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Mixed functions of strongly polar weighted homogeneous face type

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Abstract.

Let $f(\mathbf{z},\bar{\mathbf{z}})$ be a mixed polynomial with strongly non-degenerate face functions. We consider a canonical toric modification $\pi : X \to \mathbb{C}^n$ and a polar modification $\pi_{\mathbb{R}} : Y \to X$. We will show that the toric modification resolves topologically the singularity of V and the zeta function of the Milnor fibration of f is described by a formula of a Varchenko type.

§1. Introduction

Recall that a mixed polynomial $f(\mathbf{z},\bar{\mathbf{z}})$ with n complex variables variables $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$ is called a *polar weighted homogeneous polynomial* if there exist a weight vector $P = (p_1, \dots, p_n)$ and a non-zero integers d_p such that

$$f(\rho \circ \mathbf{z}, \bar{\rho} \circ \bar{\mathbf{z}}) = \rho^{d_p} f(\mathbf{z},\bar{\mathbf{z}}), \quad \rho \circ \mathbf{z} = (\rho^{p_1} z_1, \dots, \rho^{p_n} z_n),\ \rho \in \mathbb{C},\ |\rho| = 1.$$

Similarly $f(\mathbf{z},\bar{\mathbf{z}})$ is called a *radially weighted homogeneous polynomial* if there exist a weight vector $Q = (q_1, \dots, q_n)$ and a positive integer d_r such that

$$f(t \circ \mathbf{z}, t \circ \bar{\mathbf{z}}) = t^{d_r} f(\mathbf{z},\bar{\mathbf{z}}), \quad t \circ \mathbf{z} = (t^{q_1} z_1, \dots, t^{q_n} z_n),\ t \in \mathbb{R}^+$$

If f is both radially and polar weighted homogeneous, we have an associated $\mathbb{R}^+ \times S^1$-action on $\mathbb{C}^n$ by

$$(t,\rho) \circ \mathbf{z} = (t^{q_1} \rho^{p_1} z_1, \dots, t^{q_n} \rho^{p_n} z_n), \quad (t,\rho) \in \mathbb{R}^+ \times S^1.$$

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The integers d_r and d_p are called the radial and the polar degree respectively and we denote them as $d_r = \mathrm{rdeg}_Q f$ and $d_p = \mathrm{pdeg}_P f$. Usually a polar weighted homogeneous polynomial is also assumed to be radially weighted homogeneous [7]. We assume this throughout in this paper.

We say that $f(\mathbf{z}, \bar{\mathbf{z}})$ is *strongly polar weighted homogeneous* if $p_j = q_j$ for $j = 1, \ldots, n$. Then the associated $\mathbb{R}^+ \times S^1$ action on $\mathbb{C}^n$ reduces to a $\mathbb{C}^*$ action which is defined by

$$(\tau, \mathbf{z}) = (\tau, (z_1, \ldots, z_n)) \mapsto \tau \circ \mathbf{z} = (z_1 \tau^{p_1}, \ldots, z_n \tau^{p_n}), \ \tau \in \mathbb{C}^*.$$

Furthermore f is called *a strongly polar positive weighted homogeneous polynomial* if $\mathrm{pdeg}_P f > 0$.

The purpose of this paper is to generalize the result of Varchenko ([10]) to non-degenerate mixed functions of strongly polar weighted homogeneous face type (Theorem 11).

§2. Non-degeneracy and associated toric modification

Throughout this paper, we use the same notations as in [6], [9], unless we state otherwise. We recall basic terminologies for the toric modification.

2.1. Non-degenerate functions

Let $f(\mathbf{z}) = \sum_{\nu,\mu} a_{\nu,\mu} \mathbf{z}^\nu \bar{\mathbf{z}}^\mu$ be a convenient mixed analytic function. Here "mixed analytic" implies that $\sum_{\nu,\mu} a_{\nu,\mu} \mathbf{z}^\nu \mathbf{w}^\mu$ is an analytic function of $2n$ complex variables $\mathbf{z} = (z_1, \ldots, z_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$. f is convenient if it contains some monomial $z_j^{a_j} \bar{z}_j^{b_j}$ with a non zero coefficient for any $j = 1, \ldots, n$. The Newton polyhedron $\Gamma_+(f)$ is defined by the convex hull of the union $\cup_\nu \{\nu + \mu + \mathbb{R}^n_+ \mid a_{\nu,\mu} \neq 0\}$. The Newton boundary $\Gamma(f)$ is the union of the compact faces of $\Gamma_+(f)$. If f is a holomorphic function germ, $a_{\nu,\mu} = 0$ unless $\mu = (0, \ldots, 0)$ and the Newton boundary $\Gamma(f)$ coincides with the usual one. For a positive weight vector $P = {}^t(p_1, \ldots, p_n)$, we associate a linear function ℓ_P on $\Gamma(f)$ by $\ell_P(\tau) = \tau_1 p_1 + \cdots + \tau_n p_n$ for $\tau \in \Gamma(f)$. It takes a minimum value which we denote by $d(P, f)$ or $d(P)$ if f is fixed. Let $\Delta(P)$ be the face where ℓ_P takes the minimal value and put $f_P := \sum_{\nu+\mu \in \Delta(P)} a_{\nu,\mu} \mathbf{z}^\nu \bar{\mathbf{z}}^\mu$ and we call *f_P the face function of f with respect to P.*

A mixed function $f(\mathbf{z}, \bar{\mathbf{z}})$ is called *of strongly polar positive weighted homogeneous face type* if for each face Δ of dimension $n-1$, $f_\Delta(\mathbf{z}, \bar{\mathbf{z}})$ is a strongly polar positive weighted homogeneous polynomial.

Recall that f is *non-degenerate for P* (respectively *strongly non-degenerate for P*) if the polynomial mapping $f_P : \mathbb{C}^{*n} \to \mathbb{C}$ has no critical point on $f_P^{-1}(0)$ (resp. on $\mathbb{C}^{*n}$). In the case that f_P is a polar weighted homogeneous polynomial, two notions coincide ([9], Remark 4). In particular, two notions for non-degeneracy coincide for holomorphic functions.

Consider a mixed monomial $M = \mathbf{z}^\nu \bar{\mathbf{z}}^\mu$. The radial degree $\mathrm{rdeg}_P(M)$ and the polar degree $\mathrm{pdeg}_P(M)$ with respect to P are defined by

$$\mathrm{rdeg}_P(M) = \sum_{i=1}^{n} p_i(\nu_i + \mu_i), \quad \mathrm{pdeg}_P(M) = \sum_{i=1}^{n} p_i(\nu_i - \mu_i).$$

Note that the face function f_P is a radially weighted homogeneous polynomial of degree $d(P)$ by the definition.

Consider the space of positive weight vectors N^+. Recall that an equivalent relation $\sim$ on N^+ is defined by

$$\text{for } P, Q \in N^+, \ P \sim Q \iff \Delta(P) = \Delta(Q).$$

This defines a conical subdivision of N^+ which is called *the dual Newton diagram for f* and we denote it by $\Gamma^*(f)$.

2.2. Admissible subdivision and an admissible toric modification

We recall the admissible toric modification for beginner's convenience. We first take a regular simplicial subdivision Σ^* of the dual Newton diagram $\Gamma^*(f)$. Such a regular fan is called an admissible regular fan. See [6], Definition III (3.1.3). The primitive generators of one dimensional cones in Σ^* are called vertices. Namely a vertex has a unique expression as a primitive integral vector $P = {}^t(p_1, \dots, p_n)$ with $\gcd(p_1, \dots, p_n) = 1$. P is *strictly positive* if $p_j > 0$ for any j. Let $\mathcal{V}$ be the vertices of Σ^* and let $\mathcal{V}^+ \subset \mathcal{V}$ be the vertices which are strictly positive. (We denote the strict positivity by $P \gg 0$.) To each n-dimensional simplicial cone τ of Σ^*, we associate a unimodular matrix, which we denote it by τ by an abuse of notation. Thus if $P_1, \dots, P_n$ are primitive vertices of τ, we also identify τ with the unimodular matrix $(P_1, \dots, P_n) \in \mathrm{SL}(n; \mathbb{Z})$. On the other hand, as a cone, $\tau = \{\sum_{i=1}^k a_i P_i \,|\, a_i \geq 0, i = 1, \dots, n\}$. We say that Σ^* is *convenient* if the vertices of Σ^* are strictly positive except the obvious elementary ones $E_j = {}^t(0, \dots, 1, \dots 0)$ (1 is at j-th coordinates), $j = 1, \dots, n$. We assume that f is convenient and thus we assume also that Σ^* is convenient hereafter.

We denote by $\mathcal{K}$ (respectively by $\mathcal{K}_s$) the set of simplices of Σ^* (resp. s-simplices of Σ^*). Note that an s-simplex corresponds to an $(s+1)$-dimensional cone. For each $\tau = (P_1, \dots, P_n) \in \mathcal{K}_{n-1}$, we associate affine space $\mathbb{C}^n_\tau$ with the toric coordinates $\mathbf{u}_\tau = (u_{\tau 1}, \dots, u_{\tau n})$ and a toric morphism $\pi_\tau : \mathbb{C}^n_\tau \to \mathbb{C}^n$ with $\mathbf{z} = \pi_\tau(\mathbf{u}_\tau)$, $z_j = u_{\tau 1}^{p_{j1}} \cdots u_{\tau n}^{p_{jn}}$ for $j = 1, \dots, n$ where $\mathbb{C}^n$ is the base space and $\mathbf{z} = (z_1, \dots, z_n)$ is the fixed coordinates. Let X be the quotient space of the disjoint union $\amalg_\sigma \mathbb{C}^n_\sigma$ by the canonical identification $\mathbf{u}_\tau \sim \mathbf{u}_\sigma$ iff $\mathbf{u}_\tau = \pi_{\tau^{-1}\sigma}(\mathbf{u}_\sigma)$ where $\pi_{\tau^{-1}\sigma}$ is well defined on $\mathbf{u}_\sigma$. The quotient space is a complex manifold of dimension n and we have a canonical projection $\pi : X \to \mathbb{C}^n$ which is called *the associated toric modification.* Recall that π gives a birational morphism such that $\pi : X \setminus \pi^{-1}(\mathbf{0}) \to \mathbb{C}^\mathbf{n} \setminus \{\mathbf{0}\}$ is an isomorphism, as we have assumed that Σ^* is convenient. Here $\mathbf{0}$ is the origin of $\mathbb{C}^n$. It also gives a good resolution of the function germ f at the origin if $f(\mathbf{z})$ is a non-degenerate holomorphic function germ. However for a mixed non-degenerate germ, π does not give a good resolution in general ([9]).

2.3. Configuration of the exceptional divisors

We recall the configuration of exceptional divisors of $\pi : X \to \mathbb{C}^n$. For further detail, see [6], p. 73. For each vertex $P \in \mathcal{V}^+$ of Σ^*, there corresponds an exceptional divisor $\hat{E}(P)$. The restriction $\pi : X \setminus \pi^{-1}(\mathbf{0}) \to \mathbb{C}^n \setminus \{\mathbf{0}\}$ is biholomorphic and the exceptional fiber $\pi^{-1}(\mathbf{0})$ is described as:

$$\pi^{-1}(\mathbf{0}) = \cup_{P \in \mathcal{V}^+} \hat{E}(P).$$

Note that $\mathcal{V} \setminus \mathcal{V}^+ = \{E_j; j = 1, \dots, n\}$ and $\hat{E}(E_j)$ is not compact and $\pi|_{\hat{E}(E_j)} : \hat{E}(E_j) \to \{z_j = 0\}$ is biholomorphic. Let $\widetilde{V}$ be the strict transform of V to X. Recall that $E(P) := \hat{E}(P) \cap \widetilde{V}$ is non-empty if and only if $\dim \Delta(P; f) \geq 1$.

2.3.1. *Stratification.* We define *the toric stratification* and *the Milnor stratification* of the exceptional fiber $\pi^{-1}(\mathbf{0})$. For each simplex $\tau = (P_1, \dots, P_k)$ of Σ^*, we define

$$\hat{E}(\tau)^* = \cap_{i=1}^k \hat{E}(P_i) \setminus \cup_{Q \in \mathcal{V}, Q \notin \tau} \hat{E}(Q),$$
$$\widetilde{V}(\tau)^* = \hat{E}(\tau)^* \cap \widetilde{V},\ \widetilde{E}(\tau)^* = \hat{E}(\tau)^* \setminus \widetilde{V}(\tau).$$

In the case of $\tau = (P)$, we simply write $\hat{E}(P)$, $\widetilde{V}(P)^*$ and $\widetilde{E}(P)^*$. Then we consider two canonical stratifications of $\pi^{-1}(\mathbf{0})$:

$$\text{(1)} \qquad \text{Toric stratification}: \quad \mathcal{T} := \{\hat{E}(\tau)^* \,|\, \tau \cap \mathcal{V}^+ \neq \emptyset\},$$

$$\text{(2)} \qquad \text{Milnor stratification}: \mathcal{M} := \{\widetilde{E}(\tau)^*, \widetilde{V}(\tau)^* \,|\, \tau \cap \mathcal{V}^+ \neq \emptyset\}.$$

Here $\tau \cap \mathcal{V}^+ \neq \emptyset$ implies $\hat{E}(\tau) \subset \pi^{-1}(\mathbf{0})$. We call τ *the support simplex* of $\widetilde{E}(\tau), \widetilde{V}(\tau)$. If τ is a subsimplex of σ, we denote it as $\tau \prec \sigma$. The basic properties are

Proposition 1. (1) $\hat{E}(P)\cap\hat{E}(Q)\neq\emptyset$ *if and only if* (P,Q) *is a simplex of* Σ^*.

(2) *Let* $\tau=(P_1,\dots,P_k)$ *be a* k*-simplex and let* $\sigma=(P_1,\dots,P_n)$ *and* $\sigma'=(P_1,\dots,P_k,Q_{k+1},\dots,Q_n)$ *be* $(n-1)$*-simplices for which* $\tau\prec\sigma$ *and* $\tau\prec\sigma'$. *Put*

$$\hat{E}(\tau)^*_\sigma := \{\mathbf{u}_\sigma\in\mathbb{C}^n_\sigma\,|\,u_{\sigma,i}=0, i\le k,\, u_{\sigma,j}\neq 0,\, j\ge k+1\}$$
$$\hat{E}(\tau)^*_{\sigma'} := \{\mathbf{u}_{\sigma'}\in\mathbb{C}^n_{\sigma'}\,|\,u_{\sigma',i}=0, i\le k,\, u_{\sigma',j}\neq 0,\, j\ge k+1\}.$$

Then we have $\hat{E}(\tau)^*_\sigma=\hat{E}(\tau)^*_{\sigma'}$. *In particular,* $\hat{E}(\tau)^*=\hat{E}(\tau)^*_\sigma\cong\mathbb{C}^{*(n-k)}$.

(3) $\amalg_{\tau,P\in\tau}\hat{E}(\tau)^*$ *is a toric stratification of* $\hat{E}(P)$.

Proof. As a unimodular matrix, $\sigma^{-1}\sigma'$ takes the following form

$$\sigma^{-1}\sigma'=\begin{pmatrix} I_k & B\\ O & C\end{pmatrix}$$

where O is $(n-k)\times k$ zero matrix and I_k is the $k\times k$ identity matrix. From this expression, it is clear that the restriction of the morphism $\pi_{\sigma^{-1}\sigma'}:\mathbb{C}^{*n}_{\sigma'}\to\mathbb{C}^{*n}_\sigma$ gives the isomorphism $\pi_C:\hat{E}(\tau)^*_{\sigma'}\to\hat{E}(\tau)^*_\sigma$ where π_C is the toric morphism associated with the unimodular matrix C. The other assertion is obvious. See [6], Prop. III (1.3.2), Cor. III (1.3.3) for further detail. Q.E.D.

2.4. Milnor fibration

Let f be a strongly non-degenerate function which is either holomorphic or mixed analytic. We consider the Minor fibration by the second description: $f:E(\varepsilon,\delta)^*\to D^*_\delta$ where

$$E(\varepsilon,\delta)^* = B^{2n}_\varepsilon\cap f^{-1}(D^*_\delta)$$
$$B^{2n}_\varepsilon=\{\mathbf{z}\in\mathbb{C}^n\,|\,\|\mathbf{z}\|\le\varepsilon\},\ D^*_\delta:=\{\rho\in\mathbb{C}\,|\,0<|\rho|\le\delta\}.$$

The Milnor fiber is given by $F_{\eta,\varepsilon}:=f^{-1}(\eta)\cap B^{2n}_\varepsilon$ with $0\neq|\eta|\le\delta$. Note that as long as ε is smaller than the stable radius ε_0 and $\delta\ll\varepsilon$, the fibering structure does not depend on the choice of ε and δ.

Let $\pi:X\to\mathbb{C}^n$ be the associated toric modification. The restriction $\pi:X\setminus\pi^{-1}(\mathbf{0})\to\mathbb{C}^n\setminus\{\mathbf{0}\}$ is biholomorphic. Then the Milnor fibration can be replaced by $\pi^*f=f\circ\pi:\hat{E}(\varepsilon,\delta)^*\to D^*_\delta$ where

$$\hat{E}(\varepsilon,\delta)^*=\{x\in X\,|\,0<|f(\pi(x))|\le\delta\}\cap\widetilde{B}_\varepsilon$$
$$\widetilde{B}_\varepsilon=\{x\,|\,\|\pi(x)\|\le\varepsilon\}.$$

Note that $\widetilde{B}_\varepsilon$ can be understood as an ε-neighborhood of $\pi^{-1}(\mathbf{0})$. Let $\widetilde{V}$ be the strict transform of V to X. The above setting is common for holomorphic functions and mixed functions.

§3. A theorem of Varchenko

We first recall the result of Varchenko for a non-degenerate convenient holomorphic function $f(\mathbf{z})$. Consider a germ of hypersurface $V = f^{-1}(0)$. For $I \subset \{1, \dots, n\}$, let f^I be the restriction of f on the coordinate subspace $\mathbb{C}^I$ where

$$\mathbb{C}^I = \{\mathbf{z} \,|\, z_j = 0,\, j \notin I\},\ \mathbb{C}^{*I} = \{\mathbf{z} \,|\, z_j = 0 \iff j \notin I\}.$$

Let $\mathcal{S}_I$ be the set of primitive weight vectors $P = {}^t(p_i)_{i \in I}$ of the variables $\{z_i \,|\, i \in I\}$ such that $p_i > 0$ for all $i \in I$ and $\dim \Delta(P, f^I) = |I| - 1$. $P \in \mathcal{S}_I$ can be considered to be a weight vector of $\mathbf{z}$ putting $p_j = 0,\ j \notin I$. Then the result of Varchenko ([10], see also [6]) can be stated as follows.

Theorem 2. *The zeta function of the Milnor fibration of f is given by the formula*

$$\zeta(t) = \prod_I \zeta_I(t), \quad \zeta_I(t) = \prod_{P \in \mathcal{S}_I} (1 - t^{d(P,f^I)})^{-\chi(P)/d(P,f^I)}.$$

The term $\chi(P)$ is the Euler–Poincaré characteristic of the toric Milnor fiber $F(P)^*$ where

$$F(P)^* := \{\mathbf{z}^I \in \mathbb{C}^{*I} \,|\, f_P^I(\mathbf{z}^I) = 1\},$$

and it is a combinatorial invariant which satisfies the equality:

$$\chi(P) = (-1)^{|I|-1}|I|!\mathrm{Vol}_{|\mathrm{I}|}\mathrm{Cone}(\Delta(\mathrm{P};\mathrm{f}^{\mathrm{I}}), \mathbf{0}). \tag{3}$$

§4. Revisit to the proof

For the proof of Theorem 2, we use an admissible toric modification as in the proof in [6]. We will generalize this theorem for a convenient non-degenerate mixed function of strongly polar weighted homogeneous face type in the next section. For this purpose, we give a detailed description of the proof so that it can be used for a mixed function of strongly polar weighted homogeneous face type without any essential change. Let Σ^* be an admissible regular, convenient subdivision and let $\pi : X \to \mathbb{C}^n$ be the associated toric modification.

4.1. Compatibility of the charts

Let $\tau = (P_1, \dots, P_k) \in \mathcal{K}_{k-1}$ and suppose that we have two coordinate charts σ and σ' such that $\tau \prec \sigma, \sigma'$ and $\tau = \sigma \cap \sigma'$. Put $\sigma = (P_1, \dots, P_n)$ and $\sigma' = (P_1, \dots, P_k, Q_{k+1}, \dots, Q_n)$. We also assume that $\hat{E}(\tau)^* \in \mathcal{T}$. This implies $\pi(\hat{E}(\tau)) = \{\mathbf{0}\}$. Then we have

Proposition 3. *The matrix ${\sigma'}^{-1}\sigma$ takes the form*

$${\sigma'}^{-1}\sigma = \begin{pmatrix} I_k & B \\ O & C \end{pmatrix}$$

where O is the $(n-k) \times k$ zero matrix and C is a $(n-k) \times (n-k)$-unimodular matrix. Put $B = (b_{i,j})$. The toric coordinates are related by

$$(4) \qquad \begin{cases} (u_{\sigma',k+1}, \dots, u_{\sigma',n}) = \pi_C(u_{\sigma,k+1}, \dots, u_{\sigma,n}), \\ u_{\sigma',i} = u_{\sigma,i} \times \prod_{j=k+1}^{n} u_{\sigma,j}^{b_{i,j}},\ i = 1, \dots, k. \end{cases}$$

In particular, we have the commutative diagram

$$\begin{array}{ccc} \mathbb{C}^n_\sigma & \xrightarrow{p} & \hat{E}(\tau) \cap \mathbb{C}^n_\sigma \\ \downarrow \pi_{\sigma'^{-1}\sigma} & & \downarrow \pi_C \\ \mathbb{C}^n_{\sigma'} & \xrightarrow{p'} & \hat{E}(\tau) \cap \mathbb{C}^n_{\sigma'} \end{array}$$

where p, p' are the projections into $\hat{E}(\tau)$ defined by $p(\mathbf{u}_\sigma) = \mathbf{u}'_\sigma$, $p'(\mathbf{u}_{\sigma'}) = \mathbf{u}'_{\sigma'}$ where $\mathbf{u}'_\sigma = (u_{\sigma,k+1}, \dots, u_{\sigma,n})$ and $\mathbf{u}'_{\sigma'} = (u_{\sigma',k+1}, \dots, u_{\sigma',n})$.

4.2. Tubular neighborhoods of the exceptional divisors

First we fix C^∞ function $\rho(t)$ such that $\rho \equiv 1$ for $t \le R$ and monotone decreasing for $R \le t \le 2R$ and $\rho \equiv 0$ for $t \ge 2R$. The number R is large enough and will be chosen later. For $\sigma = (P_1, \dots, P_n) \in \mathcal{K}_{n-1}$, we define $\rho_\sigma(\mathbf{u}_\sigma) = \rho(\|\mathbf{u}_\sigma\|)$. For each exceptional divisor $S = \hat{E}(P)^* \in \mathcal{T}$, we consider the set of $(n-1)$-simplices $\mathcal{K}_P = \{\sigma \in \mathcal{K}_{n-1} \mid P \in \sigma\}$. For each $\sigma, \sigma' \in \mathcal{K}_P$, after ordering the vertices of σ, σ' as $\sigma = (P, P_2, \dots, P_n)$ and $\sigma' = (P, P'_2, \dots, P'_n)$, we define the *distance function* $\mathrm{dist_P}$ *from* $\hat{E}(P)$ by

$$\mathrm{dist_P} : \mathrm{X} \to \mathbb{R},\ \mathrm{dist_P}(\mathbf{w}) = \sum_{\sigma \in \mathcal{K}_\mathrm{P}} \mathrm{dist}_{\mathrm{P},\sigma}(\mathbf{w})$$
$$\mathrm{dist}_{\mathrm{P},\sigma}(\mathbf{w}) := \rho_\sigma(\mathbf{u}_\sigma(\mathbf{w}))|\mathrm{u}_{\sigma,1}(\mathbf{w})|$$

where $\mathbf{u}_\sigma(\mathbf{w})$ is the coordinate of $\mathbf{w}$ in $\mathbb{C}^n_\sigma$ and $\mathbf{u}'_\sigma := (u_{\sigma,2}, \dots, u_{\sigma,n})$. Put $B_{K\,\sigma}(P) = \{(0, \mathbf{u}'_\sigma) \mid \|\mathbf{u}'_\sigma\| \le K\}$. We assume that R is sufficiently large so that $\cup_{\sigma'\in\mathcal{K}_P} B_{R,\sigma'}(P) = \hat{E}(P)$. Note that the distance function is continuous on X and C^∞ on $X \setminus \hat{E}(P)$. We put

$$N_\varepsilon(\hat{E}(P)) := \mathrm{dist}_{\mathrm{P}}^{-1}([0,\varepsilon]).$$

Lemma 4. *Suppose that* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{C}_\sigma^n$ *with* $\alpha \notin \pi^{-1}(\mathbf{0})$ *and* $\alpha_1 \neq 0$. *Put* $\alpha(t) = (t\alpha_1, \alpha_2, \dots, \alpha_n)$ *for* $0 \le t \le 1$, $r_t = \mathrm{dist}_{\mathrm{P}}(\alpha(\mathrm{t}))$ *and* $S(r_t) := \mathrm{dist}_{\mathrm{P}}^{-1}(\mathrm{r_t})$. *Then* $\alpha(0) \in \hat{E}(P)$ *and* $\lim_{t\to+0} T_{\alpha(t)}S(\alpha(t))$ *is the real orthogonal space* $v^\perp$ *of the vector* $v := (\alpha_1, 0, \dots, 0)$. *That is, the tangent space* $T_\alpha S(\alpha)$ *converges to the real hyperplane* $v^\perp$ *when* t *goes to zero.*

This lemma states that the tubular neighborhood $\partial N_\varepsilon(\hat{E}(P))$ behaves infinitesimally as $|u_{\sigma,1}| =$ constant. For the proof, see the Appendix (§4.5).

4.2.1. *Tubular neighborhood of* $\widetilde{V}(\tau)$. Consider the stratum $\widetilde{V}(\tau)$ with $\tau = (P_1, \dots, P_k)$. Let $\mathcal{K}_\tau$ be the set of coordinate charts σ such that $\tau \prec \sigma$. We order the vertices so that $\sigma = (P_1, \dots, P_k, \dots, P_n)$. We can write

$$\begin{cases} \pi_\sigma^* f(\mathbf{u}_\sigma) = u_{\sigma,1}^{d(P_1)} \cdots u_{\sigma,k}^{d(P_k)} \widetilde{f}(\mathbf{u}_\sigma), \\ \pi^* f_\Delta = u_{\sigma,1}^{d(P_1)} \cdots u_{\sigma,k}^{d(P_k)} \widetilde{f}_\Delta(\mathbf{u}'_\sigma) \\ \text{where } \widetilde{f}(\mathbf{u}_\sigma) = \widetilde{f}_\Delta(\mathbf{u}'_\sigma) + R(\mathbf{u}_\sigma),\ \mathbf{u}'_\sigma = (u_{\sigma,k+1}, \dots, u_{\sigma,n}). \end{cases}$$

The function f_Δ is by definition the face function of the face $\Delta := \cap_{i=1}^k \Delta(P_i)$. The second term R vanishes on $\hat{E}(\tau)$. Thus the polynomial $\widetilde{f}_\Delta$ is a defining polynomial of $\tilde{V}(\tau)$ in the coordinate chart $\mathbb{C}_\sigma^n$. Take another $\sigma' \in \mathcal{K}_\tau$ and write $\sigma'^{-1}\sigma$ as in (4). Then we have

$$\prod_{i=1}^k u_{\sigma,i}^{d(P_i)} \widetilde{f}_{\tau\sigma}(\mathbf{u}'_\sigma) = \prod_{i=1}^k u_{\sigma',i}^{d(P_i)} \widetilde{f}_{\tau\sigma'}(\mathbf{u}'_{\sigma'}).$$

Thus we have

$$\widetilde{f}_{\tau\sigma'}(\mathbf{u}'_{\sigma'}) = \widetilde{f}_{\tau\sigma}(\mathbf{u}'_\sigma) \times \prod_{j=k+1}^n u_{\sigma,j}^{m_j},\ \exists m_j \in \mathbb{Z}. \tag{5}$$

Thus from now on, we fix an $(n-1)$-simplex $\sigma = \sigma(\tau)$ for each τ and put

$$\widetilde{V}_\varepsilon(\tau) = \{\mathbf{u}_\sigma \in \hat{E}(\tau) \cap \mathbb{C}_\sigma^n \,|\, |\widetilde{f}_{\tau\sigma}(\mathbf{u}'_\sigma)| \le \sqrt{\varepsilon}\}.$$

We call $\mathbb{C}_\sigma^n$ *the canonical coordinates chart of* $\hat{E}(\tau)$. Now for each $\tau = (P_1, \dots, P_k) \in \mathcal{K}$ such that $\hat{E}(\tau)^* \subset \pi^{-1}(\mathbf{0})$, we put

$$N_\varepsilon(\hat{E}(\tau)) = \cap_{j=1}^k N_\varepsilon(\hat{E}(P_j)),$$

where $N_\varepsilon(\hat{E}(\tau))$ is a tubular neighborhood of $\widetilde{E}(\tau)$.

4.2.2. *Truncated tubular neighborhoods.* Let $p_{\tau\varepsilon} : N_\varepsilon(\hat{E}(\tau)) \to \hat{E}(\tau)$ be the projection. Recall that $p_{\tau\varepsilon}$ is defined by the simple projection $\mathbf{u}_\sigma \mapsto \mathbf{u}'_\sigma$ for any chart $\mathbb{C}^n_\sigma$ with $\sigma = (P_1, \dots, P_n)$. Now we define *truncated Milnor stratification* as follows. The truncated strata and truncated tubular neighborhoods for the Milnor fibration are defined by

$$\begin{aligned} &N_\varepsilon(\widetilde{E}(\tau))^{tr} = p_{\tau\varepsilon}^{-1}(\widetilde{E}_\varepsilon(\tau)^{tr}),\\ &N_\varepsilon(\widetilde{V}(\tau))^{tr} = p_{\tau\varepsilon}^{-1}(\widetilde{V}_\varepsilon(\tau)^{tr}),\\ &\text{where } \begin{cases} \widetilde{E}_\varepsilon(\tau)^{tr} = (\hat{E}(\tau) \setminus N_\varepsilon(\widetilde{V}(\tau)) \setminus \cup_{\tau\prec\tau'}(N_\varepsilon(\hat{E}(\tau'))),\\ \widetilde{V}_\varepsilon(\tau)^{tr} = \widetilde{V}_\varepsilon(\tau) \setminus \cup_{\tau\prec\tau'} N_\varepsilon(\hat{E}(\tau')). \end{cases} \end{aligned}$$

Thus we can write $N_\varepsilon(S)^{tr} = p_{\tau\varepsilon}^{-1}(S^{tr})$, using the notations

$$S^{tr} \quad = \quad \begin{cases} \widetilde{E}_\varepsilon(\tau)^{tr}, & S = \widetilde{E}(\tau)^*,\\ \widetilde{V}_\varepsilon(\tau)^{tr}, & S = \widetilde{V}(\tau)^*. \end{cases}$$

Note that $\widetilde{E}_\varepsilon(\tau)^{tr}, \widetilde{V}_\varepsilon(\tau)^{tr}$ are relatively compact subsets of $\hat{E}(\tau)^*$ which is homotopy equivalent to $\widetilde{E}(\tau)^*$ and $\widetilde{V}(\tau)$ respectively. Put

$$N_\varepsilon(\pi^{-1}(\mathbf{0})) := \cup_{\hat{E}(\tau)\subset\pi^{-1}(\mathbf{0})} \left(N_\varepsilon(\widetilde{E}_\varepsilon(\tau))^{tr} \cup N_\varepsilon(\widetilde{V}(\tau))\right).$$

Note that $N_\varepsilon(\pi^{-1}(\mathbf{0}))$ is a homotopy equivalent cofinal system of the neighborhood of $\pi^{-1}(\mathbf{0})$. We consider the Milnor fibration over D^*_δ with $\delta \ll \varepsilon$:

$$\pi^* f : N_{\varepsilon,\delta} \to D^*_\delta, \quad N_{\varepsilon,\delta} := (\pi^* f)^{-1}(D^*_\delta) \cap N_\varepsilon(\pi^{-1}(\mathbf{0})).$$

4.3. Recipe of the proof

Step 1. First we will show that the restriction of $\pi^* f : N_{\varepsilon,\delta} \to D^*_\delta$ over each tubular neighborhood $N_\varepsilon(S)^{tr}$ is a fibration in the way that each fiber is transverse to the boundary of $N_\varepsilon(S)^{tr}$. Thus the restriction to the boundaries $\partial N_\varepsilon(S)^{tr}$ is also a fibration.

Step 2. Then using the additive formula for the Euler characteristic and the corresponding product formula for the zeta function (see [6], Chapter 1), the calculation of the zeta function of the Milnor fibration is reduced to the calculation of the Milnor fibration restricted to each ε-tubular neighborhood $N_\varepsilon(S)^{tr}$. This fibration is again a locally product of the Milnor fibration of the restriction to the normal slice of S^{tr} and the stratum S^{tr}.

Step 3. Finally we determine the set of strata which contribute to the zeta function (Lemma 5). They correspond bijectively to $\cup_I \mathcal{S}_I$.

We say that a simplex $\tau = (P_1, \dots, P_k)$ is *of a divisor type* if (up to an ordering of the vertices) $P_1 \in \mathcal{V}^+$ and the other vertices $\{P_2, \dots, P_k\}$ is a subset of the non-positive vertices $\{E_1, \dots, E_n\}$. A simplex τ of a divisor type is called to be *of a maximal dimensional face* if $\Delta(P_1) \cap \Gamma(f^I)$ is a maximal dimensional face of $\Gamma(f^I)$ (*i.e.* $\dim \Delta(P_1) = |I| - 1$) where $I = \{i \mid E_i \notin \tau\}$.

For a subset $I \subset \{1, 2, \dots, n\}$, consider the set of vertices $\mathcal{S}'_I$ of Σ^* such that there exists an $(n - |I|)$-simplex τ of a divisor type with a maximal dimensional face ($\tau \in \mathcal{K}_{n-|I|}$) whose vertices are $\{P, E_j \mid j \notin I\}$. The key assertion is the following.

Lemma 5. *Take a stratum $S \in \mathcal{M}$.*
(1) *The Milnor fibration is decomposed into the fibrations restricted on $N_\varepsilon(S)^{tr}$ for each $S \in \mathcal{M}$. This fibration is topologically determined by the corresponding face function.*
(2) *The zeta function of the normal slice is non-trivial only if $S = \widetilde{E}(\tau)^*$ and $\tau = (P_1, \dots, P_k)$ is of a divisor type.*
(3) *The zeta function of the tubular neighborhood $N_\varepsilon(S)^{tr}$ is non-trivial if and only if τ is of a maximal dimensional face.*
(4) *There is a bijective correspondence from $\mathcal{S}'_I$ to $\mathcal{S}_I$.*

Here the normal slice for $S = \widetilde{V}(\tau)$ implies normal plane of $\widetilde{V}(\tau)$ in the fixed coordinate chart $\sigma(\tau)$ and the standard metric in this affine space. The proof of Lemma 5 occupies the rest of this subsection.

Recall that in the Milnor stratification $\mathcal{M}$, there are two type of strata: $\widetilde{E}(\tau)^*$ and $\widetilde{V}(\tau)^*$ with $\tau = (P_1, \dots, P_k)$. Let $\ell(\tau) = \sharp\{i \mid d(P_i) > 0\}$ and we refer to $\ell(\tau)$ as *the strict positivity dimension* of τ. Take a $(n-1)$-simplex $\sigma = (P_1, \dots, P_n)$ having τ as a face. We write $(u_1, \dots, u_n)$ for simplicity instead of $(u_{\sigma,1}, \dots, u_{\sigma,n})$, the canonical toric coordinates of $\mathbb{C}^n_\sigma$ and $\sigma = (P_1, \dots, P_n)$.

Case 1. $S = \widetilde{E}(\tau)$

As $\hat{E}(\tau) \subset \pi^{-1}(\mathbf{0})$, we may assume that $P_1 \in \mathcal{V}^+$ so that $d(P_1) > 0$ hereafter. Put $\Delta := \cap_{i=1}^k \Delta(P_i)$. $\pi_\sigma^* f$ takes the form

$$\begin{cases} \pi^* f = U_{1,k} \widetilde{f}(\mathbf{u}), \text{ where } U_{1,k} := u_1^{d(P_1)} \cdots u_k^{d(P_k)}, \\ \pi^* f_\Delta = U_{1,k} \widetilde{f}_\Delta(\mathbf{u}), \quad \widetilde{f}(\mathbf{u}) = \widetilde{f}_\Delta(\mathbf{u}') + R(\mathbf{u}), \end{cases}$$

where R is contained in the ideal $(u_1, \dots, u_k)$ and therefore it vanishes on $\hat{E}(\tau) \cap \mathbb{C}^n_\sigma$ and $\mathbf{u}' = (u_{k+1}, \dots, u_n)$ are the coordinates of $\hat{E}(\tau) \cap \mathbb{C}^n_\sigma$. We consider the homotopy $\hat{f}_t = U_{1,k} \widetilde{f}_t$ for $0 \le t \le 1$ where $\widetilde{f}_t(\mathbf{u}) :=$

$\widetilde{f}_\Delta(\mathbf{u}') + tR(\mathbf{u})$. Note that $\hat{f}_1 = \pi^* f$ and $\hat{f}_0 = \pi^* f_\Delta$ is associated to the face function f_Δ.

4.3.1. *Smoothness.* Consider the family of the restriction of Milnor fibering $\hat{f}_t : N_\varepsilon(S^{tr}) \cap \hat{f}_t^{-1}(D_\delta^*) \to D_\delta^*$, $\delta \ll \varepsilon$ and their Milnor fibers $F_{t,\delta}(\tau) := \hat{f}_t^{-1}(\delta) \cap N_\varepsilon(S^{tr})$ and $S = \widetilde{E}(\tau)^*$. Consider (a submatrix of) the jacobian matrix

$$(\sharp) : J := \left(\frac{\partial \hat{f}_t}{\partial u_{k+1}}(P), \dots, \frac{\partial \hat{f}_t}{\partial u_n}(\mathbf{u})\right)$$
$$= U_{1,k}\left(\frac{\partial \widetilde{f}_\Delta}{\partial u_{k+1}}(\mathbf{u}') + t\frac{\partial R}{\partial u_{k+1}}(\mathbf{u}), \dots, \frac{\partial \widetilde{f}_\Delta}{\partial u_n}(\mathbf{u}) + t\frac{\partial R}{\partial u_n}(\mathbf{u})\right).$$

By the non-degeneracy, there exists $k+1 \le j \le n$ such that $\frac{\partial \widetilde{f}_\Delta}{\partial u_j}(\mathbf{u}') \ne 0$. As R and $\frac{\partial R}{\partial u_j}$, $j \ge k+1$ are constantly zero on $\hat{E}(\tau)$, this implies that $\frac{\partial \hat{f}_t}{\partial u_j}(\mathbf{u}') \ne 0$ for sufficiently small ε and $\delta \ll \varepsilon$, $i \le k$. Thus $J \ne (0, \dots, 0)$ for any $u \in F_{t,\delta}(\tau)$ with $\mathbf{u}' \in S^{tr}$, as S^{tr} is relatively compact. So $F_{t,\delta}(\tau)$ is also smooth.

4.3.2. *Transversality.* We consider the transversality of $F_{t,\delta}$ and the boundary $\partial(N_\varepsilon(S^{tr}))$ at the intersection of $B := \partial(N_\varepsilon(S^{tr})) \cap N_\varepsilon S'$ or $B' = \partial(N_\varepsilon(S^{tr})) \cap \widetilde{V}_\varepsilon(S')$ where $S' = \widetilde{E}(\tau')^*$ with

$$\tau' = (P_1, \dots, P_k, \dots, P_m).$$

Put $\Delta' = \cap_{i=1}^m \Delta(P_i)$. Let $\sigma(\tau')$ be the fixed chart for τ' and let $\mathbf{v} = (v_1, \dots, v_n)$ be the toric coordinates of $\mathbb{C}^n_{\sigma(\tau')}$ for simplicity. As we are considering a tubular neighborhood of polydisk type,

$$B = \{\mathbf{v} \mid |\mathrm{dist}_{\mathrm{P_i}}(\mathbf{v})| = \varepsilon,\ \mathrm{k}+1 \le \mathrm{i} \le \mathrm{m}\} \quad \text{or}$$
$$B' = \{\mathbf{v} \mid |\mathrm{dist}_{\mathrm{P_i}}(\mathbf{v})| = \varepsilon,\ \mathrm{k}+1 \le \mathrm{i} \le \mathrm{m},\ |\widetilde{\mathrm{f}}_{\Delta'}(\mathbf{v}'')| = \sqrt{\varepsilon}\}.$$

By (5) for some integers $m_i, i = m+1, \dots, n$,

$$f_{\Delta'}(\mathbf{v}'') = f_{\Delta'}(\mathbf{u}'') \prod_{i=m+1}^{n} u_i^{m_i}.$$

Putting $U_{k+1,m} = \prod_{j=k+1}^{m} u_j^{d(P_j)}$, we can write further that

$$(6)\quad \begin{cases} \hat{f}_t(\mathbf{u}) = U_{1,k}\widetilde{f}_t(\mathbf{u}), \\ \widetilde{f}_t(\mathbf{u}) = \widetilde{f}_\Delta(\mathbf{u}') + tR(\mathbf{u}) \\ \qquad = U_{k+1,m}\left(\widetilde{f}_{\Delta'}(\mathbf{u}'') + R'(\mathbf{u}') + t\bar{R}(\mathbf{u})\right), \\ \widetilde{f}_\Delta(\mathbf{u}') = U_{k+1,m}\left(\widetilde{f}_{\Delta'}(\mathbf{u}'') + R'(\mathbf{u}')\right), \\ \hat{f}_t(\mathbf{u}) = U_{1,k}U_{k+1,m}\left(\widetilde{f}_{\Delta'}(\mathbf{u}'') + R'(\mathbf{u}') + t\bar{R}(\mathbf{u})\right), \\ R = U_{1,k}U_{k+1,m}\bar{R}, \end{cases}$$

where $\mathbf{u}'' = (u_{m+1}, \cdots, u_n)$ and R' in the ideal generated by

$$u_{k+1}, \dots, u_m.$$

Note that $\bar{R}(\mathbf{u})$ is in the ideal generated by $\{u_i u_j \mid i \leq k < j \leq m\}$. Thus the transversality follows from the fact that the jacobian submatrix

$$\frac{\partial(\Re\hat{f}_t, \Im\hat{f}_t, \mathrm{dist}_{\mathrm{P_{k+1}}}, \dots, \mathrm{dist}_{\mathrm{P_m}})}{\partial(x_{\sigma 1}, y_{\sigma 1} \dots, x_{\sigma m}, y_{\sigma m})}, \quad u_j = x_{\sigma 1} + iy_{\sigma j}$$

has rank $m - k + 2$. For the proof of this assertion, we use the polar coordinates as follows.

Assume that $S' = V(\tau')$ is non-empty, *i.e.*, namely $\dim \Delta' \geq 1$. Put $g := \widetilde{f}_{\Delta'}(\mathbf{v}'')$. On a neighborhood of a chosen point $\mathbf{u}_0'' \in \partial\widetilde{V}_\varepsilon(\tau')^{tr}$, by the non-degeneracy of f on Δ', g can be used as a member of a coordinate chart. For example, we may assume that there exists an open neighborhood $U(\mathbf{u}^0)$, $\mathbf{u}^0 = (u_1^0, \dots, u_m^0, \mathbf{u}_0'')$ such that

$$(u_1, \dots, u_m, g, v_{m+2}, \dots, v_n)$$

is a coordinate chart on $U(\mathbf{u}^0)$ and $(g, v_{m+2}, \dots, v_n)$ is a coordinate chart of $U(\mathbf{u}^0) \cap \hat{E}(\tau')$. We use the polar coordinates for $u_{k+1}, \dots, u_m$, g. So put

$$u_1 = x_{\sigma 1} + iy_{\sigma 1},$$
$$u_j = \rho_j e^{\theta_j i},\ j = k+1, \dots, m, \quad g = \rho_g e^{\theta_g i}.$$

The tranversality can be checked using the subjacobian with respect to $(x_{\sigma 1}, y_{\sigma 1}, \rho_{k+1}, \dots, \rho_m, \rho_g)$:

Assertion 6. *Under the above notations, we have*

$$\mathrm{rank}\left(\frac{\partial(\Re\hat{\mathrm{f}}_\mathrm{t}, \Im\hat{\mathrm{f}}_\mathrm{t}, \mathrm{dist}_{\mathrm{P_{k+1}}}, \dots, \mathrm{dist}_{\mathrm{P_m}}, \rho_\mathrm{g})}{\partial(\mathrm{x}_{\sigma 1}, \mathrm{y}_{\sigma 1}, \rho_{\mathrm{k+1}}, \dots, \rho_\mathrm{m}, \rho_\mathrm{g})}\right) = \mathrm{m} - \mathrm{k} + 3.$$

For the proof, see §4.5.2 in Appendix 4.5.

Remark 7. Assume $d(P_i) > 0$ for some $2 \leq i \leq k$. Then the Milnor fiber $F_{t,\delta}$ also intersects with the boundary of the tubular neighborhood: $\mathrm{dist}_{\mathrm{P_i}} = \varepsilon$. The transversality with this boundary is treated considering it as the transversality of the Milnor fiber in the stratum $\widetilde{E}(\tau')^{tr}$, $\tau' = \tau \setminus \{P_i\}$ with $\partial N_\varepsilon(\widetilde{E}(\tau))^{tr}$. Thus this case is treated in the pair $\tau' \prec \tau$.

Thus we have observed that the Milnor fibration of $\pi^* f$ is the union of fibrations restricted on $N_\varepsilon(S)^{tr}$ and $\partial N_\varepsilon(S)^{tr}$ with $S = \widetilde{E}(\tau)^{tr}$ or $S = \widetilde{V}(\tau)^{tr}$ with $\tau = (P_1, \dots, P_m)$. Using the homotopy $\pi_\sigma^* f_t$, this restriction is equivalent to the fibration defined by $\pi_\sigma^* f_\Delta$ where $\Delta = \cap_{i=1}^k \Delta(P_i)$. This proves the first assertion (1).

4.4. Zeta functions

Next, we consider the assertion for the zeta function (2).

4.4.1. *Case 1. Stratum* $\widetilde{V}_\varepsilon(\tau)$. We first consider the stratum $\widetilde{V}_\varepsilon(\tau)$: Let $\psi_S : N_\varepsilon(\widetilde{V}_\varepsilon(\tau)) \to \widetilde{V}_\varepsilon(\tau)$ be the projection of the tubular neighborhood. At each point $x \in \widetilde{V}(\tau)^{tr}$, the Milnor fibration is homotopically defined by $\pi^* f_\Delta = \hat{f}_\Delta$ $(\Delta = \cap_{i=1}^k \Delta(P_i))$. Recall that

$$\hat{f}_\Delta(\mathbf{u}) = \prod_{i=1}^{k} u_i^{d(P_i)} \times \widetilde{f}_\Delta(\mathbf{u}').$$

Put $g = \widetilde{f}_\Delta(\mathbf{u}')$. Take a point $x \in \widetilde{V}(\tau)$. Assume $\frac{\partial \widetilde{f}_\Delta(\mathbf{u}')}{\partial u_{k+1}}(x) \neq 0$ for example. Then we may assume that locally $(g, u_{k+2}, \dots, u_n)$ is a coordinate system of a neighborhood, say $U(x)$ of $x \in \widetilde{V}(\tau)$ and also $(u_1, \dots, u_k, g, u_{k+2}, \dots, u_n)$ is a coordinate system of the open set $\cap_{i=1}^k \{|u_i| < \eta\} \cap \psi_S^{-1}(U(x))$. By the relative compactness of the truncated strata $\widetilde{V}(\tau)^{tr}$, we may also assume that $\varepsilon \ll \eta$ so that

$$\cap_{i=1}^k \{\mathrm{dist}_{\mathrm{P_i}} \leq \varepsilon\} \cap \psi_{\mathrm{S}}^{-1}(\mathrm{U(x)}) \subset \cap_{\mathrm{i}=1}^{\mathrm{k}} \{|\mathrm{u_i}| < \eta\} \cap \psi_{\mathrm{S}}^{-1}(\mathrm{U(x)}).$$

In the normal slice of x, $u_1, \dots, u_k, g$ are coordinates. The Milnor fiber restricted on $N_\varepsilon(\widetilde{V}(\tau)^{tr})$ is locally equivalent to the product of $U(x)$ and the Milnor fibration of the polynomial $h = g \prod_{i=1}^k u_i^{d(P_i)}$ (=the defining polynomial in the normal slice) in $\mathbb{C}^{k+1}$. Namely the fibration

$$\hat{f}_\Delta : N_\varepsilon(\widetilde{V}(\tau)^{tr}) \cap N_{\varepsilon,\delta} \cap \psi_S^{-1}(U(x)) \to D_\delta^*$$

is isomorphic to the product of $U(x)$ and the restriction to the normal slice:

$$\widetilde{f}_{\Delta,x} : \psi_S^{-1}(x) \cap \hat{N}_{\varepsilon,\delta} \to D_\delta^*.$$

We consider two tubular neighborhoods:

$$N_\varepsilon(\hat{E}(\tau)^{tr}) = \{\mathbf{u} \,|\, \mathrm{dist}_{\mathrm{P_i}}(\mathbf{u}) \le \varepsilon,\ \mathrm{i} = 1, \dots, \mathrm{k}\},$$
$$N_\varepsilon(\hat{E}(\tau)^{tr})' = \{\mathbf{u} \,|\, |u_i| \le \varepsilon,\ i = 1, \dots, k\}.$$

By the cofinal homotopy equivalent argument, we can consider the normal Milnor fibration in the latter space and we see easily that the fiber is given by

$$F_\delta = \{(u_1, \dots, u_k, g) \,|\, g \prod_{i=1}^{k} u_i^{d(P_i)} = \delta,\ |g| \le \sqrt{\varepsilon},\ |u_i| \le \varepsilon\}$$

and it is homotopic to $(S^1)^\ell$ where $\delta \ll \varepsilon$ and ℓ is the strict positivity dimension of τ. As $\widetilde{V}(\tau) \subset \pi^{-1}(\mathbf{0})$, $\ell \ge 1$. Thus the Euler characteristic of F_δ is also zero and the monodromy is trivial. There are no contribution from this stratum to the zeta function.

4.4.2. *Case 2. Stratum* $S = \widetilde{E}(\tau)^{tr}$. Now we consider the Milnor fibering on the tubular neighborhood over the strata $S = \widetilde{E}(\tau)^{tr}$. We have seen that the Milnor fibration of $\hat{f}$ is again isomorphic to the fibration defined by $\hat{f}_\Delta$ and the latter is locally product of the base space and the Milnor fibration of the restriction to the normal slice. This normal slice function is locally described by the product function $u_1^{d(P_1)} \cdots u_k^{d(P_k)} f_\Delta(\mathbf{u}')$. The factor $f_\Delta(\mathbf{u}')$ is a constant on the normal slice $\mathbf{u}' = \text{const}$. We know that the fiber in this normal slice is homotopic to $\gcd(d(P_1), \dots, d(P_k))$ copies of $(S^1)^{\ell-1}$ where ℓ is the strict positivity dimension of τ. See for example [6], p. 48. Therefore *the Euler characteristic of this slice Milnor fiber is non-zero if and only if* $\ell = 1$. This implies τ is of a divisor type. Assume for example that $d(P_1) > 0$ and $d(P_i) = 0$, $2 \le i \le k$. This implies $P_i = E_{\nu(i)}$ for $i = 2, \dots, k$ and $\tau = (P_1, E_{\nu(2)}, \dots, E_{\nu(k)})$ and $\Delta(P_1; f^I)$ is a face of $\Gamma(f^I)$ where $I = \{1, \dots, n\} \setminus \{\nu(2), \dots, \nu(k)\}$. The Milnor fiber F_τ^* restricted to this stratum is defined by

$$F_\tau^* = \{\mathbf{u} \,|\, u_1^{d(P_1)} \widetilde{f}_\Delta(\mathbf{u}') = \delta,\ \mathbf{u}' \in \widetilde{E}(\tau)^{tr}\}$$

is homotopically $d(P_1)$ disjoint polydisk of dimension $k-1$ defined by $\{u_1 = \delta^{1/d(P_1)}, |u_j| \le \varepsilon, j = 2, \dots, k\}$ over $(0, \mathbf{u}')$ with

$$\mathbf{u}' = (u_{k+1}, \dots, u_{\sigma n}).$$

Thus the zeta function of the Milnor fibration of the normal slice is $(1 - t^{d(P_1)})$ and thus by a standard Mayer–Vietoris argument, the zeta function of the Milnor fibration over this stratum is

$$\zeta_S(t) = (1 - t^{d(P_1)})^{\chi(\widetilde{E}(\tau)^*)}. \tag{7}$$

Now the proof of Lemma 5 reduces to:

Lemma 8. *Assume that* $\tau = (P_1, E_{\nu(1)}, \dots, E_{\nu(k)})$ *as above. Then* $\chi(\widetilde{E}(\tau)^*) \neq 0$ *only if* $\Delta := \Delta(P_1) \cap \mathbb{R}^I$ *is a face of maximal dimension of* $\Gamma(f^I)$ *where* $I = \{1, \dots, n\} \setminus \{\nu(1), \dots \nu(k)\}$.

Proof. Put $I = \{1, \dots, n\} \setminus \{\nu(1), \dots, \nu(k)\}$. Then

$$\chi(\widetilde{E}(\tau)^*) = \chi(\hat{E}(\tau)^* \setminus \widetilde{V}(\tau)) = -\chi(\widetilde{V}(\tau)^*)$$

as $\hat{E}(\tau)^* \cong \mathbb{C}^{*(n-k)}$. As $\sigma = (P_1, \dots, P_n)$ is a unimodular matrix, $P := P_1^I$ is a primitive vector and $\Delta = \Delta(P_1) \cap \mathbb{R}^I$ and f_Δ is nothing but $(f^I)_P$.

$$\begin{aligned}
\widetilde{V}(\tau)^* &= \{\mathbf{u} = (u_1, \dots, u_n) \mid \mathbf{u} \in \hat{E}(\tau)^*,\ \widetilde{f}_{P_1}(\mathbf{u}') = 0\} \\
&\overset{\pi_\sigma}{\cong} \{\mathbf{z}^I \in \mathbb{C}^{*I} \mid f_P^I(\mathbf{z}^I) = 0\} \\
&= \{\mathbf{z}^I \in \mathbb{C}^{*I} \mid f_\Delta^I(\mathbf{z}^I) = 0\}
\end{aligned}$$

where σ is a $(n-1)$-simplex with $\sigma = (P_1, \dots, P_n)$ and the Euler characteristic of the variety $\{\mathbf{z}^I \in \mathbb{C}^{*I} \mid f_\Delta^I(\mathbf{z}^I) = 0\}$ is non-zero if only if $\dim \Delta = n - k$ with $\Delta = \Delta(P_1) \cap \mathbb{R}^I$ (See for example, Theorem (5.3), [5]). Q.E.D.

4.4.3. *Correspondence of* $\mathcal{S}_I$ *and* $\mathcal{S}'_I$. For a fixed $I \subset \{1, \dots, n\}$ with $|I| = k$, let us consider the set of vertices $\mathcal{S}'_I$ which is the set of vertices $P \in \mathcal{V}^+$ such that there exists a simplex $\tau = (P_0, \dots, P_k) \in \mathcal{K}_k$, $P_0 = P$ of a divisor type with a maximal face such that $P_j = E_{\nu(j)}$ for $j = 1, \dots, k$, $I = \{\nu(i), i = 1, \dots, k\}$ and $\Delta = \Delta(P_1) \cap \mathbb{R}^I$. As τ is a regular simplex, the I component P^I of P is a primitive vector such that $\Delta(P^I, f^I) = \Delta$ and $d(P^I, f^I) = d(P, f)$. The proof of Theorem 2 is now completed by the following.

Proposition 9. *There is a one-to one correspondence of* $\mathcal{S}'_I$ *and* $\mathcal{S}_I$ *by*

$$\xi : \mathcal{S}'_I \to \mathcal{S}_I,\ P \mapsto P^I.$$

Proof. We check the surjectivity of ξ. Take a face Ξ of $\Gamma(f^I)$ of maximal dimension. Consider the set of covectors $\Sigma^*(\Xi) = \{P \mid \Delta(P) \supset \Xi\}$. It is obvious that $E_i \in \Sigma^*(\Xi)$ if $i \notin I$. Then there exists a vertex $P \gg 0$ of Σ^* such that $\{P, E_i \mid i \notin I\}$ is a simplex of $\mathcal{K}$. Then $\Delta(P) \cap$

$\mathbb{R}^I = \Xi$ and P^I is primitive. Thus $P^I \in \mathcal{S}_I$. Assume that $P, Q \in \Sigma^*(\Xi)$. The cone of $\Sigma^*(\Xi)$ has $n - |I| + 1$ dimension. The cones spanned by $\{P, E_i \,|\, i \notin I\}$ and $\{Q, E_i \,|\, i \notin I\}$ have dimension $n - |I| + 1$ and thus they must contain an open subset in their intersection. This is only possible if $P = Q$. This proves the injectivity. Q.E.D.

4.5. Appendix

4.5.1. *Proof of Lemma* 4. Put $J = \{j \,|\, \alpha_j = 0\}$ and $I = \{1, \dots, n\} \setminus J$. As $\alpha \notin \pi^{-1}(\mathbf{0})$, this implies $\{E_j \,|\, j \in J\}$ are vertices of σ. Then we restrict the argument to $\mathbb{C}^I$ and f^I. Thus we may assume for simplicity that $\alpha_j \neq 0$ for $1 \leq j \leq n$. Then using the equality $u_{\sigma',1} = u_{\sigma,1} \prod_{j=2}^n u_{\sigma,j}^{b_{1,j}}$ and putting $\rho_1 = |u_{\sigma,1}| = \sqrt{x_{\sigma,1}^2 + y_{\sigma,1}^2}$, we get

$$\begin{aligned}
\frac{\partial \mathrm{dist}_{\mathrm{P},\sigma'}(\mathbf{u}_{\sigma'})}{\partial x_{\sigma,1}}\Big|_{\mathbf{u}'_\sigma=\alpha'} &= \frac{x_{\sigma,1}}{\rho_1}\rho(\alpha_{\sigma'}) \prod_{j=2}^n |\alpha_j|^{b_{1j}} + o(\rho_1), \\
\frac{\partial \mathrm{dist}_{\mathrm{P},\sigma'}(\mathbf{u}_{\sigma'})}{\partial y_{\sigma,1}}\Big|_{\mathbf{u}'_\sigma=\alpha'} &= \frac{y_{\sigma,1}}{\rho_1}\rho(\alpha_{\sigma'}) \prod_{j=2}^n |\alpha_j|^{b_{1j}} + o(\rho_1), \\
\frac{\partial \mathrm{dist}_{\mathrm{P},\sigma'}(\mathbf{u}_{\sigma'})}{\partial x_{\sigma,j}}\Big|_{\mathbf{u}'_\sigma=\alpha'} &= o(\rho_1), \\
\frac{\partial \mathrm{dist}_{\mathrm{P},\sigma'}(\mathbf{u}_{\sigma'})}{\partial y_{\sigma,j}}\Big|_{\mathbf{u}'_\sigma=\alpha'} &= o(\rho_1), \ 2 \leq j \leq n
\end{aligned}$$

where $\alpha_{\sigma'} = (\alpha'_1, \alpha'_2, \dots, \alpha'_n) = \pi_{\sigma'^{-1}\sigma}(\alpha)$ and $u_{\sigma,j} = x_{\sigma,j} + y_{\sigma,j}\, i$. Here $o(r_1)$ is by definition a smaller term than ρ_1 when $\rho_1 \to 0$. This implies that in $\mathbb{C}_\sigma^n$ with real coordinates $(x_{\sigma,1}, y_{\sigma,1}, \dots, x_{\sigma,n}, y_{\sigma\, n})$,

$$\begin{aligned}
&\mathrm{grad\,dist_P}(\alpha(\mathrm{t})) = (\beta\frac{\mathrm{a}_1}{|\alpha_1|}, \beta\frac{\mathrm{b}_1}{|\alpha_1|}, 0, \dots, 0) + \mathrm{O}(\mathrm{t}), \text{ namely} \\
&\mathrm{grad\,dist_P}(\alpha(\mathrm{t})) \overset{\mathrm{t}\to+0}{\longrightarrow} (\beta\frac{\mathrm{a}_1}{|\alpha_1|}, \beta\frac{\mathrm{b}_1}{|\alpha_1|}, 0, \dots, 0)
\end{aligned}$$

with $\alpha_1 = a_1 + b_1 i$,

$$\beta = \rho(\alpha_\sigma) + \sum_{\sigma' \in \mathcal{K}_P, \sigma' \neq \sigma} \rho(\alpha_{\sigma'}) \prod_{j=k+1}^n |\alpha'_j|^{b_{1j}} > 0.$$

This proves the assertion.

4.5.2. *Proof of Assertion* 6. Fix a point $\mathbf{u}_0 \in \partial\tilde{V}_\varepsilon(\tau')^{tr}$. Put $U_{1,m} = \prod_{i=1}^m u_i^{d(P_i)}$ and $u_1 = x_{\sigma 1} + iy_{\sigma 1}$. Recall that

$$\hat{f}_t(\mathbf{u}) = \tilde{f}_t(\mathbf{u}) \prod_{j=1}^m u_j^{d(P_j)}, \quad \tilde{f}_t(\mathbf{u}) = \tilde{f}_{\Delta'}(\mathbf{u}'') + R'(\mathbf{u}') + t\bar{R}(\mathbf{u}).$$

Recall that $R'(\mathbf{u}')$ does not contain $u_1, \dots, u_k$ and contained in the ideal generated by $(u_{k+1}, \dots, u_m)$ and $\bar{R}$ is contained in the ideal generated by $u_i u_j$, $i \le k < j \le m$. First note that

$$\begin{aligned}
\tfrac{\partial \hat{f}_t}{\partial x_{\sigma 1}} &= U_{1,m} J_{x_{\sigma 1}}(\mathbf{u}),\ \tfrac{\partial \hat{f}_t}{\partial y_{\sigma 1}} = U_{1,m} J_{y_{\sigma 1}}(\mathbf{u}),\\
\tfrac{\partial \hat{f}_t}{\partial r_i} &= U_{1,m} J_i(\mathbf{u}), \qquad i = k+1, \dots, m,\ g,\\
J_{x_{\sigma 1}}(\mathbf{u}) &= \tfrac{d(P_1)}{u_1}\left(\widetilde{f}_{\Delta'}(\mathbf{u}'') + R'(\mathbf{u}') + t\bar{R}(\mathbf{u})\right) + t\tfrac{\partial \bar{R}(\mathbf{u})}{\partial u_1},\\
J_{y_{\sigma 2}}(\mathbf{u}) &= i\tfrac{d(P_1)}{u_1}\left(\widetilde{f}_{\Delta'}(\mathbf{u}'') + R'(\mathbf{u}') + t\bar{R}(\mathbf{u})\right) + it\tfrac{\partial \bar{R}(\mathbf{u})}{\partial u_1},\\
J_i(\mathbf{u}) &= \tfrac{d(P_i)}{r_i}\left(\widetilde{f}_{\Delta'}(\mathbf{u}'') + R'(\mathbf{u}') + t\bar{R}(\mathbf{u})\right) + \left(\tfrac{\partial \bar{R}'(\mathbf{u}')}{\partial r_i} + t\tfrac{\partial \bar{R}(\mathbf{u})}{\partial r_i}\right),\\
&\qquad\qquad k+1 \le i \le m,\\
J_g(\mathbf{u}) &= 1 + t\tfrac{\partial \bar{R}(\mathbf{u})}{\partial r_g}.
\end{aligned}$$

Note that the respective orders of $\widetilde{f}_{\Delta'}(\mathbf{u}'')$ and r_j, $k+1 \le j \le m$ are $\sqrt{\varepsilon}$ and ε. The second term of J_i, $i \ne g$ is at most ε. Thus the main term of J_i is $\frac{d(P_i)\tilde{f}_{\Delta'}(\mathbf{u}'')}{r_i}$ and the order is $\frac{1}{\sqrt{\varepsilon}}$. The order of J_g is 1. Thus we observe that

$$\begin{aligned}
X_{\mathbb{C}} \quad &:= \left(\tfrac{\partial \hat{f}_t}{\partial x_{\sigma 1}}, \tfrac{\partial \hat{f}_t}{\partial y_{\sigma 1}}, \tfrac{\partial \hat{f}_t}{\partial r_{k+1}}, \dots, \tfrac{\partial \hat{f}_t}{\partial r_m}, \tfrac{\partial \hat{f}_t}{\partial r_g}\right)\\
&\overset{proj}{\sim} \left(\tfrac{d(P_1)}{u_1}, i\tfrac{d(P_1)}{u_1}, \tfrac{d(P_m)}{r_{k+1}} \dots, \tfrac{d(P_m)}{r_m}, 0\right).
\end{aligned}$$

Here $\mathbf{a} \sim \mathbf{b}$ implies $\lim_{\varepsilon \to 0} \mathbf{a}/\|\mathbf{a}\| = \lim_{\varepsilon \to 0} \mathbf{b}/\|\mathbf{b}\|$ in the projective space. $X_{\mathbb{C}}$ is a complex vector. As a real matrix, this corresponds to $2 \times (m-k+3)$ real matrix:

$$X_{\mathbb{R}} = \begin{pmatrix} \frac{\partial \Re \hat{f}_t}{\partial x_{\sigma 1}}, \frac{\partial \Re \hat{f}_t}{\partial y_{\sigma 1}}, \frac{\partial \Re \hat{f}_t}{\partial r_{k+1}}, \dots, \frac{\Re \partial \hat{f}_t}{\partial r_m}, \frac{\partial \Re \hat{f}_t}{\partial r_g} \\ \frac{\partial \Im \hat{f}_t}{\partial x_{\sigma 1}}, \frac{\partial \Im \hat{f}_t}{\partial y_{\sigma 1}}, \frac{\partial \Im \hat{f}_t}{\partial r_{k+1}}, \dots, \frac{\partial \Im \hat{f}_t}{\partial r_m}, \frac{\partial \Im \hat{f}_t}{\partial r_g} \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

by $X_{\mathbb{C}} = X_1 + iX_2$. Let us write the first 2×2-minor of this matrix:

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} \frac{\partial \Re \hat{f}_t}{\partial x_{\sigma 1}} & \frac{\partial \Re \hat{f}_t}{\partial y_{\sigma 1}} \\ \frac{\partial \Im \hat{f}_t}{\partial x_{\sigma 1}} & \frac{\partial \Im \hat{f}_t}{\partial y_{\sigma 1}} \end{pmatrix}$$

That is,

$$(x_{11} + ix_{21}, x_{12} + ix_{22}) = (\frac{\partial \hat{f}_t}{\partial x_{\sigma 1}}, \frac{\partial \hat{f}_t}{\partial y_{\sigma 1}}).$$

By Lemma 4, we see also that

$$\left(\frac{\partial \mathrm{dist}_{\mathrm{P_i}}}{\partial x_{\sigma 1}}, \frac{\partial \mathrm{dist}_{\mathrm{P_i}}}{\partial y_{\sigma 1}}, \frac{\partial \mathrm{dist}_{\mathrm{P_i}}}{\partial r_{k+1}}, \dots, \frac{\partial \mathrm{dist}_{\mathrm{P_i}}}{\partial r_m}, \frac{\partial \mathrm{dist}_{\mathrm{P_i}}}{\partial r_g}\right) \sim (0, \dots, 1, \dots, 0)$$

for any $k + 1 \le i \le m$.

Therefore the rank of Jacobian matrix is infinitesimally equivalent to the rank of

$$\frac{\partial(\Re \hat{f}_t, \Im \hat{f}_t, \mathrm{dist}_{\mathrm{P_{k+1}}}, \dots, \mathrm{dist}_{\mathrm{P_m}}, \mathrm{r_g})}{\partial(x_{\sigma 1}, y_{\sigma 1}, r_{k+1}, \dots, r_m, r_g)}$$

$$\overset{proj}{\sim} \begin{pmatrix} \Re(d(P_1)/u_1) & \Re(i\, d(P_1)/u_1) & * \\ \Im(d(P_1)/u_1) & \Im(i\, d(P_1)/u_1) & * \\ 0 & 0 & I_{m-k+1} \end{pmatrix}$$

in the projective sense for each row vectors. Put $d(P_1)/u_1 = \alpha + i\beta$. Then the upper left 2×2-matrix A is nothing but

$$A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

and $\det A = \alpha^2 + \beta^2 \neq 0$. This is enough to see that the rank of the Jacobian is $m - k + 3$. This proves the transversality.

4.5.3. *Cofinal homotopy equivalent sequence.* Let X be a manifold and let $\{E_j\}, \{E'_j\}, j \in \mathbb{N}$ be a decreasing sequence of submanifolds such that the inclusions $\iota_{j+1} : E_{j+1} \hookrightarrow E_j$, and $\iota'_{j+1} : E'_{j+1} \hookrightarrow E'_j$ are homotopy equivalences. Suppose further that

$$E'_{j+1} \subset E_{j+1} \subset E'_j \subset E_j.$$

Then the inclusions $\xi_{j+1} : E_{j+1} \hookrightarrow E'_j$ and $\xi'_{j+1} : E'_{j+1} \subset E_{j+1}$ are homotopy equivalences as $\xi_{j+1} \circ \xi'_{j+1} = \iota'_{j+1}$ and $\xi'_j \circ \xi_{j+1} = \iota_{j+1}$. Furthermore suppose that they are total spaces of fibrations over the same base space Z with the commutativity in the following diagram:

$$\begin{array}{ccccccc} E'_{j+1} & \hookrightarrow & E_{j+1} & \hookrightarrow & E'_j & \hookrightarrow & E_j \\ \downarrow p & & \downarrow q & & \downarrow p & & \downarrow q \\ Z & = & Z & = & Z & = & Z. \end{array}$$

Then the corresponding fibrations are also homotopy equivalent. We refer this argument as *a cofinal homotopy equivalent sequence argument.*

§5. Mixed functions

5.1. Main result

Now we are ready to state our main result. First we prepare:

Proposition 10. *Assume that $f(\mathbf{z},\bar{\mathbf{z}})$ is a convenient mixed function of strongly polar positive weighted homogeneous face type. Then for any weight vector P, f_P is also a strongly polar positive weighted homogeneous polynomial with weight P.*

Proof. The assertion is proved by the descending induction on $\dim \Delta(P)$. The assertion for the case $\dim \Delta(P) = n-1$ is the definition itself. Suppose that $\dim \Delta(P) = k$ and the assertion is true for faces with $\dim \Delta \geq k+1$. In the dual Newton diagram, P is contained in the interior of a cell, say Ξ, whose vertices $Q_1,\dots,Q_s$ satisfy $\dim \Delta(Q_j) \geq k+1$ for $j=1,\dots,s$. This implies P is a linear combination $\sum_j^s a_j\, Q_j$ with $a_j \geq 0$ and $\dim \Delta(Q_j) \geq k+1$. This also implies that $\Delta(P) = \cap_{j=1}^s \Delta(Q_j)$. Write $f_P(\mathbf{z},\bar{\mathbf{z}}) = \sum_k c_k \mathbf{z}^{\nu_k}\bar{\mathbf{z}}^{\mu_k}$. As f_{Q_j} is polar weighted homogeneous polynomial with weight Q_j, we have the equality: $\mathrm{pdeg}_{Q_j} \mathbf{z}^{\nu_k}\bar{\mathbf{z}}^{\mu_k} = m_j$ for $j=1,\dots,s$ where m_j a positive integer which is independent of k. This implies f_P is polar weighted homogeneous polynomial of weight P with polar degree $\sum_{j=1}^s a_j m_j > 0$. Q.E.D.

We take a regular convenient simplicial subdivision Σ^* of $\Gamma^*(f)$ (=regular fan) and we consider the toric modification $\pi : X \to \mathbb{C}^n$ with respect to Σ^* as in §2. Let $\mathcal{S}_I$ be as in §2. Now we can generalize the theorem of Varchenko for the mixed polynomial $f(\mathbf{z},\bar{\mathbf{z}})$ as follows.

Theorem 11. *Let $f(\mathbf{z},\bar{\mathbf{z}})$ a convenient non-degenerate mixed polynomial of strongly polar positive weighted homogeneous face type. Let $V = f^{-1}(V)$ be a germ of hypersurface at the origin and let $\widetilde{V}$ be the strict stransform of V to X. Then*
(1) $\widetilde{V}$ is topologically smooth and real analytic smooth variety outside of $\pi^{-1}(\mathbf{0})$.
(2) $\widetilde{V}(\tau)^*$ is a real analytic smooth mixed variety for any $\tau \in \mathcal{K}$.
(3) The zeta function of the Milnor fibration of $f(\mathbf{z},\bar{\mathbf{z}})$ is given by the formula

$$\zeta(t) = \prod_I \zeta_I(t),\ \zeta_I(t) = \prod_{P\in\mathcal{S}_I} (1 - t^{\mathrm{pdeg}(P,f_P^I)})^{-\chi(P)/\mathrm{pdeg}(P,f_P^I)}$$

Proof. The proof is essentially the same as the proof of Theorem 2, given in the previous section. We fix a toric modification $\pi : X \to \mathbb{C}^n$ associated with a regular simplicial subdivision Σ^* as in the holomorphic case. Take a stratum $S = \widetilde{V}(\tau)^*$ or $S = \widetilde{E}(\tau)^*$ with $\tau = (P_1, \dots, P_k)$ and put $\Xi = \cap_{i=1}^k \Delta(P_i; f)$. Let us take first a toric chart $\sigma = (P_1, \dots, P_n)$ with $\tau \prec \sigma$ with toric coordinates $\mathbf{u} = (u_{\sigma 1}, \dots, u_{\sigma n})$. For simplicity, we write $u_{\sigma j} = u_j$ hereafter. The pull-back $\pi_\sigma^* f$ takes the following form. Put $r_j = \mathrm{rdeg}_{P_j} f$ and $p_j = \mathrm{pdeg}_{P_j} f$ for $j = 1, \dots, k$.

$$\pi^* f = \textstyle\prod_{j=1}^k u_j^{\frac{r_j+p_j}{2}} \bar{u}_j^{\frac{r_j-p_j}{2}} \times \widetilde{f}(\mathbf{u}, \bar{\mathbf{u}}) \tag{8}$$

$$\widetilde{f}(\mathbf{u}) = \widetilde{f}_\Delta(\mathbf{u}', \bar{\mathbf{u}}') + \widetilde{R}(\mathbf{u}, \bar{\mathbf{u}}) \tag{9}$$

where $\mathbf{u}' := (u_{k+1}, \dots, u_n)$. The term $\widetilde{f}_\Delta(\mathbf{u}', \bar{\mathbf{u}}')$ is a mixed polynomial which does not contain the variables $u_1, \dots, u_k$ by the strong polar weightedness assumption. Namely we have

$$\pi_\sigma^* f_\Delta(\mathbf{u}, \bar{\mathbf{u}}) = \prod_{j=1}^k u_j^{\frac{r_j+p_j}{2}} \bar{u}_j^{\frac{r_j-p_j}{2}} \times \widetilde{f}_\Delta(\mathbf{u}', \bar{\mathbf{u}}').$$

We will first see that $\widetilde{V}(\tau)^*$ is real analytically non-singular and $\widetilde{V}$ is topologically non-singular on this stratum. First we assume that $R(\mathbf{u}, \bar{\mathbf{u}})$ is a continuous function such that the restriction of R to $\hat{E}(\tau)^*$ is zero for a while. Thus we see that $\widetilde{V}(\tau)^* = \{(\mathbf{0}, \mathbf{u}') \mid \widetilde{f}_\Delta(\mathbf{u}', \bar{\mathbf{u}}') = 0\}$. For any $\mathbf{x}' \in \widetilde{V}(\tau)^*$, put $\hat{\mathbf{x}} = (1, \dots, 1, \mathbf{x}') \in \mathbb{C}_\sigma^{*n}$. Then $\hat{\mathbf{x}} \in \pi_\sigma^* f_\Delta^{-1}(0) = \widetilde{f}_\Delta^{-1}(0)$ and by the non-degeneracy assumption of f on the face Δ, $\hat{\mathbf{x}}$ is a non-singular point. That is, there exists a j, $k+1 \le j \le n$, such that $\frac{\partial f_\Delta}{\partial u_j}(\hat{\mathbf{x}}) \neq 0$. This implies that $\widetilde{V}(\tau)^*$ is non-singular at $\mathbf{x}'$. Now we consider $u_1, \dots, u_k$ as parameters and by implicit function theorem, we can solve $\widetilde{f}(\mathbf{u}) = 0$ in u_j so that u_j is analytic in $\{u_i; i \neq j, k+1 \le i \le n\}$ and continuous in $u_1, \dots, u_k$. This implies that $\widetilde{V}$ is topological manifold. This proves the assertions (1), (2).

Now we consider the Milnor fibration. The second term $\widetilde{R}(\mathbf{u}, \bar{\mathbf{u}})$ in (9) is a linear combination of monomials of the type $u_1^{a_1} \bar{u}_1^{b_1} \cdots u_n^{a_n} \bar{u}_n^{b_n}$ with

$$a_i + b_i > 0,\ i = 1, \dots, k,$$
$$a_j, b_j \ge 0,\ k+1 \le j \le n.$$

Here a_i, b_i might be negative for $i \le k$. See Example 5.1.3. However the inequality $a_i + b_i > 0$ is enough to see the continuity of $\widetilde{R}(\mathbf{u}, \bar{\mathbf{u}})$ and $\lim_{u_{\sigma i} \to 0} \widetilde{R}(\mathbf{u}, \bar{\mathbf{u}}) = 0$ for any $1 \le i \le k$. See also the polar coordinate expression below for further detail. Thus the function $\widetilde{R}$ is a continuous blow-analytic function in the sense of Kuo [3]. We put $\widetilde{f}_t = \widetilde{f}_\Delta + t\widetilde{R}$ as before. We use the notations:

$$\hat{f}_t = \widetilde{f}_t \prod_{j=1}^{k} u_j^{\frac{r_j+p_j}{2}} \bar{u}_j^{\frac{r_j-p_j}{2}}, \ \hat{f}_\Delta = \widetilde{f}_\Delta \prod_{j=1}^{k} u_j^{\frac{r_j+p_j}{2}} \bar{u}_j^{\frac{r_j-p_j}{2}}.$$

To show that the Milnor fibration is well-defined for any $0 \le t \le 1$ by $\hat{f}_t$ over this stratum and it is isomorphic to that of $\hat{f}_0 = \hat{f}_\Delta$, we use *the polar toric coordinates.* So put $u_j = \rho_j e^{i\theta_j}$ for $j = 1, \dots, k$. By the strong polar weighted homogenuity, the function takes the following form

$$\text{(10)} \qquad \pi_\sigma^* f(\rho, \theta, \mathbf{u}') = \rho_1^{r_1} \dots \rho_k^{r_k} e^{ip_1\theta_1} \cdots e^{ip_k\theta_k} \widetilde{f}(\rho, \theta, \mathbf{u}')$$

$$\text{(11)} \qquad \widetilde{f}(\rho, \theta, \mathbf{u}') = \widetilde{f}_\Delta(\mathbf{u}') + R(\rho, \theta, \mathbf{u}') \qquad \text{where}$$

$$\text{(12)} \qquad \pi_\sigma^* f_\Delta(\rho, \theta, \mathbf{u}') = \rho_1^{r_1} \dots \rho_k^{r_k} e^{ip_1\theta_1} \cdots e^{ip_k\theta_k} \widetilde{f}_\Delta(\mathbf{u}')$$

and $\mathbf{u}' = (u_{k+1}, \dots, u_n)$, $\rho = (\rho_1, \dots, \rho_k)$ and $\theta = (\theta_1, \dots, \theta_k)$. We put

$$\hat{f}_t = \rho_1^{r_1} \dots \rho_k^{r_k} e^{ip_1\theta_1} \cdots e^{ip_k\theta_k} \widetilde{f}_t, \ \ \widetilde{f}_t := \widetilde{f}_\Delta + tR.$$

The reminder R is an analytic function of the variables $(\rho, \theta, \mathbf{u}')$ and contained in the ideal generated by $\rho_1, \dots, \rho_k$. This implies that $R \equiv 0$ on $\widetilde{V}(\tau)$. The tubular neighborhood $N(S)$ is defined by

$$N_\varepsilon(S) : \begin{cases} \mathrm{dist}_{\mathrm{P_j}}(\mathbf{u}) \le \varepsilon, \mathrm{j} = 1, \dots, \mathrm{k}, \quad \mathrm{S} = \widetilde{\mathrm{E}}(\tau)^* \\ \mathrm{dist}_{\mathrm{P_j}}(\mathbf{u}) \le \varepsilon, \mathrm{j} = 1, \dots, \mathrm{k}, \ |\widetilde{\mathrm{f}}_\Delta| \le \sqrt{\varepsilon}, \ \mathrm{S} = \widetilde{\mathrm{V}}(\tau)^*. \end{cases}$$

The Milnor fiber $F_{t,\delta}$ of $\hat{f}_t$ in this neighborhood is defined as:

$$\rho_1^{r_1} \cdots \rho_k^{r_k} e^{ip_1\theta_1} \cdots e^{ip_k\theta_k} (\widetilde{f}_\Delta(\mathbf{u}', \bar{\mathbf{u}}') + t\widetilde{R}(\rho, \theta, \mathbf{u}', \bar{\mathbf{u}}')) = \delta$$

where $\rho = (\rho_1, \dots, \rho_k)$ and $\theta = (\theta_1, \dots, \theta_k)$, $\rho_j \le \varepsilon$, $1 \le j \le k$ and $\delta \ll \varepsilon$.

5.1.1. *Smoothness.* First, we will show the smoothness of the Milnor fiber of $\hat{f}_t$. Take any $\mathbf{u}'_0 = (u_{0,k+1}, \dots, u_{0,n}) \in S$ and $\mathbf{u}_0 \in F_{t,\delta}$ with $\mathbf{u}_0 = (u_{10}, \dots, u_{k0}, \mathbf{u}'_0)$. By the non-degeneracy assumption and the strong-polar weighted homogenuity, the jacobian matrix

$$J_{>k}\widetilde{f}_\Delta := \frac{\partial(\Re f_\Delta, \Im f_\Delta)}{\partial(x_{k+1}, y_{k+1}, \dots, x_n, y_n)}$$

has rank two. Thus

$$J_{>k}(\hat{f}_t) = \rho_1^{r_1} \cdots \rho_k^{r_k} e^{ip_1\theta_1} \cdots e^{ip_k\theta_k} (J_{>k}(\widetilde{f}_\Delta) + tJ_{>k}(R))(\mathbf{u}_0)$$

and the second term of the right side is smaller than the first. Thus $J_{>k}(\hat{f}_t)$ has rank two over an open neighborhood $U(\mathbf{u}_0')$ if $\delta \ll \varepsilon$. As S_ε^{tr} is relatively compact, we can cover by a finite such open sets.

5.1.2. *Transversality.* We consider the transversality of $F_{t,\delta}$ and the boundary $\partial N_\varepsilon(S^{tr})$ at the intersection of $B := \partial N_\varepsilon(S^{tr}) \cap N_\varepsilon S'$ or $B' = \partial N_\varepsilon(S^{tr}) \cap \widetilde{V}_\varepsilon(S')$ where $S' = \widetilde{E}(\tau')^*$ with $\tau' = (P_1, \dots, P_k, \dots, P_m)$. We use the canonical toric chart $\mathbb{C}_\sigma^n$ of τ' and let $\mathbf{u} = (u_1, \dots, u_n)$ be the coordinates for simplicity. The boundaries are described as follows.

$$B = \{\mathbf{u} \,|\, \mathrm{dist_{P_i}}(\mathbf{u}) = \varepsilon,\ \mathrm{k}+1 \le \mathrm{i} \le \mathrm{m}\} \quad \text{or}$$
$$B' = \{\mathbf{u} \,|\, \mathrm{dist_{P_i}}(\mathbf{u}) = \varepsilon,\ \mathrm{k}+1 \le \mathrm{i} \le \mathrm{m},\ |\widetilde{\mathrm{f}}_{\Delta'}(\mathbf{u}'')| = \sqrt{\varepsilon}\}.$$

The argument is almost similar as that of the holomorphic case. So we consider the case B'. Thus we assume that $S' = V(\tau')$ is non-empty, *i.e.*, namely dim $\Delta' \ge 1$. Put $g = f_{\Delta'}(\mathbf{u}'')$ where $\mathbf{u}'' = (u_{m+1}, \dots, u_n)$. If $V(\tau') \ne \emptyset$, on a neighborhood of any point $\mathbf{u}_0'' \in B'$, by the non-degeneracy of f on Δ', g can be used as a member of a real analytic complex coordinate chart. For example, we may assume that there exists an open neighborhood $U(\mathbf{u}_0'')$ such that $(u_1, \dots, u_m, g, w_{m+2}, \dots, w_n)$ is a real analytic complex coordinate chart on $U(\mathbf{u}_0'')$. Here $(g, w_{m+2}, \dots, w_n)$ is real analytic complex coordinates of $U(\mathbf{u}_0'') \cap \hat{E}(\tau')$. See the next subsection for the definition. We use further the polar coordinates

$$u_j = \rho_j e^{i\theta_j},\ j = 1, \dots, m,\ \ g = \rho_g e^{i\theta_g},$$
$$\rho = (\rho_1, \dots, \rho_m), \theta = (\theta_1, \dots, \theta_m),\ \mathbf{w} = (w_{m+2}, \dots, w_{\sigma n}).$$

The expression of (6) is now written as follows.

$$(13)\quad \begin{cases} \hat{f}_t(\mathbf{u}) = \left(\prod_{j=1}^m \rho_j^{r_j} e^{ip_j\theta_j}\right) \widetilde{f}_t(\rho, \theta, \rho_g, \theta_g, \mathbf{w}), \\ \widetilde{f}_t(\mathbf{u}) = \rho_g e^{i\theta_g} + R'(\rho_g, \theta_g, \mathbf{w}) + t\bar{R}(\rho, \theta, \rho_g, \theta_g, \mathbf{w}), \\ R = \left(\prod_{j=1}^m \rho_j^{r_j} e^{ip_j\theta_j}\right) \bar{R}. \end{cases}$$

Note that R' and $\bar{R}$ are real analytic functions of variables $\rho_g, \theta_g, \mathbf{w}$ and $\rho, \theta, \rho_g, \theta_g, \mathbf{w}$ respectively and the restriction of R'. Note that $R', \bar{R}$ are not analytic function in the variables $u_1, \dots, u_m$. Here is the advantage of using polar coordinates. Theoretically this is equivalent to consider the situation on the polar modification along $u_i = 0$, $i = 1, \dots, m$ in the sense of [9].

For simplicity, we assume that $d(P_1) = r_1 > 0$. Put $u_1 = x_{\sigma 1} + iy_{\sigma 1}$ as before. Put

$$a := \tfrac{r_1 + p_1}{2}, \quad b := \tfrac{r_1 - p_1}{2},$$
$$U_{1,m} = \prod_{j=1}^m \rho_j^{r_j} e^{ip_j\theta_j} = u_1^a \bar{u}_1^b \prod_{j=2}^m \rho_j^{r_j} e^{ip_j\theta_j}.$$

We do the same discussion as in the case of holomorphic case. Consider the Jacobian of $\hat{f}_t$ with respect to variables $\{x_{\sigma 1}, y_{\sigma 1}, \rho_{k+1}, \dots, \rho_m, \rho_g\}$. Recall that $\bar{R}$ is a Laurent series in the variable $u_1, \bar{u}_1$ but it only contains monomials $u_1^{m_1}\bar{u}_1^{n_1}$ with $m_1+n_1 \geq 1$. Thus $|\frac{\partial \bar{R}}{\partial x_{\sigma 1}}|$ and $|\frac{\partial \bar{R}}{\partial y_{\sigma 1}}|$ is bounded from above when $|u_1|$ is small enough. Thus as a complex vector,

$$(\frac{\partial \hat{f}_t}{\partial x_{\sigma 1}}, \frac{\partial \hat{f}_t}{\partial y_{\sigma 1}}, \frac{\partial \hat{f}_t}{\partial \rho_{k+1}}, \dots, \frac{\partial \hat{f}_t}{\partial \rho_g}) \overset{proj}{\sim} (\frac{a}{u_1}+\frac{b}{\bar{u}_1}, i\frac{a}{u_1}-i\frac{b}{\bar{u}_1}, \frac{r_{k+1}}{\rho_{k+1}}, \dots, \frac{r_m}{\rho_m}, 0).$$

Put $1/u_1 = \alpha + \beta i$. Then the 2×2 real matrix A of the first two coefficients (as in the holomorphic case corresponding) is

$$A = \begin{pmatrix} \alpha(a+b) & -\beta(a+b) \\ \beta(a-b) & \alpha(a-b) \end{pmatrix}$$

and we see that $\det A = (a^2 - b^2)(\alpha^2 + \beta^2) \neq 0$ as $a - b = p_1 > 0$, by the positive polar weightedness. We consider the Jacobian matrix of $\{\Re \hat{f}_t, \Im \hat{f}_t, \mathrm{dist}_{\mathrm{P_{k+1}}}, \dots, \mathrm{dist}_{\mathrm{P_m}}, |\mathrm{g}|\}$ in the variable

$$\{x_{\sigma 1}, y_{\sigma 1}, \rho_{k+1}, \dots, \rho_m, \rho_g\}.$$

Under projective equivalence for each gradient vector, we get

$$\frac{\Re\partial \hat{f}_t, \Im\partial \hat{f}_t, \mathrm{dist}_{\mathrm{P_{k+1}}}, \dots, \mathrm{dist}_{\mathrm{P_m}}, |\mathrm{g}|)}{\partial(x_{\sigma 1}, y_{\sigma 1}, \rho_{k+1}, \dots, \rho_m, \rho_g)} \overset{proj}{\sim} \begin{pmatrix} A & * \\ 0 & I_{m-k+1} \end{pmatrix}.$$

This matrix has rank $m - k + 3$ as is expected. Thus the transversality follows.

The proof of Theorem 11 is now given by the exact same argument as that of Theorem 2. By Key Lemma below, the contribution to zeta function is only from the strata $\widetilde{E}(\tau)^{tr}$ where $\tau = (P_1, \dots, P_k)$ is a simplex of maximal face type. Q.E.D.

5.1.3. *Example.* Consider the mixed function

$$f(\mathbf{z}, \bar{\mathbf{z}}) = z_1^3\bar{z}_1 + z_2^3\bar{z}_2 + z_2^5.$$

f is a non-degenerate mixed function of strongly polar weighted homogeneous type. Then an ordinary blowing up $\pi : X \to \mathbb{C}^2$ is the associated toric modification. Let us see in the chart $\sigma = (P, E_2)$ with $P = {}^t(1,1)$. Let (u, v) be the toric chart. Then we have

$$\pi_\sigma^* f(u,v) = u^3\bar{u}\left(v^3\bar{v} + 1 + \frac{u^2v^5}{\bar{u}}\right)$$

and $R = u^2v^5/\bar{u}$. We see R is a continuous blow-analytic function but not C^1 in u.

5.2. Key lemma

Let $f_1, \dots, f_n$ be complex valued real analytic functions in an open set $U \subset \mathbb{C}^n$. We say that $(f_1, \dots, f_n)$ are *real analytic complex coordinates* if $(\Re f_1, \Im f_1, \dots, \Re f_n, \Im f_n)$ are real analytic coordinates of U.

Lemma 12. *Let $f(\mathbf{g}, \bar{\mathbf{g}}) = g_1^{r_1} \bar{g}_1^{p_1} \cdots g_\ell^{r_\ell} \bar{g}_\ell^{p_\ell}$. Assume that*

$$(g_1, \dots, g_\ell)$$

are locally real analytic complex coordinates of $(\mathbb{C}^\ell, \mathbf{0})$ and $r_j \neq p_j$ for each $j = 1, \dots, \ell$. Let $\hat{q}_j = r_j + p_j$, $q_j = r_j - p_j$ and put $q_0 =$ $\gcd(q_1, \dots, q_\ell)$. Then the Milnor fibration of f exists at the origin and the Milnor fiber F is homotopic to q_0 disjoint copies of $(S^1)^{\ell-1}$ and the zeta function is given by

$$\zeta(t) = \begin{cases} (1 - t^{q_1}), & \ell = 1 \\ 1, & \ell \geq 2. \end{cases}$$

Proof. Let $q_j = r_j - p_j$ and consider the linear C^* action $(t, \mathbf{g}) \mapsto (tg_1, \dots, tg_\ell)$. Then f can be understood as a polar homogeneous polynomial in $g_1, \dots, g_\ell$. For the Milnor fibration, we can use the polydisk B'_ε which is defined by

$$B'_\varepsilon := \{(g_1, \dots, g_\ell) \,|\, |g_j| \leq \varepsilon,\, j = 1, \dots, \ell\},$$

as $\{B'_\varepsilon; \varepsilon > 0\}$ is a homotopy equivalent cofinal neighborhood system of the point $\mathbf{0}$. Note that B'_ε is diffeomorphic to the usual complex polydisk

$$\{(z_1, \dots, z_\ell) \in \mathbb{C}^\ell \,|\, |z_j| \leq \varepsilon,\, j = 1, \dots, \ell\}.$$

Using this action and polydisk B'_ε, the Milnor fibration can be identified with

$$f : B'_\varepsilon \cap f^{-1}(D^*_\delta) \to D^*_\delta,\ 0 \neq \delta \ll \varepsilon$$

where $D_\delta = \{\rho \in \mathbb{C} \,|\, |\rho| \leq \eta\}$ and $D^*_\delta = D_\delta \setminus \{0\}$. Put $g_j = r_j e^{i\theta_j}$, $j = 1, \dots, \ell$. The Milnor fiber is given by $F = f^{-1}(\delta) \cap B'_\varepsilon = \amalg_{j=1}^{p_0} F_j$ (disjoint union) and

$$\begin{aligned} F_j =& \{(g_1, \dots, g_\ell) \,|\, r_1^{\hat{q}_1} \cdots r_\ell^{\hat{q}_\ell} = \delta,\, (\theta_1 q_1 + \cdots + \theta_\ell q_\ell)/q_0 = 2\pi + \frac{2j}{q_0}\pi\} \\ &\simeq (S^1)^{\ell-1},\, j = 1, \dots, q_0. \end{aligned}$$

The monodromy map is given by the periodic map

$$h : F \to F, \quad (g_1, \dots, g_\ell) \mapsto (g_1 e^{2\pi i/q_0}, \dots, g_\ell e^{2\pi i/q_0}).$$

Thus we can see easily $F \simeq (S^1)^{\ell-1}$ and the zeta function is trivial (Theorem 9.6, [4]). Q.E.D.

5.3. An application and examples

5.3.1. *Holomorphic principal part.* Consider a holomorphic function $g(\mathbf{z})$ which is convenient and non-degenerate. Let $R(\mathbf{z},\bar{\mathbf{z}})$ be a mixed analytic function such that $\Gamma(R)$ is strictly above $\Gamma(g)$, *i.e.* $\Gamma(R) \subset \mathrm{Int}(\Gamma_+(\mathrm{f}))$. Consider a convenient non-degenerate mixed function

$$f(\mathbf{z},\bar{\mathbf{z}}) = g(\mathbf{z}) + R(\mathbf{z},\bar{\mathbf{z}}).$$

Then the Milnor fibration is determined by the principal part and therefore isomorphic to that of $g_\Gamma(\mathbf{z})$. Thus the Milnor fibration does not change by adding high order mixed monomials above the Newton boundary. The proof can be given by showing the existence of uniform radius for the Milnor fibration of $f_t := g + t\,R$, using a Curve Selection Lemma ([4, 2]). We just copy the argument in [9] for a family f_t, $0 \le t \le 1$.

5.3.2. *Mixed covering.* Consider a convenient non-degenerate mixed analytic function $f(\mathbf{z},\bar{\mathbf{z}})$ of strongly polar weighted homogeneous face type. Consider a pair of positive integers $a > b \ge 0$ and consider the covering mapping

$$\varphi: \quad \mathbb{C}^n \to \mathbb{C}^n, \quad \varphi(\mathbf{w}) = (w_1^a \bar{w}_1^b, \dots, w_n^a \bar{w}_n^b).$$

Put $g(\mathbf{w},\bar{\mathbf{w}}) = f(\varphi(\mathbf{w},\bar{\mathbf{w}})) = f(w_1^a\bar{w}_1^b, \dots, w_n^a\bar{w}_n^b)$. Consider a face function with respect to P, $f_P(\mathbf{z},\bar{\mathbf{z}})$ with $P = (p_1,\dots,p_n)$. It is a strongly polar positive weighted homogeneous polynomial with weight vector P. Thus it is a linear combination of monomials $\mathbf{z}^\nu\bar{\mathbf{z}}^\mu$ which satisfies the equalities:

$$\sum_{i=1}^n p_i(\nu_i + \mu_i) = \mathrm{rdeg}\, f \qquad \sum_{i=1}^n p_i(\nu_i - \mu_i) = \mathrm{pdeg}\, f.$$

Then we can observe that $g_P(\mathbf{w},\bar{\mathbf{w}}) = f_P(\varphi(\mathbf{w}))$ is a linear combination of mixed monomials $\mathbf{w}^{a\nu+b\mu}\bar{\mathbf{w}}^{a\mu+b\nu}$. Thus we have

$$\begin{aligned}\mathrm{rdeg}_P \mathbf{w}^{a\nu+b\mu}\bar{\mathbf{w}}^{a\mu+b\nu} &= (a+b)\mathrm{rdeg}\, f,\\ \mathrm{pdeg}_P \mathbf{w}^{a\nu+b\mu}\bar{\mathbf{w}}^{a\mu+b\nu} &= (a-b)\mathrm{pdeg}\, f\end{aligned}$$

which implies that g_P is a strongly polar weighted homogeneous polynomial of the same weight P with radial degree $\mathrm{rdeg}\, f \times (a+b)$ and polar degree $\mathrm{pdeg}\, f(\varphi(\mathbf{w})) = d(P,f)(a-b)$. g_P is also non-degenerate as $\varphi : \mathbb{C}^{*n} \to \mathbb{C}^{*n}$ is an unbranched covering of degree $(a-b)^n$. Therefore the dual Newton diagram of g is the same as that of f.

Let $\pi : X \to \mathbb{C}^n$ be an admissible toric modification. We use the same notation as in the previous section. For each $P \in \mathcal{S}_I$, consider the mapping $\varphi^I := \varphi|_{\mathbb{C}^I}$ and its restriction to $F(g_P^I)$:

$$\begin{array}{ccc} \mathbb{C}^{*I} & \xrightarrow{\varphi^I} & \mathbb{C}^{*I} \\ \uparrow \iota & & \uparrow \iota \\ F^*(g_P^I) & \xrightarrow{\varphi_P^I} & F^*(f_P^I). \end{array}$$

Here the toric Milnor fibers are defined by:

$$F^*(g_P^I) = \{\mathbf{w} \in \mathbb{C}^{*I} \,|\, g_P^I(\mathbf{w}^I, \bar{\mathbf{w}}^I) = 1\},$$
$$F^*(f_P^I) = \{\mathbf{w} \,|\, f_P^I(\mathbf{z}^I) = 1\}.$$

Put $\chi(P,f) = \chi(F^*(f_P^I))$ and $\chi(P,g) = \chi(F^*(g_P^I))$. As φ_P^I is a $(a-b)^{|I|}$-fold covering, we have

Proposition 13.

$$\chi(P,g) = (a-b)^{|I|}\chi(P,f), \tag{14}$$

$$\begin{cases} \mathrm{rdeg}_P g = (a+b)\mathrm{rdeg}_P\, f, \\ \mathrm{pdeg}_P\, g_P = (a-b)\mathrm{pdeg}_P\, f_P. \end{cases} \tag{15}$$

From this observation, we have

Theorem 14. *Let $g(\mathbf{w},\bar{\mathbf{w}}) = f(\varphi(\mathbf{w},\bar{\mathbf{w}}))$ and assume that $f(\mathbf{z},\bar{\mathbf{z}})$ is convenient non-degenerate mixed function of strongly polar positive weighted homogeneous face type. Then $g(\mathbf{w},\bar{\mathbf{w}}) = f(\varphi(\mathbf{w},\bar{\mathbf{w}}))$ is a non-degenerate mixed function of strongly polar positive weighted face type. The zeta functions of the Milnor fiberings of f and g are given by*

$$\zeta_f(t) = \prod_I \zeta_{f,I}(t),\ \zeta_{f,I} = \prod_{P\in\mathcal{S}_I} (1 - t^{\mathrm{pdeg}_P f_P})^{\chi(P,f)/\mathrm{pdeg}_P f_P}$$
$$\zeta_g(t) = \prod_I \zeta_{g,I}(t),\ \zeta_{g,I}(t) = \prod_{P\in\mathcal{S}_I} (1 - t^{\mathrm{pdeg}_P g_P})^{\chi(P,g)/\mathrm{pdeg}_P g_P}.$$

Furthermore $\zeta_g(t)$ is determined by that of $\zeta_f(t)$ using the above Proposition 13.

5.3.3. *Case $a-b=1$.* We assume that $a-b=1$. Then Theorem 14 says that $\zeta_f(t) = \zeta_g(t)$. In this case, $\varphi : \mathbb{C}^{*n} \to \mathbb{C}^{*n}$ is a homeomorphism which extends homeomorphically to $\mathbb{C}^n \to \mathbb{C}^n$. This suggests the following.

Corollary 15. *Assume $a-b=1$ in the situation of Theorem* 14.

(1) *The Milnor fibrations of $f(\mathbf{z},\bar{\mathbf{z}})$ and $g(\mathbf{w},\bar{\mathbf{w}})$ are homotopically equivalent and the links are homotopic and their zeta functions coincide.*

(2) *If in addition, $f = f_P$ is a strongly polar weighted homogeneous polynomial, the Milnor fibrations f, g are topologically equivalent and the links are homeomorphic.*

Proof. First, by Theorem 33 ([9]), f and g have stable radius for their Milnor fibrations (in the first presentation). Take a common stable radius $r_0 > 0$. For $r_0 \geq r > 0$, put

$$\begin{aligned} &B_r^* = B_r^{2n} \setminus \{\mathbf{0}\} = \{\mathbf{z} \in \mathbb{C}^n \mid \|\mathbf{z}\| \leq r, \, \mathbf{z} \neq \mathbf{0}\}, \\ &K_r(f) = f^{-1}(0) \cap S_r^{2n-1}, \, K_r(g) = g^{-1}(0) \cap S_r^{2n-1}, \\ &V_r^*(f) = f^{-1}(0) \cap B_r^*, \, V_r^*(g) = g^{-1}(0) \cap B_r^*. \end{aligned}$$

Then the following fibrations are obviously equivalent to the respective Milnor fibrations:

$$\begin{cases} f/|f| : \; B_r^* \setminus V_r^*(f) \to S^1, \\ g/|g| : \; B_r^* \setminus V_r^*(g) \to S^1. \end{cases}$$

Our homeomorphism φ preserves the values of g and f. For any $r \leq r_0$, we can find a decreasing sequence of positive real numbers $r_i < r_0, i = 1, 2, \dots$ so that $\varphi(B_{r_{2i}}^*) \supset B_{r_{2i+1}}^*$ and

$$\varphi((B_{r_{2i}}^*, V_r^*(g))) \quad \supset \quad (B_{r_{2i+1}}^*, V_{r_{2i+1}}^*(f)) \quad \supset \quad \varphi((B_{r_{2i+2}}^*, V_{r_{2i+2}}^*(g))).$$

By the cofinal homotopy equivalence sequence argument,

$$\varphi : (B_{r_{2i}}^*, V_{r_{2i}}^*(g)) \to (B_{r_{2i-1}}^*, V_{r_{2i-1},0}^*(f))$$

is a homotopy equivalence.

Assume now that $f = f_P$ is a strongly polar weighted homogeneous polynomial. Recall that we have $\mathbb{C}^*$-action defined by $t \circ \mathbf{w} = (w_1 t^{p_1}, \dots, w_n t^{p_n})$ and $t \circ \mathbf{z} = (z_1 t^{p_1}, \dots, z_n t^{p_n})$. Put $d = \mathrm{pdeg}_P f = \mathrm{pdeg}_P g$. The monodromy mapping of the global Milnor fibrations are given by

$$\begin{aligned} h_g : F(g) \to F(g), \quad & \mathbf{w} \mapsto e^{i/d} \circ \mathbf{w}, \\ h_f : F(f) \to F(f), \quad & \mathbf{z} \mapsto e^{i/d} \circ \mathbf{w}, \end{aligned}$$

where $F(g) = g^{-1}(1) \subset \mathbb{C}^n$ and $F(f) = f^{-1}(1) \subset \mathbb{C}^n$. As is easily observed, we get the equality $\varphi(h_g(\mathbf{w})) = h_f(\varphi(\mathbf{w}))$. More precisely the equivalence of the global fibration follows from the diagram:

$$\begin{array}{ccc} \mathbb{C}^n \setminus g^{-1}(0) & \xrightarrow{\varphi} & \mathbb{C}^n \setminus f^{-1}(0) \\ \downarrow g & & \downarrow f \\ \mathbb{C}^* & = & \mathbb{C}^* \end{array}$$

and φ is S^1-equivalent. Now we consider the links on the unit spheres

$$K_f = \{\mathbf{z} \in \mathbb{C}^n \mid \|\mathbf{z}\| = 1,\ f(\mathbf{z}, \bar{\mathbf{z}}) = 0\},$$
$$K_g = \{\mathbf{w} \in \mathbb{C}^n \mid \|\mathbf{w}\| = 1,\ g(\mathbf{w}, \bar{\mathbf{w}}) = 0\}.$$

The mappining φ does not keep the norm. So we need only normalize it. This follows from the following observation for $\mathbb{R}^+$-actions:

$$\varphi(t \circ \mathbf{w}) = t^{2b+1} \circ \varphi(\mathbf{w}),\ t \in \mathbb{R}^+.$$

This equality implies that φ maps a $\mathbb{R}^+$-orbit to a $\mathbb{R}^+$-orbit. The hypersurfaces $f^{-1}(0)$ and $g^{-1}(0)$ are invariant under this action. For any non-zero $\mathbf{z}$, along the orbit $\mathbf{z}(t) := t \circ \mathbf{z}$. $\|\mathbf{z}(t)\|$ is monotone increasing for $0 < t < \infty$. Let us define the normalization map $\psi : \mathbb{C}^n \setminus \{\mathbf{0}\} \to S^{2n-1}$ by $\psi(\mathbf{z}) = \mathbf{z}(\tau)$ where τ is the unique positive real number so that $\|\mathbf{z}(\tau)\| = 1$. We define the homeomorphism $\widetilde{\varphi} : K_g \to K_f$ by $\widetilde{\varphi}(\mathbf{w}) = \psi(\varphi(\mathbf{w}))$. This proves the second assertion. Q.E.D.

5.3.4. *Holomorphic case.* The most interesting case is when $f(\mathbf{z})$ is a non-degenerate holomorphic function. Obviously $f(\mathbf{z})$ is of strongly polar weighted homogeneous face type.

Theorem 16. *Let $f(\mathbf{z})$ be a convenient non-degenerate holomorphic function and let $g(\mathbf{w}, \bar{\mathbf{w}}) = f(\varphi(\mathbf{w}, \bar{\mathbf{w}}))$, $\varphi(\mathbf{w}, \bar{\mathbf{w}}) = (w_1^a \bar{w}_1^b, \dots, w_n^a \bar{w}_n^b)$. Then $(\mathbf{w}, \bar{\mathbf{w}})$ is a non-degenerate mixed function of strongly polar positive weighted homogeneous face type.*
If $a - b = 1$, the Milnor fibration of g is homotopically equivalent to that of $f(\mathbf{z})$.

The mixed functions $g(\mathbf{w}, \bar{\mathbf{w}})$ obtained from non-degenerate holomorphic functions through the pull-back by a mixed covering φ give many interesting examples of non-degenerate mixed functions of strongly polar weighted homogeneous face type.

5.4. Examples

(1) Consider the following strongly polar homogeneous polynomial (Example 59, [9])

$$g_t(\mathbf{w}, \bar{\mathbf{w}}) = -2w_1^2\bar{w}_1 + w_2^2\bar{w}_2 + tw_1^2\bar{w}_2.$$

For $t = 0$, g_0 is pull-back of $f(\mathbf{z}) = -2z_1 + z_2$ by the mixed covering mapping $\varphi : \mathbb{C}^2 \to \mathbb{C}^2$ with $\varphi(\mathbf{w}) = (w_1^2\bar{w}_1, w_2^2\bar{w}_2)$. Thus g_0 is equivalent to the trivial knot $f = 0$ and F is contractible and $\zeta(t) = (1-t)$ by Theorem 14. On the other hand, for $t > 1$, we know that g_t is non-degenerate and $\chi(F^*) = -3$, $\chi(F) = -1$ and $\zeta(t) = (1-t)^{-1}$. This shows that there are mixed functions of strongly polar weighted homogeneous face type which are not the pull-back of holomorphic functions.

(2) Consider the moduli space $\mathcal{M}(P, pab, (p+2r)ab)$ of convenient non-degenerate strongly polar weighted homogeneous polynomials

$$f(\mathbf{z}, \bar{\mathbf{z}})$$

of two variables $\mathbf{z} = (z_1, z_2)$ with weight $P = (a, b)$, $\gcd(a, b) = 1$ and $\mathrm{pdeg}_P f = pab$, $\mathrm{rdeg}_P f = (p+2r)ab$. Here r is a positive integer. Put $h_s(\mathbf{z}, \bar{\mathbf{z}}) = |z_1|^{2bs} + |z_2|^{2as}$. Consider the polynomials

$$f_s(\mathbf{z}, \bar{\mathbf{z}}) = h_{r-s}(\mathbf{z}, \bar{\mathbf{z}}) \prod_{j=1}^{p} (z_1^b - \alpha_j z_2^a) \prod_{k=1}^{s} (z_1^b - \beta_k z_2^a)(\bar{z}_1^b - \bar{\gamma}_k \bar{z}_2^a)$$

where $0 \le s \le r$ and $\alpha_1, \dots, \alpha_p, \beta_1, \gamma_1, \dots, \beta_s, \gamma_s$ are mutually distinct non-zero complex numbers. As $V(f_s) \subset \mathbb{P}^1$ consists of $p + 2s$ points, we have $\chi(F^*) = -p(p+2s)$ and $\zeta_{f_s}(t) = (1 - t^{pab})^{p+2s-2}$ (See [8]).

For any $n = 2 + m$, we can consider a join type strongly polar weighted homogeneous polynomial of $m+2$ variables $\mathbf{z}, \mathbf{w}$ with $\mathbf{w} \in \mathbb{C}^m$:

$$F_s(\mathbf{z}, \mathbf{w}, \bar{\mathbf{z}}, \bar{\mathbf{w}}) = f_s(\mathbf{z}, \bar{\mathbf{z}}) + w_1^{p+r} \bar{w}_1^r + \cdots + w_n^{p+r} \bar{w}_n^r.$$

Then by join theorem ([1]), the Milnor fiber of F_s, $s = 0, \dots, r$ have different topology. In fact, the Milnor fiber is homotopic to a bouquet of spheres and the Milnor number is $(p-1)^m (p^2 + (2s-2)p + 1)$. Thus the topology of mixed polynomials is not combinatorial invariant.

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