):

$$H_{53,1} =$$

$$[(\alpha_2\alpha_{10}\alpha_{13})(\alpha_3\alpha_{15}\alpha_{22})(\alpha_4\alpha_{16}\alpha_{11})(\alpha_6\alpha_8\alpha_{14})(\alpha_9\alpha_{20}\alpha_{19})(\alpha_{12}\alpha_{23}\alpha_{21}),$$

$$(\alpha_1\alpha_4\alpha_{14}\alpha_{20})(\alpha_5\alpha_9\alpha_{23}\alpha_{16})(\alpha_6\alpha_{19}\alpha_{12}\alpha_7)$$

$$(\alpha_8\alpha_{18}\alpha_{21}\alpha_{11})(\alpha_{10}\alpha_{13})(\alpha_{15}\alpha_{22})]$$

with $Clos(H_{53,1}) = H_{78,1}$ above;

$$H_{53,2} = [(\alpha_1\alpha_5\alpha_7)(\alpha_2\alpha_{13})(\alpha_3\alpha_{17}\alpha_{22})$$

$$(\alpha_4\alpha_{16}\alpha_{23}\alpha_9\alpha_8\alpha_{11})(\alpha_6\alpha_{12}\alpha_{14}\alpha_{21}\alpha_{20}\alpha_{19})(\alpha_{10}\alpha_{24}),$$

$$(\alpha_2\alpha_{10}\alpha_{13}\alpha_{24})(\alpha_4\alpha_8\alpha_9\alpha_{16})(\alpha_5\alpha_{18})$$

$$(\alpha_6\alpha_{12}\alpha_{21}\alpha_{20})(\alpha_{11}\alpha_{14}\alpha_{23}\alpha_{19})(\alpha_{17}\alpha_{22})]$$

with $Clos(H_{53,2}) = H_{78,2}$ above.

n=52, $H \cong 2^2(C_2 \times C_6)$ ($|H| = 48$, $i = 49$):

$$H_{52,1} = [(\alpha_1\alpha_8\alpha_{21}\alpha_5\alpha_{12}\alpha_6)(\alpha_2\alpha_{10}\alpha_{13})(\alpha_3\alpha_{15}\alpha_{22})$$

$$(\alpha_4\alpha_{18}\alpha_{16}\alpha_9\alpha_7\alpha_{20})(\alpha_{11}\alpha_{19})(\alpha_{14}\alpha_{23}),$$

$$(\alpha_1\alpha_4)(\alpha_5\alpha_9)(\alpha_6\alpha_{18})(\alpha_7\alpha_{21})(\alpha_8\alpha_{19})(\alpha_{11}\alpha_{12})(\alpha_{14}\alpha_{20})(\alpha_{16}\alpha_{23})]$$

with $Clos(H_{52,1}) = H_{78,1}$ above;

$$H_{52,2} = [(\alpha_2\alpha_{13})(\alpha_3\alpha_{22}\alpha_{17})(\alpha_4\alpha_{23}\alpha_{12}\alpha_9\alpha_{11}\alpha_{20})$$

$$(\alpha_5\alpha_7\alpha_{18})(\alpha_6\alpha_{14}\alpha_{16}\alpha_{21}\alpha_{19}\alpha_8)(\alpha_{10}\alpha_{24}),$$

$$(\alpha_1\alpha_5)(\alpha_2\alpha_{10})(\alpha_4\alpha_6)(\alpha_7\alpha_{18})(\alpha_9\alpha_{21})(\alpha_{11}\alpha_{23})(\alpha_{13}\alpha_{24})(\alpha_{14}\alpha_{19})]$$

with $Clos(H_{52,2}) = H_{78,2}$ above.

n=51, $H \cong C_2 \times \mathfrak{S}_4$ ($|H| = 48$, $i = 48$): $\operatorname{rk} N_H = 18$, $(N_H)^*/N_H \cong (\mathbb{Z}/12\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2$ and $\det(K((q_{N_H})_2)) \equiv \pm 2^6 \mod (\mathbb{Z}_2^*)^2$.

$$H_{51,1} = [(\alpha_1\alpha_6\alpha_{23}\alpha_8)(\alpha_4\alpha_{11}\alpha_{16}\alpha_7)(\alpha_5\alpha_{21}\alpha_{14}\alpha_{12})$$

$$(\alpha_9\alpha_{19}\alpha_{20}\alpha_{18})(\alpha_{10}\alpha_{13})(\alpha_{15}\alpha_{22}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_1, \alpha_6, \alpha_5, \alpha_{21}, \alpha_{23}, \alpha_{12}, \alpha_8, \alpha_{14}\}$, $\{\alpha_2, \alpha_{13}, \alpha_{10}\}$, $\{\alpha_3, \alpha_{22}, \alpha_{15}\}$, $\{\alpha_4, \alpha_7, \alpha_9, \alpha_{16}, \alpha_{11}, \alpha_{18}, \alpha_{20}, \alpha_{19}\}$;

$$H_{51,2} = [(\alpha_1\alpha_{18}\alpha_7\alpha_5)(\alpha_4\alpha_{20}\alpha_{21}\alpha_8)(\alpha_6\alpha_{16}\alpha_9\alpha_{12})$$

$$(\alpha_{10}\alpha_{24})(\alpha_{11}\alpha_{14}\alpha_{23}\alpha_{19})(\alpha_{17}\alpha_{22}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_1,\alpha_5,\alpha_7,\alpha_{18}\}$, $\{\alpha_2,\alpha_{13}\}$, $\{\alpha_3,\alpha_{22},\alpha_{17}\}$, $\{\alpha_4,\alpha_9,\alpha_{23},\alpha_{21},\alpha_6,\alpha_{11},\alpha_{16},\alpha_8,\alpha_{19},\alpha_{14},\alpha_{12},\alpha_{20}\}$, $\{\alpha_{10},\alpha_{24}\}$;

$$H_{51,3}=[(\alpha_1\alpha_{15}\alpha_{18}\alpha_{20})(\alpha_3\alpha_6\alpha_{23}\alpha_{17})(\alpha_4\alpha_{19})$$
$$(\alpha_7\alpha_{21}\alpha_{24}\alpha_{11})(\alpha_8\alpha_9\alpha_{14}\alpha_{16})(\alpha_{12}\alpha_{22}),$$
$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_1,\alpha_{15},\alpha_{21},\alpha_{24},\alpha_{18},\alpha_7,\alpha_{20},\alpha_{11}\}$, $\{\alpha_2,\alpha_{13}\}$, $\{\alpha_3,\alpha_{12},\alpha_6,\alpha_{23},\alpha_{22},\alpha_{17}\}$, $\{\alpha_4,\alpha_9,\alpha_{19},\alpha_{16},\alpha_{14},\alpha_8\}$;

$$H_{51,4}=[(\alpha_1\alpha_{17}\alpha_{16}\alpha_{19})(\alpha_4\alpha_{21}\alpha_{13}\alpha_{10})(\alpha_5\alpha_{15}\alpha_8\alpha_{14})$$
$$(\alpha_6\alpha_{24})(\alpha_7\alpha_{20}\alpha_{18}\alpha_{12})(\alpha_{11}\alpha_{23}),$$
$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_1,\alpha_{17},\alpha_{16},\alpha_{19}\}$, $\{\alpha_2,\alpha_{13},\alpha_9,\alpha_4,\alpha_{21},\alpha_{10}\}$, $\{\alpha_3,\alpha_{22}\}$, $\{\alpha_5,\alpha_{15},\alpha_8,\alpha_{14}\}$, $\{\alpha_6,\alpha_{24}\}$, $\{\alpha_7,\alpha_{18},\alpha_{23},\alpha_{11},\alpha_{12},\alpha_{20}\}$.

n=50, $H\cong 4^2C_3$ ($|H|=48$, $i=3$):

$$H_{50,1}=$$
$$[(\alpha_3\alpha_4\alpha_{22})(\alpha_5\alpha_{20}\alpha_{23})(\alpha_6\alpha_{10}\alpha_{16})(\alpha_7\alpha_9\alpha_{18})(\alpha_{11}\alpha_{17}\alpha_{12})(\alpha_{14}\alpha_{24}\alpha_{21}),$$
$$(\alpha_1\alpha_3\alpha_7\alpha_{23})(\alpha_4\alpha_{20}\alpha_6\alpha_{10})(\alpha_5\alpha_{14}\alpha_{21}\alpha_{22})$$
$$(\alpha_9\alpha_{24}\alpha_{18}\alpha_{16})(\alpha_{11}\alpha_{13})(\alpha_{12}\alpha_{17})]$$

with $Clos(H_{50,1})=H_{75,1}$ above.

n=49, $H\cong 2^4C_3$ ($|H|=48$, $i=50$): $\operatorname{rk} N_H=17$, $(N_H)^*/N_H\cong \mathbb{Z}/24\mathbb{Z}\times(\mathbb{Z}/2\mathbb{Z})^4$ and $\det(K((q_{N_H})_2))\equiv \pm 2^7\cdot 3 \mod (\mathbb{Z}_2^*)^2$.

$$H_{49,1}=$$
$$[(\alpha_1\alpha_{22}\alpha_{19})(\alpha_3\alpha_{16}\alpha_{17})(\alpha_4\alpha_{20}\alpha_9)(\alpha_7\alpha_{10}\alpha_8)(\alpha_{12}\alpha_{13}\alpha_{23})(\alpha_{14}\alpha_{18}\alpha_{21}),$$
$$(\alpha_2\alpha_{12})(\alpha_3\alpha_8)(\alpha_4\alpha_{20})(\alpha_7\alpha_{16})(\alpha_9\alpha_{11})(\alpha_{13}\alpha_{23})(\alpha_{14}\alpha_{22})(\alpha_{18}\alpha_{19}),$$
$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_1,\alpha_{22},\alpha_{21},\alpha_{10},\alpha_{19},\alpha_{14},\alpha_8,\alpha_{17},\alpha_{18},\alpha_7,\alpha_3,\alpha_{16}\}$, $\{\alpha_2,\alpha_{12},\alpha_{13},\alpha_{23}\}$, $\{\alpha_4,\alpha_{20},\alpha_{11},\alpha_9\}$;

$$H_{49,2}=$$
$$[(\alpha_1\alpha_{21}\alpha_6)(\alpha_2\alpha_{14}\alpha_4)(\alpha_3\alpha_8\alpha_7)(\alpha_9\alpha_{22}\alpha_{23})(\alpha_{11}\alpha_{19}\alpha_{20})(\alpha_{12}\alpha_{18}\alpha_{13}),$$
$$(\alpha_2\alpha_{12})(\alpha_3\alpha_8)(\alpha_4\alpha_{20})(\alpha_7\alpha_{16})(\alpha_9\alpha_{11})(\alpha_{13}\alpha_{23})(\alpha_{14}\alpha_{22})(\alpha_{18}\alpha_{19}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_1, \alpha_{21}, \alpha_6\}$, $\{\alpha_2, \alpha_{14}, \alpha_{12}, \alpha_{11}, \alpha_{23}, \alpha_{19}, \alpha_4, \alpha_{22}, \alpha_{18}, \alpha_3, \alpha_9, \alpha_{13}, \alpha_7, \alpha_{20}, \alpha_8, \alpha_{16}\}$;

$$H_{49,3} =$$

$$[(\alpha_1\alpha_6\alpha_{21})(\alpha_3\alpha_{13}\alpha_{22})(\alpha_4\alpha_{18}\alpha_7)(\alpha_8\alpha_{23}\alpha_{14})(\alpha_{10}\alpha_{24}\alpha_{17})(\alpha_{16}\alpha_{20}\alpha_{19}),$$

$$(\alpha_2\alpha_{22})(\alpha_3\alpha_{13})(\alpha_4\alpha_{18})(\alpha_7\alpha_9)(\alpha_{10}\alpha_{24})(\alpha_{11}\alpha_{20})(\alpha_{15}\alpha_{17})(\alpha_{16}\alpha_{19}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_1, \alpha_6, \alpha_{21}\}$, $\{\alpha_2, \alpha_{22}, \alpha_3, \alpha_{13}\}$, $\{\alpha_4, \alpha_{18}, \alpha_9, \alpha_7\}$, $\{\alpha_8, \alpha_{23}, \alpha_{14}, \alpha_{12}\}$, $\{\alpha_{10}, \alpha_{24}, \alpha_{15}, \alpha_{17}\}$, $\{\alpha_{11}, \alpha_{20}, \alpha_{19}, \alpha_{16}\}$.

n=48, $H \cong \mathfrak{S}_{3,3}$ ($|H| = 36$, $i = 10$): $\operatorname{rk} N_H = 18$, $(N_H)^*/N_H \cong \mathbb{Z}/18\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^2$ and $\det(K((q_{N_H})_3)) \equiv -2^2 \cdot 3^5 \mod (\mathbb{Z}_3^*)^2$.

$$H_{48,1} = [(\alpha_1\alpha_{19})(\alpha_2\alpha_{22}\alpha_{10}\alpha_{18}\alpha_3\alpha_{23})(\alpha_4\alpha_{11}\alpha_9)$$

$$(\alpha_5\alpha_{12})(\alpha_6\alpha_{14}\alpha_{24}\alpha_{17}\alpha_{20}\alpha_{16})(\alpha_7\alpha_{13}\alpha_{21}),$$

$$(\alpha_1\alpha_{19})(\alpha_2\alpha_{18}\alpha_3\alpha_{23}\alpha_{10}\alpha_{22})(\alpha_4\alpha_{20}\alpha_{14})$$

$$(\alpha_5\alpha_{12}\alpha_{15})(\alpha_6\alpha_{16}\alpha_9\alpha_{24}\alpha_{17}\alpha_{11})(\alpha_7\alpha_{21})]$$

with orbits $\{\alpha_1, \alpha_{19}\}$, $\{\alpha_2, \alpha_{18}, \alpha_{23}, \alpha_3, \alpha_{10}, \alpha_{22}\}$, $\{\alpha_4, \alpha_{11}, \alpha_{20}, \alpha_9, \alpha_{24}, \alpha_{14}, \alpha_6, \alpha_{16}, \alpha_{17}\}$, $\{\alpha_5, \alpha_{12}, \alpha_{15}\}$, $\{\alpha_7, \alpha_{21}, \alpha_{13}\}$.

n=47, $H \cong C_3 \times \mathfrak{A}_4$ ($|H| = 36$, $i = 11$):

$$H_{47,1} =$$

$$[(\alpha_2\alpha_8\alpha_9)(\alpha_3\alpha_{17}\alpha_{22})(\alpha_4\alpha_{14}\alpha_{13})(\alpha_7\alpha_{15}\alpha_{18})(\alpha_{10}\alpha_{23}\alpha_{12})(\alpha_{11}\alpha_{21}\alpha_{20}),$$

$$(\alpha_1\alpha_{10})(\alpha_2\alpha_9\alpha_4\alpha_{13}\alpha_{11}\alpha_{20})(\alpha_3\alpha_8\alpha_{17}\alpha_{14}\alpha_{22}\alpha_{21})$$

$$(\alpha_7\alpha_{15}\alpha_{18})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19}\alpha_{24})]$$

with $Clos(H_{47,1}) = H_{61,1}$ above.

n=46, $H \cong 3^2C_4$ ($|H| = 36$, $i = 9$): $\operatorname{rk} N_H = 18$ and $(N_H)^*/N_H \cong \mathbb{Z}/18\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

$$H_{46,1} =$$

$$[(\alpha_1\alpha_8\alpha_9\alpha_6)(\alpha_3\alpha_{21}\alpha_{22}\alpha_{19})(\alpha_4\alpha_{14})(\alpha_5\alpha_{16})$$

$$(\alpha_7\alpha_{20}\alpha_{11}\alpha_{23})(\alpha_{12}\alpha_{17}\alpha_{18}\alpha_{13}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_1, \alpha_8, \alpha_9, \alpha_6, \alpha_4, \alpha_{14}\}$, $\{\alpha_2, \alpha_{20}, \alpha_{18}, \alpha_{11}, \alpha_{23}, \alpha_{17}, \alpha_{13}, \alpha_{12}, \alpha_7\}$, $\{\alpha_3, \alpha_{21}, \alpha_{22}, \alpha_5, \alpha_{19}, \alpha_{16}\}$.

n=45, $H \cong \Gamma_6 a_2$ ($|H| = 32$, $i = 44$):

$$H_{45,1} =$$

$$[(\alpha_4\alpha_7)(\alpha_5\alpha_{16})(\alpha_6\alpha_{14})(\alpha_8\alpha_{11})(\alpha_9\alpha_{13})(\alpha_{12}\alpha_{24})(\alpha_{15}\alpha_{21})(\alpha_{17}\alpha_{20}),$$

$$(\alpha_1\alpha_8\alpha_{13}\alpha_{11}\alpha_{19}\alpha_{21}\alpha_9\alpha_{15})(\alpha_4\alpha_{20}\alpha_{23}\alpha_6\alpha_7\alpha_{14}\alpha_{10}\alpha_{17})$$

$$(\alpha_5\alpha_{16})(\alpha_{12}\alpha_{18}\alpha_{24}\alpha_{22}),$$

$$(\alpha_1\alpha_6\alpha_{13}\alpha_{14}\alpha_{19}\alpha_{17}\alpha_9\alpha_{20})(\alpha_4\alpha_{11}\alpha_{23}\alpha_{21}\alpha_7\alpha_{15}\alpha_{10}\alpha_8)$$

$$(\alpha_5\alpha_{16})(\alpha_{12}\alpha_{22}\alpha_{24}\alpha_{18})]$$

with $Clos(H_{45,1}) = H_{80,1}$ above.

n=44, $H \cong \Gamma_3 e$ ($|H| = 32$, $i = 11$):

$$H_{44,1} = [(\alpha_1\alpha_8\alpha_{15}\alpha_{23}\alpha_4\alpha_{14}\alpha_6\alpha_{13})(\alpha_5\alpha_{16})$$

$$(\alpha_7\alpha_{21}\alpha_{17}\alpha_{10}\alpha_{19}\alpha_{20}\alpha_{11}\alpha_9)(\alpha_{12}\alpha_{24}\alpha_{22}\alpha_{18}),$$

$$(\alpha_5\alpha_{16})(\alpha_7\alpha_{19})(\alpha_8\alpha_9)(\alpha_{10}\alpha_{14})(\alpha_{11}\alpha_{17})(\alpha_{12}\alpha_{22})(\alpha_{13}\alpha_{20})(\alpha_{21}\alpha_{23})]$$

with $Clos(H_{44,1}) = H_{80,1}$ above.

n=43, $H \cong \Gamma_7 a_2$ ($|H| = 32$, $i = 7$):

$$H_{43,1} = [(\alpha_1\alpha_6\alpha_{23}\alpha_8\alpha_{19}\alpha_{17}\alpha_{10}\alpha_{21})(\alpha_4\alpha_{11}\alpha_{13}\alpha_{20}\alpha_7\alpha_{15}\alpha_9\alpha_{14})$$

$$(\alpha_5\alpha_{16})(\alpha_{12}\alpha_{18}\alpha_{24}\alpha_{22}),$$

$$(\alpha_1\alpha_6\alpha_{13}\alpha_{14}\alpha_{19}\alpha_{17}\alpha_9\alpha_{20})(\alpha_4\alpha_{11}\alpha_{23}\alpha_{21}\alpha_7\alpha_{15}\alpha_{10}\alpha_8)$$

$$(\alpha_5\alpha_{16})(\alpha_{12}\alpha_{22}\alpha_{24}\alpha_{18})]$$

with $Clos(H_{43,1}) = H_{80,1}$ above.

n=42, $H \cong \Gamma_4 c_2$ ($|H| = 32$, $i = 31$):

$$H_{42,1} =$$

$$[(\alpha_3\alpha_{24})(\alpha_4\alpha_6)(\alpha_5\alpha_{21})(\alpha_{10}\alpha_{14})(\alpha_{11}\alpha_{17})(\alpha_{12}\alpha_{13})(\alpha_{16}\alpha_{23})(\alpha_{20}\alpha_{22}),$$

$$(\alpha_1\alpha_4\alpha_9\alpha_5)(\alpha_3\alpha_{20}\alpha_{24}\alpha_{14})(\alpha_6\alpha_{18}\alpha_{21}\alpha_7)$$

$$(\alpha_{10}\alpha_{16}\alpha_{22}\alpha_{23})(\alpha_{11}\alpha_{12})(\alpha_{13}\alpha_{17}),$$

$$(\alpha_1\alpha_3\alpha_7\alpha_{23})(\alpha_4\alpha_{20}\alpha_6\alpha_{10})(\alpha_5\alpha_{14}\alpha_{21}\alpha_{22})$$

$$(\alpha_9\alpha_{24}\alpha_{18}\alpha_{16})(\alpha_{11}\alpha_{13})(\alpha_{12}\alpha_{17})]$$

with $Clos(H_{42,1}) = H_{75,1}$ above.

n=41, $H \cong \Gamma_7 a_1$, ($|H| = 32$, $i = 6$):

$$H_{41,1,1} = [(\alpha_2\alpha_{17})(\alpha_5\alpha_{23}\alpha_{14}\alpha_{24})(\alpha_6\alpha_8\alpha_9\alpha_{18})$$
$$(\alpha_7\alpha_{16}\alpha_{10}\alpha_{13})(\alpha_{12}\alpha_{22})(\alpha_{15}\alpha_{20}\alpha_{21}\alpha_{19}),$$
$$(\alpha_1\alpha_5)(\alpha_2\alpha_{23})(\alpha_3\alpha_{14})(\alpha_7\alpha_9)(\alpha_8\alpha_{13})(\alpha_{12}\alpha_{22})(\alpha_{17}\alpha_{24})(\alpha_{19}\alpha_{20})]$$

with $Clos(H_{41,1,1}) = H_{56,1}$ above;

$$H_{41,1,2} = [(\alpha_2\alpha_{17})(\alpha_5\alpha_{23}\alpha_{14}\alpha_{24})(\alpha_6\alpha_8\alpha_9\alpha_{18})$$
$$(\alpha_7\alpha_{16}\alpha_{10}\alpha_{13})(\alpha_{12}\alpha_{22})(\alpha_{15}\alpha_{20}\alpha_{21}\alpha_{19}),$$
$$(\alpha_1\alpha_5)(\alpha_2\alpha_{24})(\alpha_3\alpha_{14})(\alpha_6\alpha_{16})(\alpha_7\alpha_{13})(\alpha_8\alpha_9)(\alpha_{10}\alpha_{18})(\alpha_{17}\alpha_{23})]$$

with $Clos(H_{41,1,2}) = H_{56,1}$ above;

$$H_{41,2} = [(\alpha_2\alpha_3\alpha_{17}\alpha_{11})(\alpha_5\alpha_{16}\alpha_{15}\alpha_6)(\alpha_7\alpha_{10}\alpha_{19}\alpha_{23})$$
$$(\alpha_8\alpha_{24}\alpha_{20}\alpha_{18})(\alpha_9\alpha_{13})(\alpha_{12}\alpha_{22}),$$
$$(\alpha_1\alpha_{14})(\alpha_3\alpha_7)(\alpha_5\alpha_8)(\alpha_9\alpha_{13})(\alpha_{10}\alpha_{24})(\alpha_{11}\alpha_{20})(\alpha_{15}\alpha_{19})(\alpha_{16}\alpha_{17})]$$

with $Clos(H_{41,2}) = H_{56,2}$ above.

n=40, $H \cong Q_8 * Q_8$ ($|H| = 32$, $i = 49$): $\operatorname{rk} N_H = 17$, $(N_H)^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^5$ and $\det(K((q_{N_H})_2)) \equiv \pm 2^{10} \mod (\mathbb{Z}_2^*)^2$.

$$H_{40,1} =$$
$$[(\alpha_1\alpha_{16})(\alpha_2\alpha_{18})(\alpha_4\alpha_9)(\alpha_5\alpha_{23})(\alpha_{10}\alpha_{12})(\alpha_{14}\alpha_{20})(\alpha_{15}\alpha_{21})(\alpha_{19}\alpha_{24}),$$
$$(\alpha_3\alpha_5)(\alpha_6\alpha_{12})(\alpha_8\alpha_{13})(\alpha_{10}\alpha_{22})(\alpha_{15}\alpha_{21})(\alpha_{16}\alpha_{19})(\alpha_{17}\alpha_{23})(\alpha_{18}\alpha_{20}),$$
$$(\alpha_1\alpha_{16})(\alpha_2\alpha_{20})(\alpha_3\alpha_6)(\alpha_5\alpha_{10})(\alpha_{12}\alpha_{23})(\alpha_{14}\alpha_{18})(\alpha_{17}\alpha_{22})(\alpha_{19}\alpha_{24}),$$
$$(\alpha_1\alpha_{14})(\alpha_2\alpha_{24})(\alpha_3\alpha_5)(\alpha_6\alpha_{10})(\alpha_{12}\alpha_{22})(\alpha_{16}\alpha_{18})(\alpha_{17}\alpha_{23})(\alpha_{19}\alpha_{20})]$$

with orbits $\{\alpha_1, \alpha_{16}, \alpha_{14}, \alpha_{19}, \alpha_{18}, \alpha_{20}, \alpha_{24}, \alpha_2\}$, $\{\alpha_3, \alpha_5, \alpha_6, \alpha_{23}, \alpha_{10}, \alpha_{12}, \alpha_{17}, \alpha_{22}\}$, $\{\alpha_4, \alpha_9\}$, $\{\alpha_8, \alpha_{13}\}$, $\{\alpha_{15}, \alpha_{21}\}$.

n=39, $H \cong 2^4 C_2$ ($|H| = 32$, $i = 27$): $\operatorname{rk} N_H = 17$, $(N_H)^*/N_H \cong \mathbb{Z}/8\mathbb{Z} \times (\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2$ and $\det(K((q_{N_H})_2)) \equiv \pm 2^9 \mod (\mathbb{Z}_2^*)^2$.

$$H_{39,1} =$$
$$[(\alpha_5\alpha_{10})(\alpha_7\alpha_{13})(\alpha_8\alpha_{12})(\alpha_9\alpha_{22})(\alpha_{11}\alpha_{23})(\alpha_{14}\alpha_{16})(\alpha_{17}\alpha_{21})(\alpha_{19}\alpha_{20}),$$
$$(\alpha_2\alpha_{12})(\alpha_3\alpha_8)(\alpha_4\alpha_{20})(\alpha_7\alpha_{16})(\alpha_9\alpha_{11})(\alpha_{13}\alpha_{23})(\alpha_{14}\alpha_{22})(\alpha_{18}\alpha_{19}),$$
$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_2, \alpha_{12}, \alpha_{13}, \alpha_3, \alpha_{18}, \alpha_8, \alpha_{23}, \alpha_{19}, \alpha_7, \alpha_{22}, \alpha_4, \alpha_{14}, \alpha_{20}, \alpha_{11}, \alpha_{16}, \alpha_9\}$, $\{\alpha_5, \alpha_{10}\}$, $\{\alpha_{17}, \alpha_{21}\}$;

$$H_{39,2} =$$

$$[(\alpha_5\alpha_{18})(\alpha_6\alpha_7)(\alpha_{10}\alpha_{17})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{13})(\alpha_{14}\alpha_{22})(\alpha_{15}\alpha_{16})(\alpha_{19}\alpha_{24}),$$
$$(\alpha_2\alpha_{12})(\alpha_3\alpha_8)(\alpha_4\alpha_{20})(\alpha_7\alpha_{16})(\alpha_9\alpha_{11})(\alpha_{13}\alpha_{23})(\alpha_{14}\alpha_{22})(\alpha_{18}\alpha_{19}),$$
$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_2, \alpha_{12}, \alpha_{13}, \alpha_{23}\}$, $\{\alpha_3, \alpha_8, \alpha_{22}, \alpha_{14}\}$, $\{\alpha_4, \alpha_{20}, \alpha_9, \alpha_{11}\}$, $\{\alpha_5, \alpha_{18}, \alpha_{24}, \alpha_6, \alpha_{19}, \alpha_7, \alpha_{15}, \alpha_{16}\}$, $\{\alpha_{10}, \alpha_{17}\}$;

$$H_{39,3} =$$

$$(\alpha_3\alpha_4)(\alpha_5\alpha_{18})(\alpha_6\alpha_{19})(\alpha_7\alpha_{24})(\alpha_8\alpha_9)(\alpha_{11}\alpha_{22})(\alpha_{14}\alpha_{20})(\alpha_{15}\alpha_{16}),$$
$$(\alpha_2\alpha_{12})(\alpha_3\alpha_8)(\alpha_4\alpha_{20})(\alpha_7\alpha_{16})(\alpha_9\alpha_{11})(\alpha_{13}\alpha_{23})(\alpha_{14}\alpha_{22})(\alpha_{18}\alpha_{19}),$$
$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_2, \alpha_{12}, \alpha_{13}, \alpha_{23}\}$, $\{\alpha_3, \alpha_4, \alpha_8, \alpha_{22}, \alpha_{14}, \alpha_{20}, \alpha_9, \alpha_{11}\}$, $\{\alpha_5, \alpha_{18}, \alpha_6, \alpha_{24}, \alpha_{19}, \alpha_7, \alpha_{15}, \alpha_{16}\}$.

n=38, $H \cong T_{24}$ ($|H| = 24$, $i = 3$):

$$H_{38,1} =$$

$$[(\alpha_2\alpha_7\alpha_{22})(\alpha_3\alpha_{18}\alpha_{13})(\alpha_5\alpha_{12}\alpha_{23})(\alpha_6\alpha_{14}\alpha_8)(\alpha_{11}\alpha_{17}\alpha_{20})(\alpha_{16}\alpha_{19}\alpha_{24}),$$
$$(\alpha_1\alpha_2\alpha_{10}\alpha_{18})(\alpha_3\alpha_{13}\alpha_{22}\alpha_7)(\alpha_4\alpha_{16}\alpha_9\alpha_{12})$$
$$(\alpha_5\alpha_{23}\alpha_{24}\alpha_{19})(\alpha_{11}\alpha_{17})(\alpha_{20}\alpha_{21})]$$

with $Clos(H_{38,1}) = H_{54,1}$ above.

n=37, $H \cong T_{24}$, ($|H| = 24$, $i = 3$):

$$H_{37,1} =$$

$$[(\alpha_4\alpha_{16}\alpha_{20})(\alpha_5\alpha_6\alpha_{13})(\alpha_7\alpha_{12}\alpha_{14})(\alpha_8\alpha_{11}\alpha_{15})(\alpha_9\alpha_{24}\alpha_{17})(\alpha_{10}\alpha_{19}\alpha_{23}),$$
$$(\alpha_1\alpha_8\alpha_{21}\alpha_{10})(\alpha_3\alpha_7\alpha_{22}\alpha_9)(\alpha_4\alpha_{13})(\alpha_6\alpha_{20})$$
$$(\alpha_{11}\alpha_{23}\alpha_{19}\alpha_{15})(\alpha_{12}\alpha_{17}\alpha_{24}\alpha_{14})]$$

with $Clos(H_{37,1}) = H_{77,1}$ above.

n=36, $H \cong C_3 \rtimes D_8$ ($|H| = 24$, $i = 8$):

$$H_{36,1} =$$

$$[(\alpha_3\alpha_{22})(\alpha_4\alpha_{11})(\alpha_7\alpha_{15})(\alpha_8\alpha_9)(\alpha_{10}\alpha_{23})(\alpha_{13}\alpha_{21})(\alpha_{14}\alpha_{20})(\alpha_{16}\alpha_{24}),$$
$$(\alpha_1\alpha_{10})(\alpha_2\alpha_9\alpha_4\alpha_{13}\alpha_{11}\alpha_{20})(\alpha_3\alpha_8\alpha_{17}\alpha_{14}\alpha_{22}\alpha_{21})$$
$$(\alpha_7\alpha_{15}\alpha_{18})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19}\alpha_{24})]$$

with $Clos(H_{36,1}) = H_{61,1}$ above.

n=35, $H \cong C_2 \times \mathfrak{A}_4$, $(|H| = 24,\ i = 13)$:

$$H_{35,1} =$$
$$[(\alpha_1\alpha_{12})(\alpha_4\alpha_{11})(\alpha_5\alpha_8)(\alpha_6\alpha_{14})(\alpha_7\alpha_{16})(\alpha_9\alpha_{19})(\alpha_{18}\alpha_{20})(\alpha_{21}\alpha_{23}),$$
$$(\alpha_1\alpha_5)(\alpha_2\alpha_{10}\alpha_{13})(\alpha_3\alpha_{15}\alpha_{22})(\alpha_4\alpha_{20}\alpha_{11}\alpha_9\alpha_{16}\alpha_{19})$$
$$(\alpha_6\alpha_{12}\alpha_{14}\alpha_{21}\alpha_8\alpha_{23})(\alpha_7\alpha_{18})]$$

with $Clos(H_{35,1}) = H_{51,1}$ above;

$$H_{35,2} =$$
$$[(\alpha_1\alpha_5)(\alpha_4\alpha_9)(\alpha_6\alpha_{21})(\alpha_7\alpha_{18})(\alpha_8\alpha_{12})(\alpha_{11}\alpha_{19})(\alpha_{14}\alpha_{23})(\alpha_{16}\alpha_{20}),$$
$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{17}\alpha_{22})(\alpha_4\alpha_{20}\alpha_{11}\alpha_9\alpha_{12}\alpha_{23})(\alpha_5\alpha_{18}\alpha_7)$$
$$(\alpha_6\alpha_8\alpha_{19}\alpha_{21}\alpha_{16}\alpha_{14})(\alpha_{10}\alpha_{24})]$$

with $Clos(H_{35,2}) = H_{51,2}$ above;

$$H_{35,3} =$$
$$[(\alpha_1\alpha_7)(\alpha_4\alpha_{19})(\alpha_6\alpha_{17})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{15})(\alpha_{12}\alpha_{22})(\alpha_{18}\alpha_{24})(\alpha_{20}\alpha_{21}),$$
$$(\alpha_1\alpha_{11}\alpha_{24}\alpha_{21}\alpha_{18}\alpha_{15})(\alpha_2\alpha_{13})(\alpha_3\alpha_{17}\alpha_{12})$$
$$(\alpha_4\alpha_{16}\alpha_8,\alpha_{19}\alpha_9\alpha_{14})(\alpha_6\alpha_{22}\alpha_{23})(\alpha_7\alpha_{20})]$$

with $Clos(H_{35,3}) = H_{51,3}$ above;

$$H_{35,4} =$$
$$[(\alpha_1\alpha_{16}\alpha_{17})(\alpha_2\alpha_4\alpha_{21})(\alpha_5\alpha_8\alpha_{15})(\alpha_7\alpha_{11}\alpha_{20})(\alpha_9\alpha_{13}\alpha_{10})(\alpha_{12}\alpha_{18}\alpha_{23}),$$
$$(\alpha_2\alpha_4\alpha_{21}\alpha_9\alpha_{13}\alpha_{10})(\alpha_3\alpha_{22})(\alpha_6\alpha_{24})(\alpha_7\alpha_{23}\alpha_{12}\alpha_{18}\alpha_{11}\alpha_{20})$$
$$(\alpha_8\alpha_{14}\alpha_{15})(\alpha_{16}\alpha_{19}\alpha_{17})]$$

with $Clos(H_{35,4}) = H_{51,4}$ above.

n=34, $H \cong \mathfrak{S}_4$ $(|H| = 24,\ i = 12)$: $\mathrm{rk}\, N_H = 17$ and $(N_H)^*/N_H \cong (\mathbb{Z}/12\mathbb{Z})^2 \times \mathbb{Z}/4\mathbb{Z}$.

$$H_{34,1} = [(\alpha_1\alpha_{18}\alpha_{15}\alpha_{22})(\alpha_2\alpha_4\alpha_{20}\alpha_{11})(\alpha_3\alpha_7)$$

$$(\alpha_8\alpha_{10}\alpha_{16}\alpha_{24})(\alpha_9\alpha_{12}\alpha_{23}\alpha_{13})(\alpha_{14}\alpha_{19}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_1,\alpha_3,\alpha_{22},\alpha_{15},\alpha_7,\alpha_{18}\}$, $\{\alpha_2,\alpha_{13},\alpha_{20},\alpha_9,\alpha_{12},\alpha_4,\alpha_{23},\alpha_{11}\}$, $\{\alpha_8,\alpha_{14},\alpha_{16},\alpha_{10},\alpha_{19},\alpha_{24}\}$;

$$H_{34,2}=[(\alpha_1\alpha_{16}\alpha_{19}\alpha_{15})(\alpha_3\alpha_5\alpha_{12}\alpha_{14})(\alpha_4\alpha_9)(\alpha_7\alpha_{20}\alpha_{23}\alpha_{22})$$

$$(\alpha_8\alpha_{11}\alpha_{17}\alpha_{18})(\alpha_{10}\alpha_{13}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_1,\alpha_{16},\alpha_{19},\alpha_{15}\}$, $\{\alpha_2,\alpha_{10},\alpha_{13}\}$, $\{\alpha_3,\alpha_{22},\alpha_{12},\alpha_{18},\alpha_{17},\alpha_{20},\alpha_7,\alpha_{11},\alpha_{23},\alpha_8,\alpha_5,\alpha_{14}\}$, $\{\alpha_4,\alpha_9\}$;

$$H_{34,3}=[(\alpha_1\alpha_6\alpha_{18}\alpha_7)(\alpha_3\alpha_{24}\alpha_{20}\alpha_{17})(\alpha_4\alpha_9)(\alpha_8\alpha_{13}\alpha_{14}\alpha_{19})$$

$$(\alpha_{10}\alpha_{15}\alpha_{12}\alpha_{23})(\alpha_{11}\alpha_{22}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_1,\alpha_{18},\alpha_6,\alpha_7\}$, $\{\alpha_2,\alpha_8,\alpha_{14},\alpha_{16},\alpha_{13},\alpha_{19}\}$, $\{\alpha_3,\alpha_{22},\alpha_{20},\alpha_{24},\alpha_{11},\alpha_{17}\}$, $\{\alpha_4,\alpha_9\}$, $\{\alpha_{10},\alpha_{12},\alpha_{15},\alpha_{23}\}$;

$$H_{34,4}=[(\alpha_1\alpha_6\alpha_{16}\alpha_{19})(\alpha_4\alpha_{14}\alpha_{23}\alpha_9)(\alpha_5\alpha_{11}\alpha_{20}\alpha_{21})$$

$$(\alpha_7\alpha_8\alpha_{12}\alpha_{18})(\alpha_{13}\alpha_{15})(\alpha_{17}\alpha_{22}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_1,\alpha_{16},\alpha_6,\alpha_{19}\}$, $\{\alpha_2,\alpha_{15},\alpha_{13}\}$, $\{\alpha_3,\alpha_{17},\alpha_{22}\}$, $\{\alpha_4,\alpha_7,\alpha_{18},\alpha_{23},\alpha_9,\alpha_{12},\alpha_8,\alpha_{14}\}$, $\{\alpha_5,\alpha_{11},\alpha_{20},\alpha_{21}\}$.

n=33, $H\cong C_7\rtimes C_3$ ($|H|=21$, $i=1$): $\operatorname{rk}N_H=18$ and $(N_H)^*/N_H\cong(\mathbb{Z}/7\mathbb{Z})^3$.

$$H_{33,1}=$$

$$[(\alpha_1\alpha_6\alpha_5)(\alpha_3\alpha_{20}\alpha_{17})(\alpha_7\alpha_{19}\alpha_{22})(\alpha_9\alpha_{24}\alpha_{16})(\alpha_{12}\alpha_{21}\alpha_{14})(\alpha_{13}\alpha_{15}\alpha_{18}),$$

$$(\alpha_1\alpha_7\alpha_6\alpha_{22}\alpha_{19}\alpha_{10}\alpha_5)(\alpha_3\alpha_{17}\alpha_4\alpha_9\alpha_{24}\alpha_{20}\alpha_{16})(\alpha_{11}\alpha_{12}\alpha_{21}\alpha_{13}\alpha_{14}\alpha_{18}\alpha_{15})]$$

with orbits $\{\alpha_1,\alpha_6,\alpha_7,\alpha_5,\alpha_{22},\alpha_{19},\alpha_{10}\}$, $\{\alpha_3,\alpha_{20},\alpha_{17},\alpha_{16},\alpha_4,\alpha_9,\alpha_{24}\}$, $\{\alpha_{11},\alpha_{12},\alpha_{21},\alpha_{14},\alpha_{13},\alpha_{18},\alpha_{15}\}$.

n=32, $H\cong Hol(C_5)$ ($|H|=20$, $i=3$): $\operatorname{rk}N_H=18$ and $(N_H)^*/N_H\cong(\mathbb{Z}/10\mathbb{Z})^2\times\mathbb{Z}/5\mathbb{Z}$.

$$H_{32,1}=[(\alpha_2\alpha_{21}\alpha_{17}\alpha_{11})(\alpha_3\alpha_{13}\alpha_9\alpha_{14})(\alpha_4\alpha_{18}\alpha_{24}\alpha_{12})$$

$$(\alpha_5\alpha_8)(\alpha_{15}\alpha_{16}\alpha_{23}\alpha_{20})(\alpha_{19}\alpha_{22}),$$

$(\alpha_2\alpha_{17}\alpha_{15}\alpha_8\alpha_{23})(\alpha_3\alpha_{14}\alpha_{10}\alpha_{13}\alpha_9)(\alpha_4\alpha_{24}\alpha_{18}\alpha_7\alpha_{12})(\alpha_5\alpha_{11}\alpha_{20}\alpha_{16}\alpha_{21})]$

with orbits $\{\alpha_2,\alpha_{21},\alpha_{17},\alpha_{11},\alpha_5,\alpha_{15},\alpha_{20},\alpha_8,\alpha_{16},\alpha_{23}\}$, $\{\alpha_3,\alpha_{13},\alpha_9,\alpha_{14},\alpha_{10}\}$, $\{\alpha_4,\alpha_{18},\alpha_{24},\alpha_{12},\alpha_7\}$, $\{\alpha_{19},\alpha_{22}\}$.

n=31, $H\cong C_3\times D_6$ ($|H|=18$, $i=3$):

$$H_{31,1}=$$

$$[(\alpha_2\alpha_3\alpha_{10})(\alpha_4\alpha_{11}\alpha_9)(\alpha_6\alpha_{20}\alpha_{24})(\alpha_7\alpha_{13}\alpha_{21})(\alpha_{14}\alpha_{16}\alpha_{17})(\alpha_{18}\alpha_{22}\alpha_{23}),$$
$$(\alpha_1\alpha_{19})(\alpha_2\alpha_{18}\alpha_{10}\alpha_{22}\alpha_3\alpha_{23})(\alpha_4\alpha_{17}\alpha_{20}\alpha_9\alpha_{14}\alpha_6)(\alpha_5\alpha_{15}\alpha_{12})$$
$$(\alpha_7\alpha_{13})(\alpha_{11}\alpha_{16}\alpha_{24})]$$

with $Clos(H_{31,1})=H_{48,1}$.

n=30, $H\cong\mathfrak{A}_{3,3}$ ($|H|=18$, $i=4$): $\operatorname{rk}N_H=16$ and $(N_H)^*/N_H\cong\mathbb{Z}/9\mathbb{Z}\times(\mathbb{Z}/3\mathbb{Z})^4$.

$$H_{30,1}=$$

$$[(\alpha_3\alpha_{21})(\alpha_4\alpha_9)(\alpha_6\alpha_{22})(\alpha_7\alpha_{10})(\alpha_{11}\alpha_{12})(\alpha_{14}\alpha_{16})(\alpha_{15}\alpha_{17})(\alpha_{18}\alpha_{24}),$$
$$(\alpha_1\alpha_{24}\alpha_{18})(\alpha_3\alpha_{14}\alpha_{22})(\alpha_4\alpha_{20}\alpha_9)(\alpha_5\alpha_7\alpha_{10})(\alpha_6\alpha_{16}\alpha_{21})(\alpha_{15}\alpha_{19}\alpha_{17}),$$
$$(\alpha_1\alpha_6\alpha_{22})(\alpha_3\alpha_{24}\alpha_{16})(\alpha_4\alpha_{20}\alpha_9)(\alpha_5\alpha_{10}\alpha_7)(\alpha_{11}\alpha_{13}\alpha_{12})(\alpha_{14}\alpha_{18}\alpha_{21})]$$

with orbits $\{\alpha_1,\alpha_{24},\alpha_{21},\alpha_{18},\alpha_6,\alpha_3,\alpha_{16},\alpha_{22},\alpha_{14}\}$, $\{\alpha_4,\alpha_9,\alpha_{20}\}$, $\{\alpha_5,\alpha_7,\alpha_{10}\}$, $\{\alpha_{11},\alpha_{12},\alpha_{13}\}$, $\{\alpha_{15},\alpha_{17},\alpha_{19}\}$.

n=29, $H\cong Q_{16}$ ($|H|=16$, $i=9$):

$$H_{29,1}=[(\alpha_1\alpha_4\alpha_{19}\alpha_7)(\alpha_5\alpha_{16})(\alpha_6\alpha_8\alpha_{17}\alpha_{21})(\alpha_9\alpha_{23}\alpha_{13}\alpha_{10})$$
$$(\alpha_{11}\alpha_{14}\alpha_{15}\alpha_{20})(\alpha_{12}\alpha_{24}),$$
$$(\alpha_1\alpha_6\alpha_{13}\alpha_{14}\alpha_{19}\alpha_{17}\alpha_9\alpha_{20})(\alpha_4\alpha_{11}\alpha_{23}\alpha_{21}\alpha_7\alpha_{15}\alpha_{10}\alpha_8)$$
$$(\alpha_5\alpha_{16})(\alpha_{12}\alpha_{22}\alpha_{24}\alpha_{18})]$$

with $Clos(H_{29,1})=H_{80,1}$ above.

n=28, $H\cong\Gamma_2 d$ ($|H|=16$, $i=6$):

$$H_{28,1}=$$

$$[(\alpha_1\alpha_6\alpha_{13}\alpha_{14}\alpha_{19}\alpha_{17}\alpha_9\alpha_{20})(\alpha_4\alpha_{11}\alpha_{23}\alpha_{21}\alpha_7\alpha_{15}\alpha_{10}\alpha_8)$$
$$(\alpha_5\alpha_{16})(\alpha_{12}\alpha_{22}\alpha_{24}\alpha_{18}),$$
$$(\alpha_1\alpha_4)(\alpha_6\alpha_{15})(\alpha_7\alpha_{19})(\alpha_8\alpha_{14})(\alpha_9\alpha_{10})(\alpha_{11}\alpha_{17})(\alpha_{13}\alpha_{23})(\alpha_{20}\alpha_{21})]$$

with $Clos(H_{28,1})=H_{80,1}$ above.

n=27, $H \cong C_2 \times Q_8$ ($|H| = 16$, $i = 12$):

$$H_{27,1} = [(\alpha_1\alpha_4\alpha_9\alpha_5)(\alpha_3\alpha_{20}\alpha_{24}\alpha_{14})(\alpha_6\alpha_{18}\alpha_{21}\alpha_7)$$

$$(\alpha_{10}\alpha_{16}\alpha_{22}\alpha_{23})(\alpha_{11}\alpha_{12})(\alpha_{13}\alpha_{17}),$$

$$(\alpha_1\alpha_3\alpha_9\alpha_{24})(\alpha_4\alpha_{14}\alpha_5\alpha_{20})(\alpha_6\alpha_{22}\alpha_{21}\alpha_{10})$$

$$(\alpha_7\alpha_{23}\alpha_{18}\alpha_{16})(\alpha_{11}\alpha_{17})(\alpha_{12}\alpha_{13}),$$

$$(\alpha_1\alpha_7)(\alpha_3\alpha_{23})(\alpha_4\alpha_6)(\alpha_5\alpha_{21})(\alpha_9\alpha_{18})(\alpha_{10}\alpha_{20})(\alpha_{14}\alpha_{22})(\alpha_{16}\alpha_{24})]$$

with $Clos(H_{27,1}) = H_{75,1}$ above.

n=26, $H \cong SD_{16}$ ($|H| = 16$, $i = 8$): $\mathrm{rk}\, N_H = 18$ and $(N_H)^*/N_H \cong (\mathbb{Z}/8\mathbb{Z})^2 \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

$$H_{26,1} = [(\alpha_1\alpha_{12}\alpha_{15}\alpha_{11})(\alpha_3\alpha_5\alpha_{18}\alpha_{17})(\alpha_4\alpha_{10})(\alpha_6\alpha_9)$$

$$(\alpha_7\alpha_{14}\alpha_{22}\alpha_8)(\alpha_{16}\alpha_{20}\alpha_{19}\alpha_{23}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_1, \alpha_{12}, \alpha_{19}, \alpha_{15}, \alpha_{23}, \alpha_{20}, \alpha_{11}, \alpha_{16}\}$, $\{\alpha_2, \alpha_{13}\}$, $\{\alpha_3, \alpha_5, \alpha_{22}, \alpha_7, \alpha_{18}, \alpha_8, \alpha_{17}, \alpha_{14}\}$, $\{\alpha_4, \alpha_{10}, \alpha_9, \alpha_6\}$.

n=25, $H \cong C_4^2$ ($|H| = 16$, $i = 2$):

$$H_{25,1} = [(\alpha_1\alpha_4\alpha_9\alpha_5)(\alpha_3\alpha_{20}\alpha_{24}\alpha_{14})(\alpha_6\alpha_{18}\alpha_{21}\alpha_7)$$

$$(\alpha_{10}\alpha_{16}\alpha_{22}\alpha_{23})(\alpha_{11}\alpha_{12})(\alpha_{13}\alpha_{17}),$$

$$(\alpha_1\alpha_3\alpha_7\alpha_{23})(\alpha_4\alpha_{20}\alpha_6\alpha_{10})(\alpha_5\alpha_{14}\alpha_{21}\alpha_{22})$$

$$(\alpha_9\alpha_{24}\alpha_{18}\alpha_{16})(\alpha_{11}\alpha_{13})(\alpha_{12}\alpha_{17})]$$

with $Clos(H_{25,1}) = H_{75,1}$ above. [1]

n=24, $H \cong Q_8 * C_4$ ($|H| = 16$, $i = 13$):

$$H_{24,1} =$$

$$[(\alpha_3\alpha_5)(\alpha_6\alpha_{12})(\alpha_8\alpha_{13})(\alpha_{10}\alpha_{22})(\alpha_{15}\alpha_{21})(\alpha_{16}\alpha_{19})(\alpha_{17}\alpha_{23})(\alpha_{18}\alpha_{20}),$$

$$(\alpha_1\alpha_{16})(\alpha_2\alpha_{18})(\alpha_4\alpha_9)(\alpha_5\alpha_{23})(\alpha_{10}\alpha_{12})(\alpha_{14}\alpha_{20})(\alpha_{15}\alpha_{21})(\alpha_{19}\alpha_{24}),$$

$$(\alpha_1\alpha_2\alpha_{24}\alpha_{14})(\alpha_3\alpha_6\alpha_{17}\alpha_{22})(\alpha_4\alpha_9)(\alpha_5\alpha_{12}\alpha_{23}\alpha_{10})$$

[1] These calculations show that for a symplectic group $G = (C_4)^2$ on a Kählerian K3 surface, the group $S^*_{(G)}/S_{(G)} = S^*_{(4,4)}/S_{(4,4)} \cong (\mathbb{Z}/8\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2$. We must correct our calculation of this group in [11, Prop. 10.1].

$$(\alpha_8\alpha_{13})(\alpha_{16}\alpha_{18}\alpha_{19}\alpha_{20})]$$

with $Clos(H_{24,1}) = H_{40,1}$ above.

n=23, $H \cong \Gamma_2 c_1$ ($|H| = 16$, $i = 3$):

$$H_{23,1} = [(\alpha_2\alpha_7\alpha_{18}\alpha_{13})(\alpha_3\alpha_9\alpha_4\alpha_{22})(\alpha_5\alpha_{10})(\alpha_8\alpha_{16}\alpha_{20}\alpha_{23})$$

$$(\alpha_{11}\alpha_{19}\alpha_{14}\alpha_{12})(\alpha_{17}\alpha_{21}),$$

$$(\alpha_2\alpha_8\alpha_3\alpha_{12})(\alpha_4\alpha_{19}\alpha_{18}\alpha_{20})(\alpha_5\alpha_{10})(\alpha_7\alpha_{14}\alpha_9\alpha_{23})$$

$$(\alpha_{11}\alpha_{22}\alpha_{16}\alpha_{13})(\alpha_{17}\alpha_{21})]$$

with $Clos(H_{23,1}) = H_{39,1}$ above;

$$H_{23,2} = [(\alpha_3\alpha_{14}\alpha_8\alpha_{22})(\alpha_4\alpha_{20}\alpha_9\alpha_{11})(\alpha_5\alpha_7\alpha_{15}\alpha_{19})$$

$$(\alpha_6\alpha_{18}\alpha_{24}\alpha_{16})(\alpha_{10}\alpha_{17})(\alpha_{12}\alpha_{13}),$$

$$(\alpha_2\alpha_{12})(\alpha_4\alpha_{11})(\alpha_5\alpha_{15})(\alpha_6\alpha_{24})(\alpha_7\alpha_{18})(\alpha_9\alpha_{20})(\alpha_{13}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with $Clos(H_{23,2}) = H_{39,2}$ above;

$$H_{23,3} = [(\alpha_2\alpha_{12})(\alpha_3\alpha_4\alpha_{22}\alpha_{11})(\alpha_5\alpha_{16}\alpha_6\alpha_7)(\alpha_8\alpha_9\alpha_{14}\alpha_{20})$$

$$(\alpha_{13}\alpha_{23})(\alpha_{15}\alpha_{18}\alpha_{24}\alpha_{19}),$$

$$(\alpha_2\alpha_{13})(\alpha_4\alpha_{20})(\alpha_5\alpha_6)(\alpha_7\alpha_{19})(\alpha_9\alpha_{11})(\alpha_{12}\alpha_{23})(\alpha_{15}\alpha_{24})(\alpha_{16}\alpha_{18})]$$

with $Clos(H_{23,3}) = H_{39,3}$ above.

n=22, $H \cong C_2 \times D_8$ ($|H| = 16$, $i = 11$): $\operatorname{rk} N_H = 16$, $(N_H)^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^4 \times (\mathbb{Z}/2\mathbb{Z})^2$, and $\det(K(q_{N_H})_2) \equiv \pm 2^{10} \mod (\mathbb{Z}_2^*)^2$.

$$H_{22,1} =$$

$$[(\alpha_4\alpha_7)(\alpha_8\alpha_{12})(\alpha_9\alpha_{18})(\alpha_{10}\alpha_{15})(\alpha_{11}\alpha_{20})(\alpha_{14}\alpha_{23})(\alpha_{16}\alpha_{19})(\alpha_{17}\alpha_{24}),$$

$$(\alpha_2\alpha_{12})(\alpha_3\alpha_8)(\alpha_4\alpha_{20})(\alpha_7\alpha_{16})(\alpha_9\alpha_{11})(\alpha_{13}\alpha_{23})(\alpha_{14}\alpha_{22})(\alpha_{18}\alpha_{19}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_2, \alpha_{12}, \alpha_{13}, \alpha_3, \alpha_8, \alpha_{23}, \alpha_{22}, \alpha_{14}\}$, $\{\alpha_4, \alpha_7, \alpha_{20}, \alpha_9, \alpha_{18}, \alpha_{16}, \alpha_{11}, \alpha_{19}\}$, $\{\alpha_{10}, \alpha_{15}\}$, $\{\alpha_{17}, \alpha_{24}\}$;

$$H_{22,2} =$$

$$[(\alpha_3\alpha_4)(\alpha_6\alpha_{15})(\alpha_8\alpha_{11})(\alpha_9\alpha_{22})(\alpha_{12}\alpha_{23})(\alpha_{14}\alpha_{20})(\alpha_{16}\alpha_{19})(\alpha_{17}\alpha_{21}),$$

$$(\alpha_2\alpha_{22})(\alpha_3\alpha_{13})(\alpha_4\alpha_{18})(\alpha_7\alpha_9)(\alpha_{10}\alpha_{24})(\alpha_{11}\alpha_{20})(\alpha_{15}\alpha_{17})(\alpha_{16}\alpha_{19}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_2,\alpha_{22},\alpha_{13},\alpha_7,\alpha_9,\alpha_3,\alpha_{18},\alpha_4\}$, $\{\alpha_6,\alpha_{15},\alpha_{21},\alpha_{17}\}$, $\{\alpha_8,\alpha_{11},\alpha_{14},\alpha_{20}\}$, $\{\alpha_{10},\alpha_{24}\}$, $\{\alpha_{12},\alpha_{23}\}$, $\{\alpha_{16},\alpha_{19}\}$;

$$H_{22,3} =$$

$$[(\alpha_1\alpha_{10})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_5\alpha_{24})(\alpha_6\alpha_{15})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{17}\alpha_{21}),$$

$$(\alpha_2\alpha_{22})(\alpha_3\alpha_{13})(\alpha_4\alpha_{18})(\alpha_7\alpha_9)(\alpha_{10}\alpha_{24})(\alpha_{11}\alpha_{20})(\alpha_{15}\alpha_{17})(\alpha_{16}\alpha_{19}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_1,\alpha_{10},\alpha_5,\alpha_{24}\}$, $\{\alpha_2,\alpha_{22},\alpha_{13},\alpha_3\}$, $\{\alpha_4,\alpha_9,\alpha_{18},\alpha_7\}$, $\{\alpha_6,\alpha_{15},\alpha_{21},\alpha_{17}\}$, $\{\alpha_8,\alpha_{14}\}$, $\{\alpha_{11},\alpha_{20}\}$, $\{\alpha_{12},\alpha_{23}\}$, $\{\alpha_{16},\alpha_{19}\}$.

n=21, $H \cong C_2^4$ ($|H| = 16$, $i = 14$): $\operatorname{rk} N_H = 15$, $(N_H)^*/N_H \cong \mathbb{Z}/8\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^6$, and $\det(K(q_{N_H})_2) \equiv \pm 2^9 \mod (\mathbb{Z}_2^*)^2$. [2]

$$H_{21,1} =$$

$$[(\alpha_2\alpha_{20})(\alpha_3\alpha_{10})(\alpha_5\alpha_6)(\alpha_8\alpha_{11})(\alpha_9\alpha_{21})(\alpha_{12}\alpha_{22})(\alpha_{17}\alpha_{23})(\alpha_{19}\alpha_{24}),$$

$$(\alpha_2\alpha_{19})(\alpha_3\alpha_5)(\alpha_6\alpha_{10})(\alpha_8\alpha_9)(\alpha_{11}\alpha_{21})(\alpha_{12}\alpha_{23})(\alpha_{17}\alpha_{22})(\alpha_{20}\alpha_{24}),$$

$$(\alpha_1\alpha_{16})(\alpha_2\alpha_{20})(\alpha_3\alpha_6)(\alpha_5\alpha_{10})(\alpha_{12}\alpha_{23})(\alpha_{14}\alpha_{18})(\alpha_{17}\alpha_{22})(\alpha_{19}\alpha_{24}),$$

$$(\alpha_1\alpha_{14})(\alpha_2\alpha_{24})(\alpha_3\alpha_5)(\alpha_6\alpha_{10})(\alpha_{12}\alpha_{22})(\alpha_{16}\alpha_{18})(\alpha_{17}\alpha_{23})(\alpha_{19}\alpha_{20})]$$

with orbits $\{\alpha_1,\alpha_{16},\alpha_{14},\alpha_{18}\}$, $\{\alpha_2,\alpha_{20},\alpha_{19},\alpha_{24}\}$, $\{\alpha_3,\alpha_{10},\alpha_5,\alpha_6\}$, $\{\alpha_8,\alpha_{11},\alpha_9,\alpha_{21}\}$, $\{\alpha_{12},\alpha_{22},\alpha_{23},\alpha_{17}\}$;

$$H_{21,2} =$$

$$[(\alpha_1\alpha_3)(\alpha_2\alpha_{23})(\alpha_5\alpha_{14})(\alpha_6\alpha_{16})(\alpha_{10}\alpha_{18})(\alpha_{12}\alpha_{20})(\alpha_{17}\alpha_{24})(\alpha_{19}\alpha_{22}),$$

$$(\alpha_1\alpha_2)(\alpha_3\alpha_{23})(\alpha_5\alpha_{17})(\alpha_6\alpha_{12})(\alpha_{10}\alpha_{22})(\alpha_{14}\alpha_{24})(\alpha_{16}\alpha_{20})(\alpha_{18}\alpha_{19}),$$

$$(\alpha_1\alpha_{16})(\alpha_2\alpha_{20})(\alpha_3\alpha_6)(\alpha_5\alpha_{10})(\alpha_{12}\alpha_{23})(\alpha_{14}\alpha_{18})(\alpha_{17}\alpha_{22})(\alpha_{19}\alpha_{24}),$$

$$(\alpha_1\alpha_{14})(\alpha_2\alpha_{24})(\alpha_3\alpha_5)(\alpha_6\alpha_{10})(\alpha_{12}\alpha_{22})(\alpha_{16}\alpha_{18})(\alpha_{17}\alpha_{23})(\alpha_{19}\alpha_{20})]$$

with orbits $\{\alpha_1,\alpha_3,\alpha_2,\alpha_{16},\alpha_{14},\alpha_{23},\alpha_6,\alpha_5,\alpha_{20},\alpha_{24},\alpha_{18},\alpha_{12},\alpha_{17},\alpha_{10},\alpha_{19},\alpha_{22}\}$.

n=20, $H \cong Q_{12}$ ($|H| = 12$, $i = 1$):

$$H_{20,1} = [(\alpha_1\alpha_{10}\alpha_{23}\alpha_{12})(\alpha_2\alpha_3\alpha_{21}\alpha_9)(\alpha_4\alpha_{22}\alpha_8\alpha_{20})(\alpha_7\alpha_{15})$$

[2] These calculations show that for a symplectic group $G = (C_2)^4$ on a Kählerian K3 surface, the group $S^*_{(G)}/S_{(G)} = S^*_{(2,2,2,2)}/S_{(2,2,2,2)} \cong \mathbb{Z}/8\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^6$. We must correct our calculation of this group in [11, Prop. 10.1].

$$(\alpha_{11}\alpha_{17}\alpha_{14}\alpha_{13})(\alpha_{16}\alpha_{19}),$$

$$(\alpha_2\alpha_4\alpha_{11})(\alpha_3\alpha_{17}\alpha_{22})(\alpha_7\alpha_{18}\alpha_{15})(\alpha_8\alpha_{14}\alpha_{21})(\alpha_9\alpha_{13}\alpha_{20})(\alpha_{16}\alpha_{24}\alpha_{19})]$$

with $Clos(H_{20,1}) = H_{61,1}$ above.

n=19, $H \cong C_2 \times C_6$ ($|H| = 12$, $i = 5$):

$$H_{19,1} =$$

$$[(\alpha_1\alpha_{23})(\alpha_2\alpha_{21})(\alpha_3\alpha_9)(\alpha_4\alpha_8)(\alpha_{10}\alpha_{12})(\alpha_{11}\alpha_{14})(\alpha_{13}\alpha_{17})(\alpha_{20}\alpha_{22}),$$

$$(\alpha_1\alpha_{10})(\alpha_2\alpha_9\alpha_4\alpha_{13}\alpha_{11}\alpha_{20})(\alpha_3\alpha_8\alpha_{17}\alpha_{14}\alpha_{22}\alpha_{21})(\alpha_7\alpha_{15}\alpha_{18})$$

$$(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19}\alpha_{24})]$$

with $Clos(H_{19,1}) = H_{61,1}$ above.

n=18, $H \cong D_{12}$ ($|H| = 12$, $i = 4$): $\operatorname{rk} N_H = 16$ and $(N_H)^*/N_H \cong (\mathbb{Z}/6\mathbb{Z})^4$.

$$H_{18,1} =$$

$$[(\alpha_2\alpha_{17})(\alpha_3\alpha_{10})(\alpha_4\alpha_{21})(\alpha_5\alpha_{18})(\alpha_8\alpha_{20})(\alpha_{11}\alpha_{23})(\alpha_{13}\alpha_{22})(\alpha_{15}\alpha_{24}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_2, \alpha_{13}, \alpha_3, \alpha_{22}, \alpha_{10}, \alpha_{17}\}$, $\{\alpha_4, \alpha_9, \alpha_{21}\}$, $\{\alpha_5, \alpha_{18}, \alpha_7\}$, $\{\alpha_8, \alpha_{14}, \alpha_{23}, \alpha_{12}, \alpha_{11}, \alpha_{20}\}$, $\{\alpha_{15}, \alpha_{24}\}$, $\{\alpha_{16}, \alpha_{19}, \}$.

n=17, $H \cong \mathfrak{A}_4$ ($|H| = 12$, $i = 3$): $\operatorname{rk} N_H = 16$ and $(N_H)^*/N_H \cong (\mathbb{Z}/12\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2$.

$$H_{17,1} =$$

$$[(\alpha_1\alpha_6\alpha_{10})(\alpha_2\alpha_{13}\alpha_{19})(\alpha_3\alpha_{18}\alpha_{14})(\alpha_4\alpha_{20}\alpha_7)(\alpha_8\alpha_{11}\alpha_{12})(\alpha_9\alpha_{22}\alpha_{23}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_1, \alpha_6, \alpha_{10}, \}$, $\{\alpha_2, \alpha_{13}, \alpha_{16}, \alpha_{19}, \}$, $\{\alpha_3, \alpha_{18}, \alpha_{22}, \alpha_{11}, \alpha_{14}, \alpha_7, \alpha_{23}, \alpha_{20}, \alpha_{12}, \alpha_8, \alpha_4, \alpha_9\}$;

$$H_{17,2} =$$

$$[(\alpha_1\alpha_{22}\alpha_{20})(\alpha_3\alpha_{11}\alpha_{15})(\alpha_4\alpha_9\alpha_{19})(\alpha_7\alpha_{21}\alpha_{12})(\alpha_8\alpha_{14}\alpha_{13})(\alpha_{18}\alpha_{24}\alpha_{23}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_1, \alpha_{22}, \alpha_{15}, \alpha_{20}, \alpha_3, \alpha_{11}\}$, $\{\alpha_2, \alpha_{13}, \alpha_8, \alpha_{14}\}$, $\{\alpha_4, \alpha_9, \alpha_{16}, \alpha_{19}\}$, $\{\alpha_7, \alpha_{21}, \alpha_{18}, \alpha_{12}, \alpha_{24}, \alpha_{23}\}$;

$$H_{17,3} =$$

$$[(\alpha_1\alpha_5\alpha_{17})(\alpha_3\alpha_7\alpha_{18})(\alpha_4\alpha_9\alpha_{13})(\alpha_6\alpha_{24}\alpha_{21})(\alpha_8\alpha_{19}\alpha_{14})(\alpha_{11}\alpha_{12}\alpha_{23}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_1, \alpha_5, \alpha_{17}\}$, $\{\alpha_2, \alpha_{13}, \alpha_4, \alpha_9\}$, $\{\alpha_3, \alpha_7, \alpha_{22}, \alpha_{18}\}$, $\{\alpha_6, \alpha_{24}, \alpha_{21}\}$, $\{\alpha_8, \alpha_{19}, \alpha_{14}, \alpha_{16}\}$, $\{\alpha_{11}, \alpha_{12}, \alpha_{20}, \alpha_{23}\}$.

n=16, $H \cong D_{10}$ ($|H| = 10$, $i = 1$): $\operatorname{rk} N_H = 16$ and $(N_H)^*/N_H \cong (\mathbb{Z}/5\mathbb{Z})^4$.

$$H_{16,1} =$$

$$[(\alpha_3\alpha_{13})(\alpha_5\alpha_{21})(\alpha_7\alpha_{18})(\alpha_8\alpha_{15})(\alpha_{10}\alpha_{14})(\alpha_{11}\alpha_{16})(\alpha_{12}\alpha_{24})(\alpha_{17}\alpha_{23}),$$

$$(\alpha_2\alpha_{17}\alpha_{15}\alpha_8\alpha_{23})(\alpha_3\alpha_{14}\alpha_{10}\alpha_{13}\alpha_9)(\alpha_4\alpha_{24}\alpha_{18}\alpha_7\alpha_{12})(\alpha_5\alpha_{11}\alpha_{20}\alpha_{16}\alpha_{21})]$$

with orbits $\{\alpha_2, \alpha_{17}, \alpha_{23}, \alpha_{15}, \alpha_8\}$, $\{\alpha_3, \alpha_{13}, \alpha_{14}, \alpha_9, \alpha_{10}\}$, $\{\alpha_4, \alpha_{24}, \alpha_{12}, \alpha_{18}, \alpha_7\}$, $\{\alpha_5, \alpha_{21}, \alpha_{11}, \alpha_{16}, \alpha_{20}\}$.

n=15, $H \cong C_3^2$ ($|H| = 9$, $i = 2$):

$$H_{15,1} =$$

$$[(\alpha_1\alpha_3\alpha_{21})(\alpha_5\alpha_{10}\alpha_7)(\alpha_6\alpha_{24}\alpha_{14})(\alpha_{11}\alpha_{12}\alpha_{13})(\alpha_{15}\alpha_{19}\alpha_{17})(\alpha_{16}\alpha_{18}\alpha_{22}),$$

$$(\alpha_1\alpha_6\alpha_{22})(\alpha_3\alpha_{24}\alpha_{16})(\alpha_4\alpha_{20}\alpha_9)(\alpha_5\alpha_{10}\alpha_7)(\alpha_{11}\alpha_{13}\alpha_{12})(\alpha_{14}\alpha_{18}\alpha_{21})]$$

with $Clos(H_{15,1}) = H_{30,1}$.

n=14, $H \cong C_8$ ($|H| = 8$, $i = 1$):

$$H_{14,1} = [(\alpha_1\alpha_{12}\alpha_{16}\alpha_{23}\alpha_{15}\alpha_{11}\alpha_{19}\alpha_{20})(\alpha_2\alpha_{13})$$

$$(\alpha_3\alpha_8\alpha_{22}\alpha_5\alpha_{18}\alpha_{14}\alpha_7\alpha_{17})(\alpha_4\alpha_6\alpha_9\alpha_{10})]$$

with $Clos(H_{14,1}) = H_{26,1}$ above.

n=13, $H \cong Q_8$ ($|H| = 8$, $i = 4$):

$$H_{13,1} = [(\alpha_1\alpha_2\alpha_{24}\alpha_{14})(\alpha_3\alpha_6\alpha_{17}\alpha_{22})(\alpha_4\alpha_9)(\alpha_5\alpha_{12}\alpha_{23}\alpha_{10})$$

$$(\alpha_8\alpha_{13})(\alpha_{16}\alpha_{18}\alpha_{19}\alpha_{20}),$$

$$(\alpha_1\alpha_{16}\alpha_{24}\alpha_{19})(\alpha_2\alpha_{20}\alpha_{14}\alpha_{18})(\alpha_3\alpha_{10}\alpha_{17}\alpha_{12})(\alpha_5\alpha_6\alpha_{23}\alpha_{22})$$

$$(\alpha_8\alpha_{13})(\alpha_{15}\alpha_{21})]$$

with $Clos(H_{13,1}) = H_{40,1}$ above.

n=12, $H \cong Q_8$ ($|H| = 8$, $i = 4$): $\operatorname{rk} N_H = 17$, $(N_H)^*/N_H \cong (\mathbb{Z}/8\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^3$, and $K((q_{N_H})_2) \cong q_\theta^{(2)}(2) \oplus q'$.

$$H_{12,1} = [(\alpha_1\alpha_9\alpha_{23}\alpha_7)(\alpha_4\alpha_{19}\alpha_{13}\alpha_{10})(\alpha_5\alpha_{12})$$

$$(\alpha_6\alpha_{21}\alpha_{20}\alpha_{15})(\alpha_8\alpha_{14}\alpha_{11}\alpha_{17})(\alpha_{16}\alpha_{24}),$$

$$(\alpha_1\alpha_{11}\alpha_{23}\alpha_8)(\alpha_4\alpha_6\alpha_{13}\alpha_{20})(\alpha_5\alpha_{16})(\alpha_7\alpha_{17}\alpha_9\alpha_{14})$$
$$(\alpha_{10}\alpha_{21}\alpha_{19}\alpha_{15})(\alpha_{12}\alpha_{24})]$$

with orbits $\{\alpha_1,\alpha_9,\alpha_{11},\alpha_{23},\alpha_{14},\alpha_7,\alpha_{17},\alpha_8\}$, $\{\alpha_4,\alpha_{19},\alpha_6,\alpha_{13},\alpha_{15},\alpha_{10},\alpha_{21},\alpha_{20}\}$, $\{\alpha_5,\alpha_{12},\alpha_{16},\alpha_{24}\}$.

n=11, $H\cong C_2\times C_4$ ($|H|=8$, $i=2$):

$$H_{11,1}=[(\alpha_2\alpha_8\alpha_3\alpha_{12})(\alpha_4\alpha_{11}\alpha_{18}\alpha_{16})(\alpha_7\alpha_{19}\alpha_9\alpha_{20})$$
$$(\alpha_{10}\alpha_{15})(\alpha_{13}\alpha_{14}\alpha_{22}\alpha_{23})(\alpha_{17}\alpha_{24}),$$
$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with $Clos(H_{11,1})=H_{22,1}$ above;

$$H_{11,2}=[(\alpha_2\alpha_3\alpha_7\alpha_4)(\alpha_6\alpha_{17}\alpha_{21}\alpha_{15})(\alpha_8\alpha_{11}\alpha_{14}\alpha_{20})$$
$$(\alpha_9\alpha_{13}\alpha_{22}\alpha_{18})(\alpha_{10}\alpha_{24})(\alpha_{16}\alpha_{19}),$$
$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with $Clos(H_{11,2})=H_{22,2}$ above;

$$H_{11,3}=[(\alpha_1\alpha_{10}\alpha_5\alpha_{24})(\alpha_2\alpha_3\alpha_{13}\alpha_{22})(\alpha_4\alpha_{18}\alpha_9\alpha_7)$$
$$(\alpha_6\alpha_{15}\alpha_{21}\alpha_{17})(\alpha_8\alpha_{14})(\alpha_{16}\alpha_{19}),$$
$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with $Clos(H_{11,3})=H_{22,3}$ above. [3]

n=10, $H\cong D_8$ ($|H|=8$, $i=3$): $\operatorname{rk} N_H=15$ and $(N_H)^*/N_H\cong(\mathbb{Z}/4\mathbb{Z})^5$.

$$H_{10,1}=$$
$$[(\alpha_2\alpha_{16})(\alpha_3\alpha_{12})(\alpha_4\alpha_{13})(\alpha_7\alpha_{11})(\alpha_8\alpha_9)(\alpha_{10}\alpha_{17})(\alpha_{14}\alpha_{19})(\alpha_{15}\alpha_{21}),$$
$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_2,\alpha_{16},\alpha_{13},\alpha_8,\alpha_{19},\alpha_9,\alpha_4,\alpha_{14}\}$, $\{\alpha_3,\alpha_{12},\alpha_{22},\alpha_{23}\}$, $\{\alpha_7,\alpha_{11},\alpha_{18},\alpha_{20}\}$, $\{\alpha_{10},\alpha_{17}\}$, $\{\alpha_{15},\alpha_{21}\}$;

$$H_{10,2}=$$
$$[(\alpha_2\alpha_{13})(\alpha_3\alpha_{24})(\alpha_4\alpha_8)(\alpha_5\alpha_{11})(\alpha_{10}\alpha_{15})(\alpha_{17}\alpha_{22})(\alpha_{18}\alpha_{23})(\alpha_{20}\alpha_{21}),$$

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[3] These calculations show that for a symplectic group $G=C_2\times C_4$ on a Kählerian K3 surface, the group $S^*_{(G)}/S_{(G)}=S^*_{(2,4)}/S_{(2,4)}\cong(\mathbb{Z}/4\mathbb{Z})^4\times(\mathbb{Z}/2\mathbb{Z})^2$. We must correct our calculation of this group in [11, Prop. 10.1].

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_2,\alpha_{13}\}$, $\{\alpha_3,\alpha_{24},\alpha_{22},\alpha_{17}\}$, $\{\alpha_4,\alpha_8,\alpha_9,\alpha_{14}\}$, $\{\alpha_5,\alpha_{11},\alpha_{21},\alpha_{20}\}$, $\{\alpha_7,\alpha_{18},\alpha_{12},\alpha_{23}\}$, $\{\alpha_{10},\alpha_{15}\}$, $\{\alpha_{16},\alpha_{19}\}$.

n=9, $H\cong C_2^3$ ($|H|=8$, $i=5$): $\operatorname{rk} N_H=14$, $(N_H)^*/N_H\cong(\mathbb{Z}/4\mathbb{Z})^2\times(\mathbb{Z}/2\mathbb{Z})^6$ and $\det(K(q_{N_H})_2)\equiv\pm 2^{10} \mod (\mathbb{Z}_2^*)^2$.

$$H_{9,1}=[(\alpha_3\alpha_8)(\alpha_4\alpha_9)(\alpha_5\alpha_{15})(\alpha_6\alpha_{24})(\alpha_7\alpha_{19})(\alpha_{11}\alpha_{20})(\alpha_{14}\alpha_{22})(\alpha_{16}\alpha_{18}),$$

$$(\alpha_2\alpha_{12})(\alpha_3\alpha_8)(\alpha_4\alpha_{20})(\alpha_7\alpha_{16})(\alpha_9\alpha_{11})(\alpha_{13}\alpha_{23})(\alpha_{14}\alpha_{22})(\alpha_{18}\alpha_{19}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_2,\alpha_{12},\alpha_{13},\alpha_{23}\}$, $\{\alpha_3,\alpha_8,\alpha_{22},\alpha_{14}\}$, $\{\alpha_4,\alpha_9,\alpha_{20},\alpha_{11}\}$, $\{\alpha_5,\alpha_{15}\}$, $\{\alpha_6,\alpha_{24}\}$, $\{\alpha_7,\alpha_{19},\alpha_{16},\alpha_{18}\}$;

$$H_{9,2}=$$

$$[(\alpha_2\alpha_3)(\alpha_4\alpha_{18})(\alpha_7\alpha_9)(\alpha_8\alpha_{12})(\alpha_{11}\alpha_{16})(\alpha_{13}\alpha_{22})(\alpha_{14}\alpha_{23})(\alpha_{19}\alpha_{20}),$$

$$(\alpha_2\alpha_{12})(\alpha_3\alpha_8)(\alpha_4\alpha_{20})(\alpha_7\alpha_{16})(\alpha_9\alpha_{11})(\alpha_{13}\alpha_{23})(\alpha_{14}\alpha_{22})(\alpha_{18}\alpha_{19}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_2,\alpha_3,\alpha_{12},\alpha_{13},\alpha_8,\alpha_{22},\alpha_{23},\alpha_{14}\}$, $\{\alpha_4,\alpha_{18},\alpha_{20},\alpha_9,\alpha_{19},\alpha_7,\alpha_{11}\alpha_{16}\}$;

$$H_{9,3}=$$

$$[(\alpha_2\alpha_4)(\alpha_3\alpha_7)(\alpha_6\alpha_{21})(\alpha_9\alpha_{13})(\alpha_{10}\alpha_{24})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{18}\alpha_{22}),$$

$$(\alpha_2\alpha_{22})(\alpha_3\alpha_{13})(\alpha_4\alpha_{18})(\alpha_7\alpha_9)(\alpha_{10}\alpha_{24})(\alpha_{11}\alpha_{20})(\alpha_{15}\alpha_{17})(\alpha_{16}\alpha_{19}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_2,\alpha_4,\alpha_{22},\alpha_{13},\alpha_{18},\alpha_9,\alpha_3,\alpha_7\}$, $\{\alpha_6,\alpha_{21}\}$, $\{\alpha_8,\alpha_{14}\}$, $\{\alpha_{10},\alpha_{24}\}$, $\{\alpha_{11},\alpha_{20}\}$, $\{\alpha_{12},\alpha_{23}\}$, $\{\alpha_{15},\alpha_{17}\}$, $\{\alpha_{16},\alpha_{19}\}$;

$$H_{9,4}=$$

$$[(\alpha_1\alpha_5)(\alpha_6\alpha_{21})(\alpha_8\alpha_{14})(\alpha_{10}\alpha_{24})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{15}\alpha_{17})(\alpha_{16}\alpha_{19}),$$

$$(\alpha_2\alpha_{22})(\alpha_3\alpha_{13})(\alpha_4\alpha_{18})(\alpha_7\alpha_9)(\alpha_{10}\alpha_{24})(\alpha_{11}\alpha_{20})(\alpha_{15}\alpha_{17})(\alpha_{16}\alpha_{19}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_1,\alpha_5\}$, $\{\alpha_2,\alpha_{22},\alpha_{13},\alpha_3\}$, $\{\alpha_4,\alpha_{18},\alpha_9,\alpha_7\}$, $\{\alpha_6,\alpha_{21}\}$, $\{\alpha_8,\alpha_{14}\}$, $\{\alpha_{10},\alpha_{24}\}$, $\{\alpha_{11},\alpha_{20}\}$, $\{\alpha_{12},\alpha_{23}\}$, $\{\alpha_{15},\alpha_{17}\}$, $\{\alpha_{16},\alpha_{19}\}$.

n=8, $H\cong C_7$ ($|H|=7$, $i=1$):

$$H_{8,1}=[(\alpha_1\alpha_5\alpha_{10}\alpha_{19}\alpha_{22}\alpha_6\alpha_7)(\alpha_3\alpha_{16}\alpha_{20}\alpha_{24}\alpha_9\alpha_4\alpha_{17})$$

$$(\alpha_{11}\alpha_{15}\alpha_{18}\alpha_{14}\alpha_{13}\alpha_{21}\alpha_{12})]$$

with $Clos(H_{8,1}) = H_{33,1}$ above.

n=7, $H \cong C_6$ ($|H| = 6$, $i = 2$):

$$H_{7,1} = [(\alpha_2\alpha_{17}\alpha_{13}\alpha_3\alpha_{10}\alpha_{22})(\alpha_4\alpha_{21}\alpha_9)(\alpha_5\alpha_7\alpha_{18})$$

$$(\alpha_8\alpha_{11}\alpha_{12}\alpha_{23}\alpha_{20}\alpha_{14})(\alpha_{15}\alpha_{24})(\alpha_{16}\alpha_{19})]$$

with $Clos(H_{7,1}) = H_{18,1}$ above.

n=6, $H \cong D_6$ ($|H| = 6$, $i = 1$): $\mathrm{rk}\, N_H = 14$ and $(N_H)^*/N_H \cong (\mathbb{Z}/6\mathbb{Z})^2 \times (\mathbb{Z}/3\mathbb{Z})^3$.

$$H_{6,1} =$$

$$[(\alpha_2\alpha_{14})(\alpha_3\alpha_{17})(\alpha_4\alpha_8)(\alpha_7\alpha_{15})(\alpha_9\alpha_{13})(\alpha_{10}\alpha_{12})(\alpha_{11}\alpha_{21})(\alpha_{16}\alpha_{19}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_2, \alpha_{14}, \alpha_9, \alpha_4, \alpha_{13}, \alpha_8\}$, $\{\alpha_3, \alpha_{17}, \alpha_{22}\}$, $\{\alpha_7, \alpha_{15}, \alpha_{18}\}$, $\{\alpha_{10}, \alpha_{12}, \alpha_{23}\}$, $\{\alpha_{11}, \alpha_{21}, \alpha_{20}\}$, $\{\alpha_{16}, \alpha_{19}\}$.

n=5, $H \cong C_5$ ($|H| = 5$, $i = 1$):

$$H_{5,1} =$$

$$[(\alpha_2\alpha_8\alpha_{17}\alpha_{23}\alpha_{15})(\alpha_3\alpha_{13}\alpha_{14}\alpha_9\alpha_{10})(\alpha_4\alpha_7\alpha_{24}\alpha_{12}\alpha_{18})(\alpha_5\alpha_{16}\alpha_{11}\alpha_{21}\alpha_{20})]$$

with $Clos(H_{5,1}) = H_{16,1}$ above.

n=4, $H \cong C_4$ ($|H| = 4$, $i = 1$): $\mathrm{rk}\, N_H = 14$ and $(N_H)^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^4 \times (\mathbb{Z}/2\mathbb{Z})^2$.

$$H_{4,1} = [(\alpha_2\alpha_{22}\alpha_{23}\alpha_6)(\alpha_3\alpha_5\alpha_9\alpha_{16})(\alpha_7\alpha_{10}\alpha_{20}\alpha_{13})(\alpha_8\alpha_{15})$$

$$(\alpha_{11}\alpha_{19}\alpha_{24}\alpha_{14})(\alpha_{12}\alpha_{21})]$$

with orbits $\{\alpha_2, \alpha_{22}, \alpha_{23}, \alpha_6\}$, $\{\alpha_3, \alpha_5, \alpha_9, \alpha_{16}\}$, $\{\alpha_7, \alpha_{10}, \alpha_{20}, \alpha_{13}\}$, $\{\alpha_8, \alpha_{15}\}$, $\{\alpha_{11}, \alpha_{19}, \alpha_{24}, \alpha_{14}\}$, $\{\alpha_{12}, \alpha_{21}\}$.

n=3, $H \cong C_2^2$ ($|H| = 4$, $i = 2$): $\mathrm{rk}\, N_H = 12$ and $(N_H)^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^6$.

$$H_{3,1} =$$

$$[(\alpha_2\alpha_{12})(\alpha_3\alpha_8)(\alpha_4\alpha_{20})(\alpha_7\alpha_{16})(\alpha_9\alpha_{11})(\alpha_{13}\alpha_{23})(\alpha_{14}\alpha_{22})(\alpha_{18}\alpha_{19}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_2, \alpha_{12}, \alpha_{13}, \alpha_{23}\}$, $\{\alpha_3, \alpha_8, \alpha_{22}, \alpha_{14}\}$, $\{\alpha_4, \alpha_{20}, \alpha_9, \alpha_{11}\}$, $\{\alpha_7, \alpha_{16}, \alpha_{18}, \alpha_{19}\}$;

$$H_{3,2} =$$

$$[(\alpha_2\alpha_{22})(\alpha_3\alpha_{13})(\alpha_4\alpha_{18})(\alpha_7\alpha_9)(\alpha_{10}\alpha_{24})(\alpha_{11}\alpha_{20})(\alpha_{15}\alpha_{17})(\alpha_{16}\alpha_{19}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_2,\alpha_{22},\alpha_{13},\alpha_3\}$, $\{\alpha_4,\alpha_{18},\alpha_9,\alpha_7\}$, $\{\alpha_8,\alpha_{14}\}$, $\{\alpha_{10},\alpha_{24}\}$, $\{\alpha_{11},\alpha_{20}\}$, $\{\alpha_{12},\alpha_{23}\}$, $\{\alpha_{15},\alpha_{17}\}$, $\{\alpha_{16},\alpha_{19}\}$;

$$H_{3,3}=$$

$$[(\alpha_1\alpha_{24})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_5\alpha_{10})(\alpha_6\alpha_{17})(\alpha_{12}\alpha_{23})(\alpha_{15}\alpha_{21})(\alpha_{16}\alpha_{19}),$$

$$(\alpha_2\alpha_{13})(\alpha_3\alpha_{22})(\alpha_4\alpha_9)(\alpha_7\alpha_{18})(\alpha_8\alpha_{14})(\alpha_{11}\alpha_{20})(\alpha_{12}\alpha_{23})(\alpha_{16}\alpha_{19})]$$

with orbits $\{\alpha_1,\alpha_{24}\}$, $\{\alpha_2,\alpha_{13}\}$, $\{\alpha_3,\alpha_{22}\}$, $\{\alpha_4,\alpha_9\}$, $\{\alpha_5,\alpha_{10}\}$, $\{\alpha_6,\alpha_{17}\}$, $\{\alpha_7,\alpha_{18}\}$, $\{\alpha_8,\alpha_{14}\}$, $\{\alpha_{11},\alpha_{20}\}$, $\{\alpha_{12},\alpha_{23}\}$, $\{\alpha_{15},\alpha_{21}\}$, $\{\alpha_{16},\alpha_{19}\}$.

n=2, $H\cong C_3$ ($|H|=3$, $i=1$): $\operatorname{rk} N_H=12$ and $(N_H)^*/N_H\cong(\mathbb{Z}/3\mathbb{Z})^6$.

$$H_{2,1}=$$

$$[(\alpha_1\alpha_{13}\alpha_7)(\alpha_2\alpha_{19}\alpha_{22})(\alpha_5\alpha_8\alpha_9)(\alpha_6\alpha_{10}\alpha_{11})(\alpha_{12}\alpha_{23}\alpha_{20})(\alpha_{15}\alpha_{18}\alpha_{16})]$$

with orbits $\{\alpha_1,\alpha_{13},\alpha_7\}$, $\{\alpha_2,\alpha_{19},\alpha_{22}\}$, $\{\alpha_5,\alpha_8,\alpha_9\}$, $\{\alpha_6,\alpha_{10},\alpha_{11}\}$, $\{\alpha_{12},\alpha_{23},\alpha_{20}\}$, $\{\alpha_{15},\alpha_{18},\alpha_{16}\}$.

n=1, $H\cong C_2$ ($|H|=2$, $i=1$): $\operatorname{rk} N_H=8$ and $(N_H)^*/N_H\cong(\mathbb{Z}/2\mathbb{Z})^8$.

$$H_{1,1}=$$

$$[(\alpha_2\alpha_{23})(\alpha_3\alpha_9)(\alpha_5\alpha_{16})(\alpha_6\alpha_{22})(\alpha_7\alpha_{20})(\alpha_{10}\alpha_{13})(\alpha_{11}\alpha_{24})(\alpha_{14}\alpha_{19})]$$

with orbits $\{\alpha_2,\alpha_{23}\}$, $\{\alpha_3,\alpha_9\}$, $\{\alpha_5,\alpha_{16}\}$, $\{\alpha_6,\alpha_{22}\}$, $\{\alpha_7,\alpha_{20}\}$, $\{\alpha_{10},\alpha_{13}\}$, $\{\alpha_{11},\alpha_{24}\}$, $\{\alpha_{14},\alpha_{19}\}$.

We obtain the following applications to Kählerian K3 surfaces.

Let X be a Kählerian K3 surface with Picard lattice S_X. In [13], we introduced *marking of X by Niemeier lattices N_k.* We remind to a reader that it is a choice of a maximal negative definite sublattice $S\subset S_X$ together with a primitive embedding $S\subset N_k$. Additionally one requires that for the set $P(X)\subset S_X$ of classes of non-singular rational curves on X one has $P(X)\cap S=P(S)$ and $P(S)=S\cap P(N_k)$ where $P(S)$ is the basis of the set $\Delta(S)$ of all (-2)-roots of S. By Theorem 1, any Kählerian K3 surface has marking by one of Niemeier lattices N_k, $k=1,2,\ldots 24$. Using Kondō's trick in [6], applying Theorem 2, one can show that any Kählerian K3 surface can be marked by one of Niemeier lattices N_k, $k=1,2,\ldots,23$; one can avoid difficult Leech lattice. See [13] for details.

Using marking of X by N_k, one can find $P(X) \cap S = P(N_k) \cap S = P(S)$ and

$$\mathrm{Aut}\,(X, S)_0 = \{\phi \in A(N_k) \mid \phi|S^{\perp}_{N_k} \textit{ is identity}\}$$

where $\mathrm{Aut}\,(X, S)_0$ is the group of symplectic automorphisms of X which act identically on the orthogonal complement to S in $H^2(X, \mathbb{Z})$. Here we identify this group with its action on S. These facts show importance of marking of X by Niemeier lattices.

Thus, we obtain the following application to Kählerian K3 surfaces X.

Let X be marked by a primitive sublattice $S \subset N = N_{23} = N(24A_1)$. Then S must satisfy Theorem 2 and $\Gamma(P(S)) \subset \Gamma(P(N_{23})) = 24\mathbb{A}_1$. Any such S gives marking of some X and $P(X) \cap S = P(S)$.

If $N_H \subset S$ where H has the type $n = 81$, 80, 79, 78, 77, 76, 74, 70, 63, 62 or 54 (equivalently, $\mathrm{rk}\, N_H = 19$), then $\mathrm{Aut}\,(X, S)_0 = H$. Otherwise, if only $N_H \subset S$ where H has the type $n = 75$, 65, 61, 56, 55, 51, 48, 46, 33, 32 or 26 (equivalently, $\mathrm{rk}\, N_H = 18$), then $\mathrm{Aut}\,(X, S)_0 = H$. Otherwise, if only $N_H \subset S$ where H has the type $n = 49$, 40, 39, 34 or 12 (equivalently, $\mathrm{rk}\, N_H = 17$), then $\mathrm{Aut}\,(X, S)_0 = H$. Otherwise, if only $N_H \subset S$ where H has the type $n = 30$, 22, 18, 17 or 16 (equivalently, $\mathrm{rk}\, N_H = 16$), then $\mathrm{Aut}\,(X, S)_0 = H$. Otherwise, if only $N_H \subset S$ where H has the type $n = 21$ or 10 (equivalently, $\mathrm{rk}\, N_H = 15$), then $\mathrm{Aut}\,(X, S)_0 = H$. Otherwise, if only $N_H \subset S$ where H has the type $n = 9$, 6 or 4 (equivalently, $\mathrm{rk}\, N_H = 14$), then $\mathrm{Aut}\,(X, S)_0 = H$. Otherwise, if only $N_H \subset S$ where H has the type $n = 3$ or 2 (equivalently, $\mathrm{rk}\, N_H = 12$), then $\mathrm{Aut}\,(X, S)_0 = H$. Otherwise, if only $N_H \subset S$ where H has the type $n = 1$ (equivalently, $\mathrm{rk}\, N_H = 8$), then $\mathrm{Aut}\,(X, S)_0 = H \cong C_2$. Otherwise, $\mathrm{Aut}\,(X, S)_0$ is trivial.

Case 22. For the Niemeier lattice N_{22}, we have

$$N = N_{22} = N(12A_2) = [12A_2,\ [2(11211122212)]] = [12A_2,$$
$$-\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7 - \epsilon_8 - \epsilon_9 - \epsilon_{10} + \epsilon_{11} - \epsilon_{12},$$
$$-\epsilon_1 - \epsilon_2 + \epsilon_3 + \epsilon_4 - \epsilon_5 + \epsilon_6 + \epsilon_7 + \epsilon_8 - \epsilon_9 - \epsilon_{10} - \epsilon_{11} + \epsilon_{12},$$
$$-\epsilon_1 + \epsilon_2 - \epsilon_3 + \epsilon_4 + \epsilon_5 - \epsilon_6 + \epsilon_7 + \epsilon_8 + \epsilon_9 - \epsilon_{10} - \epsilon_{11} - \epsilon_{12},$$
$$-\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4 + \epsilon_5 + \epsilon_6 - \epsilon_7 + \epsilon_8 + \epsilon_9 + \epsilon_{10} - \epsilon_{11} - \epsilon_{12},$$
$$-\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4 - \epsilon_5 + \epsilon_6 + \epsilon_7 - \epsilon_8 + \epsilon_9 + \epsilon_{10} + \epsilon_{11} - \epsilon_{12},$$
$$-\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 + \epsilon_5 - \epsilon_6 + \epsilon_7 + \epsilon_8 - \epsilon_9 + \epsilon_{10} + \epsilon_{11} + \epsilon_{12},$$
$$-\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 + \epsilon_6 - \epsilon_7 + \epsilon_8 + \epsilon_9 - \epsilon_{10} + \epsilon_{11} + \epsilon_{12},$$
$$-\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4 - \epsilon_5 - \epsilon_6 + \epsilon_7 - \epsilon_8 + \epsilon_9 + \epsilon_{10} - \epsilon_{11} + \epsilon_{12},$$
$$-\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 - \epsilon_5 - \epsilon_6 - \epsilon_7 + \epsilon_8 - \epsilon_9 + \epsilon_{10} + \epsilon_{11} - \epsilon_{12},$$
$$-\epsilon_1 - \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 - \epsilon_6 - \epsilon_7 - \epsilon_8 + \epsilon_9 - \epsilon_{10} + \epsilon_{11} + \epsilon_{12},$$
$$-\epsilon_1 + \epsilon_2 - \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 - \epsilon_7 - \epsilon_8 - \epsilon_9 + \epsilon_{10} - \epsilon_{11} + \epsilon_{12}]$$

(see [3, Ch. 16]) where $\epsilon_k = \epsilon_{1,k}$, $k = 1, 2, \dots, 24$. Equivalently,

$$N = N_{22} = N(12A_2) = [12A_2,\ \epsilon_1 + \epsilon_7 - \epsilon_9 + \epsilon_{10} - \epsilon_{11} - \epsilon_{12},$$
$$\epsilon_2 + \epsilon_7 - \epsilon_8 - \epsilon_9 - \epsilon_{10} + \epsilon_{11},$$
$$\epsilon_3 + \epsilon_8 - \epsilon_9 - \epsilon_{10} - \epsilon_{11} + \epsilon_{12},$$
$$\epsilon_4 - \epsilon_7 + \epsilon_8 - \epsilon_9 + \epsilon_{11} - \epsilon_{12},$$
$$\epsilon_5 + \epsilon_7 + \epsilon_8 + \epsilon_{10} + \epsilon_{11} + \epsilon_{12},$$
$$\epsilon_6 - \epsilon_7 - \epsilon_8 - \epsilon_9 + \epsilon_{10} + \epsilon_{12}]$$

for the reduced basis of the cord group. The group $A = A(N_{22})$ consists of the cyclic group $[\varphi_0]$ of order 2 where

$$\varphi_0 = (\alpha_{1,1}\alpha_{2,1})(\alpha_{1,2}\alpha_{2,2})\cdots(\alpha_{1,12}\alpha_{2,12})$$

gives non-trivial involutions on all 12 components $12\mathbb{A}_2$ and $A/[\varphi_0] = M_{12}$ is the Mathieu group M_{12} of the order 95040 on 12 components $12\mathbb{A}_2$. We have:

$$M_{12} = [\varphi_1 = (1)(2,3,4,5,6,7,8,9,10,11,12),\ \varphi_2 = (1)(2)(3)(10)$$
$$(4,8,12,9)(5,11,6,7),\ \varphi_3 = (1,2)(3,12)(4,7)(5,9)(6,10)(8,11)]$$

where $1,\dots,12$ numerate components of $12\mathbb{A}_2$. Possible lifts of the generators $\varphi_1,\varphi_2,\varphi_3$ of M_{12} to elements of $A(N_{22})$ are respectively

$$\widetilde{\varphi}_1=(\alpha_{1,1})(\alpha_{2,1})(\alpha_{1,2}\alpha_{1,3}\dots\alpha_{1,11}\alpha_{1,12})(\alpha_{2,2}\alpha_{2,3}\dots\alpha_{2,11}\alpha_{2,12}),$$

$$\widetilde{\varphi}_2=(\alpha_{1,1})(\alpha_{2,1})(\alpha_{1,2}\alpha_{2,2})(\alpha_{1,3}\alpha_{2,3})(\alpha_{1,10})(\alpha_{2,10})(\alpha_{1,4}\alpha_{1,8}\alpha_{1,12}\alpha_{1,9})$$
$$(\alpha_{2,4}\alpha_{2,8}\alpha_{2,12}\alpha_{2,9})(\alpha_{1,5}\alpha_{2,11}\alpha_{1,6}\alpha_{2,7})(\alpha_{2,5}\alpha_{1,11}\alpha_{2,6}\alpha_{1,7}),$$

$$\widetilde{\varphi}_3=(\alpha_{1,1}\alpha_{1,2}\alpha_{2,1}\alpha_{2,2})(\alpha_{1,3}\alpha_{2,12}\alpha_{2,3}\alpha_{1,12})(\alpha_{1,4}\alpha_{1,7}\alpha_{2,4}\alpha_{2,7})$$
$$(\alpha_{1,5}\alpha_{2,9}\alpha_{2,5}\alpha_{1,9})(\alpha_{1,6}\alpha_{2,10}\alpha_{2,6}\alpha_{1,10})(\alpha_{1,8}\alpha_{1,11}\alpha_{2,8}\alpha_{2,11}).$$

Thus, $A(M_{22})=[\varphi_0,\widetilde{\varphi}_1,\widetilde{\varphi}_2,\widetilde{\varphi}_3]$.

Using results described at the beginning of this Sec. 4, and GAP Progam [4], we obtain the following classification.

Classification of KahK3 conjugacy classes for $A(N_{22})$:

n=79, $H\cong\mathfrak{A}_6$ ($|H|=360$, $i=118$): $\operatorname{rk}N_H=19$ and $(N_H)^*/N_H\cong\mathbb{Z}/60\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$.

$$H_{79,1}=[(\alpha_{1,1}\alpha_{1,3}\alpha_{1,7}\alpha_{1,8}\alpha_{1,11})(\alpha_{2,1}\alpha_{2,3}\alpha_{2,7}\alpha_{2,8}\alpha_{2,11})$$
$$(\alpha_{1,4}\alpha_{1,12}\alpha_{1,5}\alpha_{1,6}\alpha_{1,10})(\alpha_{2,4}\alpha_{2,12}\alpha_{2,5}\alpha_{2,6}\alpha_{2,10}),$$
$$(\alpha_{1,1}\alpha_{1,3}\alpha_{2,11}\alpha_{2,10}\alpha_{2,7})(\alpha_{2,1}\alpha_{2,3}\alpha_{1,11}\alpha_{1,10}\alpha_{1,7})$$
$$(\alpha_{1,4}\alpha_{2,5}\alpha_{2,12}\alpha_{1,6}\alpha_{2,8})(\alpha_{2,4}\alpha_{1,5}\alpha_{1,12}\alpha_{2,6}\alpha_{1,8})]$$

with orbits (here and in what follows we show orbits with more than one elements only) $\{\alpha_{1,1},\alpha_{1,3},\alpha_{1,7},\alpha_{2,11},\alpha_{1,8},\alpha_{2,1},\alpha_{2,10},\alpha_{1,11},\alpha_{2,4},\alpha_{2,3},$ $\alpha_{2,7},\alpha_{1,10},\alpha_{2,12},\alpha_{1,5},\alpha_{2,8},\alpha_{1,4},\alpha_{2,5},\alpha_{1,6},\alpha_{1,12},\alpha_{2,6}\}$.

n=70, $H\cong\mathfrak{S}_5$ ($|H|=120$, $i=34$): $\operatorname{rk}N_H=19$ and $(N_H)^*/N_H\cong\mathbb{Z}/60\mathbb{Z}\times\mathbb{Z}/5\mathbb{Z}$.

$$H_{70,1}=[(\alpha_{1,2}\alpha_{2,3}\alpha_{1,4}\alpha_{1,5}\alpha_{2,11})(\alpha_{2,2}\alpha_{1,3}\alpha_{2,4}\alpha_{2,5}\alpha_{1,11})$$
$$(\alpha_{1,6}\alpha_{2,8}\alpha_{2,7}\alpha_{1,10}\alpha_{1,12})(\alpha_{2,6}\alpha_{1,8}\alpha_{1,7}\alpha_{2,10}\alpha_{2,12}),$$
$$(\alpha_{1,3}\alpha_{2,4})(\alpha_{2,3}\alpha_{1,4})(\alpha_{1,6}\alpha_{2,12})(\alpha_{2,6}\alpha_{1,12})$$
$$(\alpha_{1,7}\alpha_{2,8})(\alpha_{2,7}\alpha_{1,8})(\alpha_{1,9}\alpha_{1,10})(\alpha_{2,9}\alpha_{2,10})]$$

with orbits $\{\alpha_{1,2},\alpha_{2,3},\alpha_{1,4},\alpha_{1,5},\alpha_{2,11}\}$, $\{\alpha_{2,2},\alpha_{1,3},\alpha_{2,4},\alpha_{2,5},\alpha_{1,11}\}$, $\{\alpha_{1,6},\alpha_{2,8},\alpha_{2,12},\alpha_{2,7},\alpha_{1,7},\alpha_{2,6},\alpha_{1,10},\alpha_{1,8},\alpha_{2,10},\alpha_{1,12},\alpha_{1,9},\alpha_{2,9}\}$.

n=63, $H \cong M_9$ ($|H| = 72$, $i = 41$): $\operatorname{rk} N_H = 19$ and $(N_H)^*/N_H \cong \mathbb{Z}/18\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

$$H_{63,1} = [(\alpha_{1,3}\alpha_{2,3})(\alpha_{1,4}\alpha_{2,9}\alpha_{1,5}\alpha_{2,12})(\alpha_{2,4}\alpha_{1,9}\alpha_{2,5}\alpha_{1,12})$$

$$(\alpha_{1,6}\alpha_{1,7}\alpha_{1,8}\alpha_{1,10})(\alpha_{2,6}\alpha_{2,7}\alpha_{2,8}\alpha_{2,10})(\alpha_{1,11}\alpha_{2,11}),$$

$$(\alpha_{1,1}\alpha_{2,1})(\alpha_{1,3}\alpha_{2,10}\alpha_{2,4}\alpha_{2,9})(\alpha_{2,3}\alpha_{1,10}\alpha_{1,4}\alpha_{1,9})$$

$$(\alpha_{1,6}\alpha_{1,8}\alpha_{2,12}\alpha_{2,7})(\alpha_{2,6}\alpha_{2,8}\alpha_{1,12}\alpha_{1,7})(\alpha_{1,11}\alpha_{2,11})]$$

with orbits $\{\alpha_{1,1}, \alpha_{2,1}\}$, $\{\alpha_{1,3}, \alpha_{2,3}, \alpha_{2,10}, \alpha_{1,10}, \alpha_{2,6}, \alpha_{2,4}, \alpha_{1,6}, \alpha_{1,4}, \alpha_{2,7}, \alpha_{2,8}, \alpha_{1,9}, \alpha_{2,9}, \alpha_{1,7}, \alpha_{1,8}, \alpha_{1,12}, \alpha_{2,5}, \alpha_{1,5}, \alpha_{2,12}\}$, $\{\alpha_{1,11}, \alpha_{2,11}\}$;

$$H_{63,2} = [(\alpha_{1,2}\alpha_{2,2})(\alpha_{1,3}\alpha_{2,4}\alpha_{1,8}\alpha_{2,10})(\alpha_{2,3}\alpha_{1,4}\alpha_{2,8}\alpha_{1,10})$$

$$(\alpha_{1,5}\alpha_{1,9}\alpha_{1,12}\alpha_{1,7})(\alpha_{2,5}\alpha_{2,9}\alpha_{2,12}\alpha_{2,7})(\alpha_{1,11}\alpha_{2,11}),$$

$$(\alpha_{1,1}\alpha_{2,1})(\alpha_{1,3}\alpha_{2,10}\alpha_{2,4}\alpha_{2,9})(\alpha_{2,3}\alpha_{1,10}\alpha_{1,4}\alpha_{1,9})$$

$$(\alpha_{1,6}\alpha_{1,8}\alpha_{2,12}\alpha_{2,7})(\alpha_{2,6}\alpha_{2,8}\alpha_{1,12}\alpha_{1,7})(\alpha_{1,11}\alpha_{2,11})]$$

with orbits $\{\alpha_{1,1}, \alpha_{2,1}\}$, $\{\alpha_{1,2}, \alpha_{2,2}\}$, $\{\alpha_{1,3}, \alpha_{2,4}, \alpha_{2,10}, \alpha_{1,8}, \alpha_{2,9}, \alpha_{2,12}, \alpha_{2,7}, \alpha_{2,5}, \alpha_{1,6}\}$, $\{\alpha_{2,3}, \alpha_{1,4}, \alpha_{1,10}, \alpha_{2,8}, \alpha_{1,9}, \alpha_{1,12}, \alpha_{1,7}, \alpha_{1,5}, \alpha_{2,6}\}$, $\{\alpha_{1,11}, \alpha_{2,11}\}$.

n=62, $H \cong N_{72}$ ($|H| = 72$, $i = 40$): $\operatorname{rk} N_H = 19$, $(N_H)^*/N_H \cong \mathbb{Z}/36\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^2$ and $\det(K((q_{N_H})_3)) \equiv -2^2 \cdot 3^4 \mod (\mathbb{Z}_3^*)^2$.

$$H_{62,1} = [(\alpha_{1,3}\alpha_{2,12})(\alpha_{2,3}\alpha_{1,12})(\alpha_{1,4}\alpha_{2,5})(\alpha_{2,4}\alpha_{1,5})(\alpha_{1,8}\alpha_{1,10})$$

$$(\alpha_{2,8}\alpha_{2,10})(\alpha_{1,9}\alpha_{2,11})(\alpha_{2,9}\alpha_{1,11}),$$

$$(\alpha_{1,1}\alpha_{1,2}\alpha_{1,3}\alpha_{1,5}\alpha_{2,4}\alpha_{2,11})(\alpha_{2,1}\alpha_{2,2}\alpha_{2,3}\alpha_{2,5}\alpha_{1,4}\alpha_{1,11})(\alpha_{1,7}\alpha_{2,12}\alpha_{1,9})$$

$$(\alpha_{2,7}\alpha_{1,12}\alpha_{2,9})(\alpha_{1,8}\alpha_{2,10})(\alpha_{2,8}\alpha_{1,10})]$$

with orbits $\{\alpha_{1,1}, \alpha_{1,2}, \alpha_{1,3}, \alpha_{2,12}, \alpha_{1,5}, \alpha_{1,9}, \alpha_{2,4}, \alpha_{2,11}, \alpha_{1,7}\}$, $\{\alpha_{2,1}, \alpha_{2,2}, \alpha_{2,3}, \alpha_{1,12}, \alpha_{2,5}, \alpha_{2,9}, \alpha_{1,4}, \alpha_{1,11}, \alpha_{2,7}\}$, $\{\alpha_{1,8}, \alpha_{1,10}, \alpha_{2,10}, \alpha_{2,8}\}$.

n=55, $H \cong \mathfrak{A}_5$ ($|H| = 60$, $i = 5$): $\operatorname{rk} N_H = 18$ and $(N_H)^*/N_H \cong \mathbb{Z}/30\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$.

$$H_{55,1} = [(\alpha_{1,2}\alpha_{2,3}\alpha_{2,11}\alpha_{1,4}\alpha_{1,5})(\alpha_{2,2}\alpha_{1,3}\alpha_{1,11}\alpha_{2,4}\alpha_{2,5})$$

$$(\alpha_{1,6}\alpha_{2,7}\alpha_{2,9}\alpha_{1,12}\alpha_{2,8})(\alpha_{2,6}\alpha_{1,7}\alpha_{1,9}\alpha_{2,12}\alpha_{1,8}),$$

$$(\alpha_{1,1}\alpha_{1,3}\alpha_{2,4})(\alpha_{2,1}\alpha_{2,3}\alpha_{1,4})(\alpha_{1,2}\alpha_{1,5}\alpha_{2,11})(\alpha_{2,2}\alpha_{2,5}\alpha_{1,11})$$

$$(\alpha_{1,7}\alpha_{1,9}\alpha_{2,12})(\alpha_{2,7}\alpha_{2,9}\alpha_{1,12})]$$

with orbits $\{\alpha_{1,1},\alpha_{1,3},\alpha_{1,11},\alpha_{2,4},\alpha_{2,2},\alpha_{2,5}\}$, $\{\alpha_{2,1},\alpha_{2,3},\alpha_{2,11},\alpha_{1,4},\alpha_{1,2},\alpha_{1,5}\}$, $\{\alpha_{1,6},\alpha_{2,7},\alpha_{2,9},\alpha_{1,12},\alpha_{2,8}\}$, $\{\alpha_{2,6},\alpha_{1,7},\alpha_{1,9},\alpha_{2,12},\alpha_{1,8}\}$;

$$H_{55,2}=[(\alpha_{1,1}\alpha_{1,2}\alpha_{2,4}\alpha_{2,7}\alpha_{1,5})(\alpha_{2,1}\alpha_{2,2}\alpha_{1,4}\alpha_{1,7}\alpha_{2,5})$$

$$(\alpha_{1,3}\alpha_{2,9}\alpha_{1,12}\alpha_{2,11}\alpha_{2,10})(\alpha_{2,3}\alpha_{1,9}\alpha_{2,12}\alpha_{1,11}\alpha_{1,10}),$$

$$(\alpha_{1,1}\alpha_{1,3}\alpha_{2,4})(\alpha_{2,1}\alpha_{2,3}\alpha_{1,4})(\alpha_{1,2}\alpha_{1,5}\alpha_{2,11})(\alpha_{2,2}\alpha_{2,5}\alpha_{1,11})$$

$$(\alpha_{1,7}\alpha_{1,9}\alpha_{2,12})(\alpha_{2,7}\alpha_{2,9}\alpha_{1,12})]$$

with orbits $\{\alpha_{1,1},\alpha_{1,2},\alpha_{1,3},\alpha_{2,4},\alpha_{1,5},\alpha_{2,9},\alpha_{2,7},\alpha_{2,11},\alpha_{1,12},\alpha_{2,10}\}$, $\{\alpha_{2,1},\alpha_{2,2},\alpha_{2,3},\alpha_{1,4},\alpha_{2,5},\alpha_{1,9},\alpha_{1,7},\alpha_{1,11},\alpha_{2,12},\alpha_{1,10}\}$.

n=54, $H\cong T_{48}$ ($|H|=48$, $i=29$): $\operatorname{rk} N_H=19$ and $(N_H)^*/N_H\cong\mathbb{Z}/24\mathbb{Z}\times\mathbb{Z}/8\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$.

$$H_{54,1}=[(\alpha_{1,2}\alpha_{1,11})(\alpha_{2,2}\alpha_{2,11})(\alpha_{1,3}\alpha_{1,6})(\alpha_{2,3}\alpha_{2,6})(\alpha_{1,4}\alpha_{1,12})$$

$$(\alpha_{2,4}\alpha_{2,12})(\alpha_{1,9}\alpha_{1,10})(\alpha_{2,9}\alpha_{2,10}),$$

$$(\alpha_{1,1}\alpha_{2,11}\alpha_{2,1}\alpha_{1,11})(\alpha_{1,2}\alpha_{2,2})(\alpha_{1,3}\alpha_{1,6}\alpha_{2,10}\alpha_{1,8}\alpha_{2,4}\alpha_{2,12}\alpha_{2,9}\alpha_{2,7})$$

$$(\alpha_{2,3}\alpha_{2,6}\alpha_{1,10}\alpha_{2,8}\alpha_{1,4}\alpha_{1,12}\alpha_{1,9}\alpha_{1,7})]$$

with orbits $\{\alpha_{1,1},\alpha_{2,11},\alpha_{2,2},\alpha_{2,1},\alpha_{1,2},\alpha_{1,11}\}$, $\{\alpha_{1,3},\alpha_{1,6},\alpha_{2,10},\alpha_{2,9},\alpha_{1,8},\alpha_{2,7},\alpha_{2,4},\alpha_{2,12}\}$, $\{\alpha_{2,3},\alpha_{2,6},\alpha_{1,10},\alpha_{1,9},\alpha_{2,8},\alpha_{1,7},\alpha_{1,4},\alpha_{1,12}\}$.

n=48, $H\cong\mathfrak{S}_{3,3}$ ($|H|=36$, $i=10$): $\operatorname{rk} N_H=18$, $(N_H)^*/N_H\cong\mathbb{Z}/18\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}\times(\mathbb{Z}/3\mathbb{Z})^2$ and $\det(K((q_{N_H})_3))\equiv -2^2\cdot 3^5 \mod (\mathbb{Z}_3^*)^2$.

$$H_{48,1}=[(\alpha_{1,2}\alpha_{1,7})(\alpha_{2,2}\alpha_{2,7})(\alpha_{1,3}\alpha_{2,4})(\alpha_{2,3}\alpha_{1,4})$$

$$(\alpha_{1,5}\alpha_{2,12})(\alpha_{2,5}\alpha_{1,12})(\alpha_{1,9}\alpha_{2,11})(\alpha_{2,9}\alpha_{1,11}),$$

$$(\alpha_{1,1}\alpha_{1,2}\alpha_{1,3}\alpha_{1,5}\alpha_{2,4}\alpha_{2,11})(\alpha_{2,1}\alpha_{2,2}\alpha_{2,3}\alpha_{2,5}\alpha_{1,4}\alpha_{1,11})(\alpha_{1,7}\alpha_{2,12}\alpha_{1,9})$$

$$(\alpha_{2,7}\alpha_{1,12}\alpha_{2,9})(\alpha_{1,8}\alpha_{2,10})(\alpha_{2,8}\alpha_{1,10})]$$

with orbits $\{\alpha_{1,1},\alpha_{1,2},\alpha_{1,7},\alpha_{1,3},\alpha_{2,12},\alpha_{2,4},\alpha_{1,5},\alpha_{1,9},\alpha_{2,11}\}$, $\{\alpha_{2,1},\alpha_{2,2},\alpha_{2,7},\alpha_{2,3},\alpha_{1,12},\alpha_{1,4},\alpha_{2,5},\alpha_{2,9},\alpha_{1,11}\}$, $\{\alpha_{1,8},\alpha_{2,10}\}$, $\{\alpha_{2,8},\alpha_{1,10}\}$;

$$H_{48,2}=[(\alpha_{1,3}\alpha_{2,4})(\alpha_{2,3}\alpha_{1,4})(\alpha_{1,5}\alpha_{2,11})(\alpha_{2,5}\alpha_{1,11})$$

$$(\alpha_{1,6}\alpha_{1,10})(\alpha_{2,6}\alpha_{2,10})(\alpha_{1,9}\alpha_{2,12})(\alpha_{2,9}\alpha_{1,12}),$$

$$(\alpha_{1,1}\alpha_{1,2}\alpha_{1,3}\alpha_{1,5}\alpha_{2,4}\alpha_{2,11})(\alpha_{2,1}\alpha_{2,2}\alpha_{2,3}\alpha_{2,5}\alpha_{1,4}\alpha_{1,11})(\alpha_{1,7}\alpha_{2,12}\alpha_{1,9})$$

$$(\alpha_{2,7}\alpha_{1,12}\alpha_{2,9})(\alpha_{1,8}\alpha_{2,10})(\alpha_{2,8}\alpha_{1,10})]$$

with orbits $\{\alpha_{1,1},\alpha_{1,2},\alpha_{1,3},\alpha_{2,4},\alpha_{1,5},\alpha_{2,11}\}$, $\{\alpha_{2,1},\alpha_{2,2},\alpha_{2,3},\alpha_{1,4},\alpha_{2,5},\alpha_{1,11}\}$, $\{\alpha_{1,6},\alpha_{1,10},\alpha_{2,8}\}$, $\{\alpha_{2,6},\alpha_{2,10},\alpha_{1,8}\}$, $\{\alpha_{1,7},\alpha_{2,12},\alpha_{1,9}\}$, $\{\alpha_{2,7},\alpha_{1,12},\alpha_{2,9}\}$.

n=46, $H\cong 3^2C_4$ ($|H|=36$, $i=9$): $\operatorname{rk} N_H=18$ and $(N_H)^*/N_H\cong \mathbb{Z}/18\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$.

$$H_{46,1}=[(\alpha_{1,2}\alpha_{1,3}\alpha_{1,7}\alpha_{2,4})(\alpha_{2,2}\alpha_{2,3}\alpha_{2,7}\alpha_{1,4})(\alpha_{1,5}\alpha_{1,9}\alpha_{2,12}\alpha_{2,11})$$
$$(\alpha_{2,5}\alpha_{2,9}\alpha_{1,12}\alpha_{1,11})(\alpha_{1,6}\alpha_{2,6})(\alpha_{1,10}\alpha_{2,10}),$$
$$(\alpha_{1,1}\alpha_{1,3}\alpha_{2,4})(\alpha_{2,1}\alpha_{2,3}\alpha_{1,4})(\alpha_{1,2}\alpha_{1,5}\alpha_{2,11})(\alpha_{2,2}\alpha_{2,5}\alpha_{1,11})$$
$$(\alpha_{1,7}\alpha_{1,9}\alpha_{2,12})(\alpha_{2,7}\alpha_{2,9}\alpha_{1,12})]$$

with orbits $\{\alpha_{1,1},\alpha_{1,3},\alpha_{1,7},\alpha_{2,4},\alpha_{1,9},\alpha_{1,2},\alpha_{2,12},\alpha_{1,5},\alpha_{2,11}\}$, $\{\alpha_{2,1},\alpha_{2,3},\alpha_{2,7},\alpha_{1,4},\alpha_{2,9},\alpha_{2,2},\alpha_{1,12},\alpha_{2,5},\alpha_{1,11}\}$, $\{\alpha_{1,6},\alpha_{2,6}\}$, $\{\alpha_{1,10},\alpha_{2,10}\}$;

$$H_{46,2}=[(\alpha_{1,1}\alpha_{2,1})(\alpha_{1,2}\alpha_{2,3}\alpha_{1,7}\alpha_{1,4})(\alpha_{2,2}\alpha_{1,3}\alpha_{2,7}\alpha_{2,4})$$
$$(\alpha_{1,5}\alpha_{2,9}\alpha_{2,12}\alpha_{1,11})(\alpha_{2,5}\alpha_{1,9}\alpha_{1,12}\alpha_{1,11})(\alpha_{1,8}\alpha_{2,8}),$$
$$(\alpha_{1,1}\alpha_{1,3}\alpha_{2,4})(\alpha_{2,1}\alpha_{2,3}\alpha_{1,4})(\alpha_{1,2}\alpha_{1,5}\alpha_{2,11})(\alpha_{2,2}\alpha_{2,5}\alpha_{1,11})$$
$$(\alpha_{1,7}\alpha_{1,9}\alpha_{2,12})(\alpha_{2,7}\alpha_{2,9}\alpha_{1,12})]$$

with orbits $\{\alpha_{1,1},\alpha_{2,1},\alpha_{1,3},\alpha_{2,3},\alpha_{2,7},\alpha_{2,4},\alpha_{1,7},\alpha_{1,4},\alpha_{2,9},\alpha_{2,2},\alpha_{1,9},\alpha_{1,2},\alpha_{2,12},\alpha_{1,12},\alpha_{2,5},\alpha_{1,5},\alpha_{1,11},\alpha_{2,11}\}$, $\{\alpha_{1,8},\alpha_{2,8}\}$.

n=38, $H\cong T_{24}$ ($|H|=24$, $i=3$):

$$H_{38,1}=[(\alpha_{1,1}\alpha_{1,2}\alpha_{2,11})(\alpha_{2,1}\alpha_{2,2}\alpha_{1,11})(\alpha_{1,3}\alpha_{2,7}\alpha_{2,10})$$
$$(\alpha_{2,3}\alpha_{1,7}\alpha_{1,10})(\alpha_{1,4}\alpha_{2,8}\alpha_{1,9})(\alpha_{2,4}\alpha_{1,8}\alpha_{2,9}),$$
$$(\alpha_{1,2}\alpha_{2,2})(\alpha_{1,3}\alpha_{1,6}\alpha_{2,4}\alpha_{2,12})(\alpha_{2,3}\alpha_{2,6}\alpha_{1,4}\alpha_{1,12})(\alpha_{1,7}\alpha_{1,9}\alpha_{2,8}\alpha_{1,10})$$
$$(\alpha_{2,7}\alpha_{2,9}\alpha_{1,8}\alpha_{2,10})(\alpha_{1,11}\alpha_{2,11})]$$

with $Clos(H_{38,1})=H_{54,1}$.

n=34, $H\cong\mathfrak{S}_4$ ($|H|=24$, $i=12$): $\operatorname{rk} N_H=17$ and $(N_H)^*/N_H\cong(\mathbb{Z}/12\mathbb{Z})^2\times\mathbb{Z}/4\mathbb{Z}$.

$$H_{34,1}=[(\alpha_{1,2}\alpha_{2,5})(\alpha_{2,2}\alpha_{1,5})(\alpha_{1,4}\alpha_{2,8})(\alpha_{2,4}\alpha_{1,8})$$
$$(\alpha_{1,7}\alpha_{2,9})(\alpha_{2,7}\alpha_{1,9})(\alpha_{1,11}\alpha_{1,12})(\alpha_{2,11}\alpha_{2,12}),$$
$$(\alpha_{1,1}\alpha_{1,3}\alpha_{2,4})(\alpha_{2,1}\alpha_{2,3}\alpha_{1,4})(\alpha_{1,2}\alpha_{1,5}\alpha_{2,11})(\alpha_{2,2}\alpha_{2,5}\alpha_{1,11})$$
$$(\alpha_{1,7}\alpha_{1,9}\alpha_{2,12})(\alpha_{2,7}\alpha_{2,9}\alpha_{1,12})]$$

with orbits $\{\alpha_{1,1},\alpha_{1,3},\alpha_{2,4},\alpha_{1,8}\}$, $\{\alpha_{2,1},\alpha_{2,3},\alpha_{1,4},\alpha_{2,8}\}$, $\{\alpha_{1,2},\alpha_{2,5},\alpha_{1,5},\alpha_{1,11},\alpha_{2,2},\alpha_{2,11},\alpha_{1,12},\alpha_{2,12},\alpha_{2,7},\alpha_{1,7},\alpha_{1,9},\alpha_{2,9}\}$.

n=32, $H\cong Hol(C_5)$ ($|H|=20$, $i=3$): $\operatorname{rk} N_H=18$ and $(N_H)^*/N_H \cong (\mathbb{Z}/10\mathbb{Z})^2\times\mathbb{Z}/5\mathbb{Z}$.

$$H_{32,1}=[(\alpha_{1,4}\alpha_{1,9})(\alpha_{2,4}\alpha_{2,9})(\alpha_{1,5}\alpha_{2,10})(\alpha_{2,5}\alpha_{1,10})$$
$$(\alpha_{1,7}\alpha_{2,12})(\alpha_{2,7}\alpha_{1,12})(\alpha_{1,8}\alpha_{2,11})(\alpha_{2,8}\alpha_{1,11}),$$
$$(\alpha_{1,1}\alpha_{2,1})(\alpha_{1,3}\alpha_{2,10}\alpha_{2,4}\alpha_{2,9})(\alpha_{2,3}\alpha_{1,10}\alpha_{1,4}\alpha_{1,9})(\alpha_{1,6}\alpha_{1,8}\alpha_{2,12}\alpha_{2,7})$$
$$(\alpha_{2,6}\alpha_{2,8}\alpha_{1,12}\alpha_{1,7})(\alpha_{1,11}\alpha_{2,11})]$$

with orbits $\{\alpha_{1,1},\alpha_{2,1}\}$, $\{\alpha_{1,3},\alpha_{2,10},\alpha_{1,5},\alpha_{2,4},\alpha_{2,9}\}$, $\{\alpha_{2,3},\alpha_{1,10},\alpha_{2,5},\alpha_{1,4},\alpha_{1,9}\}$, $\{\alpha_{1,6},\alpha_{1,8},\alpha_{2,11},\alpha_{2,12},\alpha_{1,11},\alpha_{1,7},\alpha_{2,7},\alpha_{2,8},\alpha_{2,6},\alpha_{1,12}\}$.

n=31, $H\cong C_3\times D_6$, ($|H|=18$, $i=3$):

$$H_{31,1}=[(\alpha_{1,1}\alpha_{1,3}\alpha_{2,4})(\alpha_{2,1}\alpha_{2,3}\alpha_{1,4})(\alpha_{1,2}\alpha_{1,5}\alpha_{2,11})$$
$$(\alpha_{2,2}\alpha_{2,5}\alpha_{1,11})(\alpha_{1,7}\alpha_{1,9}\alpha_{2,12})(\alpha_{2,7}\alpha_{2,9}\alpha_{1,12}),$$
$$(\alpha_{1,1}\alpha_{1,2}\alpha_{2,12}\alpha_{2,4}\alpha_{1,5}\alpha_{1,9})(\alpha_{2,1}\alpha_{2,2}\alpha_{1,12}\alpha_{1,4}\alpha_{2,5}\alpha_{2,9})(\alpha_{1,3}\alpha_{2,11}\alpha_{1,7})$$
$$(\alpha_{2,3}\alpha_{1,11}\alpha_{2,7})(\alpha_{1,8}\alpha_{2,10})(\alpha_{2,8}\alpha_{1,10})]$$

with $Clos(H_{31,1})=H_{48,1}$ above;

$$H_{31,2}=[(\alpha_{1,1}\alpha_{1,3}\alpha_{2,4})(\alpha_{2,1}\alpha_{2,3}\alpha_{1,4})(\alpha_{1,2}\alpha_{1,5}\alpha_{2,11})$$
$$(\alpha_{2,2}\alpha_{2,5}\alpha_{1,11})(\alpha_{1,7}\alpha_{1,9}\alpha_{2,12})(\alpha_{2,7}\alpha_{2,9}\alpha_{1,12}),$$
$$(\alpha_{1,1}\alpha_{1,2}\alpha_{1,3}\alpha_{2,11}\alpha_{2,4}\alpha_{1,5})(\alpha_{2,1}\alpha_{2,2}\alpha_{2,3}\alpha_{1,11}\alpha_{1,4}\alpha_{2,5})(\alpha_{1,6}\alpha_{2,8}\alpha_{1,10})$$
$$(\alpha_{2,6}\alpha_{1,8}\alpha_{2,10})(\alpha_{1,7}\alpha_{2,12})(\alpha_{2,7}\alpha_{1,12})]$$

with $Clos(H_{31,2})=H_{48,2}$ above.

n=30, $H\cong\mathfrak{A}_{3,3}$ ($|H|=18$, $i=4$): $\operatorname{rk} N_H=16$ and $(N_H)^*/N_H\cong \mathbb{Z}/9\mathbb{Z}\times(\mathbb{Z}/3\mathbb{Z})^4$.

$$H_{30,1}=[(\alpha_{1,2}\alpha_{1,7})(\alpha_{2,2}\alpha_{2,7})(\alpha_{1,3}\alpha_{2,4})(\alpha_{2,3}\alpha_{1,4})$$
$$(\alpha_{1,5}\alpha_{2,12})(\alpha_{2,5}\alpha_{1,12})(\alpha_{1,9}\alpha_{2,11})(\alpha_{2,9}\alpha_{1,11}),$$
$$(\alpha_{1,1}\alpha_{1,2}\alpha_{1,7})(\alpha_{2,1}\alpha_{2,2}\alpha_{2,7})(\alpha_{1,3}\alpha_{1,5}\alpha_{1,9})(\alpha_{2,3}\alpha_{2,5}\alpha_{2,9})$$
$$(\alpha_{1,4}\alpha_{1,11}\alpha_{1,12})(\alpha_{2,4}\alpha_{2,11}\alpha_{2,12}),$$
$$(\alpha_{1,1}\alpha_{1,3}\alpha_{2,4})(\alpha_{2,1}\alpha_{2,3}\alpha_{1,4})(\alpha_{1,2}\alpha_{1,5}\alpha_{2,11})(\alpha_{2,2}\alpha_{2,5}\alpha_{1,11})$$
$$(\alpha_{1,7}\alpha_{1,9}\alpha_{2,12})(\alpha_{2,7}\alpha_{2,9}\alpha_{1,12})]$$

with orbits $\{\alpha_{1,1}, \alpha_{1,2}, \alpha_{1,3}, \alpha_{1,7}, \alpha_{1,5}, \alpha_{2,4}, \alpha_{1,9}, \alpha_{2,12}, \alpha_{2,11}\}$, $\{\alpha_{2,1}, \alpha_{2,2}, \alpha_{2,3}, \alpha_{2,7}, \alpha_{2,5}, \alpha_{1,4}, \alpha_{2,9}, \alpha_{1,12}, \alpha_{1,11}\}$;

$$H_{30,2} = [(\alpha_{1,3}\alpha_{2,4})(\alpha_{2,3}\alpha_{1,4})(\alpha_{1,5}\alpha_{2,11})(\alpha_{2,5}\alpha_{1,11})$$
$$(\alpha_{1,6}\alpha_{1,10})(\alpha_{2,6}\alpha_{2,10})(\alpha_{1,9}\alpha_{2,12})(\alpha_{2,9}\alpha_{1,12}),$$
$$(\alpha_{1,2}\alpha_{1,5}\alpha_{2,11})(\alpha_{2,2}\alpha_{2,5}\alpha_{1,11})(\alpha_{1,6}\alpha_{1,10}\alpha_{2,8})(\alpha_{2,6}\alpha_{2,10}\alpha_{1,8})$$
$$(\alpha_{1,7}\alpha_{2,12}\alpha_{1,9})(\alpha_{2,7}\alpha_{1,12}\alpha_{2,9}),$$
$$(\alpha_{1,1}\alpha_{1,3}\alpha_{2,4})(\alpha_{2,1}\alpha_{2,3}\alpha_{1,4})(\alpha_{1,2}\alpha_{1,5}\alpha_{2,11})(\alpha_{2,2}\alpha_{2,5}\alpha_{1,11})$$
$$(\alpha_{1,7}\alpha_{1,9}\alpha_{2,12})(\alpha_{2,7}\alpha_{2,9}\alpha_{1,12})]$$

with orbits $\{\alpha_{1,1}, \alpha_{1,3}, \alpha_{2,4}\}$, $\{\alpha_{2,1}, \alpha_{2,3}, \alpha_{1,4}\}$, $\{\alpha_{1,2}, \alpha_{1,5}, \alpha_{2,11}\}$, $\{\alpha_{2,2}, \alpha_{2,5}, \alpha_{1,11}\}$, $\{\alpha_{1,6}, \alpha_{1,10}, \alpha_{2,8}\}$, $\{\alpha_{2,6}, \alpha_{2,10}, \alpha_{1,8}\}$, $\{\alpha_{1,7}, \alpha_{2,12}, \alpha_{1,9}\}$, $\{\alpha_{2,7}, \alpha_{1,12}, \alpha_{2,9}\}$.

n=26, $H \cong SD_{16}$ ($|H| = 16$, $i = 8$): $\operatorname{rk} N_H = 18$ and $(N_H)^*/N_H \cong (\mathbb{Z}/8\mathbb{Z})^2 \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

$$H_{26,1} = [(\alpha_{1,2}\alpha_{2,2})(\alpha_{1,4}\alpha_{2,4})(\alpha_{1,5}\alpha_{2,6}\alpha_{2,11}\alpha_{2,9})(\alpha_{2,5}\alpha_{1,6}\alpha_{1,11}\alpha_{1,9})$$
$$(\alpha_{1,7}\alpha_{2,10}\alpha_{2,8}\alpha_{1,12})(\alpha_{2,7}\alpha_{1,10}\alpha_{1,8}\alpha_{2,12}),$$
$$(\alpha_{1,3}\alpha_{2,4})(\alpha_{2,3}\alpha_{1,4})(\alpha_{1,6}\alpha_{2,12})(\alpha_{2,6}\alpha_{1,12})(\alpha_{1,7}\alpha_{2,8})(\alpha_{2,7}\alpha_{1,8})$$
$$(\alpha_{1,9}\alpha_{1,10})(\alpha_{2,9}\alpha_{2,10})]$$

with orbits $\{\alpha_{1,2}, \alpha_{2,2}\}$, $\{\alpha_{1,3}, \alpha_{2,4}, \alpha_{1,4}, \alpha_{2,3}\}$, $\{\alpha_{1,5}, \alpha_{2,6}, \alpha_{2,11}, \alpha_{1,12}, \alpha_{2,9}, \alpha_{1,7}, \alpha_{2,10}, \alpha_{2,8}\}$, $\{\alpha_{2,5}, \alpha_{1,6}, \alpha_{1,11}, \alpha_{2,12}, \alpha_{1,9}, \alpha_{2,7}, \alpha_{1,10}, \alpha_{1,8}\}$.

n=18, $H \cong D_{12}$ ($|H| = 12$, $i = 4$): $\operatorname{rk} N_H = 16$ and $(N_H)^*/N_H \cong (\mathbb{Z}/6\mathbb{Z})^4$.

$$H_{18,1} = [(\alpha_{1,2}\alpha_{1,3})(\alpha_{2,2}\alpha_{2,3})(\alpha_{1,5}\alpha_{1,12})(\alpha_{2,5}\alpha_{2,12})$$
$$(\alpha_{1,6}\alpha_{1,9})(\alpha_{2,6}\alpha_{2,9})(\alpha_{1,10}\alpha_{1,11})(\alpha_{2,10}\alpha_{2,11}),$$
$$(\alpha_{1,3}\alpha_{2,4})(\alpha_{2,3}\alpha_{1,4})(\alpha_{1,6}\alpha_{2,12})(\alpha_{2,6}\alpha_{1,12})(\alpha_{1,7}\alpha_{2,8})(\alpha_{2,7}\alpha_{1,8})$$
$$(\alpha_{1,9}\alpha_{1,10})(\alpha_{2,9}\alpha_{2,10})]$$

with orbits $\{\alpha_{1,2}, \alpha_{1,3}, \alpha_{2,4}\}$, $\{\alpha_{2,2}, \alpha_{2,3}, \alpha_{1,4}\}$, $\{\alpha_{1,5}, \alpha_{1,12}, \alpha_{2,6}, \alpha_{2,9}, \alpha_{2,10}, \alpha_{2,11}\}$, $\{\alpha_{2,5}, \alpha_{2,12}, \alpha_{1,6}, \alpha_{1,9}, \alpha_{1,10}, \alpha_{1,11}\}$, $\{\alpha_{1,7}, \alpha_{2,8}\}$, $\{\alpha_{2,7}, \alpha_{1,8}\}$.

n=17, $H = \mathfrak{A}_4$ ($|H| = 12$, $i = 3$): $\operatorname{rk} N_H = 16$ and $(N_H)^*/N_H \cong (\mathbb{Z}/12\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2$.

$$H_{17,1} = [(\alpha_{1,4}\alpha_{2,7}\alpha_{1,8})(\alpha_{2,4}\alpha_{1,7}\alpha_{2,8})(\alpha_{1,5}\alpha_{1,6}\alpha_{2,10})(\alpha_{2,5}\alpha_{2,6}\alpha_{1,10})$$

$$(\alpha_{1,9}\alpha_{2,11}\alpha_{1,12})(\alpha_{2,9}\alpha_{1,11}\alpha_{2,12}),$$

$$(\alpha_{1,3}\alpha_{2,4})(\alpha_{2,3}\alpha_{1,4})(\alpha_{1,6}\alpha_{2,12})(\alpha_{2,6}\alpha_{1,12})(\alpha_{1,7}\alpha_{2,8})(\alpha_{2,7}\alpha_{1,8})$$

$$(\alpha_{1,9}\alpha_{1,10})(\alpha_{2,9}\alpha_{2,10})]$$

with orbits $\{\alpha_{1,3},\alpha_{2,4},\alpha_{1,7},\alpha_{2,8}\}$, $\{\alpha_{2,3},\alpha_{1,4},\alpha_{2,7},\alpha_{1,8}\}$, $\{\alpha_{1,5},\alpha_{1,6},\alpha_{2,10},\alpha_{2,12},\alpha_{2,9},\alpha_{1,11}\}$, $\{\alpha_{2,5},\alpha_{2,6},\alpha_{1,10},\alpha_{1,12},\alpha_{1,9},\alpha_{2,11}\}$.

n=16, $H\cong D_{10}$ ($|H|=10$, $i=1$): $\operatorname{rk} N_H=16$ and $(N_H)^*/N_H\cong(\mathbb{Z}/5\mathbb{Z})^4$.

$$H_{16,1}=[(\alpha_{1,4}\alpha_{1,6})(\alpha_{2,4}\alpha_{2,6})(\alpha_{1,5}\alpha_{2,7})(\alpha_{2,5}\alpha_{1,7})$$

$$(\alpha_{1,8}\alpha_{2,10})(\alpha_{2,8}\alpha_{1,10})(\alpha_{1,11}\alpha_{2,12})(\alpha_{2,11}\alpha_{1,12}),$$

$$(\alpha_{1,3}\alpha_{2,4})(\alpha_{2,3}\alpha_{1,4})(\alpha_{1,6}\alpha_{2,12})(\alpha_{2,6}\alpha_{1,12})(\alpha_{1,7}\alpha_{2,8})(\alpha_{2,7}\alpha_{1,8})$$

$$(\alpha_{1,9}\alpha_{1,10})(\alpha_{2,9}\alpha_{2,10})]$$

with orbits $\{\alpha_{1,3},\alpha_{2,4},\alpha_{2,6},\alpha_{1,12},\alpha_{2,11}\}$, $\{\alpha_{2,3},\alpha_{1,4},\alpha_{1,6},\alpha_{2,12},\alpha_{1,11}\}$, $\{\alpha_{1,5},\alpha_{2,7},\alpha_{1,8},\alpha_{2,10},\alpha_{2,9}\}$, $\{\alpha_{2,5},\alpha_{1,7},\alpha_{2,8},\alpha_{1,10},\alpha_{1,9}\}$.

n=15, $H\cong C_3^2$ ($|H|=9$, $i=2$):

$$H_{15,1}=[(\alpha_{1,1}\alpha_{1,2}\alpha_{1,7})(\alpha_{2,1}\alpha_{2,2}\alpha_{2,7})(\alpha_{1,3}\alpha_{1,5}\alpha_{1,9})(\alpha_{2,3}\alpha_{2,5}\alpha_{2,9})$$

$$(\alpha_{1,4}\alpha_{1,11}\alpha_{1,12})(\alpha_{2,4}\alpha_{2,11}\alpha_{2,12}),$$

$$(\alpha_{1,1}\alpha_{1,3}\alpha_{2,4})(\alpha_{2,1}\alpha_{2,3}\alpha_{1,4})(\alpha_{1,2}\alpha_{1,5}\alpha_{2,11})(\alpha_{2,2}\alpha_{2,5}\alpha_{1,11})$$

$$(\alpha_{1,7}\alpha_{1,9}\alpha_{2,12})(\alpha_{2,7}\alpha_{2,9}\alpha_{1,12})]$$

with $Clos(H_{15,1})=H_{30,1}$ above;

$$H_{15,2}=[(\alpha_{1,2}\alpha_{1,5}\alpha_{2,11})(\alpha_{2,2}\alpha_{2,5}\alpha_{1,11})(\alpha_{1,6}\alpha_{1,10}\alpha_{2,8})(\alpha_{2,6}\alpha_{2,10}\alpha_{1,8})$$

$$(\alpha_{1,7}\alpha_{2,12}\alpha_{1,9})(\alpha_{2,7}\alpha_{1,12}\alpha_{2,9}),$$

$$(\alpha_{1,1}\alpha_{1,3}\alpha_{2,4})(\alpha_{2,1}\alpha_{2,3}\alpha_{1,4})(\alpha_{1,2}\alpha_{1,5}\alpha_{2,11})(\alpha_{2,2}\alpha_{2,5}\alpha_{1,11})$$

$$(\alpha_{1,7}\alpha_{1,9}\alpha_{2,12})(\alpha_{2,7}\alpha_{2,9}\alpha_{1,12})]$$

with $Clos(H_{15,2})=H_{30,2}$ above.

n=14, $H\cong C_8$ ($|H|=8$, $i=1$):

$$H_{14,1}=$$

$$[(\alpha_{1,2}\alpha_{2,2})(\alpha_{1,3}\alpha_{1,4}\alpha_{2,3}\alpha_{2,4})(\alpha_{1,5}\alpha_{2,6}\alpha_{1,7}\alpha_{1,12}\alpha_{2,11}\alpha_{2,9}\alpha_{2,8}\alpha_{2,10})$$

$$(\alpha_{2,5}\alpha_{1,6}\alpha_{2,7}\alpha_{2,12}\alpha_{1,11}\alpha_{1,9}\alpha_{1,8}\alpha_{1,10})]$$

with $Clos(H_{14,1})=H_{26,1}$ above.

n=12, $H \cong Q_8$ ($|H| = 8$, $i = 4$): $\operatorname{rk} N_H = 17$, $(N_H)^*/N_H \cong (\mathbb{Z}/8\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^3$, and $K((q_{N_H})_2) \cong q_\theta^{(2)}(2) \oplus q'$.

$$H_{12,1} = [(\alpha_{1,1}\alpha_{1,4}\alpha_{1,10}\alpha_{1,7})(\alpha_{2,1}\alpha_{2,4}\alpha_{2,10}\alpha_{2,7})(\alpha_{1,2}\alpha_{2,6}\alpha_{1,9}\alpha_{2,12})$$
$$(\alpha_{2,2}\alpha_{1,6}\alpha_{2,9}\alpha_{1,12})(\alpha_{1,8}\alpha_{2,8})(\alpha_{1,11}\alpha_{2,11}),$$
$$(\alpha_{1,1}\alpha_{2,2}\alpha_{1,10}\alpha_{2,9})(\alpha_{2,1}\alpha_{1,2}\alpha_{2,10}\alpha_{1,9})(\alpha_{1,4}\alpha_{1,12}\alpha_{1,7}\alpha_{1,6})$$
$$(\alpha_{2,4}\alpha_{2,12}\alpha_{2,7}\alpha_{2,6})(\alpha_{1,5}\alpha_{2,5})(\alpha_{1,11}\alpha_{2,11})]$$

with orbits $\{\alpha_{1,1}, \alpha_{1,4}, \alpha_{2,2}, \alpha_{1,10}, \alpha_{1,12}, \alpha_{1,6}, \alpha_{1,7}, \alpha_{2,9}\}$, $\{\alpha_{2,1}, \alpha_{2,4}, \alpha_{1,2}, \alpha_{2,10}, \alpha_{2,12}, \alpha_{2,6}, \alpha_{2,7}, \alpha_{1,9}\}$, $\{\alpha_{1,5}, \alpha_{2,5}\}$, $\{\alpha_{1,8}, \alpha_{2,8}\}$, $\{\alpha_{1,11}, \alpha_{2,11}\}$.

n=10, $H \cong D_8$ ($|H| = 8$, $i = 3$): $\operatorname{rk} N_H = 15$ and $(N_H)^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^5$.

$$H_{10,1} = [(\alpha_{1,4}\alpha_{2,7})(\alpha_{2,4}\alpha_{1,7})(\alpha_{1,5}\alpha_{1,12})(\alpha_{2,5}\alpha_{2,12})(\alpha_{1,6}\alpha_{2,11})$$
$$(\alpha_{2,6}\alpha_{1,11})(\alpha_{1,9}\alpha_{2,10})(\alpha_{2,9}\alpha_{1,10}),$$
$$(\alpha_{1,3}\alpha_{2,4})(\alpha_{2,3}\alpha_{1,4})(\alpha_{1,6}\alpha_{2,12})(\alpha_{2,6}\alpha_{1,12})(\alpha_{1,7}\alpha_{2,8})(\alpha_{2,7}\alpha_{1,8})$$
$$(\alpha_{1,9}\alpha_{1,10})(\alpha_{2,9}\alpha_{2,10})]$$

with orbits $\{\alpha_{1,3}, \alpha_{2,4}, \alpha_{1,7}, \alpha_{2,8}\}$, $\{\alpha_{2,3}, \alpha_{1,4}, \alpha_{2,7}, \alpha_{1,8}\}$, $\{\alpha_{1,5}, \alpha_{1,12}, \alpha_{2,6}, \alpha_{1,11}\}$, $\{\alpha_{2,5}, \alpha_{2,12}, \alpha_{1,6}, \alpha_{2,11}\}$, $\{\alpha_{1,9}, \alpha_{2,10}, \alpha_{1,10}, \alpha_{2,9}\}$.

n=7, $H \cong C_6$, ($|H| = 6$, $i = 2$):

$$H_{7,1} = [(\alpha_{1,2}\alpha_{1,3}\alpha_{2,4})(\alpha_{2,2}\alpha_{2,3}\alpha_{1,4})(\alpha_{1,5}\alpha_{1,12}\alpha_{2,9}\alpha_{2,11}\alpha_{2,10}\alpha_{2,6})$$
$$(\alpha_{2,5}\alpha_{2,12}\alpha_{1,9}\alpha_{1,11}\alpha_{1,10}\alpha_{1,6})(\alpha_{1,7}\alpha_{2,8})(\alpha_{2,7}\alpha_{1,8})]$$

with $Clos(H_{7,1}) = H_{18,1}$ above.

n=6, $H \cong D_6$ ($|H| = 6$, $i = 1$): $\operatorname{rk} N_H = 14$, $(N_H)^*/N_H \cong (\mathbb{Z}/6\mathbb{Z})^2 \times (\mathbb{Z}/3\mathbb{Z})^3$.

$$H_{6,1} = [(\alpha_{1,3}\alpha_{2,4})(\alpha_{2,3}\alpha_{1,4})(\alpha_{1,5}\alpha_{2,11})(\alpha_{2,5}\alpha_{1,11})$$
$$(\alpha_{1,6}\alpha_{1,10})(\alpha_{2,6}\alpha_{2,10})(\alpha_{1,9}\alpha_{2,12})(\alpha_{2,9}\alpha_{1,12}),$$
$$(\alpha_{1,1}\alpha_{1,3}\alpha_{2,4})(\alpha_{2,1}\alpha_{2,3}\alpha_{1,4})(\alpha_{1,2}\alpha_{1,5}\alpha_{2,11})(\alpha_{2,2}\alpha_{2,5}\alpha_{1,11})$$
$$(\alpha_{1,7}\alpha_{1,9}\alpha_{2,12})(\alpha_{2,7}\alpha_{2,9}\alpha_{1,12})]$$

with orbits $\{\alpha_{1,1}, \alpha_{1,3}, \alpha_{2,4}\}$, $\{\alpha_{2,1}, \alpha_{2,3}, \alpha_{1,4}\}$, $\{\alpha_{1,2}, \alpha_{1,5}, \alpha_{2,11}\}$, $\{\alpha_{2,2}, \alpha_{2,5}, \alpha_{1,11}\}$, $\{\alpha_{1,6}, \alpha_{1,10}\}$, $\{\alpha_{2,6}, \alpha_{2,10}\}$, $\{\alpha_{1,7}, \alpha_{1,9}, \alpha_{2,12}\}$, $\{\alpha_{2,7}, \alpha_{2,9}, \alpha_{1,12}\}$;

$$H_{6,2} = [(\alpha_{1,2}\alpha_{1,7})(\alpha_{2,2}\alpha_{2,7})(\alpha_{1,3}\alpha_{2,4})(\alpha_{2,3}\alpha_{1,4})$$

$$(\alpha_{1,5}\alpha_{2,12})(\alpha_{2,5}\alpha_{1,12})(\alpha_{1,9}\alpha_{2,11})(\alpha_{2,9}\alpha_{1,11}),$$

$$(\alpha_{1,1}\alpha_{1,3}\alpha_{2,4})(\alpha_{2,1}\alpha_{2,3}\alpha_{1,4})(\alpha_{1,2}\alpha_{1,5}\alpha_{2,11})(\alpha_{2,2}\alpha_{2,5}\alpha_{1,11})$$

$$(\alpha_{1,7}\alpha_{1,9}\alpha_{2,12})(\alpha_{2,7}\alpha_{2,9}\alpha_{1,12})]$$

with orbits $\{\alpha_{1,1},\alpha_{1,3},\alpha_{2,4}\}$, $\{\alpha_{2,1},\alpha_{2,3},\alpha_{1,4}\}$, $\{\alpha_{1,2},\alpha_{1,7},\alpha_{1,5},\alpha_{1,9}, \alpha_{2,12},\alpha_{2,11}\}$, $\{\alpha_{2,2},\alpha_{2,7},\alpha_{2,5},\alpha_{2,9},\alpha_{1,12},\alpha_{1,11}\}$.

n=5, $H\cong C_5$ ($|H|=5$, $i=1$):

$$H_{5,1}=[(\alpha_{1,3}\alpha_{2,4}\alpha_{1,12}\alpha_{2,11}\alpha_{2,6})(\alpha_{2,3}\alpha_{1,4}\alpha_{2,12}\alpha_{1,11}\alpha_{1,6})$$

$$(\alpha_{1,5}\alpha_{1,8}\alpha_{2,9}\alpha_{2,10}\alpha_{2,7})(\alpha_{2,5}\alpha_{2,8}\alpha_{1,9}\alpha_{1,10}\alpha_{1,7})]$$

with $Clos(H_{5,1})=H_{16,1}$ above.

n=4, $H\cong C_4$ ($|H|=4$, $i=1$): $\operatorname{rk} N_H=14$ and $(N_H)^*/N_H\cong(\mathbb{Z}/4\mathbb{Z})^4\times(\mathbb{Z}/2\mathbb{Z})^2$.

$$H_{4,1}=[(\alpha_{1,1}\alpha_{2,1})(\alpha_{1,3}\alpha_{2,10}\alpha_{2,4}\alpha_{2,9})(\alpha_{2,3}\alpha_{1,10}\alpha_{1,4}\alpha_{1,9})$$

$$(\alpha_{1,6}\alpha_{1,8}\alpha_{2,12}\alpha_{2,7})(\alpha_{2,6}\alpha_{2,8}\alpha_{1,12}\alpha_{1,7})(\alpha_{1,11}\alpha_{2,11})]$$

with orbits $\{\alpha_{1,1},\alpha_{2,1}\}$, $\{\alpha_{1,3},\alpha_{2,10},\alpha_{2,4},\alpha_{2,9}\}$, $\{\alpha_{2,3},\alpha_{1,10},\alpha_{1,4},\alpha_{1,9}\}$, $\{\alpha_{1,6},\alpha_{1,8},\alpha_{2,12},\alpha_{2,7}\}$, $\{\alpha_{2,6},\alpha_{2,8},\alpha_{1,12},\alpha_{1,7}\}$, $\{\alpha_{1,11},\alpha_{2,11}\}$.

n=3, $H\cong C_2^2$ ($|H|=4$, $i=2$): $\operatorname{rk} N_H=12$ and $(N_H)^*/N_H\cong(\mathbb{Z}/4\mathbb{Z})^2\times(\mathbb{Z}/2\mathbb{Z})^6$.

$$H_{3,1}=[(\alpha_{1,5}\alpha_{2,11})(\alpha_{2,5}\alpha_{1,11})(\alpha_{1,6}\alpha_{1,9})(\alpha_{2,6}\alpha_{2,9})(\alpha_{1,7}\alpha_{2,8})(\alpha_{2,7}\alpha_{1,8})$$

$$(\alpha_{1,10}\alpha_{2,12})(\alpha_{2,10}\alpha_{1,12}),$$

$$(\alpha_{1,3}\alpha_{2,4})(\alpha_{2,3}\alpha_{1,4})(\alpha_{1,6}\alpha_{2,12})(\alpha_{2,6}\alpha_{1,12})(\alpha_{1,7}\alpha_{2,8})(\alpha_{2,7}\alpha_{1,8})$$

$$(\alpha_{1,9}\alpha_{1,10})(\alpha_{2,9}\alpha_{2,10})]$$

with orbits $\{\alpha_{1,3},\alpha_{2,4}\}$, $\{\alpha_{2,3},\alpha_{1,4}\}$, $\{\alpha_{1,5},\alpha_{2,11}\}$, $\{\alpha_{2,5},\alpha_{1,11}\}$, $\{\alpha_{1,6}, \alpha_{1,9}, \alpha_{2,12},\alpha_{1,10}\}$, $\{\alpha_{2,6},\alpha_{2,9},\alpha_{1,12},\alpha_{2,10}\}$, $\{\alpha_{1,7},\alpha_{2,8}\}$, $\{\alpha_{2,7},\alpha_{1,8}\}$.

n=2, $H\cong C_3$ ($|H|=3$, $i=1$): $\operatorname{rk} N_H=12$ and $(N_H)^*/N_H\cong(\mathbb{Z}/3\mathbb{Z})^6$.

$$H_{2,1}=[(\alpha_{1,1}\alpha_{1,3}\alpha_{2,4})(\alpha_{2,1}\alpha_{2,3}\alpha_{1,4})(\alpha_{1,2}\alpha_{1,5}\alpha_{2,11})(\alpha_{2,2}\alpha_{2,5}\alpha_{1,11})$$

$$(\alpha_{1,7}\alpha_{1,9}\alpha_{2,12})(\alpha_{2,7}\alpha_{2,9}\alpha_{1,12})]$$

with orbits $\{\alpha_{1,1},\alpha_{1,3},\alpha_{2,4}\}$, $\{\alpha_{2,1},\alpha_{2,3},\alpha_{1,4}\}$, $\{\alpha_{1,2},\alpha_{1,5},\alpha_{2,11}\}$, $\{\alpha_{2,2}, \alpha_{2,5},\alpha_{1,11}\}$, $\{\alpha_{1,7},\alpha_{1,9},\alpha_{2,12}\}$, $\{\alpha_{2,7},\alpha_{2,9},\alpha_{1,12}\}$.

n=1, $H \cong C_2$ ($|H| = 2$, $i = 1$): $\operatorname{rk} N_H = 8$ and $(N_H)^*/N_H \cong (\mathbb{Z}/2\mathbb{Z})^8$.

$$H_{1,1} = [(\alpha_{1,3}\alpha_{2,4})(\alpha_{2,3}\alpha_{1,4})(\alpha_{1,6}\alpha_{2,12})(\alpha_{2,6}\alpha_{1,12})(\alpha_{1,7}\alpha_{2,8})(\alpha_{2,7}\alpha_{1,8})$$
$$(\alpha_{1,9}\alpha_{1,10})(\alpha_{2,9}\alpha_{2,10})]$$

with orbits $\{\alpha_{1,3}, \alpha_{2,4}\}$, $\{\alpha_{2,3}, \alpha_{1,4}\}$, $\{\alpha_{1,6}, \alpha_{2,12}\}$, $\{\alpha_{2,6}, \alpha_{1,12}\}$, $\{\alpha_{1,7}, \alpha_{2,8}\}$, $\{\alpha_{2,7}, \alpha_{1,8}\}$, $\{\alpha_{1,9}, \alpha_{1,10}\}$, $\{\alpha_{2,9}, \alpha_{2,10}\}$.

Like for Case 23 above, we obtain the following applications to Kählerian K3 surfaces.

Let a Kählerian K3 surface X be marked by a primitive sublattice $S \subset N = N_{22} = N(12A_2)$. Then S must satisfy Theorem 2 and $\Gamma(P(S)) \subset \Gamma(P(N_{22})) = 12\mathbb{A}_2$. Any such S gives marking of some X and $P(X) \cap S = P(S)$.

If $N_H \subset S$ where H has the type $n = 79$, 70, 63, 62 or 54 (equivalently, $\operatorname{rk} N_H = 19$), then $\operatorname{Aut}(X,S)_0 = H$. Otherwise, if only $N_H \subset S$ where H has the type $n = 55$, 48, 46, 32 or 26 (equivalently, $\operatorname{rk} N_H = 18$), then $\operatorname{Aut}(X,S)_0 = H$. Otherwise, if only $N_H \subset S$ where H has the type $n = 34$ or 12 (equivalently, $\operatorname{rk} N_H = 17$), then $\operatorname{Aut}(X,S)_0 = H$. Otherwise, if only $N_H \subset S$ where H has the type $n = 30$, 18, 17 or 16 (equivalently, $\operatorname{rk} N_H = 16$), then $\operatorname{Aut}(X,S)_0 = H$. Otherwise, if only $N_H \subset S$ where H has the type $n = 10$ (equivalently, $\operatorname{rk} N_H = 15$), then $\operatorname{Aut}(X,S)_0 = H$. Otherwise, if only $N_H \subset S$ where H has the type $n = 6$ or 4 (equivalently, $\operatorname{rk} N_H = 14$), then $\operatorname{Aut}(X,S)_0 = H$. Otherwise, if only $N_H \subset S$ where H has the type $n = 3$ or 2 (equivalently, $\operatorname{rk} N_H = 12$), then $\operatorname{Aut}(X,S)_0 = H$. Otherwise, if only $N_H \subset S$ where H has the type $n = 1$ (equivalently, $\operatorname{rk} N_H = 8$), then $\operatorname{Aut}(X,S)_0 = H \cong C_2$. Otherwise, $\operatorname{Aut}(X,S)_0$ is trivial.

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