Notes on non-isolated singularities

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Abstract.

In these lectures we shall give results about non-isolated singularities of complex algebraic sets. We shall indicate many references to the reader.

We shall try to enunciate the results as clearly as possible. Detailed proofs of the results are not always given, but to complete them will be an excellent exercise for the reader. We hope that these notes will be a lead to students in the maze of the theory of singularity.

These notes follows essentially the lectures given at São Carlos in the school of 2012.

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§1.   Critical points of polynomials

1.1.   Polynomial functions

Let p_0 be a point of $\mathbb{K}^{n+1}$, where $\mathbb{K}$ is either the field of real numbers $\mathbb{R}$ or the field $\mathbb{C}$ of complex numbers. The point p_0 is called a

critical point of a $\mathbb{K}$-polynomial function f, if all the partial derivatives $\partial f/\partial X_0, \ldots, \partial f/\partial X_n$ of f vanish at p_0:

$$\partial f/\partial X_0(p_0) = \ldots = \partial f/\partial X_n(p_0) = 0.$$

If the point p_0 is a critical point of f, the value $f(p_0)$ is called a *critical value* of f.

For $\mathbb{K}$-polynomial functions one has a more precise result than Sard's Lemma (see [17] §2 and 3):

Proposition 1.1. *A $\mathbb{K}$-polynomial function has a finite number of critical values.*

Proof. We shall essentially follow the proof of J. Milnor in [18]. Let $C(f)$ be the set of critical points of the polynomial function f. The set $C(f)$ is a $\mathbb{K}$-algebraic subset of $\mathbb{K}^{n+1}$.

Let Σ_1 be the set of singular points (or *singular locus*) of $C(f)$. It is an algebraic subset of $\Sigma_0 := C(f)$ (see [18] Lemma 2.2 and [24]).

We can define by induction Σ_i as the singular locus of Σ_{i-1}. If $\Sigma_{i-1} \neq \emptyset$, Σ_i is a proper $\mathbb{K}$-algebraic subset of Σ_{i-1}:

$$\Sigma_i \subset \Sigma_{i-1}$$

and $\Sigma_{i-1} \setminus \Sigma_i$ is a manifold (see [24]).

Hilbert basis theorem (see [21] Theorem 1 p. 201) implies that the decreasing sequence of $\mathbb{K}$-algebraic sets $\Sigma_0 := C(f)$, Σ_1, ..., Σ_i, ... is finite. Therefore, we have a finite partition:

$$C(f) = (\Sigma_0 \setminus \Sigma_1) \coprod \ldots \coprod (\Sigma_{i-1} \setminus \Sigma_i) \coprod \ldots \coprod \Sigma_k,$$

which shows that the $\mathbb{K}$-algebraic subset $C(f)$ of $\mathbb{K}^{n+1}$ is the disjoint union of differences of $\mathbb{K}$-algebraic sets and each difference is a smooth manifold, since the complement in an $\mathbb{K}$-algebraic set E of the set of singular points Σ_E of E is a smooth manifold (see [24]). Now, we have (see [18] Appendix A Corollary A.4):

Lemma 1.2. *The difference $E \setminus F$ of $\mathbb{K}$-algebraic subsets of $\mathbb{K}^{n+1}$ is the disjoint union of a finite number of connected smooth manifolds.*

First, let us assume the Lemma.

By Lemma 1.2 the $\mathbb{K}$-algebraic subset $C(f)$ of $\mathbb{K}^{n+1}$ is a finite disjoint union of connected manifolds:

$$C(f) = V_1 \coprod \ldots \ldots \coprod V_\ell.$$

The restriction of f to any of these manifolds V_i, $1 \leq i \leq \ell$ is critical. Since V_i is connected, the restriction of f to V_i is constant. Therefore, the critical values of f are in finite number.

It remains to prove the Lemma.

We have to prove that the difference $E \setminus F$ of $\mathbb{K}$-algebraic subsets of $\mathbb{K}^{n+1}$ is the disjoint union of a finite number of connected smooth manifolds.

We have seen that the difference $E \setminus F$ is the finite union of smooth differences of $\mathbb{K}$-algebraic subsets of $\mathbb{K}^{n+1}$:

$$E \setminus F = (\Sigma_0 \setminus (\Sigma_1 \cap F)) \coprod \cdots \coprod (\Sigma_{i-1} \setminus (\Sigma_i \cap F)) \coprod \cdots \coprod (\Sigma_k \setminus F),$$

where $\Sigma_0 = E$ and, for $i > 0$, Σ_i is the singular locus of $\Sigma_{i-1} \not\subseteq F$.

So we may assume that the difference $E \setminus F$ is smooth.

To prove Lemma 1.2, it is enough to consider the case $\mathbb{K} = \mathbb{R}$. Let $f_1, \ldots, f_r$ be real polynomials which define F on E:

$$F = E \cap \{f_1 = \ldots = f_r = 0\}.$$

Since the base field is the field of real numbers, the subset F is also defined by the polynomial $\sum_1^r f_i^2$:

$$F = E \cap \{\sum_1^r f_i^2 = 0\}.$$

Considering the map $\varphi : E \setminus F \to \mathbb{R}$ defined by:

$$p \mapsto \frac{1}{f_1^2(p) + \ldots + f_r^2(p)},$$

The projection onto E shows that the graph G of φ is diffeomorphic to $E \setminus F$.

The set G is also a real algebraic subset of $\mathbb{R}^{n+1} \times \mathbb{R}$ defined by the intersection:

$$(E \times \mathbb{R}) \cap \{T(\sum_1^r f_i^2) = 1\},$$

where T is the coordinate of $\mathbb{R}^{n+1} \times \mathbb{R}$ given by $\{0\} \times \mathbb{R}$.

Now the space G is a closed manifold embedded in $\mathbb{R}^{n+1} \times \mathbb{R}$. For almost all points p of $\mathbb{R}^{n+1} \times \mathbb{R}$, the square of the distance function to the point p is a non-degenerate function in the sense of Morse (see [19] Theorem 6.6). The critical points of such a function are isolated. They form a real algebraic subset of $\mathbb{R}^{n+1} \times \mathbb{R}$ whose points are isolated, so it must be a finite set (see [18] Appendix A, Corollary A.2). Therefore,

by applying Morse theory [19], the space G has the homotopy type of a finite CW-complex. In particular, the number of connected components of G must be finite.

This ends the proof of the Lemma and therefore the proof of our Proposition.

1.2.   Restrictions to non-singular differences

Now, consider the restriction of a $\mathbb{K}$-polynomial function to a difference of $\mathbb{K}$-algebraic sets $E \setminus F$, which is a smooth manifold.

A result similar to Proposition 1.1 is also true. In fact, the proof is analogous to the proof that we have just done:

Proposition 1.3. *The restriction of a $\mathbb{K}$-polynomial function to a difference of $\mathbb{K}$-algebraic sets $E \setminus F$ which is non-singular has a finite number of critical values.*

Proof: Let φ the restriction of the polynomial function $f : \mathbb{K}^N \to \mathbb{K}$ to E. Let us define the critical locus $C(\varphi)$ of φ as the union of the singular locus of E and the subset of the non-singular subset of E where φ is critical. One can check that the set $C(\varphi)$ is a $\mathbb{K}$-algebraic subset of E.

We just notice that the critical locus of the restriction of φ to $E \setminus F$ is a difference of $\mathbb{K}$-algebraic sets $C(\varphi) \setminus F$. Then, one proceeds as in the proof of Proposition 1.1.

One has:

$$C(\varphi) \setminus F = (\Sigma_0 \setminus (F \cup \Sigma_1)) \coprod \cdots \coprod (\Sigma_{i-1} \setminus (F \cup \Sigma_i)) \coprod \cdots \coprod (\Sigma_k \setminus F),$$

where $\Sigma_0 = C(\varphi)$, Σ_i is the singular locus of Σ_{i-1} and $\Sigma_{k+1} = \emptyset$.

Lemma 1.2 tells that $C(\varphi) \setminus F$ is the disjoint union of a finite number of connected manifolds. The restriction of φ to each of these connected manifolds is critical, so it must be constant. This implies Proposition 1.3.

1.3.   Comments

The preceding result can be adapted to analytic functions. For instance, let $f : (\mathbb{K}^N, 0) \to (\mathbb{K}, 0)$ be the germ of a $\mathbb{K}$-analytic function. Then, one can prove that there is $\varepsilon > 0$ and an open ball $\mathring{\mathbb{B}}_\varepsilon(x)$ centered at x with radius ε, such that there exists $\eta > 0$ for which the space $f^{-1}(t) \cap \mathring{\mathbb{B}}_\varepsilon(0)$ for t, $\eta > |t| > 0$, is a $\mathbb{K}$-analytic manifold.

We usually write that there are real numbers $\varepsilon > 0$ and $\eta > 0$ such that $1 \gg \varepsilon \gg \eta > 0$ for which the space $f^{-1}(t) \cap \mathring{\mathbb{B}}_\varepsilon(0)$ is an analytic manifold for $\eta > |t| > 0$.

§2. The fibration theorem (isolated singularity case)

2.1. The local link

Let us assume that the complex polynomial function $f : \mathbb{C}^{n+1} \to \mathbb{C}$ has an isolated critical point at p_0. In this case we shall prove that we can associate a locally trivial smooth fibration to the critical point.

The difference $f^{-1}(f(p_0)) \setminus C(f)$ is a manifold. As a consequence of Proposition 1.3, the restriction of the square of the distance to p_0 to $f^{-1}(f(p_0)) \setminus C(f)$ has no critical value in the open interval $(0, \varepsilon_0)$ when ε_0 is a sufficiently small > 0 number. Therefore, for any ε, $\sqrt{\varepsilon_0} > \varepsilon > 0$, the intersection $\mathbb{S}_\varepsilon(p_0) \cap f^{-1}(f(p_0))$ of $f^{-1}(f(p_0))$ with the sphere $\mathbb{S}_\varepsilon(p_0)$ of $\mathbb{C}^{n+1}$ centered at p_0 with radius ε is a manifold $K_\varepsilon(p_0)$.

In fact, the square of the distance to the point p_0 defines a smooth map of $f^{-1}(f(p_0)) \setminus C(f)$ into the real line $\mathbb{R}$. The map induces a trivial smooth fibration onto the open interval $(0, \varepsilon_0)$. One can prove it by lifting a unit vector field by the map and integrating the lifted vector field or analogously by applying Ehresmann Lemma given below.

The fibers of this fibration are the manifolds $K_\varepsilon(p_0)$ when ε, $\sqrt{\varepsilon_0} > \varepsilon > 0$. They are diffeomorphic between each other, as well as the pairs $(\mathbb{S}_\varepsilon(p_0), K_\varepsilon(p_0))$. This is why (by abuse of language) we call $K_\varepsilon(p_0)$ the *link* of the point p_0 on the hypersurface $f^{-1}(f(p_0))$.

2.2. A fibration theorem

Let us choose ε, $\sqrt{\varepsilon_0} > \varepsilon > 0$. The Proposition 1.1 implies that there is η_ε, such that, for any η, $\eta_\varepsilon > \eta > 0$, the hypersurfaces $f^{-1}(f(p_0) + t)$ are non singular for any t, $\eta \geq |t| > 0$. Furthermore, since, by definition of $\varepsilon_0 > 0$, the sphere $\mathbb{S}_\varepsilon(p_0)$ intersects $f^{-1}(f(p_0))$ transversally in $\mathbb{C}^{n+1}$, by continuity, the hypersurfaces $f^{-1}(f(p_0) + t)$ are transverse in $\mathbb{C}^{n+1}$ to the sphere $\mathbb{S}_\varepsilon(p_0)$ for any t, $\eta \geq |t| > 0$.

We have:

Proposition 2.1. *Let us choose ε and η as above. The map:*

$$\varphi_{\varepsilon, \eta} : \mathbb{B}_\varepsilon(p_0) \cap f^{-1}(\mathbb{S}_\eta^1(f(p_0))) \to \mathbb{S}_\eta^1(f(p_0))$$

induced by the polynomial function f, is a locally trivial smooth fibration over the circle $\mathbb{S}_\eta^1(f(p_0))$.

Proof. Since, for $z \in \mathbb{S}_\eta^1(f(p_0))$, the fibers $f^{-1}(z)$ are transverse in $\mathbb{C}^{n+1}$ to the sphere $\mathbb{S}_\varepsilon(p_0)$ the restriction of $\varphi_{\varepsilon, \eta}$ to the boundary of $\mathbb{B}_\varepsilon(p_0) \cap f^{-1}(\mathbb{S}_\eta^1(f(p_0)))$ is submersive. It is also obviously surjective. Since the fibers $f^{-1}(z)$ for $z \in \mathbb{S}_\eta^1(f(p_0)$ are non singular, the restriction of $\varphi_{\varepsilon, \eta}$ to the interior $\mathring{\mathbb{B}}_\varepsilon \cap f^{-1}(\mathbb{S}_\eta^1(f(p_0)))$ is also submersive and surjective. We can conclude using Ehresmann Lemma.

2.3. Ehresmann Lemma

We have a lemma due to C. Ehresmann (see e.g. [2] (20.8) problème 4, or see also [1] Theorem 8.12 p. 84 when the boundary is empty):

Lemma 2.2. (Ehresmann Lemma) *Let $(M, \partial M)$ be a smooth manifold M with boundary ∂M. Let $\varphi : M \to N$ be a smooth map into a connected manifold N. Assume:*

(1) *The map φ is proper;*
(2) *The restriction of φ to the boundary ∂M is submersive into N;*
(3) *The restriction of φ to the interior $M - \partial M$ is submersive into N.*

Then, the map φ is a locally trivial smooth fibration.

Using this lemma, we obtain immediately a proof of Proposition 2.1.

Exercise: Prove that, if $1 \gg \varepsilon \gg \eta > 0$, the different fibrations $\varphi_{\varepsilon,\eta}$ are isomorphic (see [15] p. 80 line -7).

The fibration defined in Proposition 2.1 is called the *Milnor fibration* of the polynomial function f at p_0.

§3. Stratifications

3.1. Algebraic partition

Let E be a complex algebraic subset of $\mathbb{C}^{n+1}$. Let $S_1, \ldots, S_k$ be a finite partition of E:

$$E = S_1 \coprod \ldots \coprod S_k.$$

We say that it is a *complex algebraic partition* if the closures $\overline{S}_i$ of S_i in $\mathbb{C}^{n+1}$ and differences $\overline{S}_i \setminus S_i$ are complex algebraic sets, for $1 \le i \le k$.

Similarly let E be a real semi-algebraic subset of $\mathbb{R}^N$. Let $S_1, \ldots, S_k$ a finite partition of E:

$$E = S_1 \coprod \ldots \coprod S_k.$$

We say that this partition is a *real algebraic partition* of E if the closures $\overline{S}_i$ of S_i in $\mathbb{R}^N$ and the differences $\overline{S}_i \setminus S_i$ are semi-algebraic sets, for all i, $1 \le i \le k$.

Any set S_i in a partition $\mathcal{S}$ of E, $1 \le i \le k$, is called a *stratum* of the partition.

3.2. Algebraic stratification

Let E be a complex algebraic subset of $\mathbb{C}^{n+1}$. Let $S_1, \ldots, S_k$ be an algebraic partition of E.

We say that $(S_i)_{1 \leq i \leq k}$ is a *complex stratification* of E, if it is a complex algebraic partition of E and each S_i is a connected complex manifold and the partition satisfies the *frontier condition*, i.e. $S_i \cap \overline{S}_j \neq \emptyset$ implies $S_i \subset \overline{S}_j$ which means that $\overline{S}_j$ is the union of S_j and strata of lower dimension.

Analogously, let E be a real semi-algebraic subset of $\mathbb{R}^N$ and let $S_1, \ldots, S_k$ be an algebraic partition of E.

We say that the partition $(S_i)_{1 \leq i \leq k}$ is a *real stratification* of E, if it is a real algebraic partition and each S_i is a connected smooth manifold and the partition satisfies the *frontier condition*, i.e. $S_i \cap \overline{S}_j \neq \emptyset$ implies $S_i \subset \overline{S}_j$.

Whenever we shall consider the stratification of a real algebraic set, we shall consider a stratification by semi-algebraic sets, considering a real algebraic set as a particular semi-algebraic set.

We say that a map:

$$\varphi : E \to F$$

of a stratified set $(E, \mathcal{S})$ into a stratified set $(F, \mathcal{T})$ is a *stratified map* if for any stratum $S_i \in \mathcal{S}$, there is a stratum $T_j \in \mathcal{T}$ such that φ induces a submersive and surjective map of S_i onto T_j.

3.3. An example

Let E be a $\mathbb{K}$-algebraic set. Let $\Sigma(E)$ be the subset of singular points of E. We can define a decreasing sequence of algebraic sets defined by induction:

(1) $\Sigma_0 := E$
(2) Let $i \geq 0$, $\Sigma_{i+1} = \Sigma(\Sigma_i)$

By Hilbert basis Theorem this sequence is stationary, so there is ℓ such that $\Sigma_\ell \neq \emptyset$ and $\Sigma_{\ell+1} = \Sigma(\Sigma_\ell) = \emptyset$. Therefore, we have:

$$E = (\Sigma_0 \setminus \Sigma_1) \coprod \ldots \coprod (\Sigma_i \setminus \Sigma_{i+1}) \coprod \ldots \coprod \Sigma_\ell.$$

In this way, we have defined a partition of E defined by the strata $\Sigma_i \setminus \Sigma_{i+1}$, for $1 \leq i \leq \ell + 1$.

The strata are smooth manifolds. However, the strata might not be connected.

An algebraic partition $\mathcal{T}$ of a $\mathbb{K}$-algebraic set is said to be *finer* than the algebraic partition $\mathcal{S}$ if the closures of the strata of $\mathcal{S}$ are union of strata of $\mathcal{T}$. We also say that the partition $\mathcal{T}$ is a *refinement* of $\mathcal{S}$.

In [25] (Theorem 18.11) H. Whitney proves that:

Proposition 3.1. *Any algebraic partition of a $\mathbb{K}$-algebraic set has a refinement which is an algebraic stratification with connected strata.*

With these definitions we can define regular stratifications (see [25]), i.e. Whitney stratifications.

§4.   Whitney stratifications

4.1.   The condition (a) of Whitney

Let $\mathcal{S} = (S_i)_{1 \leq i \leq k}$ be a stratification of a $\mathbb{K}$-algebraic subset E of $\mathbb{K}^N$. Let (S_i, S_j) be a pair of strata such that $S_i \subset \overline{S}_j$. Let $p \in S_i$. We say that the pair (S_i, S_j) satisfies the condition (a) of Whitney along S_i at the point p, if, for any sequence (p_n) of points of S_j which converges to p such the sequence of tangent spaces $(T_{p_n} S_j)$ has a limit T, then the tangent space $T_p S_i$ is contained in T.

We say that, the pair (S_i, S_j) satisfies the *condition (a) of Whitney* along S_i, if it satisfies the condition (a) of Whitney along S_i at any point p of S_i.

We say that the stratification $\mathcal{S} = (S_i)_{1 \leq i \leq k}$ of E satisfies the Whitney condition (a), if for any pair (S_i, S_j) of strata such that $S_i \subset \overline{S}_j$, the pair (S_i, S_j) satisfies the condition (a) of Whitney along S_i.

Example: Consider the complex algebraic set E defined by:

$$X^2 - Y^2 Z = 0,$$

which is known as "Whitney umbrella".

The singular set of E is given by $X = Y = 0$. This is a line Σ contained in E. A stratification of E is given by $E \setminus \Sigma$ and Σ. The pair $(\Sigma, E \setminus \Sigma)$ does not satisfy the condition (a) of Whitney along Σ at the origin 0 of $\mathbb{C}^3$.

One may consider a sequence (p_n) of points of $E \setminus \Sigma$ given by $p_n = (0, y_n, 0)$, where $\lim_{n \to \infty} y_n = 0$. Then, $p = \lim_{n \to \infty} p_n = (0, 0, 0)$. We have $T_{p_n}(E \setminus \Sigma) = (X, Y) -$ plane and $T_p(\Sigma) = Z -$ axis. The limit at the point $(0, 0, 0)$ of the sequence $(T_{p_n}(E \setminus \Sigma))_{n \in \mathbb{N}}$ of tangent spaces is the $(X, Y) -$ plane and it does not contain the $Z -$ axis.

However the stratification $(E \setminus \Sigma, \Sigma \setminus \{0\}, \{0\})$ of E satisfies the Whitney condition (a).

4.2.   The condition (b) of Whitney

Let $\mathcal{S} = (S_i)_{1 \leq i \leq k}$ be a stratification of a $\mathbb{K}$-algebraic subset E of $\mathbb{K}^N$.

Let (S_i, S_j) be a pair of strata such that $S_i \subset \overline{S}_j$. Let $p \in S_i$. We say that the pair (S_i, S_j) satisfies the condition (b) of Whitney along S_i

at the point p, if for any sequence p_n of points of S_j and any sequence (q_n) of points of S_i, which converges to p, such that the sequence of tangent spaces $T_{p_n} S_j$ has a limit T and the sequence of lines $\overline{q_n p_n}$ has a limit ℓ, we have $\ell \subset T$.

We say that the pair (S_i, S_j) satisfies the *condition (b) of Whitney* along S_i if it satisfies the condition (b) of Whitney along S_i at any point p of S_i.

We say that the stratification $\mathcal{S} = (S_i)_{1 \leq i \leq k}$ of E satisfies the Whitney condition (b), if for any pair (S_i, S_j) of strata such that $S_i \subset \overline{S}_j$, the pair (S_i, S_j) satisfies the condition (b) of Whitney along S_i.

Examples: In the example given above the pair $(\Sigma, E \setminus \Sigma)$ does not satisfy the condition (b) of Whitney along Σ at the origin 0 of $\mathbb{C}^3$.

Consider the complex algebraic set F given by $X^2 - Y^2 Z^2 - Z^3 = 0$ in $\mathbb{C}^3$. The singular locus Σ is the line $X = Z = 0$. One can prove that this stratification satisfies the condition (a) of Whitney, but it does not satisfies the condition (b) at the origin 0. The sequences $p_n = (0, 1/n, -1/n^2)$ and $q_n = (0, 1/n, 0)$ give sequences of tangent planes $(T_{p_n}(F \setminus \Sigma))$ and of lines $(\overline{q_n p_n})$ whose limits T and ℓ are such that: $\ell \not\subset T$.

The stratification of F given by $(F \setminus \Sigma, \Sigma \setminus \{0\}, \{0\})$ satisfies the Whitney condition (b).

In fact, the condition (b) of Whitney implies the condition (a) of Whitney (see [16] Proposition 2.4):

Lemma 4.1. *Let E be a complex analytic subset of $\mathbb{C}^N$ (resp. a real semi-algebraic subset of $\mathbb{R}^N$). Let $\mathcal{S} = (S_1, \ldots, S_k)$ be a stratification of E. Suppose that $S_i \subset \overline{S}_j$. Let $p \in S_i$. Suppose that the pair (S_i, S_j) satisfies the (b)-condition of Whitney at p, then it satisfies the (a)-condition of Whitney at p.*

Proof. The following proof was given to me orally by D. Cheniot.

Assume that the pair (S_i, S_j) satisfies the condition (b) of Whitney at $p \in S_i$. Consider a sequence $(p_n)_{n \in \mathbb{N}}$ a sequence of points of S_j which converges to p and such that the sequence of tangent spaces $T_{p_n}(S_j)$ converge to T. We have to show that: $T_p(S_i) \subset T$. Suppose otherwise. Then, there is a line D in $\mathbb{K}^N$ through the origin, such that $D \subset T_p(S_i)$, but $D \not\subset T$.

By definition of the tangent space $T_p(S_i)$, there is a sequence of points $(q_n)_{n \in \mathbb{N}}$ of points of S_i which converges to p and such that the lines $\overline{p q_n}$ converge to D. Since the sequence (p_n) converges to p, endowing the projective space of line directions with a metric, for any $k \in \mathbb{N}$, we can find n_k such that the distance between the lines of direction $p q_k$ and

$q_k p_{n_k}$ is bounded by $1/k$. Therefore, $\overline{q_k p}_{n_k}$ converges to D, and this would contradict that (S_i, S_j) satisfies the condition (b) of Whitney at $p \in S_i$.

Because of Lemma 4.1, we often say that a stratification satisfies the condition of Whitney or a stratification is Whitney instead of "a stratification satisfies the condition (b) of Whitney".

4.3. Whitney condition

In [25] (Theorem 19.2) H. Whitney proves:

Theorem 4.2. *Any $\mathbb{K}$-algebraic set has a Whitney stratification. Any stratification has a refinement which is a Whitney stratification.*

Note: One can define partitions and stratifications of $\mathbb{K}$-analytic subsets of open subsets of $\mathbb{K}^N$. Also one can define the Whitney conditions in this context. See the original paper of H. Whitney [25].

We can also extend the notion of stratification to any complex algebraic variety V in the following way. An algebraic partition of V is a finite partition $S_1, \ldots, , S_k$ such that the closures $\overline{S}_i$ of S_i in V and the differences $\overline{S}_i \setminus S_i$ are subvarieties of V, for $1 \leq i \leq k$.

An algebraic partition of complex algebraic variety is a stratification if each S_i is a connected manifold and the partition satisfies the frontier condition, i.e. $S_i \cap \overline{S}_j \neq \emptyset$ implies $S_i \subset \overline{S}_j$.

In what follows we shall use the notion of stratification in a general sense as Whitney in [25] or Mather in [16].

4.4. Characterization of Whitney stratifications by Polar varieties

In the case the base field is the field of complex numbers $\mathbb{C}$, there are many ways to characterize Whitney stratifications. To my knowledge there are no such characterization over the real numbers.

One way is to consider polar varieties (see [22]).

Let E be an equidimensional complex algebraic subset of $\mathbb{C}^N$.

Let p_0 be a point of E. Consider the restriction of a linear projection to E:

$$\pi_k : E \to \mathbb{C}^{k+1}.$$

The restriction of π_k to the non-singular part $E \setminus \Sigma(E)$ of E has a critical set Γ_k. We have (see [14] (2.2.2)):

Proposition 4.3. *For all projections π_k in a Zariski dense open set $\Omega_k(p_0)$ of the space of linear projections of $\mathbb{C}^N$ onto $\mathbb{C}^{k+1}$, the closure $P^k = \overline{\Gamma}^k$ of Γ^k is either empty or a complex algebraic set of dimension k and the multiplicity of P^k at p_0 is the same for any $\pi_k \in \Omega_k(p_0)$.*

Definition 4.4. *For* $1 \leq k \leq \dim E$, *the germ* $P^k(E, p_0)$ *at* p_0 *of the algebraic set* P^k *defined by the projections* $\pi_k \in \Omega_k(p_0)$ *is called the polar varieties of* E *at the point* p_0 *and the multiplicity of* P^k *at* p_0 *is called the* k-*th polar multiplicity of* E *at* p_0.

When $k = \dim E$, the polar variety of dimension $\dim E$ at $p_o \in E$ coincides with E. The multiplicity of the $\dim E$-th polar variety of E at p_0 is equal to the multiplicity of E at p_0. In the case E is non-singular at p_0, the polar variety $P^k(E, p_0)$ of dimension k, for $1 \leq k < \dim E$ is empty.

The swallow tail, i.e. the discriminant of the polynomial equation of degree 4, has an empty polar curve.

In [22] B. Teissier proved the following:

Theorem 4.5. *Let* $\mathcal{S} = (S_1, \ldots, S_k)$ *be a stratification of complex algebraic set* E. *Assume that, for any pair of strata* (S_i, S_j) *such that* $S_i \subset \overline{S}_j$, *the* ℓ-*th polar multiplicity of the closure* $\overline{S}_j$ *is constant along* S_i, *when* $1 \leq \ell \leq \dim S_j$, *then* $\mathcal{S}$ *is a Whitney stratification of* E.

There is also a combinatorial way to decide if a stratification is a Whitney stratification. We shall introduce this result later because we need to understand first how much Milnor fibration Theorem can be extended.

4.5. First isotopy theorem of Thom-Mather

The main properties of Whitney stratifications are Thom-Mather isotopy Theorems ([23] and [16] Proposition 11.1 & 11.2).

The first isotopy theorem can be understood as a generalization of Ehresmann Lemma that we have used to show Milnor fibration Theorem for hypersurfaces with an isolated singular point.

In many cases we shall consider real algebraic varieties and more generally semi-algebraic spaces. In the theorem below, the space V might be a semi-algebraic space or a complex algebraic variety.

Theorem 4.6. (First isotopy theorem of Thom-Mather) *Let* $\varphi : V \to T$ *be a proper algebraic map. Assume that* V *is stratified by the Whitney stratification* $\mathcal{S} = (S_1, \ldots, S_k)$ *and that* T *is connected and non-singular. If the restrictions of* φ *to each stratum* S_i $(1 \leq i \leq k)$ *are smooth maps which are submersive into* T, *then the map* φ *is a locally trivial topological fibration.*

Notice that, since the morphism φ is proper, since it is a submersion, the restriction to each stratum of the stratification $\mathcal{S}$ is surjective onto T which is connected.

Example. Let E be the real algebraic subset of $\mathbb{R}^3$ defined by:

$$XY(X - Y)(X - TY) = 0.$$

Consider the projection onto the T-axis. Consider the "tube" given by $X^2 + Y^2 \leq 1$. The intersection:

$$V = E \cap \{X^2 + Y^2 \leq 1\} \cap \{0 < T < 1\}$$

is semi-algebraic. The projection φ onto the T-axis induces a proper map of V onto the open interval $0 < T < 1$. Let us stratify V with $S_1 = \{(0,0)\} \times \{0 < T < 1\}$, $S_2 = V \cap \{X^2 + Y^2 = 1\}$ and $S_3 = V \setminus (S_1 \cup S_2)$. The restrictions of φ to the connected components of $S_3 = V \setminus (S_1 \cup S_2)$ induce submersive and surjective maps onto the open interval $\{0 < T < 1\}$.

    The first isotopy theorem of Thom-Mather tells us that φ is a locally trivial continuous fibration, but it cannot be a smooth fibration otherwise the cross ratio of the lines:

$$XY(X - Y)(X - tY) = 0$$

which varies continuously with t would be constant.

§5. Thom Condition

5.1. Definition

    When one considers a map f between stratified sets, there is an important condition on the stratification of the source called Thom condition or a_f condition (see [23] or [16] p. 65).

    Let $\varphi : V \to W$ be an algebraic map. Let $\mathcal{S} = (S_1, \ldots, S_k)$ be a stratification of V.

    We say that the stratification $\mathcal{S}$ of V satisfies the *Thom condition* or the a_φ *condition* if, for any pair (S_i, S_j) of strata, such that $S_i \subset \overline{S}_j$, for any point $p \in S_i$ and any sequence q_n of points of S_j converging to p for which the limit of the tangent spaces at q_n to the fibers $\varphi^{-1}(\varphi(q_n)) \cap S_j$ exists and is equal to T, we have $T \supset T_p(\varphi^{-1}(\varphi(p)) \cap S_i)$.

    Example. Let $e : E \to \mathbb{C}^2$ be the blowing-up of the point 0 in the complex plane $\mathbb{C}^2$. Let stratify the map e by $E \setminus D$ and D, where D is the exceptional divisor of e and $\{0\}$ and $\mathbb{C}^2 \setminus \{0\}$. This stratified map does not satisfy the Thom condition, since the fibers of e outside 0 are points.

    Note that, pulling back the map e by e itself, the new map that we obtain can be stratified to satisfy Thom condition (see [20]).

5.2. Hironaka Theorem

A theorem of H. Hironaka will allow us to generalize Milnor fibration Theorem (see [7] Corollary 1 §5 p. 248):

Theorem 5.1. *Let $f : E \to C$ be an algebraic map from a complex algebraic variety E into a non-singular complex curve C. One can stratify the map f by Whitney stratifications such that the stratification of E satisfies Thom condition for f.*

In the case of hypersurfaces in $\mathbb{C}^{n+1}$, in [5] (Theorem 1.2.1 p. 322 - 324) we prove this theorem by using Łojasiewicz inequality.

5.3. An example

In general when the target of the map has not complex dimension one, there are no reason to obtain stratifications with Thom conditions. We have already seen the case of the blowing-up of a point in $\mathbb{C}^2$ in the preceding example.

Consider the polynomial map $F : \mathbb{C}^3 \to \mathbb{C}^2$ given by:

$$F(X, Y, Z) = (Y^2 - X^2 Z, X).$$

The critical locus is $Y = X^2 = 0$. Its image by F is $(0, 0)$.

One can check that, at the origin, one cannot find a Whitney stratification which satisfies Thom condition.

In fact, in [20] C. Sabbah proved that after a possible base change the original map gives a new map which can be stratified with Thom condition. This result has been little used in the literature.

Exercise: Prove that, for a surjective map $\Phi : \mathbb{C}^n \to \mathbb{C}^m$ with an isolated singularity $p_0 \in \mathbb{C}^n$ in the fiber $\Phi^{-1}(\Phi(p_0))$, there are open neighborhoods U of p_0 and V of $\Phi(p_0)$ which can be stratified with Thom condition.

5.4. Second isotopy theorem of Thom-Mather

Now we can formulate the second isotopy Theorem of Thom-Mather (see [16] Proposition 11,2).

Consider the following diagram:

$$
\begin{array}{ccc}
E & \xrightarrow{\;\;g\;\;} & F \\
 & \searrow{\scriptstyle f\circ g} \quad \swarrow{\scriptstyle f} & \\
 & T &
\end{array}
$$

where the algebraic map g is a stratified map, where the space (real algebraic or complex variety) E is stratified by a Whitney stratification

$S = (S_1, \ldots, S_k)$ and the space (real algebraic or complex variety) F is stratified by a Whitney stratification $S' = (S'_1, \ldots, S'_\ell)$, and T is non-singular. We say that g is a *Thom map over T* if:

(1) The maps g and f are proper;
(2) For each stratum S'_i of S' the restriction $f|S'_i$ is submersive;
(3) For any stratum S_j of S there is a stratum S'_i of S' such that $g(S_j) \subset S'_i$ and the map induced by g from S_j into S'_i is submersive;
(4) The stratification S satisfies the Thom condition for g.

Then:

Theorem 5.2. (Second isotopy theorem of Thom-Mather) *If g is a Thom map over T, the map g is topologically locally trivial over T.*

§6. The fibration theorem

6.1. Case of functions in $\mathbb{C}^{n+1}$

In §2 we have proved a fibration theorem when the point p_0 is an isolated critical point of the polynomial function $f : \mathbb{C}^{n+1} \to \mathbb{C}$.

In this section we shall drop the hypothesis that the point p_0 is an isolated critical point.

Since the critical point p_0 is no more isolated in the critical locus of f, the critical locus of f intersects any sphere centered at the point p_o. Therefore, the hypersurface $f^{-1}(f(p_0))$ does not intersect transversally in $\mathbb{C}^{n+1}$ small spheres $\mathbb{S}_\varepsilon(p_0)$. However, by Proposition 1.1, the fibers $f^{-1}(p_0 + t)$ for t small enough and $\neq 0$ are non-singular hypersurfaces. Hironaka Theorem (see 5.1 above) tells us that we can stratify the polynomial f by Whitney stratifications with Thom condition.

Let us fix a Whitney stratification $S = (S_1, \ldots, S_k)$ of $\mathbb{C}^{n+1}$ adapted to $f^{-1}(f(p_0))$ and which satisfies Thom condition relatively to the polynomial function f.

We have seen that there is $\varepsilon_i > 0$ such that the restriction to a stratum S_i, which is the difference of complex algebraic set, of the square of the distance function to the point p_0 has no critical point in an open interval $(0, \varepsilon_i)$ (see Proposition 1.3 above). Therefore, there is $\varepsilon_0 = \inf_{1 \leq i \leq k}(\varepsilon_i) > 0$, such that for ε, $\sqrt{\varepsilon_0} > \varepsilon > 0$, the sphere $\mathbb{S}_\varepsilon(p_0)$ intersects all the strata of S transversally in $\mathbb{C}^{n+1}$. Since the stratification S satisfies Thom condition relatively to f, there is $\eta > 0$ small enough, such that the non-singular hypersurfaces $f^{-1}(f(p_0 + t))$ intersect the sphere $\mathbb{S}_\varepsilon(p_0)$ transversally in $\mathbb{C}^{n+1}$, when $\eta > |t| > 0$.

We can apply Ehresmann Lemma as in §2 to claim that:

Theorem 6.1. *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a complex polynomial function. Let p_0 be point of $\mathbb{C}^{n+1}$. There is $\varepsilon_0 > 0$, such that for any ε, $\varepsilon_0 > \varepsilon > 0$, there is η_0, such that, for any $\eta > 0$, $\eta_0 > \eta > 0$, the map:*

$$\varphi_{\varepsilon,\eta} : \mathbb{B}_\varepsilon(p_0) \cap f^{-1}(\mathbb{S}_\eta^1(f(p_0))) \to \mathbb{S}_\eta^1(f(p_0))$$

induced by the polynomial function f is a locally trivial differentiable fibration.

6.2. Functions on an algebraic set

In fact, if we use Theorem 5.1 we can obtain a more general statement than Theorem 6.1, but, since we use Thom-Mather first isotopy Theorem instead of Ehresmann Lemma, we only obtain topological fibrations instead of differentiable fibrations (see [11] Theorem 1.1):

Theorem 6.2. *Let E be a complex algebraic subset of $\mathbb{C}^N$ and p_0 be a point of E. Let $f : \mathbb{C}^N \to \mathbb{C}$ be a complex polynomial. There is $\varepsilon_0 > 0$, such that, for any ε, $\varepsilon_0 > \varepsilon > 0$, there is η_ε, such that, for any $\eta > 0$, $\eta_\varepsilon > \eta > 0$, the map:*

$$\varphi_{\varepsilon,\eta} : \mathbb{B}_\varepsilon(p_0) \cap E \cap f^{-1}(\mathbb{S}_\eta^1(f(p_0))) \to \mathbb{S}_\eta^1(f(p_0))$$

induced by the polynomial function f is a locally trivial continuous fibration.

According to Hironaka Theorem 5.1 we can stratify the restriction $f_E : E \to \mathbb{C}$ of f to E to have Whitney and Thom conditions. Let $E = \coprod_{1 \le i \le k} S_i$ be this stratification. We can find $\varepsilon_0 > 0$ such that, for every stratum S_i of E which contains p_0 in its closure and for ε, $\varepsilon_0 > \varepsilon > 0$, the spheres $\mathbb{S}_\varepsilon(p_0)$ intersect the strata S_i (i=1,..., k) transversally in $\mathbb{C}^N$, because each S_i is a difference of complex algebraic sets (see Proposition 1.3 of §1 above).

Let us fix such a $\varepsilon > 0$. Because of Thom condition there is $\eta_\varepsilon > 0$ such that the fiber:

$$f^{-1}(f(p_0 + t)) \cap S_i$$

intersects transversally the sphere $\mathbb{S}_\varepsilon(p_0)$ in $\mathbb{C}^N$ whenever $\eta_\varepsilon > |t| > 0$. We can stratify the semi-algebraic set:

$$\mathbb{B}_\varepsilon(p_0) \cap E \cap f^{-1}(\mathbb{S}_\eta^1(f(p_0)))$$

with the strata:

$$S_i \cap \mathring{\mathbb{B}}_\varepsilon(p_0) \cap E \cap f^{-1}(\mathbb{S}_\eta^1(f(p_0)))$$

and $S_i \cap \mathbb{S}_\varepsilon(p_0) \cap E \cap f^{-1}(\mathbb{S}_\eta^1(f(p_0)))$, for $i = 1, \ldots, k$. Since the stratification $\mathcal{S} = (S_i)_{1 \leq i \leq k}$ of E is Whitney and satisfies Thom condition, it induces a stratification $(T_i)_{1 \leq i \leq k}$ of

$$\mathbb{B}_\varepsilon(p_0) \cap E \cap f^{-1}(\mathbb{S}_\eta^1(f(p_0))).$$

It is easy to prove that every pair of strata (T_i, T_j) such that $T_i \subset \overline{T}_j$ satisfies Whitney condition.

By using Corollary 10.5 p. 498 of [16], this partition gives a Whitney stratification.

The first isotopy Theorem of Thom-Mather 4.6 (instead of Ehresmann Lemma) implies our Theorem.

Note and Exercise: The Theorem 6.2 is true when one considers a complex analytic function on a complex analytic set. To prove it, one needs stratifications of complex analytic sets. In this case the number of strata might not be finite, but the partition is required to be locally finite. We leave the reader to develop by himself the correct theory of stratifications. The isotopy theorems are true for a general theory of stratified sets.

§7.   Relative Polar curve and general projections

7.1.   Isolated singularity relatively to a stratification

Let E be a complex algebraic subset of $\mathbb{C}^N$ and p_0 be a point of E.

Let $\mathcal{S} = (S_1, \ldots, S_k)$ be a Whitney stratification of E. The restriction $F = f \,|\, E$ of a complex polynomial f to the complex algebraic set E has an *isolated singularity at the point p_0 relatively to the stratification $\mathcal{S}$* if in an open neighborhood U of p_0 in E the fiber $f^{-1}(f(p_0))$ intersects the strata S_i transversally in $\mathbb{C}^N$ at any point of U outside p_0 (see [12]).

One can observe that if the restriction F of a complex polynomial function f to E has an isolated singular point relatively to the Whitney stratification $\mathcal{S}$, the stratification automatically satisfies Thom condition relatively to F.

7.2.   Relative polar curve

Assume that the restriction $F = f \,|\, E$ of the complex polynomial f to the complex algebraic set E has an isolated singularity at the point p_0 relatively to the stratification $\mathcal{S}$.

Let $l : \mathbb{C}^N \to \mathbb{C}$ be a linear form. Consider the restriction ℓ of l to E. The functions F and ℓ define a map:

$$\Phi = (\ell, F) : E \to \mathbb{C}^2.$$

Observe that, if the linear form l is sufficiently general, the map Φ satisfies Thom condition relatively to the Whitney stratification of E:

Proposition 7.1. *Let E be a complex algebraic set endowed with a Whitney stratification $\mathcal{S}$. If the complex polynomial function $F : E \to \mathbb{C}$ has an isolated singularity p_0 relatively to the stratification $\mathcal{S}$, there is a non-empty Zariski open set of hyperplanes Ω such that for any hyperplane $H \in \Omega$ and any equation $l = 0$ of this hyperplane, there are a open neighborhood U of p_0 in E and an open neighborhood of the point $(l(p_0), F(p_0))$ in $\mathbb{C}^2$, such that $\Phi := (\ell, F)$, where ℓ is the restriction of l to E, induces a map of U into V such that the stratification $\mathcal{S}$ induces a Thom stratification of Φ.*

In fact, if the linear form l is sufficiently general, in an open neighborhood U of p_0 in E the fibers of Φ have isolated singularities relatively to the stratification $\mathcal{S}$ in the sense that, for any point $x \in E \cap U$, the fiber $\tilde{\Phi}^{-1}(\Phi(x))$ intersects the strata $S_i \cap U$ transversally in $\mathbb{C}^N$ except at the point p_0, where $\tilde{\Phi}$ is the polynomial map:

$$(l, f) : \mathbb{C}^N \to \mathbb{C}^2.$$

Precisely, consider the restriction Φ_i of Φ to each of the strata S_i, $1 \leq i \leq k$. One can prove that if the linear form l is sufficiently general, there is an open neighborhood U of p_0 in E, such that, if $\dim S_i \geq 2$, the critical locus Γ_i of the restriction of Φ_i to $S_i \cap U$ is either the empty set $\emptyset$ or a non-singular complex analytic curve.

If $\dim S_i = 1$, the whole stratum S_i belongs to the critical locus of the restriction Φ_i of Φ to S_i. Then $\Gamma_i = S_i$.

The set:

$$\Gamma(l) = \cup_1^k \overline{\Gamma}_i,$$

which is either empty or a complex analytic curve, is called the *relative polar curve* of the linear form l relatively to the stratification $\mathcal{S}$. Notice that $\Gamma(l)$ contains the closures of the strata S_i of dimension 1 which contain p_0.

We have:.

Lemma 7.2. *The restriction of Φ to the germ at a point $p_0 \in E$ of the relative polar curve $\Gamma(l)$ is quasi-finite.*

This lemma is an immediate consequence of the fact that the fiber $\Phi^{-1}(\Phi(p_0))$ has an isolated singularity at p_0 relatively to the stratification $\mathcal{S}$. Then, the fiber of the germ of Φ restricted to $\Gamma(l)$ is either empty or equal to $\{p_0\}$.

This observation implies:

Corollary 7.3. *The germ of Φ at any point $p_0 \in E \cap U$ restricted to the germ $\Gamma(l)$ at p_0 is a germ of finite complex analytic map.*

This corollary is an immediate consequence of the lemma by using the geometric version of Weierstrass Preparation theorem given by C. Houzel in [8] which tells that a complex analytic map germ which is quasi-finite is finite, i.e. induces a proper analytic map with finite fibers.

This means that there are neighborhoods V and W of $p \in E$ in $E \cap U$ and $\Phi(p)$ in $\mathbb{C}^2$ such that Φ induces a finite analytic map, i.e. a proper complex analytic map with finite fibers, from $V \cap \Gamma(l)$ into W.

In summary, if one chooses the linear form l such that the map Φ has an isolated singularity at p_0 relatively to the stratification $\mathcal{S}$, there is a neighborhood V such that the map Φ has only isolated singularities relatively to the Whitney stratification induced by $\mathcal{S}$ in V and the singularities relatively to this stratification lie on the curve $\Gamma(l) \cap V$.

Now, let a sequence of points (q_n) of the stratum $S_j \cap V$ which converges to the point $q \neq p_0$ in the stratum $S_i \cap V$ such that the tangent spaces $T_{q_n}(\Phi^{-1}(\Phi(q_n)) \cap S_j)$ converge to T. Since Φ has an isolated singular point relatively to $\mathcal{S}$ at p_0, the tangent space $T_q(\Phi^{-1}(\Phi(q)) \cap S_i)$ is the intersection of the spaces $T_q(F^{-1}(F(q)) \cap S_i)$ and $\{l = l(q)\}$ which are transverse in $\mathbb{C}^N$.

Similarly the space $T_{q_n}(\Phi^{-1}(\Phi(q_n)) \cap S_j)$ equals:

$$T_{q_n}(F^{-1}(F(q_n)) \cap S_j) \cap \{l = l(q_n)\}.$$

We may extract a subsequence of (q_n) and assume that the sequence $T_{q_n}(F^{-1}(F(q_n)) \cap S_j)$ has a limit $\mathcal{T}$. Since the stratification $\mathcal{S}$ satisfies Thom condition for F, we have:

$$T_q(F^{-1}(F(q)) \cap S_i) \subset \mathcal{T}.$$

Since $\{l = l(q)\}$ intersects transversally $T_q(F^{-1}(F(q)) \cap S_i)$ in $\mathbb{C}^N$, it intersects $\mathcal{T}$ transversally in $\mathbb{C}^N$. This implies:

$$T_q(\Phi^{-1}(\Phi(q)) \cap S_i) \subset \mathcal{T} \cap \{l = l(q)\},$$

which shows that in V the stratification $\mathcal{S}$ induces a Thom stratification for Φ.

Remark. Because of Hironaka Theorem 5.1, in fact one can show that it is not needed to suppose that F has isolated singularities relatively to the stratification $\mathcal{S}$ (see [10] p.399 §2).

7.3. Generalized discriminant

We keep the notations of the preceding section. Then, we have:

Proposition 7.4. *There are open neighborhoods $U(p_0)$ and $V(p_0)$ of p_0 in $E \cap U$ and $\Phi(p_0)$ in $\mathbb{C}^2$ such that the map Φ induces a map $\Phi_{p_0} : U(p_0) \to V(p_0)$ such that the restriction of Φ_{p_0} to $\Gamma(l) \cap U(p_0)$ is a complex analytic finite map, and the map Φ_{p_0} induces a locally trivial topological fibration of $U(p_0) \setminus \Phi_{p_0}^{-1}(\Phi_{p_0}(\Gamma(l) \cap U(p_0)))$ onto:*

$$V(p_0) \setminus \Phi_{p_0}(\Gamma(l) \cap U(p_0)).$$

As we have seen before, Corollary 7.3 implies that there are neighborhoods U and V of p_0 in E and $\Phi(p_0)$ in $\mathbb{C}^2$ such that Φ induces a finite map $U \cap \Gamma(l) \to V$, therefore, the image $\Delta(l) = \Phi_{p_0}(\Gamma(l) \cap U)$ is a complex analytic curve of V (see e.g. [3] Proposition p. 65).

By definition inside U the restriction of Φ to any strata S_i has no critical point at any point of $\Phi^{-1}(y)$ with $y \in V \setminus \Delta(l)$. Let $\mathbb{S}_\varepsilon(p_0)$ be the sphere of $\mathbb{C}^N$ centered at the point p_0 with radius ε, such that the trace $\mathbb{B}_\varepsilon(p_0) \cap E$ in E of the ball $\mathbb{B}_\varepsilon(p_0)$, that the sphere bounds, is contained in U and any sphere $\mathbb{S}_{\varepsilon'}(p_0)$ of radius $\varepsilon \geq \varepsilon' > 0$ is transverse in $\mathbb{C}^N$ to $\Phi^{-1}(\Phi(p_0)) \cap (S_i \cap U)$, for $1 \leq i \leq k$.

Since the fiber $\Phi^{-1}(\Phi(p_0))$ has an isolated singularity at p_0 relatively to the stratification $\mathcal{S}$, there is $\eta > 0$ such that, for any $y \in \mathring{\mathbb{B}}_\eta(\Phi(p_0))$, for any i, $1 \leq i \leq k$, the space $\Phi^{-1}(y) \cap S_i$ is transverse in $\mathbb{C}^N$ to $\mathbb{S}_\varepsilon(p_0)$.

Define $U(p_0) = \mathring{\mathbb{B}}_\varepsilon(p_0) \cap E \cap \Phi^{-1}(\mathring{\mathbb{B}}_\eta(\Phi(p_0)))$ and $V(p_0) = \mathring{\mathbb{B}}_\eta(\Phi(p_0))$ and the map Φ induces a map:

$$\Phi_{p_0} : U(p_0) \to V(p_0).$$

The sets

$$S_i \cap \mathring{\mathbb{B}}_\varepsilon(p_0) \cap \Phi^{-1}(\mathring{\mathbb{B}}_\eta(\Phi(p_0)) \setminus \Delta(l))$$

and

$$S_i \cap \mathbb{S}_\varepsilon(p_0) \cap \Phi^{-1}(\mathring{\mathbb{B}}_\eta(\Phi(p_0)) \setminus \Delta(l))$$

define a Whitney stratification $\mathcal{R} = (R_i)$ of $\mathbb{B}_\varepsilon(p_0) \cap E \cap \Phi^{-1}(\mathring{\mathbb{B}}_\eta(\Phi(p_0))) \setminus \Delta(l))$ (it is easy to prove that pairs of strata such that $R_i \subset \overline{R}_j$ satisfy Whitney condition (b), then, corollary 10.5 of [16] gives the frontier property) and Φ induces a map:

$$\varphi : \mathbb{B}_\varepsilon(p_0) \cap E \cap \Phi^{-1}(V(p_0) \setminus \Delta(l)) \to V(p_0) \setminus \Delta(l).$$

Since $\mathbb{B}_\varepsilon(p_0)$ is compact, the map φ is proper and we have defined $\Gamma(l)$ and $\Delta(l)$ such that it has maximal rank on the Whitney strata of:

$$\mathbb{B}_\varepsilon(p_0) \cap E \cap \Phi^{-1}(\mathring{\mathbb{B}}_\eta(\Phi(p_0)) \setminus \Delta(l)).$$

Thom-Mather first isotopy Theorem implies that φ is a locally trivial topological fibration. Therefore, this fibration induces a locally trivial topological fibration of $U(p_0) \cap \Phi^{-1}(V(p_0) \setminus \Delta(l))$ over $V(p_0) \setminus \Delta(l)$.

Then, one may stratify the map $\Phi_{p_0} : U(p_0) \to V(p_0)$ by considering on $U(p_0)$ the strata obtained from:

$$S_i \cap U(p_0) \cap \Phi^{-1}(\Delta(l)) \setminus \Gamma_i,$$

$(\Gamma_i \setminus \{p_0\}) \cap U(p_0)$ and $\{p_0\}$ and on $V(p_0)$ the strata obtained from $V(p_0) \setminus \Delta(l)$, $\Phi((\Gamma_i \setminus \{p_0\}) \cap U(p_0))$ and $\{\Phi(p_0)\}$. These stratifications are Whitney stratifications since $\mathcal{S}$ is a Whitney stratification of E and $V(p_0)$ is non-singular.

Locally, the set $\Delta(l) \cap V(p_0)$ plays the role of a discriminant of the complex analytic map Φ_{p_0}. We shall call it the *topological discriminant* of Φ relatively to the stratification $\mathcal{S}$.

Note: One can define by induction a map $\Phi_{r+1} = (\ell_1, \ldots, \ell_r, F) : E \to \mathbb{C}^{r+1}$ for $r \leq \dim E$ where $\ell_1, \ldots, \ell_r$ are restrictions to E of general linear forms of $\mathbb{C}^N$. For any point p of E, the map Φ_{r+1} induces a map of an open neighborhood $U(p_0)$ of p_0 in E into an open neighborhood $V(p_0)$ of $\Phi(p_0)$ in $\mathbb{C}^{r+1}$.

The union of the images by Φ_{r+1} of the closures in E of the critical loci of Φ_{r+1} restricted to the strata S_i would locally define a *topological discriminant* $\Delta \cap V(p_0)$ of Φ_{r+1} relatively to $\mathcal{S}$.

If we suppose that the function F is itself induced by a general linear form l_0, we can define a *general projection* of E into $\mathbb{C}^{r+1}$ at the point p_0.

§8. Vanishing Euler characteristics

8.1. Local Vanishing fibers

Let E be a complex algebraic subset of $\mathbb{C}^N$ and p_0 be a point of E.

At the point p_0 we have defined a general projection $\pi_r : E \to \mathbb{C}^r$ for any r, $1 \leq r \leq \dim E$. At the point p_0 this general projection defines a map germ:

$$\pi_r : (E, p_0) \to (\mathbb{C}^r, \pi_r(p_0))$$

and a germ of topological discriminant $(\Delta(\pi_r), \pi_r(p_0))$. The map germ π_r induces a map:

$$p_r : U(p_0) \to V(p_0)$$

from an open neighborhood $U(p_0)$ of p_0 in E into an open neighborhood $V(p_0)$ of $\pi_r(p_0)$ in $\mathbb{C}^r$ and defines a topological discriminant Δ_r which is a closed subset of $V(p_0)$.

The map p_r induces a locally trivial topological fibration of $U(p_0) \setminus p_r^{-1}(V(p_0) \cap \Delta_r)$ onto $V(p_0) \setminus \Delta_r$. If the projection π_r is sufficiently general, in [15] we prove that the homotopy type of the fiber of this fibration does not depend on the projection π_r (see [15] (3.2.2) p. 88). Considering a sufficiently general projection, the fiber of the corresponding locally trivial fibration is called a *vanishing fiber* of dimension $\dim E - r$ of E at the point p_0.

The Euler characteristic $\chi_{\dim E - r}(E, p_0)$ of a vanishing fiber of dimension $\dim E - r$ at the point p_0 is called the *vanishing Euler characteristic* of E at p_0 of dimension $\dim E - r$ (see [15] §3).

At any point p of a complex algebraic set E we define a $(\dim E)$-uple:

$$\chi_*(E, p) = (\chi_1(E, p), \ldots, \chi_{\dim E}(E, p)).$$

We may call $\chi_*(E, p)$ the *vanishing characteristics* of E at p.

8.2. Combinatorial characterization of Whitney conditions

We can formulate the following characterization of Whitney conditions (see [15] Théorème (5.3.1)):

Theorem 8.1. *Let E be a complex algebraic subset of $\mathbb{C}^N$. Let $\mathcal{S} = (S_1, \ldots, S_k)$ be a stratification of E. Suppose that, for any pair (S_i, S_j) such that $S_i \subset \overline{S}_j$, the vanishing characteristics $\chi_*(\overline{S}_j, p)$ are constant for $p \in S_i$, then the stratification $\mathcal{S}$ is a Whitney stratification.*

By comparison with the criterion given above by Teissier of a Whitney stratification this theorem suggests that there is a relation between Vanishing characteristics and Polar varieties.

This relation exists when one considers a Whitney stratification $\mathcal{S} = (S_1, \ldots, S_k)$ of E.

It is convenient to write $\chi_{\dim E - \dim S_i - 1}(E, S_i)$ instead of:

$$\chi_{\dim E - \dim S_i - 1}(E, p)$$

when $p \in S_i$ or $\chi_{\dim S_j - \dim S_i - 1}(\overline{S}_j, S_i)$, whenever $S_i \subset \overline{S}_j$, for:

$$\chi_{\dim S_j - \dim S_i - 1}(\overline{S}_j, p)$$

when $p \in S_i$, since one can prove that, using Thom-Mather first isotopy theorem, the topology of the vanishing fibers of E (or $\overline{S}_j$) along S_i does not vary.

In particular we have (see [15] Théorème (4.1.1)):

Theorem 8.2. *Let E be an equidimensional complex algebraic subset of $\mathbb{C}^N$. Let $\mathcal{S} = (S_1, \ldots, S_k)$ be a Whitney stratification of E. Let*

us denote $d_i = \dim S_i$. Let $x \in S_i$. We have:

$$\chi_{\dim E - d_i - 1}(E, S_i) - \chi_{\dim E - d_i - 2}(E, S_i)$$
$$= \sum_{j \neq i} (-1)^{d_j - d_i - 1} m(P_{d_j - d_i - 1}(\overline{S}_j, x), x)(1 - \chi_{\dim E - d_j - 1}(E, S_j))$$

where $m(P, x)$ is the multiplicity of P at x and $d_j - d_i - 1$ is the codimension of the polar variety $P_{d_j - d_i - 1}(\overline{S}_j, x)$.

The study of Whitney stratification leads, on the algebraic geometry side, to the study of Polar Varieties and, on the topological side, to Vanishing Fibers or Vanishing Euler characteristics.

8.3. Analytic case

As we have mentioned above most of the results quoted here in the algebraic context are valid in the complex analytic situation.

The definition of the Vanishing fibers is somehow the same in the complex analytic category and they are associated to germs of complex analytic sets.

The reader is encouraged to see how the results formulated above generalize in this context.

§9. Bouquet of spheres

9.1. Statement of the theorem

In some situations, results can be formulated concerning the Vanishing Fibers associated to germs of complex algebraic sets.

The most surprising result is (see [13]):

Theorem 9.1. *The vanishing Fibers of a germ of complex analytic complete intersection have the homotopy type of bouquets of real spheres of dimensions equal to the complex dimension of each Vanishing Fiber.*

Just notice that this is a theorem similar to the result of J. Milnor in [18] (Theorem 6.5) or to the one of H. Hamm for the case of complete intersections ([4]) with isolated singlarities, but ours also concerns non-isolated singularities of complete intersections..

9.2. Hypersurfaces

In this section we shall first assume that E is a complex hypersurface defined by the polynomial function $f : \mathbb{C}^{n+1} \to \mathbb{C}$. In this section f might have non-isolated singularities.

We have seen that we can stratify the map f with Whitney stratifications satisfying Thom condition (see above Theorem 5.1 by Hironaka).

Let l be a general linear form of $\mathbb{C}^{n+1}$. It defines a map:

$$\Phi = (l, f) : \mathbb{C}^{n+1} \to \mathbb{C}^2.$$

Let p_0 be a point of E. Consider the germ of Φ at p_0. The critical locus of the germ of Φ at p depends linearly on l. So, when l varies the critical loci of Φ define a linear system. Since the ambient space is non-singular, a classical theorem of Bertini (see e. g. [6] Corollary 10.9 p. 274) implies that the singular points of a general member of this linear system lie in the fixed points of the linear system which are precisely the space of critical points of f at p_0.

Therefore in a sufficiently small neighborhood U of p_0 in $\mathbb{C}^{n+1}$ the critical space of Φ is non singular outside the hypersurface E. The closure in U of this critical space is empty or a curve Γ that we call the relative polar curve of f relatively to the linear form l.

As we have done in §7, we have:

Lemma 9.2. *If the linear form l is sufficiently general, there are $1 \gg \varepsilon \gg \eta > 0$ such that the map Φ induces:*

$$\Phi_{\varepsilon, \eta} : \mathbb{B}_\varepsilon(p_0) \cap \Phi^{-1}(\mathring{\mathbb{B}}_\eta(\Phi(p_0))) \to \mathring{\mathbb{B}}_\eta(\Phi(p_0))$$

which is a stratified map with Thom and Whitney conditions.

Proof. Consider a sufficiently small open neighborhood of p_0 in $\mathbb{C}^{n+1}$. Let $\mathcal{S} = (S_1, \ldots, S_k)$ be a Whitney stratification of U such that $S_1 = U - E$ and $E = f^{-1}(0) \cap U$ is a union of strata. We assume that $\mathcal{S}$ is also a Thom stratification for f. We know that such a stratification exists by Theorem 5.1

Let us choose l such that the critical space of Φ as we defined above is non-singular outside E and such that the restriction of l to E has an isolated singularity at p_0 relatively to the stratification $\mathcal{S}$.

As we showed above, there is a neighborhood $\mathcal{U}$ of p_0 such that the restrictions of l to the spaces $S_i \cap \mathcal{U} - \{p_0\}$ have a maximal rank. We are going to show that it induces a Thom stratification of the restriction $\Phi | \mathcal{U} - \{p_0\}$, i.e. the intersections of the strata of $\mathcal{S}$ with $\mathcal{U} - \{p_0\}$ is a Whitney stratification of $\mathcal{U} - \{p_0\}$ which is also a Thom stratification for $\Phi | \mathcal{U} - \{p_0\}$.

Let (q_n) be a sequence of points $S_j \cap \mathcal{U} - \{p_0\}$ which tends to:

$$q \in S_i \cap \mathcal{U} - \{p_0\},$$

such that the sequence of tangent spaces $T_{q_n}(f|S_j)^{-1}(f(q_n))$ converges to T. By hypothesis on l the tangent space $T_{q_n}(\Phi|S_j)^{-1}(\Phi(q_n))$ equals $T_{q_n}(f|S_j)^{-1}(f(q_n)) \cap \{l = l(q_n)\}$. Since the stratification (S_i) is a Thom

stratification for f, it induces a Thom stratification for Φ on $\mathcal{U}$. Then, we have:

$$T_q(S_i) \subset T$$

and the restriction of l to $E \cap \mathcal{U}$ having an isolated singularity at p_0 (and in fact unique singularity) relatively to the stratification $(S_i \cap \mathcal{U})$, the tangent space $T_q(S_i)$ is transverse to $\{l = l(q)\}$ in $\mathbb{C}^{n+1}$ and:

$$T_q((\Phi|S_i)^{-1}(\Phi(q))) = T_q(S_i) \cap \{l = l(q)\}.$$

Since T contains $T_q(S_i)$, it is also transverse to $\{l = l(q)\}$. in $\mathbb{C}^{n+1}$. Therefore the limit of the tangent spaces:

$$T_{q_n}((\Phi|S_j)^{-1}(\Phi(q_n)) = T_{q_n}(f^{-1}(f(q_n))) \cap \{l = l(q_n)\}$$

is $T \cap \{l = l(q)\}$ and contains the tangent space:

$$T_q((\Phi|S_i)^{-1}(\Phi(q))) = T_q(S_i) \cap \{l = l(q)\}.$$

Now by choosing $\varepsilon > 0$ and $\eta > 0$ properly such that:

$$\mathbb{B}_\varepsilon(p_0) \cap \Phi^{-1}(\mathring{\mathbb{B}}_\eta(\Phi(p_0))) \subset \mathcal{U}$$

the map $\Phi_{\varepsilon,\eta}$ induced by Φ is a stratified map with Whitney and Thom conditions.

Let Δ be the union of the image of the complex curve $\Gamma \cap \mathbb{B}_\varepsilon(p_0) \cap \Phi^{-1}(\mathring{\mathbb{B}}_\eta(\Phi(p_0)))$ by $\Phi_{\varepsilon,\eta}(p_0)$ and the trace in $\mathring{\mathbb{B}}_\eta(\Phi(p_0))$ of the line $\mathbb{C} \times \{f(p_0)\}$:

$$\Delta := [\Phi_{\varepsilon,\eta}(p_0)(\Gamma \cap \mathbb{B}_\varepsilon(p_0) \cap \Phi^{-1}(\mathring{\mathbb{B}}_\eta(\Phi(p_0))))] \cup [(\mathring{\mathbb{B}}_\eta(\Phi(p_0)) \cap \mathbb{C} \times \{f(p_0)\})].$$

As we have done in §7, we can prove that the map $\Phi_{\varepsilon,\eta}$ induces a locally trivial smooth fibration over $\mathring{\mathbb{B}}_\eta(\Phi(p_0)) \setminus \Delta$.

In particular the fibers of $\Phi_{\varepsilon,\eta}$ are transverse to the boundary $\mathbb{S}_\varepsilon(p_0)$ of $\mathbb{B}_\varepsilon(p_0)$.

Notice that the germ of the critical locus of f at p_0 which might be of dimension ≥ 1 has its image by the germ of Φ at p contained in $\mathbb{C} \times \{f(p_0)\}$.

Since the linear form l is general, if it is not empty, the space $\Gamma \setminus \{p_0\}$ is non-singular, which means that the fiber $\Phi_{\varepsilon,\eta}^{-1}(\Phi(p'))$ has an ordinary quadratic point at $p' \in \Gamma \setminus \{p_0\}$.

For t sufficiently small, since l is a sufficiently general linear form, the fiber of $\Phi_{\varepsilon,\eta}$ above the point $(l(p_0) + t, f(p_0))$, is a vanishing fiber of

dimension $\dim E - 1$ of E at p_0. Let us call (u, v) the coordinates of $\mathbb{C}^2$ such that:

$$u = l \text{ and } v = f.$$

Consider the line $u = l(p_0) + t$. By using a generalization of Ehresmann Lemma to manifolds with corners, one can prove that the space of complex dimension n:

$$\Phi_{\varepsilon,\eta}^{-1}(\{u = l(p_0) + t\} \cap \mathring{\mathbb{B}}_\eta(\Phi(0)))$$

is homeomorphic to $\Phi_{\varepsilon,\eta}^{-1}(\{u = l(p_0)\} \cap \mathring{\mathbb{B}}_\eta(\Phi(p_0)))$ so it is contractible and the space:

$$\Phi_{\varepsilon,\eta}^{-1}(\mathbb{D})$$

where $\mathbb{D}$ is a small disc of radius r of the line $\{u = l(p_0) + t\}$ centered at $(l(p_0) + t, f(p_0))$ retracts by deformation on the vanishing fiber $\Phi_{\varepsilon,\eta}^{-1}((l(p_0) + t, f(p_0)))$ of E at p_0.

The restriction ψ of $|f|$ to the space $\Phi_{\varepsilon,\eta}^{-1}(\{u = l(p_0)+t\} \cap \mathring{\mathbb{B}}_\eta(\Phi(p_0)))$ defines a smooth function:

$$\psi : \Phi_{\varepsilon,\eta}^{-1}(\{u = l(p_0) + t\} \cap \mathring{\mathbb{B}}_\eta(\Phi(p_0))) \to \{u = l(p_0) + t\} \cap \mathring{\mathbb{B}}_\eta(\Phi(p_0)).$$

Let us start with the value r, i.e. equal to the radius of $\mathbb{D}$. As the critical values of $|f|$ are given by the intersection points of the line ($\{u = l(p)+t\}$) with Δ, the function ψ induced by $|f|$ has Morse points, because the quadratic form of its second derivatives of ψ is up to a constant equal to the real part of a non-degenerate complex quadratic form (see [9] p. 30). Furthermore, the Morse indices are equal to n.

We have the lemma:

Lemma 9.3. *A CW-complex which becomes a space homotopically equivalent to a bouquet of real spheres of dimension n after attaching cells of dimension n is homotopically equivalent to a bouquet of real spheres of dimension $n - 1$.*

Therefore, the space $\Phi_{\varepsilon,\eta}^{-1}(\mathbb{D})$ is homotopically equivalent to a bouquet of real spheres of dimension $n - 1$. In fact, since the space of complex dimension n:

$$\Phi_{\varepsilon,\eta}^{-1}(\{u = l(p_0) + t\} \cap \mathring{\mathbb{B}}_\eta(\Phi(p_0)))$$

is contractible, the space $\Phi_{\varepsilon,\eta}^{-1}(\mathbb{D})$ is homotopically equivalent to a bouquet of μ spheres of dimension $n - 1$, where μ is the number of points of $\Gamma \cap \{u = l(p_0) + t\} \cap \Phi^{-1}(\mathring{\mathbb{B}}_\eta(\Phi(p_0)))$.

Since this space retracts by deformation to the vanishing fiber

$$\Phi_{\varepsilon,\eta}^{-1}((l(p_0) + t, f(p_0))),$$

this vanishing fiber is also homotopically equivalent to a bouquet of μ real spheres of dimension $n - 1$.

Thus, we have proved that the Milnor fiber at $p_0 \in E$ of the restriction of a general linear form to E has the homotopy type of a bouquet of real spheres of dimension $n-1$, in fact, we also prove that the number of spheres is μ. This Milnor fiber is also one of the vanishing fibers at the point p_0. Since we can consider the other vanishing fibers to be Milnor fibers of the restriction of a general linear form to a lower dimensional hypersurface, we have proved the Theorem 9.1 for hypersurfaces.

Notice that the proof given above can be adapted to the case of complex analytic hypersurfaces.

9.3. Complete intersections

In the case of complex algebraic complete intersections, suppose that E is defined by $f_1 = \ldots = f_s = 0$. We may replace the equations $f_1, \ldots, f_s$ by general linear combinations of the complex algebraic functions $f_1, \ldots, f_s$. Then, E is defined by f_s on the complete intersection $E_1 = \{f_1 = \ldots = f_{s-1} = 0\}$. By replacing the original equations by general linear combinations, Bertini theorem implies that the singular points of E_1 lie in the fixed points of the linear system $\sum_{i=1}^{s} \lambda_i f_i$, i.e. in $E = \{f_1 = \ldots = f_s = 0\}$.

Therefore $E_1 \setminus E$ is non-singular and we can repeat a similar argument as the one we have developed above for hypersurfaces. The Lemma 9.3 will show that by attaching cells of dimension $\dim E_1 - 1$ the Milnor fiber of a general linear form restricted to E at p gives the Milnor fiber of a general linear form restricted to E_1 at p.

Since, by induction on the number of equations, we can suppose that the Milnor fiber of a general linear form restricted to E_1 at p has the homotopy type of a bouquet of real spheres of dimension $\dim E_1 - 1$, the Milnor fiber of a general linear form restricted to E at p_0 has the homotopy type of a bouquet of spheres of dimension $\dim E - 1 = \dim E_1 - 2$. The number of spheres of the Milnor fiber of a general linear form restricted to E_1 at p plus the number of spheres of the Milnor fiber of a general linear form restricted to E at p_0 equal the intersection multiplicity μ at p_0 of the relative polar curve of f_s and $l = l(p_0)$.

We leave details to the reader (see [13]).

Problems:

Is it possible that the number of spheres of a Vanishing fiber for a complete intersection is equal to 0?

Is the number μ equal to the multiplicity of the relative polar curve of f_s?

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