Advanced Studies in Pure Mathematics 67, 2015
Variational Methods for Evolving Objects
pp. 1–58

Sobolev spaces in metric measure spaces: reflexivity and lower semicontinuity of slope

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Abstract.

In this paper we make a survey of some recent developments of the theory of Sobolev spaces $W^{1,q}(X, \mathsf{d}, \mathfrak{m})$, $1 < q < \infty$, in metric measure spaces $(X, \mathsf{d}, \mathfrak{m})$. In the final part of the paper we provide a new proof of the reflexivity of the Sobolev space based on Γ-convergence; this result extends Cheeger's work because no Poincaré inequality is needed and the measure-theoretic doubling property is weakened to the metric doubling property of the support of $\mathfrak{m}$. We also discuss the lower semicontinuity of the slope of Lipschitz functions and some open problems.

Contents

Received December 27, 2012.
Revised May 25, 2014.
2010 *Mathematics Subject Classification.* 49J52, 49M25, 49Q20, 58J35, 35K90, 31C25.
Key words and phrases. Sobolev spaces, metric measure spaces, weak gradients.

§1. Introduction

This paper is devoted to the theory of Sobolev spaces $W^{1,q}(X, \mathsf{d}, \mathfrak{m})$ on metric measure spaces $(X, \mathsf{d}, \mathfrak{m})$. It is on one hand a survey paper on the most recent developments of the theory occurred in [3], [4] (see also [5] for analogous results in the space BV of functions of bounded variation), but it contains also new results on the reflexivity of $W^{1,q}$, $1 < q < \infty$, improving those of [7]. The occasion for writing this paper has been the course given by the first author in Sapporo (July–August 2012).

In a seminal paper [7], Cheeger investigated the fine properties of Sobolev functions on metric measure spaces, with the main aim of providing generalized versions of Rademacher's theorem and, along with it, a description of the cotangent bundle. Assuming that the Polish metric measure structure $(X, \mathsf{d}, \mathfrak{m})$ is doubling and satisfies a Poincaré inequality (see Definitions 6 and 45 for precise formulations of these structural assumptions) he proved that the Sobolev spaces are reflexive and that the q-power of the slope is $L^q(X, \mathfrak{m})$-lower semicontinuous, namely

$$
\begin{aligned}
&f_h,\ f \in \mathrm{Lip}(X), \quad \int_X |f_h - f|^q \, \mathrm{d}\mathfrak{m} \to 0 \\
&\qquad\qquad \Longrightarrow \quad \liminf_{h\to\infty} \int_X |\nabla f_h|^q \, \mathrm{d}\mathfrak{m} \geq \int_X |\nabla f|^q \, \mathrm{d}\mathfrak{m}.
\end{aligned}
\tag{1.1}
$$

Here the slope $|\nabla f|$, also called local Lipschitz constant, is defined by

$$
|\nabla f|(x) := \limsup_{y\to x} \frac{|f(y) - f(x)|}{\mathsf{d}(y, x)}.
$$

These results come also as a byproduct of a generalized Rademacher's theorem, which can be stated as follows: there exist an integer N, depending on the doubling and Poincaré constants, a Borel partition $\{X_i\}_{i\in I}$ of X and Lipschitz functions f_j^i, $1 \leq j \leq N(i) \leq N$, with the property that for all $f \in \mathrm{Lip}(X)$ it is possible to find Borel coefficients c_j^i, $1 \leq j \leq N$, uniquely determined $\mathfrak{m}$-a.e. on X_i, satisfying

$$
\Bigl|\nabla\bigl(f - \sum_{j=1}^{N(i)} c_j^i(x) f_j^i\bigr)\Bigr|(x) = 0 \qquad \text{for } \mathfrak{m}\text{-a.e. } x \in X_i. \tag{1.2}
$$

It turns out that the family of norms on $\mathbb{R}^{N(i)}$

$$
\|(\alpha_1, \ldots, \alpha^i_{N(i)})\|_x := \Bigl|\nabla \sum_{j=1}^{N(i)} \alpha_j f_j^i\Bigr|(x)
$$

indexed by $x \in X_i$ satisfies, thanks to (1.2),

$$\|(c_1^i(x), \ldots, c_{N(i)}^i(x))\|_x = |\nabla f|(x) \qquad \text{for } \mathfrak{m}\text{-a.e. } x \in X_i.$$

Therefore, this family of norms provides the norm on the cotangent bundle on X_i. Since $N(i) \leq N$, using for instance John's lemma one can find Hilbertian equivalent norms $|\cdot|_x$ with bi-Lipschitz constant depending only on N. This leads to an equivalent (but not canonical) Hilbertian norm and then to reflexivity. In this paper we aim mostly at lower semicontinuity and reflexivity: we recover the latter (and separability as well) without assuming the validity of the Poincaré inequality and replacing the doubling assumption on $(X, \mathsf{d}, \mathfrak{m})$ with a weaker assumption, namely the geometric doubling of $(\operatorname{supp} \mathfrak{m}, \mathsf{d})$.

In connection with the expansion (1.2), it is also worthwhile to mention a remarkable paper [23] by Keith, where (1.2) is obtained replacing the Poincaré assumption with a more infinitesimal condition, called Lip-lip: for some constant K, for all locally Lipschitz functions f, for $\mathfrak{m}$-a.e. $x \in X$ there holds:

$$\limsup_{r \downarrow 0} \sup_{y \in B(x,r)} \frac{|f(y) - f(x)|}{r} \leq K \liminf_{r \downarrow 0} \sup_{y \in B(x,r)} \frac{|f(y) - f(x)|}{r}.$$

However, we don't know whether Keith's condition is sufficient for the lower semicontinuity of the slope.

Sobolev spaces, as well as a weak notion of norm of the gradient $|\nabla f|_{C,q}$, are built in [7] by considering the best possible approximation of f by functions f_n having a q-integrable upper gradient g_n, namely pairs (f_n, g_n) satisfying

$$|f_n(\gamma_1) - f_n(\gamma_0)| \leq \int_\gamma g_n \qquad \text{for all absolutely continuous curves } \gamma : [0,1] \to X. \tag{1.3}$$

Here, by best approximation we mean that we minimize

$$\liminf_{n \to \infty} \int_X |g_n|^q \, d\mathfrak{m}$$

among all sequences f_n that converge to f in $L^q(X, \mathfrak{m})$. It must be emphasized that even though the implication (1.1) does not involve at all weak gradients, its proof requires a fine analysis of the Sobolev spaces and, in particular, their reflexivity. At the same time, in [27] this approach was proved to be equivalent to the one based on the theory of q-upper gradients introduced in [24] and leading to a gradient that we shall denote $|\nabla f|_{S,q}$. In this theory one imposes the validity of (1.3) on "almost all curves" in the sense of [12] and uses this property to define $|\nabla f|_{S,q}$. Both approaches are described more in detail in Appendix A of this paper (see also [17] for a nice account of the theory).

More recently, the first author, N. Gigli and G. Savaré developed, motivated by a research program on metric measure spaces with Ricci curvature bounds from below, a new approach to calculus in metric measure spaces (see also [14] for the most recent developments). In particular, in [3] and [4] Sobolev spaces and weak gradients are built by a slightly different relaxation procedure, involving Lipschitz functions f_n with bounded support and their slopes $|\nabla f_n|$ instead of functions f_n with q-integrable upper gradient g_n: this leads to a weak gradient a priori larger than $|\nabla f|_{C,q}$. Still in [3] and [4], connection with the upper gradient point of view, a different notion of negligible set of curves (sensitive to the parametrization of the curves) to quantify exceptions in (1.3) was introduced, leading to a gradient a priori smaller than $|\nabla f|_{S,q}$. One of the main results of these papers is that all the four notions of gradient a posteriori coincide, and this fact is independent of doubling and Poincaré assumptions.

The paper, that as we said must be conceived mostly as a survey paper until Section 7, is organized as follows. In Section 2 we recall some preliminary tools of analysis in metric spaces, the theory of gradient flows (which plays, via energy dissipation estimates, a key role), Γ-convergence, p-th Wasserstein distance W_p, with p dual to the Sobolev exponent q, and optimal transport theory. The latter plays a fundamental role in the construction of suitable measures in the space of absolutely continuous curves via the so-called superposition principle, that allows to pass from an "Eulerian" formulation (i.e. in terms of a curve of measures or a curve of probability densities) to a "Lagrangian" one. In Section 3 we study, following very closely [4], the pointwise properties of the Hopf–Lax semigroup

$$Q_t f(x) := \inf_{y\in X} f(y) + \frac{\mathsf{d}^p(x,y)}{pt^{p-1}},$$

also emphasizing the role of the so-called asymptotic Lipschitz constant

$$\mathrm{Lip}_a(f,x) := \inf_{r>0} \mathrm{Lip}\big(f, B(x,r)\big) = \lim_{r\downarrow 0} \mathrm{Lip}\big(f, B(x,r)\big),$$

which is always larger than $|\nabla f|(x)$ and coincides with the upper semicontinuous relaxation of $|\nabla f|$ at x in length spaces.

Section 4 presents the two weak gradients $|\nabla f|_{*,q}$ and $|\nabla f|_{w,q}$, the former obtained by a relaxation and the latter by a weak upper gradient property. As suggested in the final section of [4], we work with an even stronger (a priori) gradient, where in the relaxation procedure we replace $|\nabla f_n|$ with $\mathrm{Lip}_a(f_n,\cdot)$. We present basic calculus rules and stability properties of these weak gradients.

Section 5 contains the basic facts we shall need on the gradient flow $(f_t)_{t\geq 0}$ in $L^2(X,\mathfrak{m})$ of the lower semicontinuous functional $f\mapsto \mathbf{C}_q(f) := \frac{1}{q}\int_X |\nabla f|_{*,q}^q\,\mathrm{d}\mathfrak{m}$, in particular the entropy dissipation rate

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_X \Phi(f_t)\,\mathrm{d}\mathfrak{m} = -\int_X \Phi''(f_t)|\nabla f_t|_{*,q}^q\,\mathrm{d}\mathfrak{m}$$

along this gradient flow. Notice that, in order to apply the Hilbertian theory of gradient flows, we need to work in $L^2(X,\mathfrak{m})$. Even when $\mathfrak{m}$ is finite, this requires a suitable definition (obtained by truncation) of $|\nabla f|_{*,q}$ when $q>2$ and $f\in L^2(X,\mathfrak{m})\setminus L^q(X,\mathfrak{m})$.

In Section 6 we prove the equivalence of gradients. Starting from a function f with $|\nabla f|_{w,q}\in L^q(X,\mathfrak{m})$ we approximate it by the gradient flow of f_t of $\mathbf{C}_q$ starting from f and we use the weak upper gradient property to get

$$\limsup_{t\downarrow 0}\frac{1}{t}\int_0^t\int_X \frac{|\nabla f_s|_{*,q}^q}{f_s^{p-1}}\,\mathrm{d}\mathfrak{m}\mathrm{d}s \leq \int_X \frac{|\nabla f|_{w,q}^q}{f^{p-1}}\,\mathrm{d}\mathfrak{m}$$

where $p = q/(q-1)$ is the dual exponent of q. Using the stability properties of $|\nabla f|_{*,q}$ we eventually get $|\nabla f|_{*,q}\leq |\nabla f|_{w,q}$ $\mathfrak{m}$-a.e. in X.

In Section 7 we prove that the Sobolev space $W^{1,q}(X,\mathsf{d},\mathfrak{m})$ is reflexive when $1<q<\infty$, $(\operatorname{supp}\mathfrak{m},\mathsf{d})$ is separable and doubling, and $\mathfrak{m}$ is finite on bounded sets. Instead of looking for an equivalent Hilbertian norm (whose existence is presently known only if the metric measure structure is doubling and the Poincaré inequality holds), we rather look for a discrete scheme, involving functionals $\mathcal{F}_\delta(f)$ of the form

$$\mathcal{F}_\delta(f) = \sum_i \frac{1}{\delta^q}\sum_{A_j^\delta\sim A_i^\delta} |f_{\delta,i}-f_{\delta,j}|^q\mathfrak{m}(A_i^\delta).$$

Here A_i^δ is a well chosen decomposition of $\operatorname{supp}\mathfrak{m}$ on scale δ, $f_{\delta,i} = ⨍_{A_i^\delta} f$ and the sum involves cells A_j^δ close to A_i^δ, in a suitable sense. This strategy is very close to the construction of approximate q-energies on fractal sets and more general spaces, see for instance [21], [28].

It is fairly easy to show that any Γ-limit point $\mathcal{F}_0$ of $\mathcal{F}_\delta$ as $\delta \to 0$ satisfies

$$\mathcal{F}_0(f) \le c(c_D, q) \int_X \mathrm{Lip}_a^q(f, \cdot)\, d\mathfrak{m} \quad \text{for all Lipschitz } f \text{ with bounded support,} \tag{1.4}$$

where c_D is the doubling constant of (X, d) (our proof gives $c(c_D, q) \le 6^q c_D^3$). More delicate is the proof of lower bounds of $\mathcal{F}_0$, which uses a suitable discrete version of the weak upper gradient property and leads to the inequality

$$\frac{1}{4^q} \int_X |\nabla f|_{w,q}^q \, d\mathfrak{m} \le \mathcal{F}_0(f) \qquad \forall f \in W^{1,q}(X, \mathsf{d}, \mathfrak{m}). \tag{1.5}$$

Combining (1.4), (1.5) and the equivalence of weak gradients gives

$$\frac{1}{4^q} \int_X |\nabla f|_{w,q}^q \, d\mathfrak{m} \le \mathcal{F}_0(f) \le c(c_D, q) \int_X |\nabla f|_{w,q}^q \, d\mathfrak{m} \qquad \forall f \in W^{1,q}(X, \mathsf{d}, \mathfrak{m}).$$

The discrete functionals $\mathcal{F}_\delta(f) + \sum_i |f_{\delta,i}|^q \mathfrak{m}(A_i^\delta)$ describe L^q norms in suitable discrete spaces, hence they satisfy the Clarkson inequalities; these inequalities (which reduce to the parallelogram identity in the case $q = 2$) are retained by the Γ-limit point $\mathcal{F}_0 + \|\cdot\|_q^q$. This leads to an equivalent uniformly convex norm in $W^{1,q}(X, \mathsf{d}, \mathfrak{m})$, and therefore to reflexivity. As a byproduct one obtains density of bounded Lipschitz functions in $W^{1,q}(X, \mathsf{d}, \mathfrak{m})$ and separability. In this connection, notice that the results of [3], [4] provide, even without a doubling assumption, a weaker property (but still sufficient for some applications), the so-called density in energy; on the other hand, under the assumptions of [7] one has even more, namely density of Lipschitz functions in the Lusin sense. Notice however that $\mathcal{F}_0$, like the auxiliary Hilbertian norms of [7], is not canonical: it might depend on the decomposition A_i^δ and we don't expect the whole family $\mathcal{F}_\delta$ to Γ-converge as $\delta \to 0^+$. We conclude the section with an example showing that reflexivity may fail if the metric doubling assumption is dropped.

In Section 8 we prove (1.1), following in large part the scheme of [7] (although we get the result in a more direct way, without an intermediate result in length spaces). In particular we need the Poincaré inequality to establish the bound

$$|\nabla f| \le C\, |\nabla f|_{w,q} \quad \text{for any Lipschitz function } f \text{ with bounded support,}$$

which, among other things, prevents $|\nabla f|_{w,q}$ from being trivial.

Finally, in the appendices we describe more in detail the intermediate gradients $|\nabla f|_{C,q}$ and $|\nabla f|_{S,q}$, we provide another approximation by discrete gradients also in non-doubling spaces (but our results here are not conclusive) and we list a few open problems.

§2. Preliminary notions

In this section we introduce some notation and recall a few basic facts on absolutely continuous functions, gradient flows of convex functionals and optimal transportation, see also [2], [29] as general references.

2.1. Absolutely continuous curves and slopes

Let (X, d) be a metric space, $J \subset \mathbb{R}$ a closed interval and $J \ni t \mapsto x_t \in X$. We say that (x_t) is *absolutely continuous* if

$$\mathsf{d}(x_s, x_t) \le \int_s^t g(r)\,\mathrm{d}r \qquad \forall s,\, t \in J,\ s < t$$

for some $g \in L^1(J)$. It turns out that, if (x_t) is absolutely continuous, there is a minimal function g with this property, called *metric speed*, denoted by $|\dot{x}_t|$ and given for a.e. $t \in J$ by

$$|\dot{x}_t| = \lim_{s \to t} \frac{\mathsf{d}(x_s, x_t)}{|s - t|}.$$

See [2, Theorem 1.1.2] for the simple proof.

We will denote by $C([0,1], X)$ the space of continuous curves from $[0,1]$ to (X, d) endowed with the sup norm. The set $AC^p([0,1], X) \subset C([0,1], X)$ consists of all absolutely continuous curves γ such that $\int_0^1 |\dot{\gamma}_t|^p\,\mathrm{d}t < \infty$: it is the countable union of the sets $\{\gamma : \int_0^1 |\dot{\gamma}_t|^p\,\mathrm{d}t \le n\}$, which are easily seen to be closed if $p > 1$. Thus $AC^p([0,1], X)$ is a Borel subset of $C([0,1], X)$.

We remark that the definition of absolutely continuous curve makes sense even when we consider an extended metric space (X, d), namely assuming that the distance may take the value ∞; the properties described above hold true, with minor variants, in this context (see [3] for details).

The *evaluation maps* $\mathrm{e}_t : C([0,1], X) \to X$ are defined by

$$\mathrm{e}_t(\gamma) := \gamma_t,$$

and are clearly continuous.

Given $f : X \to \mathbb{R}$ and $E \subset X$, we denote by $\mathrm{Lip}(f, E)$ the Lipschitz constant of the function f on E, namely

$$\mathrm{Lip}(f,E) := \sup_{x,\, y\in E,\, x\neq y} \frac{|f(x)-f(y)|}{\mathsf{d}(x,y)}.$$

Given $f : X \to \mathbb{R}$, we define *slope* (also called local Lipschitz constant) by

$$|\nabla f|(x) := \varlimsup_{y\to x} \frac{|f(y)-f(x)|}{\mathsf{d}(y,x)}.$$

For $f,\, g : X \to \mathbb{R}$ Lipschitz it clearly holds

$$|\nabla(\alpha f+\beta g)| \le |\alpha||\nabla f| + |\beta||\nabla g| \qquad \forall \alpha,\beta\in\mathbb{R}, \tag{2.1a}$$

$$|\nabla(fg)| \le |f||\nabla g| + |g||\nabla f|. \tag{2.1b}$$

We shall also need the following calculus lemma.

Lemma 1. *Let $f : (0,1) \to \mathbb{R}$, $q \in [1,\infty]$, $g \in L^q(0,1)$ nonnegative be satisfying*

$$|f(s)-f(t)| \le \int_s^t g(r)\,\mathrm{d}r \qquad \text{for } \mathscr{L}^2\text{-a.e. } (s,t)\in(0,1)^2.$$

Then $f \in W^{1,q}(0,1)$ and $|f'| \le g$ a.e. in $(0,1)$.

Proof. Let $N \subset (0,1)^2$ be the $\mathscr{L}^2$-negligible subset where the above inequality fails. Choosing $s \in (0,1)$, whose existence is ensured by Fubini's theorem, such that $(s,t) \notin N$ for a.e. $t \in (0,1)$, we obtain that $f \in L^\infty(0,1)$. Since the set $N_1 = \{(t,h) \in (0,1)^2 : (t,t+h) \in N \cap (0,1)^2\}$ is $\mathscr{L}^2$-negligible as well, we can apply Fubini's theorem to obtain that for a.e. h it holds $(t,h) \notin (0,1)^2 \setminus N_1$ for a.e. $t \in (0,1)$. Let $h_i \downarrow 0$ with this property and use the identities

$$\int_0^1 f(t)\frac{\phi(t-h)-\phi(t)}{h}\,\mathrm{d}t = \int_0^1 \frac{f(t+h)-f(t)}{h}\phi(t)\,\mathrm{d}t$$

with $\phi \in C^1_c(0,1)$ and $h = h_i$ sufficiently small to get

$$\left|\int_0^1 f(t)\phi'(t)\,\mathrm{d}t\right| \le \int_0^1 g(t)|\phi(t)|\,\mathrm{d}t.$$

It follows that the distributional derivative of f is a signed measure η with finite total variation which satisfies

$$-\int_0^1 f\phi'\,\mathrm{d}t = \int_0^1 \phi\,\mathrm{d}\eta, \quad \left|\int_0^1 \phi\,\mathrm{d}\eta\right| \le \int_0^1 g|\phi|\,\mathrm{d}t \ \text{ for every } \phi\in C^1_c(0,1);$$

therefore η is absolutely continuous with respect to the Lebesgue measure with $|\eta| \le g\mathscr{L}^1$. This gives the $W^{1,1}(0,1)$ regularity and, at the same time, the inequality $|f'| \le g$ a.e. in $(0,1)$. The case $q > 1$ immediately follows by applying this inequality when $g \in L^q(0,1)$. Q.E.D.

Following [18], we say that a Borel function $g : X \to [0,\infty]$ is an upper gradient of a Borel function $f : X \to \mathbb{R}$ if the inequality

$$\left|\int_{\partial\gamma} f\right| \le \int_\gamma g \tag{2.2}$$

holds for all absolutely continuous curves $\gamma : [0,1] \to X$. Here $\int_{\partial\gamma} f = f(\gamma_1) - f(\gamma_0)$, while $\int_\gamma g = \int_0^1 g(\gamma_s)|\dot\gamma_s|\,\mathrm{d}s$.

It is well-known and easy to check that the slope is an upper gradient, for locally Lipschitz functions.

2.2. Gradient flows of convex and lower semicontinuous functionals

Let H be an Hilbert space, $\Psi : H \to \mathbb{R} \cup \{+\infty\}$ convex and lower semicontinuous and $D(\Psi) = \{\Psi < \infty\}$ its finiteness domain. Recall that a gradient flow $x : (0,\infty) \to H$ of Ψ is a locally absolutely continuous map with values in $D(\Psi)$ satisfying

$$-\frac{\mathrm{d}}{\mathrm{d}t} x_t \in \partial^-\Psi(x_t) \qquad \text{for a.e. } t \in (0,\infty).$$

Here $\partial^-\Psi(x)$ is the subdifferential of Ψ, defined at any $x \in D(\Psi)$ by

$$\partial^-\Psi(x) := \{p \in H^* : \ \Psi(y) \ge \Psi(x) + \langle p, y - x\rangle \ \forall y \in H\}.$$

We shall use the fact that for all $x_0 \in \overline{D(\Psi)}$ there exists a unique gradient flow x_t of Ψ starting from x_0, i.e. $x_t \to x_0$ as $t \downarrow 0$, and that $t \mapsto \Psi(x_t)$ is nonincreasing and locally absolutely continuous in $(0,\infty)$. In addition, this unique solution exhibits a regularizing effect, namely $-\frac{\mathrm{d}}{\mathrm{d}t} x_t$ is for a.e. $t \in (0,\infty)$ the element of minimal norm in $\partial^-\Psi(x_t)$.

2.3. The space $(\mathscr{P}(X), W_p)$ and the superposition principle

Let (X, d) be a complete and separable metric space and $p \in [1,\infty)$. We use the notation $\mathscr{P}(X)$ for the set of all Borel probability measures on X. Given μ, $\nu \in \mathscr{P}(X)$, we define the Wasserstein (extended) distance $W_p(\mu,\nu) \in [0,\infty]$ between them as

$$W_p^p(\mu,\nu) := \min \int \mathsf{d}^p(x,y)\,\mathrm{d}\gamma(x,y).$$

Here the minimization is made in the class $\Gamma(\mu, \nu)$ of all probability measures γ on $X \times X$ such that $\pi^1_\sharp\gamma = \mu$ and $\pi^2_\sharp\gamma = \nu$, where $\pi^i : X \times X \to X$, $i = 1, 2$, are the coordinate projections and $f_\sharp : \mathscr{P}(Y) \to \mathscr{P}(Z)$ is the push-forward operator induced by a Borel map $f : Y \to Z$.

An equivalent definition of W_p comes from the dual formulation of the transport problem:

$$\frac{1}{p} W_p^p(\mu, \nu) = \sup_{\psi \in \mathrm{Lip}_b(X)} \int \psi \, \mathrm{d}\mu + \int \psi^c \, \mathrm{d}\nu. \tag{2.3}$$

Here $\mathrm{Lip}_b(X)$ stands for the class of bounded Lipschitz functions and the c-transform ψ^c is defined by

$$\psi^c(y) := \inf_{x \in X} \frac{\mathsf{d}^p(x, y)}{p} - \psi(x).$$

We will need the following result, proved in [26]: it shows how to associate to an absolutely continuous curve μ_t w.r.t. W_p a plan $\boldsymbol{\pi} \in \mathscr{P}(C([0, 1], X))$ representing the curve itself (see also [2, Theorem 8.2.1] for the Euclidean case). Notice that the result as stated in [26] is concerned with curves μ_t with values in the space

$$\mathscr{P}_p(X) := \left\{ \mu \in \mathscr{P}(X) : \int_X \mathsf{d}(x_0, x)^p \, \mathrm{d}\mu(x) < \infty \quad \text{for some } x_0 \in X \right\}$$

of probabilities with finite p-th moment (so that W_p is a finite distance in $\mathscr{P}_p(X)$), but the proof works also in the extended metric space $\mathscr{P}(X)$, since absolute continuity forces μ_t to belong to the component

$$\{\mu \in \mathscr{P}(X) : \; W_p(\mu, \mu_0) < \infty\}$$

at a finite distance from μ_0.

Proposition 2 (Superposition principle)**.** *Let (X, d) be a complete and separable metric space $p \in (1, \infty)$ and let*

$$\mu_t \in AC^p\big([0, T]; (\mathscr{P}(X), W_p)\big).$$

Then there exists $\boldsymbol{\pi} \in \mathscr{P}(C([0, 1], X))$, concentrated on $AC^p([0, 1], X)$, such that $(\mathrm{e}_t)_\sharp \boldsymbol{\pi} = \mu_t$ for any $t \in [0, T]$ and

$$\int |\dot\gamma_t|^p \, \mathrm{d}\boldsymbol{\pi}(\gamma) = |\dot\mu_t|^p \qquad \text{for a.e. } t \in [0, T]. \tag{2.4}$$

2.4. Γ-convergence

Definition 3. *Let (X, d) be a metric space and let $F_h : X \to [-\infty, +\infty]$. We say that F_h Γ-converge to $F : X \to [-\infty, +\infty]$ if:*

(a) *For every sequence $(u_h) \subset X$ convergent to $u \in X$ we have*

$$F(u) \le \liminf_{h\to\infty} F_h(u_h);$$

(b) *For all $u \in X$ there exists a sequence $(u_n) \subset X$ such that*

$$F(u) \ge \limsup_{h\to\infty} F_h(u_h).$$

Sequences satisfying the second property are called "recovery sequences"; whenever Γ-convergence occurs, they obviously satisfy $\lim_h F_h(u_h) = F(u)$.

The following compactness property of Γ-convergence (see for instance [10, Theorem 8.5]) is well-known.

Proposition 4. *If (X, d) is separable, any sequence of functionals $F_h : X \to [-\infty, +\infty]$ admits a Γ-convergent subsequence.*

We quickly sketch the proof, for the reader's convenience. If $\{U_i\}_{i\in\mathbb{N}}$ is a countable basis of open sets of (X, d), we may extract a subsequence $h(k)$ such that $\alpha_i := \lim_k \inf_{U_i} F_{h(k)}$ exists in $\overline{\mathbb{R}}$ for all $i \in \mathbb{N}$. Then, it is easily seen that

$$F(x) := \sup_{U_i \ni x} \alpha_i \qquad x \in X$$

is the Γ-limit of $F_{h(k)}$.

We will also need an elementary stability property of uniformly convex (and quadratic as well) functionals under Γ-convergence. Recall that a positively 1-homogeneous function $\mathcal{N}$ on a vector space V is uniformly convex with modulus ω if there exists a function $\omega : [0, \infty) \to [0, \infty)$ with $\omega > 0$ on $(0, \infty)$ such that

$$\mathcal{N}(u) = \mathcal{N}(v) = 1 \qquad \Longrightarrow \qquad \mathcal{N}\left(\frac{u+v}{2}\right) \le 1 - \omega(\mathcal{N}(u-v))$$

for all u, $v \in V$.

Lemma 5. *Let V be a normed space with the induced metric structure and let $\omega : [0, \infty) \to [0, \infty)$ be continuous, nondecreasing, positive on $(0, \infty)$. Let $\mathcal{N}_h$ be uniformly convex positively 1-homogeneous functions on V with the same modulus ω, Γ-convergent to some function $\mathcal{N}$. Then $\mathcal{N}$ is positively 1-homogeneous and uniformly convex with modulus ω.*

Proof. The verification of 1-homogeneity of $\mathcal{N}$ is trivial. Let u, $v \in V$ which satisfy $\mathcal{N}(u) = \mathcal{N}(v) = 1$. Let (u_h) and (v_h) be recovery sequences for u and v respectively, so that both $\mathcal{N}_h(u_h)$ and $\mathcal{N}_h(v_h)$ converge to 1. Hence, $u'_h = u_h/\mathcal{N}_h(u_h)$ and $v'_h = v_h/\mathcal{N}_h(v_h)$ still converge to u and v respectively. By assumption

$$\mathcal{N}_h\left(\frac{u'_h + v'_h}{2}\right) + \omega(\mathcal{N}_h(u'_h - v'_h)) \leq 1.$$

Thanks to property (a) of Γ-convergence, the monotonicity and the continuity of ω and the superadditivity of $\liminf$ we get

$$\begin{aligned}
\mathcal{N}&\left(\frac{u+v}{2}\right) + \omega\left(\mathcal{N}(u-v)\right) \\
&\leq \liminf_{h\to\infty} \mathcal{N}_h\left(\frac{u'_h + v'_h}{2}\right) + \omega\left(\liminf_{h\to\infty} \mathcal{N}_h(u'_h - v'_h)\right) \\
&\leq \liminf_{h\to\infty}\left(\mathcal{N}_h\left(\frac{u'_h + v'_h}{2}\right) + \omega(\mathcal{N}_h(u'_h - v'_h))\right) \leq 1.
\end{aligned}$$

Q.E.D.

2.5. Doubling metric measure spaces and maximal functions

From now on, $B(x,r)$ will denote the open ball centered in x of radius r and $\bar{B}(x,r)$ will denote the closed ball:

$$B(x,r) = \{y \in X \ : \ \mathsf{d}(x,y) < r\} \qquad \bar{B}(x,r) = \{y \in X \ : \ \mathsf{d}(x,y) \leq r\}.$$

If not specified, with the term ball we mean the open one.

Recall that a metric space (X, d) is doubling if there exists a natural number c_D such that every ball of radius r can be covered by at most c_D balls of halved radius $r/2$. While this condition will be sufficient to establish reflexivity of the Sobolev spaces, in the proof of lower semicontinuity of slope we shall actually need a stronger condition, involving also the reference measure $\mathfrak{m}$:

Definition 6 (Doubling m.m. spaces). *The metric measure space* $(X, \mathsf{d}, \mathfrak{m})$ *is doubling if there exists* $\tilde{c}_D \geq 0$ *such that*

$$\mathfrak{m}(B(x,2r)) \leq \tilde{c}_D \mathfrak{m}(B(x,r)) \qquad \forall x \in \operatorname{supp} \mathfrak{m},\ r > 0. \tag{2.5}$$

This condition is easily seen to be equivalent to the existence of two real positive numbers α, $\beta > 0$ which depend only on $\tilde{c}_D$ such that

$$\mathfrak{m}(B(x,r_1)) \le \beta \left(\frac{r_1}{r_2}\right)^{\alpha} \mathfrak{m}(B(y,r_2)) \tag{2.6}$$
$$\text{whenever } B(y,r_2) \subset B(x,r_1),\ r_2 \le r_1,\ y \in \operatorname{supp}\mathfrak{m}.$$

Indeed, $B(x,r_1) \subset B(y,2r_1)$, hence $\mathfrak{m}(B(x,r_1)) \le \tilde{c}_D^k \mathfrak{m}(B(y,r_2))$, where k is the smallest integer such that $2r_1 \le 2^k r_2$. Since $k \le 2 + \ln_2(r_1/r_2)$, we obtain (2.6) with $\alpha = \ln_2 \tilde{c}_D$ and $\beta = \tilde{c}_D^2$.

Condition (2.6) is stronger than the metric doubling property, in the sense that $(\operatorname{supp}\mathfrak{m}, \mathsf{d})$ is doubling whenever $(X, \mathsf{d}, \mathfrak{m})$ is. Indeed, given a ball $B(x,r)$ with $x \in \operatorname{supp}\mathfrak{m}$, let us choose recursively points $x_i \in B(x,r) \cap \operatorname{supp}\mathfrak{m}$ with $\mathsf{d}(x_i,x_j) \ge r/2$, and assume that this is possible for $i = 1, \ldots, N$. Then, the balls $B(x_i, r/4)$ are disjoint and

$$\mathfrak{m}\big(B\big(x_i, \frac{r}{4}\big)\big) \ge \tilde{c}_D^{-3} \mathfrak{m}(B(x_i, 2r)) \ge \tilde{c}_D^{-3} \mathfrak{m}(B(x,r)),$$

so that $N \le \tilde{c}_D^3$; in particular we can find a maximal finite set $\{x_i\}$ with this property, and from the maximality it follows that for every $x' \in B(x,r) \cap \operatorname{supp}\mathfrak{m}$ we have $\mathsf{d}(x_i, x') < r/2$ and so

$$B(x,r) \cap \operatorname{supp}\mathfrak{m} \subset \bigcup_i B(x_i, r/2).$$

It follows that $(\operatorname{supp}\mathfrak{m}, \mathsf{d})$ is doubling, with doubling constant $c_D \le \tilde{c}_D^3$. Conversely (but we shall not need this fact) any complete doubling metric space supports a nontrivial doubling measure (see [9, 20]).

Definition 7 (Local maximal function). *Given $q \in [1,\infty)$, $\varepsilon > 0$ and a Borel function $f : X \to \mathbb{R}$ such that $|f|^q$ is $\mathfrak{m}$-integrable on bounded sets, we define the ε-maximal function*

$$M_q^\varepsilon f(x) := \left(\sup_{0<r\le\varepsilon} \fint_{B(x,r)} |f|^q \,\mathrm{d}\mathfrak{m} \right)^{1/q} \qquad x \in \operatorname{supp}\mathfrak{m}.$$

The function $M_q^\varepsilon f(x)$ is nondecreasing w.r.t. ε, moreover $M_q^\varepsilon f(x) \to |f|(x)$ at any Lebesgue point x of $|f|^q$, namely a point $x \in \operatorname{supp}\mathfrak{m}$ satisfying

$$\lim_{r\downarrow 0} \frac{1}{\mathfrak{m}(B(x,r))} \int_{B(x,r)} |f(y)|^q \,\mathrm{d}\mathfrak{m}(y) = |f(x)|^q. \tag{2.7}$$

We recall that, in doubling metric measure spaces (see for instance [17]), under the previous assumptions on f we have that $\mathfrak{m}$-a.e. point is a Lebesgue point of $|f|^q$ (the proof is based on the so-called Vitali covering lemma). By applying this property to $|f-s|^q$ with $s \in \mathbb{Q}$ one even obtains

$$\lim_{r\downarrow 0} \frac{1}{\mathfrak{m}(B(x,r))} \int_{B(x,r)} |f(y)-f(x)|^q \, d\mathfrak{m}(y) = 0 \tag{2.8}$$

for every $x \in \operatorname{supp}\mathfrak{m}$ that is a Lebesgue point of $|f-s|^q$ for every $s \in \mathbb{Q}$. In particular it is clear that (2.8) is satisfied for $\mathfrak{m}$-a.e. $x \in \operatorname{supp}\mathfrak{m}$; we call such points q-Lebesgue points of f. We shall need a further enforcement of the q-Lebesgue point property:

Lemma 8. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a doubling metric measure space and let $f : X \to \mathbb{R}$ be a Borel function such that $|f|^q$ is $\mathfrak{m}$-integrable on bounded sets. Then, at any point x where* (2.8) *is satisfied, it holds*

$$\lim_{n\to\infty} \frac{1}{\mathfrak{m}(E_n)} \int_{E_n} |f(y)-f(x)|^q \, d\mathfrak{m}(y) = 0 \tag{2.9}$$

whenever $E_n \subset X$ are Borel sets satisfying $B(y_n, \tau r_n) \subset E_n \subset B(x, r_n)$ with $y_n \in \operatorname{supp}\mathfrak{m}$ and $r_n \to 0$, for some $\tau \in (0,1]$ independent of n. In particular $⨍_{E_n} f \, d\mathfrak{m} \to f(x)$.

Proof. Since $\mathfrak{m}$ is doubling we can use (2.6) to obtain

$$\begin{aligned}
\frac{1}{\mathfrak{m}(E_n)} & \int_{E_n} |f(y)-f(x)|^q \, d\mathfrak{m}(y) \\
&\le \frac{1}{\mathfrak{m}(B(y_n,\tau r_n))} \int_{E_n} |f(y)-f(x)|^q \, d\mathfrak{m}(y) \\
&\le \frac{1}{\mathfrak{m}(B(y_n,\tau r_n))} \int_{B(x,r_n)} |f(y)-f(x)|^q \, d\mathfrak{m}(y) \\
&\le \frac{\mathfrak{m}(B(x,r_n))}{\mathfrak{m}(B(y_n,\tau r_n))} ⨍_{B(x,r_n)} |f(y)-f(x)|^q \, d\mathfrak{m}(y) \\
&\le \beta\tau^{-\alpha} ⨍_{B(x,r_n)} |f(y)-f(x)|^q \, d\mathfrak{m}(y).
\end{aligned}$$

Since (2.8) is true by hypothesis, the last term goes to 0, and we proved (2.9). Finally, by Jensen's inequality,

$$\left| ⨍_{E_n} f \, d\mathfrak{m} - f(x) \right|^q \le ⨍_{E_n} |f - f(x)|^q \, d\mathfrak{m} \to 0.$$

Q.E.D.

§3. Hopf–Lax formula and Hamilton–Jacobi equation

Aim of this section is to study the properties of the Hopf–Lax formula in a metric space (X, d) and its relations with the Hamilton–Jacobi equation. Notice that there is no reference measure $\mathfrak{m}$ here and that not even completeness is needed for the results of this section. We fix a power $p \in (1, \infty)$ and denote by q its dual exponent.

Let $f : X \to \mathbb{R}$ be a Lipschitz function. For $t > 0$ define

$$F(t, x, y) := f(y) + \frac{\mathsf{d}^p(x, y)}{pt^{p-1}}, \tag{3.1}$$

and the function $Q_t f : X \to \mathbb{R}$ by

$$Q_t f(x) := \inf_{y \in X} F(t, x, y). \tag{3.2}$$

Notice that $Q_t f(x) \le f(x)$; on the other hand, if L denotes the Lipschitz constant of f, Young's inequality $(\mathsf{d}t^{-1/q})^p/p + (Lt^{1/q})^q/q \ge L\mathsf{d}$ gives

$$F(t, x, y) \ge f(x) - L\mathsf{d}(x, y) + \frac{\mathsf{d}(x, y)^p}{pt^{p-1}} \ge f(x) - t\frac{L^q}{q},$$

so that $Q_t f(x) \uparrow f(x)$ as $t \downarrow 0$.

Also, we introduce the functions D^+, $D^- : X \times (0, \infty) \to \mathbb{R}$ as

$$\begin{aligned} D^+(x, t) &:= \sup \limsup_{n \to \infty} \mathsf{d}(x, y_n), \\ D^-(x, t) &:= \inf \liminf_{n \to \infty} \mathsf{d}(x, y_n), \end{aligned} \tag{3.3}$$

where, in both cases, the sequences (y_n) vary among all minimizing sequences for $F(t, x, \cdot)$. We also set $Q_0 f = f$ and $D^\pm(x, 0) = 0$. Arguing as in [2, Lemma 3.1.2] it is easy to check that the map $X \times [0, \infty) \ni (x, t) \mapsto Q_t f(x)$ is continuous. Furthermore, the fact that f is Lipschitz easily yields

$$D^-(x, t) \le D^+(x, t) \le t(p \operatorname{Lip}(f))^{1/(p-1)}. \tag{3.4}$$

Proposition 9 (Monotonicity of $D^\pm$). *For all $x \in X$ it holds*

$$D^+(x, t) \le D^-(x, s) \qquad 0 \le t < s. \tag{3.5}$$

As a consequence, $D^+(x, \cdot)$ and $D^-(x, \cdot)$ are both nondecreasing, and they coincide with at most countably many exceptions in $[0, \infty)$.

Proof. Fix $x \in X$. For $t = 0$ there is nothing to prove. Now pick $0 < t < s$ and for every $\varepsilon \in (0,1)$ choose $x_{t,\varepsilon}$ and $x_{s,\varepsilon}$ minimizers up to ε of $F(t,x,\cdot)$ and $F(s,x,\cdot)$ respectively, namely such that $F(t,x,x_{t,\varepsilon})-\varepsilon \leq F(t,x,w)$ and $F(s,x,x_{s,\varepsilon}) - \varepsilon \leq F(s,x,w)$ for every $w \in X$. Let us assume that $\mathsf{d}(x,x_{t,\varepsilon}) \geq (1-\varepsilon)D^+(x,t)$ and $\mathsf{d}(x,x_{s,\varepsilon}) \leq D^-(x,s)+\varepsilon$. The minimality up to ε of $x_{t,\varepsilon}$, $x_{s,\varepsilon}$ gives

$$
\begin{aligned}
f(x_{t,\varepsilon}) + \frac{\mathsf{d}^p(x_{t,\varepsilon},x)}{pt^{p-1}} &\leq f(x_{s,\varepsilon}) + \frac{\mathsf{d}^p(x_{s,\varepsilon},x)}{pt^{p-1}} + \varepsilon \\
f(x_{s,\varepsilon}) + \frac{\mathsf{d}^p(x_{s,\varepsilon},x)}{ps^{p-1}} &\leq f(x_{t,\varepsilon}) + \frac{\mathsf{d}^p(x_{t,\varepsilon},x)}{ps^{p-1}} + \varepsilon.
\end{aligned}
$$

Adding up and using the fact that $\frac{1}{t} \geq \frac{1}{s}$ we deduce

$$
\begin{aligned}
(1-\varepsilon)^p D^+(x,t)^p \leq \mathsf{d}^p(x_{t,\varepsilon},x) &\leq \mathsf{d}^p(x_{s,\varepsilon},x) + 2p\varepsilon(t^{1-p}-s^{1-p})^{-1} \\
&\leq (D^-(x,s)+\varepsilon)^p + 2p\varepsilon(t^{1-p}-s^{1-p})^{-1}.
\end{aligned}
$$

Letting $\varepsilon \to 0$ we obtain (3.5). Combining this with the inequality $D^- \leq D^+$ we immediately obtain that both functions are nonincreasing. At a point of right continuity of $D^-(x,\cdot)$ we get

$$
D^+(x,t) \leq \inf_{s>t} D^-(x,s) = D^-(x,t).
$$

This implies that the two functions coincide out of a countable set. Q.E.D.

Next, we examine the semicontinuity properties of $D^\pm$. These properties imply that points (x,t) where the equality $D^+(x,t) = D^-(x,t)$ occurs are continuity points for both D^+ and D^-.

Proposition 10 (Semicontinuity of $D^\pm$)**.** *D^+ is upper semicontinuous and D^- is lower semicontinuous in $X \times [0,\infty)$.*

Proof. We prove lower semicontinuity of D^-, the proof of upper semicontinuity of D^+ being similar. Let (x_i,t_i) be any sequence converging to (x,t) such that the limit of $D^-(x_i,t_i)$ exists and assume that $t > 0$ (the case $t = 0$ is trivial). For every i, let (y_i^n) be a minimizing sequence of $F(t_i,x_i,\cdot)$ for which $\lim_n \mathsf{d}(y_i^n,x_i) = D^-(x_i,t_i)$, so that

$$
\lim_{n\to\infty} f(y_i^n) + \frac{\mathsf{d}^p(y_i^n,x_i)}{pt_i^{p-1}} = Q_{t_i}f(x_i).
$$

Using the continuity of Q_t we get

$$
\begin{aligned}
Q_t f(x) &= \lim_{i\to\infty}\lim_{n\to\infty} f(y_i^n) + \frac{\mathsf{d}^p(y_i^n,x_i)}{pt_i^{p-1}} \\
&\geq \limsup_{i\to\infty}\limsup_{n\to\infty} f(y_i^n) + \frac{\mathsf{d}^p(y_i^n,x)}{pt^{p-1}} \geq Q_t f(x),
\end{aligned}
$$

where the first inequality follows from the boundedness of y_i^n and the estimate

$$
\begin{aligned}
\frac{\mathsf{d}^p(y_i^n,x_i)}{pt_i^p} - \frac{\mathsf{d}^p(y_i^n,x)}{pt^p} \leq &\frac{\mathsf{d}(x,x_i)}{t_i^p}\big(\mathsf{d}(y_i^n,x_i)\vee\mathsf{d}(y_i^n,x)\big)^{p-1} \\
&+ \frac{\mathsf{d}(y_i^n,x)}{p}\cdot\frac{|t^{p-1}-t_i^{p-1}|}{(t_i t)^{p-1}}
\end{aligned}
$$

(which in turn can be proved thanks to the inequality $|a^p - b^p| \leq p|a - b|(a\vee b)^{p-1}$). Analogously

$$
\lim_{i\to\infty} D^-(x_i,t_i) = \lim_{i\to\infty}\lim_{n\to\infty}\mathsf{d}(y_i^n,x_i) \geq \limsup_{i\to\infty}\limsup_{n\to\infty}\mathsf{d}(y_i^n,x).
$$

Therefore by a diagonal argument we can find a minimizing sequence $(y_i^{n(i)})$ for $F(t,x,\cdot)$ with $\limsup_i \mathsf{d}(y_i^{n(i)},x) \leq \lim_i D^-(x_i,t_i)$, which gives the result. Q.E.D.

Proposition 11 (Time derivative of $Q_t f$). *The map $t\mapsto Q_t f$ is Lipschitz from $[0,\infty)$ to the extended metric space of continuous functions $C(X)$, endowed with the distance*

$$
\|f-g\|_\infty = \sup_{x\in X}|f(x)-g(x)|.
$$

Moreover, for all $x\in X$, it satisfies:

$$
\frac{\mathrm{d}}{\mathrm{d}t}Q_t f(x) = -\frac{1}{q}\left[\frac{D^\pm(x,t)}{t}\right]^p, \tag{3.6}
$$

for any $t>0$, with at most countably many exceptions.

Proof. Let $t<s$ and for every $\varepsilon\in(0,1)$ choose $x_{t,\varepsilon}$ and $x_{s,\varepsilon}$ minimizers up to ε of $F(t,x,\cdot)$ and $F(s,x,\cdot)$ respectively, namely such that $F(t,x,x_{t,\varepsilon})-\varepsilon \leq F(t,x,w)$ and $F(s,x,x_{s,\varepsilon})-\varepsilon\leq F(s,x,w)$ for every $w\in X$. Let us assume that $\mathsf{d}(x,x_{t,\varepsilon})\geq(1-\varepsilon)D^+(x,t)$ and $\mathsf{d}(x,x_{s,\varepsilon})\leq D^-(x,s)+\varepsilon$. We have

$$
\begin{aligned}
Q_s f(x) - Q_t f(x) &\leq F(s,x,x_{t,\varepsilon}) - F(t,x,x_{t,\varepsilon}) + \varepsilon \\
&= \frac{\mathsf{d}^p(x,x_{t,\varepsilon})}{p}\frac{t^{p-1}-s^{p-1}}{t^{p-1}s^{p-1}} + \varepsilon,
\end{aligned}
$$

$$
\begin{aligned}
Q_s f(x) - Q_t f(x) &\geq F(s, x, x_{s,\varepsilon}) - F(t, x, x_{s,\varepsilon}) - \varepsilon \\
&= \frac{\mathsf{d}^p(x, x_{s,\varepsilon})}{p} \frac{t^{p-1} - s^{p-1}}{t^{p-1} s^{p-1}} - \varepsilon.
\end{aligned}
$$

For ε small enough, dividing by $s - t$, using the definition of $x_{t,\varepsilon}$ and $x_{s,\varepsilon}$ and using the inequality $(p-1)t^{p-2} \leq \frac{s^{p-1} - t^{p-1}}{s-t} \leq (p-1)s^{p-2}$ we obtain

$$
\begin{aligned}
\frac{Q_s f(x) - Q_t f(x)}{s - t} &\leq -(1-\varepsilon)^p \frac{(D^+(x,t))^p}{q s^p} + \frac{\varepsilon}{s-t}, \\
\frac{Q_s f(x) - Q_t f(x)}{s - t} &\geq -\frac{\mathsf{d}^p(x, x_{s,\varepsilon})}{q t^p} - \frac{\varepsilon}{s-t},
\end{aligned}
$$

which gives as $\varepsilon \to 0$ that $t \mapsto Q_t f(x)$ is Lipschitz in $[\delta, T]$ for any $0 < \delta < T$ uniformly with respect to $x \in X$. Also, taking Proposition 9 into account, we get (3.6). Now notice that from (3.4) we get that $q|\frac{\mathrm{d}}{\mathrm{d}t} Q_t f(x)| \leq p^q [\mathrm{Lip}(f)]^q$ for any $x \in X$ and a.e. $t > 0$, which, together with the pointwise convergence of $Q_t f$ to f as $t \downarrow 0$, yields that $t \mapsto Q_t f \in C(X)$ is Lipschitz in $[0, \infty)$. Q.E.D.

We will bound from above the slope of $Q_t f$ at x with $|D^+(x,t)/t|^{p-1}$; actually we shall prove a more precise statement, which involves the *asymptotic Lipschitz constant*

$$
\mathrm{Lip}_a(f, x) := \inf_{r>0} \mathrm{Lip}\big(f, B(x,r)\big) = \lim_{r \downarrow 0} \mathrm{Lip}\big(f, B(x,r)\big). \tag{3.7}
$$

We collect some properties of the asymptotic Lipschitz constant in the next proposition.

Proposition 12. *Let $f : X \to \mathbb{R}$ be a Lipschitz function. Then*

$$
\mathrm{Lip}(f) \geq \mathrm{Lip}_a(f, x) \geq |\nabla f|^*(x), \tag{3.8}
$$

where $|\nabla f|^*$ is the upper semicontinuous envelope of the slope of f. In length spaces the second inequality is an equality.

Proof. The first inequality in (3.8) is trivial, while the second one follows by the fact that $\mathrm{Lip}_a(f, \cdot)$ is upper semicontinuous and larger than $|\nabla f|$. Since $|\nabla f|$ is an upper gradient of f, we have the inequality

$$
|f(y) - f(z)| \leq \int_0^{\ell(\gamma)} |\nabla f|(\gamma_t)\, dt
$$

for any curve γ with constant speed joining y to z. If (X, d) is a length space we can minimize w.r.t. γ to get

$$
\mathrm{Lip}\big(f, B(x,r)\big) \leq \sup_{B(x,3r)} |\nabla f| \leq \sup_{B(x,3r)} |\nabla f|^*.
$$

As $r \downarrow 0$ the inequality $\mathrm{Lip}_a(f,x) \le |\nabla f|^*(x)$ follows. Q.E.D.

Proposition 13 (Bound on the asymptotic Lipschitz constant of $Q_t f$)**.** *For $(x,t) \in X \times (0,\infty)$ it holds:*

$$\mathrm{Lip}_a(Q_t f, x) \le \left[\frac{D^+(x,t)}{t}\right]^{p-1}. \tag{3.9}$$

In particular $\mathrm{Lip}(Q_t(f)) \le p\mathrm{Lip}(f)$.

Proof. Fix y, $z \in X$ and $t \in (0,\infty)$. For every $\varepsilon > 0$ let $y_\varepsilon \in X$ be such that $F(t,y,y_\varepsilon) - \varepsilon \le F(t,y,w)$ for every $w \in X$ and $|\mathsf{d}(y,y_\varepsilon) - D^+(y,t)| \le \varepsilon$. Since it holds

$$\begin{aligned}
Q_t f(z) - Q_t f(y) &\le F(t,z,y_\varepsilon) - F(t,y,y_\varepsilon) + \varepsilon \\
&= f(y_\varepsilon) + \frac{\mathsf{d}^p(z,y_\varepsilon)}{pt^{p-1}} - f(y_\varepsilon) - \frac{\mathsf{d}^p(y,y_\varepsilon)}{pt^{p-1}} + \varepsilon \\
&\le \frac{(\mathsf{d}(z,y) + \mathsf{d}(y,y_\varepsilon))^p}{pt^{p-1}} - \frac{\mathsf{d}^p(y,y_\varepsilon)}{pt^{p-1}} + \varepsilon \\
&\le \frac{\mathsf{d}(z,y)}{t^{p-1}}\big(\mathsf{d}(z,y) + D^+(y,t) + \varepsilon\big)^{p-1} + \varepsilon,
\end{aligned}$$

so that letting $\varepsilon \to 0$, dividing by $\mathsf{d}(z,y)$ and inverting the roles of y and z gives

$$\mathrm{Lip}\big(Q_t f, B(x,r)\big) \le t^{1-p}\big(2r + \sup_{y\in B(x,r)} D^+(y,t)\big)^{p-1}.$$

Letting $r \downarrow 0$ and using the upper semicontinuity of D^+ we get (3.9).

Finally, the bound on the Lipschitz constant of $Q_t f$ follows directly from (3.4) and (3.9). Q.E.D.

Theorem 14 (Subsolution of HJ)**.** *For every $x \in X$ it holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}Q_t f(x) + \frac{1}{q}\mathrm{Lip}_a^q(Q_t f, x) \le 0 \tag{3.10}$$

for every $t \in (0,\infty)$, with at most countably many exceptions.

Proof. The claim is a direct consequence of Propositions 11 and 13. Q.E.D.

Notice that (3.10) is a stronger formulation of the HJ subsolution property

$$\frac{\mathrm{d}}{\mathrm{d}t}Q_t f(x) + \frac{1}{q}|\nabla Q_t f|^q(x) \le 0, \tag{3.11}$$

with the asymptotic Lipschitz constant $\mathrm{Lip}_a(Q_t f, \cdot)$ in place of $|\nabla Q_t f|$.

§4. Weak gradients

Let (X, d) be a complete and separable metric space and let $\mathfrak{m}$ be a nonnegative Borel measure in X (not even σ-finiteness is needed for the results of this section). In this section we introduce and compare two notions of weak gradient, one obtained by relaxation of the asymptotic Lipschitz constant, the other one obtained by a suitable weak upper gradient property. Eventually we will show that the two notions of gradient coincide: this will lead also to the coincidence with the other intermediate notions of gradient considered in [7], [24], [27], described in the appendix.

4.1. Relaxed slope and $|\nabla f|_{*,q}$

The following definition is a variation of the one considered in [7] (where the relaxation procedure involved upper gradients) and of the one considered in [3] (where the relaxation procedure involved slopes of Lipschitz functions). The use of the (stronger) asymptotic Lipschitz constant has been suggested in the final section of [4]: it is justified by the subsolution property (3.10) and it leads to stronger density results. In the spirit of the Sobolev space theory, these should be considered as "H definitions", since approximation with Lipschitz functions with bounded support are involved.

Definition 15 (Relaxed slope). *We say that $g \in L^q(X, \mathfrak{m})$ is a q-relaxed slope of $f \in L^q(X, \mathfrak{m})$ if there exist $\tilde{g} \in L^q(X, \mathfrak{m})$ and Lipschitz functions with bounded support f_n such that:*

(a) *$f_n \to f$ in $L^q(X, \mathfrak{m})$ and $\mathrm{Lip}_a(f_n, \cdot)$ weakly converge to $\tilde{g}$ in $L^q(X, \mathfrak{m})$;*

(b) *$\tilde{g} \le g$ $\mathfrak{m}$-a.e. in X.*

We say that g is the minimal q-relaxed slope of f if its $L^q(X, \mathfrak{m})$ norm is minimal among q-relaxed slopes. We shall denote by $|\nabla f|_{*,q}$ the minimal q-relaxed slope (also called the q-relaxed gradient).

By this definition and the sequential compactness of weak topologies, any L^q limit of Lipschitz functions f_n with bounded support and with $\int \mathrm{Lip}_a^q(f_n, \cdot)\,\mathrm{d}\mathfrak{m}$ uniformly bounded has a q-relaxed slope. On the other hand, using Mazur's lemma (see [3, Lemma 4.3] for details), the definition of q-relaxed slope would be unchanged if the weak convergence of $\mathrm{Lip}_a(f_n, \cdot)$ in (a) were replaced by the condition $\mathrm{Lip}_a(f_n, \cdot) \le g_n$ and $g_n \to \tilde{g}$ strongly in $L^q(X, \mathfrak{m})$. This alternative characterization of q-relaxed slopes is suitable for diagonal arguments and proves, together with (2.1a), that the collection of q-relaxed slopes is a closed convex set, possibly empty. Hence, thanks to the uniform convexity of $L^q(X, \mathfrak{m})$, the definition of $|\nabla f|_{*,q}$ is well posed. Also, arguing as in [3] and using once more the uniform convexity of $L^q(X,\mathfrak{m})$, it is not difficult to show the following result:

Proposition 16. *If $f \in L^q(X,\mathfrak{m})$ has a q-relaxed slope then there exist Lipschitz functions f_n with bounded support satisfying*

$$\lim_{n\to\infty} \int_X |f_n - f|^q \,\mathrm{d}\mathfrak{m} + \int_X \left|\mathrm{Lip}_a(f_n,\cdot) - |\nabla f|_{*,q}\right|^q \mathrm{d}\mathfrak{m} = 0. \tag{4.1}$$

Notice that in principle the integrability of f could be decoupled from the integrability of the gradient, because no global Poincaré inequality can be expected at this level of generality. Indeed, to increase the symmetry with the definition of weak upper gradient (which involves no integrability assumption on f), one might even consider the convergence $\mathfrak{m}$-a.e. of the approximating functions, removing any integrability assumption. We have left the convergence in L^q because this presentation is more consistent with the usual presentations of Sobolev spaces, and the definitions given in [7] and [3]. Using locality and a truncation argument, the definitions can be extended to more general classes of functions, see (5.2). In this connection, we should also mention that in [7] and [3] the approximating functions are not required to have bounded support. However, we may fix $x_0 \in X$ and a sequence of Lipschitz functions $\chi_k : X \to [0,1]$ with $\chi_k \equiv 1$ on $B_k(x_0)$, $\chi_k \equiv 1$ on $X \setminus B_{k+1}(x_0)$, $\mathrm{Lip}(\chi_k) \le 1$. Since for any locally Lipschitz function $f \in L^q(X,\mathfrak{m})$ with $\mathrm{Lip}_a(f) \in L^q(X,\mathfrak{m})$ the functions $f\chi_k$ have bounded support and satisfy

$$f\chi_k \to f \quad \text{in } L^q(X,\mathfrak{m}), \qquad \mathrm{Lip}_a(f\chi_k) \to \mathrm{Lip}_a(f) \quad \text{in } L^q(X,\mathfrak{m}),$$

a diagonal argument proves that the class of relaxed slopes is unchanged.

Lemma 17 (Pointwise minimality of $|\nabla f|_{*,q}$)**.** *Let g_1, g_2 be two q-relaxed slopes of f. Then* $\min\{g_1, g_2\}$ *is a q-relaxed slope as well. In particular, not only the L^q norm of $|\nabla f|_{*,q}$ is minimal, but also $|\nabla f|_{*,q} \le g$ $\mathfrak{m}$-a.e. in X for any q-relaxed slope g of f.*

Proof. We argue as in [7], [3]. First we notice that for every f, $g \in \mathrm{Lip}(X)$

$$\mathrm{Lip}_a(f+g,x) \le \mathrm{Lip}_a(f,x) + \mathrm{Lip}_a(g,x) \qquad \forall x \in X, \tag{4.2}$$

$$\mathrm{Lip}_a(fg,x) \le |f(x)|\,\mathrm{Lip}_a(g,x) + |g(x)|\,\mathrm{Lip}_a(f,x) \qquad \forall x \in X. \tag{4.3}$$

Indeed (4.2) is obvious; for (4.3) we have that

$$|f(z)g(z) - f(y)g(y)| \le |f(z)||g(z)-g(y)| + |g(y)||f(z)-f(y)| \quad \forall y,\, z \in X,$$

so that

$$\begin{aligned}\operatorname{Lip}(fg, B(x,r)) \leq & \sup_{z\in B(x,r)} |f(z)| \operatorname{Lip}(g, B(x,r)) \\ & + \sup_{y\in B(x,r)} |g(y)| \operatorname{Lip}(f, B(x,r)) \qquad \forall x \in X\end{aligned}$$

and we let $r \to 0$.

It is sufficient to prove that if $B \subset X$ is a Borel set, then $\chi_B g_1 + \chi_{X\setminus B} g_2$ is a q-relaxed slope of f. Let us consider the class $\mathcal{F}$ of all Borel sets B satisfying this property for any pair (g_1, g_2) of relaxed slopes of f. Clearly the class $\mathcal{F}$ is stable under complement; in addition, taking into account the closure of the class of q-relaxed slopes under L^q convergence, $B_n \uparrow B$ and $B_n \in \mathcal{F}$ implies $B \in \mathcal{F}$. Finally, it is easily seen that $\mathcal{F}$ is stable under finite disjoint unions. Hence, by Dynkin's theorem, to prove that any Borel set belongs to $\mathcal{F}$ suffices to show that open sets belong to $\mathcal{F}$.

We fix an open set B, $r > 0$ and a Lipschitz function $\phi_r : X \to [0,1]$ equal to 0 on $X \setminus B_r$ and equal to 1 on B_{2r}, where the open sets $B_s \subset B$ are defined by

$$B_s := \{x \in X : \operatorname{dist}(x, X \setminus B) > s\} \subset B.$$

Let now $f_{n,i}$, $i = 1, 2$, be Lipschitz functions with bounded support converging to f in $L^q(X, \mathfrak{m})$ as $n \to \infty$, with $\operatorname{Lip}_a(f_{n,i}, \cdot)$ weakly convergent to $\tilde{g}_i$ in $L^q(X, \mathfrak{m})$ and set $f_n := \phi_r f_{n,1} + (1 - \phi_r) f_{n,2}$. Then, $\operatorname{Lip}_a(f_n, \cdot) = \operatorname{Lip}_a(f_{n,1}, \cdot)$ on B_{2r} and $\operatorname{Lip}_a(f_n, \cdot) = \operatorname{Lip}_a(f_{n,2}, \cdot)$ on $X \setminus \overline{B_r}$; for every $x \in \overline{B_r} \setminus B_{2r}$, by applying (4.2) to $f_{n,2}$ and $\phi_r(f_{n,1} - f_{n,2})$ and by applying (4.3) to ϕ_r and$(f_{n,1} - f_{n,2})$, we can estimate

$$\begin{aligned}\operatorname{Lip}_a(f_n, x) \leq & \operatorname{Lip}_a(f_{n,2}, x) + \operatorname{Lip}(\phi_r)|f_{n,1}(x) - f_{n,2}(x)| \\ & + \phi_r\big(\operatorname{Lip}_a(f_{n,1}, x) + \operatorname{Lip}_a(f_{n,2}, x)\big).\end{aligned}$$

Since $\overline{B_r} \subset B$, by taking weak limits of a subsequence, it follows that

$$\chi_{B_{2r}} g_1 + \chi_{X\setminus \overline{B_r}} g_2 + \chi_{B\setminus B_{2r}}(g_1 + 2g_2)$$

is a q-relaxed slope of f. Letting $r \downarrow 0$ gives that $\chi_B g_1 + \chi_{X\setminus B} g_2$ is a q-relaxed slope as well.

For the second part of the statement argue by contradiction: let g be a q-relaxed slope of f and assume that $B = \{g < |\nabla f|_{*,q}\}$ is such that $\mathfrak{m}(B) > 0$. Consider the q-relaxed slope $g\chi_B + |\nabla f|_{*,q}\chi_{X\setminus B}$: its L^q norm is strictly less than the L^q norm of $|\nabla f|_{*,q}$, which is a contradiction. Q.E.D.

The previous pointwise minimality property immediately yields

$$|\nabla f|_{*,q} \le \mathrm{Lip}_a(f,\cdot) \qquad \mathfrak{m}\text{-a.e. in } X \tag{4.4}$$

for any Lipschitz function $f : X \to \mathbb{R}$ with bounded support. Since both objects are local, the inequality immediately extends by a truncation argument to all functions $f \in L^q(X,\mathfrak{m})$ with a q-relaxed slope, Lipschitz on bounded sets.

Also the proof of locality and chain rule is quite standard, see [7] and [3, Proposition 4.8] for the case $q = 2$ (the same proof works in the general case).

Proposition 18 (Locality and chain rule). *If $f \in L^q(X,\mathfrak{m})$ has a q-relaxed slope, the following properties hold.*

(a) $|\nabla h|_{*,q} = |\nabla f|_{*,q}$ *$\mathfrak{m}$-a.e. in $\{h = f\}$ whenever f has a q-relaxed slope.*

(b) $|\nabla\phi(f)|_{*,q} \le |\phi'(f)||\nabla f|_{*,q}$ *for any C^1 and Lipschitz function ϕ on an interval containing the image of f. Equality holds if ϕ is nondecreasing.*

4.2. q-weak upper gradients and $|\nabla f|_{w,q}$

Recall that the evaluation maps $\mathrm{e}_t : C([0,1],X) \to X$ are defined by $\mathrm{e}_t(\gamma) := \gamma_t$. We also introduce the restriction maps $\mathrm{restr}_t^s : C([0,1],X) \to C([0,1],X)$, $0 \le t \le s \le 1$, given by

$$\mathrm{restr}_t^s(\gamma)_r := \gamma_{(1-r)t+rs}, \tag{4.5}$$

so that restr_t^s "stretches" the restriction of the curve to $[s,t]$ to the whole of $[0,1]$.

Our definition of q-weak upper gradient is inspired by [24], [27], allowing for exceptional curves in (2.2), but with a different notion of exceptional set, compared to [24], [27]. We recall that p is the dual exponent of q.

Definition 19 (Test plans and negligible sets of curves). *We say that a probability measure $\boldsymbol{\pi} \in \mathscr{P}(C([0,1],X))$ is a p-*test plan *if $\boldsymbol{\pi}$ is concentrated on $AC^p([0,1],X)$, $\iint_0^1 |\dot\gamma_t|^p \mathrm{d}t\,\mathrm{d}\boldsymbol{\pi} < \infty$ and there exists a constant $C(\boldsymbol{\pi})$ such that*

$$(\mathrm{e}_t)_\sharp\boldsymbol{\pi} \le C(\boldsymbol{\pi})\mathfrak{m} \qquad \forall t \in [0,1]. \tag{4.6}$$

*A set $A \subset C([0,1],X)$ is said to be q-*negligible *if it is contained in a $\boldsymbol{\pi}$-negligible set for any p-test plan $\boldsymbol{\pi}$. A property which holds for every $\gamma \in C([0,1],X)$, except possibly a q-negligible set, is said to hold for q-almost every curve.*

Observe that, by definition, $C([0,1],X) \setminus AC^p([0,1],X)$ is q-negligible, so the notion starts to be meaningful when we look at subsets of $AC^p([0,1],X)$.

Remark 20. An easy consequence of condition (4.6) is that if two $\mathfrak{m}$-measurable functions f, $g : X \to \mathbb{R}$ coincide up to a $\mathfrak{m}$-negligible set and $\mathcal{T}$ is an at most countable subset of $[0,1]$, then the functions $f \circ \gamma$ and $g \circ \gamma$ coincide in $\mathcal{T}$ for q-almost every curve γ.

Moreover, choosing an arbitrary p-test plan $\boldsymbol{\pi}$ and applying Fubini's Theorem to the product measure $\mathscr{L}^1 \times \boldsymbol{\pi}$ in $(0,1) \times C([0,1];X)$ we also obtain that $f \circ \gamma = g \circ \gamma$ $\mathscr{L}^1$-a.e. in $(0,1)$ for $\boldsymbol{\pi}$-a.e. curve γ; since $\boldsymbol{\pi}$ is arbitrary, the same property holds for q-a.e. γ.

Coupled with the definition of q-negligible set of curves, there are the definitions of q-weak upper gradient and of functions which are Sobolev along q-a.e. curve.

Definition 21 (q-weak upper gradients). *A Borel function $g : X \to [0,\infty]$ is a q-weak upper gradient of $f : X \to \mathbb{R}$ if*

$$\left| \int_{\partial\gamma} f \right| \leq \int_\gamma g < \infty \qquad \text{for } q\text{-a.e. } \gamma. \tag{4.7}$$

Definition 22 (Sobolev functions along q-a.e. curve). *A function $f : X \to \mathbb{R}$ is Sobolev along q-a.e. curve if for q-a.e. curve γ the function $f \circ \gamma$ coincides a.e. in $[0,1]$ and in $\{0,1\}$ with an absolutely continuous map $f_\gamma : [0,1] \to \mathbb{R}$.*

By Remark 20 applied to $\mathcal{T} := \{0,1\}$, (4.7) does not depend on the particular representative of f in the class of $\mathfrak{m}$-measurable function coinciding with f up to a $\mathfrak{m}$-negligible set. The same Remark also shows that the property of being Sobolev along q-q.e. curve γ is independent of the representative in the class of $\mathfrak{m}$-measurable functions coinciding with f $\mathfrak{m}$-a.e. in X.

In the next proposition, based on Lemma 1, we prove that the existence of a q-weak upper gradient g implies Sobolev regularity along q-a.e. curve.

Proposition 23. *Let $f : X \to \mathbb{R}$ be $\mathfrak{m}$-measurable, and let g be a q-weak upper gradient of f. Then f is Sobolev along q-a.e. curve.*

Proof. Notice that if $\boldsymbol{\pi}$ is a p-test plan, so is $(\mathrm{restr}_t^s)_\sharp\boldsymbol{\pi}$. Hence if g is a q-weak upper gradient of f such that $\int_\gamma g < \infty$ for q-a.e. γ, then for every $t < s$ in $[0,1]$ it holds

$$|f(\gamma_s) - f(\gamma_t)| \leq \int_t^s g(\gamma_r)|\dot\gamma_r| \, \mathrm{d}r \qquad \text{for } q\text{-a.e. } \gamma.$$

Let $\boldsymbol{\pi}$ be a p-test plan: by Fubini's theorem applied to the product measure $\mathscr{L}^2 \times \boldsymbol{\pi}$ in $(0,1)^2 \times C([0,1];X)$, it follows that for $\boldsymbol{\pi}$-a.e. γ the function f satisfies

$$|f(\gamma_s) - f(\gamma_t)| \leq \left| \int_t^s g(\gamma_r)|\dot{\gamma}_r| \,\mathrm{d}r \right| \qquad \text{for } \mathscr{L}^2\text{-a.e. } (t,s) \in (0,1)^2.$$

An analogous argument shows that for $\boldsymbol{\pi}$-a.e. γ

$$\text{(4.8)} \qquad \begin{cases} |f(\gamma_s) - f(\gamma_0)| \leq \int_0^s g(\gamma_r)|\dot{\gamma}_r| \,\mathrm{d}r \\ |f(\gamma_1) - f(\gamma_s)| \leq \int_s^1 g(\gamma_r)|\dot{\gamma}_r| \,\mathrm{d}r \end{cases} \qquad \text{for } \mathscr{L}^1\text{-a.e. } s \in (0,1).$$

Since $g \circ \gamma |\dot{\gamma}| \in L^1(0,1)$ for $\boldsymbol{\pi}$-a.e. γ, by Lemma 1 it follows that $f \circ \gamma \in W^{1,1}(0,1)$ for $\boldsymbol{\pi}$-a.e. γ, and

$$\text{(4.9)} \qquad \left| \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma) \right| \leq g \circ \gamma |\dot{\gamma}| \quad \text{a.e. in } (0,1), \text{ for } \boldsymbol{\pi}\text{-a.e. } \gamma.$$

Since $\boldsymbol{\pi}$ is arbitrary, we conclude that $f \circ \gamma \in W^{1,1}(0,1)$ for q-a.e. γ, and therefore it admits an absolutely continuous representative f_γ; moreover, by (4.8), it is immediate to check that $f(\gamma_t) = f_\gamma(t)$ for $t \in \{0,1\}$ and q-a.e. γ. Q.E.D.

The last statement of the proof above and (4.9) yield the following

$$\text{(4.10)} \qquad \begin{aligned} &g_i,\ i = 1,2 \ q\text{-weak upper gradients of } f \\ &\qquad \Longrightarrow \quad \min\{g_1, g_2\} \ q\text{-weak upper gradient of } f. \end{aligned}$$

Using this stability property we can recover, as we did for relaxed slopes, a distinguished minimal object.

Definition 24 (Minimal q-weak upper gradient). *Let $f : X \to \mathbb{R}$ be a $\mathfrak{m}$-measurable function having at least a q-weak upper gradient $g_0 : X \to [0,\infty]$ such that $\{g_0 > 0\}$ is σ-finite with respect to $\mathfrak{m}$. The minimal q-weak upper gradient $|\nabla f|_{w,q}$ of f is the q-weak upper gradient characterized, up to $\mathfrak{m}$-negligible sets, by the property*

(4.11)

$$|\nabla f|_{w,q} \leq g \qquad \mathfrak{m}\text{-}a.e.\ in\ X,\ for\ every\ q\text{-}weak\ upper\ gradient\ g\ of\ f.$$

We will refer to it also as the q-weak gradient of f.

Uniqueness of the minimal weak upper gradient is obvious. For existence, since $\{g_0 > 0\}$ is σ-finite we can find a Borel and $\mathfrak{m}$-integrable function $\theta : X \to [0,\infty)$ which is positive on $\{g_0 > 0\}$ and $|\nabla f|_{w,q} :=$

$\inf_n g_n$, where g_n are q-weak upper gradients which provide a minimizing sequence in

$$\inf\left\{\int_X \theta\tan^{-1} g\,\mathrm{d}\mathfrak{m}:\ g\le g_0 \text{ is a } q\text{-weak upper gradient of } f\right\}.$$

We immediately see, thanks to (4.10), that we can assume with no loss of generality that $g_{n+1}\le g_n$. Hence, applying (4.7) to g_n and by monotone convergence, the function $|\nabla f|_{w,q}$ is a q-weak upper gradient of f and $\int_X \theta\tan^{-1} g\,\mathrm{d}\mathfrak{m}$ is minimal at $g=|\nabla f|_{w,q}$. This minimality, in conjunction with (4.10), gives (4.11).

Remark 25. Notice that the σ-finiteness assumption on $\{g_0>0\}$ automatically holds if $\int g_0^\alpha\,\mathrm{d}\mathfrak{m}<\infty$ for some $\alpha>0$. The following example shows that in order to get a minimal object we really need, unlike the theory of relaxed gradients, a σ-finiteness assumption.

Let $X=\mathbb{R}$, d the Euclidean distance, $\mathfrak{m}$ the counting measure, $f:X\to\mathbb{R}$ equal to the identity map. It is easily seen that g is a q-weak upper gradient of f (and actually an upper gradient) if and only if $g\ge 1$ $\mathscr{L}^1$-a.e. in X. In this class, there is no minimal function up to $\mathfrak{m}$-negligible sets, since we can always modify a q-weak upper gradient at a single point (thus in a set with positive $\mathfrak{m}$-measure) preserving the q-weak upper gradient property.

Next we consider the stability of q-weak upper gradients (analogous to the stability result given in [27, Lemma 4.11]). We shall actually need a slightly more general statement, which involves a weaker version of the upper gradient property (when $\varepsilon=0$ we recover the previous definition, since curves with 0 length are constant).

Definition 26 (q-weak upper gradient up to scale ε)**.** *Let $f:X\to\mathbb{R}$. We say that a Borel function $g:X\to[0,\infty)$ is a q-weak upper gradient of f up to scale $\varepsilon\ge 0$ if for q-a.e. curve $\gamma\in AC^p([0,1];X)$ such that*

$$\varepsilon<\int_0^1|\dot\gamma_t|\,\mathrm{d}t$$

it holds

$$\left|\int_{\partial\gamma} f\right|\le\int_\gamma g<\infty. \tag{4.12}$$

Theorem 27 (Stability w.r.t. $\mathfrak{m}$-a.e. convergence)**.** *Assume that f_n are $\mathfrak{m}$-measurable, $\varepsilon_n\ge 0$ and that $g_n\in L^q(X,\mathfrak{m})$ are q-weak upper gradients of f_n up to scale ε_n. Assume furthermore that $f_n(x)\to f(x)\in\mathbb{R}$ for $\mathfrak{m}$-a.e. $x\in X$, $\varepsilon_n\to\varepsilon$ and that (g_n) weakly converges to g in $L^q(X,\mathfrak{m})$. Then g is a q-weak upper gradient of f up to scale ε.*

Proof. Fix a p-test plan $\boldsymbol{\pi}$. We have to show that (4.12) holds for $\boldsymbol{\pi}$-a.e. γ with $\int_0^1 |\dot\gamma_t|\,\mathrm{d}t > \varepsilon$. Possibly restricting $\boldsymbol{\pi}$ to a smaller set of curves, we can assume with no loss of generality that

$$\int_0^1 |\dot\gamma_t|\,\mathrm{d}t > \varepsilon' \qquad \text{for } \boldsymbol{\pi}\text{-a.e. } \gamma$$

for some $\varepsilon' > \varepsilon$. We consider in the sequel integers h sufficiently large, such that $\varepsilon_h \le \varepsilon'$.

By Mazur's lemma we can find convex combinations

$$h_n := \sum_{i=N_h+1}^{N_{h+1}} \alpha_i g_i \qquad \text{with } \alpha_i \ge 0,\ \sum_{i=N_h+1}^{N_{h+1}} \alpha_i = 1,\ N_h \to \infty$$

converging strongly to g in $L^q(X, \mathfrak{m})$. Denoting by $\tilde f_n$ the corresponding convex combinations of f_n, h_n are q-weak upper gradients of $\tilde f_n$ and still $\tilde f_n \to f$ $\mathfrak{m}$-a.e. in X.

Since for every nonnegative Borel function $\varphi : X \to [0, \infty]$ it holds (with $C = C(\boldsymbol{\pi})$)

$$\begin{aligned}
\int \Big(\int_\gamma \varphi\Big)\,\mathrm{d}\boldsymbol{\pi} &= \int \Big(\int_0^1 \varphi(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t\Big)\,\mathrm{d}\boldsymbol{\pi} \\
&\le \int \Big(\int_0^1 \varphi^q(\gamma_t)\,\mathrm{d}t\Big)^{1/q}\Big(\int_0^1 |\dot\gamma_t|^p\,\mathrm{d}t\Big)^{1/p}\,\mathrm{d}\boldsymbol{\pi} &&(4.13)\\
&\le \Big(\int_0^1 \int \varphi^q\,\mathrm{d}(\mathrm{e}_t)_\sharp\boldsymbol{\pi}\,\mathrm{d}t\Big)^{1/q}\Big(\iint_0^1 |\dot\gamma_t|^p\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}\Big)^{1/p} \\
&\le \Big(C\int \varphi^q\,\mathrm{d}\mathfrak{m}\Big)^{1/q}\Big(\iint_0^1 |\dot\gamma_t|^p\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}\Big)^{1/p}, &&(4.14)
\end{aligned}$$

we obtain

$$\int\int_\gamma |h_n - g|\,\mathrm{d}\boldsymbol{\pi} \le C^{1/q}\Big(\iint_0^1 |\dot\gamma_t|^p\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}\Big)^{1/p}\|h_n - g\|_q \to 0.$$

Hence we can find a subsequence $n(k)$ such that

$$\lim_{k\to\infty} \int_\gamma |h_{n(k)} - g| \to 0 \qquad \text{for } \boldsymbol{\pi}\text{-a.e. } \gamma.$$

Since $\tilde f_n$ converge $\mathfrak{m}$-a.e. to f and the marginals of $\boldsymbol{\pi}$ are absolutely continuous w.r.t. $\mathfrak{m}$ we have also that for $\boldsymbol{\pi}$-a.e. γ it holds $\tilde f_n(\gamma_0) \to f(\gamma_0)$ and $\tilde f_n(\gamma_1) \to f(\gamma_1)$.

If we fix a curve γ satisfying these convergence properties, we can pass to the limit as $k \to \infty$ in the inequalities $|\int_{\partial\gamma} \tilde{f}_{n(k)}| \leq \int_\gamma h_{n(k)}$ to get $|\int_{\partial\gamma} f| \leq \int_\gamma g$. Q.E.D.

Combining Proposition 16 with the fact that the asymptotic Lipschitz constant is an upper gradient (and in particular a q-weak upper gradient), the previous stability property gives that $|\nabla f|_{*,q}$ is a q-weak upper gradient. Then, (4.11) gives

$$|\nabla f|_{w,q} \leq |\nabla f|_{*,q} \qquad \mathfrak{m}\text{-a.e. in } X \tag{4.15}$$

whenever $f \in L^q(X, \mathfrak{m})$ has a q-relaxed slope. The proof of the converse inequality (under no extra assumption on the metric measure structure) requires much deeper ideas, described in the next two sections.

§5. Gradient flow of $\mathbf{C}_q$ and energy dissipation

In this section we assume that (X, d) is complete and separable, and that $\mathfrak{m}$ is a finite Borel measure.

As in the previous sections, $q \in (1, \infty)$ and p is the dual exponent. In order to apply the theory of gradient flows of convex functionals in Hilbert spaces, when $q > 2$ we need to extend $|\nabla f|_{*,q}$ also to functions in $L^2(X, \mathfrak{m})$ (because Definition 15 was given for $L^q(X, \mathfrak{m})$ functions). To this aim, we denote $f^N := \max\{-N, \min\{f, N\}\}$ and set

$$\mathcal{C} := \left\{f : X \to \mathbb{R} :\ f^N \text{ has a } q\text{-relaxed slope for all } N \in \mathbb{N}\right\}. \tag{5.1}$$

Accordingly, for all $f \in \mathcal{C}$ we set

$$|\nabla f|_{*,q} := |\nabla f^N|_{*,q} \qquad \mathfrak{m}\text{-a.e. in } \{|f| < N\} \tag{5.2}$$

for all $N \in \mathbb{N}$. We can use the locality property in Proposition 18(a) to show that this definition is well posed, up to $\mathfrak{m}$-negligible sets, and consistent with the previous one. Furthermore, locality and chain rules still apply, so we shall not use a distinguished notation for the new gradient.

We define an auxiliary functional, suitable for the Hilbertian energy dissipation estimates, by

$$\mathbf{C}_q(f) := \frac{1}{q}\int_X |\nabla f|^q_{*,q}\,\mathrm{d}\mathfrak{m} \qquad \text{if } f \in L^2(X, \mathfrak{m}) \cap \mathcal{C} \tag{5.3}$$

and set to $+\infty$ if $f \in L^2(X, \mathfrak{m}) \setminus \mathcal{C}$. We note that, thanks to the sublinearity of the minimal q-relaxed slope, $\mathcal{C}$ as well as the domain of finiteness of $\mathbf{C}_q$ are vector spaces.

Theorem 28. *The functional $\mathbf{C}_q$ is convex and lower semicontinuous in $L^2(X, \mathfrak{m})$.*

Proof. The proof of convexity is elementary, so we focus on lower semicontinuity. Let (f_n) be convergent to f in $L^2(X, \mathfrak{m})$ and assume, possibly extracting a subsequence and with no loss of generality, that $\mathbf{C}_q(f_n)$ converges to a finite limit.

Assume first that all f_n are uniformly bounded, so that $f_n \to f$ also in $L^q(X, \mathfrak{m})$ (because $\mathfrak{m}$ is finite). Let $f_{n(k)}$ be a subsequence such that $|\nabla f_{n(k)}|_{*,q}$ weakly converges to g in $L^q(X, \mathfrak{m})$. Then, since we can use Proposition 16 to find Lipschitz functions g_k with bounded support satisfying

$$\lim_{k\to\infty} \int_X |f_{n(k)} - g_k|^q \, d\mathfrak{m} + \int_X \big| |\nabla f_{n(k)}|_{*,q} - \mathrm{Lip}_a(g_k, \cdot) \big|^q \, d\mathfrak{m} = 0,$$

we obtain that g is a q-relaxed slope of f and

$$\mathbf{C}_q(f) \le \frac{1}{q} \int_X |g|^q \, d\mathfrak{m} \le \liminf_{k\to\infty} \frac{1}{q} \int_X |\nabla f_{n(k)}|^q_{*,q} \, d\mathfrak{m} = \liminf_{n\to\infty} \mathbf{C}_q(f_n).$$

In the general case when $f_n \in \mathcal{C}$ we consider the functions $f_n^N := \max\{-N, \min\{f_n, N\}\}$; the pointwise inequality

$$\mathrm{Lip}_a(\max\{-N, \min\{g, N\}\}, x) \le \mathrm{Lip}_a(g, x)$$

and the $\mathfrak{m}$-a.e. minimality property of the q-relaxed slope immediately give $|\nabla f_n^N|_{*,q} \le |\nabla f_n|_{*,q}$ $\mathfrak{m}$-a.e. in X, so that the previously considered case of uniformly bounded functions gives $f^N := \max\{-N, \min\{f, N\}\}$ has q-relaxed slope for any $N \in \mathbb{N}$ and

$$\int_X |\nabla f^N|^q_{*,q} \, d\mathfrak{m} \le \liminf_{n\to\infty} \int_X |\nabla f_n^N|^q_{*,q} \, d\mathfrak{m} \le \liminf_{n\to\infty} \int_X |\nabla f_n|^q_{*,q} \, d\mathfrak{m}.$$

Passing to the limit as $N \to \infty$, the conclusion follows by monotone convergence. Q.E.D.

Remark 29. More generally, the same argument proves the $L^2(X, \mathfrak{m})$-lower semicontinuity of the functional

$$f \mapsto \int_X \frac{|\nabla f|^q_{*,q}}{|f|^\alpha} \, d\mathfrak{m}$$

in $\mathcal{C}$, for any $\alpha > 0$. Indeed, locality and chain rule allow the reduction to nonnegative functions f_n and we can use the truncation argument of Theorem 28 to reduce ourselves to functions with values in an interval

$[c, C]$ with $0 < c \leq C < \infty$. In this class, we can again use the chain rule to prove the identity

$$\int_X |\nabla f^\beta|_{*,q}^q \,\mathrm{dm} = |\beta|^q \int_X \frac{|\nabla f|_{*,q}^q}{|f|^\alpha} \,\mathrm{dm}$$

with $\beta := 1 - \alpha/q$ to obtain the result when $\alpha \neq q$. If $\alpha = q$ we use a logarithmic transformation.

Since the finiteness domain of $\mathbf{C}_q$ is dense in $L^2(X, \mathfrak{m})$ (it includes bounded Lipschitz functions), the Hilbertian theory of gradient flows (see for instance [6], [2]) can be applied to Cheeger's functional (5.3) to provide, for all $f_0 \in L^2(X, \mathfrak{m})$, a locally absolutely continuous map $t \mapsto f_t$ from $(0, \infty)$ to $L^2(X, \mathfrak{m})$, with $f_t \to f_0$ as $t \downarrow 0$, whose derivative satisfies

$$\frac{d}{dt} f_t \in -\partial^- \mathbf{C}_q(f_t) \qquad \text{for a.e. } t \in (0, \infty). \tag{5.4}$$

Having in mind the regularizing effect of gradient flows, namely the selection of elements with minimal $L^2(X, \mathfrak{m})$ norm in $\partial^- \mathbf{C}_q$, the following definition is natural.

Definition 30 (q-Laplacian). *The q-Laplacian $\Delta_q f$ of $f \in L^2(X, \mathfrak{m})$ is defined for those f such that $\partial^- \mathbf{C}_q(f) \neq \emptyset$. For those f, $-\Delta_q f$ is the element of minimal $L^2(X, \mathfrak{m})$ norm in $\partial^- \mathbf{C}_q(f)$. The domain of Δ_q will be denoted by $D(\Delta_q)$.*

It should be observed that, even in the case $q = 2$, in general the Laplacian is *not* a linear operator. For instance, if $X = \mathbb{R}^2$ endowed with the sup norm $\|(x, y)\| = \max\{|x|, |y|\}$, then

$$\mathbf{C}_2(f) = \int_{\mathbb{R}^2} \left(\left| \frac{\partial f}{\partial x} \right| + \left| \frac{\partial f}{\partial y} \right| \right)^2 dx dy.$$

Since $\mathbf{C}_2$ is not a quadratic form, its subdifferential is not linear.

Coming back to our general framework, the trivial implication

$$v \in \partial^- \mathbf{C}_q(f) \qquad \Longrightarrow \qquad \lambda^{q-1} v \in \partial^- \mathbf{C}_q(\lambda f), \quad \forall \lambda \in \mathbb{R},$$

still ensures that the q-Laplacian (and so the gradient flow of $\mathbf{C}_q$) is $(q-1)$-homogenous.

We can now write

$$\frac{\mathrm{d}}{\mathrm{d}t} f_t = \Delta_q f_t$$

for gradient flows f_t of $\mathbf{C}_q$, the derivative being understood in $L^2(X, \mathfrak{m})$, in accordance with the classical case.

Proposition 31 (Integration by parts). *For all $f \in D(\Delta_q)$, $g \in D(\mathbf{C}_q)$ it holds*

$$-\int_X g\Delta_q f\,\mathrm{d}\mathfrak{m} \le \int_X |\nabla g|_{*,q}|\nabla f|_{*,q}^{q-1}\,\mathrm{d}\mathfrak{m}. \tag{5.5}$$

Equality holds if $g = \phi(f)$ with $\phi \in C^1(\mathbb{R})$ with bounded derivative on the image of f.

Proof. Since $-\Delta_q f \in \partial^-\mathbf{C}_q(f)$ it holds

$$\mathbf{C}_q(f) - \int_X \varepsilon g\Delta_q f\,\mathrm{d}\mathfrak{m} \le \mathbf{C}_q(f+\varepsilon g), \qquad \forall g \in L^q(X,\mathfrak{m}),\ \varepsilon \in \mathbb{R}.$$

For $\varepsilon > 0$, $|\nabla f|_{*,q} + \varepsilon|\nabla g|_{*,q}$ is a q-relaxed slope of $f+\varepsilon g$ (possibly not minimal) whenever f and g have q-relaxed slope. By truncation, it is immediate to obtain from this fact that f, $g \in \mathcal{C}$ implies $f+\varepsilon g \in \mathcal{C}$ and

$$|\nabla(f+\varepsilon g)|_{*,q} \le |\nabla f|_{*,q} + \varepsilon|\nabla g|_{*,q} \qquad \mathfrak{m}\text{-a.e. in } X.$$

Thus it holds $q\mathbf{C}_q(f+\varepsilon g) \le \int_X (|\nabla f|_{*,q} + \varepsilon|\nabla g|_{*,q})^q\,\mathrm{d}\mathfrak{m}$ and therefore

$$\begin{aligned}-\int_X \varepsilon g\Delta_q f\,\mathrm{d}\mathfrak{m} &\le \frac{1}{q}\int_X (|\nabla f|_{*,q} + \varepsilon|\nabla g|_{*,q})^q - |\nabla f|_{*,q}^q\,\mathrm{d}\mathfrak{m}\\ &= \varepsilon\int_X |\nabla g|_{*,q}|\nabla f|_{*,q}^{q-1}\,\mathrm{d}\mathfrak{m} + o(\varepsilon).\end{aligned}$$

Dividing by ε and letting $\varepsilon \downarrow 0$ we get (5.5).

For the second statement we recall that $|\nabla(f+\varepsilon\phi(f))|_{*,q} = (1+\varepsilon\phi'(f))|\nabla f|_{*,q}$ for $|\varepsilon|$ small enough. Hence

$$\begin{aligned}\mathbf{C}_q(f+\varepsilon\phi(f)) - \mathbf{C}_q(f) &= \frac{1}{q}\int_X |\nabla f|_{*,q}^q\big((1+\varepsilon\phi'(f))^q - 1\big)\,\mathrm{d}\mathfrak{m}\\ &= \varepsilon\int_X |\nabla f|_{*,q}^q\phi'(f)\,\mathrm{d}\mathfrak{m} + o(\varepsilon),\end{aligned}$$

which implies that for any $v \in \partial^-\mathbf{C}_q(f)$ it holds

$$\int_X v\phi(f)\,\mathrm{d}\mathfrak{m} = \int_X |\nabla f|_{*,q}^q\phi'(f)\,\mathrm{d}\mathfrak{m},$$

and gives the thesis with $v = -\Delta_q f$. Q.E.D.

Proposition 32 (Some properties of the gradient flow of $\mathbf{C}_q$). *Let $f_0 \in L^2(X,\mathfrak{m})$ and let (f_t) be the gradient flow of $\mathbf{C}_q$ starting from f_0.*

Then the following properties hold.
(*Mass preservation*) $\int f_t \,\mathrm{d}\mathfrak{m} = \int f_0 \,\mathrm{d}\mathfrak{m}$ *for any* $t \geq 0$.
(*Maximum principle*) *If* $f_0 \leq C$ (*resp.* $f_0 \geq c$) $\mathfrak{m}$*-a.e. in* X, *then* $f_t \leq C$ (*resp* $f_t \geq c$) $\mathfrak{m}$*-a.e. in* X *for any* $t \geq 0$.
(*Energy dissipation*) *Suppose* $0 < c \leq f_0 \leq C < \infty$ $\mathfrak{m}$*-a.e. in* X *and* $\Phi \in C^2([c, C])$. *Then* $t \mapsto \int \Phi(f_t) \,\mathrm{d}\mathfrak{m}$ *is locally absolutely continuous in* $(0, \infty)$ *and it holds*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \Phi(f_t) \,\mathrm{d}\mathfrak{m} = -\int \Phi''(f_t) |\nabla f_t|_{*,q}^q \,\mathrm{d}\mathfrak{m} \qquad \textit{for a.e. } t \in (0, \infty).$$

Proof. (Mass preservation) Just notice that from (5.5) we get

$$\begin{aligned} \left| \frac{\mathrm{d}}{\mathrm{d}t} \int f_t \,\mathrm{d}\mathfrak{m} \right| &= \left| \int \mathbf{1} \cdot \Delta_q f_t \,\mathrm{d}\mathfrak{m} \right| \\ &\leq \int |\nabla \mathbf{1}|_{*,q} |\nabla f_t|_{*,q}^q \,\mathrm{d}\mathfrak{m} = 0 \quad \text{for a.e. } t > 0, \end{aligned}$$

where $\mathbf{1}$ is the function identically equal to 1, which has minimal q-relaxed slope equal to 0 by (4.4).
(Maximum principle) Fix $f \in L^2(X, \mathfrak{m})$, $\tau > 0$ and, according to the so-called implicit Euler scheme, let f^τ be the unique minimizer of

$$g \qquad \mapsto \qquad \mathbf{C}_q(g) + \frac{1}{2\tau} \int_X |g - f|^2 \,\mathrm{d}\mathfrak{m}.$$

Assume that $f \leq C$. We claim that in this case $f^\tau \leq C$ as well. Indeed, if this is not the case we can consider the competitor $g := \min\{f^\tau, C\}$ in the above minimization problem. By locality we get $\mathbf{C}_q(g) \leq \mathbf{C}_q(f^\tau)$ and the L^2 distance of f and g is strictly smaller than the one of f and f^τ as soon as $\mathfrak{m}(\{f^\tau > C\}) > 0$, which is a contradiction. Starting from f_0, iterating this procedure, and using the fact that the implicit Euler scheme converges as $\tau \downarrow 0$ (see [6], [2] for details) to the gradient flow we get the conclusion.
(Energy dissipation) Since $t \mapsto f_t \in L^2(X, \mathfrak{m})$ is locally absolutely continuous and, by the maximum principle, f_t take their values in $[c, C]$ $\mathfrak{m}$-a.e., from the fact that Φ is Lipschitz in $[c, C]$ we get the claimed absolute continuity statement. Now notice that we have $\frac{\mathrm{d}}{\mathrm{d}t} \int \Phi(f_t) \,\mathrm{d}\mathfrak{m} = \int \Phi'(f_t) \Delta_q f_t \,\mathrm{d}\mathfrak{m}$ for a.e. $t > 0$. Since Φ' belongs to $C^1([c, C])$, from (5.5) with $g = \Phi'(f_t)$ we get the conclusion. Q.E.D.

We start with the following proposition, which relates energy dissipation to a (sharp) combination of q-weak gradients and metric dissipation in W_p.

Proposition 33. *Assume that $\mathfrak{m}$ is a finite measure, let $\mu_t = f_t\mathfrak{m}$ be a curve in $AC^p([0,1],(\mathscr{P}(X),W_p))$. Assume that for some $0 < c < C < \infty$ it holds $c \le f_t \le C$ $\mathfrak{m}$-a.e. in X for any $t \in [0,1]$, and that f_0 is Sobolev along q-a.e. curve with $|\nabla f_0|_{w,q} \in L^q(X,\mathfrak{m})$. Then for all $\Phi \in C^2([c,C])$ convex it holds*

$$\begin{aligned}\int \Phi(f_0)\,\mathrm{d}\mathfrak{m} - \int \Phi(f_t)\,\mathrm{d}\mathfrak{m} \le& \frac{1}{q}\iint_0^t \big(\Phi''(f_0)|\nabla f_0|_{w,q}\big)^q f_s\,\mathrm{d}s\,\mathrm{d}\mathfrak{m}\\ &+\frac{1}{p}\int_0^t |\dot\mu_s|^p\,\mathrm{d}s \qquad \forall t>0.\end{aligned}$$

Proof. Let $\boldsymbol{\pi} \in \mathscr{P}(C([0,1],X))$ be a plan associated to the curve (μ_t) as in Proposition 2. The assumption $f_t \le C$ $\mathfrak{m}$-a.e. and the fact that $\iint_0^1 |\dot\gamma_t|^p\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma) = \int |\dot\mu_t|^p\,\mathrm{d}t < \infty$ guarantee that $\boldsymbol{\pi}$ is a p-test plan. Now notice that it holds $|\nabla\Phi'(f_0)|_{w,q} = \Phi''(f_0)|\nabla f_0|_{w,q}$ (it follows easily from the characterization (4.9)), thus we get

$$\begin{aligned}\int \Phi(f_0)\,\mathrm{d}\mathfrak{m} - \int \Phi(f_t)\,\mathrm{d}\mathfrak{m} &\le \int \Phi'(f_0)(f_0 - f_t)\,\mathrm{d}\mathfrak{m}\\
&= \int \Phi'(f_0)\circ \mathrm{e}_0 - \Phi'(f_0)\circ \mathrm{e}_t\,\mathrm{d}\boldsymbol{\pi}\\
&\le \iint_0^t \Phi''(f_0(\gamma_s))|\nabla f_0|_{w,q}(\gamma_s)|\dot\gamma_s|\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}(\gamma)\\
&\le \frac{1}{q}\iint_0^t \big(\Phi''(f_0(\gamma_s))|\nabla f_0|_{w,q}(\gamma_s)\big)^q\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}(\gamma)\\
&+\frac{1}{p}\iint_0^t |\dot\gamma_s|^p\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}(\gamma)\\
&= \frac{1}{q}\iint_0^t \big(\Phi''(f_0)|\nabla f_0|_{w,q}\big)^q f_s\,\mathrm{d}s\,\mathrm{d}\mathfrak{m} + \frac{1}{p}\int_0^t |\dot\mu_s|^p\,\mathrm{d}s.\end{aligned}$$

Q.E.D.

The key argument to achieve the identification is the following lemma which gives a sharp bound on the W_p-speed of the L^2-gradient flow of $\mathbf{C}_q$. This lemma has been introduced in [25] and then used in [13], [3] to study the heat flow on metric measure spaces.

Lemma 34 (Kuwada's lemma). *Assume that $\mathfrak{m}$ is a finite measure, let $f_0 \in L^2(X,\mathfrak{m})$ and let (f_t) be the gradient flow of $\mathbf{C}_q$ starting from f_0. Assume that for some $0 < c < C < \infty$ it holds $c \le f_0 \le C$ $\mathfrak{m}$-a.e. in X, and that $\int f_0\,\mathrm{d}\mathfrak{m} = 1$. Then the curve $t \mapsto \mu_t := f_t\mathfrak{m} \in \mathscr{P}(X)$ is* *absolutely continuous w.r.t. W_p and it holds*

$$|\dot\mu_t|^p \le \int \frac{|\nabla f_t|_{*,q}^q}{f_t^{p-1}}\,\mathrm{d}\mathfrak{m} \qquad \textit{for a.e. } t \in (0,\infty).$$

Proof. We start from the duality formula (2.3) (written with $\varphi = -\psi$)

$$\frac{W_p^p(\mu,\nu)}{p} = \sup_{\varphi\in \mathrm{Lip}_b(X)} \int_X Q_1\varphi\,\mathrm{d}\nu - \int_X \varphi\,\mathrm{d}\mu. \tag{5.6}$$

where $Q_t\varphi$ is defined in (3.1) and (3.2), so that $Q_1\varphi = \psi^c$.

We prove that the duality formula (5.6) is still true if the supremum in the right-hand side is taken over nonnegative and bounded $\varphi \in \mathrm{Lip}(X)$ with bounded support

$$\begin{aligned}\frac{W_p^p(\mu,\nu)}{p} = \sup\Big\{ \int_X Q_1\varphi\,\mathrm{d}\nu - \int_X \varphi\,\mathrm{d}\mu : \varphi\in \mathrm{Lip}(X),\ \varphi \ge 0,\\ \text{with bounded support}\Big\}.\end{aligned} \tag{5.7}$$

The duality formula (5.6) holds also if the supremum is taken over bounded nonnegative φ in $\mathrm{Lip}(X)$ up to a translation. In order to prove the equivalence it is enough to show that for every $\varphi\in\mathrm{Lip}_b(X)$ nonnegative there holds

$$\liminf_{r\to\infty}\Big\{ \int_X Q_1[\chi_r\varphi]\,\mathrm{d}\nu - \int_X \chi_r\varphi\,\mathrm{d}\mu\Big\} \ge \int_X Q_1\varphi\,d\nu - \int_X \varphi\,\mathrm{d}\mu, \tag{5.8}$$

where χ_r is a Lipschitz cutoff function which is nonnegative, identically equal to 1 in $B(x_0,r)$ and identically equal to 0 outside $B(x_0,r+1)$ for some $x_0\in X$ fixed. Since $\chi_r\varphi\le\varphi$ it follows that $\int_X \chi_r\varphi\,\mathrm{d}\mu \le \int_X\varphi\,\mathrm{d}\mu$, so that by Fatou's lemma suffices to show that $\liminf_{r\to\infty} Q_1[\chi_r\varphi]\ge Q_1\varphi$. Let $x\in X$ be fixed and let $x_r\in X$ be satisfying

$$\chi_r(x_r)\varphi(x_r) + \frac{\mathsf{d}^p(x,x_r)}{p} \le \frac{1}{r} + Q_1[\chi_r\varphi](x).$$

Since $\mathsf{d}(x_r,x)$ is obviously bounded as $r\to\infty$, the same is true for $\mathsf{d}(x_r,x_0)$, so that $\chi_r(x_r)=1$ for r large enough and $Q_1\varphi(x)\le r^{-1} + Q_1[\chi_r\varphi](x)$ for r large enough. Fix now $\varphi\in\mathrm{Lip}(X)$ nonnegative with bounded support and recall that $Q_t\varphi$ has bounded support for every $t>0$ and that (Proposition 11) the map $t\mapsto Q_t\varphi$ is Lipschitz with values in $C(X)$, in particular also as a $L^2(X,\mathfrak{m})$-valued map.

Fix also $0 \le t < s$, set $\ell = (s-t)$ and recall that since (f_t) is a gradient flow of $\mathbf{C}_q$ in $L^2(X,\mathfrak{m})$, the map $[0,\ell] \ni \tau \mapsto f_{t+\tau}$ is absolutely continuous with values in $L^2(X,\mathfrak{m})$. Therefore, since both factors are uniformly bounded, the map $[0,\ell] \ni \tau \mapsto Q_{\frac{\tau}{\ell}}\varphi f_{t+\tau}$ is absolutely continuous with values in $L^2(X,\mathfrak{m})$. In addition, the equality

$$\frac{Q_{\frac{\tau+h}{\ell}}\varphi f_{t+\tau+h} - Q_{\frac{\tau}{\ell}}\varphi f_{t+\tau}}{h} = f_{t+\tau}\frac{Q_{\frac{\tau+h}{\ell}} - Q_{\frac{\tau}{\ell}}\varphi}{h} + Q_{\frac{\tau+h}{\ell}}\varphi\frac{f_{t+\tau+h} - f_{t+\tau}}{h},$$

together with the uniform continuity of $(x,\tau) \mapsto Q_{\frac{\tau}{\ell}}\varphi(x)$ shows that the derivative of $\tau \mapsto Q_{\frac{\tau}{\ell}}\varphi f_{t+\tau}$ can be computed via the Leibniz rule.

We have:

$$\begin{aligned}\int_X Q_1\varphi\,\mathrm{d}\mu_s - \int_X \varphi\,\mathrm{d}\mu_t &= \int Q_1\varphi f_{t+\ell}\,\mathrm{d}\mathfrak{m} - \int_X \varphi f_t\,\mathrm{d}\mathfrak{m}\\ &= \int_X\int_0^\ell \frac{\mathrm{d}}{\mathrm{d}\tau}\big(Q_{\frac{\tau}{\ell}}\varphi f_{t+\tau}\big)d\tau\,\mathrm{d}\mathfrak{m} \qquad (5.9)\\ &\le \int_X\int_0^\ell -\frac{\mathrm{Lip}_a^q(Q_{\frac{\tau}{\ell}}\varphi,\cdot)}{q\ell}f_{t+\tau} + Q_{\frac{\tau}{\ell}}\varphi\Delta_q f_{t+\tau}\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{m},\end{aligned}$$

having used Theorem 14.

Observe that by inequalities (5.5) and (4.4) we have

$$\begin{aligned}&\int_X Q_{\frac{\tau}{\ell}}\varphi\Delta_q f_{t+\tau}\,\mathrm{d}\mathfrak{m}\\ &\le \int_X |\nabla Q_{\frac{\tau}{\ell}}\varphi|_{*,q}|\nabla f_{t+\tau}|_{*,q}^{q-1}\,\mathrm{d}\mathfrak{m}\\ &\le \int_X \mathrm{Lip}_a(Q_{\frac{\tau}{\ell}}\varphi)|\nabla f_{t+\tau}|_{*,q}^{q-1}\,\mathrm{d}\mathfrak{m} \qquad (5.10)\\ &\le \frac{1}{q\ell}\int_X \mathrm{Lip}_a^q(Q_{\frac{\tau}{\ell}}\varphi,\cdot)f_{t+\tau}d\mathfrak{m} + \frac{\ell^{p-1}}{p}\int_X \frac{|\nabla f_{t+\tau}|_{*,q}^q}{f_{t+\tau}^{p-1}}\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

Plugging this inequality in (5.9), we obtain

$$\int_X Q_1\varphi\,\mathrm{d}\mu_s - \int_X \varphi\,\mathrm{d}\mu_t \le \frac{\ell^{p-1}}{p}\int_0^\ell\int_X \frac{|\nabla f_{t+\tau}|_{*,q}^q}{f_{t+\tau}^{p-1}}\,\mathrm{d}\mathfrak{m}\,\mathrm{d}\tau.$$

This latter bound does not depend on φ, so from (5.7) we deduce

$$W_p^p(\mu_t,\mu_s) \le \ell^{p-1}\int_0^\ell\int_X \frac{|\nabla f_{t+\tau}|_{*,q}^q}{f_{t+\tau}^{p-1}}\,\mathrm{d}\mathfrak{m}\,\mathrm{d}\tau.$$

At Lebesgue points of $r \mapsto \int_X |\nabla f_r|_{*,q}^q/f_r^{p-1}\,\mathrm{d}\mathfrak{m}$ where the metric speed exists we obtain the stated pointwise bound on the metric speed. Q.E.D.

§6. Equivalence of gradients

In this section we assume that (X, d) is complete and separable, and that $\mathfrak{m}$ is finite on bounded sets. We prove the equivalence of weak gradients, considering first the simpler case of a finite measure $\mathfrak{m}$.

Theorem 35. *Let $f \in L^q(X, \mathfrak{m})$. Then f has a q-relaxed slope if and only if f has a q-weak upper gradient in $L^q(X, \mathfrak{m})$ and $|\nabla f|_{*,q} = |\nabla f|_{w,q}$ $\mathfrak{m}$-a.e. in X.*

Proof. One implication and the inequality $\geq$ have already been established in (4.15). We prove the converse ones first for finite measures, and then in the general case.

So, assume for the moment that $\mathfrak{m}(X) < \infty$. Up to a truncation argument and addition of a constant, we can assume that $0 < c \leq f \leq C < \infty$ $\mathfrak{m}$-a.e. for some $0 < c \leq C < \infty$. Let (g_t) be the L^2-gradient flow of $\mathbf{C}_q$ starting from $g_0 := f$ and let us choose $\Phi \in C^2([c, C])$ in such a way that $\Phi''(z) = z^{1-p}$ in $[c, C]$. Recall that $c \leq g_t \leq C$ $\mathfrak{m}$-a.e. in X and that from Proposition 32 we have

(6.1)
$$\int \Phi(g_0)\,\mathrm{d}\mathfrak{m} - \int \Phi(g_t)\,\mathrm{d}\mathfrak{m} = \int_0^t \int_X \Phi''(g_s)|\nabla g_s|_{*,q}^q \mathrm{d}\mathfrak{m}\,\mathrm{d}s \quad \forall t \in [0, \infty).$$

In particular this gives that $\int_0^\infty \int_X \Phi''(g_s)|\nabla g_s|_{*,q}^q\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s$ is finite. Setting $\mu_t = g_t \mathfrak{m}$, Lemma 34 and the lower bound on g_t give that $\mu_t \in AC^p\big((0, \infty), (\mathscr{P}(X), W_p)\big)$, so that Proposition 33 and Lemma 34 yield

$$\begin{aligned}\int \Phi(g_0)\,\mathrm{d}\mathfrak{m} - \int \Phi(g_t)\,\mathrm{d}\mathfrak{m} \leq & \frac{1}{q}\int_0^t \int_X \big(\Phi''(g_0)|\nabla g_0|_{w,q}\big)^q g_s\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s \\ &+ \frac{1}{p}\int_0^t \int_X \frac{|\nabla g_s|_{*,q}^q}{g_s^{p-1}}\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s.\end{aligned}$$

Hence, comparing this last expression with (6.1), our choice of Φ gives

$$\frac{1}{q}\iint_0^t \frac{|\nabla g_s|_{*,q}^q}{g_s^{p-1}}\mathrm{d}s\,\mathrm{d}\mathfrak{m} \leq \int_0^t \int_X \frac{1}{q}\Big(\frac{|\nabla g_0|_{w,q}}{g_0^{p-1}}\Big)^q g_s\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s.$$

Now, the bound $f \geq c > 0$ ensures $\Phi''(g_0)|\nabla g_0|_{*,q} \in L^q(X, \mathfrak{m})$. In addition, the maximum principle together with the convergence of g_s to g_0 in $L^2(X, \mathfrak{m})$ as $s \downarrow 0$ grants that the convergence is also weak* in $L^\infty(X, \mathfrak{m})$, therefore

$$\limsup_{t\downarrow 0} \frac{1}{t}\iint_0^t \frac{|\nabla g_s|_{*,q}^q}{g_s^{p-1}}\mathrm{d}s\,\mathrm{d}\mathfrak{m} \leq \int_X \frac{|\nabla g_0|_{w,q}^q}{g_0^{q(p-1)}} g_0 \mathrm{d}\mathfrak{m} = \int_X \frac{|\nabla g_0|_{w,q}^q}{g_0^{p-1}}\,\mathrm{d}\mathfrak{m}.$$

The lower semicontinuity property stated in Remark 29 with $\alpha = p-1$ then gives

$$\int_X \frac{|\nabla g_0|^q_{*,q}}{g_0^{p-1}}\,\mathrm{d}\mathfrak{m} \le \int_X \frac{|\nabla g_0|^q_{w,q}}{g_0^{p-1}}\,\mathrm{d}\mathfrak{m}.$$

This, together with the inequality $|\nabla g_0|_{w,q} \le |\nabla g_0|_{*,q}$ $\mathfrak{m}$-a.e. in X, gives the conclusion.

Finally, we consider the general case of a measure $\mathfrak{m}$ finite on bounded sets. Let $X_n = \overline{B}(x_0, n)$, $n > 1$, and notice that trivially it holds

$$|\nabla f|_{X_n,w,q} \le |\nabla f|_{w,q} \qquad \mathfrak{m}\text{-a.e. in } X_n, \tag{6.2}$$

because the class of test plans relative to X_n is smaller. Hence, if we apply the equivalence result in X_n, we can find Lipschitz functions $f_k : X_n \to \mathbb{R}$ which converge to f in $L^q(X_n, \mathfrak{m})$ and satisfy $\mathrm{Lip}_{a,X_n}(f_k, \cdot) \to |\nabla f|_{w,X_n}$ in $L^q(X_n, \mathfrak{m})$. If $\psi_n : X \to [0,1]$ is a 2-Lipschitz function identically equal to 1 on $\overline{B}(x_0, n-1)$ and with support contained in $B(0, n-\frac{1}{4})$, the functions $\psi_n f_k$ can obviously be thought as Lipschitz functions with bounded support on X and satisfy (thanks to (4.3))

$$\mathrm{Lip}_a(\psi_n f_k) \le \psi_n \mathrm{Lip}_{a,X_n}(f_k) + 2\chi_n |f_k|,$$

where χ_n is the characteristic function of $\overline{B}(0,n) \setminus B(0, n-1)$. Passing to the limit as $k \to \infty$ (notice that multiplication by ψ_n allows to turn $L^q(X_n, \mathfrak{m})$ convergence of the asymptotic Lipschitz constants to $L^q(X, \mathfrak{m})$ convergence, and similarly for f_k) it follows that $\psi_n f$ has q-relaxed slope, and that

$$|\nabla(\psi_n f)|_{*,q} \le |\nabla f|_{X_n,w,q} + 2\chi_n |f| \qquad \mathfrak{m}\text{-a.e. in } X.$$

Invoking (6.2) we obtain

$$|\nabla(\psi_n f)|_{*,q} \le |\nabla f|_{w,q} + 2\chi_n |f| \qquad \mathfrak{m}\text{-a.e. in } X.$$

Eventually we let $n \to \infty$ to conclude, by a diagonal argument, that f has a q-relaxed slope and that $|\nabla f|_{*,q} \le |\nabla f|_{w,q}$ $\mathfrak{m}$-a.e. in X. Q.E.D.

The proof of the previous result provides, by a similar argument, the following locality result.

Proposition 36. *If f has a q-weak upper gradient and $A \subset X$ is open, then denoting by $|\nabla f|_{\overline{A},w,q}$ the minimal q-relaxed slope in the metric measure space $(\overline{A}, \mathsf{d}, \mathfrak{m})$,*

$$|\nabla f|_{\overline{A},w,q} = |\nabla f|_{w,q} \qquad \mathfrak{m}\text{-}a.e.\ in\ A. \tag{6.3}$$

Proof. We already noticed that, by definition, $|\nabla f|_{\overline{A},w,q} \le |\nabla f|_{w,q}$ $\mathfrak{m}$-a.e. in $\overline{A}$. Let $B \subset A$ be an open set with $\mathrm{dist}(B, X \setminus A) > 0$ and let $\psi : X \to [0,1]$ be a Lipschitz cut-off function with support contained in A and equal to 1 on a neighbourhood of B. If $f_n \in \mathrm{Lip}(\overline{A})$ have bounded support, converge to f in $L^q(\overline{A}, \mathfrak{m})$ and satisfy $\mathrm{Lip}_a(f_n, \cdot) \to |\nabla f|_{\overline{A},*,q}$ in $L^q(\overline{A}, \mathfrak{m})$, we can consider the functions $f_n\psi$ and use (4.3) to obtain that $\psi|\nabla f|_{\overline{A},*,q} + \mathrm{Lip}(\psi)\chi|f|$ is a q-relaxed slope of f in X, where χ is the characteristic function of the set $\overline{\{\psi < 1\}}$. Since $\chi \equiv 0$ on B it follows that

$$|\nabla f|_{*,q} \le |\nabla f|_{\overline{A},*,q} \qquad \mathfrak{m}\text{-a.e. in } B.$$

Letting $B \uparrow A$ and using the identification of gradients the proof is achieved. Q.E.D.

In particular, since any open set $A \subset X$ can be written as the increasing union of open subsets A_n with $\overline{A}_n \subset A$, it will make sense to speak of the weak gradient on A of a function $f : A \to \mathbb{R}$ having a q-weak upper gradient when restricted to $\overline{A}_n$ for all n; suffices to define $|\nabla f|_{w,q} : A \to [0,\infty)$ by

$$|\nabla f|_{w,q} := |\nabla f|_{\overline{A}_n,w,q} \qquad \mathfrak{m}\text{-a.e. on } A_n \tag{6.4}$$

and the definition is well posed $\mathfrak{m}$-a.e. in X thanks to Proposition 36.

§7. Reflexivity of $W^{1,q}(X, \mathsf{d}, \mathfrak{m})$, $1 < q < \infty$

We will denote by $W^{1,q}(X, \mathsf{d}, \mathfrak{m})$ the Banach space of functions $f \in L^q(X, \mathfrak{m})$ having a q-relaxed slope, endowed with the norm

$$\|f\|^q_{W^{1,q}} = \|f\|^q_{L^q} + \||\nabla f|_{*,q}\|^q_{L^q}.$$

By a general property of normed spaces, in order to prove completeness, it suffices to show that any absolutely convergent series in $W^{1,q}(X, \mathsf{d}, \mathfrak{m})$ is convergent; if f_n satisfy $\sum_n \|f_n\|^q_{W^{1,q}} < \infty$, the completeness of $L^q(X, \mathfrak{m})$ yields that $f := \sum_n f_n$ and $g := \sum_n |\nabla f_n|_{*,q}$ converge in $L^q(X, \mathfrak{m})$, and the finite subadditivity of the relaxed gradient together with the lower semicontinuity of $\mathbf{C}_q$ give $f \in W^{1,q}(X, \mathsf{d}, \mathfrak{m})$ and $\int_X |\nabla f|^q_{*,q}\,\mathrm{d}\mathfrak{m} \le \|g\|^q_{L^q} \le (\sum_i \||\nabla f_i|_{*,q}\|_{L^q})^q$. A similar argument gives that

$$\left(\int_X |\nabla(f - \sum_{i=1}^N f_i)|^q_{*,q}\,\mathrm{d}\mathfrak{m}\right)^{1/q} \le \sum_{i=N+1}^{\infty} \||\nabla f_i|_{*,q}\|_{L^q},$$

hence $\sum_n f_n$ converges in $W^{1,q}(X, \mathsf{d}, \mathfrak{m})$.

In this section we prove that the Sobolev spaces $W^{1,q}(X,\mathsf{d},\mathfrak{m})$ are reflexive when $1<q<\infty$, (X,d) is doubling and separable, and $\mathfrak{m}$ is finite on bounded sets. Our strategy is to build, by a finite difference scheme, a family of functionals which provide a discrete approximation of Cheeger's energy. The definition of the approximate functionals relies on the existence of nice partitions of doubling metric spaces.

Lemma 37. *For every $\delta>0$ there exist $\ell_\delta\in\mathbb{N}\cup\{\infty\}$ and pairs set-point (A_i^δ,z_i^δ), $0\le i<\ell_\delta$, where $A_i^\delta\subset X$ are Borel sets and $z_i^\delta\in X$, satisfying:*

(i) *the sets A_i^δ, $0\le i<\ell_\delta$, are a partition of X and $\mathsf{d}(z_i^\delta,z_j^\delta)>\delta$ whenever $i\neq j$;*

(ii) *A_i^δ are comparable to balls centered at z_i^δ, namely*

$$B\left(z_i^\delta,\frac{\delta}{3}\right)\subset A_i^\delta\subset B\left(z_i^\delta,\frac{5}{4}\delta\right).$$

Proof. Let us fix once for all a countable dense set $\{x_k\}_{k\in\mathbb{N}}$. Then, starting from $z_0^\delta=x_0$, we proceed in this way:

- for $i\ge 1$, set recursively

$$B_i=X\setminus\bigcup_{j<i}\overline{B}(z_j^\delta,\delta);$$

- if $B_i=\emptyset$ for some $i\ge 1$, then the procedure stops. Otherwise, take $z_i^\delta=x_{k_i}$ where

$$k_i=\min\{k\in\mathbb{N}\ :\ x_k\in B_i\}.$$

We claim that for every $\varepsilon>0$ we have that

$$\bigcup_{i=0}^{\infty}B(z_i^\delta,\delta+\varepsilon)=X.$$

To show this it is sufficient to note that for every $x\in X$ we have a point x_j such that $\mathsf{d}(x_j,x)<\varepsilon$; then either $x_j=z_i^\delta$ for some i or $x_j\in\overline{B}(z_i^\delta,\delta)$ for some i. In both cases we get

$$\forall x\in X\ \exists i\in\mathbb{N}\qquad \text{such that } \mathsf{d}(z_i^\delta,x)<\delta+\varepsilon. \tag{7.1}$$

Now we define the sets A_i^δ similarly to a Voronoi diagram constructed from the starting point z_i^δ: for $i\in\mathbb{N}$ we set

$$B_i^\delta=\Big\{x\in X\ :\ \mathsf{d}(x,z_i^\delta)\le\mathsf{d}(x,z_j^\delta)+\varepsilon\quad\forall j\Big\}.$$

It is clear that B_i^δ are Borel sets whose union is the whole of X; we turn them into a Borel partition by setting

$$A_0^\delta = B_0^\delta, \qquad A_j^\delta := B_j^\delta \setminus \bigcup_{i<j} B_i^\delta, \quad j > 0.$$

We can also give an equivalent definition: $x \in A_k^\delta$ iff

$$k = \min I_x \quad \text{where} \quad I_x = \left\{ i \in \mathbb{N} \ : \ \mathsf{d}(x, z_i^\delta) \le \mathsf{d}(x, z_j^\delta) + \varepsilon \quad \forall j \in \mathbb{N} \right\}.$$

In other words, we are minimizing the quantity $\mathsf{d}(x, z_i^\delta)$ and among those indeces i who are minimizing up to ε we take the least one i_x. By this quasi minimality and (7.1) we obtain $\mathsf{d}(x, z_{i_x}^\delta) \le \inf_{i\in\mathbb{N}} \mathsf{d}(x, z_i^\delta) + \varepsilon < \delta + 2\varepsilon$. Furthermore if $\mathsf{d}(x, z_i^\delta) < \delta/2 - \varepsilon/2$ then $I_x = \{i\}$. Indeed, suppose there is another $j \in I_x$ with $j \ne i$, then $\mathsf{d}(z_j^\delta, x) \le \mathsf{d}(z_i^\delta, x) + \varepsilon \le \delta/2 + \varepsilon/2$ and so

$$\delta < \mathsf{d}(z_i^\delta, z_j^\delta) \le \mathsf{d}(z_i^\delta, x) + \mathsf{d}(z_j^\delta, x) \le \delta.$$

We just showed that

$$B\left(z_i^\delta, \frac{\delta}{2} - \frac{\varepsilon}{2}\right) \subset A_i^\delta \subset B(z_i^\delta, \delta + 2\varepsilon).$$

The dual definition gives us that A_i^δ are a partition of X, and (ii) is satisfied choosing $\varepsilon = \delta/8$. Q.E.D.

Note that this construction is quite simpler if X is locally compact, which is always the case if (X, d) is doubling and complete. In this case we can choose $\varepsilon = 0$.

We remark that partitions with additional properties have also been studied in the literature. For example, in [8] *dyadic* partitions of a doubling metric measure space are constructed.

Definition 38 (Dyadic partition). *A dyadic partition is made by a sequence* $(\ell_h) \subset \mathbb{N} \cup \{\infty\}$ *and by collections of disjoint sets (called cubes)* $\Delta^h = \{A_i^h\}_{1 \le i < \ell(h)}$ *such that for every* $h \in \mathbb{N}$ *the following properties hold:*

- $\mathfrak{m}\big(X \setminus \bigcup_i A_i^h\big) = 0$;
- *for every* $i \in \{1, \ldots, \ell_{h+1}\}$ *there exists a unique* $j \in \{1, \ldots, \ell_h\}$ *such that* $A_i^{h+1} \subset A_j^h$;
- *for every* $i \in \{1, \ldots \ell_h\}$ *there exists* $z_i^h \in X$ *such that* $B(z_i^h, a_0\delta^h) \subset A_i^h \subset B(z_i^h, a_1\delta^h)$ *for some positive constants* δ, a_0, a_1 *independent of* i *and* h.

In [8] existence of dyadic decompositions is proved, with δ, a_1 and a_0 depending on the constant $\tilde{c}_D$ in (2.5). Although some more properties of the partition might give additional information on the functionals that we are going to construct, for the sake of simplicity we just work with the partition given by Lemma 37.

In order to define our discrete gradients we give more terminology. We say that A_i^δ is a *neighbor* of A_j^δ, and we denote by $A_i^\delta \sim A_j^\delta$, if their distance is less than δ. In particular $A_i^\delta \sim A_j^\delta$ implies that $\mathsf{d}(z_i^\delta, z_j^\delta) < 4\delta$: indeed, if $\tilde{z}_i^\delta \in A_i^\delta$ and $\tilde{z}_j^\delta \in A_j^\delta$ satisfy $\mathsf{d}(\tilde{z}_i^\delta, \tilde{z}_j^\delta) < \delta'$ we have

$$\mathsf{d}(z_i^\delta, z_j^\delta) \leq \mathsf{d}(z_i^\delta, \tilde{z}_i^\delta) + \mathsf{d}(\tilde{z}_i^\delta, \tilde{z}_j^\delta) + \mathsf{d}(\tilde{z}_j^\delta, z_j^\delta) \leq \frac{10}{4}\delta + \delta'$$

and letting $\delta' \downarrow \delta$ we get

$$\mathsf{d}(z_i^\delta, z_j^\delta) \leq \frac{14}{4}\delta < 4\delta.$$

This leads us to the first important property of doubling spaces:

(7.2) In a c_D-doubling metric space (X, d), every A_i^δ has at most c_D^3 neighbors.

Indeed, we can cover $B(z_i^\delta, 4\delta)$ with c_D^3 balls with radius $\delta/2$ but each of them, by the condition $\mathsf{d}(z_i^\delta, z_j^\delta) > \delta$, can contain only one of the z_j^δ's.

Now we fix $\delta \in (0,1)$ and we consider a partition A_i^δ of $\operatorname{supp}\mathfrak{m}$ on scale δ. For every $u \in L^q(X, \mathfrak{m})$ we define the average $u_{\delta,i}$ of u in each cell of the partition by $⨍_{A_i^\delta} u \, d\mathfrak{m}$. We denote by $\mathcal{PC}_\delta(X)$, which depends on the chosen decomposition as well, the set of functions $u \in L^q(X, \mathfrak{m})$ constant on each cell of the partition at scale δ, namely

$$u(x) = u_{\delta,i} \qquad \text{for } \mathfrak{m}\text{-a.e. } x \in A_i^\delta.$$

We define a linear projection functional $\mathcal{P}_\delta : L^q(X, \mathfrak{m}) \to \mathcal{PC}_\delta(X)$ by $\mathcal{P}_\delta u(x) = u_{\delta,i}$ for every $x \in A_i^\delta$.

The proof of the following lemma is elementary.

Lemma 39. *$\mathcal{P}_\delta$ are contractions in $L^q(X, \mathfrak{m})$ and $\mathcal{P}_\delta u \to u$ in $L^q(X, \mathfrak{m})$ as $\delta \downarrow 0$ for all $u \in L^q(X, \mathfrak{m})$.*

Indeed, the contractivity of $\mathcal{P}_\delta$ is a simple consequence of Jensen's inequality and it suffices to check the convergence of $\mathcal{P}_\delta$ as $\delta \downarrow 0$ on a dense subset of $L^q(X, \mathfrak{m})$. Since $\mathfrak{m}$ is finite on bounded sets, suffices to consider bounded continuous functions with bounded support. Since bounded closed sets are compact, by the doubling property, it follows that any such function u is uniformly continuous, so that $\mathcal{P}_\delta u \to u$ pointwise as $\delta \downarrow 0$. Then, we can use the dominated convergence theorem to conclude.

We now define an approximate gradient as follows: it is constant on the cell A_i^δ for every δ, $i \in \mathbb{N}$ and it takes the value

$$|\mathcal{D}_\delta u|^q(x) := \frac{1}{\delta^q} \sum_{A_j^\delta \sim A_i^\delta} |u_{\delta,i} - u_{\delta,j}|^q \qquad \forall x \in A_i^\delta.$$

We can accordingly define the functional $\mathcal{F}_{\delta,q} : L^q(X, \mathfrak{m}) \to [0, \infty]$ by

$$\mathcal{F}_{\delta,q}(u) := \int_X |\mathcal{D}_\delta u|^q(x)\, d\mathfrak{m}(x). \tag{7.3}$$

Now, using the weak gradients, we define a functional $\mathrm{Ch} : L^q(X, \mathfrak{m}) \to [0, \infty]$ that we call *Cheeger* energy, formally similar to the one (5.3) used in Section 5, for the purposes of energy dissipation estimates and equivalence of weak gradients. Namely, we set

$$\mathrm{Ch}_q(u) := \begin{cases} \int_X |\nabla u|_{w,q}^q \, d\mathfrak{m} & \text{if } u \text{ has a } q\text{-relaxed slope} \\ +\infty & \text{otherwise.} \end{cases}$$

At this level of generality, we cannot expect that the functionals $\mathcal{F}_{\delta,q}$ Γ-converge as $\delta \downarrow 0$. However, since $L^q(X, \mathfrak{m})$ is a complete and separable metric space, from the compactness property of Γ-convergence stated in Proposition 4 we obtain that the functionals $\mathcal{F}_{\delta,q}$ have Γ-limit points as $\delta \downarrow 0$.

Theorem 40. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a metric measure space with* $(\operatorname{supp} \mathfrak{m}, \mathsf{d})$ *complete and doubling,* $\mathfrak{m}$ *finite on bounded sets. Let* $\mathcal{F}_q$ *be a* Γ*-limit point of* $\mathcal{F}_{\delta,q}$ *as* $\delta \downarrow 0$*, namely*

$$\mathcal{F}_q := \Gamma - \lim_{k \to \infty} \mathcal{F}_{\delta_k, q},$$

for some infinitesimal sequence (δ_k)*, where the* Γ*-limit is computed with respect to the* $L^q(X, \mathfrak{m})$ *distance. Then:*

(a) $\mathcal{F}_q$ *is equivalent to the Cheeger energy* Ch_q*, namely there exists* $\eta = \eta(q, c_D)$ *such that*

$$\frac{1}{\eta} \mathrm{Ch}_q(u) \le \mathcal{F}_q(u) \le \eta\, \mathrm{Ch}_q(u) \qquad \forall u \in L^q(X, \mathfrak{m}). \tag{7.4}$$

(b) *The norm on* $W^{1,q}(X, \mathsf{d}, \mathfrak{m})$ *defined by*

$$\left(\|u\|_q^q + \mathcal{F}_q(u)\right)^{1/q} \qquad \forall u \in W^{1,q}(X, \mathsf{d}, \mathfrak{m}) \tag{7.5}$$

is uniformly convex. Moreover, the seminorm $\mathcal{F}_2^{1/2}$ is Hilbertian, namely

$$\mathcal{F}_2(u+v)+\mathcal{F}_2(u-v)=2\big(\mathcal{F}_2(u)+\mathcal{F}_2(v)\big) \qquad \forall u,\, v\in W^{1,2}(X,\mathsf{d},\mathfrak{m}). \tag{7.6}$$

Corollary 41 (Reflexivity of $W^{1,q}(X,\mathsf{d},\mathfrak{m})$)**.** *Let $(X,\mathsf{d},\mathfrak{m})$ be a metric measure space with* $(\operatorname{supp}\mathfrak{m},\mathsf{d})$ *doubling and* $\mathfrak{m}$ *finite on bounded sets. The Sobolev space $W^{1,q}(X,\mathsf{d},\mathfrak{m})$ of functions $u\in L^q(X,\mathfrak{m})$ with a q-relaxed slope, endowed with the usual norm*

$$\big(\|u\|_q^q+\mathrm{Ch}_q(u)\big)^{1/q} \qquad \forall u\in W^{1,q}(X,\mathsf{d},\mathfrak{m}), \tag{7.7}$$

is reflexive.

Proof. Since the Banach norms (7.5) and (7.7) on $W^{1,q}(X,\mathsf{d},\mathfrak{m})$ are equivalent thanks to (7.4) and reflexivity is invariant, we can work with the first norm. The Banach space $W^{1,q}(X,\mathsf{d},\mathfrak{m})$ endowed with the first norm is reflexive by uniform convexity and Milman–Pettis theorem. Q.E.D.

We can also prove, by standard functional-analytic arguments, that reflexivity implies separability.

Proposition 42 (Separability of $W^{1,q}(X,\mathsf{d},\mathfrak{m})$)**.** *If $W^{1,q}(X,\mathsf{d},\mathfrak{m})$ is reflexive, then it is separable and bounded Lipschitz functions with bounded support are dense.*

Proof. The density of Lipschitz functions with bounded support follows at once from the density of this convex set in the weak topology, ensured by Proposition 16. In order to prove separability, it suffices to consider for any M a countable and $L^q(X,\mathfrak{m})$-dense subset $\mathcal{D}_M$ of

$$\mathcal{L}_M:=\left\{f\in \mathrm{Lip}(X)\cap L^q(X,\mathfrak{m}):\ \int_X |\nabla f|_{w,q}^q\,\mathrm{d}\mathfrak{m}\le M\right\},$$

stable under convex combinations with rational coefficients. The weak closure of $\mathcal{D}_M$ obviously contains $\mathcal{L}_M$, by reflexivity (because if $f_n\in\mathcal{D}_M$ converge to $f\in\mathcal{L}_M$ in $L^q(X,\mathfrak{m})$, then $f_n\to f$ weakly in $W^{1,q}(X,\mathsf{d},\mathfrak{m})$); being this closure convex, it coincides with the strong closure of $\mathcal{D}_M$. This way we obtain that the closure in the strong topology of $\cup_M\mathcal{D}_M$ contains all Lipschitz functions with bounded support. Q.E.D.

The strategy of the proof of statement (a) in Theorem 40 consists in proving the estimate from above of $\mathcal{F}_q$ with relaxed gradients and the estimate from below with weak gradients. Then, the equivalence between weak and relaxed gradients provides the result. In the estimate from below it will be useful the discrete version of the q-weak upper gradient property given in Definition 26.

In the following lemma we prove that for every $u \in L^q(X,\mathfrak{m})$ we have that $4|\mathcal{D}_\delta u|$ is a q-weak upper gradient for $\mathcal{P}_\delta u$ up to scale $\delta/2$.

Lemma 43. *Let $\gamma \in AC^p([0,1];X)$. Then we have that*

$$\begin{aligned} &|\mathcal{P}_\delta u(\gamma_b) - \mathcal{P}_\delta u(\gamma_a)| \leq 4\int_a^b |\mathcal{D}_\delta u|(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t \\ &\qquad\qquad\qquad \text{for all } a<b \text{ s.t. } \int_a^b |\dot\gamma_t|\,\mathrm{d}t > \delta/2. \end{aligned} \tag{7.8}$$

In particular $4|\mathcal{D}_\delta u|$ is a q-weak upper gradient of $\mathcal{P}_\delta u$ up to scale $\delta/2$.

Proof. It is enough to prove the inequality under the more restrictive assumption that

$$\frac{\delta}{2} \leq \int_a^b |\dot\gamma_t|\,\mathrm{d}t \leq \delta, \tag{7.9}$$

because then we can slice every interval (a,b) that is longer than $\delta/2$ into subintervals that satisfy (7.9), and we get (4.12) by adding the inequalities for subintervals and using triangular inequality.

Now we prove (4.12) for every a, $b \in [0,1]$ such that (7.9) holds. Take any time $t \in [a,b]$; by assumption, it is clear that $\mathsf{d}(\gamma_t,\gamma_a) \leq \delta$ and $\mathsf{d}(\gamma_t,\gamma_b) \leq \delta$, so that the cells relative to γ_a and γ_b are both neighbors of the one relative to γ_t. By definition then we have:

$$\begin{aligned} |\mathcal{D}_\delta u|^q(\gamma_t) &\geq \frac{1}{\delta^q}\left(|\mathcal{P}_\delta u(\gamma_b) - \mathcal{P}_\delta u(\gamma_t)|^q + |\mathcal{P}_\delta u(\gamma_t) - \mathcal{P}_\delta u(\gamma_a)|^q\right) \\ &\geq \frac{1}{2^{q-1}\delta^q}|\mathcal{P}_\delta u(\gamma_b) - \mathcal{P}_\delta u(\gamma_a)|^q. \end{aligned}$$

Taking the q-th root and integrating in t we get

$$\begin{aligned} \int_a^b |\mathcal{D}_\delta u|(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t &\geq \frac{|\mathcal{P}_\delta u(\gamma_b) - \mathcal{P}_\delta u(\gamma_a)|}{2^{1-1/q}\delta}\int_a^b |\dot\gamma_t|\,\mathrm{d}t \\ &\geq \frac{1}{2}|\mathcal{P}_\delta u(\gamma_b) - \mathcal{P}_\delta u(\gamma_a)|, \end{aligned}$$

which proves (7.8). Q.E.D.

We can now prove Theorem 40.

Proof of the first inequality in (7.4). We prove that there exists a constant $\eta_1 = \eta_1(c_D)$ such that

$$\mathcal{F}_q(u) \leq \eta_1 \int_X |\nabla f|_{*,q}^q\,\mathrm{d}\mathfrak{m} \qquad \forall u \in L^q(X,\mathfrak{m}). \tag{7.10}$$

Let $u : X \to \mathbb{R}$ be a Lipschitz function with bounded support. We prove that

$$|\mathcal{D}_\delta u|^q(x) \leq 6^q c_D^3 (\mathrm{Lip}(u, B(x, 6\delta)))^q. \tag{7.11}$$

Indeed, let us consider i, $j \in [1, \ell_\delta) \cap \mathbb{N}$ such that A_i^δ and A_j^δ are neighbors. For every $x \in A_i^\delta$, $y \in A_j^\delta$ we have that $\mathsf{d}(x, y) \leq \mathrm{diam}(A_i^\delta) + \mathrm{diam}(A_i^\delta) + \mathsf{d}(A_i^\delta, A_j^\delta) \leq (10/4 + 10/4 + 1)\delta = 6\delta$ and that $y \in B(z_i^\delta, 19\delta/4) \subset B(z_i^\delta, 5\delta)$. Hence

$$\begin{aligned} \frac{|u_{\delta,i} - u_{\delta,j}|}{\delta} &\leq \frac{1}{\delta \mathfrak{m}(A_i^\delta)\mathfrak{m}(A_j^\delta)} \int_{A_i^\delta \times A_j^\delta} |u(x) - u(y)| \, \mathrm{d}\mathfrak{m}(x) \, \mathrm{d}\mathfrak{m}(y) \\ &\leq 6 \, \mathrm{Lip}(u, B(z_i^\delta, 5\delta)). \end{aligned}$$

Thanks to the fact that the number of neighbors of A_i^h does not exceed c_D^3 (see (7.2)) we obtain

$$|\mathcal{D}_\delta u|^q(x) \leq 6^q c_D^3 (\mathrm{Lip}(u, B(x, 6\delta)))^q \qquad \forall x \in \operatorname{supp} \mathfrak{m},$$

which proves (7.11).

Integrating on X we obtain that

$$\mathcal{F}_{\delta,q}(u) \leq 6^q c_D^3 \int_X (\mathrm{Lip}(u, B(x, 6\delta)))^q \, \mathrm{d}\mathfrak{m}(x).$$

Choosing $\delta = \delta_k$, letting $k \to \infty$ and applying the dominated convergence theorem on the right-hand side as well as the definition of asymptotic Lipschitz constant (3.7) we get

$$\mathcal{F}_q(u) \leq \liminf_{k \to \infty} \mathcal{F}_{\delta_k,q}(u) \leq 6^q c_D^3 \int_X \mathrm{Lip}_a^q(u, x) \, \mathrm{d}\mathfrak{m}(x).$$

By approximation, Proposition 16 yields (7.10) with $\eta_1 = 6^q c_D^3$.

Proof of the second inequality in (7.4). We consider a sequence (u_k) which converges to u in $L^q(X, \mathfrak{m})$ with $\liminf_k \mathcal{F}_{\delta_k,q}(u_k)$ finite. We prove that u has a q-weak upper gradient and that

$$\frac{1}{4^q} \int_X |\nabla u|_{w,q}^q \, \mathrm{d}\mathfrak{m} \leq \liminf_k \mathcal{F}_{\delta_k,q}(u_k). \tag{7.12}$$

Then, (7.4) will follow easily from (7.10), (7.12), Definition 3b and the coincidence of weak and relaxed gradients.

Without loss of generality we assume that the right-hand side is finite and, up to a subsequence not relabeled, we assume that the lim inf is a limit. Hence, the sequence $f_k := |\mathcal{D}_{\delta_k} u_k|$ is bounded in $L^q(X, \mathfrak{m})$ and, by weak compactness, there exist $g \in L^q(X, \mathfrak{m})$ and a subsequence $k(h)$ such that $f_{k(h)} \rightharpoonup g$ weakly in $L^q(X, \mathfrak{m})$. By the lower semicontinuity of the q-norm with respect to the weak convergence, we have that

$$\int_X g^q \,d\mathfrak{m} \leq \liminf_{h\to\infty} \int_X f^q_{k(h)} \,d\mathfrak{m} = \lim_{k\to\infty} \mathcal{F}_{\delta_k,q}(u_k). \tag{7.13}$$

We can now apply Theorem 27 to the functions $\bar{u}_h = \mathcal{P}_{\delta_{k(h)}}(u_{k(h)})$, which converge to u in $L^q(X, \mathfrak{m})$ thanks to Lemma 39, and to the functions $g_h = 4f_{k(h)}$ which are q-weak upper gradients of $\bar{u}_h$ up to scale $\delta_{k(h)}/2$, thanks to Lemma 43. We obtain that $4g$ is a weak upper gradient of u, hence $g \geq |\nabla u|_{w,q}/4$ $\mathfrak{m}$-a.e. in X. Therefore (7.13) gives

$$\frac{1}{4^q} \int_X |\nabla u|^q_{w,q} \,d\mathfrak{m} \leq \int_X g^q \,d\mathfrak{m} \leq \lim_{k\to\infty} \mathcal{F}_{\delta_k,q}(u_k).$$

Proof of statement (b). Let $\mathcal{N}_{q,\delta} : L^q(X, \mathfrak{m}) \to [0, \infty]$ be the positively 1-homogeneous function

$$\mathcal{N}_{q,\delta}(u) = \left(\|\mathcal{P}_\delta u\|^q_q + \mathcal{F}_\delta(u)\right)^{1/q} \qquad \forall u \in L^q(X, \mathfrak{m}).$$

For $q \geq 2$ we prove that $\mathcal{N}_{q,\delta}$ satisfies the first Clarkson inequality [22]

$$\mathcal{N}^q_{q,\delta}\left(\frac{u+v}{2}\right) + \mathcal{N}^q_{q,\delta}\left(\frac{u-v}{2}\right) \leq \frac{1}{2}\left(\mathcal{N}^q_{q,\delta}(u) + \mathcal{N}^q_{q,\delta}(v)\right) \ \forall u,\, v \in L^q(X, \mathfrak{m}). \tag{7.14}$$

Indeed, let $X_\delta \subset \mathbb{N} \cup (\mathbb{N} \times \mathbb{N})$ be the (possibly infinite) set

$$X_\delta = \left([1, \ell_\delta) \cap \mathbb{N}\right) \cup \left\{(i,j) \in \left([1, \ell_\delta) \cap \mathbb{N}\right)^2 : \ A^\delta_i \sim A^\delta_j\right\}$$

and let $\mathfrak{m}_\delta$ be the counting measure on X_δ. We consider the function $\Phi_{q,\delta} : L^q(X, \mathfrak{m}) \to L^q(X_\delta, \mathfrak{m}_\delta)$ defined by

$$\begin{cases} \Phi_{q,\delta}[u](i) = (\mathfrak{m}(A^\delta_i))^{1/q} u_{\delta,i} & \forall i \in [1, \ell_\delta) \cap \mathbb{N} \\ \Phi_{q,\delta}[u]((i,j)) = (\mathfrak{m}(A^\delta_i))^{1/q} \dfrac{u_{\delta,i} - u_{\delta,j}}{\delta} & \forall (i,j) \in \left([1, \ell_\delta) \cap \mathbb{N}\right)^2 \\ & \text{s.t. } A^\delta_i \sim A^\delta_j. \end{cases}$$

It can be easily seen that $\Phi_{q,\delta}$ is linear and that

$$\|\Phi_{q,\delta}(u)\|_{L^q(X_\delta, \mathfrak{m}_\delta)} = \mathcal{N}_{q,\delta}(u) \qquad \forall u \in L^q(X, \mathfrak{m}). \tag{7.15}$$

Writing the first Clarkson inequality in the space $L^q(X_h, \mathfrak{m}_h)$ and using the linearity of $\Phi_{q,\delta}$ we immediately obtain (7.14). Let $\omega : (0,1) \to$

$(0,\infty)$ be the increasing and continuous modulus of continuity $\omega(r) = 1-(1-r^q/2^q)^{1/q}$. From (7.14) it follows that for all u, $v \in L^q(X,\mathfrak{m})$ with $\mathcal{N}_{q,\delta}(u) = \mathcal{N}_{q,\delta}(v) = 1$ it holds

$$\mathcal{N}_{q,\delta}\left(\frac{u+v}{2}\right) \le 1 - \omega(\mathcal{N}_{q,\delta}\,(u-v)).$$

Hence $\mathcal{N}_{q,\delta}$ are uniformly convex with the same modulus of continuity ω. Thanks to Lemma 5 we conclude that also the Γ-limit of these norms, namely (7.5), is uniformly convex with the same modulus of continuity.

If $q<2$ the proof can be repeated substituting the first Clarkson inequality (7.14) with the second one

$$\begin{aligned}&\Big[\mathcal{N}_{q,\delta}\Big(\frac{u+v}{2}\Big)\Big]^p + \Big[\mathcal{N}_{q,\delta}\Big(\frac{u-v}{2}\Big)\Big]^p \\ &\qquad\le \left[\frac{1}{2}\big(\mathcal{N}_{q,\delta}(u)\big)^q + \frac{1}{2}\big(\mathcal{N}_{q,\delta}(v)\big)^q\right]^{1/(q-1)}, \quad \forall u,\, v \in L^q(X,\mathfrak{m})\end{aligned}$$

where $p = q/(q-1)$, see [22]. In this case the modulus ω is $1-(1-(r/2)^p)^{1/p}$.

Finally, let us consider the case $q=2$. From the Clarkson inequality we get

$$\mathcal{F}_2\left(\frac{u+v}{2}\right) + \mathcal{F}_2\left(\frac{u-v}{2}\right) \le 2\big(\mathcal{F}_2(u) + \mathcal{F}_2(v)\big). \tag{7.16}$$

If we apply the same inequality to $u = (u'+v')/2$ and $v = (u'-v')/2$ we obtain a converse inequality and, since u' and v' are arbitrary, the equality.

We conclude this section providing a counterexample to reflexivity. We denote by ℓ_1 the Banach space of summable sequences $(x_n)_{n\ge 0}$ and by ℓ_∞ the dual space of bounded sequences, with duality $\langle\cdot,\cdot\rangle$ and norm $\|v\|_\infty$. We shall use the factorization $\ell_1 = Y_i + \mathbb{R}e_i$, where e_i, $0 \le i < \infty$, are the elements of the canonical basis of ℓ_1. Accordingly, for fixed i we write $x = x_i' + x_i e_i$ and, for $f : \ell_1 \to \mathbb{R}$ and $y \in Y_i$, we set

$$f_y(t) := f(y + te_i) \qquad t \in \mathbb{R}.$$

Proposition 44. *There exist a compact subset X of ℓ_1 and $\mathfrak{m} \in \mathscr{P}(X)$ such that, if d is the distance induced by the inclusion in ℓ_1, the space $W^{1,q}(X,\mathsf{d},\mathfrak{m})$ is not reflexive for all $q \in (1,\infty)$.*

Proof. For $i \ge 0$, we denote by $\mathfrak{m}_i$ the normalized Lebesgue measure in $X_i := [0, 2^{-i}]$ and define X to be the product of the intervals X_i

and $\mathfrak{m}$ to be the product measure. Since X is a compact subset of ℓ_1, we shall also view $\mathfrak{m}$ as a probability measure in ℓ_1 concentrated on X.

Setting $f^v(x) := \langle v, x\rangle$, we shall prove that the map $v \mapsto f^v$ provides a linear isometry between ℓ_∞, endowed with the norm

$$|v|_\infty := \left(\int_X |\langle v, x\rangle|^q \,\mathrm{d}\mathfrak{m}(x) + \|v\|_\infty^q\right)^{1/q} \tag{7.17}$$

and $W^{1,q}(X, \mathsf{d}, \mathfrak{m})$. Since the norm (7.17) is equivalent to the ℓ_∞ norm, it follows that $W^{1,q}(X, \mathsf{d}, \mathfrak{m})$ contains a non-reflexive closed subspace and therefore it is itself non-reflexive.

Since the Lipschitz constant of f^v is $\|v\|_\infty$, it is clear that $\||\nabla f^v|_{w,q}\|_{L^q} \le \|v\|_\infty$. To prove equality, suffices to show that $\int_X |\nabla f^v|_{w,q}^q \,\mathrm{d}\mathfrak{m} \ge \|v\|_\infty^q$. Therefore we fix an integer $i \ge 0$ and we prove that $\int_X |\nabla f^v|_{w,q}^q \,\mathrm{d}\mathfrak{m} \ge |v_i|^q$.

Fix a sequence (f^n) of Lipschitz functions with bounded support with f^n and $\mathrm{Lip}_a(f^n)$ strongly convergent in $L^q(X, \mathfrak{m})$ to f^v and $|\nabla f^v|_{w,q}$ respectively. Possibly refining the sequence, we can assume that

$$\sum_n \|f^n - f^v\|_q^q < \infty. \tag{7.18}$$

If we show that

$$\liminf_{n\to\infty} \int_X \mathrm{Lip}_a^q(f^n, x) \,\mathrm{d}\mathfrak{m}(x) \ge |v_i|^q \tag{7.19}$$

we are done. Denoting $\mathfrak{m} = \tilde{\mathfrak{m}}_i \otimes \mathfrak{m}_i$ the factorization of $\mathfrak{m}$ (with $\tilde{\mathfrak{m}}_i \in \mathscr{P}(Y_i)$), we can use the obvious pointwise inequalities

$$\mathrm{Lip}_a(g, y + te_i) \ge \mathrm{Lip}_a(g_y, t) \ge |\nabla g_y|(t)$$

and Fatou's lemma, to reduce the proof of (7.19) to the one-dimensional statement

$$\liminf_{n\to\infty} \int_{X_i} |\nabla f_y^n|^q(t) \,\mathrm{d}\mathfrak{m}_i(t) \ge |v_i|^q \qquad \text{for } \tilde{\mathfrak{m}}_i\text{-a.e. } y \in Y_i. \tag{7.20}$$

Since (7.18) yields

$$\int_{Y_i} \sum_n \|f_y^n - f_y^v\|_{L^q(X_i, \mathfrak{m}_i)}^q \,\mathrm{d}\tilde{\mathfrak{m}}_i(y) = \sum_n \|f^n - f^v\|_{L^q(X, \mathfrak{m})}^q < \infty,$$

we have that $f_y^n \to f_y^v$ in $L^q(X_i, \mathfrak{m}_i) = L^q(X_i, 2^i \mathscr{L}^1)$ for $\tilde{\mathfrak{m}}_i$-a.e. $y \in Y_i$. We have also $|\nabla f_y^v|(t) = |v_i|$ for any $t \in X_i$, therefore (7.20) is a consequence of the well-known lower semicontinuity in $L^q(X_i, \mathscr{L}^1)$ of $g \mapsto \int_{X_i} |g'(t)|^q \, d\mathscr{L}^1(t)$ for Lipschitz functions defined on the real line (notice also that in this context we can replace the slope with the modulus of derivative, wherever it exists). Q.E.D.

§8. Lower semicontinuity of the slope of Lipschitz functions

Let us recall, first, the formulation of the Poincaré inequality in metric measure spaces.

Definition 45. *The metric measure space $(X, \mathsf{d}, \mathfrak{m})$ supports a weak $(1,q)$-Poincaré inequality if there exist constants τ, $\Lambda > 0$ such that for every $u \in W^{1,q}(X, \mathsf{d}, \mathfrak{m})$ and for every $x \in \operatorname{supp} \mathfrak{m}$, $r > 0$ the following holds*:

$$(8.1) \qquad ⨍_{B(x,r)} \left| u - ⨍_{B(x,r)} u \right| \mathrm{d}\mathfrak{m} \le \tau\, r \left(⨍_{B(x,\Lambda r)} |\nabla u|^q_{w,q} \, \mathrm{d}\mathfrak{m} \right)^{1/q}.$$

Many different and equivalent formulations of (8.1) are possible: for instance we may replace in the right hand side $|\nabla u|^q_{w,q}$ with $|\nabla u|^q$, requiring the validity of the inequality for Lipschitz functions only. The equivalence of the two formulations has been first proved in [19], but one can also use the equivalence of weak and relaxed gradients to establish it. Other formulations involve the median, or replace the left hand side by

$$\inf_{m \in \mathbb{R}} ⨍_{B(x,r)} |u - m| \, \mathrm{d}\mathfrak{m}.$$

The following lemma contains the fundamental estimate to prove our result.

Lemma 46. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a doubling metric measure space which supports a weak $(1,q)$-Poincaré inequality with constants τ, Λ. Let $u \in W^{1,q}(X, \mathsf{d}, \mathfrak{m})$ and let $g = |\nabla u|_{w,q}$. There exists a constant $C > 0$ depending only on the doubling constant $\tilde{c}_D$ and τ such that*

$$(8.2) \qquad |u(x) - u(y)| \le C \mathsf{d}(x,y) \big(M_q^{2\Lambda \mathsf{d}(x,y)} g(x) + M_q^{2\Lambda \mathsf{d}(x,y)} g(y) \big),$$

for every Lebesgue points x, $y \in X$ of (a representative of) u.

Proof. The main estimate in the proof is the following. Denoting by $u_{z,r}$ the mean value of u on $B(z,r)$, for every $s > 0$, x, $y \in X$ such that $B(x,s) \subset B(y,2s)$ we have that

$$(8.3) \qquad |u_{x,s} - u_{y,2s}| \le C_0(\tilde{c}_D, \tau) s M_q^{2\Lambda s} g(y).$$

Since $\mathfrak{m}$ is doubling and the space supports $(1,q)$-Poincaré inequality, from (2.6) we have that

$$
\begin{aligned}
|u_{x,s} - u_{y,2s}| &\le \fint_{B(x,s)} |u - u_{y,2s}| \,\mathrm{d}\mathfrak{m} \le \beta 2^\alpha \fint_{B(y,2s)} |u - u_{y,2s}| \,\mathrm{d}\mathfrak{m} \\
&\le 2^{1+\alpha}\beta\tau s \left(\fint_{B(y,2\Lambda s)} g^q \,\mathrm{d}\mathfrak{m} \right)^{1/q}
\end{aligned}
$$

and we obtain (8.3) with $C_0 = 2^{1+\alpha}\beta\tau$.

For every $r > 0$ let $s_n = 2^{-n} r$ for every $n \ge 1$. If x is a Lebesgue point for u then $u_{x,s_n} \to u(x)$ as $n \to \infty$. Hence, applying (8.3) to $x = y$ and $s_n = 2^{-n}r$, summing on $n \ge 1$ and remarking that $M_q^{2\Lambda s_n} g \le M_q^{\Lambda r} g$, we get

(8.4)
$$
|u_{x,r} - u(x)| \le \sum_{n=0}^\infty |u_{x,s_n} - u_{x,2s_n}| \le \sum_{n=0}^\infty C_0 s_n M_q^{\Lambda r} g(x) = C_0 r M_q^{\Lambda r} g(x).
$$

For every $r > 0$, x, y Lebesgue points of u such that $B(x,r) \subset B(y,2r)$, we can use the triangle inequality, (8.3) and (8.4) to get

$$
\begin{aligned}
|u(x) - u(y)| &\le |u(x) - u_{x,r}| + |u_{x,r} - u_{y,2r}| + |u_{y,2r} - u(y)| \\
&\le C_0 r M_q^{\Lambda r} g(x) + C_0 r M_q^{2\Lambda r} g(y) + C_0 r M_q^{\Lambda r} g(y).
\end{aligned}
$$

Taking $r = \mathsf{d}(x,y)$ (which obviously implies $B(x,r) \subset B(y,2r)$) and since $M_q^\varepsilon f(x)$ is nondecreasing in ε we obtain (8.2) with $C = 2C_0$. Q.E.D.

Proposition 47. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a doubling metric measure space, supporting a weak $(1,q)$-Poincaré inequality with constants τ, Λ and with* $\operatorname{supp}\mathfrak{m} = X$ *There exists a constant $C > 0$ depending only on the doubling constant $\tilde{c}_D$ and τ such that*

$$
|\nabla u| \le C\, |\nabla u|_{w,q} \quad \mathfrak{m}\text{-a.e. in } X \tag{8.5}
$$

for any Lipschitz function u with bounded support.

Proof. We set $g = |\nabla u|_{w,q}$; we note that g is bounded and with bounded support, thus $M_q^\varepsilon g$ converges to g in $L^q(X, \mathfrak{m})$ as $\varepsilon \to 0$. Let us fix $\lambda > 0$ and a Lebesgue point x for u where (2.8) is satisfied by $M_q^\lambda g$. Let $y_n \to x$ be such that

$$
|\nabla u|(x) = \lim_{n\to\infty} \frac{|u(y_n) - u(x)|}{\mathsf{d}(y_n, x)} \tag{8.6}
$$

and set $r_n = \mathsf{d}(x, y_n)$, $B_n = B(y_n, \lambda r_n) \subset B(x, 2r_n)$. Since (8.2) of Lemma 46 holds for $\mathfrak{m}$-a.e. $y \in B_n$, from the monotonicity of $M_q^\varepsilon g$ we get

$$
\begin{aligned}
|u(x) - u(y_n)| &\le \fint_{B_n} |u(x) - u(y)| \, \mathrm{d}\mathfrak{m}(y) + \lambda r_n \operatorname{Lip}(u, B_n) \\
&\le C r_n \left(M_q^{4\Lambda r_n} g(x) + \fint_{B_n} M_q^{4\Lambda r_n} g(y) \, \mathrm{d}\mathfrak{m}(y) \right) + \lambda r_n L,
\end{aligned}
$$

where L is the Lipschitz constant of u. For n large enough $B_n \subset B(x, 1)$ and $4\Lambda r_n \le \lambda$. Using monotonicity once more we get

$$
|u(x) - u(y_n)| \le C r_n \left(M_q^\lambda g(x) + \fint_{B_n} M_q^\lambda g \, \mathrm{d}\mathfrak{m} \right) + \lambda r_n L \tag{8.7}
$$

for n large enough. Since $B(y_n, r_n) = B_n \subset B(x, 2r_n)$ and since x is a 1-Lebesgue point for $M_q^\lambda g$, we apply (2.9) of Lemma 8 to the sets B_n to get

$$
\lim_{n \to \infty} \fint_{B_n} M_q^\lambda g \, \mathrm{d}\mathfrak{m} = M_q^\lambda g(x). \tag{8.8}
$$

We now divide both sides in (8.7) by $r_n = \mathsf{d}(x, y_n)$ and let $n \to \infty$. From (8.8) and (8.6) we get

$$
|\nabla u|(x) \le 2C M_q^\lambda g(x) + \lambda L.
$$

Since this inequality holds for $\mathfrak{m}$-a.e. x, we can choose an infinitesimal sequence $(\lambda_k) \subset (0, 1)$ and use the $\mathfrak{m}$-a.e. convergence of $M_q^{\lambda_k} g$ to g to obtain (8.5). Q.E.D.

Theorem 48. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space with $\mathfrak{m}$ doubling, which supports a weak $(1, q)$-Poincaré inequality and satisfies $\operatorname{supp} \mathfrak{m} = X$. Then, for any open set $A \subset X$ it holds*

$$
\begin{aligned}
&u_n,\ u \in \operatorname{Lip}_{\mathrm{loc}}(A),\ u_n \to u \ \text{ in } L^1_{\mathrm{loc}}(A) \\
&\qquad \Longrightarrow \quad \liminf_{n \to \infty} \int_A |\nabla u_n|^q \, \mathrm{d}\mathfrak{m} \ge \int_A |\nabla u|^q \, \mathrm{d}\mathfrak{m}.
\end{aligned} \tag{8.9}
$$

In particular, understanding weak gradients according to (6.4), it holds $|\nabla u| = |\nabla u|_{w,q}$ $\mathfrak{m}$-a.e. in X for all $u \in \operatorname{Lip}_{\mathrm{loc}}(X)$.

Proof. By a simple truncation argument we can assume that all functions u_n are uniformly bounded, since $|\nabla(M \wedge v \vee -M)| \le |\nabla v|$ and $|\nabla(M \wedge v \vee -M)| \uparrow |\nabla v|$ as $M \to \infty$. Possibly extracting a subsequence we can also assume that the lim inf in the right-hand side of (8.9) is a limit and, without loss of generality, we can also assume that it is finite. Fix a bounded open set B with $\mathrm{dist}(B, X \setminus A) > 0$ and let $\psi : X \to [0,1]$ be a cut-off Lipschitz function identically equal to 1 on a neighborhood of B, with support bounded and contained in A. It is clear that the functions $v_n := u_n\psi$ and $v := u\psi$ are globally Lipschitz, $v_n \to v$ in $L^q(X, \mathfrak{m})$ and (v_n) is bounded in $W^{1,q}(X, \mathsf{d}, \mathfrak{m})$.

From the reflexivity of this space proved in Corollary 41 we have that, possibly extracting a subsequence, (v_n) weakly converges in the Sobolev space to a function w. Using Mazur's lemma, we construct another sequence $(\hat{v}_n)$ that is converging strongly to w in $W^{1,q}(X, \mathsf{d}, \mathfrak{m})$ and $\hat{v}_n$ is a finite convex combination of $v_n, v_{n+1}, \ldots$. In particular we get $\hat{v}_n \to w$ in $L^q(X, \mathfrak{m})$ and this gives $w = v$. Moreover,

$$\int_B |\nabla \hat{v}_n|^q \, d\mathfrak{m} \le \sup_{k \ge n} \int_B |\nabla v_k|^q \, d\mathfrak{m}.$$

Eventually, from Proposition 47 applied to the functions $v - \hat{v}_n$ we get:

$$\begin{aligned}
&\left(\int_B |\nabla v|^q \, d\mathfrak{m}\right)^{1/q} \\
&\quad \le \liminf_{n\to\infty} \left\{ \left(\int_B |\nabla \hat{v}_n|^q \, d\mathfrak{m}\right)^{1/q} + \left(\int_B |\nabla (v - \hat{v}_n)|^q \, d\mathfrak{m}\right)^{1/q} \right\} \\
&\quad \le \limsup_{n\to\infty} \left\{ \left(\int_B |\nabla v_n|^q \, d\mathfrak{m}\right)^{1/q} \right\} + C \limsup_{n\to\infty} \|v - \hat{v}_n\|_{W^{1,q}} \\
&\quad = \limsup_{n\to\infty} \left(\int_B |\nabla v_n|^q \, d\mathfrak{m}\right)^{1/q}.
\end{aligned}$$

Since $v_n \equiv u_n$ and $v \equiv u$ on B we get

$$\int_B |\nabla u|^q \, d\mathfrak{m} \le \limsup_{n\to\infty} \int_B |\nabla u_n|^q \, d\mathfrak{m} \le \lim_{n\to\infty} \int_A |\nabla u_n|^q \, d\mathfrak{m}$$

and letting $B \uparrow A$ gives the result. Q.E.D.

§9. Appendix A: other notions of weak gradient

In this section we consider different notions of weak gradients, all easily seen to be intermediate between $|\nabla f|_{w,q}$ and $|\nabla f|_{*,q}$, and therefore coincident, as soon as Theorem 35 is invoked. These notions inspired those adopted in [3].

9.1. q-relaxed upper gradients and $|\nabla f|_{C,q}$

In the relaxation procedure we can consider, instead of pairs $(f, \mathrm{Lip}_a f)$ (i.e. Lipschitz functions and their asymptotic Lipschitz constant), pairs (f, g) with g upper gradient of f.

Definition 49 (q-relaxed upper gradient). *We say that $g \in L^q(X, \mathfrak{m})$ is a q-relaxed upper gradient of $f \in L^q(X, \mathfrak{m})$ if there exist $\tilde{g} \in L^q(X, \mathfrak{m})$, functions $f_n \in L^q(X, \mathfrak{m})$ and upper gradient g_n of f_n such that:*

(a) *$f_n \to f$ in $L^q(X, \mathfrak{m})$ and g_n weakly converge to $\tilde{g}$ in $L^q(X, \mathfrak{m})$;*
(b) *$\tilde{g} \le g$ $\mathfrak{m}$-a.e. in X.*

We say that g is a minimal q-relaxed upper gradient of f if its $L^q(X, \mathfrak{m})$ norm is minimal among q-relaxed upper gradients. We shall denote by $|\nabla f|_{C,q}$ the minimal q-relaxed upper gradient.

Again it can be proved (see [7]) that $|\nabla f|_{C,q}$ is local, and clearly

$$|\nabla f|_{C,q} \le |\nabla f|_{*,q} \qquad \mathfrak{m}\text{-a.e. in } X \tag{9.1}$$

because any q-relaxed slope is a q-relaxed upper gradient. On the other hand, the stability property of q-weak upper gradients stated in Theorem 27 gives

$$|\nabla f|_{w,q} \le |\nabla f|_{C,q} \qquad \mathfrak{m}\text{-a.e. in } X. \tag{9.2}$$

In the end, thanks to Theorem 35, all these notions coincide $\mathfrak{m}$-a.e. in X.

Notice that one more variant of the "relaxed" definitions is the one considered in [3], with pairs $(f, |\nabla f|)$. It leads to a weak gradient intermediate between the ones on (9.1), but a posteriori equivalent, using once more Theorem 35.

9.2. q-upper gradients and $|\nabla f|_{S,q}$

Here we recall a weak definition of upper gradient, taken from [24] and further studied in [27] in connection with the theory of Sobolev spaces, where we allow for exceptions in (2.2). This definition inspired the one given in [3], based on test plans.

Recall that, for $\Gamma \subset AC([0,1], X)$, the q-modulus $\mathrm{Mod}_q(\Gamma)$ is defined by

$$\mathrm{Mod}_q(\Gamma) := \inf\left\{ \int_X \rho^q \,\mathrm{d}\mathfrak{m} : \int_\gamma \rho \ge 1 \quad \forall \gamma \in \Gamma \right\}, \tag{9.3}$$

where the infimum is taken over all non-negative Borel functions $\rho : X \to [0, +\infty]$. We say that Γ is Mod_q-negligible if $\mathrm{Mod}_q(\Gamma) = 0$. Accordingly, we say that a Borel function $g : X \to [0, \infty]$ with $\int_X g^q \, \mathrm{d}\mathfrak{m} < \infty$ is a q-upper gradient of f if there exist a function $\tilde{f}$ and a Mod_q-negligible set Γ such that $\tilde{f} = f$ $\mathfrak{m}$-a.e. in X and

$$\left|\tilde{f}(\gamma_0) - \tilde{f}(\gamma_1)\right| \le \int_\gamma g < \infty \qquad \forall \gamma \in AC([0,1], X) \setminus \Gamma.$$

Notice that the condition $\int_\gamma g < \infty$ for Mod_q-almost every curve γ is automatically satisfied, by the q-integrability assumption on g. It is not hard to prove that the collection of all q-upper gradients of f is convex and closed, so that we can call minimal q-upper gradient, and denote by $|\nabla f|_{S,q}$, the element with minimal $L^q(X, \mathfrak{m})$ norm. Furthermore, the inequality

$$|\nabla f|_{S,q} \le |\nabla f|_{C,q} \qquad \mathfrak{m}\text{-a.e. in } X \tag{9.4}$$

(namely, the fact that all q-relaxed upper gradients are q-upper gradients) follows by a stability property of q-upper gradients very similar to the one stated in Theorem 27 for q-weak upper gradients, see [27, Lemma 4.11].

Observe that for a Borel set $\Gamma \subset C([0,1], X)$ and a test plan $\boldsymbol{\pi}$, integrating on Γ w.r.t. $\boldsymbol{\pi}$ the inequality $\int_\gamma \rho \ge 1$ and then minimizing over ρ, we get

$$\boldsymbol{\pi}(\Gamma) \le (C(\boldsymbol{\pi}))^{1/q} \big(\mathrm{Mod}_q(\Gamma)\big)^{1/q} \left(\iint_0^1 |\dot{\gamma}|^p \, \mathrm{d}s \, \mathrm{d}\boldsymbol{\pi}(\gamma) \right)^{1/p},$$

which shows that any Mod_q-negligible set of curves is also q-negligible according to Definition 19. This immediately gives that any q-upper gradient is a q-weak upper gradient, so that

$$|\nabla f|_{w,q} \le |\nabla f|_{S,q} \qquad \mathfrak{m}\text{-a.e. in } X. \tag{9.5}$$

Combining (9.1), (9.4), (9.5) and Theorem 35 we obtain that also $|\nabla f|_{S,q}$ coincides $\mathfrak{m}$-a.e. with all other gradients.

§10. Appendix B: discrete gradients in general spaces

Here we provide another type of approximation via discrete gradients which doesn't even require the space (X, d) to be doubling. We don't know whether this approximation can be used to obtain the reflexivity of $W^{1,q}(X, \mathsf{d}, \mathfrak{m})$ even without doubling assumptions.

We slightly change the definition of discrete gradient: instead of taking the sum of the finite differences, that is forbidden due to the fact that the number of terms can not in general be uniformly bounded from above, we simply take the supremum among the finite differences. Let us fix a decomposition A_i^δ of $\operatorname{supp}\mathfrak{m}$ as in Lemma 37. Let $u \in L^q(X,\mathfrak{m})$ and denote by $u_{\delta,i}$ the mean of u in A_i^δ as before. We consider the discrete gradient

$$|\mathcal{D}_\delta u|_\infty(x) = \frac{1}{\delta} \sup_{A_j^\delta \sim A_i^\delta} \{|u_{\delta,i} - u_{\delta,j}|\} \qquad \forall x \in A_i^\delta.$$

Then we consider the functional $\mathcal{F}_\delta^\infty : L^q(X,\mathfrak{m}) \to [0,\infty]$ given by

$$\mathcal{F}_\delta^\infty(u) := \int_X |\mathcal{D}_\delta(u)|_\infty^q(x)\, \mathrm{d}\mathfrak{m}(x).$$

With these definitions, the following theorem holds.

Theorem 50. *Let $(X,\mathsf{d},\mathfrak{m})$ be a Polish metric measure space with $\mathfrak{m}$ finite on bounded sets. Let $\mathcal{F}_q^\infty$ be a Γ-limit point of $\mathcal{F}_{q,\delta}^\infty$ as $\delta \downarrow 0$, namely*

$$\mathcal{F}_q^\infty := \Gamma - \lim_{k\to\infty} \mathcal{F}_{q,\delta_k}^\infty,$$

where $\delta_k \to 0$ and the Γ-limit is computed with respect to the $L^q(X,\mathfrak{m})$-distance. Then the functional $\mathcal{F}_q^\infty$ is equivalent to Cheeger's energy, namely there exists a constant $\eta_\infty = \eta_\infty(q)$ such that

$$\frac{1}{\eta_\infty}\mathrm{Ch}_q(u) \le \mathcal{F}_q^\infty(u) \le \eta_\infty \mathrm{Ch}_q(u) \qquad \forall u \in L^q(X,\mathfrak{m}). \tag{10.1}$$

The proof follows closely the one of Theorem 40. An admissible choice for η_∞ is 6^q.

§11. Appendix C: some open problems

In this section we discuss some open problems.

1. Optimality of the Poincaré assumption for the lower semicontinuity of slope. As shown to us by P. Koskela, the doubling assumption, while sufficient to provide reflexivity of the Sobolev spaces $W^{1,q}(X,\mathsf{d},\mathfrak{m})$, is not sufficient to ensure the lower semicontinuity (1.1) of slope. Indeed, one can consider for instance the Von Koch snowflake $X \subset \mathbb{R}^2$ endowed with the Euclidean distance. Since X is a self-similar fractal satisfying Hutchinson's open set condition (see for instance [11]), it follows that X is Ahlfors regular of dimension $\alpha = \ln 4/\ln 3 \in (1,2)$, namely $0 < \mathscr{H}^\alpha(X) < \infty$, where $\mathscr{H}^\alpha$ denotes α-dimensional Hausdorff measure in $\mathbb{R}^2$. Using self-similarity it is easy to check that $(X,\mathsf{d},\mathscr{H}^\alpha)$ is doubling. However, since absolutely continuous curves with values in X are constant, the q-weak upper gradient of any Lipschitz function

f vanishes. Then, the equivalence of weak and relaxed gradients gives $|\nabla f|_{*,q} = 0$ $\mathscr{H}^\alpha$-a.e. on X. By Proposition 16 we obtain Lipschitz functions f_n convergent to f in $L^q(X, \mathscr{H}^\alpha)$ and satisfying

$$\lim_{n\to\infty} \int_X \mathrm{Lip}_a^q(f_n, x)\, \mathrm{d}\mathscr{H}^\alpha(x) = 0.$$

Since $\mathrm{Lip}_a(f_n, \cdot) \geq |\nabla f_n|$, if $|\nabla f|$ is not trivial we obtain a counterexample to (1.1).

One can easily show that any linear map, say $f(x_1, x_2) = x_1$, has a nontrivial slope on X at least $\mathscr{H}^\alpha$-a.e. in X. Indeed, $|\nabla f|(x) = 0$ for some $x \in X$ implies that the geometric tangent space to X at x, namely all limit points as $y \in X \to x$ of normalized secant vectors $(y-x)/|y-x|$, is contained in the vertical line $\{x_1 = 0\}$. However, a geometric rectifiability criterion (see for instance [1, Theorem 2.61]) shows that this set of points x is contained in a countable union of Lipschitz curves, and it is therefore σ-finite with respect to $\mathscr{H}^1$ and $\mathscr{H}^\alpha$-negligible.

This proves that doubling is not enough. On the other hand, quantitative assumptions weaker than the Poincaré inequality might still be sufficient to provide the result.

2. Dependence on q of the weak gradient. The dependence of $|\nabla f|_{w,q}$ on q is still open: more precisely, assuming for simplicity that $\mathfrak{m}(X)$ is finite, $f \in W^{1,q}(X, \mathsf{d}, \mathfrak{m})$ easily implies via Proposition 16 that $f \in W^{1,r}(X, \mathsf{d}, \mathfrak{m})$ for $1 < r \leq q$ and that

$$|\nabla f|_{r,*} \leq |\nabla f|_{q,*} \qquad \mathfrak{m}\text{-a.e. in } X.$$

Whether equality $\mathfrak{m}$-a.e. holds or not is an open question. As pointed out to us by Gigli, this holds if $|\nabla g|_{w,q}$ is independent of q for a dense class $\mathcal{D}$ of functions (for instance Lipschitz functions g with bounded support); indeed, if this the case, for any $g \in \mathcal{D}$ we have

$$|\nabla f|_{q,*} \leq |\nabla g|_{q,*} + |\nabla (f-g)|_{q,*} = |\nabla g|_{r,*} + |\nabla (f-g)|_{q,*}$$

and considering $g_n \in \mathcal{D}$ with $g_n \to f$ strongly in $W^{1,q}(X, \mathsf{d}, \mathfrak{m})$ we obtain the result, since convergence occurs also in $W^{1,r}(X, \mathsf{d}, \mathfrak{m})$ and therefore $|\nabla g_n|_{r,*} \to |\nabla f|_{r,*}$ in $L^r(X, \mathfrak{m})$.

Under doubling and Poincaré assumptions, we know that these requirements are met with the class $\mathcal{D}$ of Lipschitz functions with bounded support, therefore as pointed out in [7] the weak gradient is independent of q. Assuming only the doubling condition, the question is still open.[1]

[1] At the time of receiving the page proofs, a counterexample has been found in S. Di Marino, G.Speight: "The p-weak gradient depends on p", Proceedings AMS, in press.

Acknowledgements. The first author acknowledges the support of the ERC ADG GeMeThNES. The authors thank N. Gigli for useful comments on a preliminary version of the paper and the reviewer for his/her detailed comments.

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