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Models of a sudden directional diffusion

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Abstract.

We study degenerate and singular parabolic equations in one space dimension. The emphasis is put on the regularity of solutions and the creation as well as the evolution of facets. Facets are understood as flat parts of the graph of solutions being a result of extremely high singularity. The systems, which we consider, arise from the theory of crystals.

§1. Introduction

Anisotropy is a key feature of crystals. Their structure is underlined by a few directions minimizing the surface energy $\gamma : \mathbb{S}^{n-1} \to \mathbb{R}$. This energetic approach yields a law of evolution of a single crystal, the steepest descent process gives the weighted mean curvature flow, see [31], [32],

$$V = \kappa_\gamma \quad \text{on} \quad \Gamma(t), \tag{1}$$

where $\Gamma(t)$ is the surface of an evolving crystal, V is the normal speed of the free interface $\Gamma(t)$. The key element is the weighted curvature κ_γ, encoding the anisotropy. In general, this quantity is defined as follows

$$\kappa_\gamma = -\operatorname{div}_{\Gamma(t)}(\nabla_\xi \bar{\gamma}|_{\xi=\mathbf{n}}), \tag{2}$$

where $\bar{\gamma} : \mathbb{R}^n \to \mathbb{R}$ is the 1-homogeneous extension of γ, i.e.

$$\bar{\gamma}(x) = |x|\gamma(\frac{x}{|x|}) \text{ if } x \neq 0, \qquad \bar{\gamma}(0) = 0.$$

In addition, $\mathbf{n}(t)$ denotes the normal vector to $\Gamma(t)$.

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Here, we restrict our attention to the one-dimensional case. This restriction permits us to study two dimensional phenomena, i.e. our crystal occupies a planar domain. Then (2) transforms into a simpler expression

$$\kappa_\gamma = -\tau \cdot \partial_s[\nabla\bar{\gamma}|_{\xi=\mathbf{n}}],$$

where τ is a unit positively oriented tangent vector and s is the arc unit length parameter of the curve $\Gamma(t)$.

The mathematical theory of systems like (1) is still not complete, even for one-dimensional models. The main problems are of a purely analytical character and they are obstacles for the development of the theory. Hence, in order to make progress, we remove geometrical context from system (1). Such simplification leads to the following parabolic system

$$u_t - \frac{\partial}{\partial x}L(u_x) = 0, \tag{3}$$

where $L(\cdot)$ is merely monotone, and the unknown u represents the surface of crystal via its graph.

We will here briefly present an argument how eq. (1) leads to (3). Easy geometric considerations show that for the graph of function u eq. (1) becomes

$$\frac{u_t}{\sqrt{1+(u_x)^2}} = -\frac{1}{1+(u_x)^2}\,(1,u_x)\cdot\frac{\partial}{\partial x}\left(\nabla_\xi\gamma(\xi)|_{\xi=(-u_x,1)}\right),$$

where we used the 1-homogeneity of γ. After writing this equation in a more concise and abstract form we will reach,

$$u_t - a(u_x)\frac{\partial}{\partial x}L(u_x) = 0, \tag{4}$$

where monotonicity of L is a consequence of convexity of γ. After artificially having set $a(p) = 1$ in (4) we reach (3).

Basic questions, concerning the well posedness of weak solutions to (3), are well known. There are no obstacles to show the existence of unique weak/variational solutions. In the mono-dimensional case we are able to show a higher regularity of solutions. However, the understanding of qualitative features of solutions related to the singularity and the degeneration of L is still incomplete. We shall keep in mind that the same questions for the original problem (1) remain open. The understanding of all features of (3) shall give us hints to deal with (1).

The goal of this note is to present the results of the program carried out by the authors. It is devoted to the studies of models exploiting what we call a *sudden directional diffusion.* Roughly speaking these are systems (3) with $L(\cdot)$ having at least one jump. The set of jumps of $L(\cdot)$ defines the *slopes* of u which diffuse so strongly that this effect is non-local. Here, we concentrate just only on one-dimensional problems. The proofs of results, exposed here, are contained in our papers, [18], [25], [26], [27], [28], [29].

We wish also to highlight the research directions, which we find interesting. We mainly investigate systems like (3) with $L(\cdot)$ suffering jumps and having growth at infinity of no faster order than linear. We distinguish two groups of problems characterized by degeneration:

♣ the first one concerns systems with $L'(\cdot) \geq 0$, where the degeneracy set $\{L' = 0\}$ appears and plays a significant role and

♣ the second group concerns non-degenerate systems with $L'(\cdot) \geq c > 0$.

Key objects of our study are facets, i.e., flat regions of solutions with slopes that are related to the jumps of L. We expect them in both cases.

We shall point out that considering non-monotone L is quite natural. There are some attempts to study such cases, however, the known results show high instability, [17]. It is possible to study a higher order regularization of non-monotone L. It leads to an interesting theory, [7].

The organization of the paper is as follows. In Section 2, we state the existence theorem for the systems like (3). Next, in Section 3, we consider a simplification of flow (1) and mention the notion of *almost classical solutions.* In Section 4, we discuss a one-dimensional total variation flow. The definition of the almost classical solution for the equation (3) is stated too. We also point out the quadratic function should not be regarded as a regular one. Finally, we consider a total variation flow with an additive linear diffusion. We show that despite the singular term, the solution to this system enjoys the regularity typical for parabolic problems. On top of that the competing types of diffusion lead to the instantaneous facet formation which cannot disappear.

§2. The basic existence

Our main goal is to construct solutions with the best possible regularity. However, the classical spaces, like BV, do not suffice, and a description of different function structures is necessary. A consistent theory for a particular simple equation has been stated in [18]. Let us review it now. Since we are working in one dimension, then the basic regularity, stated in the classical setting, is relatively high. In [26], we have shown:

Theorem 1. *Let us suppose that $L(\cdot)$ is merely non-decreasing. We consider the initial value problem for equation (3) over an interval $I \subset \mathbb{R}$ with the zero Dirichlet data. Assume that initial datum u_0 is in $L_1(I)$ and in addition $u_{0,x} \in BV(I)$. Then, there exists a unique solution to problem (3) such that*

$$u_x \in L_\infty(0,T; BV(I)) \tag{5}$$

and there exists $\Omega \in L_2(I \times (0,T))$ such that the following identity is valid

$$(u_t, \phi)_{L_2} + (\Omega, \phi_x)_{L_2} = 0 \quad \text{for a.e. } t > 0$$

and all $\phi \in C_c^\infty(I \times [0,T))$ and

$$\Omega(x,t) \in L \circ u_x(x,t) \text{ a.e. in } I \times (0,T), \tag{6}$$

where the composition in the last line is defined for multivalued functions in the standard set way.

We recall that we used a standard notation, $(f,g)_{L_2} = \int_I fg\,dx$ and

$$\|f\|_{BV(I)} = \sup_{0=s_0<s_1<\ldots<s_N=1} \sum_{k=1}^{N} |f(s_k) - f(s_{k-1})|,$$

where the supremum taken over all decompositions of I, $s_0, \ldots, s_N$ and all $N \in \mathbb{N}$. In the case of smooth f, we just have

$$\|f\|_{BV(I)} = \int_I |f'(s)|ds.$$

In order to explain (5), we note that the map $u_x : (0,T) \mapsto BV(I)$ is required to be only weakly measurable, the strong topology is not adequate for considerations in this type of spaces. In our analysis functions from BV are mostly treated as multivalued objects such that at their discontinuous points the value is a closed interval described by the left- and right-hand side limits.

Theorem 1 is just the beginning of investigations of the regularity issue. The key point is the meaning of the solution, which in our case is related to determining (6). In general, we want to capture such a good notion of regularity that solutions enjoying it could be called *almost classical* ones. Despite the fact that our considerations are in one dimension, for general L it is too difficult to capture all interesting phenomena. This is why we concentrate our attention on particular cases of L.

§3. Caricature of a singular curvature flow

We introduced a concept of the almost classical solutions in [25]. In this reference we considered a simplification of the singular flow (1) with the interfacial energy function γ, described by a square. In other words, we adopt a square as the geometry of the unit ball. Precisely, the problem, we studied, reads:

$$(7)\qquad \begin{array}{llr} \Lambda_t = \frac{\partial}{\partial s}\frac{d}{d\phi}J(s+\Lambda_s) & \text{over} & S\times(0,T), \\ \Lambda|_{t=0} = \Lambda_0 & \text{on} & S, \end{array}$$

where S denotes the unit circle parametrized by the angle and Λ is the unknown function. In order to understand system (7), we compare it to (1). We assume that

$$\phi = \Lambda_s + s$$

represents the angle between the axis x_1 and the outer normal vector to the curve. Of course, this simplification of the geometry is so radical that we call (7) 'a caricature of the flow' for (1). Function J corresponds to the surface energy density and it is given as follows,

$$(8)\qquad J(\phi) = \frac{\pi}{4}\big(|\phi - \frac{3\pi}{4}| + |\phi - \frac{\pi}{4}| + |\phi + \frac{\pi}{4}| + |\phi + \frac{3\pi}{4}|\big).$$

We removed all geometrical terms related to the change of the length of the evolving curve. The structure of (8) implies that the optimal shape is the boundary of a square.

Here is the main result, stated in [25].

Theorem 2. *Let the initial datum Λ_0 fulfill the following conditions: $\partial_s\Lambda_0|_{t=0}\in BV(S)$, and the set Ξ_0 defined below has the following structure,*

$$\begin{aligned} &\Xi_0 \\ &= \{s\in S=[0,2\pi) : \tfrac{k\pi}{4}\in\partial_s\Lambda_0(s)+s \text{ for } k=-3,-1,1,3\} \\ &= \textstyle\bigcup_{i=1}^N [a_i,b_i]. \end{aligned}$$

We assume that $a_i\le b_i$ for $i\neq N$ and $0<a_1\le b_i<a_{i+1}$ for $i=1,\dots,N-1$ and for $i=N$ by $[a_N,b_N]$ we denote either $[a_N,b_N]$ provided that $a_N\le b_N$ or $[a_N,2\pi)\cup[0,b_N]$ if $b_N<a_N$.
In other words, we require that Ξ_0 consists of a finite number of closed intervals or isolated points.

Then, there exists a unique solution such that

$$\Lambda_s \in L_\infty(0,T;BV(S))$$

and

$$\begin{aligned}&\Xi(t)\\ &= \{s \in S = [0,2\pi) : \tfrac{k\pi}{4} \in \Lambda_s(s,t) + s \textit{ for } k = -3,-1,1,3\}\\ &= \textstyle\bigcup_{i=1}^{N(t)} [a_i(t), b_i(t)],\end{aligned}$$

where the number of components of $\Xi(t)$ *is a decreasing function of time.*

We call such solutions <u>*almost classical*</u> *ones, because for all but finitely many* $t > 0$ *they fulfill the identity*

$$\Lambda_t = \frac{d}{ds}\partial J \bar{\circ} [\Lambda_s + s] \textit{ everywhere in } S \textit{ except for a finite set.} \tag{9}$$

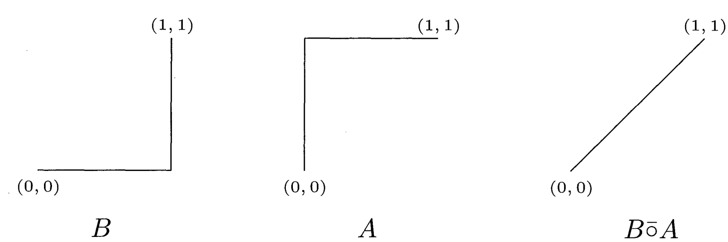
To determine the meaning of the almost classical solutions, we introduce the definition of the composition $\bar{\circ}$. It allows us to define the RHS (right hand side) of (9) in a pointwise manner. In particular, $\partial J \bar{\circ} [\Lambda_s + s]$ is a single valued function almost everywhere. The idea of the composition $\bar{\circ}$ is based on the following requirement: the composition of any monotone (multivalued) function with its inverse must be the identity. To illustrate the idea of $\bar{\circ}$, we pinpoint the following fundamental example, which is not canonical in the sense that B is not any subdifferential.

Let $B, A : [0,1] \to [0,1]$ and $B^{-1} = A$ be such that

$$B(t) = \begin{cases} 0, & t \in [0,1), \\ [0,1], & t = 1, \end{cases} \qquad A = \begin{cases} [0,1], & t = 0, \\ 1, & t \in (0,1]. \end{cases}$$

Then, we have

$$B \bar{\circ} A(t) = t.$$



The composition $\bar{\circ}$ of A and B

For the sake of self-consistency we define our new composition $\bar{\circ}$ in the Appendix. Thus, we will be able to complete the definition of the almost classical solution (9).

The result, contained in Theorem 2, permits us to draw several conclusions on closed curves evolving by (7). The main qualitative piece of information is that any sufficiently regular initial curve will eventually reach a 'minimal solution'. The geometric interpretation is that the solution reaches its asymptotic shape, i.e. the square in our case. This behavior can be illustrated by the pictures below.

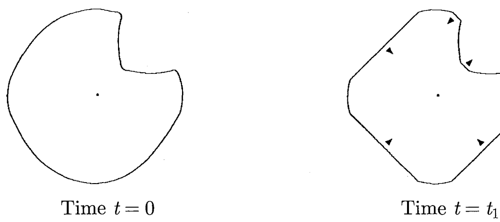


Fig. 1. Time: $t = 0, t = t_1$

The evolution is determined by the motion of facets defined by singularities of the J-function (the arrows in the above picture show the direction of the evolution). In finite time we obtain a convex domain (at time $t = t_3$ in the next picture), which becomes a square converging to a point in finite time (at time $t = t_{end}$ in the next picture).

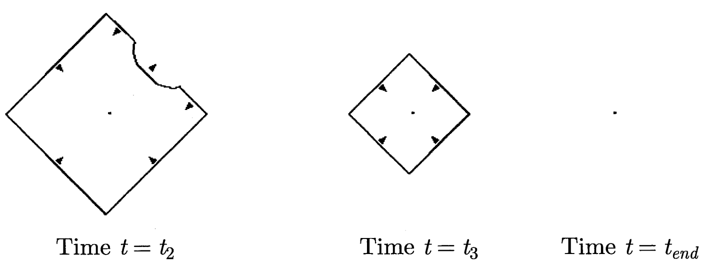


Fig. 2. Time: $t = t_2, t = t_3,\ t = t_{end}$

We want to underline that the evolution, highlighted above, hinges on a novel idea to define the singular term $\delta_0(u_x)u_{xx}$, which is in principle a product of two Dirac deltas. The non-local character of this expression will allow us to define it correctly. In addition, by the uniqueness of solutions to our system, we show that our new definition is the only admissible one. Roughly speaking, the dissipation, caused by the Dirac delta coefficient, is so strong that the changes of regularity (i.e. appearance of the facets) happen instantly.

§4. Total variation flow

The case of Theorem 2 is not the simplest reduction of flow (1). Moreover, the results in [25] do not explain completely the phenomenon of the facet creation. For this reason, we turned our attention to the analysis of the one-dimensional total variation flow,

$$u_t - \frac{\partial}{\partial x}(\operatorname{sgn} u_x) = 0 \text{ in } (0,1)\times(0,T), \tag{10}$$

with Dirichlet boundary data $u(0,t) = 0$, $u(1,t) = 0$ and initial datum u_0. This was the subject of paper [18]. Let us state the first main result of [18]:

Theorem 3. *Let us suppose that $u_0 \in L_1(0,1)$ and $u_{0,x} \in BV(0,1)$. In addition, we assume that the set*

$$\Xi(u_{0,x}) = \{x \in [0,1] : 0 \in u_{0,x}(x)\}$$

consists of N_0 connected components: closed intervals or isolated points.

Then, there exists a unique almost classical solution to (10), i.e., there is a function u such that

$$u_x \in L_\infty(0,T; BV(0,1))$$

and the set

$$\Xi(u_x(t)) = \{x \in [0,1] : 0 \in u_x(x,t)\} \tag{11}$$

consists of $N(t)$ connected components, i.e. closed intervals. Moreover, $N(t)$ is a decreasing function of time taking values in $\mathbb{N}$. Furthermore, u fulfills

$$u_t - \partial_x(\operatorname{sgn} \bar{\circ} u_x) = 0 \text{ in } (0,1)\times(0,T), \tag{12}$$

i.e. for each $t \in (0,T)$ (except for finitely many times) the equality holds everywhere except a finite spatial points $x \in (0,1)$.

The definition of almost classical solutions requires some preparation. We have to single out a subset of Ξ. Let u be such that Ξ defined in (11) consists of finite connected components. Then, we introduce

$$\Xi_{ess} \subset \Xi$$

such that

$$\Xi \setminus \Xi_{ess} = \bigcup_{k=1}^{K} [a_k, b_k] \text{ for } a_k \le b_k$$

and for each k there exists an open (relative to $[0,1]$) neighborhood O of interval $[a_k, b_k]$ such that $u|_O$ is monotone.

In order to continue our exposition we need an additional notion.

Definition 4.1. *Let us assume that* $u \in L_1(0,1)$ *and* $u_x \in BV(0,1)$ *and* $[p,q]$ *is a component of* $\Xi_{ess}(u_x(t))$. *Then, the set*

$$F[p,q] = \{(s, u_x(s,t)) : s \in [p,q]\} \tag{13}$$

will be called a facet *of the graph of function* u.

In particular, if $a_1 = 0$ then $[0,b_1] \subset \Xi \setminus \Xi_{ess}$ and if $b_K = 1$ then $[a_K, 1] \subset \Xi \setminus \Xi_{ess}$ (we shall keep in mind that Ξ and Ξ_{ess} are sums of closed sets). It is quite appropriate to say that these facets are of zero curvature. Thus, set Ξ_{ess} is composed of intervals, where function u has local extrema (minima and maxima). Geometrically speaking, here we have facets of positive (if u attains a local minimum on this facet) and negative curvature (if u attains a local maximum). To understand this notation let us look at Fig. 1. We see a facet being a local minimum.

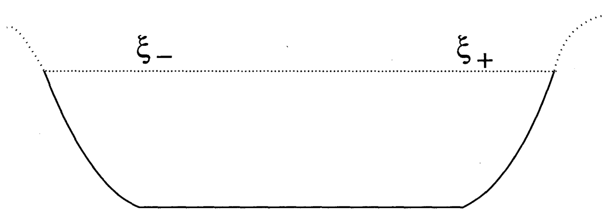


Fig. 3. A convex facet

Integrating (10) over a vicinity of $[\xi_-, \xi_+]$ we find that

$$\int_{\xi_-}^{\xi_+} u_t dx \geq 2.$$

Then comparison (10) and (1) suggests that the curvature of this facet should be positive. The same we have for local maxima.

Now, we are prepared to define composition $\bar{\circ}$ in a simple setting. For a fully general approach we refer the reader to the Appendix.

Since we are not striving for almost generality here, we restrict our attention to a definition of

$$\operatorname{sgn} \bar{\circ}\, \alpha$$

for a suitable class of multivalued operators α. We also remind an obvious observation that this object is most important in the interior of the domain we work with, see also [25], [28]. Let us also notice that $\operatorname{sgn} \bar{\circ}\, \alpha$ is a special selection of the composition of multivalued operators $\operatorname{sgn} \circ \alpha$. Any of those selections is called a Cahn–Hoffman vector field. Interestingly enough our definition of $\operatorname{sgn} \bar{\circ}\, \alpha$ is point-wise.

Definition 4.2. *Let us suppose that $\alpha = \beta_x$ is such that $\alpha \in BV(I)$ and $\Xi(\alpha)$ has a finite number of connected components, in particular Ξ_{ess} is well defined.*

Let us first consider $x \in [0,1] \setminus \Xi_{ess}(\alpha)$. Then, there exists an interval (a,b) containing x and such that either β is increasing on (a,b) or decreasing. In the first case we set

$$\operatorname{sgn} \bar{\circ}\, \alpha(x) = 1;$$

if β is decreasing on (a,b), then we set

$$\operatorname{sgn} \bar{\circ}\, \alpha(x) = -1.$$

We note that the set $[0,1] \setminus \Xi_{ess}(\alpha)$ is a finite sum of open intervals, on each of them function β is monotone. Furthermore, endpoints of $[0,1]$ cannot belong to $\Xi_{ess}(\alpha)$.

Now, let us consider $x \in \Xi_{ess}(\alpha)$, then there is $[p,q]$, a connected component of $\Xi_{ess}(\alpha)$ containing x. If $F(p,q)$ is a convex facet of β, then we set,

$$\operatorname{sgn} \bar{\circ}\, \alpha(x) = \frac{2}{q-p}x - \frac{2p}{q-p} - 1 \text{ for } x \in [p,q].$$

If $F(p,q)$ is a concave facet of β, then we set,

$$\operatorname{sgn} \bar{\circ}\, \alpha(x) = -\frac{2}{q-p}x + \frac{2p}{q-p} + 1 \text{ for } x \in [p,q].$$

The above definition provides the last missing piece of information in formula (12).

In order to obtain better qualitative information about solutions to equation (10) we proved the following result. For the sake of the clarity of its statement we introduce another shorthand. We set

$$\lambda(u_x) = \inf\{b - a : [a,b] \text{ is a connected component of } \Xi_{ess}\}.$$

Theorem 4. *Let us suppose that u_0 fulfills the assumption of Theorem 3 and $\lambda(u_{0,x}) = 0$, then there exists an almost classical solution such that $\lambda(u_x(t)) > 0$ for all $t > 0$.*

The above result is very important. It states that at positive times, neither local minima nor maxima of solution $u(\cdot, t)$ may be strict. They must be attained on a nontrivial interval. Of course, this observation does not apply to the initial state. This theorem also shows that the class of functions having only non-degenerate facets is typical. Each initially degenerate essential facet momentarily evolves into a nontrivial interval. Furthermore, shrinking an essential facet to a point is impossible. In order to explain this phenomenon let us analyze the following nonlinear elliptic operator defined by a subdifferential. Namely, we show here that polynomials of type x^2 can not be viewed as a regular function in the setting of nonlinear semigroup generated by (10). We first recall the basic definition. We say that $w \in \partial\mathcal{J}(u)$ iff $w \in L_2(0,1)$ and for all $h \in L_2(0,1)$ the following inequality holds,

$$\mathcal{J}(u+h) - \mathcal{J}(u) \geq (w,h)_2.$$

Here $(f,g)_2$ stands for the inner product in $L_2(0,1)$. We also say that $v \in D(\partial\mathcal{J})$, i.e. v belongs to the domain of $\partial\mathcal{J}$ iff $\partial\mathcal{J}(v) \neq \emptyset$. In our case $\mathcal{J}$ is given as follows

$$\mathcal{J}(u) = \begin{cases} \int_0^1 |Du| & \text{if } u \in D(\mathcal{J}) \equiv \\ & \{u \in BV[0,1],\ u(0)=0,\ u(1)=0\}, \\ +\infty & \text{if } L_2(0,1) \setminus D(\mathcal{J}), \end{cases}$$

where $\int_0^1 |Du|$ is the total variation of measure Du.

An astute reader may wonder if this functional is lower semicontinuous. In fact, it is not because an integral over the boundary is missing. However, for the sake of illustrating the problem we prefer to consider this simpler functional despite that it may look deficient.

Lemma 1. *Function $\frac{1}{2}x^2 - \frac{1}{2}x$ does not belong to $D(\partial\mathcal{J})$.*

Proof. Let us suppose the contrary. If $\frac{1}{2}x^2 - \frac{1}{2}x$ belonged to $D(\partial\mathcal{J})$, then there would exist $w \in L_2(0,1)$ such that for all $\phi \in C_0^\infty(0,1)$ and $s \in \mathbb{R}$

$$\int_{(0,1)} (|x - \frac{1}{2} + s\phi_x| - |x - \frac{1}{2}|)dx \geq s\int_{(0,1)} w\phi dx. \tag{14}$$

We may restrict our attention to ϕ such that $\phi \in C_0^\infty(-\delta + \frac{1}{2}, \delta + \frac{1}{2})$ and

$$\operatorname{supp}\phi_x \subset [-\delta + \frac{1}{2}, -\delta/2 + \frac{1}{2}] \cup [\delta/2 + \frac{1}{2}, \delta + \frac{1}{2}]$$

for $\delta > 0$ sufficiently small. In addition, we may require that,

$$\phi_x(s) > 0 \text{ for } s \in (-\delta + \tfrac{1}{2}, -\delta/2 + \tfrac{1}{2}),$$
$$\phi_x(s) < 0 \text{ for } s \in (\delta/2 + \tfrac{1}{2}, \delta + \tfrac{1}{2})$$

and

$$\phi(s) = 1 \text{ for } s \in (-\delta/2 + \frac{1}{2}, \delta/2 + \frac{1}{2}).$$

Next, let us observe that for this choice of ϕ we have

$$|x - \frac{1}{2} + s\phi_x(x)| - |x - \frac{1}{2}| = s\phi_x(x) \operatorname{sgn}(x - \frac{1}{2})$$

for $|s\phi_x(x)| \leq \delta/2$; we keep in mind that $\phi_x(s) = 0$ for $s \in (-\delta/2 + \frac{1}{2}, \delta/2 + \frac{1}{2})$.

Thus, for such ϕ and s, the RHS of (14) equals

$$\int_{(-\delta/2+\frac{1}{2}, \delta/2+\frac{1}{2})} (|x - \frac{1}{2} + s\phi_x(x)| - |x - \frac{1}{2}|)dx =$$
$$\int_{(-\delta+\frac{1}{2}, -\delta/2+\frac{1}{2})} s\phi_x \cdot (-1)dx + \int_{(\delta/2+\frac{1}{2}, \delta+\frac{1}{2})} s\phi_x \cdot (1)dx = -2s.$$

Hence, we get

$$-2s \geq s \int_{(-\delta+\frac{1}{2}, \delta+\frac{1}{2})} w\phi dx,$$

what implies for $s > 0$ that

$$2 \leq -\int_{(-\delta+\frac{1}{2}, \delta+\frac{1}{2})} w\phi dx \leq \int_{(-\delta+\frac{1}{2}, \delta+\frac{1}{2})} |w| dx \to 0 \quad \text{as} \quad \delta \to 0.$$

The convergence of the RHS to zero, as $\delta > 0$ goes to zero follows from $w \in L_2(0,1)$. Thus, we have reached a contradiction. Hence, $\frac{1}{2}x^2 - \frac{1}{2}x$ can not belong to $D(\partial \mathcal{J})$. Q.E.D.

§5. The flow of TV regularized by additive diffusion

The analysis of the one-dimensional total variation flow is relatively easy. In general, we know the evolution of the initial state. The dissipation acts only on facets, so the main issue in [18] was to explain analytical features, not to 'find' them. The next system we study is the following *regularization* of (10). We consider

$$u_t - (u_x + \operatorname{sgn} u_x)_x = 0 \text{ in } I \times (0,T) \tag{15}$$

with Dirichlet boundary conditions. It is evident that we have, at least in principle, two competing types of diffusion. One tries to form facets, the second one smears them out. Here is our basic result.

Theorem 5. *Let us suppose that $u_0 \in H^1(I)$ and it satisfies the Dirichlet boundary condition, then u, a unique solution to* (15) *given by Theorem* 1, *is such that $u, u_t, u_{xx} \in L_2(I \times (0,T))$. In addition,*

$$\sup_{t\in[\delta,T]} \|u_t(\cdot,t)\|_{L_2(I)}, \qquad \sup_{t\in[\delta,T]} \|u_{xx}(\cdot,t)\|_{L_2(I)}$$

are bounded for any $\delta > 0$. Actually, the bound has the form $C\delta^{-1/2}$.

The above result improves regularity of solutions given by Theorem 1. This extra smoothness implies that $u_x \in C(0,T; C^{1/2}(I))$, so the vector normal to the graph of the solution at fixed time is well-defined and continuous in x. This continuity of u_x is particularly important, because it gives a natural boundary condition at the end points of facets. As a result, we may easily split any solution into two parts: the monotone one and the flat one. On the monotone part we consider just the linear heat equation and on the flat part we analyze the motion of facets. However, the position of the facet endpoint is one of the unknown in this problem. Hence, system (15) becomes a free boundary problem. This approach allows us to obtain many interesting qualitative properties of solutions. We keep saying that facets are the key point of our investigations. The result below explains why they are irreplaceable in our analysis.

Theorem 6. *Let us suppose that u is a solution to* (15) *with initial condition $u_0 \in W^2_2(I)$ satisfying the compatibility condition. We also assume that for all $t > 0$ $u(\cdot,t)$ has only a finite number of inflection points. If function $u(\cdot,t)$ attains a local minimum at $x_0 \notin \partial I$, then there exist $\xi^- < \xi^+$ such that $x_0 \in [\xi^-, \xi^+]$ and $u([\xi_-, \xi_+], t) = \{u(x_0,t)\}$. The same assertion holds for local maxima.*

Proof. This theorem is so important for our program that we repeat its proof from [27]. We show that for any $t > 0$, a local minimum of function $u(\cdot,t)$ can not be strict, but it must be attained on a flat region, i.e. a facet $[\xi^-, \xi^+]$ with $\xi^- < \xi^+$.

Let us assume that it is not true and for $t > 0$ a minimizer is a single point, say $m(t)$, then the function $u(\cdot,t)$ is strictly decreasing for $m(t) - \epsilon < x < m(t)$ and strictly increasing for $m(t) + \epsilon > x > m(t)$. Then, we integrate equation (15) over $[m(t) - \delta, m(t) + \delta]$, where $\delta > 0$

and pass to the limit as $\delta \to 0$,

$$\int_{m(t)^-}^{m(t)^+} u_t dx - \int_{m(t)^-}^{m(t)^+} (L\bar{\circ} u_x)_x dx = 0. \tag{16}$$

In order to compute the second term in the LHS (left hand side) of (16), we recall that by Theorem 5, $u_x(\cdot,t) \in C^{1/2}(I)$ for each $t > 0$, in particular, u_x is continuous, so at a minimum the derivative must be zero at endpoints. This leads to the following conclusion,

$$L\bar{\circ} u_x(m(t)^-) = -1 \text{ and } L\bar{\circ} u_x(m(t)^+) = 1.$$

Subsequently, $\int_{m(t)^-}^{m(t)^+} (L\bar{\circ} u_x)_x dx = 2$ which implies $\int_{m(t)^-}^{m(t)^+} u_t dx = 2$, but by assumption

$$m(t)^+ = \lim_{\delta \to 0^+} (m(t) + \delta) = m(t) = \lim_{\delta \to 0^+} (m(t) - \delta) = m(t)^-.$$

We reached a contradiction, because we know from Theorem 5 that $u_t(\cdot,t)$ is integrable. As a result, we conclude that the local minimum at x_0 is not strict and must be attained on a nondegenerate interval $[\xi^-, \xi^+]$. Q.E.D.

Let us return to the physical interpretation. The graph of a solution, u, at $t > 0$ is regarded as an interface in a phase transition model, [30]. Thus, our results imply that facets must appear in a propagating front, here they are the local maximum/minimum of the graph, see Fig. 2.

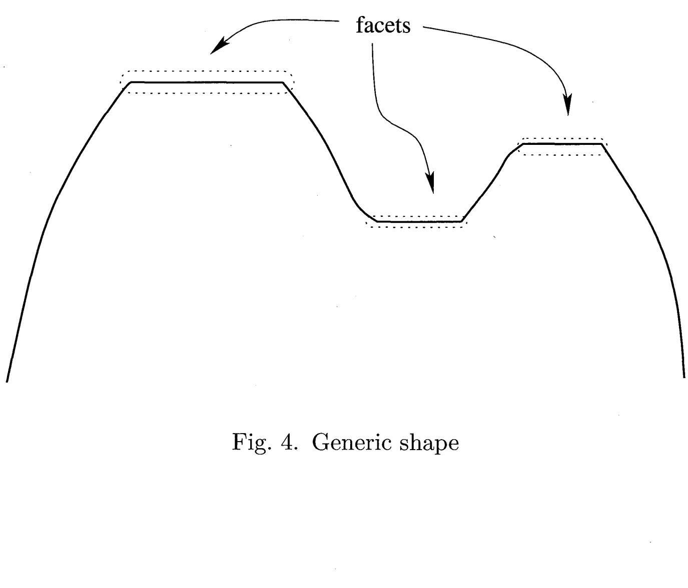


Fig. 4. Generic shape

However, facets may never appear in the middle of the graph of a monotone function, see Fig. 3. This may happen only at the initial time or at the time of facets merging. These are the only instances of appearance of this kind of facets.

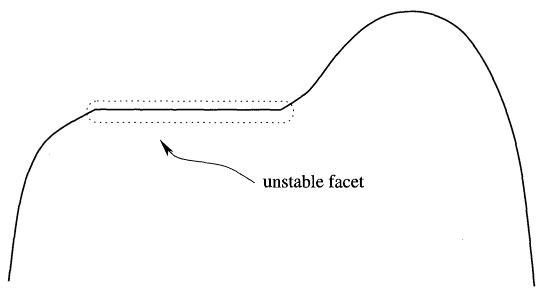


Fig. 5. Nonadmissible shape

While comparing systems (10) and (15), we notice one important difference. Namely, in the case of the flow combining the isotropic diffusion and the TV flow, we are able to construct states such that a facet can shrink. On the other hand, this is impossible in the case of the pure TV flow. Let us also mention that if both systems are considered without forcing, like here, then it is not possible to observe breaking phenomena. This is expected in the presence of a sufficiently large external force (the RHS is not zero). Partial results in this direction are obtained by [15], [16], [22].

At the end of our note we say a few words about the stationary solution to problem (15). In [24] the following problem was investigated

$$-(u_x + \operatorname{sgn} u_x)_x = f \text{ in } (0,1) \tag{17}$$

with say zero Dirichlet boundary conditions. The aim of this paper is to make a classification of possible types of solutions. In particular, the following unexpected result was proved there.

Theorem 7. *Consider* (17) *with zero Dirichlet boundary conditions. If $f \in L_\infty$ and fulfills*

$$-1 \le \int_0^x f(s)ds \le 1 \qquad \text{for all } x \in (0,1), \tag{18}$$

then the only solution is $u \equiv 0$.

A proof of this result is elementary, based on the definition of the weak solutions. The theorem points that to break such stable structure like the facet we shall pump sufficiently large force, otherwise our act causes no effect.

§ Appendix

We present the complete definition of the composition $\bar{\circ}$ from Section 3, which appeared first in formula (9).

Definition 5.1. For a multivalued operator A given as

$$A = \Lambda_s + s,$$

where Λ is the solution yielded by Theorem 2 and ∂J being a subdifferential of (8), we define the *composition* $\partial J \bar{\circ} A$, as follows.

To begin with, we decompose the domain of A into three disjoint parts $[a, b) = \mathcal{D}_r \cup \mathcal{D}_f \cup \mathcal{D}_s$, where

$$\begin{aligned} &\mathcal{D}_s = \{s \in [0, 2\pi) : A(s) = [c_s, d_s] \text{ and } c_s < d_s\}; \\ &\mathcal{D}_f = \{\textstyle\bigcup_k (a_k, b_k) : A|_{(a_k,b_k)} = c_k, \text{ where } c_k \text{ is a constant}\}; \\ &\mathcal{D}_r = [0, 2\pi) \setminus (\mathcal{D}_s \cup \mathcal{D}_f). \end{aligned} \tag{19}$$

Then, the composition is defined in three steps:

1. For each $s \in \mathcal{D}_r$ the set $A(s)$ is a singleton, thus the composition is given in the classical way

$$\partial J \bar{\circ} A(s) = \partial J(A(s)) \qquad \text{for} \quad s \in \mathcal{D}_r. \tag{20}$$

2. In the case $s \in \mathcal{D}_f$, the definition is "unnatural". For a given set $(a_k, b_k) \subset \mathcal{D}_f$ we have $A|_{(a_k,b_k)} = c_k$. If $\partial J(c_k)$ is single-valued, then for $s \in (a_k, b_k)$ we have,

$$\partial J \bar{\circ} A(s) = \{\frac{dJ}{d\phi}(c_k)\}.$$

However, if $\partial J(c_k)$ is multivalued, i.e. $\partial J(c_k) = [\alpha_k, \beta_k]$, then the definition is not immediate. We have to consider four cases related to the behavior of multifunction A in a neighborhood of interval (a_k, b_k). The properties of solutions imply the necessity to consider the following four cases (for small $\epsilon > 0$):

(i) A is increasing, i.e. $A(s) < c_k$ for $s \in (a_k - \epsilon, a_k)$ and $A(s) > c_k$ for $s \in (b_k, b_k + \epsilon)$;

(ii) A is decreasing, i.e. $A(s) > c_k$ for $s \in (a_k - \epsilon, a_k)$ and $A(s) < c_k$ for $s \in (b_k, b_k + \epsilon)$;

(iii) A is convex, i.e. $A(s) > c_k$ for $s \in (a_k - \epsilon, a_k)$ and $A(s) > c_k$ for $s \in (b_k, b_k + \epsilon)$;

(iv) A is concave, i.e. $A(s) < c_k$ for $s \in (a_k - \epsilon, a_k)$ and $A(s) < c_k$ for $s \in (b_k, b_k + \epsilon)$.

Let us emphasize that these are the only possibilities, because we explicitly excluded all functions with oscillatory behavior. Indeed, the set $\Xi(\phi)$ is permitted to have only a finite number of components.

In the case (i), we put

$$\partial J \bar{\circ} A(t) = x_k(t - b_k) + y_k(t - a_k) \qquad \text{for } t \in (a_k, b_k), \tag{21}$$

where $x_k = \frac{\alpha_k}{a_k - b_k}$ and $y_k = \frac{\beta_k}{b_k - a_k}$.

For case (ii), we put

$$\partial J \bar{\circ} A(t) = x_k(t - b_k) + y_k(t - a_k) \qquad \text{for } t \in (a_k, b_k), \tag{22}$$

where $x_k = \frac{\beta_k}{a_k - b_k}$ and $y_k = \frac{\alpha_k}{b_k - a_k}$.

When we deal with case (iii), we set

$$\partial J \bar{\circ} A(t) = \beta_k \qquad \text{for } t \in (a_k, b_k). \tag{23}$$

Finally, if (iv) holds, then we put

$$\partial J \bar{\circ} A(t) = \alpha_k \qquad \text{for } t \in (a_k, b_k). \tag{24}$$

3. In the last case, i.e., if $s \in \mathcal{D}_s$, our definition is just a consequence of the first two steps. Since set $\mathcal{D}_s$ consists of a countable number of points, we consider each of them separately. We have $A(d_k) = [e_k, f_k]$ with $e_k \neq f_k$, then

$$\partial J \bar{\circ} A(d_k) = [\limsup_{t \to d_k^-} \partial J \bar{\circ} A(t), \liminf_{t \to d_k^+} \partial J \bar{\circ} A(t)]. \tag{25}$$

Definition 5.1 is complete.

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